4.8. Let X & Y be discrete random variables with joint pmf

$$f(x,y) = \begin{cases} c\dfrac{2^{x+y}}{x!\,y!} & x = 0,1,2,\dots,\ y = 0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

(a) Find c.

In order for f to be a pmf, we need

$$\sum_{x,y} f(x,y) = 1 \quad \text{or} \quad \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} c\frac{2^{x+y}}{x!\,y!} = 1$$

$$c\left(\sum_{x=0}^{\infty}\frac{2^x}{x!}\right)\left(\sum_{y=0}^{\infty}\frac{2^y}{y!}\right) = 1$$

$$c \cdot e^2 \cdot e^2 = 1$$

$$\Rightarrow c = e^{-4}$$

(b) $$f_X(x) = \sum_y f(x,y) = \sum_{y=0}^{\infty} e^{-4}\frac{2^{x+y}}{x!\,y!} = e^{-4}\frac{2^x}{x!}\sum_{y=0}^{\infty}\frac{2^y}{y!}$$

$$f_X(x) = e^{-2}\frac{2^x}{x!}$$

$$f_Y(y) = \sum_x f(x,y) = \sum_{x=0}^{\infty} e^{-4}\frac{2^{x+y}}{x!\,y!} = e^{-4}\frac{2^y}{y!}\sum_{x=0}^{\infty}\frac{2^x}{x!}$$

$$f_Y(y) = e^{-2}\frac{2^y}{y!}$$

(c) X & Y are independent because

$$f(x,y) = e^{-4}\frac{2^{x+y}}{x!\,y!} = \left(e^{-2}\frac{2^x}{x!}\right)\left(e^{-2}\frac{2^y}{y!}\right) = f_X(x)f_Y(y),$$

which is the definition of independence.

4.10 X = # of hearts Y = # of black cards

(a) $f(x,y)$ is defined by the following table of values

$f(x,y)$	$x=0$	$x=1$	$x=2$
$y=0$	$\frac{78}{1326}$	$\frac{169}{1326}$	$\frac{78}{1326}$
$y=1$	$\frac{338}{1326}$	$\frac{338}{1326}$	0
$y=2$	$\frac{325}{1326}$	0	0

, $f(x,y)=0$ otherwise

Total # of draws $= \binom{52}{2} = \frac{52 \cdot 51}{2} = 1326$

$X=0 \cap Y=0 \Rightarrow$ a diamond was drawn both times

ways to draw 2 diamonds $= \binom{13}{2} = \frac{13 \cdot 12}{2} = 78$

$X=1 \cap Y=0 \Rightarrow$ a heart & a diamond were drawn

ways to get 1 heart & 1 diamond $= \binom{13}{1}\binom{13}{1} = 13^2 = 169$

$X=2 \cap Y=0 \Rightarrow$ 2 hearts were drawn

ways to draw 2 hearts $= \binom{13}{2} = 78$

$X=0 \cap Y=1 \Rightarrow$ 1 black & 1 diamond were drawn

ways to get 1 black & 1 diamond $= \binom{26}{1}\binom{13}{1} = 26 \cdot 13 = 338$

$X=1 \cap Y=1 \Rightarrow$ 1 black & 1 heart were drawn

ways to get 1 black & 1 heart $= \binom{26}{1}\binom{13}{1} = 26 \cdot 13 = 338$

$X=0 \cap Y=2 \Rightarrow$ 2 blacks were drawn

ways to get 2 blacks $= \binom{26}{2} = \frac{26 \cdot 25}{2} = 325$

Any other values of X and Y are impossible, so $f(x,y)=0$ anywhere else

4.10 (b) $F(x,y)$ can be written as the following table

$$F(x,y) = \sum_{x' \le x} \sum_{y' \le y} f(x',y')$$

$F(x,y)$	$x \in (-\infty,0)$	$[0,1)$	$[1,2)$	$[2,\infty)$
$y \in (-\infty,0)$	0	0	0	0
$[0,1)$	0	$\frac{78}{1326}$	$\frac{247}{1326}$	$\frac{325}{1326}$
$[1,2)$	0	$\frac{416}{1326}$	$\frac{923}{1326}$	$\frac{1001}{1326}$
$[2,\infty)$	0	$\frac{741}{1326}$	$\frac{1248}{1326}$	1

each entry computed by summing to the left and up from corresponding entry in table (a) of $f(x,y)$

(c) $$f_1(x) = \sum_y f(x,y) = \sum_{y=0}^{2} f(x,y) = \begin{cases} \frac{741}{1326} & x = 0 \\ \frac{507}{1326} & x = 1 \\ \frac{78}{1326} & x = 2 \\ 0 & \text{o.w.} \end{cases}$$

sum along columns in (a)

$$f_2(y) = \sum_x f(x,y) = \sum_{x=0}^{2} f(x,y) = \begin{cases} \frac{325}{1326} & y = 0 \\ \frac{676}{1326} & y = 1 \\ \frac{325}{1326} & y = 2 \\ 0 & \text{o.w.} \end{cases}$$

sum along rows in (a)

(d) X & Y are not independent: $f(0,0) = \frac{78}{1326} \neq \frac{741 \cdot 325}{1326^2} = f_1(0) f_2(0)$

(e) $$P(Y=1 \mid X=1) = \frac{P(X=1 \cap Y=1)}{P(X=1)} = \frac{f(1,1)}{f_1(1)} = \frac{338}{507}$$

from (a) & (c)

(f) $$P(Y=y \mid X=1) = \frac{P(X=1 \cap Y=y)}{P(X=1)} = \frac{f(1,y)}{f_1(1)}$$

$$= \begin{cases} \frac{169}{507} & y = 0 \\ \frac{338}{507} & y = 1 \\ 0 & \text{o.w.} \end{cases}$$

from (a) & (c)

4.10 (g) $P(Y=y \mid X=x)$ can be written in a table as follows

$P(Y=y \mid X=x)$	$x=0$	$x=1$	$x=2$
$y=0$	$\frac{78}{741}$	$\frac{169}{507}$	1
$y=1$	$\frac{338}{741}$	$\frac{338}{507}$	0
$y=2$	$\frac{325}{741}$	0	0

, $P(Y=y \mid X=x)=0$, o/w,

where entries are computed from (a) & (c) using the definition

$$P(Y=y \mid X=x) = \frac{P(X=x \cap Y=y)}{P(X=x)} = \frac{f(x,y)}{f_1(x)}$$

if $P(X=x)\neq 0$, otherwise 0.

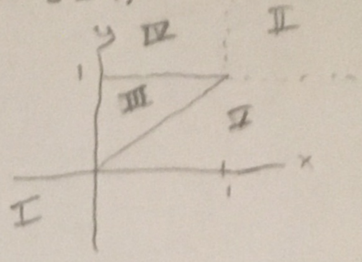
(h) $P(X+Y\leq z) = \begin{cases} P(X+Y\leq 0) & z=0 \quad X,Y\geq 0, \text{ so need both } 0 \\ P(X+Y\leq 1) & z=1 \quad \text{One} =1 \text{ and other} = 0, \text{ or both } 0 \\ P(X+Y\leq 2) & z=2 \quad \text{always draw 2 cards so prob. is 1} \end{cases}$

$= \begin{cases} P(X=0 \cap Y=0) & z=0 \\ P(X=1 \cap Y=0) + P(X=0 \cap Y=1) + P(X=0 \cap Y=0) & z=1 \\ 1 & z=2 \end{cases}$

$= \begin{cases} f(0,0) & z=0 \\ f(0,0)+f(1,0)+f(0,1) & z=1 \\ 1 & z=2 \end{cases}$

$= \begin{cases} \frac{78}{1326} & z=0 \\ \frac{585}{1326} & z=1 \\ 1 & z=2 \end{cases}$

4.20 X & Y have joint pdf. $f(x,y) = 8xy \quad 0 \le x \le y \le 1.$

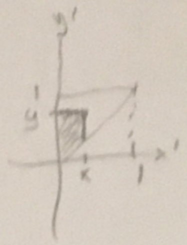


(a) $F(x,y) = P(X \le x \cap Y \le y)$

I. $x<0$ or $y<0$. Then $P(X \le x \cap Y \le y) = 0 = F(x,y)$

II. $x>1$ and $y>1$. Then $P(X \le x \cap Y \le y) = 1 = F(x,y)$

III. $0 \le y \le 1$ and $x \le y$. Then



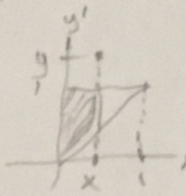
$$F(x,y) = P(X \le x \cap Y \le y) = \int_{\substack{x' \le x \\ y' \le y}} f(x',y')\,dx'dy' = \int_0^x \int_{x'}^y 8x'y'\,dy'dx'$$

$$= \int_0^x 4x'(y^2 - x'^2)\,dx'$$

$$= 2x'^2y^2 - x'^4 \Big|_0^x$$

$$= 2x^2y^2 - x^4$$

IV $y>1$ and $x \le 1$ Then

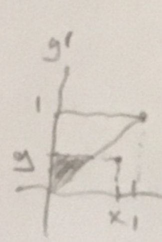


$$F(x,y) = P(X \le x \cap Y \le y) = \int_{\substack{x' \le x \\ y' \le y}} f(x',y')\,dx'dy' = \int_0^x \int_{x'}^{\overset{y=1}{1}} 8x'y'\,dy'dx'$$

= same as in case III but with $y=1$

$$= 2x^2 - x^4$$

V $0 \le y \le 1$ and $x > y$. Then



$$F(x,y) = P(X \le x \cap Y \le y) = \int_{\substack{x' \le x \\ y' \le y}} f(x',y')\,dx'dy' = \int_0^{\overset{x=y}{y}} \int_{x'}^y 8x'y'\,dy'dx'$$

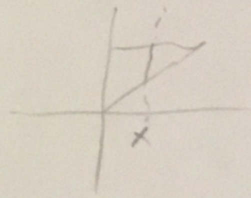
= Same as in case III but with $x=y$

$$= y^4$$

4.20 (a) Altogether

$$F(x,y) = \begin{cases} 0 & x<0 \text{ or } y<0 \\ 1 & x>1 \text{ and } y>1 \\ 2x^2y^2 - x^4 & 0 \le y \le 1 \text{ and } x \le y \\ y^4 & 0 \le y \le 1 \text{ and } x > y \\ 2x^2 - x^4 & y > 1 \text{ and } x \le 1 \end{cases}$$

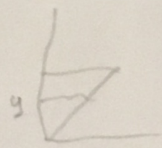
(b) $f(y|x) = \dfrac{f(x,y)}{f_X(x)}$



$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} 0 & x<0 \\ \int_x^1 8xy\,dy & 0 \le x \le 1 \\ 0 & x>1 \end{cases}$$

$$= \begin{cases} 4xy^2\Big|_x^1 & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases} = \begin{cases} 4x(1-x^2) & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

so $f(y|x) = \begin{cases} \dfrac{8xy}{4x(1-x^2)} & 0 \le x \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$



(c) $f(x|y) = \dfrac{f(x,y)}{f_Y(y)}$, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \begin{cases} \int_0^y 8xy\,dx & 0 \le x \le 1 \\ 0 & \text{o.w} \end{cases}$

$$= \begin{cases} 4y^3 & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

so $f(y|x) = \begin{cases} \dfrac{8xy}{4y^3} & 0 \le x \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$

(d) $P(X \le 0.5 \mid Y = 0.75) = \int_{-\infty}^{0.5} f(x \mid y=0.75)\,dx = \int_0^{0.5} \dfrac{8x \cdot 0.75}{4 \cdot 0.75^3}\,dx$

$$= \frac{1}{0.5625} x^2 \Big|_0^{0.5}$$

$$= \frac{0.5^2}{0.75^2} = \frac{4}{9}$$

4.20 (e) $P(X \le 0.5 \mid Y \le 0.75) = \dfrac{P(X \le 0.5 \cap Y \le 0.75)}{P(Y \le 0.75)}$

$$= \frac{F(0.5, 0.75)}{\int_{-\infty}^{0.75} f_Y(y)\,dy}$$

from (a)

$$= \frac{2 \cdot 0.5^2 \cdot 0.75^2 - 0.5^4}{\int_0^{0.75} 4y^3\,dy}$$

from (c)

$$= \frac{0.5 \cdot 0.75^2 - 0.5^4}{0.75^4} = \frac{2}{3}\,\frac{0.75^2 - 0.5^3}{0.75^3}$$

$$= \frac{2}{3}\left(\frac{4}{3} - \left(\frac{2}{3}\right)^3\right) = \frac{56}{81}$$

22. Let $X_1, \ldots, X_n$ be a random sample from a population with pdf $f(x) = 3x^2$, $0 < x < 1$.

(a) Joint pdf of the sample is

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i) = 3^n \prod_{i=1}^{n} x_i^2 \quad \text{if } 0 < x_i < 1 \ \forall i = 1, \ldots n$$

and 0 otherwise

(b) $$P(X_1 < 0.5) = \int_{-\infty}^{0.5} f(x_1)\,dx_1 = \int_{0}^{0.5} 3x_1^2\,dx_1 = x_1^3 \Big|_0^{0.5} = \frac{1}{8}$$

(c) All vars. are independent, so the events $X_i < 0.5$ and $X_j < 0.5$ are independent $\forall i \neq j$, so

$$P\left(\bigcap_{i=1}^{n} X_i < 0.5\right) = \prod_{i=1}^{n} P(X_i < 0.5) = \left(P(X_1 < 0.5)\right)^n$$ because

all vars. are identically distributed, implying that

$P(X_i < 0.5) = P(X_1 < 0.5) \ \forall i = 1, \ldots, n.$

Then

$$P\left(\bigcap_{i=1}^{n} X_i < 0.5\right) = \left(\frac{1}{8}\right)^n = \frac{1}{8^n}$$

4.26 X_1, X_2, X_3 have joint pdf $f(x_1,x_2,x_3)=6$,

$0<x_1<x_2<x_3<1$.

(a) $$f(x_1,x_3)=\int_{-\infty}^{\infty} f(x_1,x_2,x_3)\,dx_2 = \begin{cases} \int_{x_1}^{x_3} 6\,dx_2 & 0<x_1<x_3<1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 6(x_3-x_1) & 0<x_1<x_3<1 \\ 0 & \text{o.w.} \end{cases}$$

(b) $$f(x_2,x_3)=\int_{-\infty}^{\infty} f(x_1,x_2,x_3)\,dx_1 = \begin{cases} \int_{0}^{x_2} 6\,dx_1 & 0<x_2<x_3<1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 6x_2 & 0<x_2<x_3<1 \\ 0 & \text{o.w.} \end{cases}$$

(c) $$f(x_2 \mid x_1,x_3) = \frac{f(x_1,x_2,x_3)}{f(x_1,x_3)} \leftarrow \text{from (a)}$$

$$= \begin{cases} \frac{1}{x_3-x_1} & 0<x_1<x_2<x_3<1 \\ 0 & \text{o.w.} \end{cases}$$

(d) $$f(x_1 \mid x_2,x_3) = \frac{f(x_1,x_2,x_3)}{f(x_2,x_3)} \leftarrow \text{from (b)}$$

$$= \begin{cases} \frac{1}{x_2} & 0<x_1<x_2<x_3<1 \\ 0 & \text{o.w.} \end{cases}$$

(e) $$f(x_1,x_2 \mid x_3) = \frac{f(x_1,x_2,x_3)}{f(x_3)}$$

Need $f(x_3)=\int_{-\infty}^{\infty} f(x_2,x_3)\,dx_2$

$$= \int_0^{x_3} 6x_2\,dx_2 = 3x_3^2 \text{ if } 0<x_3<1$$

0 o.w.

$$= \begin{cases} \frac{2}{x_3^2} & 0<x_1<x_2<x_3<1 \\ 0 & \text{o.w.} \end{cases}$$

4.27. X_1, X_2 is a random sample. pmf $= f(x) = \begin{cases} 0.2 & x=1 \text{ or } x=3 \\ 0.6 & x=2 \\ 0 & \text{o.w.} \end{cases}$

(a) $f(x_1,x_2) = f(x_1)f(x_2)$, and can be written as the following table

$f(x_1,x_2)$	$x_1=$ 1	2	3
$x_2=$ 1	0.04	0.12	0.04
2	0.12	0.36	0.12
3	0.04	0.12	0.04

(b) $F(x_1,x_2) = P(X_1 \le x_1 \text{ and } X_2 \le x_2)$

$$= \sum_{\substack{x_1' \le x_1 \\ x_2' \le x_2}} f(x_1,x_2)$$

computing for every value of x_1, x_2

$F(x_1,x_2)$	$x_1 \in$ $(-\infty,1)$	$[1,2)$	$[2,3)$	$[3,\infty)$
$x_2 \in$ $(-\infty,1)$	0	0	0	0
$[1,2)$	0	0.04	0.16	0.20
$[2,3)$	0	0.16	0.64	0.80
$[3,\infty)$	0	0.20	0.80	1

(c) $P(X_1+X_2 \le 4) = \sum_{x_1+x_2\le 4} f(x_1,x_2) = \sum_{x_1=1}^{3}\sum_{x_2=1}^{4-x_1} f(x_1,x_2)$

$$= \sum_{x_2=1}^{3} f(1,x_2) + \sum_{x_2=1}^{2} f(2,x_2) + \sum_{x_2=1}^{1} f(3,x_2)$$

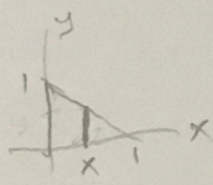
$$= f(1,1) + f(1,2) + f(1,3) + f(2,1) + f(2,2) + f(3,1)$$

$$= 0.04 + 0.12 + 0.04 + 0.12 + 0.36 + 0.04$$

$$= 0.72$$

4.30. Let X & Y have pdf $f(x,y) = 60x^2y$ if $0<x,\ 0<y$
$x+y<1$ ($\Rightarrow 1-x>0$)

(a) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \int_0^{1-x} 60x^2y\,dy & 0<x<1 \\ 0 & \text{o.w.} \end{cases}$

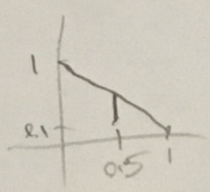


$= \begin{cases} 30x^2(1-x)^2 & 0<x<1 \\ 0 & \text{o.w.} \end{cases}$

(b) $f(y|x) = \dfrac{f(x,y)}{f_X(x)} = \begin{cases} \dfrac{60x^2y}{30x^2(1-x)^2} & 0<x,\ 0<y,\ x+y<1 \\ 0 & \text{o.w.} \end{cases}$

$= \begin{cases} \dfrac{2y}{(1-x)^2} & 0<x,\ 0<y\ \ x+y<1 \\ 0 & \text{o.w.} \end{cases}$

(c) $P(Y>0.1 \mid X=0.5) = \int_{0.1}^{\infty} f(y|0.5)\,dy = \int_{0.1}^{0.5} 8y\,dy = 4y^2\Big|_{0.1}^{0.5}$



Need $0<y$
$y<\frac{1}{2}=1-x$

$= 4\cdot\left(\frac{1}{4}-\frac{1}{10}\right)$

$= 0.6$