Math 6417 Homework 2

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A continuous function $\sigma : \mathbf{R} \to \mathbf{R}$ is called **sigmoidal** if there exists $T > 0$ such that

$$\sigma(t) = \begin{cases} 1 & t \geq T, \\ 0 & t \leq -T. \end{cases} \tag{1}$$

Let σ be sigmoidal in the following problems.

Question 1.

Let $y \in \mathbf{R}^n$, and $(\theta, \phi) \in \mathbf{R}^2$. For $x \in \mathbf{R}^n$, define

$$\sigma_\lambda(x; \theta, \phi) = \sigma\left(\lambda\left(y^T x + \theta\right) + \phi\right). \tag{2}$$

Then

$$\sigma_\lambda(x; \theta, \phi) \to \gamma(x) = \begin{cases} 1 & y^T x + \theta > 0 \\ 0 & y^T x + \theta < 0 \\ \sigma(\phi) & y^T x + \theta = 0 \end{cases} \qquad \text{as } \lambda \to \infty. \tag{3}$$

Proof. If $y^T x + \theta = 0$, then $\sigma_\lambda(x; \phi, \theta) = \sigma(\phi)$ for all λ, and the result is clear. Otherwise, let $s = \operatorname{sgn}(y^T x + \theta)$. Then

$$\lambda \geq \frac{T - s\phi}{|y^T x + \theta|} \tag{4}$$

implies that $\lambda\left(y^T x + \theta\right) + \phi \geq T$ if $s = 1$, and $\lambda\left(y^T x + \theta\right) + \phi \leq -T$ if $s = -1$. Then (4) implies that $\sigma_\lambda(x; \theta, \phi) = 1$ if $s = 1$, and $\sigma_\lambda(x; \theta, \phi) = 0$ if $s = -1$. The result follows. $\square$

Question 2.

Let $y \in \mathbf{R}^n$, let $\Pi_{y,\theta} = \left\{x \mid y^T x + \theta = 0\right\}$, and let $H_{y,\theta} = \left\{x \mid y^T x + \theta > 0\right\}$. If μ is a finite Borel measure on $[0,1]^n$ such that

$$\int_{[0,1]^n} \sigma_\lambda(x) \, \mathrm{d}\mu(x) = 0 \qquad \text{for all } (\lambda, \theta, \phi) \in \mathbf{R}^3, \tag{5}$$

then

$$\sigma(\phi)\mu(\Pi_{y,\theta}) + \mu(H_{y,\theta}) = 0 \qquad \text{for all } (\lambda, \theta, \phi) \in \mathbf{R}^3. \tag{6}$$

Proof. Fix $(\theta, \phi) \in \mathbf{R}^2$. For any $\lambda \in \mathbf{R}$, the function $\sigma_\lambda(\,\cdot\,; \theta, \phi)$ is dominated by the constant function $C(x) = \max\limits_{t \in [-T,T]} |\sigma(t)|$. Since σ is continuous, σ_λ is continuous as well, so σ_λ is integrable on $[0,1]^n$. By the previous problem, σ_λ converges to γ pointwise as $\lambda \to \infty$. Thus, the Dominated Convergence Theorem implies that

$$0 = \lim_{\lambda \to \infty} \int_{[0,1]^n} \sigma_\lambda(x) \, \mathrm{d}\mu(x) = \int_{[0,1]^n} \gamma(x) \, \mathrm{d}\mu(x) = \sigma(\phi)\mu(\Pi_{y,\theta}) + \mu(H_{y,\theta}). \tag{7}$$

$\square$

Question 3.

Suppose that μ satisfies (5). Then $\mu = 0$.

Proof. Define the linear functional $F : L^\infty(\mathbf{R}) \to \mathbf{R}$ by

$$F(h) = \int_{[0,1]^n} h\left(y^T x\right)\ \mathrm{d}\mu(x) \tag{8}$$

First, let $h = \chi_{[\theta,\infty)}$ for some $\theta \in \mathbf{R}$. $\square$