

# Math 6418 Homework 1

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Define

$$k_\varepsilon(x) = \frac{1}{\varepsilon} \chi_{[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]}(x). \quad (1)$$

1.

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For  $u \in \mathcal{D}'(\mathbf{R})$ , define  $u * k_\varepsilon \in \mathcal{D}'(\mathbf{R})$  by

$$\langle u * k_\varepsilon, \varphi \rangle = \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi)(y) \, dy, \quad \varphi \in \mathcal{D}(\mathbf{R}). \quad (2)$$

Note that the integral is defined because  $u * R\varphi$  is continuous. Furthermore,  $u * k_\varepsilon$  is a distribution because it is linear and continuous.

## Linearity

We can verify linearity easily using the linearity of convolution with a test function, the linearity of integration, and the linearity of reflection. If  $\alpha, \beta \in \mathbf{R}$  and  $\varphi, \psi \in \mathcal{D}(\mathbf{R})$ , then

$$\langle u * k_\varepsilon, \alpha\varphi + \beta\psi \rangle = \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R(\alpha\varphi + \beta\psi))(y) \, dy \quad (3)$$

$$= \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * (\alpha R\varphi + \beta R\psi))(y) \, dy \quad (4)$$

$$= \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} [\alpha(u * R\varphi)(y) + \beta(u * R\psi)(y)] \, dy \quad (5)$$

$$= \alpha \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi)(y) \, dy + \beta \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\psi)(y) \, dy \quad (6)$$

$$= \alpha \langle u * k_\varepsilon, \varphi \rangle + \beta \langle u * k_\varepsilon, \psi \rangle, \quad (7)$$

so  $u * k_\varepsilon$  is linear.

## Continuity

Let  $\varphi_n \rightarrow \varphi$  in  $\mathcal{D}(\mathbf{R})$ . Then clearly  $R\varphi_n \rightarrow R\varphi$  in  $\mathcal{D}(\mathbf{R})$ , and by the continuity of  $u$ ,  $u * R\varphi_n \rightarrow u * R\varphi$  pointwise. If  $\{u * R\varphi_n\}_{n=1}^\infty$  is uniformly bounded on  $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ , then the bounded convergence theorem implies that

$$\frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi_n)(y) \, dy \rightarrow \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi)(y) \, dy; \quad (8)$$

that is,  $\langle u * k_\varepsilon, \varphi_n \rangle \rightarrow \langle u * k_\varepsilon, \varphi \rangle$ . Thus,  $u * k_\varepsilon$  is continuous.

To show that  $\{u * R\varphi_n\}$  is uniformly bounded on  $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ , suppose for the sake of contradiction that it is not. Then for all  $m > 0$ , there exists  $\varphi_{n_m}$  and  $x_m \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$  such that

$$m \leq |(u * \varphi_{n_m})(x_m)| = |\langle u, \tau_{x_m} R\varphi_{n_m} \rangle|.$$

There is a convergent subsequence  $\{x_{m_k}\}_{k=1}^\infty$  such that  $x_{m_k} \rightarrow x \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ . The convergence of  $\{x_{m_k}\}$  and the convergence of  $\{\varphi_{n_{m_k}}\}$  in  $\mathcal{D}(\mathbf{R})$  ensure that  $\tau_{x_{m_k}} R\varphi_{n_{m_k}} \rightarrow \tau_x R\varphi$  in  $\mathcal{D}(\mathbf{R})$  as  $k \rightarrow \infty$ . Then, by the continuity of  $u$ ,

$$\infty = \lim_{k \rightarrow \infty} |\langle u, \tau_{x_{m_k}} R\varphi_{n_{m_k}} \rangle| = |\langle u, \tau_x R\varphi \rangle| < \infty,$$

which is a contradiction.

### Extension

Definition (2) is a good definition of convolution at least in the sense that it reduces to convolution with  $k_\varepsilon$  for regular distributions. Indeed, suppose that  $f \in L^1_{\text{loc}}(\mathbf{R})$ . Then for any  $\varphi \in \mathcal{D}(\mathbf{R})$ ,

$$\langle f * k_\varepsilon, \varphi \rangle = \int_{-\infty}^{\infty} (f * k_\varepsilon)(x) \varphi(x) dx \quad (9)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) k_\varepsilon(x - y) \varphi(x) dy dx \quad (10)$$

$$= \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} f(y) \varphi(x) dy dx. \quad (11)$$

Using the change of variables  $y' = y - x$ , we get

$$\langle f * k_\varepsilon, \varphi \rangle = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} f(y' + x) \varphi(x) dy' dx \quad (12)$$

$$= \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \int_{-\infty}^{\infty} f(y' + x) \varphi(x) dx dy'. \quad (13)$$

Using the change of variables  $x' = y' + x$ , we get

$$\langle f * k_\varepsilon, \varphi \rangle = \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \int_{-\infty}^{\infty} f(x') \varphi(x' - y') dx' dy' \quad (14)$$

$$= \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (f * R\varphi)(y') dy', \quad (15)$$

which agrees with our definition of  $f * k_\varepsilon$  in (2) if we view  $f$  as a distribution.

## 2.

Consider  $\delta_0 * k_\varepsilon$  using our definition of convolution from (2). Since  $\delta_0$  is supposed to be the identity for the convolution operator, we expect that  $\delta_0 * k_\varepsilon = k_\varepsilon$  (viewing  $k_\varepsilon$  as a distribution).

This turns out to be the case. According to the definition in (2), for any  $\varphi \in \mathcal{D}(\mathbf{R})$ ,

$$\langle \delta_0 * k_\varepsilon, \varphi \rangle = \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (\delta_0 * R\varphi)(y) dy \quad (16)$$

$$= \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} R\varphi(y) dy \quad \text{because } \delta_0 \text{ is identity for convolution} \quad (17)$$

$$= \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \varphi(y) dy \quad \text{using change of variables } y \mapsto -y \quad (18)$$

$$= \langle k_\varepsilon, \varphi \rangle. \quad (19)$$

Thus,  $\delta_0 * k_\varepsilon = k_\varepsilon$ , viewing  $k_\varepsilon$  as a distribution.

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**3.**

Since  $\int k_\varepsilon = 1$  all  $\varepsilon$ , and  $k_\varepsilon(x) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  if  $x \neq 0$ , it would seem that  $k_\varepsilon$  behaves like  $\delta_0$  as  $\varepsilon \rightarrow 0$ . Thus, it would make sense that  $u * k_\varepsilon \rightarrow u * \delta_0 = u$ . That is, it would make sense that  $u * k_\varepsilon \rightarrow u$  as  $\varepsilon \rightarrow 0$  in the topology of  $\mathcal{D}'(\mathbf{R})$ .

In fact, this turns out to be the case. Let  $\varphi \in \mathcal{D}(\mathbf{R})$ . Since  $u * R\varphi$  is  $C^\infty$  and therefore continuous, it has an antiderivative  $\psi$ . Then

$$\lim_{\varepsilon \rightarrow 0} \langle u * k_\varepsilon, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi)(y) \, dy \quad (20)$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{\psi\left(\frac{\varepsilon}{2}\right) - \psi\left(-\frac{\varepsilon}{2}\right)}{\varepsilon} = \psi'(0) \quad (21)$$

$$= (u * R\varphi)(0) = \langle u, \tau_0 R\varphi \rangle \quad (22)$$

$$= \langle u, \varphi \rangle. \quad (23)$$

Hence,  $u * k_\varepsilon \rightarrow u$  in  $\mathcal{D}'(\mathbf{R})$  as  $\varepsilon \rightarrow 0$ .

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**4.**

For  $u, v \in \mathcal{D}'(\mathbf{R})$  we are tempted to define  $u * v \in \mathcal{D}'(\mathbf{R})$  by

$$\langle u * v, \varphi \rangle = \langle u, Rv * \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbf{R}). \quad (24)$$

Unfortunately, although  $Rv * \varphi$  is  $C^\infty$ , it may not be compactly supported. For example, if  $Rv = 1$ , then  $Rv * \varphi = \int \varphi$ , which is not compactly supported as long as  $\int \varphi \neq 0$ . Thus, it may not make sense to take the action of  $u$  on  $Rv * \varphi$ .

Furthermore, this definition would obviously be linear in  $\varphi$  by the linearity of convolution and of  $u$ , but to prove continuity, we would need to prove that  $Rv * \varphi_n \rightarrow Rv * \varphi$  in  $\mathcal{D}(\mathbf{R})$  whenever  $\varphi_n \rightarrow \varphi$  in  $\mathcal{D}(\mathbf{R})$ . So, not only would we need that  $Rv * \varphi_n$  is compactly supported, but we would need that each  $Rv * \varphi_n$  is supported in the same compact set.

If we could provide some restriction on  $v$  so that  $Rv$  could allay these concerns, then there would be some hope of defining  $u * v$  as above. Alternatively, we might be able to restrict  $u$  so that its action could be extended to a function that is not compactly supported. This is essentially what we did in part 1. with  $k_\varepsilon$ . Using the fact the  $Rk_\varepsilon = k_\varepsilon$ , a simple change of variables shows that the proposed definition above and the definition given for convolution with  $k_\varepsilon$  are one and the same, assuming that the action of  $k_\varepsilon$  is defined by integration (in which case we must use the compact support of  $k_\varepsilon$  to deal with the fact that  $Rv * \varphi$  might not be compactly supported).

On the bright side, if  $f, g \in L^1(\mathbf{R})$  are viewed as distributions, then  $f * g$  as defined above is the same as the usual convolution of  $f$  with  $g$ , which I will denote  $f \star g$  to avoid confusion. Indeed, for any  $\varphi \in \mathcal{D}(\mathbf{R})$ ,

$$\langle f * g, \varphi \rangle = \langle f, Rg * \varphi \rangle \quad (25)$$

$$= \int_{-\infty}^{\infty} f(x) (Rg \star \varphi)(x) \, dx \quad (* \text{ reduces to } \star \text{ for } Rg \text{ and } \varphi) \quad (26)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) Rg(y) \varphi(x - y) \, dy \, dx \quad (27)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) g(-y) \varphi(x - y) \, dy \, dx. \quad (28)$$

If we let  $y' = x - y$ , then we get

$$\langle f * g, \varphi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y' - x)\varphi(y') \, dy' \, dx \quad (29)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y' - x)\varphi(y') \, dx \, dy' \quad (30)$$

$$= \int_{-\infty}^{\infty} (f \star g)(y')\varphi(y') \, dy' \quad (31)$$

$$= \langle f \star g, \varphi \rangle. \quad (32)$$

Thus, as distributions, we have  $f * g = f \star g$ , as claimed.