Math 6417 Homework 4

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December 9, 2023

Question 1.

Define the **Fourier transform operator** $\mathscr{F} : L^1(\mathbf{R}) \to L^\infty(\mathbf{R})$ by

$$\mathscr{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f(x) \,\mathrm{d}x. \tag{1}$$

1.1) We note that the function $x \mapsto e^{iyx} f(x)$ is clearly integrable if f is, so the integral in (1) exists for all y. We show that $\mathscr{F}(f) \in L^\infty(\mathbf{R})$ as claimed, and $\|\mathscr{F} f\|_{L^\infty} \le \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$. Indeed, for $y \in \mathbf{R}$,

$$\begin{aligned} |\mathscr{F}(f)(y)| &= \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} e^{iyx} f(x) \,\mathrm{d}x \right| && (2)\\ &\le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| e^{iyx} f(x) \right| \,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}. && (3) \end{aligned}$$

Therefore, $\|\mathscr{F} f\|_{L^\infty} \le \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$.

1.2) Suppose that $f \in C^2(\mathbf{R})$, and $f, f', f'' \in L^1(\mathbf{R})$, and $f(x), f'(x), f''(x) \to 0$ as $x \to \pm\infty$. Then there exists a constant C such that $|y^2 \mathscr{F}(f)(y)| \le C$ for all $y \in \mathbf{R}$. Furthermore, $\mathscr{F}(f) \in L^1(\mathbf{R})$.

Proof. Since $f'' \in L^1(\mathbf{R})$, we can take its Fourier transform, which yields

$$\mathscr{F}(f'')(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f''(x) \,\mathrm{d}x. \tag{4}$$

We can integrate by parts because $f', f \in L^1(\mathbf{R})$ and are continuous, and $f(x), f'(x) \to 0$ as $x \to \pm\infty$. This gives

$$\begin{aligned} \mathscr{F}(f'')(y) &= \frac{1}{\sqrt{2\pi}} \left[f'(x) e^{iyx} \Big|_{-\infty}^{\infty} - iy \int_{-\infty}^{\infty} e^{iyx} f'(x) \,\mathrm{d}x \right] && (5)\\ &= \frac{iy}{\sqrt{2\pi}} \left[-f(x) e^{iyx} \Big|_{-\infty}^{\infty} + iy \int_{-\infty}^{\infty} e^{iyx} f(x) \,\mathrm{d}x \right] && (6)\\ &= -y^2 \mathscr{F}(f)(y). && (7) \end{aligned}$$

By the reasoning in 1.1), it follows that

$$|y^2 \mathscr{F}(f)(y)| = |\mathscr{F}(f'')(y)| \le \frac{1}{\sqrt{2\pi}} \|f''\|_{L^1} \tag{8}$$

for all $y \in \mathbf{R}$.

Thus, if $C = \frac{1}{\sqrt{2\pi}} \|f''\|_{L^1}$, then $|\mathscr{F}(f)(y)| \le \frac{C}{y^2}$ for all $y \in \mathbf{R}$. On the other hand, $\mathscr{F}(f) \in L^\infty(\mathbf{R})$ by part 1.1), so $\mathscr{F}(f)$ is dominated by the integrable function

$$\phi(y) = \begin{cases} \|\mathscr{F}(f)\|_{L^\infty} & y \in [-1, 1], \\ \frac{C}{y^2} & \text{otherwise.} \end{cases} \tag{9}$$

By the integral comparison test, $\mathscr{F}(f) \in L^1(\mathbf{R})$. $\square$

1.3) Formally, $\mathscr{F}^2(f)(y) = f(-y)$.

Proof. We note that if $f \in C^1 \cap L^1(\mathbf{R})$, and $f' \in L^1(\mathbf{R})$, and $f(x) \to 0$ as $x \to \pm\infty$, then we can use integration by parts to show that

$$\mathscr{F}(f')(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f'(x) \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[e^{iyx} f(x) \Big|_{-\infty}^{\infty} - iy \int_{-\infty}^{\infty} e^{iyx} f(x) \, \mathrm{d}x \right] \tag{10}$$

$$= -iy\mathscr{F}(f)(y). \tag{11}$$

On the other hand, let $f \in L^1(\mathbf{R})$, and define $g(x) = ixf(x)$. If $g \in L^1(\mathbf{R})$ as well, then

$$\frac{\mathrm{d}}{\mathrm{d}y} \frac{1}{\sqrt{2\pi}} \mathscr{F}(f)(y) = \frac{\mathrm{d}}{\mathrm{d}y} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f(x) \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial y} \left[e^{iyx} f(x) \right] \, \mathrm{d}x \tag{12}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} ixf(x) \, \mathrm{d}x = \mathscr{F}(g)(y). \tag{13}$$

If we take $f(x) = e^{-ax^2}$, then f satisfies the above assumptions. Since $f'(x) = -2axf(x)$,

$$2ai \frac{\mathrm{d}}{\mathrm{d}y} \mathscr{F}(f)(y) = 2ai\mathscr{F}(i(\cdot)f(\cdot))(y) = \mathscr{F}(-2a(\cdot)f(\cdot))(y) = \mathscr{F}(f')(y) = -iy\mathscr{F}(f)(y). \tag{14}$$

Hence, $\mathscr{F}(f)(y)$ is the unique solution of the IVP

$$u' = -\frac{y}{2a} u, \qquad u(0) = \mathscr{F}(f)(0). \tag{15}$$

The general solution of the differential equation is

$$u(y) = u(0) e^{-\frac{y^2}{4a}}. \tag{16}$$

Since

$$\mathscr{F}(f)(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} \, \mathrm{d}x = \frac{1}{\sqrt{2a}}, \tag{17}$$

it follows that

$$\mathscr{F}(f)(y) = \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}. \tag{18}$$

Thus, if $\phi_a(x) = e^{-ax^2}$, then, formally,

$$\mathscr{F}(1)(y) = \mathscr{F}\left(\lim_{a \to 0^+} \phi_a \right)(y) = \lim_{a \to 0^+} \mathscr{F}(\phi_a)(y) = \lim_{a \to 0^+} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}. \tag{19}$$

We would like to interpret the last limit formally as a constant multiple of the Dirac delta function. Clearly,

$$\lim_{a \to 0^+} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}} = \begin{cases} 0 & y \neq 0, \\ \infty & y = 0. \end{cases} \tag{20}$$

At the same time, for any $a > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}} \, \mathrm{d}y = \frac{1}{\sqrt{2a}} \sqrt{4a\pi} = \sqrt{2\pi}, \tag{21}$$

so it makes sense formally that we should have $\mathscr{F}(1)(y) = \sqrt{2\pi}\delta(y)$.

Now, if we consider applying the Fourier transform twice to a function f, we get

$$
\begin{aligned}
\mathscr{F}\mathscr{F}(f)(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixy}e^{izx}f(z)\,\mathrm{d}z\,\mathrm{d}x &(22)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix(y+z)}\,\mathrm{d}x\,\mathrm{d}z &(23)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)\mathscr{F}(1)(y+z)\,\mathrm{d}z &(24)\\
&= \int_{-\infty}^{\infty} f(z)\delta(y+z)\,\mathrm{d}z &(25)\\
&= \int_{-\infty}^{\infty} f(z-y)\delta(z)\,\mathrm{d}z &(26)\\
&= f(-y). &(27)
\end{aligned}
$$

$\square$

1.4) Define $g(y) = f(-y)$ for some function f. Based on the formal result from part 1.3), we see immediately that

$$\mathscr{F}^4(f)(y) = \mathscr{F}^2(\mathscr{F}^2(f))(y) = \mathscr{F}^2(g)(y) = g(-y) = f(y). \tag{28}$$

Since f was arbitrary, it follows formally that $\mathscr{F}^4 = I$, the identity operator.

1.5) Let $p(x) = x^4$. By the Spectral Mapping Theorem,

$$p(\sigma(\mathscr{F})) = \sigma(p(\mathscr{F})). \tag{29}$$

Since $p(\mathscr{F}) = \mathscr{F}^4 = I$, the spectrum of $p(\mathscr{F})$ is just $\sigma(I) = \{1\}$, as the operator $I - \lambda I = (1-\lambda)I$ is invertible, with inverse $\frac{1}{1-\lambda}I$, if and only if $\lambda \neq 1$. Therfore, if $\lambda \in \sigma(\mathscr{F})$, then $p(\lambda) = 1$, that is, $\lambda^4 = 1$. The possible solutions of this equation are $1, -1, i, -i$, so $\sigma(\mathscr{F}) \subseteq \{1, -1, i, -i\}$.

1.6) If we reuse the result in equation (18) with $a = \frac{1}{2}$, we see that if $f(x) = e^{-\frac{1}{2}x^2}$, then

$$\mathscr{F}(f)(y) = e^{-\frac{1}{2}y^2} \tag{30}$$

as well. Thus, $\mathscr{F}f = f$, so f is an eigenfunction of $\mathscr{F}$ with corresponding eigenvalue 1.

Question 2.

On this question, we will reuse the notation from Question 2 of Homework 3.

Let $\dot{L}^2(-\pi,\pi) = \{f \in L^2(-\pi,\pi) : f = \bar{f} \text{ and } \mathrm{mean}(f) = 0\}$, where $\mathrm{mean}(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f$. Consider the following problem.

$$\text{Let } f \in \dot{L}^2(-\pi,\pi). \text{ Find } u \in H \text{ such that } -u'' = f, \tag{31}$$

where H is the space defined in Homework 3.

2.1) Let $f \in \dot{L}^2(-\pi,\pi)$. Then $f \in L^2(-\pi,\pi)$, and, recalling from Homework 3, there exists $\{f_j\} \subset \mathbf{C}$ such that

$$f = \sum_j f_j e_j, \qquad f_j = (f, e_j). \tag{32}$$

Since $e_0 = \text{constant}$, we have $f_0 = (f, e_0) \propto \text{mean}(f) = 0$, so $f_0 = 0$. Furthermore, by an argument we used several times in Homework 3, the fact that $f = \bar{f}$ implies that $f_{-j} = \bar{f}_j$. Lastly, by Parseval's identity,

$$\sum_{j\neq 0} j^{-2}|f_j|^2 \leq \sum_{j\neq 0} |f_j|^2 = \|f\|_2^2 < \infty, \tag{33}$$

so $f \in H^{-1}$ from Homework 3 because $\{f_j\}_{j\neq 0} \in S_{H^{-1}}$. Therefore, $\dot{L}^2(-\pi, \pi) \subseteq H^{-1}$.

We claim that for $f \in \dot{L}^2(-\pi, \pi)$ and $u \in H$,

$$-u'' = f \qquad \iff \qquad B(u, v) = f(v) \quad \forall v \in H, \tag{34}$$

where we define the action of f on v in the same way as in Homework 3, and

$$B(u, v) = \sum_{j\neq 0} j^2 u_j \bar{v}_j, \qquad \{u_j\} = \varphi(u), \quad \{v_j\} = \varphi(v), \tag{35}$$

where $\varphi : H \to S_H$ is define as in Homework 3. We use essentially the same formal argument that we used on 2.5) in Homework 3.

Suppose that $-u'' = f$, and let $\{f_j\} = \psi(f)$, $\{u_j\} = \varphi(u)$. Formally differentiating the Fourier series for u, we have

$$-\sum_{j\neq 0} f_j e_j = -f = u'' = \sum_{j\neq 0} -j^2 u_j e_j. \tag{36}$$

Therefore, $f_j = j^2 u_j$ for all j, and for any $v \in H$,

$$B(u, v) = \sum_{j\neq 0} j^2 u_j \bar{v}_j = \sum_{j\neq 0} f_j \bar{v}_j = f(v). \tag{37}$$

On the other hand, suppose that $B(u, v) = f(v)$ for all $v \in H$. Clearly, $e_j + e_{-j} \in H$, and $e_j - e_{-j} \in H$, so

$$j^2 u_j + j^2 u_{-j} = f_j + f_{-j}, \qquad j^2 u_j - j^2 u_{-j} = f_j - f_{-j}, \tag{38}$$

which implies that $j^2 u_j = f_j$ for all j. By the same formal differentiation reasoning, it follows that $-u'' = f$. Additionally, we have $u \in H$ because

$$\bar{u}_{-j} = \frac{\bar{f}_{-j}}{j^2} = \frac{f_j}{j^2} = u_j, \qquad \sum_{j\neq 0} j^2 |u_j|^2 = \sum_{j\neq 0} j^2 \left|\frac{f_j}{j^2}\right|^2 \leq \sum_{j\neq 0} |f_j|^2 < \infty, \tag{39}$$

which implies that $\{u_j\} \in S_H$.

The function B is bilinear because for any $\alpha, \beta \in \mathbf{R}$, and any $u, v, w \in H$,

$$B(\alpha u + \beta v, w) = \sum_{j\neq 0} j^2(\alpha u_j + \beta v_j)\bar{w}_j = \alpha \sum_{j\neq 0} j^2 u_j \bar{w}_j + \beta \sum_{j\neq 0} j^2 v_j \bar{w}_j = \alpha B(u, w) + \beta B(v, w), \tag{40}$$

and

$$B(w, \alpha u + \beta v) = \sum_{j\neq 0} j^2 w_j \overline{\alpha u_j + \beta v_j} = \alpha \sum_{j\neq 0} j^2 w_j \bar{u}_j + \beta \sum_{j\neq 0} j^2 w_j \bar{v}_j = \alpha B(w, u) + \beta B(w, v) \tag{41}$$

because $\varphi(\alpha u + \beta v) = \alpha\varphi(u) + \beta\varphi(v)$.

The function B is also continuous because for any $u, v \in H$,

$$|B(u, v)| = \left|\sum_{j\neq 0} j^2 u_j \bar{v}_j\right| \leq \|u\|_H \|v\|_H \tag{42}$$

by the Cauchy-Schwarz inequality.

Lastly, B is coercive because for any $u \in H$,

$$B(u,u) = \sum_{j\neq 0} j^2|u_j|^2 = \|u_j\|_H^2. \tag{43}$$

Hence, the Lax-Milgram Theorem implies that, given $f \in \dot{L}^2(-\pi,\pi) \subseteq H^{-1} \subseteq H^*$, there is exists a unique $u \in H$ such that $B(u,v) = f(v)$ for all $v \in H$. That is, there exists a unique $u \in H$ such that $-u'' = f$.

2.2) Let $T : \dot{L}^2(-\pi,\pi) \to H$ denote the solution operator of (31), which exists by 2.1). Then T is compact as an operator on $\dot{L}^2(-\pi,\pi)$.

Proof. Given $f \in \dot{L}^2(-\pi,\pi)$, there exists $u \in H$ such that

$$\|Tf\|_H = \|u\|_H, \tag{44}$$

and $B(u,v) = f(v)$ for all $v \in H$. In particular, if we take $v = u$, we obtain

$$\|u\|_H^2 = f(u) = \sum_{j\neq 0} f_j \bar{u}_j \leq \|u\|_H \|f\|_{L^2}, \tag{45}$$

which implies that $\|Tf\|_H \leq \|f\|_{L^2}$, so T is bounded.

As we showed in 2.1), $Tf = u$ if and only if $f_j = j^2 u_j$, where $\{u_j\} = \varphi(u)$, and $\{f_j\} = \psi(f)$. Thus, $u_j = \frac{f_j}{j^2}$.

Let A be a bounded set in $\dot{L}^2(-\pi,\pi)$. Then $T(A)$ is bounded in H because T is a bounded operator from $\dot{L}^2(-\pi,\pi)$ to H. Then there exists $M > 0$ such that $\|u\|_H \leq M$ for all $u \in T(A)$.

Thus, for any $J > 0$ and any $u \in T(A)$,

$$\sum_{|j|>J} |u_j|^2 \leq \frac{1}{J^2} \sum_{|j|>J} j^2 |u_j^2| \leq \frac{M}{J^2}, \tag{46}$$

where $\{u_j\} = \varphi(u)$. We recall from Homework 3 that

$$\|u\|_{L^2}^2 = \sum_{j\neq 0} |u_j|^2, \tag{47}$$

so the tail of the series form of the L^2 norm of u is uniformly small for $u \in T(A)$, which is what allows us to establish the compactness of T.

Let $\{u^n\}$ be a sequence in $T(A)$. The space H is a Hilbert space by Homework 3, and $\{u^n\}$ is bounded because $T(A)$ is bounded, so there exists a weakly convergent subsequence of $\{u^n\}$, call it $\{u^{n_k}\}$. Then $\{u_j^{n_k}\}_k$ converges for all j, where $\{u_j^{n_k}\}$ are the Fourier coefficients of u^{n_k} with respect to $\{e_j\}$. In particular, each such sequence is Cauchy.

Let $\varepsilon > 0$ be given. By (46), we can choose J large enough that

$$\sum_{|j|>J} |u_j^{n_k}|^2 < \varepsilon^2 \tag{48}$$

for all k. We can also choose N large enough that $k, \ell > N$ implies that $|u_j^{n_k} - u_j^{n_\ell}|^2 < \frac{\varepsilon^2}{2J}$ for all $|j| \leq J$. Then

$$\|u^{n_k} - u^{n_\ell}\|_{L^2}^2 = \sum_{|j|\leq J} |u_j^{n_k} - u_j^{n_\ell}|^2 + \sum_{|j|>J} |u_j^{n_k} - u_j^{n_\ell}|^2 \tag{49}$$

$$\leq \varepsilon^2 + 2\sum_{|j|>J} |u_j^{n_k}|^2 + 2\sum_{|j|>J} |u_j^{n_\ell}|^2 \tag{50}$$

$$\leq 5\varepsilon^2, \tag{51}$$

so $k, \ell > N$ implies that $\|u^{n_k} - u^{n_\ell}\|_H < \sqrt{5}\varepsilon$.

This implies that $\{u^{n_k}\}_k$ is Cauchy in $L^2(-\pi,\pi)$. Since $L^2(-\pi,\pi)$ is a Hilbert space, it follows that $\{u^{n_k}\}$ converges to some $u \in L^2(-\pi,\pi)$. It remains to show that $u \in \dot{L}^2(-\pi,\pi)$.

We recall from Homework 3 that $\{u_j^{n_k}\}_{j\neq 0} = \varphi(u^{n_k})$, so $\overline{u_j^{n_k}} = u_{-j}^{n_k}$ for all k and all $j \neq 0$, and $u_0^{n_k} = 0$ for all k. Taking the limit as $k \to \infty$, we get $\bar{u}_j = \bar{u}_{-j}$ for all $j \neq 0$, and $u_0 = 0$. Thus, $\bar{u} = u$ by the same reasoning used several times in Homework 3, and $\mathrm{mean}(u) \propto (u, e_0) = u_0 = 0$; therefore, $u \in \dot{L}^2(-\pi,\pi)$ by definition.

Thus, every sequence in the bounded set $T(A)$ has a convergent subsequence, so $T(A)$ is pre-compact in H. This implies that T is compact by definition. □

2.3) The operator T is self-adjoint as an operator on $\dot{L}^2(-\pi,\pi)$.

Proof. Let $f, g \in \dot{L}^2(-\pi,\pi)$, and define $u = Tf$, and $v = Tg$. Then, by the same reasoning as in 2.1), if $\{u_j\} = \psi(u)$, $\{v_j\} = \psi(v)$, $\{f_j\} = \psi(f)$, and $\{g_j\} = \psi(g)$, then

$$u_j = \frac{f_j}{j^2}, \qquad v_j = \frac{g_j}{j^2}. \tag{52}$$

Thus,

$$(Tf, g) = (u, g) = \sum_{j\neq 0} u_j \bar{g}_j = \sum_{j\neq 0} f_j \bar{v}_j = (f, v) = (f, Tg), \tag{53}$$

so T is self-adjoint. □

2.4) Suppose that $Tf = \lambda f$ for $f \in H$, with $f \neq 0$. Note that since T is self-adjoint, we must have $\lambda \in \mathbf{R}$. By the reasoning in 2.1) giving the formula for T in terms of Fourier coefficients, we must have $j^{-2} f_j = \lambda f_j$ for all $j \neq 0$, where $\{f_j\} = \psi(f)$. Then either $f_j = 0$ or $\lambda = j^{-2}$ for all $j \neq 0$. We cannot have $f_j = 0$ for all j, because then $f = 0$ by the linearity of ψ^{-1}.

Thus, there exists some $k > 0$ such that $f_k \neq 0$, which implies that $\lambda = k^{-2}$. The equation $\lambda f_j = j^{-2} f_j$ for all j implies that $f_j = 0$ for all $j \neq \pm k$. Since $f_{-k} = \bar{f}_k$, it follows that $f = f_k e_k + \bar{f}_k e_{-k}$. Supposing that $f_j = a + ib$, we must have

$$f(x) = \frac{1}{\sqrt{2\pi}}\left((a+ib)e^{ikx} + (a-ib)e^{-ikx}\right) \tag{54}$$

$$= \frac{1}{\sqrt{2\pi}}\big(a\cos(kx) - b\sin(kx) + ib\cos(kx) + ia\sin(kx) \tag{55}$$

$$+ a\cos(kx) - b\sin(kx) - ib\cos(kx) - ia\sin(kx)\big) \tag{56}$$

$$= \frac{1}{\sqrt{2\pi}}\left(2a\cos(kx) - 2b\sin(kx)\right). \tag{57}$$

Thus, if λ is an eigenvalue of T, then $\lambda = k^{-2}$ for some integer $k > 0$, and the corresponding eigenvectors must be linear combinations of $\cos(kx)$ and $\sin(kx)$; that is, the corresponding eigenspace is

$\text{span}\{\cos(kx), \sin(kx)\}$. We note that $\cos(kx)$ and $\sin(kx)$ are indeed in $\dot{L}^2(-\pi,\pi)$ because they have mean zero on $(-\pi,\pi)$.

It is not hard to see by reversing the above logic that the converse is true, namely, that k^{-2} is an eigenvalue of T for all integers $k \neq 0$, and its corresponding eigenspace is $\text{span}\{\cos(kx), \sin(kx)\}$.

Thus, the eigenvalues of T are $\{\lambda_k = k^{-2}\}_{k>0}$, with corresponding eigenspaces $\{\text{span}\{\cos(kx), \sin(kx)\}\}_{k>0}$.

2.5) Let $c \in \mathbf{C}$ be given. Then for any $j > 0$, there exists $a, b \in R$ such that $ce_j + \bar{c}e_{-j} = a\cos(jx) + b\sin(jx)$; indeed, by the calculation in 2.4), we just need to choose $a = \sqrt{\frac{2}{\pi}}\Re(c)$, and $b = -\sqrt{\frac{2}{\pi}}\Im(c)$. Therefore, given $u \in \dot{L}^2(-\pi,\pi)$, the partial sum of the Fourier series for u can be written as a linear combination of elements of the set $\mathcal{B} = \{\cos(jx), \sin(jx)\}_{j>0} \subseteq \dot{L}^2(-\pi,\pi)$. Since $L^2(-\pi,\pi)$ is a Hilbert space, the Fourier series of u converges to u, so u is the limit of a sequence of elements of $\text{span}(\mathcal{B})$.

In other words, $\mathcal{B}$ is a basis for $\dot{L}^2(-\pi,\pi)$. In fact, it is an orthogonal basis, as we show now. Let $\{c_j^k\}_j = \psi(\cos(kx))$, and let $\{s_j^k\}_j = \psi(\sin(kx))$. Then, by the Euler formula relating e^{ijx} to $\sin(x)$ and $\cos(x)$,

$$c_j^k = \begin{cases} \sqrt{\frac{\pi}{2}} & j = |k| \\ 0 & \text{otherwise}, \end{cases} \qquad s_j^k = \begin{cases} -i\sqrt{\frac{\pi}{2}}\text{sgn}(j) & j = |k| \\ 0 & \text{otherwise}, \end{cases} \tag{58}$$

where $\text{sgn}(j)$ is the sign of j. Hence, for $k, \ell > 0$,

$$(\cos(kx), \cos(\ell x)) = \sum_{j\neq 0} c_j^k \overline{c_j^\ell} = \begin{cases} 0 & k \neq \ell \\ \frac{\pi}{2} & k = \ell, \end{cases} \tag{59}$$

$$(\cos(kx), \sin(\ell x)) = \sum_{j\neq 0} c_j^k \overline{s_j^\ell} = \begin{cases} 0 & k \neq \ell \\ \frac{\pi}{2}(-i+i) = 0 & k = \ell \end{cases} = 0, \tag{60}$$

$$(\sin(kx), \sin(\ell x)) = \sum_{j\neq 0} s_j^k \overline{s_j^\ell} = \begin{cases} 0 & k \neq \ell \\ \frac{\pi}{2} & k = \ell. \end{cases} \tag{61}$$

Therefore, $\mathcal{B}$ is orthogonal in H. Moreover, we can clearly modify the elements of $\mathcal{B}$ so that they are orthonormal; if we set $\mathcal{B}' = \left\{\sqrt{\frac{2}{\pi}}\cos(jx), \sqrt{\frac{2}{\pi}}\sin(jx)\right\}$, then $\mathcal{B}'$ is an orthonormal basis for $\dot{L}^2(-\pi,\pi)$.

Finally, we note that $\mathcal{B}'$ is the orthonormal basis that diagonalizes T in the sense of the spectral theorem for self-adjoint, compact operators. Let $c^k = \sqrt{\frac{2}{\pi}}\cos(kx)$, and let $s^k = \sqrt{\frac{2}{\pi}}\sin(kx)$. Define $u^k = T(c^k)$, and $v^k = T(s^k)$, and define $\{u_j^k\} = \psi(u^k)$, and $\{v_j^k\} = \psi(v^k)$. By our formula for T in terms of Fourier coefficients and by (58),

$$v_{\pm k}^k = k^{-2}s_{\pm k}^k \implies T(s^k) = \lambda_k s^k, \qquad u_{\pm k}^k = k^{-2}c_{\pm k}^k \implies T(c^k) = \lambda_k c^k. \tag{62}$$

Furthermore, given $f \in \dot{L}^2(-\pi,\pi)$, if $u = Tf$, $\{f_j\} = \psi(f)$, and $\{u_j\} = \varphi(u)$, then, on the one hand,

$$f(x) = \sum_{j\neq 0} f_j e_j(x) = \sum j > 0 f_j e_j(x) + \bar{f}_j e_{-j}(x) = \sum_{j>0} \Re(f_j)\sqrt{\frac{2}{\pi}}\cos(jx) - \Im(f_j)\sqrt{\frac{2}{\pi}}\sin(jx) \tag{63}$$

$$= \sum_{j>0} \Re(f_j)c^j(x) - \Im(f_j)s^j(x), \tag{64}$$

so that $\left(f, c^j\right) = \Re(f_j)$, and $(f, s^j) = -\Im(f_j)$ by the orthonormality of $\mathcal{B}'$. On the other hand,

$$Tf(x) = \sum_{j\neq 0} \frac{f_j}{j^2} e_j(x) = \sum_{j>0} j^{-2}(f_j e_j(x) + \bar{f}_j e_{-j}(x)) \tag{65}$$

$$= \sum_{j>0} \Re(f_j) j^{-2} \sqrt{\frac{2}{\pi}} \cos(jx) - \Im(f_j) j^{-2} \sqrt{\frac{2}{\pi}} \sin(jx) \tag{66}$$

$$= \sum_{j>0} \lambda_j (f, c^j) c^j(x) + \lambda_j (f, s^j) s^j(x), \tag{67}$$

which matches the diagonalization formula guaranteed by the spectral theorem for self-adjoint, compact operators.