2.5 A discrete random variable has pmf $f(x)$.

(a) If $f(x) = K(\frac{1}{2})^x$, $x = 1, 2, 3$, and zero otherwise, then we must have $\sum_x f(x) = 1$

$$K\left(\tfrac{1}{2}\right) + K\left(\tfrac{1}{2}\right)^2 + K\left(\tfrac{1}{2}\right)^3 = 1$$

$$\Rightarrow K = \frac{8}{7}$$

(b) Let $f(x) = K\left(\left(\frac{1}{2}\right)^x - \frac{1}{2}\right)$, $x = 0, 1, 2$, zero, otherwise. This cannot be a pmf, because $f(2) = K\left(\left(\frac{1}{2}\right)^2 - \frac{1}{2}\right) = -K/4$ but $f(0) = K\left(\left(\frac{1}{2}\right)^0 - \frac{1}{2}\right) = \frac{K}{2}$. If $K > 0$, then $f(2) = -K/4 < 0$, and if $K < 0$, then $f(0) = \frac{K}{2} < 0$. If $K = 0$, then $f(x) = 0$, and $\sum_x f(x) = 0 \neq 1$. No matter what K is, f will either violate $f(x) \geq 0$ or $\sum_x f(x) = 1$.

2.8 A nonnegative-integer-valued r.v. X has cdf

$$F(x) = \begin{cases} 1 - \frac{1}{2^{x+1}} & x = 0, 1, 2 \ldots \\ 0 & \text{otherwise} \end{cases}$$

(a) The pmf f of X is given by

$f(x) = P(X = x) = 0$ if $x \neq 0, 1, 2, \ldots$

$f(0) = P(X=0) = P(X \leq 0) - \underbrace{\cancel{P(X<0)}}_{0}$ because $X \geq 0$

$= F(0) = \frac{1}{2}$

If $x = 1, 2, \ldots$

$f(x) = P(X \leq x) - P(X < x)$

$= P(X \leq x) - P(X \leq x-1) - \underbrace{\cancel{P(x-1 < X < x)}}_{0}$ because X is integral and there are no integers in $(x-1, x)$.

$= F(x) - F(x-1)$

$= 1 - \frac{1}{2^{x+1}} - 1 + \frac{1}{2^x} = \frac{1}{2^x}\left(1 - \frac{1}{2}\right)$

$= \frac{1}{2^{x+1}}$

So, since $f(0) = \frac{1}{2} = \frac{1}{2^{0+1}}$,

$$f(x) = \begin{cases} \frac{1}{2^{x+1}}, & x = 0, 1, 2 \ldots \\ 0 & \text{otherwise} \end{cases}$$

2.8 (b)

$$P(10 \leq X \leq 20) = P(X \leq 20) - P(X < 10)$$

$$= P(X \leq 20) - P(X \leq 9) - P(\cancel{9 < X < 10})^{0}$$

$$= F(20) - F(9)$$

$$= \frac{1}{2^{21}} - \frac{1}{2^{10}}$$

$$= \frac{1}{2^{10}}\left(\frac{1}{2^{11}} - 1\right)$$

$$= \frac{2^{11} - 1}{2^{21}} \approx 9.761 \cdot 10^{-4}$$

(c) $$P(X \text{ even}) = P\left(\bigcup_{n=0}^{\infty} (X = 2n)\right)$$

$$= \sum_{n=0}^{\infty} P(X = 2n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n}$$

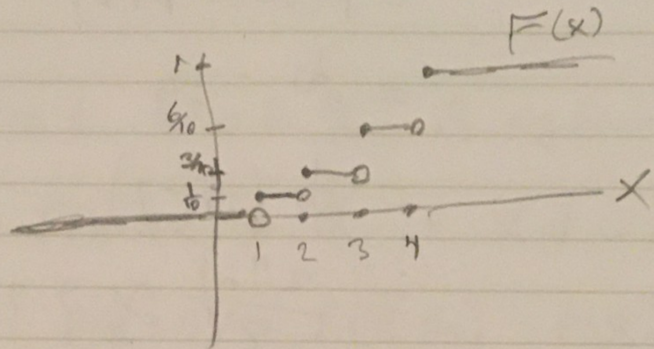
$$= \frac{1}{2} \frac{1}{1 - \frac{1}{4}}$$

$$= \frac{2}{3}$$

2.10 (a) $F(x) = 0.05x(1+x)$ if $x = 1, 2, 3, 4$.

Between 1, 2, 3, 4, $F'(x) = 0$, and to the left of 1, $P(X < x) = 0$ if $x < 1$ and $P(X > x) = 1$ if $x > 4$.

So



(b) $f(x) = P(X = x) = 0$ if $x \neq 1, 2, 3, 4$

$f(1) = P(X = 1) = P(X \le 1) - \cancel{P(X < 1)}^{0}$

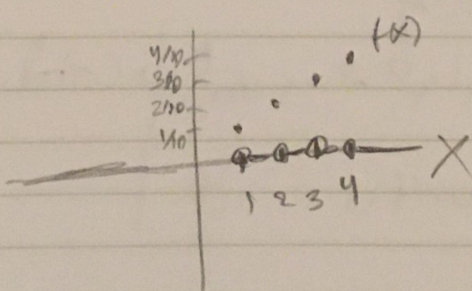
$= F(1) = \frac{1}{10}$

$f(2) = P(X = 2) = P(X \le 2) - P(X \le 1) - \cancel{P(1 < X < 2)}^{0}$

$= F(2) - F(1) = \frac{3}{10} - \frac{1}{10} = \frac{2}{10}$

$f(3) = P(X = 3) = F(3) - F(2) = \frac{6}{10} - \frac{3}{10} = \frac{3}{10}$

$f(4) = P(X = 4) = F(4) - F(3) = 1 - \frac{6}{10} = \frac{4}{10}$



2.10 (c)

$$E[X] = \sum_x x f(x)$$

$$= 1f(1) + 2f(2) + 3f(3) + 4f(4)$$

$$= \frac{1}{10} + \frac{4}{10} + \frac{9}{10} + \frac{16}{10}$$

$$= 3$$

2.14 (a) is $F(x) = e^{-x}$, $0 \leq x < \infty$ a CDF?

- if $x > y$, then $F(x) = e^{-x} < e^{-y} = F(y)$, so F is decreasing which means that F is not a CDF

(b) $F(x) = e^{x}$, $-\infty < x \leq 0$, $F(x) = 1$ $x > 0$.

- if $x \leq y$, then $F(x) \leq F(y)$, so F is non-decreasing
- ~~the~~ $\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} e^{x} = 0$ and $\lim_{x \to \infty} F(x) = \lim_{x \to \infty} 1 = 1$
- $\lim_{x \to 0^-} F(x) = 1 = \lim_{x \to 0^+} F(x) = F(0)$ so F is continuous everywhere (since e^x and 1 are continuous) and $\therefore$ right-continuous everywhere.

So F is a CDF

2.14 (e) $F(x) = 1 - e^{-x}$, $-1 \le x < \infty$.

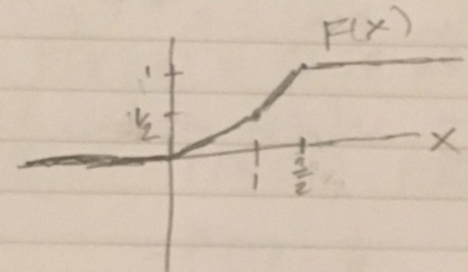
$F(-1) = 1 - e < 0 = F(-2)$, but $-2 < -1$ and

$F(-2) > F(-1)$, so F is not nondecreasing,

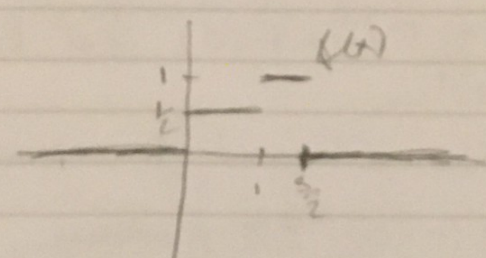
and F is not a CDF.

2.17 $F(x) = \begin{cases} x/2 & 0 < x \le 1 \\ x - 1/2 & 1 < x \le \frac{3}{2} \end{cases}$

(a)



(b) $f(x) = F'(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 < x < 1 \\ 1 & 1 < x < \frac{3}{2} \\ 0 & x > \frac{3}{2} \end{cases}$



(c) $P(X \leq \frac{1}{2}) = F(\frac{1}{2}) = \frac{1}{4}$

(d) $P(X \geq \frac{1}{2}) = \overset{0 \text{ because } F \text{ is continuous}}{\cancel{P(X = \frac{1}{2})}} + P(X > \frac{1}{2})$

$= 1 - P(X \leq \frac{1}{2}) = \frac{3}{4}$

(e) $P(X \leq 1.25) = F(1.25) = 0.75$

(f) $P(X = 1.25) = 0$ because F is continuous