

Bifurcation Analysis of a Discrete-Time Prey-Predator Model

by Pravaiz Naik, Zohreh Eskandari, Hossein Shahkari and
Kolade Owolabi

24 April 2024

Presented by Jacob Hauck

Outline

- ▶ Description and interpretation of a discrete-time predator-prey model

Outline

- ▶ Description and interpretation of a discrete-time predator-prey model
- ▶ Determination of fixed points

Outline

- ▶ Description and interpretation of a discrete-time predator-prey model
- ▶ Determination of fixed points
- ▶ Bifurcation analysis
 - ▶ Period-doubling bifurcation
 - ▶ Neimark-Sacker bifurcation

Outline

- ▶ Description and interpretation of a discrete-time predator-prey model
- ▶ Determination of fixed points
- ▶ Bifurcation analysis
 - ▶ Period-doubling bifurcation
 - ▶ Neimark-Sacker bifurcation
- ▶ Numerical investigations
 - ▶ Bifurcation diagram of period-doubling bifurcation
 - ▶ Phase portrait changes at Neimark-Sacker bifurcation

Model Description

$$x_p(n+1) = x_p(n) \left[1 + r \left(1 - \frac{x_p(n)}{k} \right) - ay_p(n) \right]$$

$$y_p(n+1) = y_p(n) \left[1 - b + \frac{cx_p(n)}{y_p(n)} \right]$$

► $n \in \mathbb{Z}$: discrete time step

Model Description

$$x_p(n+1) = x_p(n) \left[1 + r \left(1 - \frac{x_p(n)}{k} \right) - ay_p(n) \right]$$

$$y_p(n+1) = y_p(n) \left[1 - b + \frac{cx_p(n)}{y_p(n)} \right]$$

- $n \in \mathbb{Z}$: discrete time step
- $x_p(n)$: number of prey; $y_p(n)$: number of predators

Model Description

$$x_p(n+1) = x_p(n) \left[1 + r \left(1 - \frac{x_p(n)}{k} \right) - ay_p(n) \right]$$

$$y_p(n+1) = y_p(n) \left[1 - b + \frac{cx_p(n)}{y_p(n)} \right]$$

- $n \in \mathbb{Z}$: discrete time step
- $x_p(n)$: number of prey; $y_p(n)$: number of predators
- $r > 0$: intrinsic growth rate of prey

Model Description

$$x_p(n+1) = x_p(n) \left[1 + r \left(1 - \frac{x_p(n)}{k} \right) - ay_p(n) \right]$$

$$y_p(n+1) = y_p(n) \left[1 - b + \frac{cx_p(n)}{y_p(n)} \right]$$

- $n \in \mathbb{Z}$: discrete time step
- $x_p(n)$: number of prey; $y_p(n)$: number of predators
- $r > 0$: intrinsic growth rate of prey
- $k > 0$: carrying capacity of prey

Model Description

$$x_p(n+1) = x_p(n) \left[1 + r \left(1 - \frac{x_p(n)}{k} \right) - ay_p(n) \right]$$

$$y_p(n+1) = y_p(n) \left[1 - b + \frac{cx_p(n)}{y_p(n)} \right]$$

- $n \in \mathbb{Z}$: discrete time step
- $x_p(n)$: number of prey; $y_p(n)$: number of predators
- $r > 0$: intrinsic growth rate of prey
- $k > 0$: carrying capacity of prey
- $a > 0$: predation rate

Model Description

$$x_p(n+1) = x_p(n) \left[1 + r \left(1 - \frac{x_p(n)}{k} \right) - ay_p(n) \right]$$

$$y_p(n+1) = y_p(n) \left[1 - b + \frac{cx_p(n)}{y_p(n)} \right]$$

- ▶ $n \in \mathbb{Z}$: discrete time step
- ▶ $x_p(n)$: number of prey; $y_p(n)$: number of predators
- ▶ $r > 0$: intrinsic growth rate of prey
- ▶ $k > 0$: carrying capacity of prey
- ▶ $a > 0$: predation rate
- ▶ $b > 0$: death rate of predators

Model Description

$$x_p(n+1) = x_p(n) \left[1 + r \left(1 - \frac{x_p(n)}{k} \right) - ay_p(n) \right]$$

$$y_p(n+1) = y_p(n) \left[1 - b + \frac{cx_p(n)}{y_p(n)} \right]$$

- $n \in \mathbb{Z}$: discrete time step
- $x_p(n)$: number of prey; $y_p(n)$: number of predators
- $r > 0$: intrinsic growth rate of prey
- $k > 0$: carrying capacity of prey
- $a > 0$: predation rate
- $b > 0$: death rate of predators
- $c > 0$: conversion rate (of prey into predators)

Fixed Points

$(0, 0)$ is a fixed point (total extinction), and so is $(k, 0)$ (predator extinction), but these are not the focus of the analysis.

Fixed Points

$(0, 0)$ is a fixed point (total extinction), and so is $(k, 0)$ (predator extinction), but these are not the focus of the analysis.

Ecologically, an important fixed points occur when $x_p > 0$, and $y_p > 0$, when predator and prey are in equilibrium.

Fixed Points

$(0, 0)$ is a fixed point (total extinction), and so is $(k, 0)$ (predator extinction), but these are not the focus of the analysis.

Ecologically, an important fixed points occur when $x_p > 0$, and $y_p > 0$, when predator and prey are in equilibrium.

There is one such fixed point:

$$\mathcal{P}_* = \left(\frac{rkb}{ack + br}, \frac{crk}{ack + br} \right).$$

Period-doubling Bifurcations

On the time scale \mathbb{Z} , fixed points are also 1-periodic solutions.
In general, if $x(n)$ is a solution of

$$x(n+1) = f(x(n))$$

such that $x(n+p) = x(n)$, where p is the smallest integer that makes this true, then $x_0 = x(0)$ is called a **periodic point of minimal period p** .

Period-doubling Bifurcations

On the time scale \mathbb{Z} , fixed points are also 1-periodic solutions. In general, if $x(n)$ is a solution of

$$x(n+1) = f(x(n))$$

such that $x(n+p) = x(n)$, where p is the smallest integer that makes this true, then $x_0 = x(0)$ is called a **periodic point of minimal period p** .

A **period-doubling bifurcation** occurs when the stability of a fixed point changes and a pair of periodic points of minimal period 2 emerge.

Period-doubling Bifurcations

On the time scale \mathbb{Z} , fixed points are also 1-periodic solutions. In general, if $x(n)$ is a solution of

$$x(n+1) = f(x(n))$$

such that $x(n+p) = x(n)$, where p is the smallest integer that makes this true, then $x_0 = x(0)$ is called a **periodic point of minimal period p** .

A **period-doubling bifurcation** occurs when the stability of a fixed point changes and a pair of periodic points of minimal period 2 emerge.

See Section 3.4 of *Dynamics and Bifurcations*.

Period-doubling Bifurcation in the Predator-Prey Model

Using the predation rate a as a bifurcation parameter, there is a period-doubling bifurcation at the parameter value

$$a_{\text{PD}} = -\frac{br(br - 2b - 2r + 4)}{ck(br - 2b + 4)}.$$

Furthermore, the bifurcation is supercritical (subcritical) if $\widehat{\beta}_{\text{PD}}^{pp} > 0$ (< 0), where

$$\widehat{\beta}_{\text{PD}}^{pp} = \frac{16r(b - 2)^3(r + 2)}{(br - 2b + 4)^2 k^2 c^2 (br - 4)}.$$

Recall: supercritical \iff stable \rightarrow unstable, subcritical \iff unstable \rightarrow stable.

Period-Doubling Bifurcation – Method

One-dimensional case (from *Dynamics and Bifurcations*):

Let $f \in C^3$ with

$$f(0) = 0, \quad f'(0) = -1, \quad (f^2)'''(0) \neq 0.$$

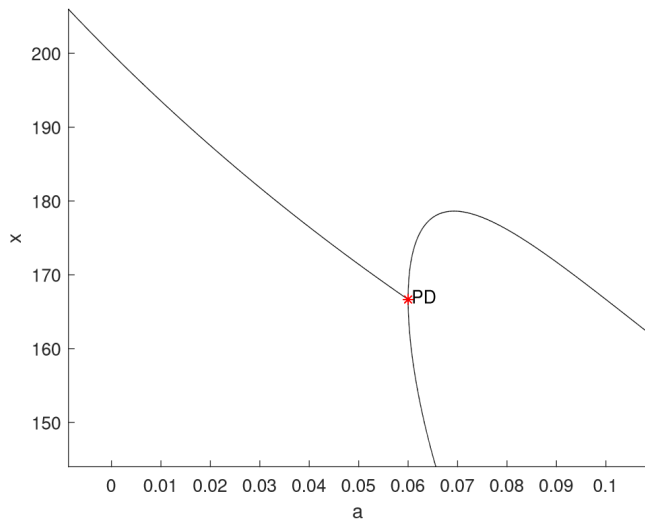
If $F(\lambda, x)$ is a perturbation of f such that

$$F(0, x) = f(x), \quad F(\lambda, 0) = 0, \quad \frac{\partial F}{\partial \lambda}(\lambda, 0) = -(1 + \lambda),$$

then the discrete equation $x_{n+1} = F(\lambda, x_n)$ undergoes a period-doubling bifurcation at $\lambda = 0$.

Apply a similar result to higher-dimensional equations – this involves Jacobian matrix and third-order partial derivatives.

Period-Doubling Bifurcation Diagram



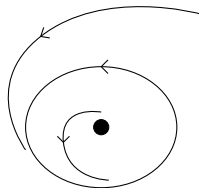
A subcritical period-doubling bifurcation

Neimark-Sacker Bifurcations

In a **Neimark-Sacker bifurcation** the fixed point changes stability type and a closed invariant curve containing the fixed point emerges with opposite stability.



Before Bifurcation



After Bifurcation

Neimark-Sacker Bifurcation in the Predator-Prey Model

A Neimark-Sacker bifurcation occurs with respect to a when

$$a = a_{\text{NS}} = \frac{-r(br - b - r)}{ck(r - 1)}.$$

The bifurcation is supercritical (subcritical) if $\widehat{\sigma_{\text{NS}}^{pp}} < 0$ (> 0).

What is $\widehat{\sigma_{\text{NS}}^{pp}}$? This is a value that depends on the parameters and is related to the following result...

Neimark-Sacker Bifurcation – Method

From *Elements of Applied Bifurcation Theory* by Y.A. Kuznetsov:

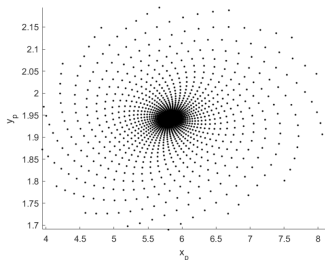
In the two-dimensional discrete system $x_{n+1} = f(\lambda, x_n)$, let $\mu_{\pm}(\lambda) = r(\lambda)e^{\pm i\theta(\lambda)}$ be the eigenvalues of the Jacobian near $\lambda = 0$. If

$$r(0) = 1, \quad r'(0) \neq 0, \quad e^{ik\theta(0)} \neq 1 \text{ for } k = 1, 2, 3, 4,$$

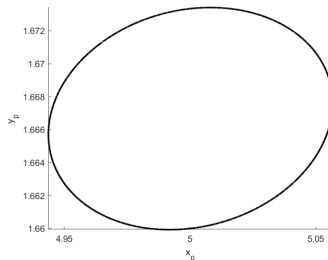
then the system undergoes a Neimark-Sacker bifurcation, which is supercritical (subcritical) if $\sigma = \Re(e^{-i\theta(0)}c_1(0)) < 0$ (> 0).

Here, $c_1(0)$ is a complicated function of the first, second, and third derivatives of f at $\lambda = 0$ and at the critical point.

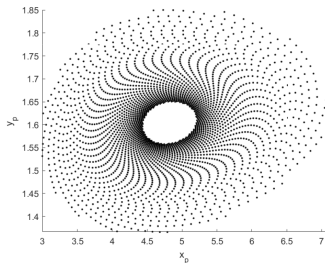
Phase Portraits Near the Neimark-Sacker Bifurcation



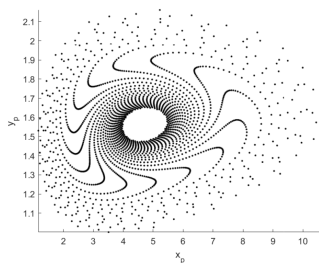
(a)



(b)



(c)



(d)

(a) $a < a_{NS}$, (b) $a = a_{NS}$, (c) $a > a_{NS}$, (d) $a > a_{NS}$

Thank You!