Math 5604 Homework 1

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Problem 1.

Consider the IVP

$$y' = 3 + e^{-t} - y, \quad t > 0; \qquad y(0) = 1. \tag{1}$$

1.1) Multiplying both sides by the integrating factor e^t gives

$$y'e^t + ye^t = 3e^t + 1. \tag{2}$$

The left-hand side is $(ye^t)'$, so integrating on both sides gives

$$ye^t = 3e^t + t + C, \tag{3}$$

for some constant C, so $y(t) = 3 + (t + C)e^{-t}$. The initial condition $y(0) = 1$ implies that $C = -2$, so

$$y(t) = 3 + (t - 2)e^{-t}. \tag{4}$$

1.2) (**a**) To discretize the IVP on $[0, 2]$ using the forward Euler method, we need to have an evenly-spaced set of time samples $\{t_i\}_{i=0}^n$ defined by

$$t_i = \begin{cases} 0 & i = 0 \\ t_{i-1} + k, & i \geq 1, \end{cases}, \qquad i = 0, 1, \ldots, n. \tag{5}$$

The value k is the step size and is chosen so that $t_n = 2$; that is, $k = \frac{2}{n}$. We will attempt to find an approximation $\{y_i\}_{i=0}^n$ of the values $\{y(t_i)\}_{i=0}^n$. To find $\{y_i\}$, we create and solve a system of equations from the ODE by approximating $y'(t_i)$ by the forward difference $y'(t_i) \approx \frac{y(t_{i+1}) - y(t_i)}{k}$, where $i < n$. Since we know that $y(0) = 1$ from the initial condition, we are led to the scheme

$$\begin{cases} y_0 = 1 \\ \frac{y_{i+1} - y_i}{k} = 3 + e^{-t_i} - y_i, \quad 0 \leq i < n, \end{cases} \tag{6}$$

which allows to write an explicit recursive formula for y_i:

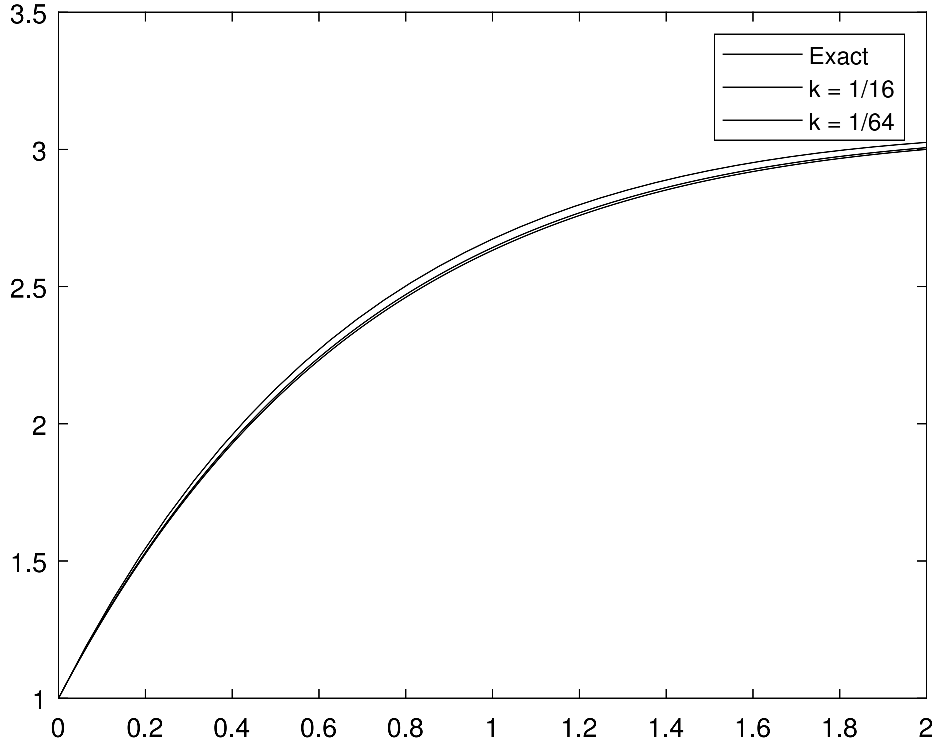
$$\begin{cases} y_0 = 1 \\ y_{i+1} = y_i + k(3 + e^{-t_i} - y_i), \quad 0 \leq i < n. \end{cases} \tag{7}$$

(**b**) According the output from `problem1_output.txt`, the numerical value of $y(2)$ is 3.012754.

(**c**) Below is the plot generated by `problem1_calculations.m`.

1.3) (**a**) To discretize the IVP on $[0, 2]$ using the backward Euler method, we need to have an evenly-spaced set of time samples $\{t_i\}_{i=0}^n$ defined by

$$t_i = \begin{cases} 0 & i = 0 \\ t_{i-1} + k, & i \geq 1, \end{cases}, \qquad i = 0, 1, \ldots, n. \tag{8}$$



The value k is the step size and is chosen so that $t_n = 2$; that is, $k = \frac{2}{n}$. We will attempt to find an approximation $\{y_i\}_{i=0}^n$ of the values $\{y(t_i)\}_{i=0}^n$. To find $\{y_i\}$, we create and solve a system of equations from the ODE by approximating $y'(t_i)$ by the backward difference $y'(t_i) \approx \frac{y(t_i)-y(t_{i-1})}{k}$, where $i > 0$. Since we know that $y(0) = 1$ from the initial condition, we are led to the scheme

$$\begin{cases} y_0 = 1 \\ \frac{y_i - y_{i-1}}{k} = 3 + e^{-t_i} - y_i, \quad 0 < i \le n, \end{cases} \tag{9}$$

which allows to write an explicit recursive formula for y_i:

$$\begin{cases} y_0 = 1 \\ y_i = \frac{y_{i-1} + k(3 + e^{-t_i})}{1+k}, \quad 0 < i \le n. \end{cases} \tag{10}$$

(**b**)

(**c**)

1.4)

Problem 2.

Consider the IVP

$$y' = \frac{3t^2 + 10t + 1}{2(y+1)}, \quad t > 0; \qquad y(0) = -2. \tag{11}$$

2.1) Multiplying both sides by $2(y+1)$ gives

$$2(y+1)(y+1)' = 3t^2 + 10t + 1. \tag{12}$$

The left-hand side is $\left((y+1)^2\right)'$, so integrating on both sides gives

$$(y+1)^2 = t^3 + 5t^2 + t + C \tag{13}$$

for some constant C. The initial condition $y(0) = -2$ implies that $C = 1$. Therefore,

$$y(t) = -1 \pm \sqrt{t^3 + 5t^2 + t + 1}. \tag{14}$$

The initial condition forces us to choose a negative sign after taking the square root; thus,

$$y(t) = -1 - \sqrt{t^3 + 5t^2 + t + 1}. \tag{15}$$

2.2) To discretize the IVP on $[0,1]$ using the backward Euler method, we need to have an evenly-spaced set of time samples $\{t_i\}_{i=0}^n$ defined by

$$t_i = \begin{cases} 0 & i = 0 \\ t_{i-1} + k, & i \geq 1, \end{cases}, \qquad i = 0, 1, \dots, n. \tag{16}$$

The value k is the step size and is chosen so that $t_n = 1$; that is, $k = \frac{1}{n}$. We will attempt to find an approximation $\{y_i\}_{i=0}^n$ of the values $\{y(t_i)\}_{i=0}^n$. To find $\{y_i\}$, we create and solve a system of equations from the ODE by approximating $y'(t_i)$ by the backward difference $y'(t_i) \approx \frac{y(t_i) - y(t_{i-1})}{k}$, where $i > 0$. Since we know that $y(0) = 1$ from the initial condition, we are led to the scheme

$$\begin{cases} y_0 = 1 \\ \frac{y_i - y_{i-1}}{k} = \frac{3t_i^2 + 10t_i + 1}{2(y_i + 1)}, \quad 0 < i \leq n, \end{cases} \tag{17}$$

which allows to write an implicit recursive formula for y_i:

$$\begin{cases} y_0 = 1 \\ 2(y_i + 1)(y_i - y_{i-1}) - k(3t_i^2 + 10t_i + 1) = 0, \quad 0 < i \leq n. \end{cases} \tag{18}$$

We can solve the implicit equation for y_i numerically using Newton's method. Indeed, if we set

$$f_i(y) = 2(y+1)(y - y_{i-1}) - k(3t_i^2 + 10t_i + 1), \qquad 0 < i \leq n, \tag{19}$$

then finding y_i is equivalent to finding the root of f_i. Newton's method is easy to apply once we note that $f_i'(y) = 2(y - y_{i-1}) + 2(y+1)$.