Math 5601 Midterm Project

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Throughout this project, we consider the IVP

$$
\begin{aligned}
y' &= f(t,y), & a \le t \le b \qquad (1)\\
y(a) &= g_a, & g_a \in \mathbf{R}. \qquad (2)
\end{aligned}
$$

We also use the mesh with sample points $t_j = a + jh$, with $t_0 = a$, where $h > 0$ is the step size. Lastly, we assume that f is L-Lipschitz in y uniformly for $t \in [a,b]$ (so that the solution of (1-2) is unique).

Problem 1.

Using the Taylor expansion for y about t_j, we get

$$y(t_{j+1}) = y(t_j) + hy'(t_j) + \frac{h^2}{2}y''(t_j) + \mathcal{O}(h^3). \tag{3}$$

Similarly, expanding y about t_{j+1} gives

$$y(t_j) = y(t_{j+1}) - hy'(t_{j+1}) + \frac{h^2}{2}y''(t_{j+1}) + \mathcal{O}(h^3). \tag{4}$$

Further expanding y'' about t_j, we get

$$
\begin{aligned}
y(t_j) &= y(t_{j+1}) - hy'(t_{j+1}) + \frac{h^2}{2}\left(y''(t_j) + \mathcal{O}(h)\right) + \mathcal{O}(h^3) && (5)\\
&= y(t_{j_1}) - hy'(t_{j+1}) + \frac{h^2}{2}y''(t_j) + \mathcal{O}(h^3). && (6)
\end{aligned}
$$

Rearranging (6) and (3) and substituting from (1), we get

$$
\begin{aligned}
\frac{y(t_{j+1}) - y(t_j)}{h} &= y'(t_j) + \frac{h}{2}y''(t_j) + \mathcal{O}(h^2) &&= f(t_j, y(t_j)) + \frac{h}{2}y''(t_j) + \mathcal{O}(h^2), && (7)\\
\frac{y(t_{j+1}) - y(t_j)}{h} &= y'(t_{j+1}) - \frac{h}{2}y''(t_j) + \mathcal{O}(h^2) &&= f(t_{j+1}, y(t_{j+1})) - \frac{h}{2}y''(t_j) + \mathcal{O}(h^2). && (8)
\end{aligned}
$$

If we take the average of both sides of (7) and (8), then we finally obtain

$$\frac{y(t_{j+1}) - y(t_j)}{h} = \frac{f(t_{j+1}, y(t_{j+1})) + f(t_j, y(t_j))}{2} + \mathcal{O}(h^2). \tag{9}$$

Thus, if $y_j = y(t_j)$ and we compute y_{j+1} using the trapezoidal scheme, that is, as the solution of

$$y_{j+1} = y_j + h \cdot \frac{f(t_{j+1}, y_{j+1}) + f(t_j, y_j)}{2}, \tag{10}$$

then y_{j+1} (assuming the solution of (10) is unique) will satisfy the estimate

$$|y_{j+1} - y(t_{j+1})| = \frac{h}{2} \cdot |f(t_{j+1}, y(t_{j+1})) - f(t_{j+1}, y_{j+1})| + \mathcal{O}(h^3). \tag{11}$$

Using the Lipschitz property of f, we obtain

$$|y_{j+1} - y(t_{j+1})| \le \frac{hL}{2} \cdot |y_{j+1} - y(t_{j+1})| + \mathcal{O}(h^3), \tag{12}$$

so

$$|y_{j+1} - y(t_{j+1})| \cdot \left(1 - \frac{hL}{2}\right) \le \mathcal{O}(h^3). \tag{13}$$

As $h \to 0$, the quantity $1 - \frac{hL}{2} \to 1$; therefore,

$$|y_{j+1} - y(t_{j+1})| = \mathcal{O}(h^3). \tag{14}$$

That is, the *local truncation error* of the trapezoidal scheme is of order 3, which means that the accuracy of the method as a whole is of order 2.

Problem 2.

Consider the Taylor expansion of y about t_{j+1} at the points t_{j-1}, t_j and t_{j+1}:

$$y(t_{j-1}) = y(t_{j+1}) - 2hy'(t_{j+1}) + 2h^2y''(t_{j+1}) + \mathcal{O}(h^3) \tag{15}$$

$$y(t_j) = y(t_{j+1}) - hy'(t_{j+1}) + \frac{h^2}{2}y''(t_{j+1}) + \mathcal{O}(h^3) \tag{16}$$

$$y(t_{j+1}) = y(t_{j+1}). \tag{17}$$

If we form the linear combination $3y(t_{j+1}) - 4y(t_j) + y(t_{j-1})$, then we get

$$\begin{aligned} 3y(t_{j+1}) - 4y(t_j) + y(t_{j-1}) = \quad & 3y(t_{j+1}) && (18)\\ & - 4y(t_{j+1}) + 4hy'(t_{j+1}) - 2y''(t_{j+1})h^2 && (19)\\ & + y(t_j) - 2hy'(t_j) + 2y''(t_j)h^2 + \mathcal{O}(h^3). && (20) \end{aligned}$$

Therefore, canceling terms and substituting from (1), we have

$$3y(t_{j+1}) - 4y(t_j) + y(t_{j-1}) = hf(t_{j+1}, y(t_{j+1})) + \mathcal{O}(h^3) \tag{21}$$

Thus, if we know that $y_{j-1} = y(t_{j-1})$, and $y(t_j) = y_j$ and we compute y_{j+1} using the two-step backward differentiation scheme, that is, as the solution of

$$\frac{3y_{j+1} - 4t_j + y_{j-1}}{2h} = hf(t_{j+1}, y_{j+1}), \tag{22}$$

then the local truncation error $|y_{j+1} - y(t_{j+1})|$ will satisfy

$$|y_{j+1} - y(t_{j+1})| = h|f(t_{j+1}, y_{j+1}) - f(t_{j+1}, y(t_{j+1}))| + \mathcal{O}(h^3). \tag{23}$$

By the Lipschitz property of f,

$$|y_{j+1} - y(t_{j+1})| \le hL|y_{j+1} - y(t_{j1+})| + \mathcal{O}(h^3), \tag{24}$$

so

$$|y_{j+1} - y(t_{j+1})|(1 - hL) \le \mathcal{O}(h^3). \tag{25}$$

As $h \to 0$, the quantity $(1 - hL) \to 1$; therefore,

$$|y_{j+1} - y(t_{j+1})| = \mathcal{O}(h^3). \tag{26}$$

That is, the *local trunction error* of the two-step backward differentiation scheme is of order 3, and the accuracy of the method as a whole is of order 2.

Problem 3.

Consider the Taylor expansions of $y(t_{j+1})$, $y(t_j)$, $y(t_{j-1})$, and $y(t_{j-2})$ about t_{j+1}:

$$y(t_{j+1}) = y(t_{j+1}) \tag{27}$$

$$y(t_j) = y(t_{j+1}) - hy'(t_{j+1}) + \frac{h^2}{2}y''(t_{j+1}) - \frac{h^3}{6}y'''(t_{j+1}) + \mathcal{O}(h^4) \tag{28}$$

$$y(t_{j-1}) = y(t_{j+1}) - 2hy'(t_{j+1}) + 2h^2y''(t_{j+1}) - \frac{4h^3}{3}y'''(t_{j+1}) + \mathcal{O}(h^4) \tag{29}$$

$$y(t_{j-2}) = y(t_{j+1}) - 3hy'(t_{j+1}) + \frac{9h^2}{2}y''(t_{j+1}) - \frac{9h^3}{2}y'''(t_{j+1}) + \mathcal{O}(h^4) \tag{30}$$

Then, for $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbf{R}$,

$$\beta_1 y(t_{j+1}) + \beta_2 y(t_j) + \beta_3 y(t_{j-1}) + \beta_4 y(t_{j-2}) = \tag{31}$$

$$(\beta_1 + \beta_2 + \beta_3 + \beta_4)y(t_{j+1}) \tag{32}$$

$$- (\beta_2 + 2\beta_3 + 3\beta_4)y'(t_{j+1})h \tag{33}$$

$$+ \frac{1}{2}(\beta_2 + 4\beta_3 + 9\beta_4)y''(t_{j+1})h^2 \tag{34}$$

$$- \frac{1}{6}(\beta_2 + 8\beta_3 + 27\beta_4)y'''(t_{j+1})h^3 + \mathcal{O}(h^4). \tag{35}$$

To cancel the lower-order terms, we must choose β_2, β_3, and β_4 such that

$$\begin{aligned} -1 &= \beta_2 + \beta_3 + \beta_4 \\ 0 &= \beta_2 + 4\beta_3 + 9\beta_4 \\ 0 &= \beta_2 + 8\beta_3 + 27\beta_4, \end{aligned} \tag{36}$$

then we get

$$\frac{y(t_{j+1}) + \beta_2 y(t_j) + \beta_3 y(t_{j-1}) + \beta_4 y(t_{j-2})}{-(\beta_2 + 2\beta_3 + 3\beta_4)h} = y'(t_{j+1}) + \mathcal{O}(h^4) = f(t_{j+1}, y(t_{j+1})) + \mathcal{O}(h^3). \tag{37}$$

To satisfy (36), we must have $4\beta_3 + 9\beta_4 = 8\beta_3 + 27\beta_4$, so $\beta_3 = -\frac{9}{2}\beta_4$. Then $\beta_2 = 18\beta_4 - 9\beta_4 = 9\beta_4$, and $-1 = 9\beta_4 - \frac{9}{2}\beta_4 + \beta_4 = \frac{11}{2}\beta_4$, so $\beta_4 = -\frac{2}{11}$. Then $\beta_3 = \frac{9}{11}$, and $\beta_2 = -\frac{18}{11}$. Lastly, $-(\beta_2 + 2\beta_3 + 3\beta_4) = \frac{6}{11}$. If we set

$$\alpha_1 = \frac{1}{-(\beta_2 + 2\beta_3 + 3\beta_4)} = \frac{11}{6} \tag{38}$$

$$\alpha_2 = \frac{\beta_2}{-(\beta_2 + 2\beta_3 + 3\beta_4)} = -\frac{18}{6} \tag{39}$$

$$\alpha_3 = \frac{\beta_3}{-(\beta_2 + 2\beta_3 + 3\beta_4)} = \frac{9}{6} \tag{40}$$

$$\alpha_4 = \frac{\beta_4}{-(\beta_2 + 2\beta_3 + 3\beta_4)} = -\frac{2}{6}, \tag{41}$$

then by (37),

$$\frac{\alpha_1 y(t_{j+1}) + +\alpha_2 y(t_j) + \alpha_3 y(t_{j-1}) + \alpha_4 y(t_{j-2})}{h} = f(t_{j+1}, y(t_{j+1})) + \mathcal{O}(h^3). \tag{42}$$

If we had $y_{j-2} = y(t_{j-2})$, $y_{j-1} = y(t_{j-1})$, and $y_j = y(t_j)$, and we computed y_{j+1} as the solution of

$$\frac{\alpha_1 y(t_{j+1}) + +\alpha_2 y(t_j) + \alpha_3 y(t_{j-1}) + \alpha_4 y(t_{j-2})}{h} = f(t_{j+1}, y_{j+1}), \tag{43}$$

then $|y_{j+1} - y(t_{j+1})$ would satisfy

$$|y_{j+1} - y(t_{j+1})| = |f(t_{j+1}, y_{j+1}) - f(t_{j+1}, y(t_{j+1}))| + \mathcal{O}(h^3). \tag{44}$$

Using the Lipschitz property of f, we obtain

$$|y_{j+1} - y(t_{j+1})|(1 - hL) \leq \mathcal{O}(h^4). \tag{45}$$

As $h \to 0$, the quantity $1 - hL \to 1$; therefore,

$$|y_{j+1} - y(t_{j+1})| = \mathcal{O}(h^3). \tag{46}$$

That is, the implicit scheme

$$\frac{\alpha_1 y(t_{j+1}) + +\alpha_2 y(t_j) + \alpha_3 y(t_{j-1}) + \alpha_4 y(t_{j-2})}{h} = f(t_{j+1}, y_{j+1}) \tag{47}$$

with $\alpha_1 = \frac{11}{6}$, $\alpha_2 = -\frac{18}{6}$, $\alpha_3 = \frac{9}{6}$, and $\alpha_4 = -\frac{2}{6}$ has 3rd-order accuracy. Since we had to choose these values of α to cancel higher-order terms, these must be the coefficients in the third-order backward differentiation scheme.

We now consider Newton's method for finding the root of a function f:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{48}$$

Problem 4.

Suppose that f has a root z of multiplicity $m \geq 2$. Then, by definition, there exists a function r such $r(z) \neq 0$, and $f(x) = (x-z)^m r(x)$. Then $f'(x) = m(x-z)^{m-1} r(x) + (x-z)^m r'(x) = (x-z)^{m-1}(mr(x) + (x-z)r'(x))$. Then we can still safely define Newton's method despite the fact that $f'(z) = 0$ by setting

$$g(x) = x - \frac{f'(x)}{f(x)} = x - \frac{(x-z)^m r(x)}{(x-z)^{m-1}(mr(x) + (x-z)r'(x))} = x - \frac{(x-z)r(x)}{mr(x) + (x-z)r'(x)} \tag{49}$$

and observing that the denominator in the last expression is nonzero when $x = z$ because $r(z) \neq 0$. Then Newton's method becomes $x_{k+1} = g(x_k)$.

To apply the theory of convergence in the project description, we need to compute

$$g'(x) = 1 - \frac{(r(x) + (x-z)r'(x))(mr(x) + (x-z)r'(x)) - (x-z)r(x)(mr'(x) + r'(x) + (x-z)r''(x))}{(mr(x) + (x-z)r'(x))^2}$$

so that

$$g'(z) = 1 - \frac{m(r(z))^2}{(mr(z))^2} = 1 - \frac{1}{m} \tag{50}$$

since $r(z) \neq 0$. Since $g'(z) \neq 0$ if $m \geq 2$, but $|g'(z)| < 1$, it follows by the convergence theorem in the project description that Newton's method has *linear* convergence in this case.

Problem 5.

In the case that f has a root z of multiplicity $m \geq 2$, we saw that Newton's method defined by $x_{k+1} = g(x_k)$, where

$$g(x) = x - \frac{(x-z)r(x)}{mr(x) + (x-z)r'(x)} \tag{51}$$

had a linear convergence rate to the root z of f. We can fix this simply by adjusting Newton's method to

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}, \tag{52}$$

that is, by replacing g by g_m, where

$$g_m(x) = x - m\frac{(x-z)r(x)}{mr(x) + (x-z)r'(x)}. \tag{53}$$

This method has at least quadratic convergence by the convergence theorem in the project description because

$$g_m'(x) = 1 - m\frac{(r(x) + (x-z)r'(x))(mr(x) + (x-z)r'(x)) - (x-z)r(x)(mr'(x) + r'(x) + (x-z)r''(x))}{(mr(x) + (x-z)r'(x))^2}$$

so that

$$g_m'(z) = 1 - m\frac{m(r(z))^2}{(mr(z))^2} = 0. \tag{54}$$

Then the iteration $x_{k+1} = g_m(x_k)$ converges at least quadratically to the root z of f by the convergence theorem in the project description.