1. Let (X, d) be a metric space. If $x, y \in X$ and $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} x_n = y$$

then $x = y$.

Proof. Let $\varepsilon > 0$. Then $\exists N_x$ s.t. $n > N_x \to d(x, x_n) < \frac{\varepsilon}{2}$ and $\exists N_y$ s.t. $n > N_y \to d(y, x_n) < \frac{\varepsilon}{2}$. Then if $n > \max\{N_x, N_y\}$ both $d(x, x_n) < \frac{\varepsilon}{2}$ and $d(y, x_n) < \frac{\varepsilon}{2}$, and

$$d(x,y) \leq d(x, x_n) + d(x_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $d(x,y) \leq \varepsilon \ \forall \varepsilon > 0$. This implies $d(x,y) = 0$, which implies $x = y$. $\square$

2. Let $f \in L^1(\mathbb{R})$ and $x_0 \in \mathbb{R}$. Define g by

$$g(x) = f(x + x_0).$$

Then $\hat{g}(\xi) = e^{ix_0\xi}\hat{f}(\xi)$.

Proof. By definition of F.T. for L^1 functions

$$\hat{g}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} g(x)\,dx = \int_{\mathbb{R}} e^{-ix\xi} f(x + x_0)\,dx.$$

Let $u = x + x_0$, $du = dx$, so

$$\hat{g}(\xi) = \int_{\mathbb{R}} e^{-i(u - x_0)\xi} f(u)\,du = e^{ix_0\xi} \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx$$

$$= e^{ix_0\xi}\hat{f}(\xi). \ \square$$

3. Let $r(\xi) = \frac{\sin \xi}{\xi}$. Let $\chi(x)$ be the characteristic function on $[-1,1]$.

Then $\hat{\chi}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} \chi(x)\,dx = \int_{-1}^{1} e^{-ix\xi}\,dx = \frac{1}{-i\xi}\left(e^{-i\xi} - e^{i\xi}\right)$

$$= \frac{2\sin(\xi)}{\xi}$$

$\therefore\ r(\xi) = \frac{1}{2}\hat{\chi}(\xi)$, and by Parseval's Formula

$$\int_{\mathbb{R}} |\hat{\chi}(\xi)|^2\,d\xi = 2\pi \int_{\mathbb{R}} |\chi(x)|^2\,dx = 2\pi \int_{-1}^{1} dx = 4\pi,$$

So $$\int_{\mathbb{R}} |r(\xi)|^2\,d\xi = \frac{1}{4}\int_{\mathbb{R}} |\hat{\chi}(\xi)|^2\,d\xi = \pi.$$

4. Let $f \in L^1(\mathbb{R})$ and $\lambda > 0$. Define g by

$$g(x) = f(\lambda x).$$

Then $\hat{g}(\xi) = \lambda^{-1}\hat{f}\left(\frac{\xi}{\lambda}\right)$.

Proof. Simple substitution shows that $g \in L^1(\mathbb{R})$ also. Then

$$\hat{g}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} g(x)\,dx = \int_{\mathbb{R}} e^{-ix\xi} f(\lambda x)\,dx$$

Let $u = \lambda x$, $du = \lambda dx$. Then

$$\hat{g}(\xi) = \lambda^{-1}\int_{\mathbb{R}} e^{-i\frac{u}{\lambda}\xi} f(u)\,du = \lambda^{-1}\int_{\mathbb{R}} e^{-ix(\xi/\lambda)} f(x)\,dx$$

$$= \lambda^{-1}\hat{f}\left(\frac{\xi}{\lambda}\right)$$