Math 6108 Homework 1

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Problem 1.

Problem 2.

If $P \in \mathbf{R}^{n\times n}$ is a projection, then $I - P$ is projection.

Proof. Since P is a projection, $P^2 = P$. Thus,

$$(I-P)^2 = (I-P)(I-P) = I^2 - IP - PI + P^2 = I - 2P + P = I - P,$$

so $I - P$ is also a projection. □

Problem 3.

Problem 4.

Problem 5.

Problem 6.

If a matrix $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) has the form

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix},$$

where $x_1, x_2, \dots, x_n \in \mathbf{R}$ (or $\mathbf{C}$) then A is called a Vandermonde matrix. The determinant of A is given by

$$\det(A) = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Proof. The determinant is preserved by adding a scalar multiple of one column to another. Let C_i denote the

ith column of A. If we perform the following sequence of column operations, which preserve the determinant,

$$\begin{aligned} C_n &\leftarrow C_n - x_1 C_{n-1}, \\ C_{n-1} &\leftarrow C_{n-1} - x_1 C_{n-2}, \\ &\vdots \\ C_3 &\leftarrow C_3 - x_1 C_2, \\ C_2 &\leftarrow C_2 - x_1 C_1, \end{aligned}$$

then we find that

$$\det(A) = \det\left(\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & x_2 - x_1 & x_2(x_2 - x_1) & x_2^2(x_2 - x_1) & \dots & x_2^{n-2}(x_2 - x_1) \\ 1 & x_3 - x_1 & x_3(x_3 - x_1) & x_3^2(x_3 - x_1) & \dots & x_3^{n-2}(x_3 - x_1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n - x_1 & x_n(x_n - x_1) & x_n^2(x_n - x_1) & \dots & x_n^{n-2}(x_n - x_1) \end{bmatrix}\right).$$

Using the Laplace expansion for the determinant on the first row of the matrix on the right-hand side we get

$$\det(A) = \det\left(\begin{bmatrix} x_2 - x_1 & x_2(x_2 - x_1) & x_2^2(x_2 - x_1) & \dots & x_2^{n-2}(x_2 - x_1) \\ x_3 - x_1 & x_3(x_3 - x_1) & x_3^2(x_3 - x_1) & \dots & x_3^{n-2}(x_3 - x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n - x_1 & x_n(x_n - x_1) & x_n^2(x_n - x_1) & \dots & x_n^{n-2}(x_n - x_1) \end{bmatrix}\right).$$

Factoring $x_j - x_1$ out of the ith row of the above matrix on the right-hand side, for $j = 2, 3, \dots, n$, and applying the multilinearity of the determinant gives

$$\det(A) = \prod_{j=2}^{n}(x_j - x_1) \det\left(\begin{bmatrix} 1 & x_2 & x_2^2 & \dots & x_2^{n-2} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-2} \end{bmatrix}\right)$$

□

Problem 7.

Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathbf{C}^n$ be a set of nonzero, orthogonal vectors. Then $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are linearly independent.

Proof. Let $c_1, c_2, \dots, c_k \in \mathbf{C}$ satisfy

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_k\mathbf{x}_k = 0.$$

Taking the inner product of both sides with $\mathbf{x}_i$, for $i = 1, 2, \dots, k$, gives

$$c_i(\mathbf{x}_i, \mathbf{x}_i) = (0, \mathbf{x}_i) = 0, \qquad i = 1, 2, \dots, k.$$

Since $\mathbf{x}_i \neq 0$ for $i = 1, 2, \dots, k$, it follows that $(\mathbf{x}_i, \mathbf{x}_i) \neq 0$, and $c_i = 0$ for $i = 1, 2, \dots, k$. Therefore, $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are linearly independent. □