1.59(ii)

σ and μ are always rd-continuous, and ρ is not necessarily rd-continuous.

1. Suppose that $t \in \mathbb{T}$ is right-dense. Given $\varepsilon > 0$, we can find $s_1, s_2 \in \mathbb{T}$ such that $t < s_1 < s_2 < t + \varepsilon$. Set $\delta = s_1 - t$. If $s > t$, then $s - t < \delta$ implies that $\sigma(s) - t \leq \sigma(s_1) - t \leq s_2 - t < \varepsilon$. Therefore, $\lim_{s \to t^+} \sigma(s) = t$.

2. Now suppose that $t \in \mathbb{T}$ is left-dense. Given $\varepsilon > 0$, if $s < t$ and $t - s < \varepsilon$, then $0 \leq t - \sigma(s) \leq t - s < \varepsilon$. Therefore, $\lim_{s \to t^-} \sigma(s) = t$.

If t is left-dense, then by 2., $\lim_{s \to t^-} \sigma(s) = t$, which is finite.

If t is right-dense, then $\sigma(t) = t$, so σ is right-continuous by 1. If, in addition, t is left-scattered or $t = \inf \mathbb{T}$, then σ is trivially left-continuous at t; otherwise, t is left-dense, in which case σ must be left-continuous at t by 2. Thus, σ is continuous at t.

The rd-continuity of μ follows from that of σ, as $\mu(t) = \sigma(t) - t$; the identity function $t \mapsto t$ is rd-continuous, and rd-continuity is preserved under linear combination.

To see that ρ may not be rd-continuous, consider the time-scale $\mathbb{T} = \{0\} \cup [1, 2]$. In this case, we have

$$\rho(t) = \begin{cases} 0 & \text{if } t \in \{0, 1\}, \\ t & \text{if } t \in (1, 2]. \end{cases}$$

Then the point $1 \in \mathbb{T}$ is right-dense, but clearly ρ is not continuous at 1, as $\rho(1) = 0$, but $\lim_{s \to t^+} \rho(s) = 1$.

1.59(iii)

rd-continuous implies regulated, so σ and μ are always regulated. We can show that ρ is also always regulated by virtually the same argument from above.

1. Suppose that $t \in \mathbb{T}$ is right-dense. Given $\varepsilon > 0$, if $s > t$ and $s - t < \varepsilon$, then $0 \leq \rho(s) - t \leq s - t < \varepsilon$. Therefore, $\lim_{s \to t^+} \rho(s) = t$.

2. Now suppose that $t \in \mathbb{T}$ is left-dense. Given $\varepsilon > 0$, we can find $s_1, s_2 \in \mathbb{T}$ such that $t - \varepsilon < s_1 < s_2 < t$. Set $\delta = t - s_2$. If $s > t$, then $s - t < \delta$ implies that $t - \rho(s) \leq t - \rho(s_2) \leq t - s_1 < \varepsilon$. Therefore, $\lim_{s \to t^-} \rho(s) = t$.

Together, 1. and 2. imply that ρ is regulated.