Math 6108 Homework 4

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Problem 1.

Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - y + 2z \\ 2x + y \\ -x - 2y + 2z \end{bmatrix}.$$

1. T is a linear transformation.

   Proof. Let $\mathbf{x} = (x, y, z)^T, \mathbf{u} = (u, v, w)^T \in \mathbf{R}^3$, and let $a \in \mathbf{R}$. Then

$$\begin{aligned} T(a\mathbf{x} + \mathbf{u}) &= \begin{bmatrix} ax + u - (ay + v) + 2(az + w) \\ 2(ax + u) + ay + v \\ -(ax + u) - (ay + v) + 2(az + w) \end{bmatrix} \\ &= a\begin{bmatrix} x - y + 2z \\ 2x + y \\ -x - y + 2z \end{bmatrix} + \begin{bmatrix} u - v + 2w \\ 2u + v \\ -u - v + 2w \end{bmatrix} \\ &= aT(\mathbf{x}) + T(\mathbf{u}). \end{aligned}$$

   This shows that T is a linear transformation. $\square$

2. Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbf{R}^3$. Then

$$[T]_E = \begin{bmatrix} [T(\mathbf{e}_1)]_E & [T(\mathbf{e}_2)]_E & [T(\mathbf{e}_3)]_E \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}.$$

   Calculating the row echelon form of $[T]_E$, we have

$$[T]_E \xrightarrow[\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1}]{} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 + R_2]{} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Since the pivots are in the first two columns, it follows that

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right\}$$

   is a basis for the column space of $[T]_E$. Then the vectors whose coordinates are given by the elements of B form a basis for range(T). Since we are using the standard basis E, these vectors are actually the same as those in B. Thus, B is also a basis for range(T). Then rank$(T) = \#B = 2$.

3. Reusing the row echelon form computed in part 2., a vector $(x, y, z)^T \in \mathbf{R}^3$ is in $\mathrm{Null}([T]_E)$ if and only if
$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0,$$
or $3y = 4z$, and $x = \frac{4}{3}z - 2z = -\frac{2}{3}z$. That is, $\mathbf{v} \in \mathrm{Null}([T]_E)$ if and only if
$$\mathbf{v} = z \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}, \qquad z \in \mathbf{R}.$$
Thus, $\mathrm{Null}([T]_E) = \mathrm{span}\{(2, 4, 1)^T\}$, so $N = \{(2, 4, 1)^T\}$ forms a basis for $\mathrm{Null}([T]_E)$ (because a set with only one nonzero element is always linearly independent). The vectors whose coordinates in E are the elements of N form a basis for $\mathrm{Null}(T)$. Since we are using the standard basis E, these vectors are the same as those in N. Hence, N is also a basis for $\mathrm{Null}(T)$, and $\mathrm{Nullity}(T) = \#N = 1$.

Problem 2.

Let $T : V \to V$ be a linear transformation of an n-dimensional vector space V. Let B be any basis for V. Then $[T]_B = I_n$ if and only if T is the identity mapping.

Proof. Suppose that $[T]_B = I_n$. By the definition of the standard matrix, for all $\mathbf{v} \in V$,
$$[T(\mathbf{v})]_B = [T]_B[\mathbf{v}]_B = I_n[\mathbf{v}]_B = [\mathbf{v}]_B.$$
It follows from the uniqueness of coordinates that $T(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$. Thus, T is the identity mapping.

Conversely, suppose that T is the identity mapping. Then $T(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then, using block matrix multiplication, we have
$$\begin{aligned} I_n &= \begin{bmatrix} [\mathbf{v}_1]_B & \cdots & [\mathbf{v}_n]_B \end{bmatrix} \\ &= \begin{bmatrix} [T(\mathbf{v}_1)]_B & \cdots & [T(\mathbf{v}_n)]_B \end{bmatrix} \\ &= \begin{bmatrix} [T]_B[\mathbf{v}_1]_B & \cdots & [T]_B[\mathbf{v}_n]_B \end{bmatrix} \\ &= [T]_B \begin{bmatrix} [\mathbf{v}_1]_B & \cdots & [\mathbf{v}_n]_B \end{bmatrix} \\ &= [T]_B I_n \\ &= [T]_B. \end{aligned}$$
□

Problem 3.

Let $V = \mathrm{span}\{e^x \sin(2x), e^x \cos(2x)\}$, and let $D : V \to V$ be the differentiation operator, defined by
$$\begin{aligned} D(ae^x \sin(2x) + be^x \cos(2x)) &= ae^x \sin(2x) + 2ae^x \cos(2x) + be^x \cos(2x) - 2be^x \sin(2x) \\ &= (a - 2b)e^x \sin(2x) + (2a + b)e^x \cos(2x). \end{aligned}$$
Noting that $B = \{e^x \sin(2x), e^x \cos(2x)\}$ is linearly independent because the Wronskian of B at $x = 0$ is given by
$$W(0) = \det\left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right) = -2 \neq 0,$$

it follows that B is a basis for V. Then

$$[D]_B = \left[[D(e^x\sin(2x))]_B \quad [D(e^x\cos(2x))]_B\right] = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

It is easy to see that $[D]_B$ is invertible, with

$$[D]_B^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Then D is also invertible, with $[D^{-1}]_B = [D]_B^{-1}$. The vector $e^x\sin(2x) \in V$ has coordinates $[e^x\sin(2x)]_B = (1,0)^T$. Thus,

$$[D^{-1}(e^x\sin(2x))]_B = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

This implies that

$$D\left(\frac{1}{5}e^x\sin(2x) - \frac{2}{5}e^x\cos(2x)\right) = D(D^{-1}(e^x\sin(2x))) = e^x\sin(2x).$$

Since $\int e^x\sin(2x)$ is given by $f(x) + C$, where f is any function such that $D(f) = e^x\sin(2x)$, and C is an arbitrary constant, it follows that

$$\int e^x\sin(2x) = \frac{1}{5}e^x\sin(2x) - \frac{2}{5}e^x\cos(2x) + C.$$

Problem 4.

Let $T : V \to W$ be an invertible linear transformation between vector spaces V and W, and let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of vectors in V.

1. If B is linearly independent, then $T(B)$ is linearly independent in W.

   Proof. Let $c_1, \ldots, c_n \in \mathbb{F}$, the underlying field for V and W. Suppose that

   $$c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n) = 0.$$

   By the linearity of T, we have

   $$T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = 0.$$

   Since T^{-1} is also linear, we must have $T^{-1}(0) = 0$, so

   $$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0,$$

   which implies that $c_1 = c_2 = \cdots = c_n = 0$ by the linear independence of B. Thus, $T(B)$ is linearly independent. □

2. If $\operatorname{span}(B) = V$, then $\operatorname{span}(T(B)) = W$.

   Proof. Let $\mathbf{w} \in W$. Then there exists $c_1, c_2, \ldots, c_n \in \mathbb{F}$, the field underlying V and W, such that

   $$T^{-1}(\mathbf{w}) = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

   because B spans V. Applying T to both sides and using the linearity of T shows that

   $$\mathbf{w} = c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n).$$

   Thus $\mathbf{w} \in \operatorname{span}(T(B))$, and $W \subseteq \operatorname{span}(T(B))$ because $\mathbf{w} \in W$ was arbitrary. Certainly $T(B) \subseteq W$, so $W = \operatorname{span}(T(B))$. □

3. If B is a basis for V, then $T(B)$ is a basis for W.

Proof. If B is a basis for V, then B is linearly independent, and $\mathrm{span}(B) = V$. By part 1. $T(B)$ is linearly independent, and by part 2. $\mathrm{span}(T(B)) = W$. This means that B is a basis for W by definition. $\square$

Problem 5.

Let $T : V \to W$ be a linear transformation.

1. The range of T, defined by $T(V) = \{T(\mathbf{v}) \mid \mathbf{v} \in V\}$, is a subspace of W.

Proof. We begin by noting that $T(V)$ is nonempty because, for example, $T(0) = 0$, so $0 \in T(V)$.

Now, let $\mathbf{w}_1$ and $\mathbf{w}_2 \in W$, and let $a \in \mathbb{F}$, the field underlying V and W. Then there exist $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $\mathbf{w}_1 = T(\mathbf{v}_1)$, and $\mathbf{w}_2 = T(\mathbf{v}_2)$. Thus,

$$\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2),$$

so $\mathbf{w}_1 + \mathbf{w}_2 \in T(V)$, as $\mathbf{v}_1 + \mathbf{v}_2 \in V$. Additionally,

$$a\mathbf{w}_1 = aT(\mathbf{v}_1) = T(a\mathbf{v}_1),$$

so $a\mathbf{w}_1 \in T(V)$, as $a\mathbf{v}_1 \in V$. This shows that $T(V)$ is a subspace of W. $\square$

2. The null space of T, defined by $\mathrm{Null}(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = 0\}$, is a subspace of V.

Proof. We begin by noting that $\mathrm{Null}(T)$ is nonempty because $T(0) = 0$, so $0 \in \mathrm{Null}(T)$.

Now, let $\mathbf{v}_1, \mathbf{v}_2 \in \mathrm{Null}(T)$. Then

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) = 0,$$

so $\mathbf{v}_1 + \mathbf{v}_2 \in \mathrm{Null}(T)$. Let $a \in \mathbb{F}$, the field underlying V and W. Then

$$T(a\mathbf{v}_1) = aT(\mathbf{v}_1) = 0,$$

so $a\mathbf{v}_1 \in \mathrm{Null}(T)$. This shows that $\mathrm{Null}(T)$ is a subspace of V. $\square$