Stat 6841 Homework 3

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Problem 1.

(a) We need to calculate $P(N(2) = 0)$ because the event that no customers come in the first two hours (8am to 10am) is the same as $N(2) = 0$. Since $N(2)$ is Poisson distributed with mean $2 \cdot 3 = 6$, we have

$$P(N(2) = 0) = e^{-6}.$$

(b) Let W be the amount of time Oscar has to wait for a customer to arrive. If any customers have arrived already, then Oscar doesn't have to wait at all, so $W = 0$ given $N(2) > 0$. If no customers have arrived when Oscar starts work at 10am, then the amount of time he has to wait for a customer to arrive is exponentially distributed with rate 3 (due to the memoryless property of Poisson process event times). That is, W is exponential with rate 3 given that $N(2) = 0$. Thus, by part (a),

$$\begin{aligned} F(t) = P(W \leq t) &= P(W \leq t | N(2) = 0)P(N(2) = 0) + P(W \leq t | N(2) > 0)P(N(2) > 0) \\ &= (1 - e^{-3t})e^{-6} + 1 \cdot (1 - e^{-6}) \\ &= 1 - e^{-3t-6}, \qquad t \geq 0 \end{aligned}$$

is the CDF of W.

Problem 2.

N/A – already did this one in Homework 2.

Problem 3.

(a) If $N(t)$ is the total number of signals transmitted up to time t, then $N_1(t)$ and $N_2(t)$ are obtained from $N(t)$ by thinning; therefore, $\{N_(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent Poisson processes with rates $\lambda_1 = p\lambda$ and $\lambda_2 = (1 - p)\lambda$. Thus, $(N_1(t), N_2(t))$ are independent Poisson random variables with means $p\lambda t$ and $(1 - p)\lambda t$, so the joint PMF is given by

$$f(n_1, n_2) = \frac{(p\lambda t)^{n_1}((1-p)\lambda t)^{n_2} e^{-p\lambda} e^{-(1-p)\lambda}}{n_1! n_2!} = \frac{p^{n_1}(1-p)^{n_2}(\lambda t)^{n_1+n_2} e^{-\lambda}}{n_1! n_2!}, \qquad n_1 \geq 0,\, n_2 \geq 0.$$

(b) We note that for $k > 0$, we have $L \geq k$ if and only if $S_k^2 < S_1^1$, where S_n^j is the time until the nth signal is transmitted if $j = 1$ and lost if $j = 2$. By the Poisson race formula,

$$P(L \geq k) = P(S_k^2 < S_1^1) = \sum_{i=k}^{k} \binom{k}{i} (1-p)^i p^{k-i} = (1-p)^k, \qquad k > 0.$$

Also, $P(L = 0)$ is the probability that the first signal is successfully transmitted, which is given as p. Then the PMF of L is given by

$$P(L = k) = P(L \geq k) - P(L \geq k+1) = (1-p)^k - (1-p)^{k+1} = (1-(1-p))(1-p)^k = p(1-p)^k, \qquad k \geq 0,$$

if we note that $P(L = 0) = p = p(1-p)^0$, making the above also correct for $k = 0$. This is the PMF of a geometric random variable with parameter p.

Problem 4.

(a) For $u > 0$, using the independence of the Y_i from each other and from $N(t)$, we have

$$\begin{aligned}
f(u) = E\left[u^S\right] = E\left[u^{\sum_{i=1}^{N(t)} Y_i}\right] &= E\left[E\left[\prod_{i=1}^{N(t)} u^{Y_i}\middle| N(t)\right]\right] \\
&= \sum_{n=0}^{\infty} E\left[\prod_{i=1}^{n} u^{Y_i}\right] P(N(t) = n) \\
&= \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \prod_{i=1}^{n} E[u^{Y_i}] \\
&= \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} (g(u))^n \\
&= e^{-\lambda t + \lambda t g(u)} \sum_{n=0}^{\infty} \frac{(\lambda t g(u))^n e^{-\lambda t g(u)}}{n!} \\
&= e^{\lambda t (g(u) - 1)}.
\end{aligned}$$

(b) Differentiating f gives

$$f'(u) = \lambda t g'(u) e^{\lambda t(g(u)-1)},$$

and

$$f''(u) = \left[\lambda t g''(u) + (\lambda t g'(u))^2\right] e^{\lambda t(g(u)-1)}.$$

Then

$$E[S] = f'(1) = \lambda t g'(1) e^{\lambda t(g(1)-1)} = \lambda t E[Y_i],$$

and

$$\begin{aligned}
\mathrm{Var}[S] = E[S^2] - E[S]^2 &= E[S(S-1)] + E[S] - E[S]^2 \\
&= f''(1) + \lambda t E[Y_i] - \lambda^2 t^2 E[Y_i]^2 \\
&= \lambda t g''(1) + (\lambda t g'(1))^2 + \lambda t E[Y_i] - \lambda^2 t^2 E[Y_i]^2 \\
&= \lambda t E[Y_i^2] - \lambda t E[Y_i] + \lambda^2 t^2 E[Y_i]^2 + \lambda t E[Y_i] - \lambda^2 t^2 E[Y_i]^2 \\
&= \lambda t (\mathrm{Var}[Y_i] + E[Y_i]^2).
\end{aligned}$$

Problem 5.

We want to show that the two following definitions are equivalent.

1. A counting process $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ if
   (a) $N(0) = 0$,
   (b) the process has independent increments,
   (c) $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$ for any $t \geq 0$,

(d) and $P(N(t+h) - N(t) \geq 2) = o(h)$ for any $t \geq 0$.

2. A counting process $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ if

   (a) $N(0) = 0$,

   (b) the process has independent increments,

   (c) $N(t+s) - N(t)$ is Poisson-distributed with mean λs for any $t \geq 0$, $s > 0$.

Proof. First, we show that 2. implies 1. Let $\{N(t) : t \geq 0\}$ be a counting process that satisfies properties 2.(a), 2.(b), and 2.(c). Then the process also satisfies 1.(a) and 1(b) trivially. Let $t \geq 0$, and let $h > 0$ be given. By 2.(c), $N(t+h) - N(t)$ is Poisson-distributed with rate λh, so

$$\begin{aligned} P(N(t+h) - N(t) = 1) &= \lambda h e^{-\lambda h} \\ &= \lambda h \left(1 - \lambda h + o(h)\right) \\ &= \lambda h + o(h), \end{aligned}$$

so the process satisfies 1.(c). Furthermore,

$$\begin{aligned} P(N(t+h) - N(t) \geq 2) &= 1 - \left[P(N(t+h) - N(t) = 0) + P(N(t+h) - N(t)) = 1)\right] \\ &= 1 - \left[e^{-\lambda h} + \lambda h e^{-\lambda h}\right] \\ &= 1 - 1 + \lambda h + o(h) - \lambda h + o(h) \\ &= o(h) \end{aligned}$$

so the process satisfies 1.(d). Thus, 2. implies 1.

Second, we show that 1. implies 2. Let $\{N(t) : t \geq 0\}$ be a counting process that satisfies 1.(a), 1.(b), 1.(c), and 1.(d). Then the process trivially satisfies 2.(a) and 2.(b). Let $t \geq 0$ and $s > 0$ be given. Divide the interval $(t, t+s]$ into n subintervals of equal length $h = \frac{s}{n}$ each. Let $M_i = N(t+h) - N(t+(i-1)h)$ for $i = 1, 2, \ldots, n$ be the number of events in each subinterval. Using a telescoping sum, we have

$$N(t+s) - N(t) = \sum_{i=1}^{n} M_i.$$

Let $\mathcal{S}$ be the set of all sequences of length n whose entries are nonnegative integers that sum to $k \geq 0$. Then

$$\begin{aligned} P(N(t+s) - N(t) = k) &= P\left(\sum_{i=1}^{n} M_i = k\right) \\ &= \sum_{(m_1, m_2, \ldots, m_n) \in \mathcal{S}} P(M_1 = m_1, M_2 = m_2, \ldots, M_n = m_n). \end{aligned}$$

It follows from independence of increments that $M_1 = m_1$, $M_2 = m_2$, $\ldots M_n = m_n$ are independent, so

$$P(N(t+s) - N(t) = k) = \sum_{(m_1, m_2, \ldots, m_n) \in \mathcal{S}} P(M_1 = m_1) P(M_2 = m_2) \cdots P(M_n = m_n).$$

Let $\mathcal{S}'$ be the subset of $\mathcal{S}$ such that for $(m_1, m_2, \ldots, m_n) \in \mathcal{S}'$ we have $m_i \leq 1$ for $i = 1, 2, \ldots, n$. We observe that $\mathcal{S}'$ contains $\binom{n}{k}$ elements because each m_i in an element of $\mathcal{S}'$ must be either 0 or 1, and there must be exactly k that are 1 in order for the sum of the m_i to be k. Furthermore, by 1.(c) and 1.(d), we have

$$P(M_i = 0) = 1 - P(M_i = 1) - P(M_i \geq 2) = 1 - \lambda h + o(h), \qquad P(M_i = 1) = \lambda h + o(h).$$

It follows that

$$\begin{aligned}P(N(t+s)-N(t)=k) &= \sum_{(m_1,\dots,m_n)\in\mathcal{S}'} P(M_1=m_1)\cdots P(M_n=m_n) + \sum_{(m_1,\dots,m_n)\in\mathcal{S}\setminus\mathcal{S}'} P(M_1=m_1)\cdots P(M_n=m_n)\\ &= \binom{n}{k}\left(\lambda h+o(h)\right)^k\left(1-\lambda h+o(h)\right)^{n-k}+R_n,\end{aligned}$$

where

$$R_n = \sum_{(m_1,\dots,m_n)\in\mathcal{S}\setminus\mathcal{S}'} P(M_1=m_1)\cdots P(M_n=m_n).$$

Replacing h by $\frac{s}{n}$, we have

$$P(N(t+s)-N(t)=k) = \binom{n}{k}\left(\frac{\lambda s}{n}+o\left(\frac{1}{n}\right)\right)^k\left(1-\frac{\lambda s}{n}+o\left(\frac{1}{n}\right)\right)^{n-k}+R_n.$$

We now take the limit as $n\to\infty$ on both sides. We begin by showing that $R_n\to 0$ as $n\to\infty$. Let $\mathcal{S}_{p,q}$ denote the set of all sequences $(m_1,\dots,m_n)\in\mathcal{S}\setminus\mathcal{S}'$ such that $p=\#\{i : m_i\geq 2\}$ and $q=\#\{i : m_i=1\}$. We note that the values of p and q for which $\mathcal{S}_{p,q}$ is nonempty are determined by $p\geq 1$ and $2p+q\leq k$. In particular, the number of nonempty sets $\mathcal{S}_{p,q}$ is independent of n.

Now we compute $\#\mathcal{S}_{p,q}$. There are $\binom{n}{p+q}$ ways to select the nonzero m_i. For each such choice there are furthermore $\binom{p+q}{p}$ ways to choose which m_i are greater than 1. Lastly, we can think about choosing the values of the nonzero m_i as distributing k items among the $p+q$ slots so that q slots have 1 item each and the remaining p slots have 2 or more items each. Thus, we need to distribute $2p+q$ items to meet these requirements – 2 for each of the p slots and 1 for each of the q slots – then we can decide freely how to distribute the remaining $k-2p-q$ items among the p slots that may have 2 or more items. There are $\binom{k-2p-q+p-1}{p-1}$ ways to select the $k-2p-q$ slots from among the p choices with replacement allowed, where order does not matter. Hence,

$$\begin{aligned}\#\mathcal{S}_{p,q} &= \binom{n}{p+q}\binom{p+q}{p}\binom{k-p-q-1}{p-1}\\ &= \frac{n!}{(n-p-q)!(p+q)!}\binom{p+q}{p}\binom{k-p-q-1}{p-1}\\ &= \frac{n(n-1)\cdots(n-p-q+1)}{(p+q)!}\binom{p+q}{p}\binom{k-p-q-1}{p-1}\\ &= O(n^{p+q}),\end{aligned}$$

as the values of k,p,q are independent of n. Writing

$$R_n = \sum_{\substack{p\geq 1,\\ 2p+q\leq k}}\ \sum_{(m_1,\dots,m_n)\in\mathcal{S}_{p,q}} P(M_1=m_1)\cdots P(M_n=m_n),$$

we observe that each term in the sum of $\mathcal{S}_{p,q}$ has q factors of $\frac{\lambda s}{n}+o\left(\frac{1}{n}\right)$ by 1.(c), and p factors of $o\left(\frac{1}{n}\right)$ by 1.(d). The remaining factors are probabilities and, therefore, are bounded above by 1. Hence,

$$\begin{aligned}\sum_{(m_1,\dots,m_n)\in\mathcal{S}_{p,q}} P(M_1=m_1)\cdots P(M_n=m_n) &\leq \#\mathcal{S}_{p,q}\left(\frac{\lambda s}{n}+o\left(\frac{1}{n}\right)\right)^q\left(o\left(\frac{1}{n}\right)\right)^p\\ &\leq O\left(n^{p+q}\right)\left(O\left(\frac{1}{n}\right)\right)^q\left(o\left(\frac{1}{n}\right)\right)^p\\ &= O\left(n^{p+q}\right)o\left(\frac{1}{n^{p+q}}\right)\\ &= o(1)\end{aligned}$$

if $p \geq 1$. Since there are a fixed number of terms (independent of n) in the sum over $p \geq 1$ and $2p + q \leq k$, it follows that $R_n = o(1)$ as $n \to \infty$. In other words, $R_n \to 0$ as $n \to \infty$.

Now we handle the first term, starting with the right factor. There exists a sequence $\{a_n\}$ such that $a_n \to 0$ as $n \to \infty$, and

$$\left(1 - \frac{\lambda s}{n} + o\left(\frac{1}{n}\right)\right)^n = \left(1 - \frac{\lambda s}{n} + \frac{a_n}{n}\right)^n.$$

Then

$$\left(1 - \frac{\lambda s}{n} + o\left(\frac{1}{n}\right)\right)^n = \left(1 - \frac{\lambda s}{n}\right)^n + \sum_{i=1}^n \frac{n^{\underline{i}}}{i!}\left(1 - \frac{\lambda s}{n}\right)^{n-i} \frac{a_n^i}{n^i},$$

where $n^{\underline{i}} = n(n-1)\cdots(n-i+1)$ is the falling factorial of n. Clearly, $\frac{n^{\underline{i}}}{n^i} \leq 1$, and, for n large enough, $\left(1 - \frac{\lambda s}{n}\right)^{n-i} \leq 1$. Let $a = \sup_n |a_n| < \infty$. Given $\varepsilon > 0$, there exists N such that

$$\left|\sum_{i=N}^n \frac{a_n^i}{i!}\right| \leq \sum_{i=N}^\infty \frac{a^i}{i!} < \frac{\varepsilon}{2}.$$

On the other hand, we must have

$$\sum_{i=1}^N \frac{a_n^i}{i!} \to 0 \quad \text{as } n \to \infty$$

because $a_n^i \to 0$ as $n \to \infty$ for each of the finitely-many terms in the sum. Then for n sufficiently large,

$$\left|\sum_{i=1}^n \frac{n^{\underline{i}}}{i!}\left(1 - \frac{\lambda s}{n}\right)^{n-i} \frac{a_n^i}{n^i}\right| \leq \sum_{i=1}^N \frac{a_n^i}{i!} + \sum_{i=N}^\infty \frac{a^i}{i!} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus,

$$\lim_{n\to\infty}\left(1 - \frac{\lambda s}{n} + o\left(\frac{1}{n}\right)\right)^n = \lim_{n\to\infty}\left(1 - \frac{\lambda s}{n}\right)^n = e^{-\lambda s}.$$

Lastly, we observe that

$$\left(1 - \frac{\lambda s}{n} + o\left(\frac{1}{n}\right)\right)^{-k} \to 1$$

as $n \to \infty$ because of the continuity of $x \mapsto x^{-k}$ at 1. This handles the right factor:

$$\lim_{n\to\infty}\left(1 - \frac{\lambda s}{n} + o\left(\frac{1}{n}\right)\right)^{n-k} = e^{-\lambda s}.$$

The left factor is a bit simpler. We have

$$\binom{n}{k}\left(\frac{\lambda s}{n} + o\left(\frac{1}{n}\right)\right)^k = \frac{n^{\underline{k}}(\lambda s)^k}{n^k k!} + \sum_{i=1}^k \frac{n^{\underline{k}}}{k!}\binom{k}{i}\frac{(\lambda s)^i}{n^i}\left(o\left(\frac{1}{n}\right)\right)^{k-i} = \frac{n^{\underline{k}}(\lambda s)^k}{n^k k!} + o\left(\frac{n^{\underline{k}}}{n^k}\right).$$

Since $n^{\underline{k}}$ is a polynomial of degree k in n with leading coefficient 1, it follows that $\frac{n^{\underline{k}}}{n^k} \to 1$ as $n \to \infty$. This completes the argument for the left factor:

$$\lim_{n\to\infty}\binom{n}{k}\left(\frac{\lambda s}{n} + o\left(\frac{1}{n}\right)\right)^k = \frac{(\lambda s)^k}{k!}.$$

This completes the proof, as we finally obtain

$$P(N(t+s) - N(t) = k) = \frac{(\lambda s)^k e^{-\lambda s}}{k!},$$

which implies that $N(t+s) - N(t)$ is Poisson-distributed with mean λs, which proves 2.(c). □

6. Textbook: 36, p. 370

(a) Using the independence of the X_i from each other and from $N(t)$ and the fact that $E[X_i] = \frac{1}{\mu}$ because X_i is exponential with rate μ, we have

$$\begin{aligned}
E[S(t)] = E\left[S(0)\prod_{i=1}^{N(t)} X_i\right] &= sE\left[E\left[\prod_{i=1}^{N(t)} X_i \middle| N(t)\right]\right] \\
&= s\sum_{k=0}^{\infty} E\left[\prod_{i=1}^{n} X_i\right] P(N(t) = n) \\
&= s\sum_{k=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \prod_{i=1}^{n} E[X_i] \\
&= s\sum_{k=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \frac{1}{\mu^n} \\
&= se^{\frac{\lambda t}{\mu} - \lambda t}\sum_{k=0}^{\infty} \left(\frac{\lambda t}{\mu}\right)^n \frac{e^{-\frac{\lambda t}{\mu}}}{n!} \\
&= se^{\lambda t\left(\frac{1}{\mu} - 1\right)}.
\end{aligned}$$

(b) Note that

$$S^2(t) = s^2 \prod_{i=1}^{N(t)} X_i^2,$$

so by nearly the same calculation as in part (a) and the fact that $E[X_i^2] = \frac{2}{\mu^2}$ because X_i is exponential with rate μ, we have

$$\begin{aligned}
E[S^2(t)] &= s^2\sum_{k=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \prod_{i=1}^{n} E[X_i^2] \\
&= s^2 e^{\frac{2\lambda t}{\mu^2} - \lambda t}\sum_{k=0}^{\infty} \left(\frac{2\lambda t}{\mu^2}\right)^n \frac{e^{-\frac{2\lambda t}{\mu^2}}}{n!} \\
&= s^2 e^{\lambda t\left(\frac{2}{\mu^2} - 1\right)}.
\end{aligned}$$

7. Textbook: 58, p. 374

We note that $\{N(t) : t \geq 0\}$, with $N(t) = N_1(t) + N_2(t)$, is a Poisson process with rate $\lambda_1 + \lambda_2$, and by our superposition theorem the $P(T_i = j) = \frac{\lambda_j}{\lambda_1 + \lambda_2}$, where T_i is the type of the ith event in the process $\{N(t) : t \geq 0\}$.

Let V_i be the amount of the ith claim in the process $\{N(t) : t \geq 0\}$. Then, by Bayes' Law,

$$P(T_i = 1 | V_i = 4000) = \frac{f_{V_i|T_i}(4000|1)P(T_i = 1)}{f_{V_i}(4000)},$$

where $f_{V_i|T_i}(v|t)$ is the conditional PDF of V_i given T_i, and $f_{V_i}(v)$ is the PDF of V_i. Since V_i is exponential with mean 1000 given $T_i = 1$ and V_i is exponential with mean 5000 given $T_i = 2$, it follows that

$$f_{V_i|T_i}(v|1) = \frac{1}{1000}e^{-\frac{1}{1000}v}, \qquad v \geq 0,$$

and

$$f_{V_i|T_i}(v|2) = \frac{1}{5000}e^{-\frac{1}{5000}v}, \qquad v \geq 0.$$

By the law of total probability,

$$\begin{aligned} f_{V_i}(v) &= f_{V_i|T_i}(v|1)P(T_i = 1) + f_{V_i|T_i}(v|2)P(T_i = 2) \\ &= \frac{1}{1000}e^{-\frac{1}{1000}v}\frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{1}{5000}e^{-\frac{1}{5000}v}\frac{\lambda_2}{\lambda_1+\lambda_2} \\ &= \frac{1}{1100}e^{-\frac{1}{1000}v} + \frac{1}{50000}e^{-\frac{1}{5000}v}. \end{aligned}$$

Hence,

$$\begin{aligned} P(T_i = 1|V_i = 4000) &= \frac{\frac{1}{1000}e^{-\frac{4000}{1000}} \cdot \frac{10}{11}}{\frac{1}{1100}e^{-\frac{4000}{1000}} + \frac{1}{50000}e^{-\frac{4000}{5000}}} \\ &= \frac{\frac{1}{11}e^{-4}}{\frac{1}{11}e^{-4} + \frac{1}{500}e^{-\frac{4}{5}}} \\ &= \frac{1}{1 + \frac{11}{500}e^{\frac{16}{5}}} \\ &\approx 0.6495. \end{aligned}$$

8. Textbook: 64, p. 375

(a) Given $N(t) = n > 0$, we have

$$X = \sum_{i=1}^{n}(t - S_i),$$

where S_i is the time of the ith arrival. If $N(t) = 0$, then $X = 0$. Then

$$E[X|N(t)] = E\left[\sum_{i=1}^{N(t)}(t - S_i)\middle|N(t)\right] = tN(t) - \sum_{i=1}^{N(t)} E[S_i|N(t)].$$

Given $N(t)$, the arrival times $S_1, \dots, S_{N(t)}$ are distributed the same as the order statistics $U_{(1)}, \dots, U_{(N(t))}$ of i.i.d. random variables $U_1, \dots, U_{N(t)}$ that are uniformly distributed on $[0, t]$. Thus, since the order statistics are a permutation of the original variables,

$$E[X|N(t)] = tN(t) - \sum_{i=1}^{N(t)} E[U_{(i)}] = tN(t) - \sum_{i=1}^{N(t)} E[U_i] = tN(t) - \frac{t}{2}N(t) = \frac{tN(t)}{2}.$$

(b) Using the formula for X given $N(t)$ from above,

$$\mathrm{Var}[X|N(t)] = \mathrm{Var}\left[\sum_{i=1}^{N(t)}(t - S_i)\middle|N(t)\right] = \mathrm{Var}[tN(t)|N(t)] + \mathrm{Var}\left[\sum_{i=1}^{N(t)} S_i\middle|N(t)\right].$$

Since $\mathrm{Var}[tN(t)|N(t)] = 0$, and the S_i given $N(t)$ have the same distribution as the order statistics from part (a), we have

$$\mathrm{Var}[X|N(t)] = \sum_{i=1}^{N(t)} \mathrm{Var}\left[U_{(i)}\right].$$

Once again, the order statistics are just a permutation of the original variables, so their sum is the same as the sum of the originals, giving

$$\mathrm{Var}[X|N(t)] = \sum_{i=1}^{N(t)} \mathrm{Var}\left[U_i\right] = N(t)\frac{t^2}{12}.$$

(c) We compute $\mathrm{Var}[X]$ by conditioning on $N(t)$:

$$\mathrm{Var}[X] = E[\mathrm{Var}[X|N(t)]] + \mathrm{Var}[E[X|N(t)]] = E\left[\frac{t^2N(t)}{12}\right] + \mathrm{Var}\left[\frac{tN(t)}{2}\right] = \frac{\lambda t^3}{12} + \frac{\lambda t^3}{4} = \frac{\lambda t^3}{3}.$$

9. Textbook: 80, p. 378

(a) The formula is true for $n = 1$ because

$$F(t_1) = P(S_1 \le t_1) = 1 - P(S_1 > t_1) = 1 - P(N(t_1) = 0) = 1 - e^{-m(t_1)}, \qquad t_1 > 0,$$

is the CDF of S_1, so the PDF of S_1 is

$$f_{S_1}(t_1) = F'(t_1) = e^{-m(t_1)}m'(t_1) = e^{-m(t_1)}\lambda(t_1), \qquad t_1 > 0,$$

as $m'(t) = \lambda(t)$ by definition.

Suppose for induction that the formula holds for some $n \ge 1$. Let $t_1, \dots, t_{n+1}$ be given, and let $h_1, \dots, h_{n+1} > 0$ be given. Define the event $A_i = \{t_i \le S_i \le t_i + h_i\}$.

If $t_{i+1} \le t_i$ for some $i = 1, \dots, n$, then

$$P\left(\bigcap_{i=1}^{n+1} A_i\right) \le P(A_i \cap A_{i+1}) = P(t_i \le S_i \le t_i + h_i,\, t_{i+1} \le S_{i+1} \le t_{i+1} + h_{i+1}).$$

Since $t_i \le S_i \le t_i + h_i$ and $t_{i+1} \le S_{i+1} \le t_{i+1} + h_{i+1}$ implies that two events (i and $i+1$) occur in the interval $[t_{i+1}, t_{i+1} + h_{i+1}]$, it follows that

$$P\left(\bigcap_{i=1}^{n+1} A_i\right) \le o(h_{i+1}).$$

This implies that the joint PDF of $S_1, \dots, S_{n+1}$ vanishes if $t_{i+1} \le t_i$. Thus, we consider only the case that $t_i < t_{i+1}$ for $i = 1, \dots, n$. Furthermore, if $t_1 < 0$, then for sufficiently small h_1 the event A_1 would require $S_1 < 0$, which is impossible, so the PDF must vanish also if $t_1 < 0$, and we need only consider the case $t_1 > 0$.

We observe that

$$\begin{aligned} P\left(\bigcap_{i=1}^{n+1} A_i\right) &= P\left(A_{n+1} \cap \bigcap_{i=1}^{n} A_i\right) = P\left(A_{n+1} \middle| \bigcap_{i=1}^{n} A_i\right) P\left(\bigcap_{i=1}^{n} A_i\right) \\ &= P\left(A_{n+1} \middle| \bigcap_{i=1}^{n} A_i\right)(f_{S_1,\dots,S_n}(t_1, \dots, t_n)h_1h_2\dots, h_n + o(|h|)), \end{aligned}$$

where $h = (h_1, \dots, h_n)^T$. For h_n small enough that $t_n + h_n < t_{n+1}$, and h_{n-1} small enough that $t_{n-1} < t_{n-1} + h_{n-1}$, the event A_{n+1} given the events $A_i, \dots A_n$ is equivalent to the event that no events occur between t_n+h_n and t_{n+1} and that 2 or more events do not occur between t_n and t_n+h_n and that at least one event occurs between t_{n+1} and $t_{n+1}+h_{n+1}$. That is,

$$P\left(\bigcap_{i=1}^{n+1} A_i\right) = P(N(t_n+h_n)-N(t_n)\le 1, N(t_{n+1}-t_n-h_n)-N(t_n+h_n)=0, N(t_{n+1}+h_{n+1})-N(t_{n+1})\ge 1)$$
$$\cdot(f_{S_1,\dots,S_n}(t_1,\dots,t_n)h_1\dots h_n+o(|h|)).$$

Since h_n and h_{n+1} are sufficiently small, the independence of increments property implies that

$$P\left(\bigcap_{i=1}^{n+1} A_i\right) = P(N(t_n+h_n)-N(t_n)\le 1)P(N(t_{n+1}-t_n-h_n)-N(t_n+h_n)=0)P(N(t_{n+1}+h_{n+1})-N(t_{n+1})\ge 1)$$
$$\cdot(f_{S_1,\dots S_n}(t_1,\dots t_n)h_1\dots h_n+o(|h|)).$$

From (one) definition of an inhomogeneous Poisson process, we have

$$P(N(t_n+h_n)-N(t_n)\le 1) = 1-P(N(t_n+h_n)-N(t_n)\ge 2) = 1-o(h_n).$$

Thus,

$$P\left(\bigcap_{i=1}^{n+1} A_i\right) = (1-o(h_n))P(N(t_{n+1}-t_n-h_n)-N(t_n+h_n)=0)(1-P(N(t_{n+1}+h_{n+1})-N(t_{n+1})=0))$$
$$\cdot(f_{S_1,\dots,S_n}(t_1,\dots,t_n)h_1\dots h_n++o(|h|)).$$

Additionally,

$$P(N(t_{n+1}-t_n-h_n)-N(t_n+h_n)=0) = e^{-\int_{t_n+h_n}^{t_{n+1}}\lambda(s)\,\mathrm{d}s},$$

and

$$P(N(t_{n+1}+h_{n+1})-N(t_{n+1})=0) = e^{-\int_{t_{n+1}}^{t_{n+1}+h_{n+1}}\lambda(s)\,\mathrm{d}s}.$$

It follows that

$$P\left(\bigcap_{i=1}^{n+1} A_i\right) = (1-o(h_n))e^{-\int_{t_n+h_n}^{t_{n+1}}\lambda(s)\,\mathrm{d}s}\left(1-e^{\int_{t_{n+1}}^{t_{n+1}+h_{n+1}}\lambda(s)\,\mathrm{d}s}\right)(f_{S_1,\dots,S_n}(t_1,\dots,t_n)h_1\dots h_n+o(|h|)).$$

Then

$$\begin{aligned}
f_{S_1,\dots,S_n,S_{n+1}}(t_1,\dots,t_n,t_{n+1}) &= \lim_{(h_1,\dots,h_{n+1})\to 0}\frac{P\left(\bigcap_{i=1}^{n+1}A_i\right)}{h_1h_2\dots h_nh_{n+1}}\\
&= \lim_{(h_1\dots,h_{n+1})\to 0}(1-o(h_n))e^{-\int_{t_n+h_n}^{t_{n+1}}\lambda(s)\,\mathrm{d}s}\frac{1-e^{-\int_{t_{n+1}}^{t_{n+1}+h_{n+1}}\lambda(s)\,\mathrm{d}s}}{h_{n+1}}\\
&\qquad\cdot\frac{f_{S_1,\dots,S_n}(t_1,\dots,t_n)h_1\dots h_n+o(|h|)}{h_1\dots h_n}\\
&= \lim_{(h_1\dots,h_{n+1})\to 0}e^{-\int_{t_n}^{t_{n+1}}\lambda(s)\,\mathrm{d}s}f_{S_1,\dots S_n}(t_1,\dots,t_n)\frac{1-e^{-\int_{t_{n+1}}^{t_{n+1}+h_{n+1}}\lambda(s)\,\mathrm{d}s}}{h_{n+1}}.
\end{aligned}$$

Using L'Hopital's rule, we get

$$\begin{aligned}
f_{S_1,\dots,S_{n+1}}(t_1,\dots,t_{n+1}) &= f_{S_1,\dots,S_n}(t_1,\dots,t_n)e^{-\int_{t_n}^{t_{n+1}}\lambda(s)\,\mathrm{d}s}\lim_{h_{n+1}\to 0}\frac{1-e^{-\int_{t_{n+1}}^{t_{n+1}+h_{n+1}}\lambda(s)\,\mathrm{d}s}}{h_{n+1}}\\
&= \lambda(t_1)\dots\lambda(t_n)e^{-\int_0^{t_n}\lambda(s)\,\mathrm{d}s-\int_{t_n}^{t_{n+1}}\lambda(s)\,\mathrm{d}s}\lim_{h_{n+1}\to 0}\lambda(t_{n+1}+h_{n+1})e^{-\int_{t_{n+1}}^{t_{n+1}+h_{n+1}}\lambda(s)\,\mathrm{d}s}\\
&= \lambda(t_1)\dots\lambda(t_n)\lambda(t_{n+1})e^{-\int_0^{t_{n+1}}\lambda(s)\,\mathrm{d}s}\\
&= \lambda(t_1)\dots\lambda(t_n)\lambda(t_{n+1})e^{-m(t_{n+1})}.
\end{aligned}$$

Thus, the formula for the joint PDF of $S_1, \ldots, S_{n+1}$ holds. This means that it holds for all n by induction.

(b) Using a similar argument to that in the book, we note that for $0 < s_1 < \cdots < s_n < t$, the event that $S_1 = s_1$, $S_2 = s_2$, $\ldots S_n = s_n$ and $N(t) = n$ is equivalent to the event that $T_1 = s_1$, $T_2 = s_2 - s_1$, $\ldots$, $T_n = s_n - s_{n-1}$, $T_{n+1} > t - s_n$, where T_i in our case is the amount of time until the next event occurs starting at time $t = s_{i-1}$, with $s_0 = 0$.

Since the density of T_i at $s_i - s_{i-1}$ is given by $e^{-\int_{s_{i-1}}^{s_i} \lambda(s)\ \mathrm{d}s} = \lambda(s_i)e^{-m(s_i)+m(s_{i-1})}$, and

$$P(T_{n+1} > t - s_n) = 1 - P(T_{n+1} \leq t - s_n) = e^{-\int_{s_n}^{t} \lambda(s)\ \mathrm{d}s} = e^{-m(t)+m(s_n)},$$

we have

$$\begin{aligned} f_{S_1,\ldots,S_n|N(t)}(s_1, \ldots, s_n|n) &= \frac{\lambda(s_1)e^{-m(s_1)+m(s_0)} \ldots \lambda(s_n)e^{-m(s_n)+m(s_{n-1})}e^{-\int_{s_n}^{t} \lambda(s)\ \mathrm{d}s}}{P(N(t) = n)} \\ &= \frac{\lambda(s_1)\cdots\lambda(s_n)e^{-m(t)}n!}{e^{m(t)}(m(t))^n} \\ &= \frac{n!\lambda(s_1)\cdots\lambda(s_n)}{(m(t))^n}, \qquad 0 < s_1 < \cdots < s_n < t. \end{aligned}$$

(c) Let $X_1, \ldots, X_n$ be independent random variables with identical distribution given by the PDF

$$f(s) = \frac{\lambda(s)}{m(t)}, \qquad s \in [0, t].$$

Then the joint PDF of the order statistics $X_{(1)}, \ldots, X_{(n)}$ is given by

$$f(s_1, \ldots, s_n) = n!f(s_1)\cdots f(s_n) = \frac{n!\lambda(s_1)\cdots\lambda(s_n)}{(m(t))^n}, \qquad 0 < s_1 < \cdots < s_n < t.$$

Thus, the distribution of $S_1, \ldots, S_n$ given $N(t) = n$ is the same as the distribution of the order statistics $X_{(1)}, \ldots, X_{(n)}$.

10. Textbook: 88, p. 379

Let X_n the amount withdrawn on the nth withdrawal. Then the total daily withdrawal is

$$Y = \sum_{n=1}^{N(15)} X_n.$$

To find $P(Y < 6000)$, we condition on $N(15)$:

$$\begin{aligned} P(Y < 6000) &= \sum_{n=1}^{\infty} P(Y < 6000|N(15) = n)P(N(15) = n) \\ &= \sum_{n=1}^{\infty} P(Y < 6000|N(15) = n)\frac{180^n e^{-180}}{n!}. \end{aligned}$$

Given $N(15) = n$, the Central Limit Theorem means that $Z = (Y/n - E[X_n])/\sqrt{\mathrm{Var}[X_n]/n}$ roughly follows the standard normal distribution, if n is sufficiently large. Then

$$P(Y < 6000|N(15) = n) = P\left(Z < \frac{6000/n - 30}{50/\sqrt{n}} \middle| N(15) = n\right) \approx \Phi\left(120/\sqrt{n} - 0.6\sqrt{n}\right),$$

where Φ is the CDF of the standard normal distribution. Let F be the CDF of $N(15)$, which is Poisson-distributed with mean 180. Numerical calculation of F using Python shows that

$$F(70) < 10^{-18}, \qquad 1 - F(320) < 10^{-18},$$

so if we sum only the terms from $n = 70$ to $n = 320$ in the infinite sum for $P(Y < 6000)$, we will incur an error less than machine precision ($\sim 10^{-18}$). Furthermore, by restricting to $n \geq 70$, we will ensure that our normal approximation of Y is fairly accurate. This gives

$$P(Y < 6000) \approx \sum_{n=70}^{320} \Phi(120/\sqrt{n} - 0.6\sqrt{n}) \frac{180^n e^{-180}}{n!} \approx 0.7805.$$