Chaby Shev Inequality. Let X be a r.v. with variance σ^2 and $E[X] = \mu$. Then

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2},$$

Proof. $|X-\mu| \geq k\sigma$ iff $(X-\mu)^2 \geq k^2\sigma^2$, so $P(|X-\mu| \geq k\sigma) = P((X-\mu)^2 \geq k^2\sigma^2)$.

If $u(x) = (x-\mu)^2$ and $c = k^2\sigma^2$, then by Theorem 2.4.6

$$P(u(X) \geq c) \leq \frac{E[u(X)]}{c}$$

or

$$P((X-\mu)^2 \geq k^2\sigma^2) \leq \frac{E[(X-\mu)^2]}{k^2\sigma^2} = \frac{1}{k^2},$$ since

$$\sigma^2 = Var[X] = E[(X-\mu)^2].$$

Therefore

$$P(|X-\mu| \geq k\sigma) = P((X-\mu)^2 \geq k^2\sigma^2) \geq \frac{1}{k^2}.$$

2.23. Let X be a random variable with pmf $f(x) = \frac{x}{8}$ if $x = 1, 2, 5$ and $f(x) = 0$ otherwise.

(a) $E[X] = \sum_x x f(x) = 1 \cdot f(1) + 2f(2) + 5f(5)$

$$= \frac{1}{8} + \frac{4}{8} + \frac{25}{8}$$

$$= \frac{30}{8} = \frac{15}{4} = 3.75$$

(b) $Var[X] = E[X^2] - E[X]^2$,

$E[X^2] = \sum_x x^2 f(x) = 1f(1) + 4f(2) + 25f(5)$

$$= \frac{1}{8} + \frac{8}{8} + \frac{125}{8}$$

$$= \frac{134}{8} = \frac{67}{4}$$

$\therefore Var[X] = \frac{67}{4} - \left(\frac{15}{4}\right)^2 = \frac{268}{16} - \frac{225}{16} = \frac{43}{16} = 2.6875$

(c) $E[2X+3] = 2E[X]+3 = 2 \cdot \frac{15}{4} + 3 = \frac{15}{2} + \frac{6}{2} = \frac{21}{2} = 10.5$

2.24 Let X be continuous with pdf $f(x) = \begin{cases} 3x^2 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

(a) $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\, 3x^2\,dx = \frac{3}{4} x^4 \Big|_0^1 = \frac{3}{4}$.

(b) $Var[X] = E[X^2] - E[X]^2$, and

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\, 3x^2\,dx = \frac{3}{5} x^5 \Big|_0^1 = \frac{3}{5},$$ so

$$Var[X] = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} = 0.0375$$

(c) $E[X^r] = \int_{-\infty}^{\infty} x^r f(x)\,dx = 3\int_0^1 x^r x^2\,dx = \frac{3}{3+r} x^{3+r}\Big|_0^1 = \frac{3}{3+r}$

(d) $E[3X - 5X^2 + 1] = 3E[X] - 5E[X^2] + 1$

$$= 3 \cdot \frac{3}{4} - 5 \cdot \frac{3}{5} + 1 = \frac{1}{4}$$

2.25. Let X be continuous with pdf $f(x) = \begin{cases} x^{-2} & 1 < x < \infty \\ 0 & \text{otherwise} \end{cases}$.

(a) If $E[X]$ exists it would need to be

$$\int_{-\infty}^{\infty} f(x) \cdot x \, dx = \int_{1}^{\infty} x \, x^{-2} \, dx = \int_{1}^{\infty} x^{-1} dx = \lim_{x \to \infty} \ln x,$$

and the limit does not exist, so $E[X]$ does not exist.

(b) If $E[X^{-1}]$ exists, then it equals

$$\int_{-\infty}^{\infty} f(x) x^{-1} dx = \int_{1}^{\infty} x^{-2} x^{-1} dx = \int_{1}^{\infty} x^{-3} dx = -\frac{1}{2} x^{-2} \Big|_{1}^{\infty} = \frac{1}{2}, \text{ so}$$

$E[X^{-1}]$ exists.

(c) If $E[X^K]$ exists, then it must equal

$$\int_{-\infty}^{\infty} x^K f(x) \, dx = \int_{1}^{\infty} x^K x^{-2} dx = \int_{1}^{\infty} x^{K-2} dx, \text{ but in}$$

general $\int_{1}^{\infty} x^p dx$ converges iff $p < -1$, so $E[X^K]$ exists iff $K - 2 < -1$, or iff $K < 1$.

2.34 Let $M_X(t) = \frac{1}{8}e^t + \frac{1}{4}e^{2t} + \frac{5}{8}e^{5t}$ be the MGF of X.

(a) Let Y be a random variable with pmf $f(y) = \begin{cases} \frac{y}{8} & y = 1,2,5 \\ 0 & \text{otherwise.} \end{cases}$

Then the MGF of Y $M_Y(t) = E[e^{Yt}]$

$$= \sum_y e^{yt} f(y)$$

$$= \frac{1}{8}e^t + \frac{1}{4}e^{2t} + \frac{5}{8}e^{5t}$$

$$= M_X(t).$$

Therefore, the distribution of X is the same as the distribution of Y because $M_X = M_Y$.

(b) $P(X=2) = P(Y=2) = f(2) = \frac{1}{4}$.

2.36 Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} e^{-x-2} & -2 < x \\ 0 & \text{otherwise.} \end{cases}$$

(a) If $M_X(t)$ is the MGF of X, then

$$M_X(t) = E[e^{Xt}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-2}^{\infty} e^{tx} e^{-x-2}\,dx$$

$$= \int_{-2}^{\infty} e^{x(t-1)-2}\,dx$$

$$= \frac{1}{t-1} e^{x(t-1)-2} \Big|_{-2}^{\infty}$$

$$= \frac{1}{1-t} e^{-2t} \quad \text{if } t-1<0$$

or $t<1$

(b) $$E[X] = M_X'(0) = \frac{-2e^{-2t}(1-t) - e^{-2t}\cdot(-1)}{(1-t)^2}\Bigg|_{t=0}$$

$$= -1$$

$$E(X^2] = M_X''(0) = \frac{d}{dt}\left(\frac{-2e^{-2t}}{1-t} + \frac{e^{-2t}}{(1-t)^2}\right)_{t=0}$$

$$= \frac{d}{dt}\left(e^{-2t}\left((1-t)^{-2} - 2(1-t)^{-1}\right)\right)_{t=0}$$

$$= \left[-2e^{-2t}\left((1-t)^{-2} - 2(1-t)^{-1}\right) + e^{-2t}\left(2(1-t)^{-3} - 2(1-t)^{-2}\right)\right]_{t=0}$$

$$= -2(-1) = 2$$