Math 312
Worksheet for September 15

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Exercise 1) Let S be a nonempty subset of the real numbers that has a greatest lower bound α and a least upper bound β. Let T be the set defined as $T = -S = \{x = -s \text{ for } s \in S\}$. Make conjectures about $\delta = \text{infT}$ and $\gamma = \text{supT}$ using the information about S. Then prove that your conjectures are correct.

Conjectures:

$$\delta = -\beta$$
$$\gamma = -\alpha$$

Proof:

$\delta = -\beta$

$$\begin{aligned} &\text{Let } t \in T \\ &-t \in S \\ \implies &-t < \beta \\ &\implies t > -\beta = \delta \end{aligned}$$

Therefore, δ is a lower bound of T.

Assume some bound $\delta + d$ is a lower bound of T.

$$\begin{aligned} \forall t \in T, &\beta + d < t \\ &\beta - d > -t \end{aligned}$$

Which implies $\beta - d$ is an upper bound of S which contradicts that β is the supremum of S. Therefore, δ is the infimum of T.

$\gamma = -\alpha$

$$\begin{aligned} &\text{Let } t \in T \\ &-t \in S \\ \implies &-t > \alpha \\ &\implies t < -\alpha = \gamma \end{aligned}$$

Therefore, γ is an upper bound of T.

Assume some bound $\gamma - d$ is an upper bound of T.

$$\begin{aligned} \forall t \in T, &\alpha - d > t \\ &\alpha + d < -t \end{aligned}$$

Which implies $\alpha + d$ is a lower bound of S which contradicts that α is the infimum of S. Therefore, γ is the supremum of T. $\square$

Exercise 2) Consider the sets $S = \{x \in \mathbb{R} | x > 0$ and $sin(\frac{1}{x}) = 0\}$ and $T = \{x \in \mathbb{R} | e^x = cos(x)$. What kind of bounds do they have?

set S:

Hypothesis: sup$S = \frac{1}{\pi}$, inf$S = 0$

Proof:

$sin(\frac{1}{x}) = 0$ when $x = \frac{1}{t\pi}$ with $t > 0 \wedge t \in \mathbb{Z}$

Let $f(t) = \frac{1}{t\pi}$, with $f(t)$ being a strictly decreasing for $t > 0$. Upper Bound:

$$f(1) = \frac{1}{\pi}$$

Since 1 is the lowest value of t, $\frac{1}{\pi}$ is the LUB.

Lower bound:

Note that $0 < f(t)$ for all $t > 0$, and hence is a lower bound.

Assume for contradiction $\epsilon > 0$ is a lower bound.

Let $a, b \in \mathbb{Z} > 0$ such that $\epsilon \geq \frac{a}{b}$.

$$\frac{a}{b} > \frac{1}{b}$$
$$\implies \frac{a}{b} > \frac{1}{b\pi} \text{ with } \frac{1}{b\pi} \in S$$

Therefore ϵ is not a lower bound.

Hence 0 is the GLB.

Set T:

Hypothesis: sup$S = 0$, inf$S$ DNE

Proof:

$0 \leq e^x \leq 1$ and $-1 \leq cos(x) \leq 1$ when $x \leq 0$.

Upper Bound:

Since e^x strictly increases at $x \geq 0$, and $e^0 = 1 \wedge cos(0) = 1$, 0 is the LUB.

Lower Bound:

Assume for contradiction $x < 0$ is the lower bound of T.

$$2\pi\lceil x \rceil - \frac{\pi}{2} \leq 2\pi\lceil x \rceil \leq x$$

Note that $cos(2\pi\lceil x \rceil - \frac{\pi}{2}) = 0$ and $cos(2\pi\lceil x \rceil) = 1$, and that $0 \leq e^x \leq 1$ when $x < 0$.

Hence, by the intermediate value theorem there is a value n with $2\pi\lceil x \rceil - \frac{\pi}{2} \leq n \leq 2\pi\lceil x \rceil$ such that $cos(n) = e^n$.

Therefore, x is not a lower bound, so T is not bounded below. $\square$

Exercise 3) Consider the sequence $\{g_n\}$ defined by $g_1 = 1$ and $g_{n+1} = 1 + \frac{1}{g_n}$ for $n \geq 1$.

(b) Prove by induction that $g_n \leq 2$.

Proof by Math Induction.

Hypothesis: $1 \leq g_n \leq 2$ for $n \geq 1$.

Base Case: $n = 1$

$$g_1 = 1$$

It holds true for $n = 1$.

Assume it holds true for some number, $n \geq 1$, does it hold for $n + 1$?

$$g_{n+1} = 1 + \frac{1}{g_n}$$

First we will find the bound of $\frac{1}{g_n}$.

Consider the function $f(x) = \frac{1}{x}$.

$$f'(x) = -\frac{1}{x^2}$$

Note there are no inflection points between 1 and 2, the function is strictly decreasing in this range. Therefore,

$$\begin{aligned} &f(2) \leq f(g_n) \leq f(1) \\ \implies &\frac{1}{2} \leq \frac{1}{g_n} \leq 1 \\ \implies &1 + \frac{1}{2} \leq 1 + \frac{1}{g_n} \leq 2 \\ \implies &1 \leq g_{n+1} \leq 2 \end{aligned}$$

Therefore it holds for $n + 1$, so by THE PRINCIPAL OF MATHEMATICAL INDUCTION it holds for all $n \geq 1$. $\square$

(c) Observe (with technology) that this sequence appears to hop back and forth across the number $\phi = \frac{(1+\sqrt{5})}{2} \approx 1.618$, the golden ratio. Prove algebraically that:

(i) if $g_n > \phi$, then $g_{n+1} < \phi$.

$$\begin{aligned} g_n &> \frac{1+\sqrt{5}}{2} \\ \frac{2}{1+\sqrt{5}} &> \frac{1}{g_n} \\ \frac{1-\sqrt{5}}{1-\sqrt{5}} * \frac{2}{1+\sqrt{5}} &> \frac{1}{g_n} \\ 1 + \frac{1-\sqrt{5}}{2} &> 1 + \frac{1}{g_n} \\ \frac{1+\sqrt{5}}{2} &> g_{n+1} \end{aligned}$$

(ii) if $g_n < \phi$, then $g_{n+1} > \phi$.

$$\begin{aligned}
g_n &< \frac{1+\sqrt{5}}{2} \\
\frac{2}{1+\sqrt{5}} &< \frac{1}{g_n} \\
\frac{1-\sqrt{5}}{1-\sqrt{5}} * \frac{2}{1+\sqrt{5}} &< \frac{1}{g_n} \\
1 + \frac{1-\sqrt{5}}{2} &< 1 + \frac{1}{g_n} \\
\frac{1+\sqrt{5}}{2} &< g_{n+1}
\end{aligned}$$