Math 6108 Homework 6

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Problem 1.

Let V be a vector space over $\mathbb{C}$. If $\|\cdot\|$ is a norm on V, then there exists an inner product $\langle\cdot,\cdot\rangle$ on V that induces $\|\cdot\|$ if and only if

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2) \quad \text{for all } \mathbf{x},\mathbf{y} \in V. \tag{1}$$

Proof. Suppose that there is an inner product $\langle\cdot,\cdot\rangle$ that induces $\|\cdot\|$. Then for all $\mathbf{x},\mathbf{y} \in V$,

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle + \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle \\
&= \langle \mathbf{x},\mathbf{x}\rangle + \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle + \langle \mathbf{y},\mathbf{y}\rangle + \langle \mathbf{x},\mathbf{x}\rangle + \langle \mathbf{x},-\mathbf{y}\rangle + \langle -\mathbf{y},\mathbf{x}\rangle + \langle -\mathbf{y},-\mathbf{y}\rangle \\
&= 2\langle \mathbf{x},\mathbf{x}\rangle + 2\langle \mathbf{y},\mathbf{y}\rangle + \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle - \langle \mathbf{x},\mathbf{y}\rangle - \langle \mathbf{y},\mathbf{x}\rangle \\
&= 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)
\end{aligned}$$

because $\|\cdot\|$ is induced by $\langle\cdot,\cdot\rangle$.

Conversely, suppose that (1) holds. We note that V is also a vector space over $\mathbb{R}$ if we use the same addition operator and a scalar multiplication operator given by restricting the scalar multiplication from $\mathbb{C}$ to $\mathbb{R}$. This can be verified by checking the vector space axioms. The axioms relating only to the addition operator are automatically satisfied because we are using the same addition. Then the axioms relating to the scalar multiplication remain. Let $a,b \in \mathbb{R}$, and $\mathbf{x},\mathbf{y} \in V$.

1. **(Closure)** Since $a \in \mathbb{C}$, it follows that $a\mathbf{x} \in V$.
2. **(Associativity of field and scalar multiplication)** Since $a,b \in \mathbb{R}$, we also have $ab \in \mathbb{R}$. On the other hand, $a,b,ab \in \mathbb{C}$, so $a(b\mathbf{x}) = (ab)\mathbf{x}$.
3. **(Multiplicative identity)** We note that $1 \in \mathbb{R}$, and $1\mathbf{x} = \mathbf{x}$.
4. **(Distributivity over vector addition)** Since $a \in \mathbb{C}$, we have $a(\mathbf{x}+\mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.
5. **(Distributivity over field addition)** Since $a,b \in \mathbb{R}$, we also have $a+b \in \mathbb{R}$. On the other hand, $a,b,a+b \in \mathbb{C}$, so $(a+b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.

Furthermore, we also see that $\|\cdot\|$ is a norm for V as a vector space over $\mathbb{R}$. In particular, positive definiteness is retained because it does not depend on the scalar multiplication, and the triangle inequality is retained because it depends only on the vector addition operator, which is the same. For the homogeneity property, we note that if $a \in \mathbb{R}$, and $\mathbf{x} \in V$, then $a \in \mathbb{C}$, so

$$\|a\mathbf{x}\| = |a|\|\mathbf{x}\|.$$

Thus, we can apply the theorem we proved in class; namely, the function $\langle\cdot,\cdot\rangle_R$ defined by

$$\langle \mathbf{x},\mathbf{y}\rangle_R = \frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\right), \qquad \mathbf{x},\mathbf{y} \in V,$$

is an inner product for V, as a vector space over $\mathbb{R}$, that induces $\|\cdot\|$.

Now define $\langle\cdot,\cdot\rangle$ by

$$\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle_R + i\langle i\mathbf{x}, \mathbf{y}\rangle_R, \qquad \mathbf{x}, \mathbf{y} \in V.$$

Then $\langle\cdot,\cdot\rangle$ is an inner product for V, as a vector space over $\mathbb{C}$, that induces $\|\cdot\|$. We begin by observing that for $\mathbf{x} \in V$,

$$\langle \mathbf{x}, \mathbf{x}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle_R + \frac{i}{4}\left(\|i\mathbf{x} + \mathbf{x}\|^2 - \|i\mathbf{x} - \mathbf{x}\|^2\right) = \langle \mathbf{x}, \mathbf{x}\rangle_R = \|x\|^2$$

because $\|i\mathbf{x} - \mathbf{x}\| = |i|\|\mathbf{x} - i^{-1}\mathbf{x}\| = \|\mathbf{x} + i\mathbf{x}\|$. This implies that $\langle\cdot,\cdot\rangle$ is positive definite and, if it is an inner product for V over $\mathbb{C}$, that it induces $\|\cdot\|$.

Let $\mathbf{x}, \mathbf{y} \in V$. Then

$$\langle \mathbf{x}, i\mathbf{y}\rangle = \langle \mathbf{x}, i\mathbf{y}\rangle_R + \frac{i}{4}\left(\|i\mathbf{x} + i\mathbf{y}\|^2 - \|i\mathbf{x} - i\mathbf{y}\|^2\right) = \langle \mathbf{x}, \mathbf{y}\rangle_R$$

□

Problem 2.

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^n$ be orthonormal. Let $A \in \mathbb{R}^n$. If $A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n$ are also orthonormal, then A is orthogonal.

Proof. Let $X = \begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n\end{bmatrix}$, and let $B = \begin{bmatrix}A\mathbf{x}_1 & A\mathbf{x}_2 & \cdots & A\mathbf{x}_n\end{bmatrix}$. Then $B = AX$. Since the columns of B and X are orthonormal, they are both orthogonal matrices. Therefore,

$$A = BX^T.$$

Since $(BX^T)^T(BX^T) = XB^TBX^T = I$ and $(BX^T)(BX^T)^T = BX^TXB^T = I$ by the orthogonality of B and X, it follows that A is invertible with $A^{-1} = (BX^T)^T = A^T$. This implies that A is orthogonal. □

Problem 3.

The algorithm for the Gram-Schmidt process is described in Algorithm 1. An implementation in Python is provided in Listing 1. We note that this implementation detects linear dependence of the columns of A as a part of the Gram-Schmidt process by checking if the produced orthogonal vectors are zero (well, almost zero, to account for numerical rounding error). This is possible because the columns of A are linearly dependent if and only the Gram-Schmidt process produces a zero vector at some point. This is easy to prove.

For $j < i$, each $\mathbf{b}_j$ is a linear combination of the first j columns of A. We can prove this by induction. For the base case, $\mathbf{b}_1 = \|\mathbf{a}_1\|^{-1}\mathbf{a}_1$. For some $1 \leq k < i-1$, suppose for induction that $\mathbf{b}_j = \sum_{m=1}^{j} c_{jm}\mathbf{a}_m$ for $1 \leq j \leq k$ and some constants c_{jm}. Then

$$\mathbf{b}_{k+1} = \mathbf{a}_{k+1} - \sum_{p=1}^{k}\langle \mathbf{b}_p, \mathbf{a}_{k+1}\rangle \sum_{m=1}^{p} c_{pm}\mathbf{a}_m$$

which completes the proof by induction.

Suppose that $\mathbf{b}_i = \mathbf{0}$. Then

$$\mathbf{0} = \mathbf{b}_i = \mathbf{a}_i - \sum_{p=1}^{i-1}\langle \mathbf{b}_p, \mathbf{a}_i\rangle \sum_{m=1}^{p} c_{pm}\mathbf{a}_m.$$

The coefficient of $\mathbf{a}_i$ is non-zero, so a non-trivial linear combination of the columns of A is $\mathbf{0}$, meaning that the columns of A are linearly dependent.

Conversely, if the columns of A are linearly dependent, then there exists $c_1, \dots, c_m$ not all equal to zero such that

$$\sum_{i=1}^{m} c_i \mathbf{a}_i = \mathbf{0}.$$

Let k be the largest integer such that $c_k \neq 0$. Then

$$\mathbf{0} = \sum_{i=1}^{k} c_i \mathbf{a}_i = c_k \mathbf{b}_k + \sum_{p=1}^{k-1} \langle \mathbf{b}_p, \mathbf{a}_k \rangle \mathbf{b}_p + \sum_{i=1}^{k-1} \left(c_i \mathbf{b}_i + c_i \sum_{p=1}^{i-1} \langle \mathbf{b}_p, \mathbf{a}_i \rangle \mathbf{b}_p \right).$$

The coefficient of $\mathbf{b}_k$ is nonzero, so it follows that the columns of B are linearly dependent. Since the columns of B are also orthogonal because of the Gram-Schmidt process, one of them must be zero.

The command `python -m gs` can be used to run the tests, which verify that the function works across a range of input types that cover every code path. The output from running these tests is given in Listing 2.

Algorithm 1: Gram-Schmidt Orthogonalization

Input: Matrix $A \in \mathbb{R}^{n \times m}$ with linearly independent columns $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{R}^n$

Output: Matrix $B \in \mathbb{R}^{n \times m}$, whose columns $\mathbf{b}_1, \dots, \mathbf{b}_m \in \mathbb{R}^n$ are the orthogonal vectors obtained by applying the Gram-Schmidt process to the columns of A

$c \leftarrow 1$;
repeat
  $\mathbf{b}_c \leftarrow \mathbf{a}_c - \sum_{p=1}^{c-1} \langle \mathbf{b}_p, \mathbf{a}_c \rangle \mathbf{b}_p$; // Sum is 0 by convention if $c = 1$
  $\mathbf{b}_c \leftarrow \frac{\mathbf{b}_c}{\|\mathbf{b}_c\|}$
until $c = n$;

Listing 1: Python implementation of the Gram-Schmidt process

```python
import numpy as np


class LinearDependenceError(BaseException):
    """Exception class that is raised to indicate linearly dependent input vectors
    """
    pass


def gram_schmidt(a, eps_d=1e-10):
    """
    Perform the Gram-Schmidt orthogonalization process on the columns of
    a matrix, returning orthogonalized vectors as the columns of a new matrix.

    :param a: n x m matrix with linearly independent columns. Raises
        SingularMatrixError if columns of a are linearly dependent or almost
        linearly dependent (see eps_d).
    :param eps_d: Tolerance for approximate linear independence (minimum norm of
        the computed orthogonal columns). Default = 10^{-10}
    :return: n x m matrix b whose columns are orthogonal (orthonormal, if
        normalize == True) vectors obtained by performing Gram-Schmidt
        orthogonalization on the columns of a.
```

```python
    """

    # ==== Input Validation ====

    # Ensure input has the correct data type
    a = np.array(a, dtype=float)
    assert len(a.shape) == 2

    # Early check for linear dependence
    if a.shape[1] > a.shape[0]:
        raise LinearDependenceError('Matrix has linearly dependent columns '
                                    '(more columns than rows)')

    # ==== Run Gram-Schmidt process ====

    # Initialization

    # The first step for each column is copying the corresponding column from a,
    # so we initialize the output equal to a. Since we already copied the input
    # with np.array(), we can use that memory for our output matrix
    b = a

    # Normalize the first column of b
    norm = np.linalg.norm(b[:, 0])

    # Check for approximate linear dependence before possible divide-by-zero
    if norm < eps_d:
        raise LinearDependenceError('Matrix has linearly dependent or almost '
                                    'linearly dependent columns because first '
                                    'column is almost 0')
    b[:, 0] /= norm

    # Iteration
    for col in range(1, a.shape[1]):
        # Recall that b[:, col] == a[:, col] because of initialization

        # Subtract out previous orthonormal columns
        b[:, col] -= b[:, :col] @ (b[:, :col].T @ b[:, col])

        # Normalize new column
        norm = np.linalg.norm(b[:, col])

        # Check for approximate linear dependence before possible divide-by-zero
        if norm < eps_d:
            raise LinearDependenceError('Aborting orthogonalization; matrix has '
                                        'linearly dependent or almost linearly '
                                        'dependent columns')
        b[:, col] /= norm

    # Return orthogonal columns
    return b
```

```python


# Test example
if __name__ == '__main__':
    # Set RNG seed for reproducible results
    np.random.seed(2024)

    print('Test 1: random square matrix')
    a = np.random.random((5, 5))
    print('Input matrix')
    print(a)
    print()
    print('Orthonormalized matrix')
    b = gram_schmidt(a)
    print(b)
    print()
    print('Implementation worked?', np.allclose(b.T @ b, np.eye(5)))
    print()

    print('Test 2: random non-square matrix')
    a = np.random.random((5, 3))
    print('Input matrix')
    print(a)
    print()
    print('Orthonormalized matrix')
    b = gram_schmidt(a)
    print(b)
    print()
    print('Implementation worked?', np.allclose(b.T @ b, np.eye(3)))
    print()

    print('Test 3: random matrix with too many columns')
    a = np.random.random((3, 5))
    print('Input matrix')
    print(a)
    print()
    try:
        gram_schmidt(a)  # should raise an error
    except LinearDependenceError as e:
        print(e)
    print()

    print('Test 4: matrix with first column 0')
    a = np.array([
        [0, 1, 2],
        [0, 3, 4],
        [0, 5, 6]
    ])
    print('Input matrix')
    print(a)
    print()
```

```
    try:
        gram_schmidt(a)  # should raise an error
    except LinearDependenceError as e:
        print(e)
    print()

    print('Test 5: singular matrix')
    a = np.array([
        [1, 2, -1],
        [2, 5, -3],
        [3, 3, 0]
    ])
    print('Input matrix')
    print(a)
    print()
    try:
        gram_schmidt(a)  # should raise an error
    except LinearDependenceError as e:
        print(e)
```

Listing 2: Output for test cases

```
>python -m gs
Test 1: random square matrix
Input matrix
[[0.58801452 0.69910875 0.18815196 0.04380856 0.20501895]
 [0.10606287 0.72724014 0.67940052 0.4738457  0.44829582]
 [0.01910695 0.75259834 0.60244854 0.96177758 0.66436865]
 [0.60662962 0.44915131 0.22535416 0.6701743  0.73576659]
 [0.25799564 0.09554215 0.96090974 0.25176729 0.28216512]]

Orthonormalized matrix
[[ 0.66075857  0.10599106 -0.32621454 -0.56072372 -0.36240445]
 [ 0.11918405  0.62410254  0.17611003 -0.29186203  0.69288743]
 [ 0.02147069  0.73799494  0.08061366  0.44264935 -0.50245942]
 [ 0.68167657 -0.1644837  -0.10793795  0.62668109  0.32230789]
 [ 0.28991262 -0.16604366  0.91892338 -0.10834125 -0.17950537]]

Implementation worked? True

Test 2: random non-square matrix
Input matrix
[[0.76825393 0.7979234  0.5440372 ]
 [0.38270763 0.38165095 0.28582739]
 [0.74026815 0.23898683 0.4377217 ]
 [0.8835387  0.28928114 0.78450686]
 [0.75895366 0.41778538 0.22576877]]

Orthonormalized matrix
[[ 0.47270878  0.73926285  0.17771326]
 [ 0.23548107  0.33566518  0.12907525]
 [ 0.45548905 -0.37843189 -0.14903895]
```

```
 [ 0.54364382 -0.44335429  0.57756706]
 [ 0.46698629 -0.03233593 -0.77198527]]

Implementation worked? True

Test 3: random matrix with too many columns
Input matrix
[[0.42009814 0.06436369 0.59643269 0.83732372 0.89248639]
 [0.20052744 0.50239523 0.89538184 0.25592093 0.86723234]
 [0.01648793 0.55249695 0.52790539 0.92335039 0.24594844]]

Matrix has linearly dependent columns (more columns than rows)

Test 4: matrix with first column 0
Input matrix
[[0 1 2]
 [0 3 4]
 [0 5 6]]

Matrix has linearly dependent or almost linearly dependent columns because first column is almost 0

Test 5: singular matrix
Input matrix
[[ 1  2 -1]
 [ 2  5 -3]
 [ 3  3  0]]

Aborting orthogonalization; matrix has linearly dependent or almost linearly dependent columns
```