# Math 5601 Final Project

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Consider the following second-order ODE with Dirichlet boundary conditions:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(c(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right) = f(x), \qquad a \leq x \leq b, \tag{1}$$

$$u(a) = g_a, \quad u(b) = g_b. \tag{2}$$

**Problem 1.**

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Consider the second-order ODE (1).

**(a)** Suppose we have the boundary conditions

$$u'(a) = p_a, \qquad u(b) = g_b. \tag{3}$$

Multiplying the ODE by a test function $v$ and integrating by parts, we have

$$\int_a^b f(x)v(x)\,\mathrm{d}x = \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(c(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)v(x)\,\mathrm{d}x = c(x)u'(x)v(x)\Big|_a^b - \int_a^b c(x)u'(x)v'(x)\,\mathrm{d}x. \tag{4}$$

If we choose $v(b) = 0$, then we are led to the weak formulation: find $u \in C^1([a,b])$ with $u(b) = g_b$ such that

$$-c(a)p_a v(a) - \int_a^b c(x)u'(x)v'(x)\,\mathrm{d}x = \int_a^b f(x)v(x)\,\mathrm{d}x \tag{5}$$

for all $v \in C^1([a,b])$ with $v(b) = 0$.

**(b)** Suppose we have the boundary conditions

$$u'(a) = p_a, \qquad u'(b) + q_b u(b) = p_b. \tag{6}$$

Multiplying the ODE by a test function $v$ and integrating by parts, we have

$$\int_a^b f(x)v(x)\,\mathrm{d}x = \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(c(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)v(x)\,\mathrm{d}x = c(x)u'(x)v(x)\Big|_a^b - \int_a^b c(x)u'(x)v'(x)\,\mathrm{d}x. \tag{7}$$

**Problem 2.**

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**Problem 3.**

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If $u \in C^2[a,b]$, then

$$\|u - I_h u\|_\infty \leq \frac{1}{8}h^2\|u''\|_\infty, \tag{8}$$

$$\|(u - I_h u)'\|_\infty \leq \frac{1}{2}h\|u''\|_\infty. \tag{9}$$

*Proof.* Consider the interior $(x_i, x_{i+1})$ of the $i$th element of $[a, b]$, where $1 \leq i \leq N$. For $x \in (x_i, x_{i+1})$, Taylor's Theorem implies that there is some $\xi \in (x_i, x_{i+1})$ such that

$$u(x) = u(x_i) + (x - x_i)u'(x_i) + \frac{1}{2}(x - x_i)^2 u''(\xi). \tag{10}$$

$\square$