Math 5601 Final Project

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Consider the following second-order ODE with Dirichlet boundary conditions:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(c(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right) = f(x), \qquad a \leq x \leq b, \tag{1}$$

$$u(a) = g_a, \quad u(b) = g_b. \tag{2}$$

Problem 1.

Consider the second-order ODE (1). Multiplying by $v \in H^1([a,b])$ and integrating by parts gives

$$\int_a^b fv = c(b)u'(b)v(b) - c(a)u'(a)v(a) - \int_a^b cu'v. \tag{3}$$

(a) Suppose we have the boundary conditions

$$u'(a) = p_a, \qquad u(b) = g_b. \tag{4}$$

Equation (3) still holds, and we can impose the condition $v(b) = 0$ because we already know that $u(b) = p_b$. Since $u'(a) = p_a$, equation (3) becomes

$$\int_a^b fv = -c(a)p_a v(a) - \int_a^b cu'v' \tag{5}$$

for all $v \in H^1([a,b])$ such that $v(b) = 0$, which is our weak formulation of (1) with the given boundary conditions.

(b) Suppose we have the boundary conditions

$$u'(a) = p_a, \qquad u'(b) + q_b u(b) = p_b. \tag{6}$$

Equation (3) still holds. Since $u'(b) = p_b - q_b u(b)$, and $u'(a) = p_a$, we get

$$\int_a^b fv = c(b)(p_b - q_b u(b))v(b) - c(a)p_a v(a) - \int_a^b cu'v' \tag{7}$$

for all $v \in H^1([a,b])$, which is our weak formulation of (1) with the given boundary conditions.

(c) Suppose we have the boundary conditions

$$u'(a) = p_a, \qquad u'(b) = p_b. \tag{8}$$

Equation (3) still holds. Since $u'(a) = p_a$, and $u'(b) = p_b$, we get

$$\int_a^b fv = c(b)p_b v(b) - c(a)p_a v(a) - \int_a^b cu'v' \tag{9}$$

for all $v \in H^1([a,b])$, which is our weak formulation of (1) with the given boundary conditions.

We note that solutions of this formulation are not unique. Indeed, if $u \in H^1([a,b])$ satisfies (9) for all $v \in H^1([a,b])$, then so does $u + \alpha$, where $\alpha \in \mathbf{R}$ is any real number because $(u+\alpha)' = u'$ regardless of what α is, and the weak formulation depends only on u'.

Problem 2.

Consider the Poisson equation

$$\nabla \cdot (c\nabla u) = f \text{ in } D. \tag{10}$$

Using integration by parts, we have

$$\int_D fv = \int_D \nabla \cdot (c\nabla u)v = \int_{\partial D} cv\nabla u \cdot n \,\mathrm{d}S - \int_D c\nabla u \cdot \nabla v, \tag{11}$$

where $\mathrm{d}S$ is the surface measure on ∂D, and $v \in H^1\left(\overline{D}\right)$.

(a) Suppose that we have the boundary condition

$$u = g \text{ on } \partial D. \tag{12}$$

Equation (11) still holds. Since we know the value of u on ∂D, we can set $v = 0$ on ∂D. Then we get

$$\int_D fv = -\int_D c\nabla u \cdot \nabla v \tag{13}$$

for all $v \in H^1\left(\overline{D}\right)$ such that $v = 0$ on ∂D, which is our weak formulation of (10) with the given boundary condition.

(b) Suppose that we have the boundary condition

$$\nabla u \cdot n + qu = p \text{ on } \partial D, \tag{14}$$

where n is the outward unit normal vector to ∂D, and p and q are functions on ∂D. Equation (11) still holds. Since $\nabla u \cdot n = p - qu$ on ∂D, it follows that

$$\int_D fv = \int_{\partial D} cv(p - qu) \,\mathrm{d}S - \int_D c\nabla u \cdot \nabla v \tag{15}$$

for all $v \in H^1\left(\overline{D}\right)$, which is our weak formulation of (10) with the given boundary condition.

Problem 3.

If $u \in C^2[a, b]$, then

$$\|u - I_h u\|_\infty \le \frac{1}{8}h^2\|u''\|_\infty, \tag{16}$$

$$\|(u - I_h u)'\|_\infty \le \frac{1}{2}h\|u''\|_\infty. \tag{17}$$

Proof. Consider the interval $[x_i, x_{i+1}]$, where $1 \le i \le N$. Restricted to this interval, $I_h u$ is the degree-1 Lagrange polynomial interpolation of u on with nodes x_i and x_{i+1}. By the error formula for Lagrange polynomial approximation in the slides,

$$u(x) - I_h u(x) = \frac{f''(\xi(x))(x - x_i)(x - x_{i+1})}{2} \tag{18}$$

for some $\xi(x) \in [x_i, x_{i+1}]$. Then

$$|u(x) - I_h u(x)| \le \|f''\|_\infty \cdot \frac{1}{2}(x - x_i)(x_{i+1} - x). \tag{19}$$

The function $g(x) = (x - x_i)(x_{i+1} - x)$ is a downward-opening parabola, so it achieves maximum halfway between its roots x_i and x_{i+1}. Therefore,

$$|u(x) - I_h u(x)| \leq \|f''\|_\infty \cdot \frac{\left(\frac{x_i + x_{i+1}}{2} - x_i\right)\left(x_{i+1} - \frac{x_i + x_{i+1}}{2}\right)}{2} \tag{20}$$

$$= \|f''\|_\infty \frac{(x_{i+1} - x_i)^2}{8} = \frac{h^2}{8}\|f''\|_\infty. \tag{21}$$

Since this holds for all $x \in [x_i, x_{i+1}]$ and all $1 \leq i \leq N$, it holds for all $x \in [a, b]$. Therefore, the inequality (16) follows.

Let $1 \leq i \leq N$, and let $x \in [x_i, x_{i+1}]$. By the Mean Value Theorem, there exists $c \in [x_i, x_{i+1}]$ such that

$$u'(c) = \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} = (I_h u)'(x). \tag{22}$$

Then $(I_h u)'$ is the degree-0 Lagrange polynomial interpolation of u' with node c. By the error formula for Lagrange polynomial approximation in the slides,

$$u'(x) - (I_h u)'(x) = u''(\xi(x))(x - c) \tag{23}$$

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