Math 312
Worksheet for November 11

Jacob Harkins
jah6863@psu.edu

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Exercise 1) Use the $\epsilon - \delta$ definition of continuity to prove the function $f(x) = \frac{x-2}{x-5}$ is continuous at $a = -2$.

First find δ.

Let $|\delta| < 1$

$$|\frac{x-2}{x-5} - \frac{4}{7}| = |\frac{7x - 14 - 4x + 20}{7x - 35}|$$
$$= |\frac{3x+6}{7x-35}| = |\frac{3}{7} \cdot \frac{x+2}{x-5}|$$

Since $|\delta| < 1$, $x - 5 > 0$

$$\implies |\frac{3}{7}\frac{x+2}{x-5}| < |x+2| < \epsilon$$

So $\delta = \min(\epsilon, 1)$

Proof:

Let $\epsilon > 0$, $\delta = \min(\epsilon, 1)$

Case 1: $\epsilon < 1$

$$0 < |x+2| < \delta = \epsilon$$
$$\implies \epsilon > |x+2| > |\frac{x+2}{x-5}| = |\frac{x-2}{x-5} - \frac{4}{7}| > 0$$

Case 2: $\epsilon \geq 1$

$$0 < |x+2| < \delta = 1$$
$$\implies 1 > |x+2| > |\frac{x+2}{x-5}| = |\frac{x-2}{x-5} - \frac{4}{7}| = |f(x) - f(a)|$$

Therefore it holds true for both cases, and the limit $f(x)$ as $x \to -2 = f(-2)$. So the function is continuous at -2. □

Exercise 2) Use the $\epsilon - \delta$ definition of continuity to prove that the function $f(x) = 7x^2 + 5x - 14$ is continuous for all real numbers a.

First find δ.

Let $|x+a|<1$

$$
\begin{aligned}
|\frac{f(x)-f(a)}{x-a}| &= |\frac{7x^2+5x-14-7a^2-5a+14}{x-a}| \\
&= |\frac{7x^2+5x-7a^2-5a}{x-a}| = |\frac{7(x+a)(x-a)+5(x-a)}{x-a}| \\
&= |7(x-a+2a)+5| < |12+14a| = C
\end{aligned}
$$

$$\implies \delta = \frac{\epsilon}{12+14a}$$

Proof:
Let $\epsilon > 0$, $\delta = \min(\frac{\epsilon}{12+14a}, 1)$.
Case 1: $\frac{\epsilon}{12} \leq 1$

$$
\begin{aligned}
& 0 < |x-a| < \delta = \frac{\epsilon}{12+14a} \\
\implies & \frac{\epsilon}{12+14a} > |x-a| \\
& \epsilon > |(12+14a)(x-a)| \\
& \quad = |(7(1+2a)+5)(x-a)| > |(7(x-a+2a)+5)(x-a)| \\
& \quad = |(7(x+a)+5)(x-a)| = |7x^2-7a^2+5x-5a-14+14| = |f(x)-f(a)|
\end{aligned}
$$

Case 2: $\frac{\epsilon}{12} > 1$
$\implies \delta = 1$

$$
\begin{aligned}
& 0 < |x-a| < \delta = 1 < \frac{\epsilon}{12+14a} \\
\implies & \frac{\epsilon}{12+14a} > |x-a| \\
& \epsilon > |(12+14a)(x-a)| \\
& \quad = |(7(1+2a)+5)(x-a)| > |(7(x-a+2a)+5)(x-a)| \\
& \quad = |(7(x+a)+5)(x-a)| = |7x^2-7a^2+5x-5a-14+14| = |f(x)-f(a)|
\end{aligned}
$$

Exercise 3) Use the $\epsilon - \delta$ definition of continuity to prove that if f and g are continuous at a, and k, s are any two nonzero real numbers then the function $h = kf + sg$ defined as $h(x) = kf(x) + sg(x)$ is also continuous at a.

Let $\epsilon > 0$ such that $\epsilon_1 = \frac{\epsilon}{2k}$ and $\epsilon_2 = \frac{\epsilon}{2s}$. Since $f(x)$ and $g(x)$ are continuous, there exists a δ_1 and δ_2 such that $|x-a| < \delta_1 \implies |f(x)-f(a)| < \epsilon_1$ and $|x-a| < \delta_2 \implies |g(x)-g(a)| < \epsilon_2$.

Therefore let $\delta = \min(\delta_1, \delta_2)$. Note this δ satisfies both of the above equations.

Let $|x-a| < \delta$

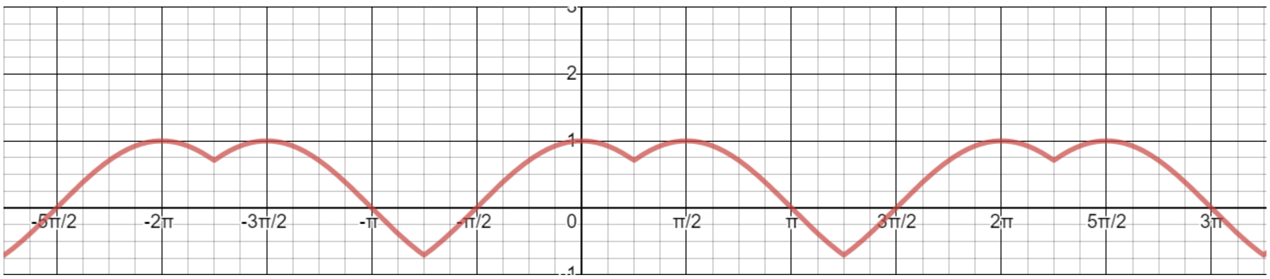
$$
\begin{aligned}
& |h(x)-h(a)| \\
& = |kf(x)+sg(x)-kf(a)-sg(a)| \\
& \leq |kf(x)-kf(a)| + |sg(x)-sg(a)| \\
& \leq k|f(x)-f(a)| + s|g(x)-g(a)| < k\frac{\epsilon}{2k} + s\frac{\epsilon}{2s} = \epsilon
\end{aligned}
$$

Therefore the function is continuous. □

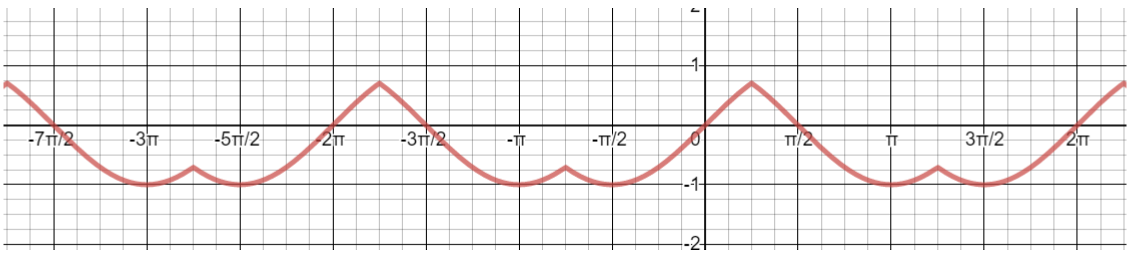
Exercise 4) If $f : I \to R$ and $g : I \to R$ are functions defined on I, then we can form new functions $\max\{f,g\}$ and $\min\{f,g\}$ in the "obvious" way: $\max\{f,g\}(x) = \max\{f(x), g(x)\}$ $\forall x \in I$, and similarly for $\min\{f,g\}(x)$

(a) Sketch graphs of $\max\{f, g\}$ and $\min\{f, g\}$ if $f(x) = sin(x)$ and $g(x) = cos(x)$ on $I = [-2\pi, 2\pi]$.

Max:



Min:



(b) Show that $\max\{f(x), g(x)\} = \frac{|f(x)-g(x)|+f(x)+g(x)}{2}$. Give the formula for the min (no need to prove this one).

Proof by cases:

Case 1: $f > g$

$$\begin{aligned} \frac{|f(x) - g(x)| + f(x) + g(x)}{2} &= \frac{f(x) - g(x) + f(x) + g(x)}{2} \\ &= \frac{2f(x)}{2} = f(x) \end{aligned}$$

Case 2: $g > f$

$$\begin{aligned} \frac{|f(x) - g(x)| + f(x) + g(x)}{2} &= \frac{g(x) - f(x) + f(x) + g(x)}{2} \\ &= \frac{2g(x)}{2} = g(x) \end{aligned}$$

Case 3: $g = f$

$$\begin{aligned} \frac{|f(x) - f(x)| + f(x) + f(x)}{2} &= \frac{f(x) - f(x) + f(x) + f(x)}{2} \\ &= \frac{2f(x)}{2} = f(x) \end{aligned}$$

The minimum function is $\frac{g(x)+f(x)-|f(x)-g(x)|}{2}$

(c) Show that if f and g are continuous on I, then so are $\max\{f, g\}$ and $\min\{f, g\}$. Don't use the $\epsilon - \delta$ definition of continuity.(Hint: Show that if $g : I \to R$ is continuous on I, then so is $h(x) = |g(x)|$.)

First show that if $g : I \to R$ is continous on I, then so is $h(x) = |g(x)|$.

Proof: note that $|x| = \sqrt{x^2}$

$g(x) * g(x)$ is continuous and strictly non-negative.

$\sqrt{g(x) * g(x)}$ is continuous and well defined at all points.

Therefore, $|g(x)| = \sqrt{g(x)^2}$ is continuous

Now prove original statement.

Since $f(x)$ and $g(x)$ are continuous, $f(x) - g(x)$ is continuous.

$\implies |f(x) - g(x)|$ is continuous

$\implies |f(x) - g(x)| + f(x) + g(x)$ is continuous.

and since continuity is preserved over multiplication by constants,

$\frac{|f(x)-g(x)|+f(x)+g(x)}{2}$ is continuous.

Similarly for min,

Since $f(x)$ and $g(x)$ are continuous, $f(x) - g(x)$ is continuous.

$\implies |f(x) - g(x)|$ is continuous

$\implies -|f(x) - g(x)| + f(x) + g(x)$ is continuous.

and since continuity is preserved over multiplication by constants,

$\frac{-|f(x)-g(x)|+f(x)+g(x)}{2}$ is continuous.

$\square$