

Math 5604 Homework 8

Jacob Hauck

April 24, 2024

Problem 1.

Consider the BVP

$$\begin{aligned}\Delta u &= -2\pi^2 \sin(\pi(x+y)) = f(x,y), & 0 < x < 1, \quad 0 < y < 1 \\ u(0,y) &= \sin(\pi y) = g_\ell(y), \quad u(1,y) = \sin(\pi(1+y)) = g_r(y), & 0 \leq y \leq 1 \\ u(x,0) &= \sin(\pi x) = g_b(x), \quad u(x,1) = \sin(\pi(1+x)) = g_t(x), & 0 \leq x \leq 1.\end{aligned}$$

The exact solution of this equation is given by $u(x,y) = \sin(\pi(x+y))$.

- (a) Consider a grid of sample points $\{(x_i, y_j)\}$ on the domain $[0,1]^2$, where $i = 0, 1, \dots, M$, and $j = 0, 1, \dots, N$. If the points are evenly spaced horizontally by $h_x = \frac{1}{M}$ and vertically by $h_y = \frac{1}{N}$, then $x_i = ih_x$, and $y_j = jh_y$.

We approximate $u(x_i, y_j)$ by $u_{i,j}$. Using a centered-difference scheme to approximate Δu on the interior and applying the boundary conditions on the boundary points, we are led to the numerical scheme

$$\begin{aligned}\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} &= f(x_i, y_j), \\ i &= 1, 2, \dots, M-1, \quad j = 1, 2, \dots, N-1,\end{aligned}\tag{1}$$

and

$$\begin{aligned}u_{0,j} &= g_\ell(y_j), & u_{M,j} &= g_r(y_j), & j &= 0, 1, \dots, N, \\ u_{i,0} &= g_b(x_i), & u_{i,N} &= g_t(x_i), & i &= 0, 1, \dots, M.\end{aligned}$$

In order to solve this linear system, we need to reshape the matrix of unknowns $\{u_{i,j}\}_{i=1,j=1}^{M-1,N-1}$ into a vector U and rewrite the corresponding equations (1) as a matrix-vector system, substituting the known boundary values where applicable.

We use row-wise ordering to reshape the matrix of unknowns; that is, we define the block vector of rows of the unknown matrix

$$U = \begin{bmatrix} U^{(1)} \\ U^{(2)} \\ \vdots \\ U^{(N-1)} \end{bmatrix}, \quad U^{(j)} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{M-1,j} \end{bmatrix}, \quad j = 1, 2, \dots, N-1.$$

We can rewrite the equations (1) into a matrix-vector system, expressing the matrix A and vector b in block form corresponding to the blocks of U :

$$A = \begin{bmatrix} A^{(1,1)} & \dots & A^{(1,N-1)} \\ \vdots & \ddots & \vdots \\ A^{(N-1,1)} & \dots & A^{(N-1,N-1)} \end{bmatrix}, \quad b = \begin{bmatrix} b^{(1)} \\ b^{(2)} \\ \vdots \\ b^{(N-1)} \end{bmatrix}.$$

We remark that the block $A^{(j,j')}$ expresses the dependence of equations $(1, j), (2, j), \dots, (M-1, j)$ on the unknowns in row j' of the unknown matrix $\{u_{i,j}\}$.

We construct A and b one block row at a time. Consider the blocks $A^{(1,j')}$ for $j' = 1, 2, \dots, N-1$, the first row of blocks of A . As mentioned, these blocks correspond to equations $(1, 1), (2, 1), \dots, (M-1, 1)$. Substituting in boundary conditions, we see that

$$\begin{aligned} (1, 1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right)u_{1,1} + \frac{1}{h_x^2}u_{2,1} + \frac{1}{h_y^2}u_{1,2} = f(x_1, y_1) - \frac{g_\ell(y_1)}{h_x^2} - \frac{g_b(x_1)}{h_y^2} \\ i=2, \dots, M-2 &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right)u_{i,1} + \frac{1}{h_x^2}u_{i-1,1} + \frac{1}{h_x^2}u_{i+1,1} + \frac{1}{h_y^2}u_{i,2} = f(x_i, y_1) - \frac{g_b(x_i)}{h_y^2} \\ (M-1, 1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right)u_{M-1,1} + \frac{1}{h_x^2}u_{M-2,1} + \frac{1}{h_y^2}u_{M-1,2} = f(x_{M-1}, y_1) - \frac{g_r(y_1)}{h_x^2} - \frac{g_b(x_{M-1})}{h_y^2}. \end{aligned}$$

Thus, each equation depends only on rows 1 and 2 of the matrix $\{u_{i,j}\}$, so only blocks $A^{(1,1)}$ and $A^{(1,2)}$ are nonzero. Examining these dependencies, we get

$$A^{(1,1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ & & \ddots & & \\ & & & \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \quad A^{(1,2)} = \begin{bmatrix} \frac{1}{h_y^2} & & & \\ & \ddots & & \\ & & & \frac{1}{h_y^2} \end{bmatrix},$$

where blanks indicate zero entries. The block $b^{(1)}$ corresponding to the right hand sides of equations $(1, 1), (2, 1), \dots, (M-1, 1)$ we can read off easily:

$$b^{(1)} = \begin{bmatrix} f(x_1, y_1) - \frac{g_\ell(y_1)}{h_x^2} - \frac{g_b(x_1)}{h_y^2} \\ f(x_2, y_1) - \frac{g_b(x_2)}{h_y^2} \\ f(x_3, y_1) - \frac{g_b(x_3)}{h_y^2} \\ \vdots \\ f(x_{M-2}, y_1) - \frac{g_b(x_{M-2})}{h_y^2} \\ f(x_{M-1}, y_1) - \frac{g_r(y_1)}{h_x^2} - \frac{g_b(x_{M-1})}{h_y^2} \end{bmatrix}.$$

Now consider the blocks $A^{(j,j')}$ for $j = 2, 3, \dots, N-2$, and $j' = 1, 2, \dots, N-1$. These correspond to equations $(1, j), (2, j), \dots, (M-1, j)$. Substituting boundary conditions, we have

$$\begin{aligned} (1, j) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right)u_{1,j} + \frac{1}{h_x^2}u_{2,j} + \frac{1}{h_y^2}u_{1,j-1} + \frac{1}{h_y^2}u_{1,j+1} = f(x_1, y_j) - \frac{g_\ell(y_j)}{h_x^2} \\ i=2, \dots, M-2 &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right)u_{i,j} + \frac{1}{h_x^2}u_{i-1,j} + \frac{1}{h_x^2}u_{i+1,j} + \frac{1}{h_y^2}u_{i,j-1} + \frac{1}{h_y^2}u_{i,j+1} = f(x_i, y_j) \\ (M-1, j) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right)u_{M-1,j} + \frac{1}{h_x^2}u_{M-2,j} + \frac{1}{h_y^2}u_{M-1,j-1} + \frac{1}{h_y^2}u_{M-1,j+1} = f(x_{M-1}, y_j) - \frac{g_r(y_j)}{h_x^2}. \end{aligned}$$

Thus, each equation depends only on rows $j-1, j$, and $j+1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(j,j-1)}$, $A^{(j,j)}$, and $A^{(j,j+1)}$ are nonzero. Examining these dependencies gives

$$A^{(j,j)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ & & \ddots & & \\ & & & \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \quad A^{(j,j-1)} = A^{(j,j+1)} = \begin{bmatrix} \frac{1}{h_y^2} & & & \\ & \ddots & & \\ & & & \frac{1}{h_y^2} \end{bmatrix}.$$

The block $b^{(j)}$ can be read off from the right hand sides easily:

$$b^{(j)} = \begin{bmatrix} f(x_1, y_j) - \frac{g_\ell(y_j)}{h_x^2} \\ f(x_2, y_j) \\ f(x_3, y_j) \\ \vdots \\ f(x_{M-2}, y_j) \\ f(x_{M-1}, y_j) - \frac{g_r(y_j)}{h_x^2} \end{bmatrix}.$$

Finally, consider the blocks $A^{(N-1, j')}$, $j' = 1, 2, \dots, N-1$. These correspond to equations $(1, N-1), (2, N-1), \dots, (M-1, N-1)$. Substituting boundary conditions, we have

$$\begin{aligned} (1, N-1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1, N-1} + \frac{1}{h_x^2} u_{2, N-1} + \frac{1}{h_y^2} u_{1, N-2} = f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \\ (i, N-1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i, N-1} + \frac{1}{h_x^2} u_{i-1, N-1} + \frac{1}{h_x^2} u_{i+1, N-1} + \frac{1}{h_y^2} u_{i, N-2} = f(x_i, y_{N-1}) - \frac{g_t(x_i)}{h_y^2} \\ (M-1, N-1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M-1, N-1} + \frac{1}{h_x^2} u_{M-2, N-1} + \frac{1}{h_y^2} u_{M-1, N-2} \\ &= f(x_{M-1}, y_{N-1}) - \frac{g_r(y_{N-1})}{h_x^2} - \frac{g_t(x_{M-1})}{h_y^2}. \end{aligned}$$

Thus, each equation depends only on rows $N-2$ and $N-1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(N-1, N-2)}$ and $A^{(N-1, N-1)}$ are nonzero. Examining these dependencies gives

$$A^{(N-1, N-1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ & & \ddots & & \\ & & & \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \quad A^{(N-1, N-2)} = \begin{bmatrix} \frac{1}{h_y^2} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \frac{1}{h_y^2} \end{bmatrix}.$$

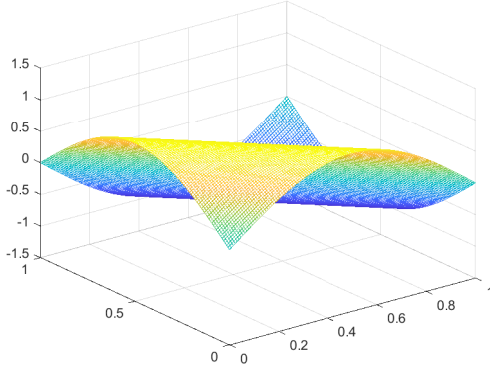
The block $b^{(N-1)}$ can be read off from the right hand sides of the equations easily:

$$b^{(N-1)} = \begin{bmatrix} f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \\ f(x_2, y_{N-1}) - \frac{g_t(x_2)}{h_y^2} \\ f(x_3, y_{N-1}) - \frac{g_t(x_3)}{h_y^2} \\ \vdots \\ f(x_{M-2}, y_{N-1}) - \frac{g_t(x_{M-2})}{h_y^2} \\ f(x_{M-1}, y_{N-1}) - \frac{g_r(y_{N-1})}{h_x^2} - \frac{g_t(x_{M-1})}{h_y^2} \end{bmatrix}.$$

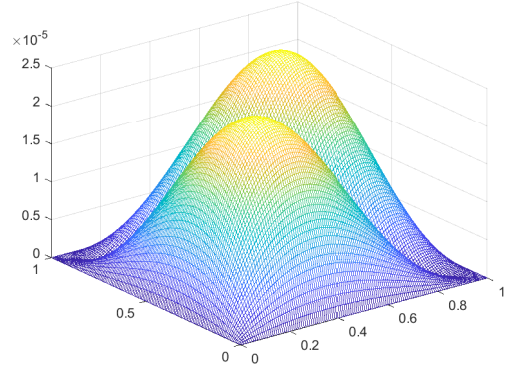
Therefore, the entire system of equations (i, j) is equivalent to the matrix-vector equation $AU = b$.

- (b) See `problem1.m` for the implementation of the scheme in (a).
- (c) See Figure 1 for the mesh plots of the numerical solution and the error function.

Problem 2.



(a) Numerical solution



(b) Numerical error function

Figure 1: Numerical solution and error function mesh plots with $h_x = h_y = \frac{1}{128}$

Consider the BVP

$$\begin{aligned} \Delta u &= -2\pi^2 \sin(\pi x) \sin(\pi y) = f(x, y), & 0 < x < 1, \quad 0 < y < 2 \\ u(0, y) &= 2 = g_\ell(y), \quad u(1, y) = 2 = g_r(y), & 0 \leq y \leq 2 \\ u(x, 0) &= 2 = g_b(x), \quad u(x, 1) = 2 = g_t(x), & 0 \leq x \leq 1. \end{aligned}$$

The exact solution of this equation is given by $u(x, y) = 2 + \sin(\pi x) \sin(\pi y)$.

- (a) Consider a grid of sample points $\{(x_i, y_j)\}$ on the domain $[0, 1] \times [0, 2]$, where $i = 0, 1, \dots, M$, and $j = 0, 1, \dots, N$. If the points are evenly spaced horizontally by $h_x = \frac{1}{M}$ and vertically by $h_y = \frac{2}{N}$, then $x_i = ih_x$, and $y_j = jh_y$.

We approximate $u(x_i, y_j)$ by $u_{i,j}$. Using a centered-difference scheme to approximate Δu on the interior and applying the boundary conditions on the boundary points, we are led to the numerical scheme

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} = f(x_i, y_i), \quad (i, j)$$

$$i = 1, 2, \dots, M-1, \quad j = 1, 2, \dots, N-1,$$

and

$$\begin{aligned} u_{0,j} &= g_\ell(y_j), & u_{M,j} &= g_r(y_j), & j &= 0, 1, \dots, N, \\ u_{i,0} &= g_b(x_i), & u_{i,N} &= g_t(x_i), & i &= 0, 1, \dots, M. \end{aligned}$$

In order to solve this linear system, we need to reshape the matrix of unknowns $\{u_{i,j}\}_{i=1,j=1}^{M-1,N-1}$ into a vector U and rewrite the corresponding equations (i, j) as a matrix-vector system, substituting the known boundary values where applicable.

We use column-wise ordering to reshape the matrix of unknowns; that is, we define the block vector of columns of the unknown matrix

$$U = \begin{bmatrix} U^{(1)} \\ U^{(2)} \\ \vdots \\ U^{(M-1)} \end{bmatrix}, \quad U^{(i)} = \begin{bmatrix} u_{i,1} \\ u_{i,2} \\ \vdots \\ u_{i,N-1} \end{bmatrix}, \quad i = 1, 2, \dots, M-1.$$

We can rewrite the equations (i, j) into a matrix-vector system, expressing the matrix A and vector b in block form corresponding to the blocks of U :

$$A = \begin{bmatrix} A^{(1,1)} & \dots & A^{(1,M-1)} \\ \vdots & \ddots & \vdots \\ A^{(M-1,1)} & \dots & A^{(M-1,M-1)} \end{bmatrix}, \quad b = \begin{bmatrix} b^{(1)} \\ b^{(2)} \\ \vdots \\ b^{(M-1)} \end{bmatrix}.$$

We remark that the block $A^{(i,i')}$ expresses the dependence of equations $(i, 1), (i, 2), \dots, (i, N-1)$ on the unknowns in column i' of the unknown matrix $\{u_{i,j}\}$.

We construct A and b one block row at a time. Consider the blocks $A^{(1,i')}$ for $i' = 1, 2, \dots, M-1$, the first row of blocks of A . As mentioned, these blocks correspond to equations $(1, 1), (1, 2), \dots, (1, N-1)$. Substituting in boundary conditions, we see that

$$\begin{aligned} (1, 1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,1} + \frac{1}{h_x^2} u_{2,1} + \frac{1}{h_y^2} u_{1,2} = f(x_1, y_1) - \frac{g_\ell(y_1)}{h_x^2} - \frac{g_b(x_1)}{h_y^2} \\ j=2, \dots, N-2 &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,j} + \frac{1}{h_x^2} u_{2,j} + \frac{1}{h_y^2} u_{1,j-1} + \frac{1}{h_y^2} u_{1,j+1} = f(x_1, y_j) - \frac{g_\ell(y_j)}{h_x^2} \\ (1, N-1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,N-1} + \frac{1}{h_x^2} u_{2,N-1} + \frac{1}{h_y^2} u_{1,N-2} = f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2}. \end{aligned}$$

Thus, each equation depends only on columns 1 and 2 of the matrix $\{u_{i,j}\}$, so only blocks $A^{(1,1)}$ and $A^{(1,2)}$ are nonzero. Examining these dependencies, we get

$$A^{(1,1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & & & \\ \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & & \\ & & \ddots & & \\ & & & \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \quad A^{(1,2)} = \begin{bmatrix} \frac{1}{h_x^2} & & & \\ & \ddots & & \\ & & & \frac{1}{h_x^2} \end{bmatrix},$$

where blanks indicate zero entries. The block $b^{(1)}$ corresponding to the right hand sides of equations $(1, 1), (1, 2), \dots, (1, N-1)$ we can read off easily:

$$b^{(1)} = \begin{bmatrix} f(x_1, y_1) - \frac{g_\ell(y_1)}{h_x^2} - \frac{g_b(x_1)}{h_y^2} \\ f(x_1, y_2) - \frac{g_\ell(y_2)}{h_x^2} \\ f(x_1, y_3) - \frac{g_\ell(y_3)}{h_x^2} \\ \vdots \\ f(x_1, y_{N-2}) - \frac{g_\ell(y_{N-2})}{h_x^2} \\ f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \end{bmatrix}.$$

Now consider the blocks $A^{(i,i')}$ for $i = 2, 3, \dots, N-2$, and $i' = 1, 2, \dots, N-1$. These correspond to equations $(i, 1), (i, 2), \dots, (i, N-1)$. Substituting boundary conditions, we have

$$\begin{aligned} (i, 1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,1} + \frac{1}{h_x^2} u_{i-1,1} + \frac{1}{h_x^2} u_{i+1,1} + \frac{1}{h_y^2} u_{i,2} = f(x_i, y_1) - \frac{g_b(x_i)}{h_y^2} \\ j=2, \dots, N-2 &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,j} + \frac{1}{h_x^2} u_{i-1,j} + \frac{1}{h_x^2} u_{i+1,j} + \frac{1}{h_y^2} u_{i,j-1} + \frac{1}{h_y^2} u_{i,j+1} = f(x_i, y_j) \\ (i, N-1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,N-1} + \frac{1}{h_x^2} u_{i-1,N-1} + \frac{1}{h_x^2} u_{i+1,N-1} + \frac{1}{h_y^2} u_{i,N-2} = f(x_i, y_{N-1}) - \frac{g_t(x_i)}{h_y^2}. \end{aligned}$$

Thus, each equation depends only on columns $i - 1$, i , and $i + 1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(i,i-1)}$, $A^{(i,i)}$, and $A^{(i,i+1)}$ are nonzero. Examining these dependencies gives

$$A^{(i,i)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & & \\ \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & \\ & & \ddots & \\ & & \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \quad A^{(i,i-1)} = A^{(i,i+1)} = \begin{bmatrix} \frac{1}{h_x^2} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \frac{1}{h_x^2} \end{bmatrix}.$$

The block $b^{(i)}$ can be read off from the right hand sides easily:

$$b^{(i)} = \begin{bmatrix} f(x_i, y_1) - \frac{g_b(x_i)}{h_y^2} \\ f(x_i, y_2) \\ f(x_i, y_3) \\ \vdots \\ f(x_i, y_{N-2}) \\ f(x_i, y_{N-1}) - \frac{g_t(x_i)}{h_y^2} \end{bmatrix}.$$

Finally, consider the blocks $A^{(M-1,i')}$, $i' = 1, 2, \dots, M - 1$. These correspond to equations $(M - 1, 1), (M - 1, 2), \dots, (M - 1, N - 1)$. Substituting boundary conditions, we have

$$\begin{aligned} (M - 1, 1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M-1,1} + \frac{1}{h_x^2} u_{M-2,1} + \frac{1}{h_y^2} u_{M-1,2} = f(x_1, y_{N-1}) - \frac{g_r(y_1)}{h_x^2} - \frac{g_b(x_{M-1})}{h_y^2} \\ (M-1, j)_{j=2, \dots, N-2} &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M-1,j} + \frac{1}{h_x^2} u_{M-2,j} + \frac{1}{h_y^2} u_{M-1,j-1} + \frac{1}{h_y^2} u_{M-1,j+1} = f(x_{M-1}, y_j) - \frac{g_r(y_j)}{h_x^2} \\ (M - 1, N - 1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M-1,N-1} + \frac{1}{h_x^2} u_{M-2,N-1} + \frac{1}{h_y^2} u_{M-1,N-2} \\ &= f(x_{M-1}, y_{N-1}) - \frac{g_r(y_{N-1})}{h_x^2} - \frac{g_t(x_{M-1})}{h_y^2}. \end{aligned}$$

Thus, each equation depends only on columns $M - 2$ and $M - 1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(M-1,M-2)}$ and $A^{(M-1,M-1)}$ are nonzero. Examining these dependencies gives

$$A^{(N-1,N-1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & & \\ \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & \\ & & \ddots & \\ & & \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \quad A^{(N-1,N-2)} = \begin{bmatrix} \frac{1}{h_x^2} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \frac{1}{h_x^2} \end{bmatrix}.$$

The block $b^{(M-1)}$ can be read off from the right hand sides of the equations easily:

$$b^{(M-1)} = \begin{bmatrix} f(x_{M-1}, y_1) - \frac{g_r(y_1)}{h_x^2} - \frac{g_b(x_{M-1})}{h_y^2} \\ f(x_{M-1}, y_2) - \frac{g_r(y_2)}{h_x^2} \\ f(x_{M-1}, y_3) - \frac{g_r(y_3)}{h_x^2} \\ \vdots \\ f(x_{M-1}, y_{N-2}) - \frac{g_r(y_{N-2})}{h_x^2} \\ f(x_{M-1}, y_{N-1}) - \frac{g_r(y_{N-1})}{h_x^2} - \frac{g_t(x_{M-1})}{h_y^2} \end{bmatrix}.$$

Therefore, the entire system of equations (i, j) is equivalent to the matrix-vector equation $AU = b$.

(b) See `problem2.m` for the implementation of the scheme in (a).

Problem 3.

Consider the BVP

$$\begin{aligned}\Delta u &= -2\pi^2 \sin(\pi(x+y)) = f(x,y), & 0 < x < 1, \quad 0 < y < 1 \\ u(0,y) &= \sin(\pi y) = g_\ell(y), \quad u_x(1,y) = \pi \cos(\pi(1+y)) = g_r(y), & 0 \leq y \leq 1 \\ u_y(x,0) &= \pi \cos(\pi x) = g_b(x), \quad u(x,1) = \sin(\pi(1+x)) = g_t(x), & 0 \leq x \leq 1.\end{aligned}$$

The exact solution of this equation is given by $u(x,y) = \sin(\pi(x+y))$.

(a) Consider a grid of sample points $\{(x_i, y_j)\}$ on the domain $[0,1]^2$, where $i = 0, 1, \dots, M$, and $j = 0, 1, \dots, N$. If the points are evenly spaced horizontally by $h_x = \frac{1}{M}$ and vertically by $h_y = \frac{1}{N}$, then $x_i = ih_x$, and $y_j = jh_y$.

We approximate $u(x_i, y_j)$ by $u_{i,j}$. For the Neumann boundary conditions on the right and bottom boundaries, we use a centered difference scheme with ghost points to approximate the derivatives, and extend the PDE to the boundaries to eliminate the ghost points, as follows:

$$\frac{u_{M+1,j} - u_{M-1,j}}{2h_x} = g_r(y_j), \quad j = 1, 2, \dots, N-1, \quad \frac{u_{i,1} - u_{i,-1}}{2h_y} = g_b(x_i), \quad i = 1, 2, \dots, M-1.$$

Using a centered-difference scheme to approximate Δu on the interior and the points on the parts of the boundary that have a Neumann condition, we obtain the following scheme:

$$\begin{aligned}\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} &= f(x_i, y_j), \\ i = 1, 2, \dots, M, \quad j = 0, 2, \dots, N-1.\end{aligned} \tag{i,j}$$

For the left and top boundaries, we have Dirichlet boundary conditions, giving

$$u_{0,j} = g_\ell(y_j), \quad j = 0, 1, \dots, N, \quad u_{i,N} = g_t(x_i), \quad i = 0, 1, \dots, M.$$

In order to solve this linear system, we need to reshape the matrix of unknowns $\{u_{i,j}\}_{i=1,j=0}^{M,N-1}$ into a vector U and rewrite the corresponding equations (i,j) as a matrix-vector system, substituting the known boundary values and ghost point relationships where applicable.

We use row-wise ordering to reshape the matrix of unknowns; that is, we define the block vector of rows of the unknown matrix

$$U = \begin{bmatrix} U^{(0)} \\ U^{(1)} \\ \vdots \\ U^{(N-1)} \end{bmatrix}, \quad U^{(j)} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{M,j} \end{bmatrix}, \quad j = 0, 1, \dots, N-1.$$

We can rewrite the equations (i,j) into a matrix-vector system, expressing the matrix A and vector b in block form corresponding to the blocks of U :

$$A = \begin{bmatrix} A^{(0,0)} & \dots & A^{(0,N-1)} \\ \vdots & \ddots & \vdots \\ A^{(N-1,0)} & \dots & A^{(N-1,N-1)} \end{bmatrix}, \quad b = \begin{bmatrix} b^{(0)} \\ b^{(1)} \\ \vdots \\ b^{(N-1)} \end{bmatrix}.$$

We remark that the block $A^{(j,j')}$ expresses the dependence of equations $(1,j), (2,j), \dots, (M,j)$ on the unknowns in row j' of the unknown matrix $\{u_{i,j}\}$.

We construct A and b one block row at a time. Consider the blocks $A^{(0,j')}$ for $j' = 1, 2, \dots, N-1$, the first row of blocks of A . As mentioned, these blocks correspond to equations $(1,0), (2,0), \dots, (M,0)$. Substituting in boundary conditions, we see that

$$\begin{aligned} (1,0) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,0} + \frac{1}{h_x^2} u_{2,0} + \frac{2}{h_y^2} u_{1,1} = f(x_1, y_0) - \frac{g_\ell(y_0)}{h_x^2} + \frac{2}{h_y} g_b(x_1) \\ (i,0)_{i=2,\dots,M-1} &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,0} + \frac{1}{h_x^2} u_{i-1,0} + \frac{1}{h_x^2} u_{i+1,0} + \frac{2}{h_y^2} u_{i,1} = f(x_i, y_0) + \frac{2}{h_y} g_b(x_i) \\ (M,0) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M,0} + \frac{2}{h_x^2} u_{M-1,0} + \frac{2}{h_y^2} u_{M-1,1} = f(x_M, y_0) - \frac{2}{h_x} g_r(y_0) + \frac{2}{h_y} g_b(x_M). \end{aligned}$$

Thus, each equation depends only on rows 0 and 1 of the matrix $\{u_{i,j}\}$, so only blocks $A^{(0,0)}$ and $A^{(0,1)}$ are nonzero. Examining these dependencies, we get

$$A^{(1,1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & \\ & & \ddots & \\ & & \frac{2}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \quad A^{(1,2)} = \begin{bmatrix} \frac{2}{h_y^2} & & & \\ & \ddots & & \\ & & \frac{2}{h_y^2} & \end{bmatrix},$$

where blanks indicate zero entries. The block $b^{(0)}$ corresponding to the right hand sides of equations $(1,0), (2,0), \dots, (M-1,0)$ we can read off easily:

$$b^{(0)} = \begin{bmatrix} f(x_1, y_0) - \frac{g_\ell(y_0)}{h_x^2} + \frac{2}{h_y} g_b(x_1) \\ f(x_2, y_0) + \frac{2}{h_y} g_b(x_2) \\ f(x_3, y_0) + \frac{2}{h_y} g_b(x_3) \\ \vdots \\ f(x_{M-1}, y_0) + \frac{2}{h_y} g_b(x_{M-1}) \\ f(x_M, y_0) - \frac{2}{h_x} g_r(y_0) + \frac{2}{h_y} g_b(x_M) \end{bmatrix}.$$

Now consider the blocks $A^{(j,j')}$ for $j = 1, 2, 3, \dots, N-2$, and $j' = 0, 1, \dots, N-1$. These correspond to equations $(1,j), (2,j), \dots, (M,j)$. Substituting boundary conditions, we have

$$\begin{aligned} (1,j) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,j} + \frac{1}{h_x^2} u_{2,j} + \frac{1}{h_y^2} u_{1,j-1} + \frac{1}{h_y^2} u_{1,j+1} = f(x_1, y_j) - \frac{g_\ell(y_j)}{h_x^2} \\ (i,j)_{i=2,\dots,M-1} &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,j} + \frac{1}{h_x^2} u_{i-1,j} + \frac{1}{h_x^2} u_{i+1,j} + \frac{1}{h_y^2} u_{i,j-1} + \frac{1}{h_y^2} u_{i,j+1} = f(x_i, y_j) \\ (M,j) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M,j} + \frac{2}{h_x^2} u_{M-1,j} + \frac{1}{h_y^2} u_{M,j-1} + \frac{1}{h_y^2} u_{M,j+1} = f(x_M, y_j) - \frac{2}{h_x} g_r(y_j). \end{aligned}$$

Thus, each equation depends only on rows $j-1, j$, and $j+1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(j,j-1)}$, $A^{(j,j)}$, and $A^{(j,j+1)}$ are nonzero. Examining these dependencies gives

$$A^{(j,j)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & \\ & & \ddots & \\ & & \frac{2}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \quad A^{(j,j-1)} = A^{(j,j+1)} = \begin{bmatrix} \frac{1}{h_y^2} & & & \\ & \ddots & & \\ & & \frac{1}{h_y^2} & \end{bmatrix}.$$

The block $b^{(j)}$ can be read off from the right hand sides easily:

$$b^{(j)} = \begin{bmatrix} f(x_1, y_j) - \frac{g_\ell(y_j)}{h_x^2} \\ f(x_2, y_j) \\ f(x_3, y_j) \\ \vdots \\ f(x_{M-1}, y_j) \\ f(x_M, y_j) - \frac{2}{h_x} g_r(y_j) \end{bmatrix}.$$

Finally, consider the blocks $A^{(N-1, j')}$, $j' = 0, 1, \dots, N-1$. These correspond to equations $(1, N-1), (2, N-1), \dots, (M, N-1)$. Substituting boundary conditions, we have

$$\begin{aligned} (1, N-1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1, N-1} + \frac{1}{h_x^2} u_{2, N-1} + \frac{1}{h_y^2} u_{1, N-2} = f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \\ (i, N-1)_{i=2, \dots, M-1} &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i, N-1} + \frac{1}{h_x^2} u_{i-1, N-1} + \frac{1}{h_x^2} u_{i+1, N-1} + \frac{1}{h_y^2} u_{i, N-2} = f(x_i, y_{N-1}) - \frac{g_t(x_i)}{h_y^2} \\ (M, N-1) &\implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M, N-1} + \frac{2}{h_x^2} u_{M-1, N-1} + \frac{1}{h_y^2} u_{M, N-2} \\ &= f(x_M, y_{N-1}) - \frac{2}{h_x} g_r(y_{N-1}) - \frac{g_t(x_M)}{h_y^2}. \end{aligned}$$

Thus, each equation depends only on rows $N-2$ and $N-1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(N-1, N-2)}$ and $A^{(N-1, N-1)}$ are nonzero. Examining these dependencies gives

$$A^{(N-1, N-1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ & & \ddots & & \\ & & & \frac{2}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \quad A^{(N-1, N-2)} = \begin{bmatrix} \frac{1}{h_y^2} & & & \\ & \ddots & & \\ & & & \frac{1}{h_y^2} \end{bmatrix}.$$

The block $b^{(N-1)}$ can be read off from the right hand sides of the equations easily:

$$b^{(N-1)} = \begin{bmatrix} f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \\ f(x_2, y_{N-1}) - \frac{g_t(x_2)}{h_y^2} \\ f(x_3, y_{N-1}) - \frac{g_t(x_3)}{h_y^2} \\ \vdots \\ f(x_{M-1}, y_{N-1}) - \frac{g_t(x_{M-1})}{h_y^2} \\ f(x_M, y_{N-1}) - \frac{2}{h_x} g_r(y_{N-1}) - \frac{g_t(x_M)}{h_y^2} \end{bmatrix}.$$

Therefore, the entire system of equations (i, j) is equivalent to the matrix-vector equation $AU = b$.

(b) See `problem3.m` for the implementation of the scheme in (a).