Math 6417 Homework 3

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Question 1.

Let $B(\cdot,\cdot)$ be a continuous, bilinear form on a real Hilbert space H. Suppose that B is coercive in the sense that there is some $\alpha > 0$ such that $B(x,x) \geq \alpha\|x\|^2$ for all $x \in H$.

1.1) Let $y \in H$. Then the map $f_y : H \to \mathbf{R}$ defined by $f_y(x) = B(x,y)$ is a bounded linear functional on H. Consequently, there exists a unique $w \in H$ such that $B(x,y) = f_y(x) = (x,w)$ for all $x \in H$.

Proof. Firstly, it is clear that f_y is linear; indeed, given $a_1, a_2 \in \mathbf{R}$ and $x_1, x_2 \in H$,

$$f_y(a_1x_1 + a_2x_2) = B(a_1x_1 + a_2x_2, y) = a_1B(x_1,y) + a_2B(x_2,y) = a_1f_y(x_1) + a_2f_y(x_2) \tag{1}$$

by the bilinearity of B.

Secondly, $B(\cdot, y) = f_y$ must be continuous because B is continuous. Hence, f_y is bounded.

Thirdly, by the Riesz representation theorem, there exists a unique $w \in H$ such that $B(x,y) = f_y(x) = (x,w)$ for all $x \in H$. □

1.2) Given $y \in H$, by 1.1), there is a unique w such that $B(x,y) = (x,w)$ for all $x \in H$; this defines a function $A : H \to H$, where $Ay = w$. Then A is a bounded, linear operator on H, that is, $A \in B(H)$.

Proof. Let $a_1, a_2 \in \mathbf{R}$ and $y_1, y_2 \in H$. Then for all $x \in H$,

$$\begin{aligned}(x, A(a_1y_1 + a_2y_2)) &= B(x, a_1y_1 + a_2y_2) = a_1B(x,y_1) + a_2B(x,y_2) = a_1(x,Ay_1) + a_2(x,Ay_2)\\ &= (x, a_1Ay_1 + a_2Ay_2).\end{aligned} \tag{2}$$

Thus, $w = A(a_1y_1 + a_2y_2)$ and $w' = a_1Ay_1 + a_2Ay_2$ satisfy the property that $B(x, a_1y_1 + a_2y_2) = (x,w) = (x,w')$ for all $x \in H$. Since there is only one element of H that can satisfy this property by the Riesz representation theorem, it follows that $w = w'$, that is, $A(a_1y_1 + a_2y_2) = a_1Ay_1 + a_2Ay_2$. Therefore, A is linear.

Note that B is continuous if and only if (see, e.g., Theorem 8.10 assumption (a) in Arbogast and Bona) there exists some $M > 0$ such that

$$|B(x,y)| \leq M\|x\|\|y\|, \quad \text{for all } x, y \in H. \tag{3}$$

Let $y \in H$. Then

$$\|Ay\| = \left|\left(\frac{Ay}{\|Ay\|}, Ay\right)\right| = \left|B\left(\frac{Ay}{\|Ay\|}, y\right)\right| \leq M\|y\|. \tag{4}$$

Since y was arbitrary, it follows that A is bounded, and $\|A\| \leq M$. Thus, A is a bounded, linear operator on H. □

1.3) A is bounded below in the sense that there exists $\gamma > 0$ such that $\|Ay\| \geq \gamma\|y\|$ for all $y \in H$.

Proof. This follows from the coercivity of B: for all $y \in H$,

$$\|Ay\|\|y\| \geq |(y, Ay)| = |B(y,y)| \geq \alpha\|y\|^2, \tag{5}$$

so $\|Ay\| \geq \alpha\|y\|$ for all $y \in H$, as claimed. □

1.4) A is one-to-one, and the range of A is closed.

Proof. Let $y_1, y_2 \in H$, and suppose that $Ay_1 = Ay_2$. Then, by the previous part,

$$\|y_1 - y_2\| \leq \frac{1}{\gamma}\|A(y_1 - y_2)\| = \frac{1}{\gamma}\|Ay_1 - Ay_2\| = 0. \tag{6}$$

Therefore, $y_1 = y_2$. This shows that A is one-to-one.

Let $R(A)$ denote the range of A. To show that $R(A)$ is closed, we need to show that if $\{w_n\} \subset R(A)$ is any sequence that converges to w, then $w \in R(A)$. To this end, let $\{w_n\} \subseteq R(A)$ be a convergent sequence, and let w be its limit. Since $w_n \in R(A)$ for all n, there exists $y_n \in H$ such that $w_n = Ay_n$ for all n. We can use the coercivity of B to show that $\{y_n\}$ is convergent.

Indeed, for all m, n and all $x \in H$, the definition of A implies that $|B(x, y_n - y_m)| = |(x, w_n - w_m)| \leq \|x\|\|w_n - w_m\|$. Since $\{w_n\}$ converges, it is Cauchy; hence,

$$\begin{aligned}
&\forall\varepsilon > 0 : \exists N : n, m > N \to \|w_n - w_m\| < \varepsilon && (7)\\
\Longrightarrow\ &\forall\varepsilon > 0 : \exists N : n, m > N \to \forall x \in H : |B(x, y_n - y_m)| \leq \|x\|\|w_n - w_m\| < \|x\|\varepsilon && (8)\\
\Longrightarrow\ &\forall\varepsilon > 0 : \exists N : n, m > N \to \alpha\|y_n - y_m\|^2 \leq |B(y_n - y_m, y_n - y_m)| < \|y_n - y_m\|\varepsilon && (9)\\
\Longrightarrow\ &\forall\varepsilon > 0 : \exists N : n, m > N \to \|y_n - y_m\| < \frac{\varepsilon}{\alpha}, && (10)
\end{aligned}$$

which implies that $\{y_n\}$ is Cauchy. Therefore, there exists $y \in H$ such that $y_n \to y$ as $n \to \infty$. Let $x \in H$, and let $\varepsilon > 0$ be given. By the continuity of B and the inner product and the convergence of $\{y_n\}$ and $\{w_n\}$, there exists n large enough that $|B(x, y - y_n)| < \frac{\varepsilon}{2}$, and $|(x, w - w_n)| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}
|B(x,y) - (x,w)| &= |B(x, y - y_n) + B(x, y_n) - (x, w_n) - (x, w - w_n)| \\
&\leq |B(x, y - y_n)| + |(x, w - w_n)| < \varepsilon.
\end{aligned} \tag{11}$$

Since $\varepsilon > 0$ was arbitrary and $x \in H$ was arbitrary, it follows that $B(x, y) = (x, w)$ for all $x \in H$. This implies that $w = Ay$ by the definition of A, and $w \in R(A)$. Since the convergent sequence $\{w_n\} \subseteq R(A)$ was arbitrary, and its limit $w \in R(A)$, it follows that $R(A)$ is closed. □

1.5) A is onto.

Proof. Suppose that $x \in R(A)^\perp$, that is, $(x, w) = 0$ for all $w \in R(A)$. This implies that $(x, Ay) = 0$ for all $y \in H$, which is equivalent to saying that $B(x, y) = 0$ for all $y \in H$. In particular, if we choose $y = x$, then $\|x\|^2 \leq \frac{1}{\alpha}|B(x,x)| = 0$. Therefore, $x = 0$. This shows that $R(A)^\perp = \{0\}$ because x was arbitrary.

Let $y \in H$. Since $R(A)$ is a closed subspace of H by (1.4), there exists a best approximation $w \in R(A)$ of y, which satisfies the property $(y - w, x) = 0$ for all $x \in R(A)$ (Theorem 3.7 and Corollary 3.8 in Arbogast and Bona). That is, $y - w \in R(A)^\perp$. Since $R(A)^\perp = \{0\}$ by the above, it follows that $y - w = 0$, and $y = w \in R(A)$. Since y was arbitrary and $R(A) \subseteq H$, it follows that $R(A) = H$, that is, A is onto. □

1.6) A is invertible.

Proof. By the previous two parts, A is bijective, so it has a set-theoretic inverse function A^{-1}. By 1.2), A is bounded. Therefore, by the open mapping theorem, A maps open sets to open sets, which means that the preimage of an open set under A^{-1} is open, that is, A^{-1} is continuous. Therefore, A is invertible. $\square$

1.7) Given $f \in H^*$, the Riesz representation theorem implies that there exists a unique $w \in H$ such that $f(x) = (x, w)$ for all $x \in H$, and we can view H^* and H as the same under the correspondence $f \leftrightarrow w$.

1.8) Consider the equation $B(x, y) = f(x)$ for all $x \in H$, where $f \in H^*$. By the remark in part 1.7), we can choose $w \in H$ such that $f(x) = (x, w)$ for all $x \in H$. Then the equation is equivalent to $B(x, y) = (x, w)$ for all $x \in H$. If y is a solution of this equation, then, by the definition of A, we must have $Ay = w$. Using the invertibility of A, we obtain $y = A^{-1}w$ as the unique solution of the equation. Viewing f and w as the same under the correspondence in 1.7), we might also write $y = A^{-1}f$.

Question 2.