The Fréchet Derivative

Jacob Hauck

Math 6418

Motivation

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$$f(x) - f(x_0) \approx A_{x_0}(x - x_0)$$

for x close to x_0 (point-slope form)

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Say $X = Y = \mathbf{R}$, then A_{x_0} is given by multiplication by a number $a_{x_0} \in \mathbf{R}$, and a natural choice for a_{x_0} is $f'(x_0)$, as

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How can we generalize to the case that $X, Y \neq \mathbf{R}$? Can't divide by h, so rewrite as

$$\frac{|f(x_0 + h) - f(x_0) - f'(x_0)h|}{|h|} \to 0 \quad \text{as} \quad h \to 0.$$

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A function $f : U \to Y$ is called **Fréchet differentiable** at $x \in U$ if there exists $A \in B(X, Y)$ such that

$$\frac{\|f(x+h) - f(x) - Ah\|_Y}{\|h\|_X} \to 0 \quad \text{as} \quad \|h\|_X \to 0,$$

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The Fréchet derivative is the same as the usual derivative if $f \in C^1(\mathbf{R})$.

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4. Fréchet differentiable $\implies$ locally Lipschitz
   $f : X \to Y$ is **locally Lipschitz** at $x \in X$ if there exists $\delta > 0$ and $L > 0$ such that

$$\|f(x) - f(y)\|_Y \leq L\|x - y\|_X \qquad \text{if} \quad \|x - y\| < \delta.$$

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   For any $\varepsilon > 0$, can choose δ small enough so that $L = \|Df(x)\|_{B(X,Y)} + \varepsilon$ works.

Examples – Linear Operators

Example 1. Suppose that $f : X \to Y$ is actually linear: $f(x) = Ax$, where $A \in B(X,Y)$. Then, as expected,

$$Df(x) = A \qquad \text{for all } x \in X.$$

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Example 2. Suppose that $X = H$ is a Hilbert space over $\mathbf{R}$, and $f(x) = (x, Ax)$, where $A \in B(H, H)$.

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$$\frac{|f(x+h) - f(x) - ((A * +A)x, h)|}{\|h\|_X} = \frac{|(h, Ah)|}{\|h\|_X}$$

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and $\|Ah\|_X \to 0$ as $h \to 0$.

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Let $n = 2$, and define $\tilde{h} = h_1 e_1$. Then

$$|f(x+h) - f(x) - \nabla f(x)^T h| = |f(x+h) - f(x) - \partial_1 f(x) h_1 - \partial_2 f(x) h_2|$$

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Dividing both sides by $\|h\|$ and applying definition of partial derivative and continuity...

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Second term:

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Third term:

$$|\partial_2 f(x+\tilde{h}) - \partial_2 f(x)| \cdot \frac{|h_2|}{\|h\|} \to 0 \quad \text{as} \quad h \to 0$$

by the continuity of $\partial_2 f$.

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Then, if $J(x) \in \mathbf{R}^{m \times n}$ is the Jacobian matrix of f,

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} \implies J(x) = \begin{bmatrix} \nabla f_1(x)^T \\ \vdots \\ \nabla f_m(x)^T \end{bmatrix},$$

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Each component $\to 0$ as $h \to 0$ because ∇f_1 is the Fréchet derivative of f_1; therefore $Df(x) = J(x)$, interpreting the matrix as an operator.

Examples – Integral of a Power

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Let $h \in L^p(\mathbf{R})$; then

$$\begin{aligned} f(\varphi + h) - f(\varphi) &= \int [(\varphi + h)^p - \varphi^p] \\ &= \int \left[p\varphi^{p-1}h + \binom{p}{2}\varphi^{p-2}h^2 + \cdots + h^p \right] \end{aligned}$$

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Note that $\varphi^{p-k}h^k$ is integrable for $k = 0, 1, \ldots, p$:

$$\int |\varphi|^{p-k}|h|^k \leq \left(\int |\varphi|^p \right)^{\frac{1}{u}} \left(\int |h|^p \right)^{\frac{1}{v}} = \|\varphi\|^{p-k}_{L^p(\mathbf{R})} \|h\|^k_{L^p(\mathbf{R})}$$

if $u = \frac{p}{p-k}$ and $v = \frac{p}{k}$.

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Therefore,

$$\frac{1}{\|h\|_{L^p(\mathbf{R})}}\Big|f(\varphi+h)-f(\varphi)-\int p\varphi^{p-1}h\Big|$$
$$=\frac{1}{\|h\|}\left|\int\left[\binom{p}{2}\varphi^{p-2}h^2+\cdots+h^p\right]\right|$$

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&\le \frac{1}{\|h\|}\left[\binom{p}{2}\|\varphi\|^{p-2}_{L^p(\mathbf{R})}\|h\|^2_{L^p(\mathbf{R})}+\cdots+\|h\|^p_{L^p(\mathbf{R})}\right]
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&\leq \binom{p}{2}\int|\varphi|^{p-2}|h^2|+\cdots+\int|h|^p \\
&\leq \frac{1}{\|h\|}\left[\binom{p}{2}\|\varphi\|_{L^p(\mathbf{R})}^{p-2}\|h\|_{L^p(\mathbf{R})}^2+\cdots+\|h\|_{L^p(\mathbf{R})}^p\right] \\
&\leq C(\varphi,p)\left(\|h\|+\|h\|^2+\cdots+\|h\|^{p-2}\right)
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which $\to 0$ as $\|h\| \to 0$.

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\end{aligned}$$

which $\to 0$ as $\|h\| \to 0$.

Hence,

$$Df(\varphi)h = \int p\varphi^{p-1}h$$

Directional Derivatives

We can also define a directional derivative of an operator $f : X \to Y$.

Gateaux Derivative

Let $h \in X$ be a unit vector. Then $A \in B(X, Y)$ is called the **Gateaux derivative** of f at $x \in X$ **in the direction** h if

$$\frac{\|f(x + th) - f(x) - tAh\|_Y}{|t|} \to 0 \quad \text{as} \quad t \to 0,$$

which is also denoted by

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Question: if f is Gateaux differentiable (G.d. in every direction), is f also Fréchet differentiable? Conversely?

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