Math 312
Worksheet for September 8

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May 20, 2024

Exercise 1) Write the definition for the statement "The set S is not bounded below" where S is a subset of the real numbers.

The set S is not bounded below if there does not exist a real number, m, such that $m \leq s$ for all $s \in S$.

Exercise 2) Show that if S is bounded and $f(x) = ax + b$, then f is bounded on S.

Let u be an upper bound of S and l be a lower bound of S.
Let $c \in S$ such that $l \leq c \leq u$
case 1: $a \geq 0$

$$\begin{aligned} &l \leq c \leq u \\ &la \leq ca \leq ua \\ &la + b \leq ca + b \leq ua + b \end{aligned}$$

Therefore, the function f is bounded on S when $a \geq 0$.
case 2: $a < 0$

$$\begin{aligned} &l \leq c \leq u \\ &la \geq ca \geq ua \\ &la + b \geq ca + b \geq ua + b \end{aligned}$$

Therefore, the function f is bounded on S when $a < 0$. $\square$

Exercise 3)

(a) Let $S = [-5, 2], T = (0, 7)$. Find the sets $S \cup T$ and $S \cap T$. Prove that they are bounded by finding their least upper bounds and greatest lower bounds.

$$\begin{aligned} &S \cup T = [-5, 7) \\ &S \cap T = (0, 2] \\ &\text{GLB } = -5 \\ &\text{LUB } = 7 \end{aligned}$$

(b) Let $S = (-1, 2].T = [0, 3]$. Find the sets $S \cup T$ and $S \cap T$. Prove that they are bounded by finding their least upper bounds and greatest lower bounds.

$$\begin{aligned} &S \cup T = (-1, 3] \\ &S \cap T = [0, 2] \\ &\text{GLB } = -1 \\ &\text{LUB } = 3 \end{aligned}$$

(c) Let $S = (a, b), T = (c, d)$ be two intervals such that $a < c < b < d$. Find the sets $S \cup T$ and $S \cap T$. Prove that they are bounded by finding their least upper bounds and greatest lower bounds.

$$\begin{aligned} S \cup T &= (a, d) \\ S \cap T &= (c, b) \\ \text{GLB } &= a \\ \text{LUB } &= d \end{aligned}$$

Exercise 4) Let $S \subset \mathbb{R}$ be bounded (above) and let $\beta = \sup S$. Prove that S is not bounded away from β.

Let U be the set of all upper bounds of S.

Assume for contradiction some $x < \beta$ is an upper bound of S.

By definition of supremum,

$$\begin{aligned} \beta = &\ \min U \\ \implies &\ \beta \leq x \end{aligned}$$

This forms a contradiction since $x < \beta$ and $\beta \leq x$. $\square$
