5.22 Let X & Y have joint pdf

$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{o.w.} \end{cases}$$

$$M(t,s) = E[e^{tX+sY}] = \int e^{tx+sy} f(x,y)\,dx\,dy$$

$$= \int_0^\infty \int_0^y e^{tx+sy} e^{-y}\,dx\,dy = \int_0^\infty e^{(s-1)y} \frac{1}{t}[e^{tx}]_0^y\,dy$$

$$= \int_0^\infty e^{(s-1)y} \cdot \frac{1}{t}(e^{ty} - 1)\,dy = \frac{1}{t}\int_0^\infty (e^{(s+t-1)y} - e^{(s-1)y})\,dy$$

If $s+t-1 < 0$ and $s-1 < 0$, then $M(t,s)$ exists, and

$$M(t,s) = \frac{1}{t} \cdot \left[\frac{1}{s+t-1} e^{(s+t-1)y} - \frac{1}{s-1} e^{(s-1)y}\right]_0^\infty$$

$$= \frac{1}{t}\left(\frac{1}{s-1} - \frac{1}{s+t-1}\right) \qquad s < 1,\ t < 1-s$$

= 1 / ((s - 1) * (s + t - 1))

5.23 Let $(X, Y) \sim$ Bivariate Normal $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. Then
$X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, and $\rho = corr(X, Y)$.

Proof: By Example 5.5.2, the joint MGF of (X, Y) is

$$M(t,s) = E[e^{tX + sY}] = e^{\mu_1 t + \mu_2 s + \frac{1}{2}(\sigma_1^2 t^2 + \sigma_2^2 s^2 + 2\rho\sigma_1\sigma_2 ts)}$$

Then the MGF of X is $M(t,0) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2}$, which is the MGF of a Normal (μ_1, σ_1^2) R.V. by Theorem 3.3.5.

The MGF of Y is $M(0,s) = e^{\mu_2 s + \frac{1}{2}\sigma_2^2 s^2}$, which is the MGF of a Normal(μ_2, σ_2^2) R.V. by theorem 3.3.5.

Therefore, $X \sim N(\mu_1, \sigma_1^2)$, and $Y \sim N(\mu_2, \sigma_2^2)$.

By the discussion at the start of 5.5,

$$E[XY] = \frac{\partial^2 M}{\partial t \partial s}(0,0) =$$

$$\frac{\partial}{\partial s}\left((\mu_1 + \sigma_1^2 t + \rho\sigma_1\sigma_2 s) M(t,s)\right)\Big|_{(t,s)=(0,0)}$$

$$= \rho\sigma_1\sigma_2 M(0,0) + (\mu_2 + \sigma_2^2 s + \rho\sigma_1\sigma_2 t)(\mu_1 + \sigma_1^2 t + \rho\sigma_1\sigma_2 s) \cdot M(s,t)\Big|_{(s,t)=(0,0)}$$

$$= \rho\sigma_1\sigma_2 + \mu_2\mu_1$$

so $Cov(X,Y) = E[XY] - E[X]E[Y] = \rho\sigma_1\sigma_2 + \mu_1\mu_2 - \mu_1\mu_2$
$= \rho\sigma_1\sigma_2$, and

$$corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var[X]}\sqrt{Var[Y]}} = \frac{\rho\sigma_1\sigma_2}{\sigma_1 \cdot \sigma_2} = \rho.$$

5.25 Let X_1 and X_2 be independent, $X_i \sim N(\mu_i, \sigma_i^2)$,

$Y_1 = X_1$, $Y_2 = X_1 + X_2$.

(a) The MGF of X_i is $M_i(t_i) = e^{\mu_i t_i + \frac{1}{2}\sigma_i^2 t_i^2}$, and because X_1 & X_2 are independent their joint MGF $M(t_1,t_2)$ factors: $M(t_1,t_2) = M_1(t_1)M_2(t_2)$

$$= e^{\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}(\sigma_1^2 t_1^2 + \sigma_2^2 t_2^2)}$$

The joint MGF of Y_1, Y_2, then, is

$$N(t_1,t_2) = E[e^{t_1 Y_1 + t_2 Y_2}] = E[e^{t_1 X_1 + t_2 X_1 + t_2 X_2}]$$

$$= E\left[e^{(t_1+t_2)X_1 + t_2 X_2}\right] = M(t_1+t_2, t_2)$$

$$= e^{\mu_1(t_1+t_2) + \mu_2 t_2 + \frac{1}{2}(\sigma_1^2(t_1+t_2)^2 + \sigma_2^2 t_2^2)}$$

$$= e^{\mu_1 t_1 + (\mu_1+\mu_2)t_2 + \frac{1}{2}(\sigma_1^2 t_1^2 + (\sigma_1^2+\sigma_2^2)t_2^2 + 2\sigma_1^2 t_1 t_2)}$$

Which is the MGF of a Bivariate Normal$(\mu_1', \mu_2', \sigma_1'^2, \sigma_2'^2, \rho)$

(b) R.V. with $\mu_1' = \mu_1$, $\mu_2' = \mu_1 + \mu_2$, $\sigma_1'^2 = \sigma_1^2$, $\sigma_2'^2 = \sigma_1^2 + \sigma_2^2$,

$$\rho = \frac{\sigma_1^2}{\sigma_1'\sigma_2'} = \frac{\sigma_1'^2}{\sigma_1'\sigma_2'} = \frac{\sigma_1'}{\sigma_2'} = \frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}} = \operatorname{corr}(Y_1, Y_2).$$

$E[Y_1] = \mu_1$ and $E[Y_2] = \mu_1 + \mu_2$

(c) By Theorem 5.4.8.,

$$Y_2 \mid Y_1 = y_1 \sim N\left(\mu_1 + \mu_2 + \frac{\sigma_1'}{\sigma_2'}\frac{\sigma_2'}{\sigma_1'}(y_1 - \mu_1), \sigma_2'^2(1-\rho^2)\right)$$

$$\sim N\left(\mu_2 + y_1, \sigma_2'^2\left(1 - \frac{\sigma_1'^2}{\sigma_2'^2}\right)\right) \sim N(\mu_2 + y_1, \sigma_2'^2 - \sigma_1'^2)$$

$$\sim N(y_1 + \mu_2, \sigma_2^2)$$