Math 6330 Homework 3

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1.89

Let $\mathbb{T} = \mathbf{Z}$, and define $f(t) = g(t) = t^2$ for $t \in \mathbb{T}$. Viewing f as a function on $\mathbf{R}$, we have $f'(t) = 2t$. As we have calculated before, $g^\Delta(t) = 2t + 1$. Lastly, $(f \circ g)(t) = t^4$, so

$$(f \circ g)^\Delta(t) = (t+1)^4 - t^4 = 4t^3 + 6t^2 + 4t + 1.$$

According to Theorem 1.87, there exists $c \in [2, \sigma(t)] = [2, 3]$ such that

$$(f \circ g)^\Delta(2) = f'(g(c))g^\Delta(2).$$

Using the formulas for $(f \circ g)^\Delta$, f', and g^Δ, this means that

$$4 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 + 1 = 2c^2 \cdot (2 \cdot 2 + 1)$$

or

$$65 = 10c^2 \implies c = \pm\sqrt{\frac{13}{2}}.$$

Since $c \in [2, 3]$, it follows that $c = \sqrt{\frac{13}{2}}$. Note that $\sqrt{\frac{13}{2}} \in [2, 3]$, as promised, because

$$8 \le 13 \le 18 \implies 4 \le \frac{13}{2} \le 9 \implies 2 \le \sqrt{\frac{13}{2}} \le 3.$$

1.95

Let $\mathbb{T} = \mathbb{N}_0$, $\nu(t) = t^2$, $\widetilde{\mathbb{T}} = \nu(\mathbb{T})$, and $w(t) = 2t^2 + 3$. Note that $\widetilde{\mathbb{T}} = \{n^2 \mid n \in \mathbb{N}_0\}$, so, given $t = n^2 \in \widetilde{\mathbb{T}}$, we must have $\widetilde{\sigma}(t) = (n+1)^2 = (\sqrt{t} + 1)^2$. On the one hand, we have

$$(w \circ \nu)^\Delta(t) = (2t^4 + 2)^\Delta = 2(t+1)^4 + 2 - (2t^4 + 2) = 2(t+1)^4 - 2t^4.$$

On the other hand, we have

$$\nu^\Delta(t) = (t+1)^2 - t^2 = 2t + 1, \qquad w^{\widetilde{\Delta}}(t) = \frac{w^{\widetilde{\sigma}}(t) - w(t)}{\widetilde{\sigma}(t) - t} = \frac{2(\sqrt{t}+1)^4 - 2t^2}{(\sqrt{t}+1)^2 - t}$$

because every point of $\widetilde{\mathbb{T}}$ is right-scattered. Thus,

$$\left(w^{\widetilde{\Delta}} \circ \nu\right)(t)\nu^\Delta(t) = \frac{2(t+1)^4 - 2t^4}{(t+1)^2 - t}(2t+1) = 2(t+1)^4 - 2t^4 = (w \circ \nu)^\Delta(t),$$

which agrees with the chain rule.

1.96

Let $\mathbb{T} = \mathbb{N}$, and define $\nu : \mathbb{T} \to \mathbf{R}$ by $\nu(t) = -\frac{1}{t}$. Then ν is strictly increasing on $\mathbb{T}$, but $\nu(\mathbb{T})$ is not a time-scale. Indeed, $0 \in \overline{\nu(\mathbb{T})}$, but $0 \notin \nu(\mathbb{T})$, so $\nu(\mathbb{T})$ is not closed.

1.100

Let $\mathbb{T} = \left\{ \frac{n}{2} \mid n \in \mathbb{N}_0 \right\}$, and consider the integral

$$\int_0^t 2\tau(2\tau - 1)\Delta\tau.$$

If we define $\nu(t) = 2t$ and $f(t) = t(2t-1)$, then

$$\nu^\Delta(t) = 2, \qquad \nu^{-1}(t) = \frac{t}{2}, \qquad f(t)\nu^\Delta(t) = 2t(2t-1), \qquad \left(f \circ \nu^{-1}\right)(t) = \frac{t}{2}(t-1).$$

By the substitution rule, we have

$$\int_0^t 2\tau(2\tau-1)\Delta\tau = \int_0^t f(\tau)\nu^\Delta(\tau)\Delta\tau = \int_{\nu(0)}^{\nu(t)} \left(f \circ \nu^{-1}\right)(s)\widetilde{\Delta}s = \frac{1}{2}\int_0^{2t} s(s-1)\widetilde{\Delta}s.$$

Note that on $\widetilde{\mathbb{T}} = \mathbb{N}_0$, an antiderivative of $s(s-1) = s^{(2)}$ is $\frac{1}{3}s^{(3)}$, so

$$\int_0^t 2\tau(2\tau-1)\Delta\tau = \frac{1}{2}\int_0^{2t} s(s-1)\widetilde{\Delta}s = \frac{1}{6}s^{(3)}\Big|_0^{2t} = \frac{t(2t-1)(2t-2)}{3}$$

1.106

Let $\mathbb{T} = \overline{q^{\mathbf{Z}}}$, where $q > 1$, and consider

$$\int_0^t s^n \Delta s,$$

where $n \geq 1$ is an integer. For $k \in \mathbf{Z}$, we have

$$\int_0^t s^n\Delta s = \int_0^{q^k} s^n\Delta s + \int_{q^k}^t s^n\Delta s.$$

Since $|s^n| \leq q^{nk}$ on $[0, q^k]$, by Theorem 1.76 (viii),

$$\left|\int_0^{q^k} s^n\Delta s\right| \leq \int_0^{q^k} q^{nk}\Delta s = q^{(n+1)k}.$$

As $k \to -\infty$, $q^{(n+1)k} \to 0$ because $n \geq 1$ and $q > 1$. Hence,

$$\int_0^t s^n\Delta s = \lim_{k\to-\infty}\int_{q^k}^t s^n\Delta s.$$

For all $k < \log_q(t)$, we have $q^k < t$. Furthermore, assuming $k < \log_q(t)$, every point in $[q^k, t] \cap \mathbb{T}$ is isolated, so by Theorem 1.79 (ii) (recall that $\mu(q^j) = (q-1)q^j$),

$$\begin{aligned}\int_{q^k}^t s^n \Delta s &= \sum_{j=k}^{\log_q(t)-1} q^{jn}(q-1)q^j = (q-1)\sum_{j=k}^{\log_q(t)-1} \left(q^{n+1}\right)^j \\ &= (q-1)\frac{\left(q^{n+1}\right)^{\log_q(t)} - \left(q^{n+1}\right)^k}{q^{n+1}-1} \\ &= \frac{t^{n+1} - q^{k(n+1)}}{\sum\limits_{\mu=0}^n q^\mu},\end{aligned}$$

where the last equality follows from the well-known fact that $x^{n+1} - 1 = (x-1)\sum\limits_{\mu=0}^n x^\mu$ for any x and any integer $n \geq 0$ (we can cancel the $(q-1)$ factor because $q > 1$ by hypothesis).

Therefore,

$$\int_0^t s^n \Delta s = \lim_{k\to-\infty} \int_{q^k}^t s^n \Delta s = \lim_{k\to-\infty} \frac{t^{n+1} - q^{k(n+1)}}{\sum\limits_{\mu=0}^n q^\mu} = \frac{t^{n+1}}{\sum\limits_{\mu=0}^n q^\mu}$$

where we have again used the fact that $q^{k(n+1)} \to 0$ as $k \to -\infty$ because $n \geq 1$ and $q > 1$.

1.107

Let $\mathbb{T} = [0,1] \cup [3,4]$. We can find $h_k(\cdot, 0)$ for $k \in \{0,1,2,3\}$ by using the recursive definition:

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau, \quad k = 0,1,2,\ldots, \qquad h_0(t,s) = 1.$$

Thus, $h_0(t,0) = 1$. To find $h_1(t,0)$, we compute

$$h_1(t,0) = \int_0^t h_0(\tau,0)\Delta\tau = \int_0^t \Delta\tau = t.$$

To find $h_2(t,0)$, we note that if $t \in [0,1]$, then

$$h_2(t,0) = \int_0^t h_1(\tau,0)\Delta\tau = \int_0^t \tau\Delta\tau = \int_0^t \tau \,\mathrm{d}\tau = \frac{t^2}{2}.$$

If $t = 3$, then

$$h_2(3,0) = \int_0^3 h_1(\tau,0)\Delta\tau = \int_0^1 \tau\Delta\tau + \int_1^3 \tau\Delta\tau = \frac{1}{2} + \int_1^{\sigma(1)} \tau\Delta\tau = \frac{1}{2} + 1\cdot\mu(1) = \frac{5}{2}.$$

If $t \in [3,4]$, then

$$h_2(t,0) = \int_0^t h_1(\tau,0)\Delta\tau = \int_0^3 \tau\Delta\tau + \int_3^t \tau \,\mathrm{d}\tau = \frac{5}{2} + \frac{t^2}{2} - \frac{9}{2} = \frac{t^2}{2} - 2.$$

Thus,

$$h_2(t,0) = \begin{cases} \frac{t^2}{2} & t \in [0,1] \\ \frac{t^2}{2} - 2 & t \in [3,4]. \end{cases}$$

To find $h_3(t,0)$, we note that if $t \in [0,1]$, then

$$h_3(t,0) = \int_0^t h_2(\tau,0)\Delta\tau = \int_0^t \frac{\tau^2}{2}\,\mathrm{d}\tau = \frac{t^3}{6}.$$

If $t = 3$, then

$$h_3(t,0) = \int_0^1 h_2(\tau,0)\Delta\tau + \int_1^3 h_2(\tau,0)\Delta\tau = \frac{1}{6} + \int_1^{\sigma(1)} \frac{\tau^2}{2}\Delta\tau = \frac{1}{6} + \frac{1}{2}\cdot\mu(1) = \frac{7}{6}.$$

If $t \in [3,4]$, then

$$\begin{aligned}
h_3(t,0) &= \int_0^3 h_2(\tau,0)\Delta\tau + \int_3^t h_2(\tau,0)\Delta\tau = \frac{7}{6} + \int_3^t \left(\frac{t^2}{2} - 2\right)\,\mathrm{d}\tau \\
&= \frac{7}{6} + \left[\frac{\tau^3}{6} - 2\tau\right]_3^t = \frac{t^3}{6} - 2t + \frac{7}{6} - \frac{27}{6} + 6 \\
&= \frac{t^3}{6} - 2t + \frac{8}{3}.
\end{aligned}$$

Thus,

$$h_3(t,0) = \begin{cases} \frac{t^3}{6} & t \in [0,1] \\ \frac{t^3}{6} - 2t + \frac{8}{3} & t \in [3,4]. \end{cases}$$