The Fréchet Derivative

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Motivation

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Say $X = Y = \mathbf{R}$, then A_{x_0} is given by multiplication by a number $a_{x_0} \in \mathbf{R}$, and a natural choice for a_{x_0} is $f'(x_0)$, as

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so

$$f(x_0 + h) - f(x_0) = f'(x_0)h + \omega(h)h,$$

where $\omega(h) \to 0$ as $h \to 0$.

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A function $f : U \to Y$ is called **Fréchet differentiable** at $x \in U$ if there exists $A \in B(X, Y)$ such that

$$\frac{\|f(x+h) - f(x) - Ah\|_Y}{\|h\|_X} \to 0 \quad \text{as} \quad \|h\|_X \to 0,$$

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The Fréchet derivative is the same as the usual derivative if $f \in C^1(\mathbf{R})$.