# Math 5604 Homework 8

Jacob Hauck

April 19, 2024

**Problem 1.**

Consider the BVP

$$\begin{aligned}
\Delta u = -2\pi^2 \sin(\pi(x+y)) = f(x,y), \qquad & & 0 < x < 1, \quad 0 < y < 1 \\
u(0,y) = \sin(\pi y) = g_\ell(y), \quad u(1,y) = \sin(\pi(1+y)) = g_r(y), & & 0 \le y \le 1 \\
u(x,0) = \sin(\pi x) = g_b(x), \quad u(x,1) = \sin(\pi(1+x)) = g_t(x), & & 0 \le x \le 1.
\end{aligned}$$

The exact solution of this equation is given by $u(x,y) = \sin(\pi(x+y))$.

**(a)** Consider a grid of sample points $\{(x_i, y_j)\}$ on the domain $[0,1]^2$, where $i = 0,1,\dots,M$, and $j = 0,1,\dots,N$. If the points are evenly spaced horizontally by $h_x = \frac{1}{M}$ and vertically by $h_y = \frac{1}{N}$, then $x_i = ih_x$, and $y_j = jh_y$.

We approximate $u(x_i, y_j)$ by $u_{i,j}$. Using a centered-difference scheme to approximate $\Delta u$ on the interior and applying the boundary conditions on the boundary points, we are led to the numerical scheme

$$\begin{aligned}
\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} = f(x_i, y_i), \\
i = 1,2,\dots,M-1,\ j = 1,2,\dots N-1,
\end{aligned} \tag{$(i,j)$}$$

and

$$\begin{aligned}
u_{0,j} = g_\ell(y_j), \qquad u_{M,j} = g_r(y_j), \qquad j = 0,1,\dots,N, \\
u_{i,0} = g_b(x_i), \qquad u_{i,N} = g_t(x_i), \qquad i = 0,1,\dots M.
\end{aligned}$$

In order to solve this linear system, we need to reshape the matrix of unknowns $\{u_{i,j}\}_{i=1,j=1}^{M-1,N-1}$ into a vector $U$ and rewrite the corresponding equations $(i,j)$ as a matrix-vector system, substituting the known boundary values where applicable.

We use row-wise ordering to reshape the matrix of unknowns; that is, we define the block vector of rows of the unknown matrix

$$U = \begin{bmatrix} U^{(1)} \\ U^{(2)} \\ \vdots \\ U^{(N-1)} \end{bmatrix}, \qquad U^{(j)} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{M-1,j} \end{bmatrix}, \quad j = 1,2,\dots,N-1.$$

We can rewrite the equations $(i,j)$ into a matrix-vector system, expressing the matrix $A$ and vector $b$ in block form corresponding to the blocks of $U$:

$$A = \begin{bmatrix} A^{(1,1)} & \dots & A^{(1,N-1)} \\ \vdots & \ddots & \vdots \\ A^{(N-1,1)} & \dots & A^{(N-1,N-1)} \end{bmatrix}, \qquad b = \begin{bmatrix} b^{(1)} \\ b^{(2)} \\ \vdots \\ b^{(N-1)} \end{bmatrix}.$$

We remark that the block $A^{(j,j')}$ expresses the dependence of equations $(1,j),(2,j),\dots,(M-1,j)$ on the unknowns in row $j'$ of the unknown matrix $\{u_{i,j}\}$.

We construct $A$ and $b$ one block row at a time. Consider the blocks $A^{(1,j')}$ for $j'=1,2,\dots,N-1$, the first row of blocks of $A$. As mentioned, these blocks correspond to equations $(1,1),(2,1),\dots,(M-1,1)$. Substituting in boundary conditions, we see that

$$\begin{aligned}
(1,1) &\implies \left(-\frac{2}{h_x^2}-\frac{2}{h_y^2}\right)u_{1,1}+\frac{1}{h_x^2}u_{2,1}+\frac{1}{h_y^2}u_{1,2}=f(x_1,y_1)-\frac{g_\ell(y_1)}{h_x^2}-\frac{g_b(x_1)}{h_y^2}\\
\underset{i=2,\dots,M-2}{(i,1)} &\implies \left(-\frac{2}{h_x^2}-\frac{2}{h_y^2}\right)u_{i,1}+\frac{1}{h_x^2}u_{i-1,1}+\frac{1}{h_x^2}u_{i+1,1}+\frac{1}{h_y^2}u_{i,2}=f(x_i,y_1)-\frac{g_b(x_i)}{h_y^2}\\
(M-1,1) &\implies \left(-\frac{2}{h_x^2}-\frac{2}{h_y^2}\right)u_{M-1,1}+\frac{1}{h_x^2}u_{M-2,1}+\frac{1}{h_y^2}u_{M-1,2}=f(x_{M-1},y_1)-\frac{g_r(y_1)}{h_x^2}-\frac{g_b(x_{M-1})}{h_y^2}.
\end{aligned}$$

Thus, each equation depends only on rows 1 and 2 of the matrix $\{u_{i,j}\}$, so only blocks $A^{(1,1)}$ and $A^{(1,2)}$ are nonzero. Examining these dependencies, we get

$$A^{(1,1)}=\begin{bmatrix}-\frac{2}{h_x^2}-\frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2}-\frac{2}{h_y^2} & \frac{1}{h_x^2} & \\ & & \ddots & \\ & & \frac{1}{h_x^2} & -\frac{2}{h_x^2}-\frac{2}{h_y^2}\end{bmatrix}, \qquad A^{(1,2)}=\begin{bmatrix}\frac{1}{h_y^2} & & \\ & \ddots & \\ & & \frac{1}{h_y^2}\end{bmatrix},$$

where blanks indicate zero entries. The block $b^{(1)}$ corresponding to the right hand sides of equations $(1,1),(2,1),\dots,(M-1,1)$ we can read off easily:

$$b^{(1)}=\begin{bmatrix} f(x_1,y_1)-\frac{g_\ell(y_1)}{h_x^2}-\frac{g_b(x_1)}{h_y^2}\\ f(x_2,y_1)-\frac{g_b(x_2)}{h_y^2}\\ f(x_3,y_1)-\frac{g_b(x_3)}{h_y^2}\\ \vdots\\ f(x_{M-2},y_1)-\frac{g_b(x_{M-2})}{h_y^2}\\ f(x_{M-1},y_1)-\frac{g_r(y_1)}{h_x^2}-\frac{g_b(x_{M-1})}{h_y^2}\end{bmatrix}.$$

Now consider the blocks $A^{(j,j')}$ for $j=2,3,\dots,N-2$, and $j'=1,2,\dots,N-1$. These correspond to equations $(1,j),(2,j),\dots,(M-1,j)$. Substituting boundary conditions, we have

$$\begin{aligned}
(1,j) &\implies \left(-\frac{2}{h_x^2}-\frac{2}{h_y^2}\right)u_{1,j}+\frac{1}{h_x^2}u_{2,j}+\frac{1}{h_y^2}u_{1,j-1}+\frac{1}{h_y^2}u_{1,j+1}=f(x_1,y_j)-\frac{g_\ell(y_j)}{h_x^2}\\
\underset{i=2,\dots,M-2}{(i,j)} &\implies \left(-\frac{2}{h_x^2}-\frac{2}{h_y^2}\right)u_{i,j}+\frac{1}{h_x^2}u_{i-1,j}+\frac{1}{h_x^2}u_{i+1,j}+\frac{1}{h_y^2}u_{i,j-1}+\frac{1}{h_y^2}u_{i,j+1}=f(x_i,y_j)\\
(M-1,j) &\implies \left(-\frac{2}{h_x^2}-\frac{2}{h_y^2}\right)u_{M-1,j}+\frac{1}{h_x^2}u_{M-2,j}+\frac{1}{h_y^2}u_{M-1,j-1}+\frac{1}{h_y^2}u_{M-1,j+1}=f(x_{M-1},y_j)-\frac{g_r(y_j)}{h_x^2}.
\end{aligned}$$

Thus, each equation depends only on rows $j-1$, $j$, and $j+1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(j,j-1)}$, $A^{(j,j)}$, and $A^{(j,j+1)}$ are nonzero. Examining these dependencies gives

$$A^{(j,j)}=\begin{bmatrix}-\frac{2}{h_x^2}-\frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2}-\frac{2}{h_y^2} & \frac{1}{h_x^2} & \\ & & \ddots & \\ & & \frac{1}{h_x^2} & -\frac{2}{h_x^2}-\frac{2}{h_y^2}\end{bmatrix}, \qquad A^{(j,j-1)}=A^{(j,j+1)}=\begin{bmatrix}\frac{1}{h_y^2} & & \\ & \ddots & \\ & & \frac{1}{h_y^2}\end{bmatrix}.$$

The block $b^{(j)}$ can be read off from the right hand sides easily:

$$b^{(j)} = \begin{bmatrix} f(x_1, y_j) - \frac{g_\ell(y_j)}{h_x^2} \\ f(x_2, y_j) \\ f(x_3, y_j) \\ \vdots \\ f(x_{M-2}, y_j) \\ f(x_{M-1}, y_j) - \frac{g_r(y_j)}{h_x^2} \end{bmatrix}.$$

Finally, consider the blocks $A^{(N-1,j')}$, $j' = 1, 2, \ldots, N-1$. These correspond to equations $(1, N-1), (2, N-1), \ldots, (M-1, N-1)$. Substituting boundary conditions, we have

$$\begin{aligned}
(1, N-1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,N-1} + \frac{1}{h_x^2} u_{2,N-1} + \frac{1}{h_y^2} u_{1,N-2} = f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \\
\underset{i=2,\ldots,M-2}{(i,N-1)} &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,N-1} + \frac{1}{h_x^2} u_{i-1,N-1} + \frac{1}{h_x^2} u_{i+1,N-1} + \frac{1}{h_y^2} u_{i,2} = f(x_i, y_{N-1}) - \frac{g_t(x_i)}{h_y^2} \\
(M-1, N-1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M-1,N-1} + \frac{1}{h_x^2} u_{M-2,N-1} + \frac{1}{h_y^2} u_{M-1,N-2} \\
&\qquad = f(x_{M-1}, y_{N-1}) - \frac{g_r(y_{N-1})}{h_x^2} - \frac{g_t(x_{M-1})}{h_y^2}.
\end{aligned}$$

Thus, each equation depends only on rows $N-2$ and $N-1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(N-1,N-2)}$ and $A^{(N-1,N-1)}$ are nonzero. Examining these dependencies gives

$$A^{(N-1,N-1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & \\ & & \ddots & \\ & & \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(N-1,N-2)} = \begin{bmatrix} \frac{1}{h_y^2} & & \\ & \ddots & \\ & & \frac{1}{h_y^2} \end{bmatrix}.$$

The block $b^{(N-1)}$ can be read off from the right hand sides of the equations easily:

$$b^{(N-1)} = \begin{bmatrix} f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \\ f(x_2, y_{N-1}) - \frac{g_t(x_2)}{h_y^2} \\ f(x_3, y_{N-1}) - \frac{g_t(x_3)}{h_y^2} \\ \vdots \\ f(x_{M-2}, y_{N-1}) - \frac{g_t(x_{M-2})}{h_y^2} \\ f(x_{M-1}, y_{N-1}) - \frac{g_r(y_{N-1})}{h_x^2} - \frac{g_t(x_{M-1})}{h_y^2} \end{bmatrix}.$$

Therefore, the entire system of equations $(i, j)$ is equivalent to the matrix-vector equation $AU = b$.

**Problem 2.**

Consider the BVP

$$\begin{aligned}
\Delta u = -2\pi^2 \sin(\pi x)\sin(\pi y) &= f(x, y), && 0 < x < 1, \quad 0 < y < 2 \\
u(0, y) = 2 = g_\ell(y), \quad u(1, y) &= 2 = g_r(y), && 0 \le y \le 2 \\
u(x, 0) = 2 = g_b(x), \quad u(x, 1) &= 2 = g_t(x), && 0 \le x \le 1.
\end{aligned}$$

The exact solution of this equation is given by $u(x, y) = 2 + \sin(\pi x)\sin(\pi y)$.

**(a)** Consider a grid of sample points $\{(x_i, y_j)\}$ on the domain $[0,1] \times [0,2]$, where $i = 0, 1, \ldots, M$, and $j = 0, 1, \ldots, N$. If the points are evenly spaced horizontally by $h_x = \frac{1}{M}$ and vertically by $h_y = \frac{2}{N}$, then $x_i = ih_x$, and $y_j = jh_y$.

We approximate $u(x_i, y_j)$ by $u_{i,j}$. Using a centered-difference scheme to approximate $\Delta u$ on the interior and applying the boundary conditions on the boundary points, we are led to the numerical scheme

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} = f(x_i, y_i), \qquad i = 1, 2, \ldots, M-1,\ j = 1, 2, \ldots N-1, \tag{$i,j$}$$

and

$$\begin{aligned} u_{0,j} &= g_\ell(y_j), & u_{M,j} &= g_r(y_j), & j &= 0, 1, \ldots, N, \\ u_{i,0} &= g_b(x_i), & u_{i,N} &= g_t(x_i), & i &= 0, 1, \ldots M. \end{aligned}$$

In order to solve this linear system, we need to reshape the matrix of unknowns $\{u_{i,j}\}_{i=1,j=1}^{M-1,N-1}$ into a vector $U$ and rewrite the corresponding equations $(i,j)$ as a matrix-vector system, substituting the known boundary values where applicable.

We use column-wise ordering to reshape the matrix of unknowns; that is, we define the block vector of columns of the unknown matrix

$$U = \begin{bmatrix} U^{(1)} \\ U^{(2)} \\ \vdots \\ U^{(M-1)} \end{bmatrix}, \qquad U^{(i)} = \begin{bmatrix} u_{i,1} \\ u_{i,2} \\ \vdots \\ u_{i,N-1} \end{bmatrix}, \quad i = 1, 2, \ldots, M-1.$$

We can rewrite the equations $(i,j)$ into a matrix-vector system, expressing the matrix $A$ and vector $b$ in block form corresponding to the blocks of $U$:

$$A = \begin{bmatrix} A^{(1,1)} & \ldots & A^{(1,M-1)} \\ \vdots & \ddots & \vdots \\ A^{(M-1,1)} & \ldots & A^{(M-1,M-1)} \end{bmatrix}, \qquad b = \begin{bmatrix} b^{(1)} \\ b^{(2)} \\ \vdots \\ b^{(M-1)} \end{bmatrix}.$$

We remark that the block $A^{(i,i')}$ expresses the dependence of equations $(i,1), (i,2), \ldots, (i,N-1)$ on the unknowns in column $i'$ of the unknown matrix $\{u_{i,j}\}$.

We construct $A$ and $b$ one block row at a time. Consider the blocks $A^{(1,i')}$ for $i' = 1, 2, \ldots, M-1$, the first row of blocks of $A$. As mentioned, these blocks correspond to equations $(1,1), (1,2), \ldots, (1,N-1)$. Substituting in boundary conditions, we see that

$$\begin{aligned} (1,1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,1} + \frac{1}{h_x^2} u_{2,1} + \frac{1}{h_y^2} u_{1,2} = f(x_1, y_1) - \frac{g_\ell(y_1)}{h_x^2} - \frac{g_b(x_1)}{h_y^2} \\ \underset{j=2,\ldots,N-2}{(1,j)} &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,j} + \frac{1}{h_x^2} u_{2,j} + \frac{1}{h_y^2} u_{1,j-1} + \frac{1}{h_y^2} u_{1,j+1} = f(x_1, y_j) - \frac{g_\ell(y_j)}{h_x^2} \\ (1,N-1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,N-1} + \frac{1}{h_x^2} u_{2,N-1} + \frac{1}{h_y^2} u_{1,N-2} = f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2}. \end{aligned}$$

Thus, each equation depends only on columns 1 and 2 of the matrix $\{u_{i,j}\}$, so only blocks $A^{(1,1)}$ and

$A^{(1,2)}$ are nonzero. Examining these dependencies, we get

$$A^{(1,1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & & \\ \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & \\ & & \ddots & \\ & & \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(1,2)} = \begin{bmatrix} \frac{1}{h_x^2} & & \\ & \ddots & \\ & & \frac{1}{h_x^2} \end{bmatrix},$$

where blanks indicate zero entries. The block $b^{(1)}$ corresponding to the right hand sides of equations $(1,1), (1,2), \dots, (1, N-1)$ we can read off easily:

$$b^{(1)} = \begin{bmatrix} f(x_1, y_1) - \frac{g_\ell(y_1)}{h_x^2} - \frac{g_b(x_1)}{h_y^2} \\ f(x_1, y_2) - \frac{g_\ell(y_2)}{h_x^2} \\ f(x_1, y_3) - \frac{g_\ell(y_3)}{h_x^2} \\ \vdots \\ f(x_1, y_{N-2}) - \frac{g_\ell(y_{N-2})}{h_x^2} \\ f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \end{bmatrix}.$$

Now consider the blocks $A^{(i,i')}$ for $i = 2, 3, \dots, N-2$, and $i' = 1, 2, \dots, N-1$. These correspond to equations $(i,1), (i,2), \dots, (i, N-1)$. Substituting boundary conditions, we have

$$\begin{aligned} (i,1) \implies& \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,1} + \frac{1}{h_x^2} u_{i-1,1} + \frac{1}{h_x^2} u_{i+1,1} + \frac{1}{h_y^2} u_{i,2} = f(x_i, y_1) - \frac{g_b(x_i)}{h_y^2} \\ \underset{j=2,\dots,N-2}{(i,j)} \implies& \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,j} + \frac{1}{h_x^2} u_{i-1,j} + \frac{1}{h_x^2} u_{i+1,j} + \frac{1}{h_y^2} u_{i,j-1} + \frac{1}{h_y^2} u_{i,j+1} = f(x_i, y_j) \\ (i, N-1) \implies& \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,N-1} + \frac{1}{h_x^2} u_{i-1,N-1} + \frac{1}{h_x^2} u_{i+1,N-1} + \frac{1}{h_y^2} u_{i,N-2} = f(x_i, y_{N-1}) - \frac{g_t(x_i)}{h_y^2}. \end{aligned}$$

Thus, each equation depends only on columns $i-1$, $i$, and $i+1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(i,i-1)}$, $A^{(i,i)}$, and $A^{(i,i+1)}$ are nonzero. Examining these dependencies gives

$$A^{(i,i)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & & \\ \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & \\ & & \ddots & \\ & & \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(i,i-1)} = A^{(i,i+1)} = \begin{bmatrix} \frac{1}{h_x^2} & & \\ & \ddots & \\ & & \frac{1}{h_x^2} \end{bmatrix}.$$

The block $b^{(i)}$ can be read off from the right hand sides easily:

$$b^{(i)} = \begin{bmatrix} f(x_i, y_1) - \frac{g_b(x_i)}{h_y^2} \\ f(x_i, y_2) \\ f(x_i, y_3) \\ \vdots \\ f(x_i, y_{N-2}) \\ f(x_i, y_{N-1}) - \frac{g_t(x_i)}{h_y^2} \end{bmatrix}.$$

Finally, consider the blocks $A^{(M-1,i')}$, $i' = 1, 2, \dots, M-1$. These correspond to equations $(M -$

$1,1),(M-1,2),\ldots,(M-1,N-1)$. Substituting boundary conditions, we have

$$(M-1,1) \implies \left(-\frac{2}{h_x^2}-\frac{2}{h_y^2}\right)u_{M-1,1}+\frac{1}{h_x^2}u_{M-2,1}+\frac{1}{h_y^2}u_{M-1,2}=f(x_1,y_{N-1})-\frac{g_r(y_1)}{h_x^2}-\frac{g_b(x_{M-1})}{h_y^2}$$

$$\underset{j=2,\ldots,N-2}{(M-1,j)} \implies \left(-\frac{2}{h_x^2}-\frac{2}{h_y^2}\right)u_{M-1,j}+\frac{1}{h_x^2}u_{M-2,j}+\frac{1}{h_y^2}u_{M-1,j-1}+\frac{1}{h_y^2}u_{M-1,j+1}=f(x_{M-1},y_j)-\frac{g_r(y_j)}{h_x^2}$$

$$\begin{aligned}(M-1,N-1) \implies &\left(-\frac{2}{h_x^2}-\frac{2}{h_y^2}\right)u_{M-1,N-1}+\frac{1}{h_x^2}u_{M-2,N-1}+\frac{1}{h_y^2}u_{M-1,N-2}\\ &=f(x_{M-1},y_{N-1})-\frac{g_r(y_{N-1})}{h_x^2}-\frac{g_t(x_{M-1})}{h_y^2}.\end{aligned}$$

Thus, each equation depends only on columns $M-2$ and $M-1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(M-1,M-2)}$ and $A^{(M-1,M-1)}$ are nonzero. Examining these dependencies gives

$$A^{(N-1,N-1)}=\begin{bmatrix}-\frac{2}{h_x^2}-\frac{2}{h_y^2} & \frac{1}{h_y^2} & & \\ \frac{1}{h_y^2} & -\frac{2}{h_x^2}-\frac{2}{h_y^2} & \frac{1}{h_y^2} & \\ & & \ddots & \\ & & \frac{1}{h_y^2} & -\frac{2}{h_x^2}-\frac{2}{h_y^2}\end{bmatrix}, \qquad A^{(N-1,N-2)}=\begin{bmatrix}\frac{1}{h_x^2} & & \\ & \ddots & \\ & & \frac{1}{h_x^2}\end{bmatrix}.$$

The block $b^{(M-1)}$ can be read off from the right hand sides of the equations easily:

$$b^{(M-1)}=\begin{bmatrix}f(x_{M-1},y_1)-\frac{g_r(y_1)}{h_x^2}-\frac{g_b(x_{M-1})}{h_y^2}\\ f(x_{M-1},y_2)-\frac{g_r(y_2)}{h_x^2}\\ f(x_{M-1},y_3)-\frac{g_r(y_3)}{h_x^2}\\ \vdots \\ f(x_{M-1},y_{N-2})-\frac{g_r(y_{N-2})}{h_x^2}\\ f(x_{M-1},y_{N-1})-\frac{g_r(y_{N-1})}{h_x^2}-\frac{g_t(x_{M-1})}{h_y^2}\end{bmatrix}.$$

Therefore, the entire system of equations $(i,j)$ is equivalent to the matrix-vector equation $AU=b$.

**Problem 3.**

Consider the BVP

$$\begin{aligned}\Delta u &= -2\pi^2\sin(\pi(x+y)) = f(x,y), && 0<x<1,\quad 0<y<1\\ u(0,y)=\sin(\pi y)=g_\ell(y),\quad u_x(1,y)&=\pi\cos(\pi(1+y))=g_r(y), && 0\le y\le 1\\ u_y(x,0)=\pi\cos(\pi x)=g_b(x),\quad u(x,1)&=\sin(\pi(1+x))=g_t(x), && 0\le x\le 1.\end{aligned}$$

The exact solution of this equation is given by $u(x,y)=\sin(\pi(x+y))$.

**(a)** Consider a grid of sample points $\{(x_i,y_j)\}$ on the domain $[0,1]^2$, where $i=0,1,\ldots,M$, and $j=0,1,\ldots,N$. If the points are evenly spaced horizontally by $h_x=\frac{1}{M}$ and vertically by $h_y=\frac{1}{N}$, then $x_i=ih_x$, and $y_j=jh_y$.

We approximate $u(x_i,y_j)$ by $u_{i,j}$. For the Neumann boundary conditions on the right and bottom boundaries, we use a centered difference scheme with ghost points to approximate the derivatives, and extend the PDE to the boundaries to eliminate the ghost points, as follows:

$$\frac{u_{M+1,j}-u_{M-1,j}}{2h_x}=g_r(y_j),\quad j=1,2,\ldots,N-1,\qquad \frac{u_{i,1}-u_{i,-1}}{2h_y}=g_b(x_i),\quad i=1,2,\ldots M-1.$$

Using a centered-difference scheme to approximate $\Delta u$ on the interior and the points on the parts of the boundary that have a Neumann condition, we obtain the following scheme:

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} = f(x_i, y_i), \quad i = 1, 2, \ldots, M,\ j = 0, 2, \ldots N-1. \tag{$i,j$}$$

For the left and top boundaries, we have Dirichlet boundary conditions, giving

$$u_{0,j} = g_\ell(y_j), \quad j = 0, 1, \ldots, N, \qquad u_{i,N} = g_t(x_i), \quad i = 0, 1, \ldots, M.$$

In order to solve this linear system, we need to reshape the matrix of unknowns $\{u_{i,j}\}_{i=1,j=0}^{M,N-1}$ into a vector $U$ and rewrite the corresponding equations $(i,j)$ as a matrix-vector system, substituting the known boundary values and ghost point relationships where applicable.

We use row-wise ordering to reshape the matrix of unknowns; that is, we define the block vector of rows of the unknown matrix

$$U = \begin{bmatrix} U^{(0)} \\ U^{(1)} \\ \vdots \\ U^{(N-1)} \end{bmatrix}, \qquad U^{(j)} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{M,j} \end{bmatrix}, \quad j = 0, 1, \ldots, N-1.$$

We can rewrite the equations $(i,j)$ into a matrix-vector system, expressing the matrix $A$ and vector $b$ in block form corresponding to the blocks of $U$:

$$A = \begin{bmatrix} A^{(0,0)} & \cdots & A^{(0,N-1)} \\ \vdots & \ddots & \vdots \\ A^{(N-1,0)} & \cdots & A^{(N-1,N-1)} \end{bmatrix}, \qquad b = \begin{bmatrix} b^{(0)} \\ b^{(1)} \\ \vdots \\ b^{(N-1)} \end{bmatrix}.$$

We remark that the block $A^{(j,j')}$ expresses the dependence of equations $(1,j), (2,j), \ldots, (M,j)$ on the unknowns in row $j'$ of the unknown matrix $\{u_{i,j}\}$.

We construct $A$ and $b$ one block row at a time. Consider the blocks $A^{(0,j')}$ for $j' = 1, 2, \ldots, N-1$, the first row of blocks of $A$. As mentioned, these blocks correspond to equations $(1,0), (2,0), \ldots, (M,0)$. Substituting in boundary conditions, we see that

$$\begin{aligned}
(1,0) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,0} + \frac{1}{h_x^2} u_{2,0} + \frac{2}{h_y^2} u_{1,1} = f(x_1, y_0) - \frac{g_\ell(y_0)}{h_x^2} + \frac{2}{h_y} g_b(x_1) \\
\underset{i=2,\ldots,M-1}{(i,0)} &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,0} + \frac{1}{h_x^2} u_{i-1,0} + \frac{1}{h_x^2} u_{i+1,0} + \frac{2}{h_y^2} u_{i,1} = f(x_i, y_0) + \frac{2}{h_y} g_b(x_i) \\
(M,0) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M,0} + \frac{2}{h_x^2} u_{M-1,0} + \frac{2}{h_y^2} u_{M-1,1} = f(x_M, y_0) - \frac{2}{h_x} g_r(y_0) + \frac{2}{h_y} g_b(x_M).
\end{aligned}$$

Thus, each equation depends only on rows 0 and 1 of the matrix $\{u_{i,j}\}$, so only blocks $A^{(0,0)}$ and $A^{(0,1)}$ are nonzero. Examining these dependencies, we get

$$A^{(1,1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & \\ & & \ddots & \\ & & \frac{2}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(1,2)} = \begin{bmatrix} \frac{2}{h_y^2} & & \\ & \ddots & \\ & & \frac{2}{h_y^2} \end{bmatrix},$$

where blanks indicate zero entries. The block $b^{(0)}$ corresponding to the right hand sides of equations $(1,0),(2,0),\dots,(M-1,0)$ we can read off easily:

$$b^{(0)} = \begin{bmatrix} f(x_1,y_0) - \frac{g_\ell(y_0)}{h_x^2} + \frac{2}{h_y}g_b(x_1) \\ f(x_2,y_0) + \frac{2}{h_y}g_b(x_2) \\ f(x_3,y_0) + \frac{2}{h_y}g_b(x_3) \\ \vdots \\ f(x_{M-1},y_0) + \frac{2}{h_y}g_b(x_{M-1}) \\ f(x_M,y_0) - \frac{2}{h_x}g_r(y_0) + \frac{2}{h_y}g_b(x_M) \end{bmatrix}.$$

Now consider the blocks $A^{(j,j')}$ for $j = 1,2,3,\dots,N-2$, and $j' = 0,1,\dots,N-1$. These correspond to equations $(1,j),(2,j),\dots,(M,j)$. Substituting boundary conditions, we have

$$\begin{aligned}
(1,j) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right)u_{1,j} + \frac{1}{h_x^2}u_{2,j} + \frac{1}{h_y^2}u_{1,j-1} + \frac{1}{h_y^2}u_{1,j+1} = f(x_1,y_j) - \frac{g_\ell(y_j)}{h_x^2} \\
\underset{i=2,\dots,M-1}{(i,j)} &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right)u_{i,j} + \frac{1}{h_x^2}u_{i-1,j} + \frac{1}{h_x^2}u_{i+1,j} + \frac{1}{h_y^2}u_{i,j-1} + \frac{1}{h_y^2}u_{i,j+1} = f(x_i,y_j) \\
(M,j) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right)u_{M,j} + \frac{2}{h_x^2}u_{M-1,j} + \frac{1}{h_y^2}u_{M,j-1} + \frac{1}{h_y^2}u_{M,j+1} = f(x_M,y_j) - \frac{2}{h_x}g_r(y_j).
\end{aligned}$$

Thus, each equation depends only on rows $j-1$, $j$, and $j+1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(j,j-1)}$, $A^{(j,j)}$, and $A^{(j,j+1)}$ are nonzero. Examining these dependencies gives

$$A^{(j,j)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & \\ & & \ddots & \\ & & \frac{2}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(j,j-1)} = A^{(j,j+1)} = \begin{bmatrix} \frac{1}{h_y^2} & & \\ & \ddots & \\ & & \frac{1}{h_y^2} \end{bmatrix}.$$

The block $b^{(j)}$ can be read off from the right hand sides easily:

$$b^{(j)} = \begin{bmatrix} f(x_1,y_j) - \frac{g_\ell(y_j)}{h_x^2} \\ f(x_2,y_j) \\ f(x_3,y_j) \\ \vdots \\ f(x_{M-1},y_j) \\ f(x_M,y_j) - \frac{2}{h_x}g_r(y_j) \end{bmatrix}.$$

Finally, consider the blocks $A^{(N-1,j')}$, $j' = 0,1,\dots,N-1$. These correspond to equations $(1,N-1),(2,N-1),\dots,(M,N-1)$. Substituting boundary conditions, we have

$$\begin{aligned}
(1,N-1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right)u_{1,N-1} + \frac{1}{h_x^2}u_{2,N-1} + \frac{1}{h_y^2}u_{1,N-2} = f(x_1,y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \\
\underset{i=2,\dots,M-1}{(i,N-1)} &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right)u_{i,N-1} + \frac{1}{h_x^2}u_{i-1,N-1} + \frac{1}{h_x^2}u_{i+1,N-1} + \frac{1}{h_y^2}u_{i,2} = f(x_i,y_{N-1}) - \frac{g_t(x_i)}{h_y^2} \\
(M,N-1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right)u_{M,N-1} + \frac{2}{h_x^2}u_{M-1,N-1} + \frac{1}{h_y^2}u_{M,N-2} \\
&\qquad = f(x_M,y_{N-1}) - \frac{2}{h_x}g_r(y_{N-1}) - \frac{g_t(x_M)}{h_y^2}.
\end{aligned}$$

Thus, each equation depends only on rows $N-2$ and $N-1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(N-1,N-2)}$ and $A^{(N-1,N-1)}$ are nonzero. Examining these dependencies gives

$$A^{(N-1,N-1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & \\ & & \ddots & \\ & & \frac{2}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(N-1,N-2)} = \begin{bmatrix} \frac{1}{h_y^2} & & \\ & \ddots & \\ & & \frac{1}{h_y^2} \end{bmatrix}.$$

The block $b^{(N-1)}$ can be read off from the right hand sides of the equations easily:

$$b^{(N-1)} = \begin{bmatrix} f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \\ f(x_2, y_{N-1}) - \frac{g_t(x_2)}{h_y^2} \\ f(x_3, y_{N-1}) - \frac{g_t(x_3)}{h_y^2} \\ \vdots \\ f(x_{M-1}, y_{N-1}) - \frac{g_t(x_{M-1})}{h_y^2} \\ f(x_M, y_{N-1}) - \frac{2}{h_x} g_r(y_{N-1}) - \frac{g_t(x_M)}{h_y^2} \end{bmatrix}.$$

Therefore, the entire system of equations $(i,j)$ is equivalent to the matrix-vector equation $AU = b$.