Math 5604 Homework 9

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Consider the 1D heat equation

$$u_t = \frac{1}{2}u_{xx} + f(x,t), \qquad 0 < x < 1,\ t > 0,$$

with source term

$$f(x,t) = \left(\frac{\pi^2}{2} - 1\right)\sin\left(\pi\left(x + \frac{1}{2}\right)\right)$$

and Dirichlet boundary conditions given by

$$u(0,t) = e^{-t}, \qquad u(1,t) = -e^{-t}, \qquad t > 0$$

and initial condition

$$u(x,0) = \sin\left(\pi\left(x + \frac{1}{2}\right)\right), \qquad 0 \le x \le 1.$$

The exact solution is $u(x,t) = e^{-t}\sin\left(\pi\left(x + \frac{1}{2}\right)\right)$.

Problem 1.

(a) Discretizing this equation on the time interval $[0,1]$ using a central difference method in space and the forward Euler method in time with the space sample points $x_i = ih$ for $i = 0, 1, \dots, M$ and time sample points $t_n = nk$ for $n = 0, 1, \dots N$, where $h = \frac{1}{M}$, and $k = \frac{1}{N}$, we obtain

$$\begin{aligned}
\frac{u_{i+1}^n - u_i^n}{k} &= \frac{1}{2}\cdot\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2} + f(x_i, t_n), \qquad i = 1, 2, \dots, M-1,\ n = 0, 1, \dots, N-1,\\
u_0^n &= e^{-t_n}, \quad u_M^n = -e^{-t_n}, \qquad n = 0, 1, \dots, N,\\
u_i^0 &= \sin\left(\pi\left(x + \frac{1}{2}\right)\right), \qquad i = 0, 1, \dots, M.
\end{aligned}$$

where $u_i^n \approx u(x_i, t_n)$. We can rewrite this system as

$$U^{n+1} = U^n + k(AU^n + b^n),$$

where

$$U^n = \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{M-1}^n \end{bmatrix}, \qquad A = \frac{1}{2h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 - 2 & \end{bmatrix}, \qquad b^n = \begin{bmatrix} f(x_1, t_n) + \frac{e^{-t_n}}{2h^2} \\ f(x_2, t_n) \\ \vdots \\ f(x_{M-2}, t_n) \\ f(x_{M-1}, t_n) - \frac{e^{-t_n}}{2h^2} \end{bmatrix}.$$

This linear recurrence is implemented in `problem.m` using a simple for loop.

(b) The forward Euler method is unstable if $k > ch^2$ for some constant c. We determine this constant empirically to be roughly 1 by using the bisection method in `stability_test.m`. By setting h^2 a little bit larger than k, we can still see the first-order convergence in k; see Table 1.

Observing second-order convergence in space is easier given the constraint $k < h^2$. We simply calculate the error for various values of h with k fixed and much smaller than the smallest value of h^2; see Table 2. These tables are generated by running `problem1_calculations.m`.

k	L^∞ error	L^∞ rate	L^2 error	L^2 rate

Table 1: First-order convergence in time of the forward Euler method. Note that $h^2 = 2k$.

k	L^∞ error	L^∞ rate	L^2 error	L^2 rate

Table 2: Second-order convergence in space of the forward Euler method. Note that $k =$