1. $A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

$A^0 = I$, $A^1 = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $A^2 = \omega^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $A^3 = \omega^3 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

and $A^4 = \omega^4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \omega^4 I$

So $A^n = A^{4m+r}$, some $0 \le r < 4$, $m \ge 0$, and

$$A^n = A^{4m+r} = (A^4)^m A^r = (\omega^4 I)^m A^r = \omega^{4m} A^r$$

Therefore

$$A^n = \omega^n \cdot \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & n \equiv 0 \mod 4 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & n \equiv 1 \mod 4 \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & n \equiv 2 \mod 4 \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & n \equiv 3 \mod 4 \end{cases}$$

So $[A^n]_{11} = [A^n]_{22}$, and $[A^n]_{12} = -[A^n]_{21}$; $\therefore [e^{At}]_{11} = [e^{At}]_{22}$ and $[e^{At}]_{12} = -[e^{At}]_{21}$.

$[A^n]_{11} \neq 0$ iff n is even, in which case it is 1 if $n \equiv 0 \mod 4$

and -1 if $n \equiv 2 \mod 4$, so

$$[e^{At}]_{11} = \sum_{n=0}^{\infty} \frac{t^n}{n!} [A^n]_{11} = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} (-1)^k \omega^{2k} = \cos(\omega t)$$

And $[A^n]_{12} \neq 0$ iff n is odd, in which case it is 1 if $n \equiv 1 \mod 4$ and -1 if $n \equiv 3 \mod 4$, so

$$\{e^{At}\}_{12} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \{A^n\}_{12} = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} (-1)^k \omega^{2k+1} = \sin(\omega t)$$

Then $e^{At} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$.

2.

$$M\left(\begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) = M\left(\begin{bmatrix} x\cos\omega t + y\sin\omega t \\ -x\sin\omega t + y\cos\omega t \end{bmatrix}\right)$$

$$= x\cos\omega t + y\sin\omega t + i(-x\sin\omega t + y\cos\omega t)$$

$$= x(\cos\omega t - i\sin\omega t) + iy(\cos\omega t - i\sin\omega t)$$

$$= xe^{-i\omega t} + iye^{-i\omega t}$$

$$= e^{-i\omega t}(x+iy)$$

$$= e^{-i\omega t} M\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$