Math 6330 Homework 1

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1.3 (iv)

Let $\mathbb{T} = \{\sqrt{n} \mid n \in \mathbb{N}_0\}$. Then for $t = \sqrt{n} \in \mathbb{T}$,

- the next point to the right of $\sqrt{n}$ is $\sqrt{n+1}$, so $\sigma(t) = \sigma(\sqrt{n}) = \sqrt{n+1} = \sqrt{t^2+1}$,
- the next point to the left of $\sqrt{n}$ is $\sqrt{n-1}$ if $n > 0$. If $n = 0$, then there is no point in $\mathbb{T}$ to the left of $t = 0$, so

$$\begin{aligned} \rho(t) = \rho(\sqrt{n}) &= \begin{cases} \sqrt{n-1} & n > 0 \\ 0 & n = 0 \end{cases} \\ &= \begin{cases} \sqrt{t^2-1} & t > 0 \\ 0 & t = 0. \end{cases} \end{aligned}$$

- $\mu(t) = \sigma(t) - t = \sqrt{t^2+1} - t$.

Every point in $\mathbb{T}$ is right-scattered because $\sigma(t) = \sqrt{t^2+1} > t$. If $t > 0$, then t is left-scattered because $\rho(t) = \sqrt{t^2-1} < t$. The point $0 \in \mathbb{T}$ is not left-scattered because $\rho(0) = 0$, and it is not left-dense either because $0 = \inf \mathbb{T}$.

1.4 (ii)

Let $\mathbb{T} = \{0\} \cup [1, 2]$. Then $\mathbb{T}$ is a time-scale, and $1 \in \mathbb{T}$ does not satisfy $\rho(\sigma(1)) = 1$. Indeed, $\sigma(1) = 1$, and $\rho(1) = 0$, so $\rho(\sigma(1)) = 0 \neq 1$.

Given any time-scale $\mathbb{T}$ and $t \in \mathbb{T}$, then $\rho(\sigma(t)) = t$ if and only if t is not left-scattered or t is right-scattered.

Proof. Suppose that t is left-scattered and not right-scattered. Then $\sigma(t) = t$, so $\rho(\sigma(t)) = \rho(t) \neq t$. Hence, $\rho(\sigma(t)) = t$ implies that t is not left-scattered or t is right-scattered.

Conversely, if t is right-scattered, then $\sigma(t) \in \mathbb{T}$ is left-scattered with $\rho(\sigma(t)) = t$. If t is not right-scattered and not left-scattered, then $\rho(t) = t$ and $\sigma(t) = t$, so $\rho(\sigma(t)) = t$. $\square$

1.14 (i)

Define $f : \mathbb{T} \to \mathbf{R}$ by $f(t) = t^2$. Then $f^\Delta(t) = t + \sigma(t)$.

Proof. Let $t \in \mathbb{T}$, and let $\varepsilon > 0$ be given. Set $\delta = \varepsilon$. Then for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$,

$$\begin{aligned} |f(\sigma(t)) - f(s) - (t + \sigma(t))(\sigma(t) - s)| &= |\sigma(t)^2 - s^2 - (t + \sigma(t))(\sigma(t) - s)| \\ &= |ts + \sigma(t)s - s^2 - t\sigma(t)| \\ &= |(s - t)(\sigma(t) - s)| \\ &\leq \varepsilon|\sigma(t) - s|, \end{aligned}$$

so $f^{\Delta}(t) = t + \sigma(t)$ by definition. $\square$

1.19 (i) (derivative at 0 only)

Let $\mathbb{T} = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \cup \{0\}$, and define $f : \mathbb{T} \to \mathbf{R}$ by $f(t) = \sigma(t)$. Recalling from class that $\sigma(t) = \frac{t}{1-t}$ when $t \neq 1$, we see that $\sigma(0) = 0$, so 0 is right-dense, and

$$\lim_{s \to 0} \frac{f(0) - f(s)}{0 - s} = \lim_{s \to 0} \frac{0 - \frac{s}{1-s}}{0 - s} = \lim_{s \to 0} \frac{1}{1 - s} = 1.$$

Therefore, by Theorem 1.16, $f^{\Delta}(0) = 1$.

1.19 (ii)

Let $\mathbb{T} = \{\sqrt{n} \mid n \in \mathbb{N}_0\}$, and define $f : \mathbb{T} \to \mathbf{R}$ by $f(t) = t^2$. Recall from 1.3 (iv) that $\sigma(t) = \sqrt{t^2 + 1}$, and every point in $\mathbb{T}$ is right-scattered. Note that every point $t \in \mathbb{T}$ is (topologically) isolated, so f is continuous on $\mathbb{T}$. Therefore, by Theorem 1.16, f is differentiable everywhere on $\mathbb{T}$, and for $t \in \mathbb{T}$,

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{\left(\sqrt{t^2 + 1}\right)^2 - t^2}{\sqrt{t^2 + 1} - t} = \sqrt{t^2 + 1} + t.$$

1.19 (iii)

Let $\mathbb{T} = \left\{ \frac{n}{2} \mid n \in \mathbb{N}_0 \right\}$, and define $f : \mathbb{T} \to \mathbf{R}$ by $f(t) = t^2$. Then for $t = \frac{n}{2} \in \mathbb{T}$, the next point to the right of t is $\frac{n+1}{2} = t + \frac{1}{2}$. Hence, $\sigma(t) = t + \frac{1}{2}$. Moreover, every point in $\mathbb{T}$ is right-scattered, and every point in $\mathbb{T}$ is (topologically) isolated, so f is continuous on $\mathbb{T}$. By Theorem 1.16, for $t \in \mathbb{T}$,

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{\left(t + \frac{1}{2}\right)^2 - t^2}{t + \frac{1}{2} - t} = 2\left(t + \frac{1}{4}\right) = 2t + \frac{1}{2}.$$

1.21 (iv)

Suppose that $f : \mathbb{T} \to \mathbf{R}$ is differentiable at $t \in \mathbb{T}$, and $f(t)f(\sigma(t)) \neq 0$. Then $\frac{1}{f}$ is differentiable at t, and

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

Proof. We know from Theorem 1.16 that f is continuous at t. Since $f(t) \neq 0$ by assumption, it follows that f is bounded away from 0 in a neighborhood of t. That is, there exists $C > 0$ and $\delta_0 > 0$ such that for all $s \in (t - \delta_0, t + \delta_0) \cap \mathbb{T}$, we have $|f(s)| \geq C$.

Let $\varepsilon > 0$ be given, and set

$$\varepsilon^* = \varepsilon \left(\frac{1}{C|f(\sigma(t))|} + \frac{\left|f^{\Delta}(t)\right|}{C\left|f(t)f(\sigma(t))\right|} \right)^{-1}.$$

Since f is continuous and delta-differentiable at t, we can choose $\delta \in (0, \delta_0]$ such that for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$,

1. $|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon^* |\sigma(t) - s|$,

2. $|f(t) - f(s)| \leq \varepsilon^*$.

Note also that $|f(s)| \geq C$ for all $s \in (t-\delta, t+\delta) \cap \mathbb{T}$ because $\delta \leq \delta_0$.

Then

$$\begin{aligned}
&\left|\frac{1}{f(\sigma(t))} - \frac{1}{f(s)} - \left(-\frac{f^\Delta(t)}{f(t)f(\sigma(t))}\right)(\sigma(t) - s)\right| \\
&\qquad = \left|\frac{f(t)f(s) - f(t)f(\sigma(t)) + f(s)f^\Delta(t)(\sigma(t)-s)}{f(t)f(\sigma(t))f(s)}\right| \\
&\qquad = \left|\frac{f(t)\big[f(s) - f(\sigma(t)) + f^\Delta(t)(\sigma(t)-s)\big] + (f(s) - f(t))f^\Delta(t)(\sigma(t)-s)}{f(t)f(\sigma(t))f(s)}\right| \\
&\qquad \leq \frac{\varepsilon^*|\sigma(t)-s|}{\big|f(\sigma(t))f(s)\big|} + \frac{\varepsilon^*\left|f^\Delta(t)\right| \cdot |\sigma(t)-s|}{\big|f(t)f(\sigma(t))f(s)\big|} \\
&\qquad \leq \left(\frac{1}{C|f(\sigma(t))|} + \frac{\left|f^\Delta(t)\right|}{C\big|f(t)f(\sigma(t))\big|}\right)\varepsilon^*|\sigma(t)-s| \\
&\qquad = \varepsilon|\sigma(t)-s|,
\end{aligned}$$

so

$$\left(\frac{1}{f}\right)^\Delta(t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))}$$

by definition. $\square$

1.22

Let x, y and z be delta-differentiable at t. Then xyz is delta-differentiable at t, and

$$(xyz)^\Delta = x^\Delta yz + xy^\Delta z + xyz^\Delta \quad \text{at } t.$$

Proof. By the product rule, yz is delta-differentiable at t. By the product rule again, $xyz = x(yz)$ is also delta-differentiable at t. Furthermore, at t, the product rule gives (putting σ always on the second term)

$$(xyz)^\Delta = (x(yz))^\Delta = x^\Delta yz + x^\sigma(yz)^\Delta = x^\Delta yz + x^\sigma y^\Delta z + x^\sigma y^\sigma z^\Delta,$$

as desired. $\square$

1.26

(i) Let $\mathbb{T} = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$, and let $f(t) = \sigma(t)$. Recall from class that

$$\sigma(t) = \begin{cases} \frac{t}{1-t} & \text{if } 0 \leq t < 1, \\ 1 & \text{if } t = 1. \end{cases}$$

Since we $1 \notin \mathbb{T}^\kappa$, we can get $f^\Delta = \sigma^\Delta$ by taking $f(t) = \frac{t}{1-t}$. We can apply the product rule easily once we know $\left(\frac{1}{1-t}\right)^\Delta$. By Theorem 1.24,

$$\left(\frac{1}{1-t}\right)^\Delta = \frac{1}{(\sigma(t)-1)(t-1)}.$$

Hence,

$$f^\Delta(t) = \frac{\sigma(t)}{(\sigma(t)-1)(t-1)} + \frac{1}{1-t} = \frac{1}{(\sigma(t)-1)(t-1)} = \frac{1}{-t-(t-1)} = \frac{1}{1-2t}, \quad t \in \mathbb{T}^\kappa.$$

(ii) Let $\mathbb{T} = \{\sqrt{n} \mid n \in \mathbb{N}_0\}$, and let $f(t) = t^2$. Recall from 1.3 (iv) that $\sigma(t) = \sqrt{t^2+1}$. Then, by Theorem 1.24,

$$f^\Delta(t) = \sigma(t)t^0 + (\sigma(t))^0 t = t + \sigma(t) = t + \sqrt{t^2+1},$$

which agrees with the calculation in 1.19 (ii).

(iii) Let $\mathbb{T} = \left\{\frac{n}{2} \mid n \in \mathbb{N}_0\right\}$, and let $f(t) = t^2$. Recall from 1.19 (iii) that $\sigma(t) = t + \frac{1}{2}$. Then, by Theorem 1.24,

$$f^\Delta(t) = t + \sigma(t) = 2t + \frac{1}{2},$$

which agrees with the calculation in 1.19 (iii).

(iv) Let $\mathbb{T} = \{\sqrt[3]{n} \mid n \in \mathbb{N}_0\}$, and let $f(t) = t^3$. If $t = \sqrt[3]{n} \in \mathbb{T}$, then

$$\sigma(t) = \sigma(\sqrt[3]{n}) = \sqrt[3]{n+1} = \sqrt[3]{t^3+1}.$$

By Theorem 1.24, we have

$$\begin{aligned} f^\Delta(t) &= \sum_{\nu=0}^{2} (\sigma(t))^\nu t^{2-\nu} = t^2 + \sigma(t)t + (\sigma(t))^2 \\ &= t^2 + (t^3+1)^{\frac{1}{3}} t + (t^3+1)^{\frac{2}{3}}. \end{aligned}$$

1.28

(i) Let $\mathbb{T} = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$, and let $f(t) = \sigma(t)$. Recall from 1.26 (i) that $\sigma(t) = \frac{t}{1-t}$, and $f^\Delta(t) = \frac{1}{1-2t}$ for $t \in \mathbb{T}^\kappa$. By the quotient rule,

$$f^{\Delta\Delta}(t) = \left(\frac{1}{1-2t}\right)^\Delta = -\frac{-2}{(1-2t)(1-2\sigma(t))} = \frac{2}{(1-2t)\left(1-\frac{2t}{1-t}\right)} = \frac{2(1-t)}{(1-2t)(1-3t)}, \qquad t \in \mathbb{T}^{\kappa^2}.$$

(ii) Let $\mathbb{T} = \{\sqrt{n} \mid n \in \mathbb{N}_0\}$, and let $f(t) = t^2$. Recall from 1.26 (ii) that $\sigma(t) = \sqrt{t^2+1}$, and $f^\Delta(t) = t + \sqrt{t^2+1}$. Since every point $t \in \mathbb{T}$ is right-scattered, and f^Δ is continuous (because every point $t \in \mathbb{T}$ is also topologically isolated), we can use Theorem 1.16 to find that

$$\begin{aligned} f^{\Delta\Delta}(t) &= \left(t + \sqrt{t^2+1}\right)^\Delta = 1 + \frac{\sqrt{\sigma(t)^2+1} - \sqrt{t^2+1}}{\sqrt{t^2+1}-t} = 1 + \left(\sqrt{t^2+1}+1\right)\left(\sqrt{t^2+2} - \sqrt{t^2+1}\right) \\ &= \sqrt{(t^2+1)(t^2+2)} + \sqrt{t^2+2} - t^2 - \sqrt{t^2+1}. \end{aligned}$$

(iii) Let $\mathbb{T} = \left\{ \frac{n}{2} \mid n \in \mathbb{N}_0 \right\}$, and let $f(t) = t^2$. Recall from 1.26 (iii) that $f^\Delta(t) = 2t + \frac{1}{2}$. Then by linearity and Theorem 1.24,

$$f^{\Delta\Delta}(t) = 2.$$

(iv) Let $\mathbb{T} = \{\sqrt[3]{n} \mid n \in \mathbb{N}_0\}$, and let $f(t) = t^3$. Recall from 1.26 (iv) that $\sigma(t) = \sqrt[3]{t^3+1}$, and $f^\Delta(t) = t^2 + (t^3+1)^{\frac{1}{3}} t + (t^3+1)^{\frac{2}{3}}$. Since every point of $\mathbb{T}$ is right-scattered and f^Δ is continuous on $\mathbb{T}$ (because every point of $\mathbb{T}$ is topologically isolated), we can use Theorem 1.16 to find that

$$\begin{aligned} f^{\Delta\Delta}(t) &= \frac{f^\Delta(\sigma(t)) - f^\Delta(t)}{\sigma(t) - t} \\ &= \frac{(t^3+1)^{\frac{2}{3}} + (t^3+2)^{\frac{1}{3}}(t^3+1)^{\frac{1}{3}} + (t^3+2)^{\frac{2}{3}} - t^2 - (t^3+1)^{\frac{1}{3}} t - (t^3+1)^{\frac{2}{3}}}{\sqrt[3]{t^3+1} - t} \\ &= \frac{(t^3+2)^{\frac{1}{3}}(t^3+1)^{\frac{1}{3}} + (t^3+2)^{\frac{2}{3}} - t^2 - (t^3+1)^{\frac{1}{3}} t}{\sqrt[3]{t^3+1} - t} \end{aligned}$$

1.36 (i)

Suppose the μ is differentiable, and f^{Δ^σ} and f^{σ^Δ} both exist. If f is twice differentiable, then

$$(f^\sigma)^\Delta = (f + \mu f^\Delta)^\Delta = f^\Delta + \mu^\Delta (f^\Delta)^\sigma + \mu f^{\Delta\Delta},$$

and

$$f^{\Delta^\sigma} = f^\Delta + \mu f^{\Delta\Delta}.$$

Substituting the second equation into the first gives

$$f^{\sigma^\Delta} = f^{\Delta^\sigma} + \mu^\Delta f^{\Delta^\sigma} = (1 + \mu^\Delta) f^{\Delta^\sigma}.$$