Math 312
Worksheet for October 13

Jacob Harkins
jah6863@psu.edu

May 20, 2024

Exercise 1) (a) Let $\{a_n\}$ be a sequence of positive numbers. Use the $\epsilon - N$ definition of limit to prove that "If $a_n \to 3$, then $\sqrt{a_n} \to \sqrt{3}$.

First prove $|\sqrt{a_n} - \sqrt{3}| < |a_n - 3|$

$$|\sqrt{a_n} - \sqrt{3} * \frac{\sqrt{a_n} + \sqrt{3}}{\sqrt{a_n} + \sqrt{3}}| = \frac{a_n - 3}{\sqrt{a_n} + \sqrt{3}} < |a_n - 3|$$

Let $\epsilon > 0$

$$\begin{aligned} &\forall \epsilon > 0, \exists N \text{ such that } \forall n > N, |a_n - 3| < \epsilon \\ \implies &\epsilon > |a_N - 3| > |\sqrt{a_N} - \sqrt{3}| \\ \implies &|\sqrt{a_n} - \sqrt{3}| \to 0 \,\forall n > N \end{aligned}$$

(b) Generlize the result from part (a). That is: Let $\{a_n\}$ be a sequence of positive numbers. Use the $\epsilon - N$ definition of limit to prove that "If $a_n \to L > 0$, then $\sqrt{a_n} \to \sqrt{L}$."

Case 1: $\sqrt{a_n} + \sqrt{L} > 1$

$$|\sqrt{a_n} - \sqrt{L} * \frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}} = \frac{a_n - L}{\sqrt{a_n} + \sqrt{L}} < |a_n - L|$$

Let $\epsilon > 0$

$$\begin{aligned} &\forall \epsilon > 0, \exists N \text{ such that } \forall n > N, |a_n - L| < \epsilon \\ \implies &\epsilon > |a_N - L| > |\sqrt{a_N} - \sqrt{L}| \\ \implies &|\sqrt{a_n} - \sqrt{L}| \to 0 \,\forall n > N \end{aligned}$$

Case 2: $\sqrt{a_n} + \sqrt{L} \leq 1$: ?

Exercise 2) Start with $k = 1$. Why does there exist infinitely many terms of the sequence for which $|x_n - x_0| < \frac{1}{10}$? Pick one and call it x_{n_1}. Then keep increasing k to find other elements of the subsequence.
Proof by Mathematical Induction:

Base Case: $k = 1$

$$\begin{aligned} &|x_{n_1} - x_0| < \frac{1}{10} \\ &\exists N \forall n > N\, |x_n - x_0| < \frac{1}{10} \\ \implies &n_1 = N \end{aligned}$$

So it holds for $k = 1$.

Assume this holds for some n_k.Does this work for n_{k+1}?, lets find out!

$$\text{By definition of limit we know that for } \epsilon = \frac{1}{10^{k+1}} \ \exists N.$$
$$\text{Let } n_{k+1} = max(N, n_{k+1})$$

Therefore, there exists an N such that $|n_{k+1} - x_0| < \frac{1}{10^{k+1}}$.

By The Principal of Mathematical Induction this holds for all $k > 0$. $\square$

Exercise 3) Consider the sequence of the form

$$x_n = \frac{an^p + 10n - 3}{n^{10} + 2n^3}$$

where a and p are any two real numbers. For each of the following, find values of a and p for which the given phenomenon occurs.

(a) The sequence diverges to $+\infty$

$$p = 11$$
$$a = 1$$

(b) The sequence converges to 100

$$p = 10$$
$$a = 100$$

(c) The sequence converges to 0

$$p = 10$$
$$a = 0$$

(d) The sequence diverges to $-\infty$

$$p = 11$$
$$a = -1$$

Exercise 4) Suppose $I = a, b$ be a closed interval with $a \neq b$, and let $A = \{t = |x| \big| x \in I\}$, be the set of absolute values of the elements of I. The numbers in I can be negative, the numbers in A are never negative. For each of the cases below, either give an example, or show that no such example exists.

(a) $\max A = |a|$ and $\min A = |b|$

$$I = [-1, -2]$$
$$A = [2, 10]$$
$$a = -10, b = -2$$
$$maxA = 10$$
$$minA = 2$$

(b) $\max A = |b|$ and $\min A = |a|$

$$\begin{aligned} I &= [0, 5] \\ A &= [0, 5] \\ a &= 0, b = 5 \\ maxA &= 5 \\ minA &= 0 \end{aligned}$$

(c) $\max A = |a|$ and $\min A \neq |b|$

$$\begin{aligned} I &= [-10, 5] \\ A &= [0, 10] \\ a &= -10, b = 5 \\ maxA &= 10 \\ minA &= 0 \end{aligned}$$

(d) $\max A = |b|$ and $\min A \neq |a|$

$$\begin{aligned} I &= [-1, 5] \\ A &= [0, 5] \\ a &= -1, b = 5 \\ maxA &= 5 \\ minA &= 0 \end{aligned}$$

(e) $\max A \neq |a|, |b|$ Proof By Cases:

Let $x \in I$

Case 1: $|a| > |b|$

$$\begin{aligned} &\implies a \leq x \leq b \\ &\implies |a| \geq |x| \geq |b| \\ &\implies \max A = |a| \end{aligned}$$

Case 2: $|b| > |a|$

$$\begin{aligned} &\implies a \leq x \leq b \\ &\implies |b| \geq |x| \geq |a| \\ &\implies \max A = |b| \end{aligned}$$

Case 1: $|a| = |b|$

$$\begin{aligned} &\implies a \leq x \leq b \\ &\implies |x| \leq |a| = |b| \\ &\implies \max A = |a| \wedge \max A = |b| \end{aligned}$$

$\square$