Math 6108 Homework 2

Jacob Hauck

September 4, 2024

Problem 1.

1. Let $S = \{(x, y) \in \mathbf{R}^2 \mid x \geq 0, y \geq 0\}$. Then S is not a subspace of $\mathbf{R}^2$ because $(1, 0) \in S$, but $-(1, 0) = (-1, 0) \notin S$; that is, S is not closed under scalar multiplication in $\mathbf{R}^2$.
2. Let $S = \{f \in \mathcal{P}^3 \mid f(4) = 0\}$. Then S is a subspace of $\mathcal{P}^3$. To see this, let $f, g \in S$, and let $a \in \mathbf{R}$. Then $(f + g)(4) = f(4) + g(4) = 0$, so $f + g \in S$, and $(af)(4) = af(4) = 0$, so $af \in S$. Finally, $\mathbf{0} \in S$ because $\mathbf{0}(4) = 0$, so S is nonempty. This shows that S is a subspace of $\mathcal{P}^3$.
3. Let $S = \{f \in \mathcal{P}^4 \mid f(4) = 2\}$. Then S is not a subspace of $\mathcal{P}^4$ because $\mathbf{0} \notin S$, as $\mathbf{0}(4) = 0 \neq 2$. A subspace must contain the zero element of the larger space.
4. Let $S = \{f \in \mathcal{F} \mid f'(x) + f(x) = 2\}$. Then S is not a subspace of $\mathcal{F}$ because $\mathbf{0} \notin S$, as $\mathbf{0}'(x) + \mathbf{0}(x) = 0 \neq 2$.
5. Let $S = \{f \in \mathcal{F} \mid f''(x) - 2f(x) = 0\}$. Then S is a subspace of $\mathcal{F}$. To see this, let $f, g \in S$, and let $a \in \mathbf{R}$. Then $(f + g)''(x) - 2(f + g)(x) = f''(x) - 2f(x) + g''(x) - 2g(x) = 0$, so $f + g \in S$, and $(af)''(x) - 2(af)(x) = a(f''(x) - 2f(x)) = 0$, so $af \in S$. This shows that S is a subspace of $\mathcal{F}$.

Problem 2.

Let $\mathcal{C}$ and $\mathcal{D}$ denote the sets of all continuous and differentiable functions. Then $\mathcal{C}$ and $\mathcal{D}$ are subspaces of $\mathcal{F}$. This is because continuity and differentiability are preserved under addition and scalar multiplication. That is, if f, g are continuous functions, and $a \in \mathbf{R}$, then $f + g$ and af are continuous functions. Similarly, if f, g are differentiable, and $a \in \mathbf{R}$, then $f + g$ and af are differentiable. Since $\mathbf{0} \in \mathcal{C}$, and $\mathbf{0} \in \mathcal{D}$, it follows that $\mathcal{C}$ and $\mathcal{D}$ are subspaces of $\mathcal{F}$.

Problem 3.

Let V be a vector space over a field $\mathbb{F}$, and let $S \subseteq V$ be nonempty. Then S is subspace of V if and only if S is closed under linear combinations.

Proof. Let S be a subspace of V, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v_n} \in S$, let $c_1, c_2, \ldots, c_n \in \mathbf{R}$, and let $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v}_n$. If $\mathbf{u}_i = c_i\mathbf{v}_i$, then $\mathbf{u}_i \in S$ for $i = 1$ to n because subspaces are closed under scalar multiplication. Furthermore, $\mathbf{v} = \mathbf{u}_1 + \mathbf{u_2} + \cdots + \mathbf{u}_n$; therefore, $\mathbf{v} \in S$ because subspaces are closed under addition (we have to apply this inductively). Thus, S is closed under linear combinations.

Suppose that S is closed under linear combinations. If $\mathbf{v}_1, \mathbf{v}_2 \in S$, then $\mathbf{v}_1 + \mathbf{v}_2$ is a linear combination of elements of S and therefore also in S. Similarly, if $a \in \mathbb{F}$, then $a\mathbf{v_1}$ is a linear combination of elements of S and therefore also in S. Hence, S is a subspace of V. $\square$

Problem 4.

Let S_1 and S_2 be subspaces of a vector space V over a field $\mathbb{F}$. Then $S_1 \cap S_2$ is a subspace of V.

Proof. Since S_1 and S_2 are subspaces they both contain $\mathbf{0} \in S$. Therefore, $\mathbf{0} \in S_1 \cap S_2$, so $S_1 \cap S_2$ is nonempty.

Let $\mathbf{v}_1, \mathbf{v}_2 \in S_1 \cap S_2$, and let $a \in \mathbb{F}$. Then for $i \in \{1, 2\}$, $\mathbf{v}_1, \mathbf{v}_2 \in S_i$, so $\mathbf{v}_1 + \mathbf{v}_2 \in S_i$, and $a\mathbf{v}_1 \in S_i$ because S_i is subspace of V. Therefore, $\mathbf{v}_1 + \mathbf{v_2} \in S_1 \cap S_2$, and $a\mathbf{v}_1 \in S_1 \cap S_2$.

This shows that $S_1 \cap S_2$ is a subspace of V. □

Problem 5.

1. Let $S = \{1 + x^2, 1 + x, x^2 - x\} \subseteq \mathcal{P}^2$. Then S is linearly dependent because

   $$c_1(x^2 - x) + c_2(1 + x^2) + c_3(1 + x) = 0$$

   if we choose $c_1 = 1$, $c_2 = -1$, and $c_3 = 1$. Furthermore, $x^2 - x = (1)(1 + x^2) + (-1)(1 + x)$.

2. Let $S = \{\cos(x), \sin(x), 1\} \subseteq \mathcal{F}$. Then S is linearly independent. To see this, let $c_1, c_2, c_3 \in \mathbf{R}$, and suppose that

   $$c_1 \cos(x) + c_2 \sin(x) + c_3 = 0.$$

   This must be true for all $x \in \mathbf{R}$ by the definition of scalar multiplication and addition in $\mathcal{F}$; in particular, if $x = 0$, then we obtain $c_1 = -c_3$, and if $x = \pi$, then we obtain $c_1 = c_3$. Thus, $c_1 = c_3 = 0$. This means that $c_2 \sin(x) = 0$. Taking $x = \frac{\pi}{2}$, we get $c_2 = 0$. Thus, $c_1 = c_2 = c_3 = 0$. This means that S is linearly independent.

Problem 6.

Let W_1 and W_2 be subspaces of a vector space R of dimensions $\dim(W_1) = m$ and $\dim(W_2) = p$. Define

$$W_1 + W_2 = \{\mathbf{w}_1 + \mathbf{w}_2 : \mathbf{w}_1 \in W_1 \text{ and } \mathbf{w}_2 \in W_2\}.$$

Then $W_1 + W_2$ is a subspace of R, and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

Proof. We begin by showing that $W_1 + W_2$ is a subspace. Noting that $\mathbf{0} \in W_1$ and $\mathbf{0} \in W_2$, it follows that $\mathbf{0} + \mathbf{0} \in W_1 + W_2$, so $W_1 + W_2$ is nonempty. Now let $\mathbf{v}_1, \mathbf{v}_2 \in W_1 + W_2$, and let $a \in \mathbb{F}$, the field for R. Then there exist $\mathbf{w}_{11} \in W_1, \mathbf{w}_{12} \in W_2$, and $\mathbf{w}_{21} \in W_1, \mathbf{w}_{22} \in W_2$ such that

$$\mathbf{v}_1 = \mathbf{w}_{11} + \mathbf{w}_{12}, \qquad \mathbf{v}_2 = \mathbf{w}_{21} + \mathbf{w}_{22}.$$

Then

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{w}_{11} + \mathbf{w}_{12} + \mathbf{w}_{21} + \mathbf{w}_{22} = (\mathbf{w}_{11} + \mathbf{w}_{21}) + (\mathbf{w}_{12} + \mathbf{w}_{22}) \in W_1 + W_2$$

because W_1 and W_2 are closed under addition. Similarly,

$$a\mathbf{v}_1 = a(\mathbf{w}_{11} + \mathbf{w}_{12}) = a\mathbf{w}_{11} + a\mathbf{w}_{12} \in W_1 + W_2$$

because W_1 and W_2 are closed under scalar multiplication. Thus, $W_1 + W_2$ is a subspace of R.

Since $W_1 \cap W_2$ is a subspace of W_1, it must have a lower dimension because any basis for W_1 would span $W_1 \cap W_2 \subseteq W_1$. Similarly, $\dim(W_1 \cap W_2) \leq W_2$, as well. Let $k = \dim(W_1 \cap W_2)$. Then $k \leq m$, and $k \leq p$.

There is a basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for $W_1 \cap W_2$. We would like to extend this basis with vectors $\{\mathbf{w}_{11}, \mathbf{w}_{21}, \ldots, \mathbf{w}_{(m-k)1}\} \subseteq W_1$ to form a basis for W_1. This can be done inductively. If $m = k$, then it is trivial. Otherwise, $m > k$, so B does not span W_1. This implies that there is a vector $\mathbf{w}_{11} \in W_1$ that cannot be expressed as a linear combination of elements of B; thus, if

$$c_0\mathbf{w}_{11} + c_1\mathbf{v}_1 + \cdots + \mathbf{v}_k = 0,$$

then $c_0 = 0$, because $c_0 \neq 0$ implies that $\mathbf{w}_{11} \in \operatorname{span}(B)$, contrary to our assumption. Since $c_0 = 0$, it follows that $c_1 = c_2 = \cdots = c_k = 0$ by the linear independence of B. This shows that $B \cup \{\mathbf{w}_{11}\}$ is linearly independent. We can repeat this argument inductively to construct a linearly independent set $C = B \cup \{\mathbf{w}_{11}, \mathbf{w}_{21}, \ldots, \mathbf{w}_{(m-k)1}\} \subseteq W_1$.

If there were another vector $\mathbf{w} \in W_1$ not in $\operatorname{span}(C)$, then we could add $\mathbf{w}$ to C to obtain a linearly independent subset of W_1 of size $m+1$, which is impossible. Therefore, C spans W_1 and must be a basis for W_1.

In a similar way, we can choose vectors $\{\mathbf{w}_{21}, \mathbf{w}_{22}, \ldots, \mathbf{w}_{(p-k)2}\}$ such that $D = B \cup \{\mathbf{w}_{21}, \mathbf{w}_{22}, \ldots, \mathbf{w}_{(p-k)2}\}$ is a basis for W_2.

We claim that $C \cup D$ is a basis for $W_1 + W_2$. To see that $C \cup D$ is linearly independent, suppose there were coefficients $c_{10}, \ldots c_{k0}, c_{11}, \ldots c_{(m-k)1}, c_{12}, \ldots, c_{(p-k)2}$ such that

$$c_{10}\mathbf{v}_1 + \cdots + c_{k0}\mathbf{v}_k + c_{11}\mathbf{w}_{11} + \ldots c_{(m-k)1}\mathbf{w}_{(m-k)1} + c_{12}\mathbf{w}_{12} + \cdots + c_{(p-k)2}\mathbf{w}_{(p-k)2} = 0. \tag{1}$$

Then we would have $\mathbf{w} = c_{12}\mathbf{w}_{12} + \cdots + c_{(p-k)2}\mathbf{w}_{(p-k)w} \in W_1$. Since $\mathbf{w} \in W_2$ by construction, we would have $\mathbf{w} \in W_1 \cap W_2$ Then there would be coefficients $d_1, \ldots d_k$ such that $\mathbf{w} = d_1\mathbf{v}_1 + \cdots + d_k\mathbf{v}_k$. Substituting into (1), we would have

$$(c_{10} + d_1)\mathbf{v}_1 + \cdots + (c_{k0} + d_k)\mathbf{v}_k + c_{11}\mathbf{w}_{11} + \ldots c_{(m-k)1}\mathbf{w}_{(m-k)1} = 0,$$

which implies that $c_{11} = c_{21} = \cdots = c_{(m-k)1} = 0$ by the linear independence of C. Substituting this into (1) gives

$$c_{10}\mathbf{v}_1 + \cdots + c_{k0}\mathbf{v}_k + c_{12}\mathbf{w}_{12} + \cdots + c_{(p-k)2}\mathbf{w}_{(p-k)2} = 0,$$

from which we deduce that $c_{10} = c_{20} = \cdots = c_{k0} = c_{12} = c_{22} = \cdots = c_{(p-k)2} = 0$ by the linear independence of D. Therefore, $C \cup D$ is linearly independent.

To see that $C \cup D$ spans $W_1 + W_2$, let $\mathbf{w}_1 + \mathbf{w}_2 \in W_1 + W_2$, where $\mathbf{w}_1 \in W_1$, and $\mathbf{w}_2 \in W_2$. Then, because C is a basis for W_1 and D is a basis for W_2, there exist coefficients $c_{101}, c_{201}, \ldots, c_{k01}, c_{102}, c_{202}, \ldots, c_{k02}$, $c_{11}, c_{21}, \ldots c_{(m-k)1}$, and $c_{12}, c_{22}, \ldots c_{(p-k)2}$ such that

$$\begin{aligned}
\mathbf{w}_1 &= c_{101}\mathbf{v}_1 + \cdots + c_{k01}\mathbf{v}_k + c_{11}\mathbf{w}_{11} + \cdots + c_{(m-k)1}\mathbf{w}_{(m-k)1}, \\
\mathbf{w}_2 &= c_{102}\mathbf{v}_2 + \cdots + c_{k02}\mathbf{v}_k + c_{12}\mathbf{w}_{12} + \cdots + c_{(m-k)2}\mathbf{w}_{(m-k)2},
\end{aligned}$$

which implies that

$$\mathbf{w}_1 + \mathbf{w}_2 = (c_{101} + c_{102})\mathbf{v}_1 + \cdots + (c_{k01} + c_{k02})\mathbf{v}_k + c_{11}\mathbf{w}_11 + \cdots + c_{(m-k)1}\mathbf{w}_{(m-k)1} + c_{12}\mathbf{w}_{12} + \cdots + c_{(p-k)2}\mathbf{w}_{(p-k)2},$$

which is in the $\operatorname{span}(C \cup D)$. Since $\mathbf{w}_1 + \mathbf{w}_2 \in W_1 + W_2$ was arbitrary, it follows that $W_1 + W_2 \subseteq C \cup D$.

On the other hand, if $\mathbf{w} \in C \cup D$, then there exist $c_{10}, \ldots c_{k0}$, $c_{11}, \ldots, c_{(m-k)1}$, and $c_{12}, \ldots, c_{(p-k)2}$ such that

$$\mathbf{w} = (c_{10}\mathbf{v}_1 + \cdots + c_{k0}\mathbf{v}_k + c_{11}\mathbf{w}_{11} + \cdots + c_{(m-k)1}\mathbf{w}_{(m-k)1}) + (c_{12}\mathbf{w}_{12} + \cdots + c_{(p-k)2}\mathbf{w}_{(p-k)2}) \in W_1 + W_2$$

by the construction of of the vectors $\mathbf{w}_{ij}$. Thus, $\operatorname{span}(C \cup D) \subseteq W_1 + W_2$. Then $\operatorname{span}(C \cup D) = W_1 + W_2$.

This shows that $C \cup D$ is a basis for $W_1 + W_2$. Since $|C \cup D| = k + m - k + p - k = m + p - k$, it follows that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

$\square$

Problem 7.

1. The set
$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$
is not a basis for $\mathcal{M}_2$ because it does not span $\mathcal{M}_2$. Indeed, the vector
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
cannot be expressed as linear combination of elements of B. To see why, suppose that there were scalars c_1, c_2, c_3 such that
$$A = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$
From the top left component of both sides we get $c_1+c_2+c_3 = 1$, but from the bottom right component we get $c_1 + c_2 + c_3 = 0$. This is a contradiction, so $A \notin \operatorname{span}\{B\}$.

Problem 8.

Let S be the set of all symmetric matrices in $\mathbf{R}^{n\times n}$.

1. S is a subspace of $\mathbf{R}^{n\times n}$.

   Proof. Clearly, S is nonempty (it contains, for example, the zero matrix). Let $A, B \in S$, and let $a \in \mathbf{R}$. Then $(A + B)^T = A^T + B^T = A + B$, so $A + B \in S$. Furthermore, $(aA)^T = aA^T$, so $aA \in S$. This shows that S is a subspace of $\mathbf{R}^{n\times n}$. $\square$

2. As we did in class, let E_{ij} denote the matrix with all zero entries except for the entry in the ith row and jth column, which is 1. For $1 \le i \le j < n$, define $A_{ij} = E_{ij} + E_{ji}$. Then $B = \{A_{ij} \mid 1 \le i \le j \le n\}$ is a basis for S.

   Proof. We need to show that 1. $\operatorname{span} B = S$ and 2. B is linearly independent.

   (a) Since $E_{ij}^T = E_{ji}$ for all $i, j = 1, \dots, n$, it follows that $A_{ij}^T = E_{ij}^T + E_{ji}^T = E_{ji} + E_{ij} = A_{ij}$, so $A_{ij} \in S$ for all $1 \le i \le j \le n$. Thus, $\operatorname{span}(B) \subseteq S$, as subspaces are closed under linear combination by Problem 3. On the other hand, let $C = (c_{ij}) \in S$. Then
   $$\sum_{1\le i\le j\le n} \alpha_{ij} c_{ij} A_{ij} = \sum_{1\le i<j\le n} c_{ij}(E_{ij} + E_{ji}) + \sum_{i=1}^{n} c_{ii} E_{ii} = \sum_{i,j=1}^{n} c_{ij} E_{ij} = C,$$
   where $\alpha_{ij} = 1$ if $i < j$, and $\alpha_{ii} = \frac{1}{2}$, for $1 \le i \le n$. This shows that $C \in \operatorname{span}(B)$. Since C was arbitrary, it follows that $S \subseteq \operatorname{span}(B)$. Therefore, $S = \operatorname{span}(B)$.

   (b) Suppose that
   $$\sum_{1\le i\le j\le n} c_{ij} A_{ij} = 0$$
   for some $c_{ij} \in \mathbf{R}$, for $1 \le i \le j \le n$. Then
   $$\sum_{1\le i<j\le n} c_{ij}(E_{ij} + E_{ji}) + \sum_{i=1}^{n} 2c_{ii} E_{ii} = 0.$$
   Since the entry in the ith row and jth column on the left side is c_{ij} if $i < j$ and $2c_{ij}$ if $i = j$, it follows that $c_{ij} = 0$ for $1 \le i \le j \le n$. Therefore, B is linearly independent.

□

Problem 9.

The polynomials 2, $1+t$, $t+t^2$ form a basis for $\mathcal{P}^2$.

Proof. Let $S = \{2, 1+t, t+t^2\}$. We need to show that $\operatorname{span}(S) = \mathcal{P}^2$ and that S is linearly independent.

1. Each element of S is an element of $\mathcal{P}^2$. Thus, every linear combination of elements of S is an element of $\mathcal{P}^2$ (by the same argument used in Problem 3.). This implies that $\operatorname{span}(S) \subseteq \mathcal{P}^2$. On the other hand, let $p_0 + p_1 t + p_2 t^2 \in \mathcal{P}^2$ be an arbitrary element. Then

$$p_0 + p_1 t + p_2 t^2 = p_2(t+t^2) + (p_1 - p_2)(1+t) + \frac{p_0 - p_1 + p_2}{2}(2),$$

   so $p_0 + p_1 t + p_2 t^2 \in \operatorname{span}(S)$. This implies that $\mathcal{P}^2 \subseteq \operatorname{span}(S)$. Therefore, $\operatorname{span}(S) = \mathcal{P}^2$.

2. Let $c_1, c_2, c_3 \in \mathbf{R}$, and suppose that

$$0 = c_1(2) + c_2(1+t) + c_3(t+t^2) = c_3 t^3 + (c_2 + c_3)t + (2c_1 + c_2).$$

   Since this equation holds for all t, it follows (by, say, differentiation) that the coefficients of each power of t are equal on both sides. In particular, we must have $c_3 = 0$, $c_2 + c_3 = 0$, and $2c_1 + c_2 = 0$. Together these equations imply that $c_1 = c_2 = c_3 = 0$. Hence, S is linearly independent.

This shows that S is a basis for $\mathcal{P}^2$. □

The coordinates of $3 + t + 2t^2$ in S is the vector $\mathbf{c} \in \mathbf{R}^3$ such that $\mathbf{c} = (c_1, c_2, c_3)^T$, and

$$3 + t + 2t^2 = c_1(2) + c_2(1+t) + c_3(t+t^2).$$

Equating the coefficients of the powers of t on both sides gives

$$\begin{aligned} 2 &= \phantom{2c_1 + c_2 +{}} c_3 \\ 1 &= \phantom{2c_1 +{}} c_2 + c_3 \\ 3 &= 2c_1 + c_2, \end{aligned}$$

from which we deduce that $c_3 = 2$, $c_2 = 1 - 2 = -1$, and $c_1 = \frac{3+1}{2} = 2$. That is, $\mathbf{c} = (2, -1, 2)^T$.