Math 5601 Homework 8

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Problem 1.

Consider the formula

$$\int_0^h f(x)\,\mathrm{d}x \approx h\left[Af(0) + Bf\left(\frac{h}{3}\right) + Cf(h)\right] \tag{1}$$

(a) Formula (1) is exact for all polynomials of degree 2 or less if and only if it is exact for $f(x) = 1$, $f(x) = x$, and $f(x) = x^2$. Therefore, to be exact for all polynomial of degree 2 or less, we need to choose A, B, C such that

$$\int_0^h 1\,\mathrm{d}x = h = h(A + B + C), \tag{2}$$

$$\int_0^h x\,\mathrm{d}x = \frac{h^2}{2} = h\left[\frac{Bh}{3} + Ch\right], \tag{3}$$

$$\int_0^h x^2\,\mathrm{d}x = \frac{h^3}{3} = h\left[\frac{Bh^2}{9} + Ch^2\right]. \tag{4}$$

Then

$$A + B + C = 1, \tag{5}$$

$$\frac{B}{3} + C = \frac{1}{2}, \tag{6}$$

$$\frac{B}{9} + C = \frac{1}{3}. \tag{7}$$

The last two equation imply that $B = \frac{3}{4}$, and $C = \frac{1}{4}$. Together with the first equation, this gives $A = 0$.

(b) Suppose that the trapezoid rule for $\int_0^2 f(x)\,\mathrm{d}x$ gives the approximation $\frac{1}{2}$, but formula (1) gives $\frac{1}{4}$. If $f(0) = 3$, then $f\left(\frac{2}{3}\right) = 1$.

Proof. The trapezoid rule for $\int_0^2 f(x)\,\mathrm{d}x$ is $f(0)\cdot\frac{2-0}{2} + f(2)\cdot\frac{2-0}{2} = f(0) + f(2)$. Thus, $f(0) + f(2) = \frac{1}{2}$, so $f(2) = -\frac{5}{2}$. Using (1) with $h = 2$, we must have

$$\frac{1}{4} = 2\cdot\left[0\cdot f(0) + \frac{3}{4}f\left(\frac{2}{3}\right) + \frac{1}{4}f(2)\right] = 2\cdot\left[\frac{3}{4}f\left(\frac{2}{3}\right) - \frac{5}{8}\right]. \tag{8}$$

Hence, $1 = 6f\left(\frac{2}{3}\right) - 5$, which implies that $f\left(\frac{2}{3}\right) = 1$. □

Problem 2.

Algorithm 1: Forward Phase of Gaussian Elimination

Data: $A = \{a_{ij}\}$, an $n \times n$ nonsingular matrix.
Data: $b = \{b_i\}$, an $n \times 1$ column vector.
Result: Matrix A and vector b after forward phase of Gaussian elimination; that is, A and b are modified so that A is upper triangular, but the solution x of $Ax = b$ is the same as before.

1 **for** $k = 1, 2, \ldots, n-1$ **do**
2   **for** $i = k+1, \ldots, n$ **do**
3         $m_{ik} \leftarrow \frac{a_{ik}}{a_{kk}}$;
4   **for** $j = k, \ldots, n$ **do**
5             $a_{ij} \leftarrow a_{ij} - m_{ik} a_{kj}$;
6   **end**
7         $b_i \leftarrow b_i - m_{ik} b_k$;
8   **end**
9 **end**

(a) Recall our algorithm (version 2) for the forward phase of Gaussian elimination (Algorithm 1). We notice the multiplications and divisions occur on lines 3, 5, and 7 in the algorithm. Thus, the total number of multiplications and divisions is the total number of times these instructions are performed. For a given value of k, where $1 \leq k \leq n-1$, the loop starting on line 2 is performed $n-(k+1)+1 = n-k$ times. Therefore, lines 3 and 7 are performed

$$N_{3,7} = \sum_{k=1}^{n-1} (n-k) = \sum_{k=1}^{n-1} k = \frac{(n-1)n}{2} \tag{9}$$

times. For a given value of k and i, the loop at line 4 is executed $n-k+1$ times. Therefore, line 5 is performed a total of

$$N_5 = \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} (n-k+1) = \sum_{k=1}^{n-1} (n-k)(n-k+1) = \sum_{k=1}^{n-1} k(k+1) = \frac{(2n-1)n(n-1)}{6} + \frac{(n-1)n}{2} \tag{10}$$

times. Finally, the total number of multiplications and divisions is

$$N_{3,7} + N_5 = \frac{1}{3}n^3 + \frac{5(n-1)n}{6} = \frac{1}{3}n^3 + \mathcal{O}(n^2). \tag{11}$$

(b) Recall the algorithm for the backward phase of Gaussian elimination (Algorithm 2). We see that one

Algorithm 2: Backward Phase of Gaussian Elimination

Data: $A = \{a_{ij}\}$, an $n \times n$ nonsingular, upper triangular matrix.
Data: $b = \{b_i\}$, an $n \times 1$ column vector.
Result: The solution $x = \{x_i\}$ of $Ax = b$ as an $n \times 1$ column vector.

for $i = n, n-1, \ldots, 1$ **do**
    $x_i \leftarrow b_i$;
    for $j = i+1, \ldots, n$ **do** `% Loop does no iterations when` $i = n$
        $x_i \leftarrow x_i - a_{ij} x_j$;
    end
    $x_i \leftarrow \frac{x_i}{a_{ii}}$;
end

multiplication/division occurs on lines 4 and 6, so the total number of multiplications and division is equal to the total number of times these instructions are performed. Clearly, the instruction on line 6 is performed n times.

For a given value of i, where $1 \le i \le n$, the loop at line performs $n - (i+1) + 1 = n - i$ iterations, so line 4 is executed a total of

$$N_4 = \sum_{i=1}^{n} (n-i) = \sum_{i=0}^{n-1} i = \frac{(n-1)n}{2} \tag{12}$$

times. Thus, the total number of multiplications and divisions is

$$n + N_4 = n + \frac{(n-1)n}{2} = \frac{n^2+n}{2}. \tag{13}$$