Math 6108 Homework 1

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August 26, 2024

Problem 1.

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, unit vectors, then $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal.

Proof. Since

$$\begin{aligned}(\mathbf{u}+\mathbf{v},\mathbf{u}-\mathbf{v}) &= (\mathbf{u},\mathbf{u}) - (\mathbf{u},\mathbf{v}) + (\mathbf{v},\mathbf{u}) - (\mathbf{v},\mathbf{v}) \\ &= \|\mathbf{u}\|^2 - (\mathbf{u},\mathbf{v}) + (\mathbf{u},\mathbf{v}) - \|\mathbf{v}\|^2 = 1 - 1 = 0,\end{aligned}$$

it follows that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal. □

Problem 2.

If $P \in \mathbf{R}^{n\times n}$ is a projection, then $I - P$ is projection.

Proof. Since P is a projection, $P^2 = P$. Thus,

$$(I-P)^2 = (I-P)(I-P) = I^2 - IP - PI + P^2 = I - 2P + P = I - P,$$

so $I - P$ is also a projection. □

Problem 3.

Let U and V be $n \times n$ unitary matrices. Then UV is an $n \times n$ unitary matrix.

Proof. It suffices to show that $(UV)^*(UV) = I$. This is the case because

$$(UV)^*(UV) = V^*U^*UV = V^*IV = V^*V = I$$

by the unitarity of U and V. □

Problem 4.

Let $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ be $n \times n$ matrices. Suppose that $AB = \{c_{ij}\}$, and $BA = \{d_{ij}\}$. Then, by definition,

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, \qquad d_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj}.$$

Hence, by the definition of trace,

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} = \sum_{k=1}^{n} d_{kk} = \operatorname{tr}(BA).$$

Problem 5.

Let $A \in \mathbf{C}^{n\times n}$ be a matrix whose columns $\{\mathbf{a}_i\}_{i=1}^n$ form an orthogonal set.

1. A^*A is a diagonal matrix.

   Proof. Since
$$A^*A = \begin{bmatrix}\mathbf{a}_1^* \\ \mathbf{a}_2^* \\ \vdots \\ \mathbf{a}_n^*\end{bmatrix}\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n\end{bmatrix},$$
   if b_{ij} is the entry of A^*A in the ith row and jth column, then $b_{ij} = \mathbf{a}_i^*\mathbf{a}_j = (\mathbf{a}_i, \mathbf{a}_j) = 0$ if $i \neq j$. Thus, A^*A is diagonal. □

2. AA^* is not necessarily diagonal.

   Proof. Suppose that
$$A = \begin{bmatrix}1 & 2 \\ -1 & 2\end{bmatrix}.$$
   Then $(\mathbf{a}_1, \mathbf{a}_2) = 1\cdot 2 - 1 \cdot 2 = 0$, so the columns of A form an orthogonal set, but
$$AA^* = \begin{bmatrix}1 & 2 \\ -1 & 2\end{bmatrix}\begin{bmatrix}1 & -1 \\ 2 & 2\end{bmatrix} = \begin{bmatrix}5 & 3 \\ 3 & 5\end{bmatrix},$$
   which is clearly not diagonal. □

Problem 6.

If a matrix $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) has the form
$$A = \begin{bmatrix}1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1}\end{bmatrix},$$
where $x_1, x_2, \dots, x_n \in \mathbf{R}$ (or $\mathbf{C}$) then A is called a Vandermonde matrix. The determinant of A is given by
$$\det(A) = \prod_{1\le i<j\le n}(x_j - x_i). \tag{1}$$

Proof. We use induction. For the base case[1], consider $n = 2$. Then
$$A = \begin{bmatrix}1 & x_1 \\ 1 & x_2\end{bmatrix}, \qquad x_1, x_2 \in \mathbf{R} \text{ (or } \mathbf{C}),$$
so
$$\det(A) = x_2 - x_1 = \prod_{1\le i<j\le 2}(x_j - x_i);$$

[1] We can also say that (1) is true for $n = 1$ under the convention that the product over the empty set is 1, and that a 1×1 Vandermonde matrix is given by $A = \begin{bmatrix}1\end{bmatrix}$, but this complicates the induction step a bit by generating two cases.

that is, (1) holds for $n = 2$.

Now suppose for induction that, for some $n \geq 2$, the determinant of any Vandermonde matrix in $\mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) is given by (1). Let $A \in \mathbf{R}^{(n+1)\times(n+1)}$ (or $\mathbf{C}^{(n+1)\times(n+1)}$). The determinant of A is preserved by adding a scalar multiple of one column to another. Let C_i denote the ith column of A. If we perform the following sequence of column operations, which preserve the determinant,

$$\begin{aligned} C_{n+1} &\leftarrow C_{n+1} - x_1 C_n, \\ C_n &\leftarrow C_n - x_1 C_{n-1}, \\ &\vdots \\ C_3 &\leftarrow C_3 - x_1 C_2, \\ C_2 &\leftarrow C_2 - x_1 C_1, \end{aligned}$$

then we find that

$$\det(A) = \det\left(\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & x_2 - x_1 & x_2(x_2 - x_1) & x_2^2(x_2 - x_1) & \dots & x_2^{n-1}(x_2 - x_1) \\ 1 & x_3 - x_1 & x_3(x_3 - x_1) & x_3^2(x_3 - x_1) & \dots & x_3^{n-1}(x_3 - x_1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+1} - x_1 & x_{n+1}(x_{n+1} - x_1) & x_{n+1}^2(x_{n+1} - x_1) & \dots & x_{n+1}^{n-1}(x_{n+1} - x_1) \end{bmatrix}\right).$$

Using the Laplace expansion for the determinant on the first row of the matrix on the right-hand side we get

$$\det(A) = \det\left(\begin{bmatrix} x_2 - x_1 & x_2(x_2 - x_1) & x_2^2(x_2 - x_1) & \dots & x_2^{n-1}(x_2 - x_1) \\ x_3 - x_1 & x_3(x_3 - x_1) & x_3^2(x_3 - x_1) & \dots & x_3^{n-1}(x_3 - x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n+1} - x_1 & x_{n+1}(x_{n+1} - x_1) & x_{n+1}^2(x_{n+1} - x_1) & \dots & x_{n+1}^{n-1}(x_{n+1} - x_1) \end{bmatrix}\right).$$

Factoring $x_j - x_1$ out of the ith row of the above matrix on the right-hand side, for $j = 2, 3, \dots, n+1$, and applying the multilinearity of the determinant gives

$$\det(A) = \prod_{j=2}^{n+1}(x_j - x_1)\det\left(\begin{bmatrix} 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}\right).$$

The right factor in the product above is the determinant of a Vandermonde matrix in $\mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$), so, by the induction hypothesis, it follows that

$$\det(A) = \prod_{j=2}^{n+1}(x_j - x_1)\prod_{2\leq i<j\leq n+1}(x_j - x_i) = \prod_{1\leq i<j\leq n+1}(x_j - x_i),$$

which proves the claim by induction. $\square$

Problem 7.

Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathbf{C}^n$ be a set of nonzero, orthogonal vectors. Then $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are linearly independent.

Proof. Let $c_1, c_2, \ldots, c_k \in \mathbf{C}$ satisfy

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = 0.$$

Taking the inner product of both sides with $\mathbf{x}_i$, for $i = 1, 2, \ldots, k$, gives

$$c_i(\mathbf{x}_i, \mathbf{x}_i) = (0, \mathbf{x}_i) = 0, \qquad i = 1, 2, \ldots, k.$$

Since $\mathbf{x}_i \neq 0$ for $i = 1, 2, \ldots, k$, it follows that $(\mathbf{x}_i, \mathbf{x}_i) \neq 0$, and $c_i = 0$ for $i = 1, 2, \ldots, k$. Therefore, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly independent. $\square$