Math 312
Worksheet for October 20

Jacob Harkins
jah6863@psu.edu

May 20, 2024

Exercise 1) Let $\{a_n\}$ be a sequence of integer numbers. Prove that if this sequence has a limit (it converges), then its terms become eventually constant (from a certain index N up).

$$\text{Let } \epsilon = \frac{1}{10000}$$
$$\text{since } \lim a_n \to L, \exists N \text{ such that}$$
$$|a_n - L| < \frac{1}{10000} \;\; \forall n > N$$
$$\implies a_n = L \text{ (since the difference of two integer numbers is an integer)}$$

Therefore, when $n > N$, all the terms are constant. □

Exercise 2) (a) Let $\{x_n\}$ be a bounded sequence with $\inf x_n = \alpha$ and $\sup x_n = \beta$. Assume that $\alpha \notin \{x_n\}$, and $\beta \notin \{x_n\}$. Prove that the sequence diverges.

Assume the sequence converges.

Let $\epsilon < \frac{\beta-\alpha}{4}$, assume $x_n \to L$

$\implies N$ such that $\forall n > N$, $|x_n - L| < \epsilon$

Let S be the set $\{x_n\}\ \forall n \leq N$

Note that S is finite,

$\implies S$ has a max/min

$\implies$ without loss of generality, either $|\beta - L| > \epsilon \ \vee \ |\alpha - L| > \epsilon$

There exists a point, $x_t < \beta \ \wedge x_t > L + \epsilon$

Because S is finite, the supremum, β, is an element of S.

This contradicts the original statement,

Therefore by contradiction, the sequence diverges.

(b) Using the result from part (a) explain how one knows that the sequence $\{x_n = \arctan(-1)^n n\}_1^\infty$ diverges without a lot of calculations.

The function can be written as

$$-.78^n n$$

the supremum $= \infty$ and infinum $= -\infty$

since the sup, inf $\notin$ the set, the sequence diverges.

Exercise 3) Consider the sequence $\{x_n\}$ defined by $x_n = (-1)^n \frac{n}{n+1}$.

(a) Find the sub sequences $\{x_{2k}\}$ and $\{x_{2k-1}\}$. What are their limits?. (no Proofs needed). What does the answer imply about convergence of $\{x_n\}$? sub sequences,

$$\{x_{2k}\} = \frac{n}{n+1}\{x_{2k-1}\} = -\frac{n}{n+1}$$

So the $\lim n \to \infty = 1, -1$ for the sub sequences $\{x_{2k}\}$ and $\{x_{2k-1}\}$ respectively.

Therefore, $\{x_n\}$ diverges.

(b) Let $\{x_{n_k}\}$ be any sub sequence of $\{x_n\}$. Show that if $\{x_{n_k}\}$ converges, then it must converge either to 1 or to -1.

Let A be the sequence of positive values, and B be the sequence of negative values, and C be the sub sequence given by $\{x_{n_k}\}$.

Proof by cases:

Case 1: C has an infinitely many terms from A and finite from B.

Therefore, there exists an N such that all the terms of C past the Nth are from A.

Then $\{x_{n_k} | k > N\} \subseteq A$, and converges to 1.

Therefore, C converges to 1.

Case 2: C has an infinitely many terms from B and finite from A.

Therefore, there exists an N such that all the terms of C past the Nth are from B.

Then $\{x_{n_k} | k > N\} \subseteq B$, and converges to -1.

Therefore, C converges to -1.

Case 3: C has infinitely many terms from B and infinitely many from A.

Then there exists 2 subsets of C such that they converge to -1, and 1.

Therefore C diverges.

Exercise 4) (a) Is the sequence $\left\{x_n = \frac{2n^2}{3n+9}\right\}_{n=1}^{\infty}$ monotone? If yes, is it increasing/decreasing? Prove your conjecture.

The sequence is monotone increasing

Proof:

Note $\{x_n\} > 0 \ \forall n$

$$\frac{\frac{2(n+1)^2}{3(n+1)+9}}{\frac{2n^2}{3n+9}}$$
$$= \frac{(3n+9)(2n^2+4n+2)}{(2n^2)(3n+12)}$$
$$= \frac{6n^3+30n^2+42n+18}{6n^3+24n^2}$$

Since the numerator is greater then the denominator $\forall n > 1$, the sequence is monotone increasing.

(b) Using the definition, prove that the sequence diverges to $+\infty$.

Since the sequence is positive, and monotone increasing, it is unbounded above. This implies that the $\lim\{x_n\} \to \infty$. $\square$

(c) After how many terms will the sequence pass $M = 10,000$?

after 15003 terms, the sequence passes 10,000.

Exercise 5) Guess the value for each of the following limits; prove your answers using definition 3.1 page 133.

(c) $\lim_{x\to 1} \frac{x^2+x-2}{x-1}$

Conjecture: $\lim x \to 1 = 3$

Proof:

Note that $\frac{x^2+x-2}{x-1} = x - 2$

$$|x + 2 - 3| < \epsilon$$
$$|x - 1| < \epsilon$$
$$|x - 1| < \delta = \epsilon$$

Let $\epsilon > 0$, Let $\delta = \epsilon \implies \delta > 0$.

$$0 < |x - 1| < \delta$$
$$|x - 1| < \epsilon$$
$$|(x + 2) - 3| < \epsilon$$

Therefore $\lim_{x\to 1} \frac{x^2+x-2}{x-1} = 3$ $\square$

Exercise Bonus Find a sequence $\{x_n\}$ that has a maximum β and a minimum α (not just sup and inf), but no sub sequence converging to either one of them. Don't just write down the sequence and let the reader guess that it works. List its max and min and explain why it meets the requirements.

Let $\{x_n\}$ be $\{1, -1, 0, 0, 0, 0, ...\}$.

No sub sequence converges to the max, 1, nor the min, -1.
