# Math 5601 Homework 9

Jacob Hauck

November 15, 2023

**Problem 1.**

Let $A$ be a nonsingular matrix, and let $A^{(2)}$ be the matrix from the lecture slides in the second step of Gaussian elimination. Then there exists $s \geq 2$ such that $a_{2s}^{(2)} \neq 0$.

*Proof.* Suppose on the contrary. By the Gaussian elimination process, we know that $a_{21}^{(2)} = 0$. If there is no $s \geq 2$ such that $a_{2s}^{(2)} \neq 0$, then the whole second row of $A^{(2)}$ is zero. Hence, expanding by cofactors along the second row, we see that the determinant of $A^{(2)}$ is

$$\det\left(A^{(2)}\right) = 0 \cdot \det(B_1) + 0 \cdot \det(B_2) + \cdots + 0 \cdot \det(B_n) = 0, \tag{1}$$

where $B_i$ is the cofactor corresponding to $a_{2i}^{(2)}$. Then $A^{(2)}$ is singular.

This is a contradiction because $A^{(2)}$ was obtained from $A$ by elementary row operations, and $A$ was nonsingular, and applying row operations to a nonsingular matrix must result in a nonsingular matrix. □

**Problem 2.**

Let $A = \{a_{ij}\}$, and consider the SOR iteration for solving $Ax = b$:

$$x_i^{(k+1)} = (1-\sigma)x_i^{(k)} + \frac{\sigma}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij}x_j^{(k)}\right], \qquad i = 1, 2, \dots, n. \tag{2}$$

If $L$ is the lower-triangular part of $A$ and $U$ is the upper-triangular part, and $D$ is the diagonal, so that $A = L + U + D$, then the SOR iteration becomes

$$\begin{aligned}
x^{(k+1)} &= (1-\sigma)x^{(k)} + \sigma D^{-1}\left[b - Lx^{(k+1)} - Ux^{(k)}\right] && (3)\\
\implies (D + \sigma L)x^{(k+1)} &= ((1-\sigma)D - \sigma U)x^{(k)} + \sigma b && (4)\\
\implies x^{(k+1)} &= \left(L + \frac{1}{\sigma}D\right)^{-1}\left(\frac{1}{\sigma}D - D - U\right)x^{(k)} + \left(L + \frac{1}{\sigma}D\right)^{-1} b && (5)\\
&= -\left(L + \frac{1}{\sigma}D\right)^{-1}\left(D - \frac{1}{\sigma}D + U\right)x^{(k)} + \left(L + \frac{1}{\sigma}D\right)^{-1} b. && (6)
\end{aligned}$$

Define $M = L + \frac{1}{\sigma}D$, and $N = -\left(D - \frac{1}{\sigma}D + U\right)$. Then

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \tag{7}$$

and

$$A = L + D + U = \left(L + \frac{1}{\sigma}D\right) + \left(D - \frac{1}{\sigma}D + U\right) = M - N. \tag{8}$$

Therefore, SOR is an iterative method that uses the $M$ and $N$ defined above.