Math 312
Worksheet for September 22

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Exercise 1) Consider the sequence defined recursively as $x_n = \frac{x_(n-1)}{1+x_(n-1)}$ for $n \geq 2$. The values of the terms depend on the values of the seed term, $x_1 > 0$.

(a) Use the following different five seed values for x_1 to construct five different sequences: $x_1 = \frac{1}{2}, x_1 = \frac{3}{4}, x_1 = 1, x_1 = \frac{5}{4}, x_1 = 2000$. Calculate at least up to x_{10}. Use a format that makes them easy to read/compare.

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$
$\frac{3}{4}$	$\frac{3}{7}$	$\frac{3}{10}$	$\frac{3}{13}$	$\frac{3}{16}$	$\frac{3}{19}$	$\frac{3}{22}$	$\frac{3}{25}$	$\frac{3}{28}$	$\frac{3}{31}$
$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$
$\frac{5}{4}$	$\frac{5}{9}$	$\frac{5}{14}$	$\frac{5}{19}$	$\frac{5}{24}$	$\frac{5}{29}$	$\frac{5}{34}$	$\frac{5}{39}$	$\frac{5}{44}$	$\frac{5}{49}$
$\frac{2000}{1}$	$\frac{2000}{2001}$	$\frac{2000}{4001}$	$\frac{2000}{6001}$	$\frac{2000}{8001}$	$\frac{2000}{10001}$	$\frac{2000}{12001}$	$\frac{2000}{14001}$	$\frac{2000}{16001}$	$\frac{2000}{18001}$

(b) Does the convergence seem to depend on the seed value or not? This is just a conjecture, write a brief explanation to support your answer.

No, the fractions numerator stays the same while the denominator increases each time, so it will approach 0 regardless of the seed value.

(c) Starting from the recursive formula, find a way to write x_n using x_1 and n.

$$\frac{x_1}{1 + (n-1)x_1}$$

(d) By mathematical induction prove that the formula you found in part (c) is correct for all natural numbers, $n \geq 2$.

without loss of generality assume the seed value does not affect the convergence of the series.

Base Case: $n = 2$

$$x_2 = \frac{x_1}{1 + x_1}$$

It holds true for the base case.

Assume there is some n which this holds true for. Does it hold true for $n+1$? Lets find out!

$$\begin{aligned}
x_{n+1} &= \frac{x_n}{1+x_n} \\
&= \frac{\frac{x_1}{1+(n-1)x_1}}{1+\frac{x_1}{1+(n-1)x_1}} \\
&= \frac{x_1}{1+(n-1)x_1} * \frac{1+(n-1)x_1}{1+nx_1} \\
&= \frac{x_1}{1+nx_1}
\end{aligned}$$

So it holds true for $n+1$. Therefore, by The Principal of Math Induction it is true for all $n \geq 2$. $\square$

(e) If the sequence converges, find its limit and prove that it is correct, using $\epsilon - N$ definition of limit. If it does not, explain why.

Let $\epsilon > 0$ be given. Is there an N such that

$$|c_n - 0| < \epsilon$$

for all $n > N$?

$$|x_n - 0| = |\frac{x_1}{1+(n-1)x_1}| < \frac{x_1}{(n-1)x_1} = \frac{1}{(n-1)} < \epsilon$$

So we need $\frac{1}{\epsilon} + 1 < n$. Thus, $N = \frac{1}{\epsilon} + 1$.

When $n > N$, $|c_n - 0| < \epsilon$. This proves that 0 is the limit.

Exercise 2) Suppose someone incorrectly decided that

$$\lim_{n\to\infty} \frac{6n+1}{5n+7} = \frac{1}{7}$$

(a) We believe they are now trying to prove this. Show why they will not be able to prove that for EVERY $\epsilon > 0$ there exists an N such that...

Let $\epsilon > 0$ be given. Is there an N such that

$$|\frac{6n+1}{5n+7} - \frac{1}{7} < \epsilon$$

for all $n > N$?

$$|\frac{6n+1}{5n+7} - \frac{1}{7}| = |\frac{38n}{35n+7}| < \frac{38n}{35n} = \frac{38}{35} < \epsilon$$

This implies that if $\epsilon < \frac{38}{35}$ there is no N such that for $n > N$ $|\frac{6n+1}{5n+7} - \frac{1}{7}| < \epsilon$.

Which implies that the sequence does not converge to $\frac{1}{7}$.

(b) Find the correct limit and prove that it is indeed the correct one.

The limit is $\frac{6}{5}$.

Proof:

Let $\epsilon > 0$ be given. Is there an N such that

$$|\frac{6n+1}{5n+7} - \frac{6}{5}| < \epsilon$$

for all $n > N$?

$$|\frac{6n+1}{5n+7} - \frac{6}{5}| = \frac{37}{25n+35} < \frac{37}{25n} < \epsilon$$

So we need $\frac{37}{25\epsilon} < n$. Thus $N = \frac{37}{25\epsilon}$.
When $n > N$, $|\frac{6n+1}{5n+7} - \frac{6}{5}| < \epsilon$. This proves that $\frac{6}{5}$ is indeed the limit.

Exercise 3) Prove the following part of Theorem 2.5, page 83: If $a_n \to a$ and c is any constant, then $ca_n \to ca$. (The result is trivial if $c = 0$ so assume $c \neq 0$).

Let $\epsilon > 0$
since $a_n \to a$, $\exists N \ \forall n > N$

$$|a_n - a| < \frac{\epsilon}{|c|}$$
$$|c||a_n - a| < \epsilon$$
$$|c(a_n - a)| < \epsilon$$

Therefore $ca_n \to ca$. $\square$