# Math 312
# Worksheet for November 20

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**Exercise 1)** Explain why the function $g$ in Example 8, page 142, satisfies the squeezing inequality

$$-|2x| \leq g(x) \leq |2x|$$

Use this to give another proof that $\lim_{x\to 0} g(x) = 0$.

This works because $g(x)$ is bounded by the inequality $|g(x)| \leq |2x| \; \forall x$.

Proof:

Let $\delta = \frac{\epsilon}{2}$.

$$|x - 0| < \delta = \frac{\epsilon}{2}$$
$$= |2x| < \epsilon$$

Therefore the limit is 0.

If $x$ is negative, $|\frac{x}{2}| \leq |2x|$, so the limit from the left must equal 0 by the squeeze theorem.

If $x$ is positive, $|2x\sin(\frac{1}{x})| \leq |2x|$, so the limit from the right must equal 0 by the sequeeze theorem.

Therefore, since the lim from the left is equal to the lim from the right, the limit must be 0. □

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**Exercise 2)** Let $f$ be the function whose graph consists of the two line segments joining (0,0), (1,1), (2,-1). Show that $f$ is continuous at $x = 0$ and at $x = 1$.

Equation of the function:

$$F(x) = \begin{cases} x & 0 \leq 1 \\ -2x+3 & 1 < x \leq 2 \end{cases} \tag{1}$$

Since we have shown in class that the function $f(x) = x$ is continuous at all points, and the point $(0,0)$ is defined by this function and in our bounds, $f$ is continuous at $x = 0$.

Proof for $x = 1$. We want to show that both the functions $f(x) = x$ and $f(x) = -2x+3$ exist and have the same limit at $x = 1$.

$f(x) = x$:

Let $\delta = \epsilon$.

$$|x - 1| < \delta = \epsilon$$

So the limit is 1, and the function equals 1 at $x = 1$.

$f(x) = -2x+3$:

Let $\delta = \frac{\epsilon}{2}$.

$$|x - 1| < \delta = \frac{\epsilon}{2}$$
$$= |2x - 2| = |-2x + 2| = |-2x + 3 - 1| < \epsilon$$

So the limit is 1, and the function equals 1 at $x = 1$.

Therefore, the function $F(x)$ is continous at 1. □

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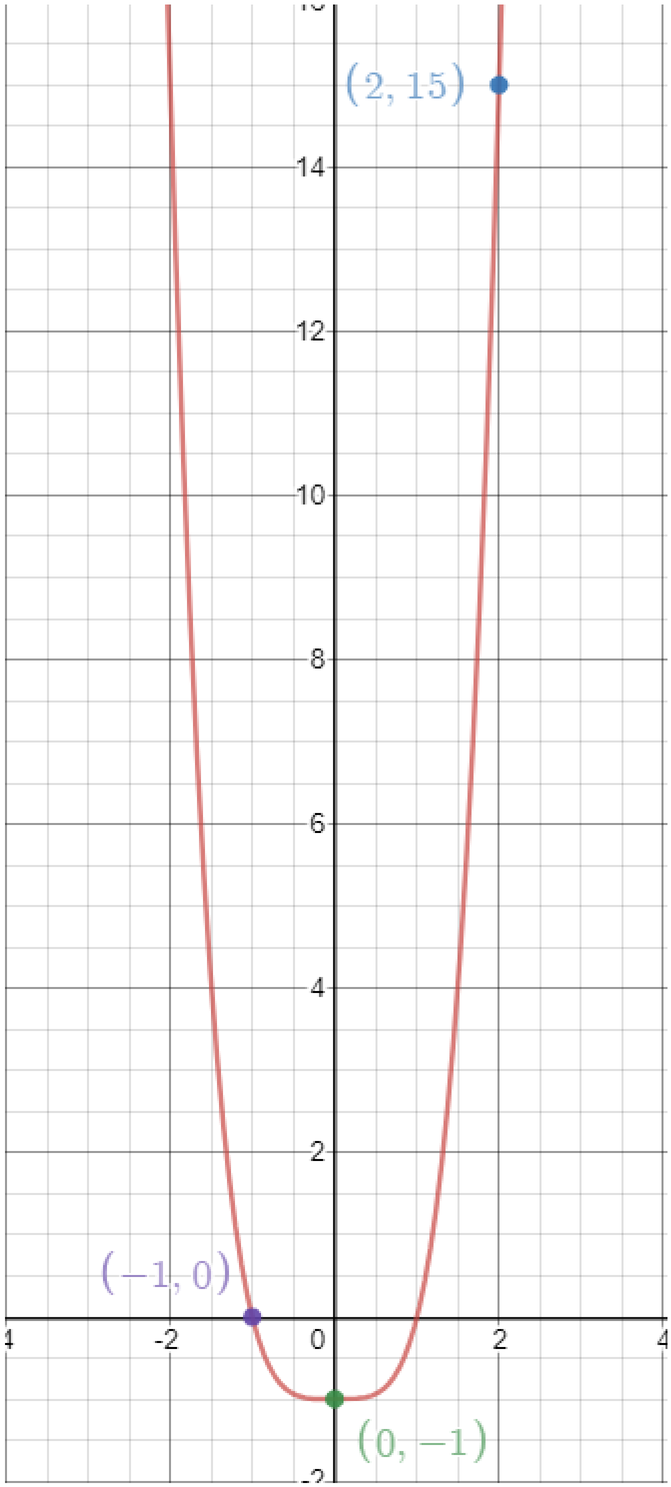
**Exercise 3)** Does the function $f(x) = x^4 - 1$ achieve maximum and/or minimum values on the interval (-1,2)? If the answer is "yes" find them. If it is "no", explain why. Does the Extreme Value Theorem apply in this case? A detailed graph can be used as supporting evidence.

Yes, at 0 $f = -1$ which is a min, and at 2 $f = 15$ which is a local max because of the boundary. The Extreme Value Theorem applies because on the interval $(-1,2)$ $f(0) < f(a)\ \forall a \neq 0$ and $f(2) > f(b)\ \forall b \neq 2$. This can be seen in the graph below.




Disregard the values from $x < -1$ and $x > 2$ as I did not know how to hide them on the calculator but do not effect the answer.

**Exercise 4)** Let $I$ be an open interval containing zero, and $f : I \to \mathbb{R}$ any function that is bounded on $I$. Define a new function, $g : I \to \mathbb{R}$ by $g(x) = xf(x)$.

(a) Show that $g$ is continuous at $x = 0$.

Let $\delta = \frac{\epsilon}{M}$ and recall that $f(x) < M$ for some $M \geq 0 \in \mathbb{R}$, $\forall x \in I$ since it is bounded.

$$\frac{\epsilon}{M} > |x - 0|$$
$$\Longrightarrow \epsilon > |xM| > |xf(x)| = |xf(x) - x0|$$

Therefore, the function is continuous at $x = 0$. □

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**Exercise 5)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x)$ is a rational number for every real number input $x$. Show that $f$ must be a constant function.

Assume for contradiction there exists such a function $f(x)$, that is not constant.
Let $a$ and $b$ be two real numbers with $a < b$ and $f(a) \neq f(b)$.
Let $\alpha = \min(f(a), f(b))$ and $\beta = \max(f(a), f(b))$.
Let $C$ be equal to $\frac{\beta - \alpha)\pi}{4} + \alpha$.
Note that $C$ is irrational and $\alpha < C < \beta$.
By The Intermediate Value Theorem there exists a $c$ such that $f(c) = C$, and $a < c < b$.
Therefore, this forms a contradiction because $f(x)$ must take on irrational values. So $f(x)$ must be a constant function. □

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**Exercise BONUS** The Intermediate Value Theorem's statement from class reads: "Let $f$ be a continuous function on a closed interval $[a, b]$, with $f(a) \not\to f(b)$. Let $u$ be any intermediate value between $f(a)$ and $f(b)$.

Then there exists a number $c$ in $[a, b]$ such that $f(c) = u$."

The proof seen in class used the set $S = \{x \in [a, b] | f(x) < u\}$, showed that $S$ has a sup(let $\beta = \sup S$), and then concluded that $f(\beta) = u$.

The goal of this exercise is to provide a different (but structurally similar) proof, using the set $T = \{x \in [a, b] | f(x) > u\}$. Prove that, because of the properties of $T$, there exists $\alpha = \inf T$, and then prove that $f(\alpha) = u$.

**Proof**

Note that $T$ is not empty, because $f(b) > u$,
Since the sequence is continuous, $T$ has a sup, $\beta$, and an inf, $\alpha$.
There is a sequence $\{x_n\}$ which converges to $\alpha$.
Since $f$ is continuous, the sequence $\{y_n = f(x_n)\}$ converges to $f(\alpha)$.
$\forall n.\ f(x_n) > u$, so $f(\alpha) \geq u$.

Assume that $f(\alpha) > u$.
$\Longrightarrow$ the existence of an interval around $\alpha$, so that $f(\alpha \pm \delta) > u$.
This would mean every value in the interval would be in our original set, $S$, and form a contradiction because $\alpha = \inf T$.
Therefore $\alpha$ must be equal to $u$. □

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