Math 6417 Homework 3

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Question 1.

Let $B(\cdot, \cdot)$ be a continuous, bilinear form on a real Hilbert space H. Suppose that B is coercive in the sense that there is some $\alpha > 0$ such that $B(x, x) \geq \alpha \|x\|^2$ for all $x \in H$.

1.1) Let $y \in H$. Then the map $f_y : H \to \mathbf{R}$ defined by $f_y(x) = B(x, y)$ is a bounded linear functional on H. Consequently, there exists a unique $w \in H$ such that $B(x, y) = f_y(x) = (x, w)$ for all $x \in H$.

Proof. Firstly, it is clear that f_y is linear; indeed, given $a_1, a_2 \in \mathbf{R}$ and $x_1, x_2 \in H$,

$$f_y(a_1x_1 + a_2x_2) = B(a_1x_1 + a_2x_2, y) = a_1B(x_1, y) + a_2B(x_2, y) = a_1f_y(x_1) + a_2f_y(x_2) \tag{1}$$

by the bilinearity of B.

Secondly, $B(\cdot, y) = f_y$ must be continuous because B is continuous. Hence, f_y is bounded.

Thirdly, by the Riesz representation theorem, there exists a unique $w \in H$ such that $B(x, y) = f_y(x) = (x, w)$ for all $x \in H$. □

1.2) Given $y \in H$, by 1.1), there is a unique $w \in H$ such that $B(x, y) = (x, w)$ for all $x \in H$; this defines a function $A : H \to H$, where $Ay = w$. Then A is a bounded, linear operator on H, that is, $A \in B(H)$.

Proof. There are two steps to this proof: showing that A is linear, and showing that A is bounded.

Step 1: linearity

Let $a_1, a_2 \in \mathbf{R}$ and $y_1, y_2 \in H$. Then for all $x \in H$,

$$\begin{aligned}(x, A(a_1y_1 + a_2y_2)) &= B(x, a_1y_1 + a_2y_2) = a_1B(x, y_1) + a_2B(x, y_2) = a_1(x, Ay_1) + a_2(x, Ay_2) \\ &= (x, a_1Ay_1 + a_2Ay_2).\end{aligned} \tag{2}$$

Thus, $w = A(a_1y_1 + a_2y_2)$ and $w' = a_1Ay_1 + a_2Ay_2$ satisfy the property that $B(x, a_1y_1 + a_2y_2) = (x, w) = (x, w')$ for all $x \in H$. By the Riesz representation theorem, there is only one element of H that can satisfy this property; therefore, $w = w'$, or $A(a_1y_1 + a_2y_2) = a_1Ay_1 + a_2Ay_2$. Thus, A is linear.

Step 2: boundedness

Note that B is continuous if and only if (see, e.g., Theorem 8.10 assumption (a) in Arbogast and Bona) there exists some $M > 0$ such that

$$|B(x, y)| \leq M\|x\|\|y\|, \quad \text{for all } x, y \in H. \tag{3}$$

Let $y \in H$. Then

$$\|Ay\| = \left|\left(\frac{Ay}{\|Ay\|}, Ay\right)\right| = \left|B\left(\frac{Ay}{\|Ay\|}, y\right)\right| \leq M\|y\|. \tag{4}$$

Since y was arbitrary, it follows that A is bounded, and $\|A\| \leq M$. □

1.3) A is bounded below in the sense that there exists $\gamma > 0$ such that $\|Ay\| \geq \gamma\|y\|$ for all $y \in H$.

Proof. This follows from the coercivity of B: for all $y \in H$,

$$\|Ay\|\|y\| \geq |(y, Ay)| = |B(y, y)| \geq \alpha\|y\|^2, \tag{5}$$

so $\|Ay\| \geq \alpha\|y\|$ for all $y \in H$, as claimed. □

1.4) A is one-to-one, and the range of A is closed.

Proof. Let $y_1, y_2 \in H$, and suppose that $Ay_1 = Ay_2$. Then, by the previous part,

$$\|y_1 - y_2\| \leq \frac{1}{\gamma}\|A(y_1 - y_2)\| = \frac{1}{\gamma}\|Ay_1 - Ay_2\| = 0. \tag{6}$$

Therefore, $y_1 = y_2$. This shows that A is one-to-one.

Let $R(A)$ denote the range of A, and let $\{w_n\} \subseteq R(A)$ be a convergent sequence in the range of A. By the definition of $R(A)$, there exists $y_n \in H$ such that $w_n = Ay_n$.

Let $\varepsilon > 0$ be given. Since $\{w_n\}$ is convergent, it is also Cauchy, so we can choose N such that $n, m > N$ implies that $\|w_n - w_m\| < \varepsilon$. By the linearity of A, we have $A(y_n - y_m) = w_n - w_m$; hence,

$$\begin{aligned} \alpha\|y_n - y_m\|^2 &\leq B(y_n - y_m, y_n - y_m) = (y_n - y_m, w_n - w_m) \leq \|y_n - y_m\| \cdot \|w_n - w_m\| && (7) \\ &\leq \varepsilon\|y_n - y_m\| && (8) \end{aligned}$$

if $n, m > N$. Thus, $n, m > N$ implies that $\|y_n - y_m\| < \frac{\varepsilon}{\alpha}$. This implies that $\{y_n\}$ Cauchy.

Since H is complete, there exists $y \in H$ such that $y_n \to y$ as $n \to \infty$. By the continuity of A,

$$Ay = \lim_{n\to\infty} Ay_n = \lim_{n\to\infty} w_n = w. \tag{9}$$

This means that $w \in R(A)$. Since the convergent sequence $\{w_n\} \subseteq R(A)$ was arbitrary, and its limit $w \in R(A)$, it follows that $R(A)$ is closed. □

1.5) A is onto.

Proof. Suppose that $x \in R(A)^\perp$, that is, $(x, w) = 0$ for all $w \in R(A)$. This implies that $(x, Ay) = 0$ for all $y \in H$, which is equivalent to saying that $B(x, y) = 0$ for all $y \in H$. In particular, if we choose $y = x$, then $\|x\|^2 \leq \frac{1}{\alpha}B(x, x) = 0$. Therefore, $x = 0$. This shows that $R(A)^\perp = \{0\}$ because x was arbitrary.

Let $y \in H$. Since $R(A)$ is a closed subspace of H by (1.4), there exists a best approximation $w \in R(A)$ of y, which satisfies the property $(y - w, x) = 0$ for all $x \in R(A)$ (Theorem 3.7 and Corollary 3.8 in Arbogast and Bona). That is, $y - w \in R(A)^\perp$. Since $R(A)^\perp = \{0\}$ by the above, it follows that $y - w = 0$, and $y = w \in R(A)$. Since y was arbitrary and $R(A) \subseteq H$, it follows that $R(A) = H$, that is, A is onto. □

1.6) A is invertible.

Proof. By the previous two parts, A is bijective, so it has a set-theoretic inverse function A^{-1}. By 1.2), A is bounded and linear, and by 1.5) it is surjective. Therefore, by the open mapping theorem, A maps open sets to open sets, which means that the preimage of an open set under A^{-1} is open, that is, A^{-1} is continuous. Therefore, A is invertible. □

1.7) Given $f \in H^*$, the Riesz representation theorem implies that there exists a unique $w \in H$ such that $f(x) = (x, w)$ for all $x \in H$, and we can view H^* and H as the same under the correspondence $f \leftrightarrow w$.

1.8) Consider the equation $B(x, y) = f(x)$ for all $x \in H$, where $f \in H^*$. By the remark in part 1.7), we can choose $w \in H$ such that $f(x) = (x, w)$ for all $x \in H$. Then the equation is equivalent to $B(x, y) = (x, w)$ for all $x \in H$. If y is a solution of this equation, then, by the definition of A, we must have $Ay = w$. Using the invertibility of A, we obtain $y = A^{-1}w$ as the unique solution of the equation. Viewing f and w as the same under the correspondence in 1.7), we might also write $y = A^{-1}f$.

Question 2.

Let $e_j \in L^2(-\pi, \pi)$ be defined by $e_j(x) = \frac{1}{\sqrt{2\pi}} e^{ijx}$ for $j \in \mathbf{Z}$. Define

$$H = \left\{ f \in L^2(-\pi, \pi) : f = \bar{f} \text{ and } f = \sum_{j \neq 0} f_j e_j \text{ for some } \{f_j\} \subseteq \mathbf{C} \text{ such that } \sum_{j \neq 0} j^2 |f_j|^2 < \infty \right\}, \tag{10}$$

and

$$H^{-1} = \left\{ f = \sum_{j \neq 0} f_j e_j : \sum_{j \neq 0} j^{-2} |f_j|^2 < \infty \text{ and } f = \bar{f} \right\}. \tag{11}$$

2.1) H and H^{-1} are real Hilbert spaces when equipped with the inner products

$$(f, g)_H = \sum_{j \neq 0} j^2 f_j \bar{g}_j, \qquad (f, g)_{H^{-1}} = \sum_{j \neq 0} j^{-2} f_j \bar{g}_j. \tag{12}$$

Proof. Before we can show that H and H^{-1} are real Hilbert spaces, we need to study their definitions. For the definition H, there isn't much trouble – its elements are just elements of $L^2(-\pi, \pi)$. For the definition H^{-1}, the series might not converge to an element of $L^2(-\pi, \pi)$ (say, $f_j = 1$ for all j), so understanding what the convergence of the infinite sum means is more difficult.

As we will see, we can understand H in terms of a corresponding space of coefficient sequences, which is isomorphic to H as a real vector space. By analogy, then, we can understand H^{-1} in terms of its corresponding space of coefficient sequences, which will suffice for our purposes.

To understand these spaces of coefficient sequences, we introduce the following Lemma showing that H is a real vector space isomorphic to its space of coefficient sequences.

Lemma 1. *Define*

$$S_H = \left\{ \{f_j\}_{j \neq 0} \subseteq \mathbf{C} : \sum_{j \neq 0} j^2 |f_j|^2 < \infty \text{ and } \forall j \in \boldsymbol{Z} \setminus \{0\}, f_j = \bar{f}_{-j} \right\}. \tag{13}$$

Then S_H is a real vector space, and there is an isomorphism (of real vector spaces) $\varphi : H \to S_H$ such that if $\varphi(f) = \{f_j\}$, then

$$f = \sum_{j \neq 0} f_j e_j. \tag{14}$$

Proof. We need to prove that φ is well-defined, bijective, and, after showing that H and S_H are real vector spaces, that φ is a linear mapping between them.

Step 1: definition of φ

Let $f \in H$. Then $f \in L^2(-\pi,\pi)$. Recalling from our lecture that $\{e_j\}_{j\in\mathbf{Z}}$ is an orthonormal basis for $L^2(-\pi,\pi)$, it follows that there exists exactly one sequence of coefficients $\{g_j\} \in \ell^2(\mathbf{Z})$ such that

$$f = \sum_{j\in\mathbf{Z}} g_j e_j. \tag{15}$$

On the other hand, since $f \in H$, there must be a sequence of coefficients $\{f_j\} \subseteq \mathbf{C}$ such that

$$f = \sum_{j\neq 0} f_j e_j, \qquad \sum_{j\neq 0} j^2 |f_j|^2 < \infty, \tag{16}$$

It follows by the uniqueness of $\{g_j\}$ that $g_0 = 0$, and $g_j = f_j$ for all $j \neq 0$. Furthermore, since $f = \bar{f}$, it follows that

$$\bar{f}(x) = \frac{1}{\sqrt{2\pi}} \overline{\sum_{j\neq 0} f_j e^{ijx}} = \frac{1}{\sqrt{2\pi}} \sum_{j\neq 0} \bar{f}_j e^{-ijx} = \frac{1}{\sqrt{2\pi}} \sum_{j\neq 0} \bar{f}_{-j} e^{ijx} = f(x). \tag{17}$$

That is, $f = \sum_{j\neq 0} \bar{f}_{-j} e_j$; by the uniqueness of $\{g_j\}$ again, we must have $\bar{f}_{-j} = g_j = f_j$ for all $j \neq 0$. Hence, $\{f_j\} \in S_H$. Since $\{f_j\}$ is uniquely determined by f by the uniqueness of $\{g_j\}$, we can define a function $\varphi : H \to S_H$ by $\varphi(f) = \{f_j\}$.

Step 2: φ is a one-to-one correspondence

First, suppose that $\varphi(f) = \{f_j\} = \varphi(g)$ for some $f, g \in H$ and $\{f_j\} \in S_H$. Then, by the definition of φ,

$$f = \sum_{j\neq 0} f_j e_j = g, \tag{18}$$

so φ is one-to-one.

Second, let $\{f_j\} \in S_H$. Since $L^2(-\pi,\pi)$ is a Hilbert space, its Riesz-Fischer map $F : L^2(-\pi,\pi) \to \ell^2(\mathbf{Z})$ corresponding to the orthonormal basis $\mathcal{B}$ is an isomorphism. If we set $g_j = f_j$ for $j \neq 0$, and $g_0 = 0$, then we have

$$\sum_{j\in\mathbf{Z}} |g_j|^2 = \sum_{j\neq 0} |f_j|^2 \leq \sum_{j\neq 0} j^2 |f_j|^2 < \infty \tag{19}$$

because $\{f_j\} \in S_H$. Therefore, $\{g_j\} \in \ell^2(\mathbf{Z})$, and $f = F^{-1}(\{g_j\}) \in L^2(-\pi,\pi)$. By the definition of F and the fact that $\{e_j\}$ is an orthonormal basis, we have

$$f = F^{-1}(\{g_j\}) = \sum_{j\in\mathbf{Z}} g_j e_j = \sum_{j\neq 0} f_j e_j. \tag{20}$$

Since $\{f_j\} \in S_H$, we have $f_j = \bar{f}_{-j}$, so

$$\bar{f}(x) = \frac{1}{\sqrt{2\pi}} \overline{\sum_{j\neq 0} f_j e^{ijx}} = \frac{1}{\sqrt{2\pi}} \sum_{j\neq 0} \bar{f}_j e^{-ijx} = \frac{1}{\sqrt{2\pi}} \sum_{j\neq 0} \bar{f}_{-j} e^{ijx} = \frac{1}{\sqrt{2\pi}} \sum_{j\neq 0} f_j e^{ijx} = f(x), \tag{21}$$

so $f = \bar{f}$. Since $\{f_j\} \in S_H$, we also have $\sum_{j\neq 0} j^2 |f_j|^2 < \infty$. Therefore, $f \in H$ by definition.

Finally, $\varphi(f) = \{f_j\}$ because $f = \sum_{j\neq 0} f_j e_j$, and $\varphi(f)$ is, by definition, the unique sequence of coefficients in S_H that make that statement true.

Thus, φ is one-to-one and onto.

Step 3: H and S_H are real vector spaces

H is a nonempty subset (it contains 0) of the real vector space $L^2(-\pi, \pi)$, and S_H is a nonempty subset (it contains 0) of the real vector space $\ell^2(\mathbf{Z} \setminus \{0\})$. Thus, it suffices to show that H and S_H are subspaces of these two vector spaces.

Let $\{f_j\}, \{g_j\} \in S_H$, and let $\alpha, \beta \in \mathbf{R}$. Then $\{h_j\} = \alpha\{f_j\} + \beta\{g_j\} \in S_H$ because $\bar{h}_{-j} = \alpha\bar{f}_{-j} + \beta\bar{g}_{-j} = \alpha f_j + \beta g_j = h_j$, and

$$\sum_{j \neq 0} j^2 |h_j|^2 = \sum_{j \neq 0} j^2 |\alpha f_j + \beta g_j|^2 \leq 2\alpha^2 \sum_{j \neq 0} j^2 |f_j|^2 + 2\beta^2 \sum_{j \neq 0} j^2 |g_j|^2 < \infty \tag{22}$$

since $|\alpha f_j + \beta g_j|^2 \leq 2(\alpha^2 |f_j|^2 + \beta^2 |g_j|^2)$ for all j. Thus, S_H is a real vector subspace of the space of all sequences of complex numbers.

Now let $f, g \in H$, and $\alpha, \beta \in \mathbf{R}$. Set $\{f_j\} = \varphi(f)$, and $\{g_j\} = \varphi(g)$. Then $\{h_j\} = \alpha\{f_j\} + \beta\{g_j\} \in S_H$, so we can define $h = \varphi^{-1}(\{h_j\})$. By the definition of h, we have

$$\alpha f + \beta g = \alpha \sum_{j \neq 0} f_j e_j + \beta \sum_{j \neq 0} g_j e_j = \sum_{j \neq 0} (\alpha f_j + \beta g_j) e_j = h \in H. \tag{23}$$

Therefore, H is a (real) vector subspace of $L^2(-\pi, \pi)$.

Step 4: φ is linear

Let $f, g \in H$, and let $\alpha, \beta \in \mathbf{R}$. Define $\{f_j\} = \varphi(f)$, and $\{g_j\} = \varphi(g)$. Then

$$\alpha f + \beta g = \alpha \sum_{j \neq 0} f_j e_j + \beta \sum_{j \neq 0} g_j e_j = \sum_{j \neq 0} (\alpha f_j + \beta g_j) e_j. \tag{24}$$

This implies that $\alpha\varphi(f) + \beta\varphi(g) = \{\alpha f_j + \beta g_j\} = \varphi(\alpha f + \beta g)$ by the definition of φ. Thus, φ is linear. □

The Lemma establishes that H is essentially the same as S_H as a vector space. If we define by analogy

$$S_{H^{-1}} = \left\{ \{f_j\}_{j \neq 0} \subseteq \mathbf{C} : \sum_{j \neq 0} j^{-2} |f_j|^2 < \infty \text{ and } \forall j \in \mathbf{Z} \setminus \{0\}, f_j = \bar{f}_{-j} \right\}, \tag{25}$$

then we can understand H^{-1} to be a real vector space isomorphic to $S_{H^{-1}}$. Note that $S_{H^{-1}}$ is indeed a real vector space; like S_H, it is a nonempty subset (it contains 0) of the vector space of all sequences of complex numbers, and if $\{f_j\}, \{g_j\} \in S_{H^{-1}}$, and $\alpha, \beta \in \mathbf{R}$, then $\{h_j\} = \alpha\{f_j\} + \beta\{g_j\} \in S_H$ because $\bar{h}_{-j} = \alpha\bar{f}_{-j} + \beta\bar{g}_{-j} = \alpha f_j + \beta g_j = h_j$, and

$$\sum_{j \neq 0} j^{-2} |h_j|^2 = \sum_{j \neq 0} j^{-2} |\alpha f_j + \beta g_j|^2 \leq 2\alpha^2 \sum_{j \neq 0} j^{-2} |f_j|^2 + 2\beta^2 \sum_{j \neq 0} j^{-2} |g_j|^2 < \infty \tag{26}$$

since $|\alpha f_j + \beta g_j|^2 \leq 2(\alpha^2 |f_j|^2 + \beta^2 |g_j|^2)$ for all j.

We notice that it is possible for the series in the definition of H^{-1} to converge for some $\{f_j\} \in S_{H^{-1}}$. In particular, if $\{f_j\} \in \ell^2(\mathbf{Z} \setminus \{0\})$, then the series converges to a function $f \in L^2(-\pi, \pi)$. Moreover, by the same reasoning in Lemma 1, the function f is uniquely determined by the coefficients $\{f_j\}$, and vice versa.

Thus, when $f \in L^2(-\pi, \pi)$, and the coefficients $\{f_j\}$ of f with respect to the orthonormal basis $\{e_j\}$ belong to $\ell^2(\mathbf{Z} \setminus \{0\})$ with $f_0 = 0$ and $f_j = \bar{f}_{-j}$, it makes sense to view f as an element of H^{-1}, and we can define a function $\psi : H^{-1} \to S_{H^{-1}}$ by $\psi(f) = \{f_j\}_{j \neq 0}$. By the exact same reasoning in Lemma 1, we can easily verify that ψ is one-to-one and linear. This gives us an interpretation of at least some of the elements of H^{-1}; from now on, we will simply assume that ψ can be extended to an isomorphism between H^{-1} and $S_{H^{-1}}$.

Now we turn to the issue of equipping H and H^{-1} with inner products. The given inner products are defined in terms of the sequence representations of elements of H and H^{-1}; this is well-defined due to the (actual) isomorphism between H and S_H and the (assumed) isomorphism between H^{-1} and $S_{H^{-1}}$. It also allows us to work with $S_{H^{-1}}$ without needing to worry about how to interpret its elements – we will only need to work with the sequence representations in $S_{H^{-1}}$.

We need to show that H and H^{-1} are real inner product spaces when equipped with $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_{H^{-1}}$ as inner products. We wrap this into the following Lemma.

Lemma 2. *For $G \in \{H, H^{-1}\}$, define*

$$\rho = \rho_G = \begin{cases} \varphi & G = H, \\ \psi & G = H^{-1}, \end{cases} \qquad s = s_G = \begin{cases} 1 & G = H, \\ -1 & G = H^{-1}. \end{cases} \tag{27}$$

Then G is a real inner product space with inner product $(\cdot,\cdot)_G$ defined by

$$(f,g)_G = \sum_{j\neq 0} j^{2s} f_j \bar{g}_j, \qquad f, g \in G, \tag{28}$$

where $\{f_j\} = \rho(f), \{g_j\} = \rho(g) \in S_G$ are the coefficients of f and g in S_G.

Proof. As we have already remarked, the uniqueness of $\{f_j\}$ and $\{g_j\}$ implies that $(f,g)_G$ is well-defined. We still need to show that the series converges to a real number, and that $(\cdot,\cdot)_G$ is symmetric, linear in the first argument, and positive definite.

Step 1: $(f,g)_G$ is a real number

Let $f, g \in G$ with $\{f_j\} = \rho(f)$, and $\{g_j\} = \rho(g)$. Then the series for $(f,g)_G$ converges absolutely because

$$\sum_{j\neq 0} j^{2s} |f_j| \cdot |\bar{g}_j| \leq \left(\sum_{j\neq 0} j^{2s}|f_j|^2\right)^{\frac{1}{2}} \left(\sum_{j\neq 0} j^{2s}|g_j|^2\right)^{\frac{1}{2}} < \infty \tag{29}$$

by the Cauchy-Schwarz inequality.

Next, $(f,g)_G \in \mathbf{R}$ because

$$\overline{(f,g)_G} = \overline{\sum_{j\neq 0} j^{2s} f_j \bar{g}_j} = \sum_{j\neq 0} j^{2s} \bar{f}_j g_j = \sum_{j\neq 0} j^{2s} f_{-j}\bar{g}_{-j} = \sum_{j\neq 0} j^{2s} f_j \bar{g}_j = (f,g)_G, \tag{30}$$

and only real numbers are equal to their own complex conjugate.

Step 2: $(\cdot,\cdot)_G$ is symmetric

Let $f, g \in G$, and let $\{f_j\} = \rho(f)$, and $\{g_j\} = \rho(g)$. Then

$$(f,g)_G = \sum_{j\neq 0} j^{2s} f_j \bar{g}_j = \sum_{j\neq 0} j^{2s} f_{-j}\bar{g}_{-j} = \sum_{j\neq 0} j^{2s} g_j \bar{f}_j = (g,f)_G. \tag{31}$$

Therefore, $(\cdot,\cdot)_G$ is symmetric.

Step 3: $(\cdot,\cdot)_G$ is linear in the first argument

Let $f, g, h \in G$, and let $\{f_j\} = \rho(f)$, $\{g_j\} = \rho(g)$, and $\{h_j\} = \rho(h)$. Let $\alpha, \beta \in \mathbf{R}$. Then $\rho(\alpha f + \beta g) = \alpha\rho(f) + \beta\rho(g)$ because ρ is linear, so

$$(\alpha f + \beta g, h)_G = \sum_{j\neq 0} j^{2s}(\alpha f_j + \beta g_j)\bar{h}_j = \alpha \sum_{j\neq 0} j^{2s} f_j \bar{h}_j + \beta \sum_{j\neq 0} j^{2s} g_j \bar{h}_j = \alpha(f,h)_G + \beta(g,h)_G. \tag{32}$$

Therefore, $(\cdot,\cdot)_G$ is linear in the first argument.

Step 4: $(\cdot,\cdot)_G$ is positive definite

Let $f \in G$, and let $\{f_j\} = \rho(f)$. Then

$$(f,f)_G = \sum_{j\neq 0} j^{2s}|f_j|^2 \geq 0 \tag{33}$$

because each term of the series is nonnegative. Moreover, if $f = 0$, then $|f_j|^2 = 0$ (by the linearity of ρ), so each term is 0, and $(f,f)_G = 0$. Conversely, if $(f,f)_G = 0$, then, since each term of the series for $(f,f)_G$ is nonnegative, it must be that each term is 0. This implies that $f_j = 0$ for all $j \neq 0$, that is, $\{f_j\} = 0$ in S_G. Therefore $f = \rho^{-1}(0) = 0$ by the linearity of ρ^{-1}.

This shows that $(\cdot,\cdot)_G$ is positive definite. □

Now we know that H and H^{-1} are real inner product spaces, so there is only one more thing to show in order to prove that they are real Hilbert spaces: they need to be complete with respect to the norms $\|\cdot\|_H$ and $\|\cdot\|_{H^{-1}}$ induced by their inner products. For the sake of organization, we put this proof inside one last Lemma.

Lemma 3. *For $G \in \{H, H^{-1}\}$ and $s = s_G$, $\rho = \rho_G$ defined as in Lemma 2, the space G is complete with respect to the norm $\|\cdot\|_G$ induced by the inner product $(\cdot,\cdot)_G$ from Lemma 2.*

Proof. Let $\{f^n\}_{n=1}^\infty$ be a Cauchy sequence in G with respect to $\|\cdot\|_G$. We need to show that there exists $f \in G$ such that $f^n \to f$ as $n \to \infty$ in $\|\cdot\|_G$. To do this, we first identify a candidate element, then we show that the sequence converges to the candidate.

Step 1: identifying a candidate limit

Given $\varepsilon > 0$, we can choose N such that $n, m > N$ implies that

$$\varepsilon^2 > \|f^n - f^m\|_G^2 = \sum_{j\neq 0} j^{2s}|f_j^n - f_j^m|^2, \tag{34}$$

where $\{f_j^n\} = \rho(f^n)$. Since each term in the summation is nonnegative, this means that $|j^s f_j^n - j^s f_j^m| < \varepsilon$ for all $n, m > N$. Then $\{j^s f_j^n\}_{n=1}^\infty$ is Cauchy for all $j \neq 0$. Thus, by the completeness of $\mathbf{C}$, the sequence $\{j^s f_j^n\}$ converges to a limit $j^s f_j \in \mathbf{C}$ as $n \to \infty$.

Since $j^s \bar{f}_j^n = j^s f_{-j}^n$ for all j and all n, taking the limit as $n \to \infty$ and using the continuity of the complex conjugate function, we get $j^s \bar{f}_j = j^s f_{-j}$ for all $j \neq 0$. Since $j^s \neq 0$, it follows that $\bar{f}_j = f_{-j}$ for all $j \neq 0$.

Furthermore, by the convergence of $\{j^s f_j^n\}$ to $j^s f_j$, for all $J > 0$, we can choose n large enough that $|j^s f_j - j^s f_j^n|^2 \leq \frac{1}{2J}$ for all $0 < |j| \leq J$. Then

$$\sum_{0<|j|\leq J} j^{2s}|f_j|^2 = \sum_{0<|j|\leq J} |j^s f_j - j^s f_j^n + j^s f_j^n|^2 \leq 2\sum_{0<|j|\leq J} |j^s f_j - j^s f_j^n|^2 + 2\sum_{0<|j|\leq J} j^{2s}|f_j^n|^2 \tag{35}$$

$$\leq 2 + 2\|f^n\|_G^2. \tag{36}$$

Since $\{f^n\}$ is Cauchy in $\|\cdot\|_G$, it must be bounded; that is, there exists $M > 0$ such that $\|f^n\|_G \leq M$ for all n. Then

$$\sum_{0<|j|\leq J} j^{2s}|f_j|^2 \leq 2 + 2M^2 \tag{37}$$

for all $J > 0$. This implies that

$$\sum_{j\neq 0} j^{2s}|f_j|^2 \leq 2 + 2M^2 < \infty. \tag{38}$$

Therefore, $\{f_j\} \in S_G$, and we can define our candidate limit as $f = \rho^{-1}(\{f_j\})$.

Step 2: showing that the candidate is the limit

Let $\varepsilon > 0$ be given. Since $\{f^n\}$ is Cauchy, we can choose N such that $n, m > N$ implies that $\|f^n - f^m\|_G^2 < \varepsilon$. By the convergence of $\{j^s f_j^n\}$, for any $J > 0$, we can choose $m_J > N$ such that for all $0 < |j| \le J$ we have $|j^s f_j - j^s f_j^{m_J}|^2 < \frac{\varepsilon}{2J}$. Then

$$\sum_{0<|j|\le J} j^{2s}|f_j - f_j^n|^2 = \sum_{0<|j|\le J} |(j^s f_j - j^s f_j^{m_J}) + (j^s f_j^{m_J} - j^s f_j^n)|^2 \tag{39}$$

$$\le 2\sum_{0<|j|\le J} |j^s f_j - j^s f_j^{m_J}|^2 + 2\sum_{0<|j|\le J} j^{2s}|f^{m_J} - f_j^n|^2 \tag{40}$$

$$\le 2\varepsilon + 2\|f^{m_J} - f^n\|_G^2 \le 4\varepsilon \tag{41}$$

if $n > N$. Since this estimate is independent of J, it follows that

$$\|f - f^n\|_G^2 = \sum_{j\ne 0} j^{2s}|f_j - f_j^n| \le 4\varepsilon \tag{42}$$

if $n > N$. Therefore, $f^n \to f$ as $n \to \infty$ in $\|\cdot\|_G$. ☐

The three Lemmas above establish that H and H^{-1} are Hilbert spaces with the given inner products. ☐

2.2) Define the bilinear form B on H by

$$B(f, g) = \sum_{j\ne 0} (ij + j^2) f_j \bar{g}_j, \tag{43}$$

where $\{f_j\} = \varphi(f)$, and $\{g_j\} = \varphi(g)$. Then B satisfies the hypotheses of the Lax-Milgram theorem.

Proof. As with the inner products, the isomorphism φ ensures that $B(f, g)$ is well-defined in terms of $\{f_j\} = \varphi(f)$ and $\{g_j\} = \varphi(g)$. We need to show that the series for B converges and that B is actually bilinear over $\mathbf{R}$. Then, we need to show that B satisfies the hypotheses of the Lax-Milgram theorem, that is, that B is continuous and coercive.

Step 1: B is well-defined and bilinear

Let $f, g \in H$ with $\{f_j\} = \varphi(f)$ and $\{g_j\} = \varphi(g)$. Then the series for $B(f, g)$ converges absolutely because

$$\sum_{j\ne 0} |(ij + j^2) f_j \bar{g}_j| \le \left(\sum_{j\ne 0} j\sqrt{1 + j^2}|f_j|^2\right)^{\frac{1}{2}} \left(\sum_{j\ne 0} j\sqrt{1 + j^2}|g_j|^2\right)^{\frac{1}{2}} \le \sqrt{2}\|f\|_H\|g\|_H < \infty \tag{44}$$

by the Cauchy-Schwarz inequality (note that $|ij + j^2| = j\sqrt{1 + j^2} \le \sqrt{2}j^2$ for all $j \ne 0$).

Let $f, g, h \in H$, and let $\alpha, \beta \in \mathbf{R}$. Set $\{f_j\} = \varphi(f)$, $\{g_j\} = \varphi(g)$, and $\{h_j\} = \varphi(h)$. By the linearity of φ, we have $\varphi(\alpha f + \beta g) = \alpha\varphi(f) + \beta\varphi(g)$; therefore,

$$B(\alpha f + \beta g, h) = \sum_{j\ne 0} (ij + j^2)(\alpha f_j + \beta g_j)\bar{h}_j = \alpha \sum_{j\ne 0} (ij + j^2) f_j \bar{h}_j + \beta \sum_{j\ne 0} (ij + j^2) g_j \bar{h}_j \tag{45}$$

$$= \alpha B(f, h) + \beta B(g, h). \tag{46}$$

Similarly,

$$B(h, \alpha f + \beta g) = \sum_{j \neq 0} (ij + j^2) h_j \overline{(\alpha f_j + \beta g_j)} = \alpha \sum_{j \neq 0} (ij + j^2) h_j \bar{f}_j + \beta \sum_{j \neq 0} (ij + j^2) h_j \bar{g}_j \tag{47}$$

$$= \alpha B(h, f) + \beta B(h, g). \tag{48}$$

Thus, B is bilinear.

Step 2: B is continuous and coercive

We have practically already shown that B is continuous: by (44),

$$|B(f, g)| \leq \sqrt{2} \|f\|_H \|g\|_H \tag{49}$$

for all $f, g \in H$ with $\{f_j\} = \varphi(f)$ and $\{g_j\} = \varphi(g)$. This implies that B is continuous because B is bilinear.

For coercivity, observe that

$$B(f, f) = \sum_{j \neq 0} (ij + j^2) |f_j|^2 = \sum_{j \neq 0} j^2 |f_j|^2 = \|f\|_H^2 \tag{50}$$

because $f_{-j} = \bar{f}_j$. Thus, B is coercive. □

2.3) $H^{-1} \subseteq H^*$ if we assign the following action to $f \in H^{-1}$:

$$f(g) = \sum_{j \neq 0} f_j \bar{g}_j, \qquad g \in H, \tag{51}$$

where $\{g_j\} = \varphi(g)$, and $\{f_j\} = \psi(f)$.

Proof. Let $f \in H^{-1}$. Note that, because H is a real Hilbert space, an element of H^* should be a real-valued, bounded linear functional. Thanks to the isomorphisms φ and ψ, the action of $f \in H^{-1}$ is well-defined, but we still need to show that the series converges to a real number. Then, we need to show that f is linear over $\mathbf{R}$ and bounded. Then we will have $f \in H^*$, completing the proof.

Step 1: $f(g)$ converges to a real number

Let $f \in H^{-1}$ and $g \in H$, with $\{f_j\} = \psi(f)$, and $\{g_j\} = \varphi(g)$. Then the series for $f(g)$ converges absolutely because

$$\sum_{j \neq 0} |f_j| |\bar{g}_j| \leq \left(\sum_{j \neq 0} j^{-2} |f_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j \neq 0} j^2 |g_j|^2 \right)^{\frac{1}{2}} = \|f\|_{H^{-1}} \cdot \|g\|_H \tag{52}$$

by the Cauchy-Schwarz inequality.

Next, $f(g) \in \mathbf{R}$ because

$$\overline{f(g)} = \overline{\sum_{j \neq 0} f_j \bar{g}_j} = \sum_{j \neq 0} \bar{f}_j g_j = \sum_{j \neq 0} f_{-j} \bar{g}_{-j} = \sum_{j \neq 0} f_j \bar{g}_j = f(g), \tag{53}$$

and only real numbers equal their own complex conjugate.

Step 2: f is linear and bounded

Let $f \in H^{-1}$ and $g, h \in H$ with $\{f_j\} = \psi(f)$, and $\{g_j\} = \varphi(g)$, $\{h_j\} = \varphi(h)$. Let $\alpha, \beta \in \mathbf{R}$. Then, by the linearity of φ, we have $\varphi(\alpha g + \beta h) = \alpha \varphi(g) + \beta \varphi(h)$. Hence,

$$f(\alpha g + \beta h) = \sum_{j \neq 0} f_j \overline{(\alpha g_j + \beta h_j)} = \alpha \sum_{j \neq 0} f_j \bar{g}_j + \beta \sum_{j \neq 0} f_j \bar{h}_j = \alpha f(g) + \beta f(h). \tag{54}$$

Therefore, f is linear.

We have practically already shown that f is bounded. By (52), we have

$$|f(g)| \leq \|f\|_{H^{-1}} \cdot \|g\|_H \tag{55}$$

for all $g \in G$. Therefore, f is bounded.

Hence, $f \in H^*$ for all $f \in H^{-1}$, using the action we have assigned to f to view it as a functional on H. Thus, $H^{-1} \subseteq H^*$. ☐

2.4) For every $f \in H^{-1}$, there exists $u \in H$ such that

$$B(x,u) = f(x) \quad \text{for all } x \in H. \tag{56}$$

Proof. After all the work on the previous parts, this is a simple application of the Lax-Milgram theorem. Since B satisfies the assumptions of the theorem by 2.2, it follows by the Lax-Milgram theorem that for all $f \in H^*$, there exists $u \in H$ such that (56) is true. Since $H^{-1} \subseteq H^*$ by 2.3, given $f \in H^{-1}$, we have $f \in H^*$, so we can find $u \in H$ such that (56) holds. ☐

2.5) Suppose that u satisfies (56). Then, formally, u solves the ODE

$$-(u' + u'') = f. \tag{57}$$

Proof. Set $\{u_j\} = \varphi(u)$, and $\{f_j\} = \psi(f)$. Formally,

$$u'(x) = \sum_{j\neq 0} iju_j e^{ijx}, \qquad u''(x) = -\sum_{j\neq 0} j^2 u_j e^{ijx}. \tag{58}$$

Define $\{u_j'\} = \{iju_j\}$, and $\{u_j''\} = \{-j^2 u_j\}$. Then $\{u_j'\}, \{u_j''\} \in S_{H^{-1}}$ because $\{u_j\} \in S_H$ implies that

$$\sum_{j\neq 0} j^{-2}|iju_j|^2 \leq \sum_{j\neq 0} j^2|u_j|^2 < \infty, \qquad \sum_{j\neq 0} j^{-2}|-j^2u_j|^2 \leq \sum_{j\neq 0} j^2|u_j|^2 < \infty, \tag{59}$$

and $\overline{u_{-j}'} = \overline{-iju_{-j}} = iju_j = u_j'$, and $\overline{u_{-j}''} = \overline{-j^2u_{-j}} = -j^2u_j = u_j''$.

Thus, for all $x \in H$, if $\{x_j\} = \varphi(x)$, then

$$\sum_{j\neq 0} f_j\bar{x}_j = f(x) = B(x,u) = \sum_{j\neq 0}(ij + j^2)x_j\bar{u}_j = \sum_{j\neq 0}(-ij + j^2)x_{-j}\bar{u}_{-j} = -\sum_{j\neq 0}(u_j' + u_j'')\bar{x}_j. \tag{60}$$

We can choose $\{x_j\} \in S_H$ such that $x_j = 0$ if $j \neq \pm k$ and $x_{\pm k} = 1$. Then (60) implies that

$$f_{-k} + f_k = -(u_{-k}' + u_k' + u_{-k}'' + u_k''). \tag{61}$$

We can also choose $\{x_j\} \in S_H$ such that $x_j = 0$ if $j \neq \pm k$ and $x_{\pm k} = \pm i$. Then (60) implies that

$$-f_{-k} + f_k = -(-u_{-k}' + u_k' - u_{-k}'' + u_k''). \tag{62}$$

Together, (61) and (62) imply that $f_j = -(u_j' + u_j'')$ for all $j \neq 0$. This implies that $f = -(u' + u'')$ by the linearity of ψ.

☐

This can be made into a classical ODE solution by adding assumptions. For example, if $u \in C^2[-\pi,\pi]$, and u is periodic, then the Fourier series of u and u' are continuous, and u' and u'' are piecewise smooth, meaning that the Fourier series of u and u' may be differentiated term-by-term.

2.6) Any bounded set in H is pre-compact in $L^2(-\pi,\pi)$.

Proof. Let A be a bounded set in H; that is, there is some $M > 0$ such that $\|g\|_H^2 \le M$ for all $g \in A$. We recall that $\{e_j\}$ is an orthonormal basis in $L^2(-\pi,\pi)$. Hence, for any $f \in L^2(-\pi,\pi)$ (see Arbogast and Bona, Theorem 3.18 part (iii)),

$$\|f\|_{L^2(-\pi,\pi)}^2 = \sum_j |f_j|^2, \qquad f_j = (f, e_j). \tag{63}$$

If $f \in H$ as well, then $\{f_j\} = \varphi(f)$, so

$$\|f\|_{L^2(-\pi,\pi)}^2 = \sum_j |f_j|^2 \le \sum_{j\neq 0} j^2|f_j|^2 = \|f\|_H^2. \tag{64}$$

This implies that A is bounded in $L^2(-\pi,\pi)$ in addition to H. Then $\overline{A}$ is also bounded in $L^2(-\pi,\pi)$, because the closure of a bounded set is bounded. We need to show that $\overline{A}$ is also compact.

To this end, let $\{f^n\}$ be a sequence in $\overline{A}$. Since $\overline{A}$ is bounded in $L^2(-\pi,\pi)$, and $L^2(-\pi,\pi)$ is a Hilbert space, there exists a weakly convergent subsequence $\{f^{n_k}\}$ of $\{f^n\}$. That is, there exists $f \in L^2(-\pi,\pi)$ such that $f_j^{n_k} \to f_j$ for all j, where $f_j^{n_k} = (f^{n_k}, e_j)$, and $f_j = (f, e_j)$. If we can show that $f^{n_k} \to f$ strongly as well, then we are done.

Before that, however, we need the following fact: for all $g \in A$ and all $J > 0$,

$$\sum_{|j|>J} |g_j|^2 \le \frac{1}{J^2}\sum_{|j|>J} j^2|g_j|^2 \le \frac{M}{J^2}, \qquad \{g_j\} = \varphi(g). \tag{65}$$

Therefore, given $\varepsilon > 0$, we can choose $J > 0$ such that for all $g \in A$,

$$\sum_{|j|>J} |g_j|^2 < \varepsilon, \qquad \{g_j\} = \varphi(g). \tag{66}$$

Now we show that $\{f^{n_k}\}$ converges strongly. Since $L^2(-\pi,\pi)$ is complete, we only need to show $\{f^{n_k}\}$ is Cauchy in $\|\cdot\|_{L^2(-\pi,\pi)}$.

Let $\varepsilon > 0$ be given. Then we can choose $J > 0$ such that for all $g \in A$,

$$\sum_{|j|>J} |g_j|^2 < \varepsilon, \qquad \{g_j\} = \varphi(g). \tag{67}$$

Since $\{f_j^{n_k}\}_{k=1}^\infty$ is convergent for all j, it is also Cauchy for all j. Thus, we can choose K large enough that $k, \ell > K$ implies that $|f_j^{n_k} - f_j^{n_\ell}|^2 < \frac{\varepsilon}{2J+1}$ for all $0 \le |j| \le J$. Then

$$\|f^{n_k} - f^{n_\ell}\|_{L^2(-\pi,\pi)}^2 = \sum_{|j|\le J} |f_j^{n_k} - f_j^{n_\ell}|^2 + \sum_{|j|>J} |f_j^{n_k} - f_j^{n_\ell}|^2 \le \varepsilon + \sum_{|j|>J} |f_j^{n_k} - f_j^{n_\ell}|^2 \tag{68}$$

if $k, \ell > K$. Since $f^{n_k} \in \overline{A}$, for all k, we can choose $g^k \in A$ such that $\|f^{n_k} - g^k\|_{L^2(-\pi,\pi)}^2 < \varepsilon$. Hence, if $g_j^k = \varphi(g^k)$, then

$$\begin{aligned}
\|f^{n_k} - f^{n_\ell}\|_{L^2(-\pi,\pi)}^2 &\le \varepsilon + \sum_{|j|>J} |(f_j^{n_k} - g_j^k) + (g_j^k - g_j^\ell) + (g_j^\ell - f_j^{n_\ell})|^2 && (69)\\
&\le \varepsilon + 3\sum_{|j|>J} |f_j^{n_k} - g_j^k|^2 + 3\sum_{|j|>J} |f^{n_\ell} - g_j^\ell|^2 + 3\sum_{|j|>J} |g_j^k - g_j^\ell|^2 && (70)\\
&\le \varepsilon + 3\|f^{n_k} - g^k\|_{L^2(-\pi,\pi)}^2 + 3\|f^{n_\ell} - g^\ell\|_{L^2(-\pi,\pi)}^2 && (71)\\
&\quad + 6\sum_{|j|>J} |g_j^k|^2 + 6\sum_{|j|>J} |g_j^\ell|^2 && (72)\\
&\le 19\varepsilon && (73)
\end{aligned}$$

if $k, \ell > K$. This implies that $\{f^{n_k}\}$ is Cauchy in $\|\cdot\|_{L^2(-\pi,\pi)}$. Therefore, $\{f^{n_k}\}$ converges in $L^2(-\pi, \pi)$.

Thus, any sequence in $\overline{A}$ has a convergent subsequence. This implies that $\overline{A}$ is compact, and A is pre-compact. $\square$