6.2 Let $X \sim \text{Unif}(0,1)$

(a) pdf of $Y = X^{1/4}$

CDF is $F_Y(y) = P(Y \le y) = P(X^{1/4} \le y) = P(X \le y^4)$

$$= F_X(y^4) = \begin{cases} 0 & y < 0 \\ y^4 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

So pdf is

$$f_Y(y) = F_Y'(y) = \begin{cases} 4y^3 & 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

(b) $W = e^{-X}$

CDF is $F_W(w) = P(W \le w) = P(e^{-X} \le w)$

If $w \le 0$, then $P(e^{-X} \le w) = 0$

If $w > 0$, then $P(e^{-X} \le w) = P(X \ge -\ln(w))$

$$= 1 - P(X \le -\ln(w))$$

So the CDF is

$$F_W(w) = \begin{cases} 0 & w \le 0 \\ 1 & -\ln(w) < 0 \\ 1+\ln(w) & 0 \le -\ln(w) \le 1 \\ 0 & -\ln(w) > 1 \end{cases} = \begin{cases} 0 & w < e^{-1} \\ 1+\ln(w) & e^{-1} \le w \le 1 \\ 1 & w > 1 \end{cases}$$

and the pdf is

$$f_W(w) = F_W'(w) = \begin{cases} \frac{1}{w} & e^{-1} \le w \le 1 \\ 0 & \text{o.w.} \end{cases}$$

(c) $Z = 1 - e^{-X} = 1 - W$

CDF is

$$F_Z(z) = P(Z \le z) = P(1 - W \le z) = P(W \ge 1-z) = 1 - P(W \le 1-z)$$

$$= 1 - F_W(1-z)$$

$$= 1 - \begin{cases} 0 & 1-z < e^{-1} \\ 1 + \ln(1-z) & e^{-1} \le 1-z \le 1 \\ 1 & 1-z > 1 \end{cases} = \begin{cases} 0 & z < 0 \\ -\ln(1-z) & 0 \le z \le 1-e^{-1} \\ 1 & \end{cases}$$

So the CDF is

$$f_Z(z) = F_Z'(z) = \begin{cases} \frac{1}{1-z} & 0 \le z \le 1-e^{-1} \\ 0 & \text{o.w.} \end{cases}$$

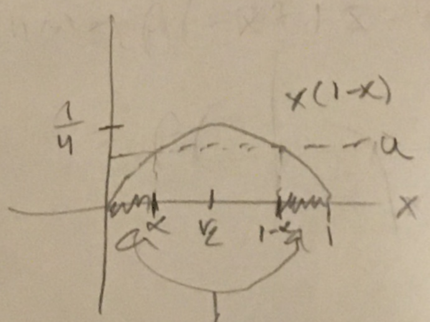
(d) $U = X(1-X)$

CDF is

$$F_U(u) = P(U \le u) = P(X(1-X) \le u)$$

$$= 0 \text{ if } u < 0$$

$x(1-x)$ for $x \in [0,1]$



$\{X(1-X) < u\}$

max of $X(1-X)$ is $\frac{1}{4}$ so

$P(X(1-X) \le u) = 1$ if $u > \frac{1}{4}$

If $0 \le u \le \frac{1}{4}$, then by diagram

$P(X(1-X) \le u) = 2\alpha$

α = smallest root of $x(1-x) = u$

where $\alpha(1-\alpha) = u$ or $\alpha^2 - \alpha + u = 0$

or $\alpha = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4u}$ choose $-$

$\alpha = \frac{1}{2} - \frac{1}{2}\sqrt{1-4u}$

$P(X(1-X) \le u) = 1 - \sqrt{1-4u} \quad 0 \le u \le \frac{1}{4}$

so $F_U(u) = \begin{cases} 0 & u < 0 \\ 1 - \sqrt{1+4u} & 0 \le u \le \frac{1}{4} \\ 1 & u > \frac{1}{4} \end{cases}$

and the pdf is $F_U'(u) = \begin{cases} \frac{2}{\sqrt{1-4u}} & 0 \le u \le \frac{1}{4} \\ 0 & \text{o.w.} \end{cases}$

6.11 $X \sim Bin(n,p)$ $Y = n - X$. Then $Y \in \{0, 1, \ldots, n\}$, and

$$P(Y=k) = P(n - X = k) = P(X = n-k)$$

$$= \binom{n}{n-k} p^{n-k}(1-p)^{k} = f(k), \text{ the pmf of}$$

Y

6.14 X & Y have joint pdf $4e^{-2(x+y)}$ $0<x<\infty$, $0<y<\infty$.

(a) $W = X + Y$, CDF

$$F_W(w) = P(W \le w) = P(X+Y \le w) = 0 \text{ if } w<0,$$

otherwise $= \int_0^w \int_0^{w-x} 4e^{-2(x+y)} \, dy \, dx$

$$= \int_0^w 4e^{-2x} \frac{1}{-2} e^{-2y} \Big|_0^{w-x} dx$$

$$= \int_0^w 2e^{-2x}\left(1 - e^{-2(w-x)}\right) dx$$

$$= \int_0^w \left(2e^{-2x} - 2e^{-2w}\right) dx$$

$$= 1 - e^{-2w} - 2we^{-2w}$$

(b) Let $u = \frac{x}{y}$, $v = x$, then $0<x<\infty$, $0<y<\infty \Rightarrow$
$0<u<\infty$ & $0<v<\infty$

$$x = v, \quad y = \frac{v}{u}$$

$f_{UV}(u,v) = f_{XY}(v, \frac{v}{u})|J|$, $\quad |J| = \left| \begin{matrix} 1 & 0 \\ \frac{1}{u} & -\frac{v}{u^2} \end{matrix} \right| = \frac{v}{u^2}$

$= 4e^{-2(v+\frac{v}{u})}$, $0<u<\infty, 0<v<\infty$

(c) $f_U(u) = \int_0^\infty f_{UV}(u,v)\,dv = \int_0^\infty 4 e^{-2v(1+\frac{1}{u})} \frac{v}{u^2}\,dv$

$dv \quad v$

$$= 4\left[\frac{v}{u^2} \frac{e^{-2v(1+\frac{1}{u})}}{-2(1+\frac{1}{u})}\Bigg|_0^\infty - \int_0^\infty \frac{e^{-2v(1+\frac{1}{u})}}{-2(1+\frac{1}{u})}\frac{dv}{u^2}\right]$$

$$= \frac{2}{u^2+u}\int_0^\infty e^{-2v(1+\frac{1}{u})}\,dv = \frac{2}{u^2+u}\left(\frac{e^{-2v(1+\frac{1}{u})}}{-2(1+\frac{1}{u})}\Bigg|_0^\infty\right)$$

$$= \frac{1}{u^2+u}\cdot\frac{1}{1+\frac{1}{u}} = \frac{u}{(u^2+u)(u+1)} = \frac{u}{u^3+2u^2+u} = \frac{1}{u^2+2u+1}$$

$$= \frac{1}{(u+1)^2},\quad 0<u<\infty$$

6.18 X & Y have joint pdf $f(x,y) = e^{-y}$ $0<x<y<\infty$

(a) Let $s = x+y$, $t = x$, then $x = t$, $y = s-t$

$$|J| = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 1$$

$$f_{ST}(s,t) = f(t, s-t)\,|J| = e^{t-s}$$

$0<x<y<\infty$

$\Updownarrow$

$0<t<s-t<\infty$

or $0<2t<s<\infty$

If $t>0$,

(b) $f_T(t) = \int_{2t}^\infty f_{ST}(s,t)\,ds = \int_{2t}^\infty e^{t-s}\,ds = e^t\left[-e^{-s}\right]_{2t}^\infty$

$= e^{-t},\ t>0$

If $s>0$,

(c) $f_S(s) = \int_0^{s/2} f_{ST}(s,t)\,dt = \int_0^{s/2} e^{t-s}\,dt = e^{-s}\,e^t\Big|_0^{s/2}$

$= e^{-s}(e^{s/2}-1)$

$= e^{-s/2} - e^{-s},\ s>0$

6.24 Let $X_1, \ldots, X_{10}$ be a random sample with $X_i \sim Exp(2)$

(a) since X_i are independent, the MGF of $Y = \sum_{i=1}^{10} X_i$

is

$$M_Y(t) = M_{X_1}(t) \cdot \ldots \cdot M_{X_{10}}(t) = \left[M_{X_1}(t)\right]^{10}$$

$$= \frac{1}{(1-2t)^{10}}$$ since $M_{X_1}(t) = \frac{1}{1-2t}$ is the MGF of

Exp(2)

(b) From above, $Y \sim Gamma(10, 2)$, so its pdf is

(by comparing MGF)

$$f_Y(y) = \frac{1}{\Gamma(10) 2^{10}} y^9 e^{-y/2}$$

6.27 $Y_i \sim Logn(\mu_i, \sigma_i^2)$ $i = 1, \ldots, n$ are independent.

(a) $$\prod_{i=1}^{n} Y_i = \prod_{i=1}^{n} e^{\ln(Y_i)} = e^{\sum_{i=1}^{n} \ln(Y_i)}$$

since $\ln(Y_i) \sim N(\mu_i, \sigma_i^2)$ for $i = 1, \ldots n$, the sum

$S = \sum_{i=1}^{n} \ln(Y_i) \sim N(B\mu, B\Sigma B^T)$ where

$$\Sigma = \begin{pmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \ddots \; \sigma_n^2 \end{pmatrix}$$ covariances are 0 by independence, is the variance matrix of

$$X = \begin{pmatrix} \ln(Y_1) \\ \ln(Y_2) \\ \vdots \\ \ln(Y_n) \end{pmatrix}$$, and $\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$ is $E[X]$,

and $S = BX$, where $B = (\overbrace{1 \; 1 \; \ldots \; 1}^{n \text{ 1's}})$. Then

$B\mu = \sum_{i=1}^{n} \mu_i$ and $B \Sigma B^T = (\sigma_1^2 \; \sigma_2^2 \; \ldots \; \sigma_n^2) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^{n} \sigma_i^2$

so $S \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$, and then e^S is distributed

$\text{Logn}\left(\sum_{i=1}^{n}\mu_i, \sum_{i=1}^{n}\sigma_i^2\right)$ by defn, but $e^S = \prod_{i=1}^{n} X_i$.

(b) $$\prod_{i=1}^{n} X_i^a = \prod_{i=1}^{n} e^{a\ln X_i} = e^{a\sum_{i=1}^{n}\ln(X_i)}$$

as before, $S = \sum_{i=1}^{n}\ln(X_i) \sim \text{Normal}\left(\sum_{i=1}^{n}\mu_i, \sum_{i=1}^{n}\sigma_i^2\right)$,

so $aS \sim N\left(a\sum_{i=1}^{n}\mu_i, a^2\sum_{i=1}^{n}\sigma_i^2\right)$ and

$$\prod_{i=1}^{n} X_i^a = e^{aS} \sim \text{Logn}\left(a\sum_{i=1}^{n}\mu_i, a^2\sum_{i=1}^{n}\sigma_i^2\right)$$

(c) $$\frac{X_1}{X_2} = e^{\ln X_1 - \ln X_2}$$

$$\ln X_1 - \ln X_2 \sim N\left(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2\right)$$

so $$\frac{X_1}{X_2} \sim \text{Logn}\left(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2\right)$$

(d) $E\left[\prod_{i=1}^{n} X_i\right] = e^{\sum_{i=1}^{n}\mu_i + \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2}$ since $\prod_{i=1}^{n} X_i \sim \text{Logn}\left(\sum_{i=1}^{n}\mu_i, \sum_{i=1}^{n}\sigma_i^2\right)$

6.31 (a) $X_i \sim \text{Exp}(1)$

pdf of $X_{(1)}$ is by 6.5.5

$$f_1(x_1) = n(1-F(x_1))^{n-1} f(x_1) \qquad 0 < x_1$$

$$= n e^{-(n-1)x_1} e^{-x_1} = n e^{-nx_1}$$

(b) pdf of $X_{(n)}$ is by 6.5.6

$$f_n(x_n) = n[F(x_n)]^{n-1} f(x_n) = n(1-e^{-x_n})^{n-1} e^{-x_n}$$