Math 312
Worksheet for December 10

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Exercise 1) Prove the following statement. Let f be a differentiable function on an open interval I on which $f'(x) \neq 0$. The function f is increasing on I if and only if $f'(x) > 0$ on I.

Proof: For the $\longrightarrow$ direction, if f is increasing on I, then $f'(x) > 0$ on I.

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

If $t > x$, $f(t) > f(x)$, so $f(t) - f(x) > 0$ while $t - x > 0$. So the fraction is positive.

If $t < x$, $f(t) < f(x)$, so $f(t) - f(x) < 0$ while $t - x < 0$. So the fraction is positive.

Therefore $f'(x) = \lim_{t \to x} \frac{f(t)-f(x)}{t-x} \geq 0$, but by hypothesis $f'(x) \neq 0$, so $f'(x) > 0$.

For the $\longleftarrow$ direction: If $f'(x) > 0$, then f is increasing.

Since $f'(x) > 0$ and $f'(x) = \lim_{t \to x} \frac{f(t)-f(x)}{t-x}$, the fraction is positive for all values of t and x on I.

Case 1: Assume $f(t) - f(x) > 0$ while $t - x > 0$.

$\implies t > x,\ f(t) > f(x)$. So f is increasing.

Case 2: Assume $f(t) - f(x) < 0$ while $t - x < 0$.

$\implies t < x,\ f(t) < f(x)$. So f is increasing.

Exercise 2) Using the limit definition of derivative, given $f(x) = \cos(x)$, find $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \cos(x)}{h} - \frac{\sin(x)\sin(h)}{h} \\
&= \lim_{h \to 0} \cos(x)\frac{\cos(h) - 1}{h} - \frac{\sin(x)\sin(h)}{h}
\end{aligned}$$

Using $\lim_{h \to 0} \frac{\cos(h)-1}{h} = 0$ and $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$:

$$\begin{aligned}
&= \lim_{h \to 0} \cos(x)\frac{\cos(h) - 1}{h} - \frac{\sin(x)\sin(h)}{h} \\
&= \cos(x) * 0 - \sin(x) = -\sin(x)
\end{aligned}$$

Exercise 3) consider the function

$$f(x) = \begin{cases} (x+\alpha)^2 & x < 0 \\ x^2 + \alpha & x \geq 0 \end{cases} \tag{1}$$

(a) Find the values of α for which the function is continuous at $a = 0$.

$\alpha = 1, 0$

$f_0(x) = \begin{cases} (x)^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$

$f_1(x) = \begin{cases} (x+1)^2 & x < 0 \\ x^2 + 1 & x \geq 0 \end{cases}$

(b) Are the continuous functions obtained in part (a) differentiable at $a = 0$? Explain how the conclusion is reached. Nicely prepared graphs can help the explanation.

Yes

$\lim_{x\to 0} f_0(x) = 0$

$\lim_{x\to 0} f_1(x) = 1$

Exercise 4) Let f, g, h be three functions that are differentiable on the whole real line.

(a) Use the product rule for two functions (not the limit definition of derivative) to prove that

$$(f(x)g(x)h(x))' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

.

$$\begin{aligned} f(x) * (g(x)h(x)) &= f'(x) * (g(x)h(x)) + f(x) * (g'(x)h(x) + g(x)h'(x)) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \end{aligned}$$

(b) State a "three deep chain rule" for the composition of three functions.

$$\big(f(g(h(x))\big)' =$$

$$f'(g(h(x)))g'(h(x))h'(x)$$

(c) State a formula for the second derivative of the composition of two functions.

$$\big(f(g(x)\big)" =$$

$$\begin{aligned} f' &= f'(g(x))g'(x) \\ f" &= f"(g(x))g'(x)^2 + g"(x)f'(g(x)) \end{aligned}$$

Exercise 5) Given the function $f(x) = 3x^2 + 5x + 7$ on the interval [1,7], find the value c whose existence is stated in the Mean Value Theorem. Is it unique? Explain.

$$\begin{aligned} f' &= 6x + 5 \\ f'(c) &= 6c + 5 = \frac{f(b) - f(a)}{b - a} = 29 \\ c &= 4 \end{aligned}$$

It is unique because the algebra required is unique.

Exercise 6 Show that the function $f(x) = x^3 - 3x + k$ cannot have two zeros in $[0, \frac{1}{2}]$ for any value of k.

Assume for contradiction there is a k such that $f(x)$ has two zeros on the interval $[0, \frac{1}{2}]$

$$\begin{aligned} f' =& 3x^2 - 3 \\ f" =& 6x \end{aligned}$$

Since there is only one inflection point on the interval $[0, \frac{1}{2}]$, there can only be 1 zero.

Therefore, the function can not have two zeros for any value of k on the interval $[0, \frac{1}{2}]$.

Exercise 7 Let $f : \mathbb{R} \to \mathbb{R}$ have continuous derivatives $f', f"$ and suppose that $f(0) = 0, f(1) = 1, f(2) = 2$. Prove that $f"(x_0) = 0$ for at least one x_0 in $(0, 2)$.

Proof: Let $g(x) = f(x) - x$.

$$\begin{aligned} g'(x) &= f'(x) - 1 \\ g"(x) &= f"(x) - 0 \\ g(0) &= 0 \\ g(2) &= 0 \end{aligned}$$

By Rolle's Theorem, $g"(x) = 0$ at some point on the interval $[0, 2]$. Since $g"(x) = f"(x)$, $f"(x)$ must equal 0 at some point on the interval $[0, 2]$.
