Math 6417 Homework 4

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Question 1.

Define the **Fourier transform operator** $\mathscr{F} : L^1(\mathbf{R}) \to L^\infty(\mathbf{R})$ by

$$\mathscr{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f(x) \, \mathrm{d}x. \tag{1}$$

1.1) We note that the function $x \mapsto e^{iyx} f(x)$ is clearly integrable if f is, so the integral in (1) exists for all y. We show that $\mathscr{F}(f) \in L^\infty(\mathbf{R})$ as claimed, and $\|\mathscr{F} f\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$. Indeed, for $y \in \mathbf{R}$,

$$\begin{aligned} |\mathscr{F}(f)(y)| &= \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} e^{iyx} f(x) \, \mathrm{d}x \right| &(2)\\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| e^{iyx} f(x) \right| \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}. &(3) \end{aligned}$$

Therefore, $\|\mathscr{F} f\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$.

1.2) Suppose that $f \in C^2(\mathbf{R})$, and $f, f', f'' \in L^1(\mathbf{R})$, and $f(x), f'(x), f''(x) \to 0$ as $x \to \pm\infty$. Then there exists a constant C such that $|y^2 \mathscr{F}(f)(y)| \leq C$ for all $y \in \mathbf{R}$. Furthermore, $\mathscr{F}(f) \in L^1(\mathbf{R})$.

Proof. Since $f'' \in L^1(\mathbf{R})$, we can take its Fourier transform, which yields

$$\mathscr{F}(f'')(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f''(x) \, \mathrm{d}x. \tag{4}$$

We can integrate by parts because $f', f \in L^1(\mathbf{R})$ and are continuous, and $f(x), f'(x) \to 0$ as $x \to \pm\infty$. This gives

$$\begin{aligned} \mathscr{F}(f'')(y) &= \frac{1}{\sqrt{2\pi}} \left[f'(x) e^{iyx} \Big|_{-\infty}^{\infty} - iy \int_{-\infty}^{\infty} e^{iyx} f'(x) \, \mathrm{d}x \right] &(5)\\ &= \frac{iy}{\sqrt{2\pi}} \left[-f(x) e^{iyx} \Big|_{-\infty}^{\infty} + iy \int_{-\infty}^{\infty} e^{iyx} f(x) \, \mathrm{d}x \right] &(6)\\ &= -y^2 \mathscr{F}(f)(y). &(7) \end{aligned}$$

By the reasoning in 1.1), it follows that

$$|y^2 \mathscr{F}(f)(y)| = |\mathscr{F}(f'')(y)| \leq \frac{1}{\sqrt{2\pi}} \|f''\|_{L^1} \tag{8}$$

for all $y \in \mathbf{R}$.

Thus, if $C = \frac{1}{\sqrt{2\pi}} \|f''\|_{L^1}$, then $|\mathscr{F}(f)(y)| \leq \frac{C}{y^2}$ for all $y \in \mathbf{R}$. On the other hand, $\mathscr{F}(f) \in L^\infty(\mathbf{R})$ by part 1.1), so $\mathscr{F}(f)$ is dominated by the integrable function

$$\phi(y) = \begin{cases} \|\mathscr{F}(f)\|_{L^\infty} & y \in [-1, 1], \\ \frac{C}{y^2} & \text{otherwise.} \end{cases} \tag{9}$$

By the integral comparison test, $\mathscr{F}(f) \in L^1(\mathbf{R})$. $\square$

1.3) Formally, $\mathscr{F}^2(f)(y) = f(-y)$.

Proof. We note that if $f \in C^1 \cap L^1(\mathbf{R})$, and $f' \in L^1(\mathbf{R})$, and $f(x) \to 0$ as $x \to \pm\infty$, then we can use integration by parts to show that

$$\mathscr{F}(f')(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f'(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[e^{iyx} f(x) \Big|_{-\infty}^{\infty} - iy \int_{-\infty}^{\infty} e^{iyx} f(x)\,\mathrm{d}x \right] \tag{10}$$

$$= -iy\mathscr{F}(f)(y). \tag{11}$$

On the other hand, let $f \in L^1(\mathbf{R})$, and define $g(x) = ixf(x)$. If $g \in L^1(\mathbf{R})$ as well, then

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathscr{F}(f)(y) = \frac{\mathrm{d}}{\mathrm{d}y} \int_{-\infty}^{\infty} e^{iyx} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\partial}{\partial y} \left[e^{iyx} f(x) \right]\,\mathrm{d}x \tag{12}$$

$$= \int_{-\infty}^{\infty} e^{iyx} ixf(x)\,\mathrm{d}x = \mathscr{F}(g)(y). \tag{13}$$

If we take $f(x) = e^{-ax^2}$, then f satisfies the above assumptions. Since $f'(x) = -2axf(x)$,

$$2ai\frac{\mathrm{d}}{\mathrm{d}y}\mathscr{F}(f)(y) = 2ai\mathscr{F}(i(\cdot)f(\cdot))(y) = \mathscr{F}(-2a(\cdot)f(\cdot))(y) = \mathscr{F}(f')(y) = -iy\mathscr{F}(f)(y). \tag{14}$$

Hence, $\mathscr{F}(f)(y)$ is the unique solution of the IVP

$$u' = -\frac{y}{2a}u, \qquad u(0) = \mathscr{F}(f)(0). \tag{15}$$

The general solution of the differential equation is

$$u(y) = u(0)e^{-\frac{y^2}{4a}}. \tag{16}$$

Since

$$\mathscr{F}(f)(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2}\,\mathrm{d}x = \frac{1}{\sqrt{2a}}, \tag{17}$$

it follows that

$$\mathscr{F}(f)(y) = \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}. \tag{18}$$

Thus, if $\phi_a(x) = e^{-ax^2}$, then, formally,

$$\mathscr{F}(1)(y) = \mathscr{F}\left(\lim_{a\to 0^+} \phi_a \right)(y) = \lim_{a\to 0^+} \mathscr{F}(\phi_a)(y) = \lim_{a\to 0^+} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}. \tag{19}$$

We would like to interpret the last limit formally as a constant multiple of the Dirac delta function. Clearly,

$$\lim_{a\to 0^+} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}} = \begin{cases} 0 & y \neq 0, \\ \infty & y = 0. \end{cases} \tag{20}$$

At the same time, for any $a > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}\,\mathrm{d}y = \frac{1}{\sqrt{2a}}\sqrt{4a\pi} = \sqrt{2\pi}, \tag{21}$$

so it makes sense formally that we should have $\mathscr{F}(1)(y) = \sqrt{2\pi}\delta(y)$.

Now, if we consider applying the Fourier transform twice to a function f, we get

$$
\begin{aligned}
\mathscr{F}\mathscr{F}(f)(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixy}e^{izx}f(z)\,\mathrm{d}z\,\mathrm{d}x && (22)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix(y+z)}\,\mathrm{d}x\,\mathrm{d}z && (23)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)\mathscr{F}(1)(y+z)\,\mathrm{d}z && (24)\\
&= \int_{-\infty}^{\infty} f(z)\delta(y+z)\,\mathrm{d}z && (25)\\
&= \int_{-\infty}^{\infty} f(z-y)\delta(z)\,\mathrm{d}z && (26)\\
&= f(-y). && (27)
\end{aligned}
$$

$\square$

1.4) Define $g(y) = f(-y)$ for some function f. Based on the formal result from part 1.3), we see immediately that

$$\mathscr{F}^4(f)(y) = \mathscr{F}^2(\mathscr{F}^2(f))(y) = \mathscr{F}^2(g)(y) = g(-y) = f(y). \quad (28)$$

Since f was arbitrary, it follows formally that $\mathscr{F}^4 = I$, the identity operator.

1.5) Let $p(x) = x^4$. By the Spectral Mapping Theorem,

$$p(\sigma(\mathscr{F})) = \sigma(p(\mathscr{F})). \quad (29)$$

Since $p(\mathscr{F}) = \mathscr{F}^4 = I$, the spectrum of $p(\mathscr{F})$ is just $\sigma(I) = \{1\}$, as the operator $I - \lambda I = (1-\lambda)I$ is invertible, with inverse $\frac{1}{1-\lambda}I$, if and only if $\lambda \neq 1$. Therfore, if $\lambda \in \sigma(\mathscr{F})$, then $p(\lambda) = 1$, that is, $\lambda^4 = 1$. The possible solutions of this equation are $1, -1, i, -i$, so $\sigma(\mathscr{F}) \subseteq \{1, -1, i, -i\}$.

1.6) If we reuse the result in equation (18) with $a = \frac{1}{2}$, we see that if $f(x) = e^{-\frac{1}{2}x^2}$, then

$$\mathscr{F}(f)(y) = e^{-\frac{1}{2}y^2} \quad (30)$$

as well. Thus, $\mathscr{F}f = f$, so f is an eigenfunction of $\mathscr{F}$ with corresponding eigenvalue 1.