Math 6417 Homework 1

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Question 1.

Let f be continuous on $[0,1]\times\mathbf{R}$ and satisfy $|f(x,u)-f(x,v)|\leq L|u-v|$ for all $x\in[0,1]$ and $u,v\in\mathbf{R}$, where $0\leq L<8$.

For $\alpha,\beta\in\mathbf{R}$, consider the boundary value problem

$$\begin{aligned} -u''(x) = f(x,u(x)) \quad &\text{if } x\in(0,1) \\ u(0)=\alpha \qquad & u(1)=\beta. \end{aligned} \tag{1}$$

This problem has one and only one solution $u\in C^2[0,1]$.

Indeed, define

$$G(x,\xi)=\begin{cases} \xi(1-x) & 0\leq\xi\leq x\leq 1 \\ x(1-\xi) & 0\leq x\leq\xi\leq 1 \end{cases} \tag{2}$$

and also consider the integral equation

$$u(x)=\alpha(1-x)+\beta x+\int_0^1 G(x,\xi)f(\xi,u(\xi))\,\mathrm{d}\xi \quad \text{if } x\in[0,1]. \tag{3}$$

We show that if $u\in C^2[0,1]$, then u solves (1) if and only if u solves (3), and that there is a unique solution $u\in C^2[0,1]$ of (3) by the Banach Fixed Point Theorem. Then the claim follows.

(i) If $u\in C^2[0,1]$, then u is a solution of (1) if and only if u is a solution of (3).

Proof. Suppose that $u\in C^2[0,1]$ is a solution of (1). Then, using integration by parts,

$$\begin{aligned} \int_0^1 G(x,\xi)f(\xi,u(\xi))\,\mathrm{d}\xi &= -\int_0^x \xi(1-x)u''(\xi)\,\mathrm{d}\xi - \int_x^1 x(1-\xi)u''(\xi)\,\mathrm{d}\xi \\ &= -(1-x)\left[\xi u'(\xi)\Big|_0^x - \int_0^x u'(\xi)\,\mathrm{d}\xi\right] - x\left[(1-\xi)u'(\xi)\Big|_x^1 + \int_x^1 u'(\xi)\,\mathrm{d}\xi\right] \\ &= -(1-x)xu'(x)+(1-x)(u(x)-u(0)) \\ &\quad + x(1-x)xu'(x) - x(u(1)-u(x)) \\ &= -\alpha(1-x)-\beta x+u(x) \end{aligned}$$

for any $x\in[0,1]$. Therefore, u solves (3).

Conversely, suppose that $u\in C^2[0,1]$ is a solution of (3). Then differentiating both sides of (3) implies that

$$u'(x)=\beta-\alpha+\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \xi(1-x)f(\xi,u(\xi))\,\mathrm{d}\xi+\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 x(1-\xi)f(\xi,u(\xi))\,\mathrm{d}\xi \tag{4}$$

for $x \in (0, 1)$. Since the integrands in both integrals above are obviously continuous and have a continuous partial derivative with respect to x on $[0, 1]^2$, the action of the derivative on the integrals gives

$$\begin{aligned} u'(x) &= \beta - \alpha + x(1-x)f(x, u(x)) - \int_0^x \xi f(\xi, u(\xi))\, \mathrm{d}\xi - x(1-x)f(x, u(x)) + \int_x^1 (1-\xi)f(\xi, u(\xi))\, \mathrm{d}\xi \\ &= \beta - \alpha - \int_0^x \xi f(\xi, u(\xi))\, \mathrm{d}\xi + \int_x^1 (1-\xi)f(\xi, u(\xi))\, \mathrm{d}\xi \end{aligned} \tag{5}$$

for $x \in (0, 1)$. Since f is continuous, the integrands in the above integrals are continuous, and the Fundamental Theorem of Calculus implies that

$$u''(x) = -xf(x, u(x)) - (1-x)f(x, u(x)) = -f(x, u(x)) \tag{6}$$

for $x \in (0, 1)$. Lastly, note that the definition of G implies that $G(0, \xi) = 0 = G(1, \xi)$ for all $\xi \in [0, 1]$. Thus, $u(0) = \alpha$, and $u(1) = \beta$, so u solves (1). □

(ii) There is one and only one solution $u \in C^2[0, 1]$ of (3).

Proof. For $u \in C^2[0, 1]$, define

$$Au(x) = \alpha(1-x) + \beta x + \int_0^1 G(x, \xi)f(\xi, u(\xi))\, \mathrm{d}\xi. \tag{7}$$

Then $Au \in C^2[0, 1]$ because, by the same calculation in (4-5) of part **(i)**,

$$(Au)''(x) = -f(x, u(x)), \tag{8}$$

which is continuous for $x \in [0, 1]$ by hypothesis. Thus, $A : C^2[0, 1] \to C^2[0, 1]$. Equip $C^2[0, 1]$ with the uniform metric

$$\rho(u, v) = \max_{x \in [0,1]} |u(x) - v(x)| \tag{9}$$

□