Hw 6

3.3.1) a): not linear: consider,

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$
$$f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$$
$$f\left(\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}\right) = f\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$
$$f\begin{pmatrix}1\\0\end{pmatrix} + f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} \neq$$

b) $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}y\\x\end{pmatrix}$

$f\begin{pmatrix}x_0\\y_0\end{pmatrix} = \begin{pmatrix}y_0\\x_0\end{pmatrix}$ $\qquad$ $f\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}y_1\\x_1\end{pmatrix}$

$$f\begin{pmatrix}x_0\\y_0\end{pmatrix} + f\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}y_0+y_1\\x_0+x_1\end{pmatrix} = f\begin{pmatrix}x_0+x_1\\y_0+y_1\end{pmatrix}$$

Let $\alpha \in \mathbb{R}$

$$f\left(\alpha\begin{pmatrix}x_0\\y_0\end{pmatrix}\right) = \begin{pmatrix}\alpha y_0\\\alpha x_0\end{pmatrix}$$
$$\alpha f\begin{pmatrix}x_0\\y_0\end{pmatrix} = \alpha\begin{pmatrix}y_0\\x_0\end{pmatrix} = \begin{pmatrix}\alpha y_0\\\alpha x_0\end{pmatrix}$$

So f is a linear function

c) $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\xy\end{pmatrix}$

$f\begin{pmatrix}x_0\\y_0\end{pmatrix} = \begin{pmatrix}0\\x_0y_0\end{pmatrix}$ $\qquad$ $f\begin{pmatrix}x_1\\x_1\end{pmatrix} = \begin{pmatrix}0\\x_1y_1\end{pmatrix}$

$$f\begin{pmatrix}x_0\\y_0\end{pmatrix} + f\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}0\\x_0y_0+x_1y_1\end{pmatrix}$$
$$f\begin{pmatrix}x_0+x_1\\y_0+y_1\end{pmatrix} = \begin{pmatrix}0\\(x_0+x_1)(y_0+y_1)\end{pmatrix} \neq$$ so f is not linear.

d) $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x^2\\y^2\end{pmatrix}$

Let $\alpha \in \mathbb{R} \land \alpha \neq 0$

$$f\left(\alpha\begin{pmatrix}x_0\\y_0\end{pmatrix}\right) = f\begin{pmatrix}\alpha x_0\\\alpha y_0\end{pmatrix} = \begin{pmatrix}\alpha^2 x_0^2\\\alpha^2 y_0^2\end{pmatrix}$$
$$\neq$$
$$\alpha f\begin{pmatrix}x_0\\y_0\end{pmatrix} = \alpha\begin{pmatrix}x_0^2\\y_0^2\end{pmatrix} = \begin{pmatrix}\alpha x_0^2\\\alpha y_0^2\end{pmatrix}$$ So f is not linear.

e) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \sin(y) \end{pmatrix}$

$f\begin{pmatrix} 1 \\ \pi/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $f\begin{pmatrix} 0 \\ \pi/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$f\begin{pmatrix} 1 \\ \pi/2 \end{pmatrix} + f\begin{pmatrix} 0 \\ \pi/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\neq$

$f\left(\begin{pmatrix} 1 \\ \pi/2 \end{pmatrix} + \begin{pmatrix} 0 \\ \pi/2 \end{pmatrix}\right) = f\begin{pmatrix} 1 \\ \pi \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ So f is not linear

~~f)~~

f) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$

$f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0+y_0 \\ x_0-y_0 \end{pmatrix}$, $f\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_1-y_1 \end{pmatrix}$

$f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + f\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0+y_0 \\ x_0-y_0 \end{pmatrix} + \begin{pmatrix} x_1+y_1 \\ x_1-y_1 \end{pmatrix} = \begin{pmatrix} x_0+y_0+x_1+y_1 \\ x_0-y_0+x_1-y_1 \end{pmatrix}$

$f\begin{pmatrix} x_0+x_1 \\ y_0+y_1 \end{pmatrix} = \begin{pmatrix} x_0+x_1+y_0+y_1 \\ x_0+x_1-y_0-y_1 \end{pmatrix}$ $=$

Let $\alpha \in \mathbb{R}$

$f\left(\alpha\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) = \begin{pmatrix} \alpha x_0+\alpha y_0 \\ \alpha x_0-\alpha y_0 \end{pmatrix}$ $=$

$\alpha f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \alpha x_0+\alpha y_0 \\ \alpha x_0-\alpha y_0 \end{pmatrix}$ So f is linear

3.4.1 $f(p) = f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$

Reflection: $(x_1, x_2) \Rightarrow (x_1, -x_2)$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1x_1 & 0x_2 \\ 0x_1 & -1x_2 \end{pmatrix}$$

$a_{11} = 1$ $\quad a_{12} = 0$ $\quad a_{21} = 0$ $\quad a_{22} = -1$

Rotation:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta)x_1 & -\sin(\theta)x_2 \\ \sin(\theta)x_1 & \cos(\theta)x_2 \end{pmatrix}$$

$a_{11} = \cos(\theta)$ $\quad a_{12} = -\sin(\theta)$ $\quad a_{21} = \sin(\theta)$ $\quad a_{22} = \cos(\theta)$

Projection:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos(-45) & -\sin(-45) \\ \sin(-45) & \cos(-45) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2}\,x_1 + 1/\sqrt{2}\,x_2 \\ 1/\sqrt{2}\,x_1 - 1/\sqrt{2}\,x_2 \end{pmatrix}$$

$a_{11} = 1/\sqrt{2}$ $\quad a_{12} = 1/\sqrt{2}$ $\quad a_{21} = 1/\sqrt{2}$ $\quad a_{22} = -1/\sqrt{2}$

3.4.2 rotation: $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

reflection: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ -\sin(\theta) & -\cos(\theta) \end{pmatrix}$$