

Math 6417 Homework 3

Jacob Hauck

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Question 1.

Let $B(\cdot, \cdot)$ be a continuous, bilinear form on a real Hilbert space H . Suppose that B is coercive in the sense that there is some $\alpha > 0$ such that $|B(x, x)| \geq \alpha \|x\|^2$ for all $x \in H$.

- 1.1) Let $y \in H$. Then the map $f_y : H \rightarrow \mathbf{R}$ defined by $f_y(x) = B(x, y)$ is a bounded linear functional on H . Consequently, there exists a unique $w \in H$ such that $B(x, y) = f_y(x) = (x, w)$ for all $x \in H$.

Proof. Firstly, it is clear that f_y is linear; indeed, given $a_1, a_2 \in \mathbf{R}$ and $x_1, x_2 \in H$,

$$f_y(a_1x_1 + a_2x_2) = B(a_1x_1 + a_2x_2, y) = a_1B(x_1, y) + a_2B(x_2, y) = a_1f_y(x_1) + a_2f_y(x_2) \quad (1)$$

by the bilinearity of B .

Secondly, $B(\cdot, y) = f_y$ must be continuous because B is continuous. Hence, f_y is bounded.

Thirdly, by the Riesz representation theorem, there exists a unique $w \in H$ such that $B(x, y) = f_y(x) = (x, w)$ for all $x \in H$. \square

- 1.2) Given $y \in H$, by 1.1), there is a unique $w \in H$ such that $B(x, y) = (x, w)$ for all $x \in H$; this defines a function $A : H \rightarrow H$, where $Ay = w$. Then A is a bounded, linear operator on H , that is, $A \in B(H)$.

Proof. There are two steps to this proof: showing that A is linear, and showing that A is bounded.

Step 1: linearity

Let $a_1, a_2 \in \mathbf{R}$ and $y_1, y_2 \in H$. Then for all $x \in H$,

$$\begin{aligned} (x, A(a_1y_1 + a_2y_2)) &= B(x, a_1y_1 + a_2y_2) = a_1B(x, y_1) + a_2B(x, y_2) = a_1(x, Ay_1) + a_2(x, Ay_2) \\ &= (x, a_1Ay_1 + a_2Ay_2). \end{aligned} \quad (2)$$

Thus, $w = A(a_1y_1 + a_2y_2)$ and $w' = a_1Ay_1 + a_2Ay_2$ satisfy the property that $B(x, a_1y_1 + a_2y_2) = (x, w) = (x, w')$ for all $x \in H$. By the Riesz representation theorem, there is only one element of H that can satisfy this property; therefore, $w = w'$, or $A(a_1y_1 + a_2y_2) = a_1Ay_1 + a_2Ay_2$. Thus, A is linear.

Step 2: boundedness

Note that B is continuous if and only if (see, e.g., Theorem 8.10 assumption (a) in Arbogast and Bona) there exists some $M > 0$ such that

$$|B(x, y)| \leq M \|x\| \|y\|, \quad \text{for all } x, y \in H. \quad (3)$$

Let $y \in H$. Then

$$\|Ay\| = \left| \left(\frac{Ay}{\|Ay\|}, Ay \right) \right| = \left| B \left(\frac{Ay}{\|Ay\|}, y \right) \right| \leq M \|y\|. \quad (4)$$

Since y was arbitrary, it follows that A is bounded, and $\|A\| \leq M$. \square

1.3) A is bounded below in the sense that there exists $\gamma > 0$ such that $\|Ay\| \geq \gamma\|y\|$ for all $y \in H$.

Proof. This follows from the coercivity of B : for all $y \in H$,

$$\|Ay\|\|y\| \geq |(y, Ay)| = |B(y, y)| \geq \alpha\|y\|^2, \quad (5)$$

so $\|Ay\| \geq \alpha\|y\|$ for all $y \in H$, as claimed. \square

1.4) A is one-to-one, and the range of A is closed.

Proof. Let $y_1, y_2 \in H$, and suppose that $Ay_1 = Ay_2$. Then, by the previous part,

$$\|y_1 - y_2\| \leq \frac{1}{\gamma}\|A(y_1 - y_2)\| = \frac{1}{\gamma}\|Ay_1 - Ay_2\| = 0. \quad (6)$$

Therefore, $y_1 = y_2$. This shows that A is one-to-one.

Let $R(A)$ denote the range of A , and let $\{w_n\} \subseteq R(A)$ be a convergent sequence in the range of A . By the definition of $R(A)$, there exists $y_n \in H$ such that $w_n = Ay_n$.

Let $\varepsilon > 0$ be given. Since $\{w_n\}$ is convergent, it is also Cauchy, so we can choose N such that $n, m > N$ implies that $\|w_n - w_m\| < \varepsilon$. By the linearity of A , we have $A(y_n - y_m) = w_n - w_m$; hence,

$$\begin{aligned} \alpha\|y_n - y_m\|^2 &\leq |B(y_n - y_m, y_n - y_m)| = |(y_n - y_m, w_n - w_m)| \leq \|y_n - y_m\| \cdot \|w_n - w_m\| \\ &\leq \varepsilon\|y_n - y_m\| \end{aligned} \quad (7) \quad (8)$$

if $n, m > N$. Thus, $n, m > N$ implies that $\|y_n - y_m\| < \frac{\varepsilon}{\alpha}$. This implies that $\{y_n\}$ is Cauchy.

Since H is complete, there exists $y \in H$ such that $y_n \rightarrow y$ as $n \rightarrow \infty$. Let $x \in H$, and let $\varepsilon > 0$ be given. By the continuity of B and the inner product and the convergence of $\{y_n\}$ and $\{w_n\}$, there exists n large enough that $|B(x, y - y_n)| < \frac{\varepsilon}{2}$, and $|(x, w - w_n)| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} |B(x, y) - (x, w)| &= |B(x, y - y_n) + B(x, y_n) - (x, w_n) - (x, w - w_n)| \\ &\leq |B(x, y - y_n)| + |(x, w - w_n)| < \varepsilon. \end{aligned} \quad (9)$$

Since $\varepsilon > 0$ was arbitrary and $x \in H$ was arbitrary, it follows that $B(x, y) = (x, w)$ for all $x \in H$. This implies that $w = Ay$ by the definition of A , which means that $w \in R(A)$. Since the convergent sequence $\{w_n\} \subseteq R(A)$ was arbitrary, and its limit $w \in R(A)$, it follows that $R(A)$ is closed. \square

1.5) A is onto.

Proof. Suppose that $x \in R(A)^\perp$, that is, $(x, w) = 0$ for all $w \in R(A)$. This implies that $(x, Ay) = 0$ for all $y \in H$, which is equivalent to saying that $B(x, y) = 0$ for all $y \in H$. In particular, if we choose $y = x$, then $\|x\|^2 \leq \frac{1}{\alpha}|B(x, x)| = 0$. Therefore, $x = 0$. This shows that $R(A)^\perp = \{0\}$ because x was arbitrary.

Let $y \in H$. Since $R(A)$ is a closed subspace of H by (1.4), there exists a best approximation $w \in R(A)$ of y , which satisfies the property $(y - w, x) = 0$ for all $x \in R(A)$ (Theorem 3.7 and Corollary 3.8 in Arbogast and Bona). That is, $y - w \in R(A)^\perp$. Since $R(A)^\perp = \{0\}$ by the above, it follows that $y - w = 0$, and $y = w \in R(A)$. Since y was arbitrary and $R(A) \subseteq H$, it follows that $R(A) = H$, that is, A is onto. \square

1.6) A is invertible.

Proof. By the previous two parts, A is bijective, so it has a set-theoretic inverse function A^{-1} . By 1.2), A is bounded. Therefore, by the open mapping theorem, A maps open sets to open sets, which means that the preimage of an open set under A^{-1} is open, that is, A^{-1} is continuous. Therefore, A is invertible. \square

- 1.7) Given $f \in H^*$, the Riesz representation theorem implies that there exists a unique $w \in H$ such that $f(x) = (x, w)$ for all $x \in H$, and we can view H^* and H as the same under the correspondence $f \leftrightarrow w$.
- 1.8) Consider the equation $B(x, y) = f(x)$ for all $x \in H$, where $f \in H^*$. By the remark in part 1.7), we can choose $w \in H$ such that $f(x) = (x, w)$ for all $x \in H$. Then the equation is equivalent to $B(x, y) = (x, w)$ for all $x \in H$. If y is a solution of this equation, then, by the definition of A , we must have $Ay = w$. Using the invertibility of A , we obtain $y = A^{-1}w$ as the unique solution of the equation. Viewing f and w as the same under the correspondence in 1.7), we might also write $y = A^{-1}f$.

Question 2.

Let $e_j \in L^2(-\pi, \pi)$ be defined by $e_j(x) = \frac{1}{\sqrt{2\pi}} e^{ijx}$ for $j \in \mathbf{Z}$. Define

$$H = \left\{ f \in L^2(-\pi, \pi) : f = \bar{f} \text{ and } f = \sum_{j \neq 0} f_j e_j \text{ for some } \{f_j\} \subseteq \mathbf{C} \text{ such that } \sum_{j \neq 0} j^2 |f_j|^2 < \infty \right\}, \quad (10)$$

and

$$H^{-1} = \left\{ f(x) = \sum_{j \neq 0} f_j e_j(x) : \sum_{j \neq 0} j^{-2} |f_j|^2 < \infty \text{ and } f = \bar{f} \right\}. \quad (11)$$

- 2.1) H and H^{-1} are real Hilbert spaces when equipped with the inner products

$$(f, g)_H = \sum_{j \neq 0} j^2 f_j \bar{g}_j, \quad (f, g)_{H^{-1}} = \sum_{j \neq 0} j^{-2} f_j \bar{g}_j. \quad (12)$$

Proof. Before we can show that H and H^{-1} are real Hilbert spaces, we need to study their definitions. For the definition H , there isn't much trouble – its elements are just elements of $L^2(-\pi, \pi)$. For the definition H^{-1} , the series might not converge to an element of $L^2(-\pi, \pi)$ (say, $f_j = 1$ for all j), so understanding what the convergence of the infinite sum means is more difficult.

As we will see, we can understand H in terms of a corresponding space of coefficient sequences, which is isomorphic to H as a real vector space. By analogy, then, we can understand H^{-1} in terms of its corresponding space of coefficient sequences, which will suffice for our purposes.

To understand these spaces of coefficient sequences, we introduce the following Lemma showing that H is a real vector space isomorphic to its space of coefficient sequences.

Lemma 1. Define

$$S_H = \left\{ \{f_j\}_{j \neq 0} \subseteq \mathbf{C} : \sum_{j \neq 0} j^2 |f_j|^2 < \infty \text{ and } \forall j \in \mathbf{Z} \setminus \{0\}, f_j = \bar{f}_{-j} \right\}. \quad (13)$$

Then S_H is a real vector space, and there is an isomorphism (of real vector spaces) $\varphi : H \rightarrow S_H$ such that if $\varphi(f) = \{f_j\}$, then

$$f = \sum_{j \neq 0} f_j e_j. \quad (14)$$

Proof. We need to prove that φ is well-defined, bijective, and, after showing that H and S_H are real vector spaces, that φ is a linear mapping between them.

Step 1: definition of φ

Let $f \in H$. Then $f \in L^2(-\pi, \pi)$. Recalling from our lecture that $\{e_j\}_{j \in \mathbf{Z}}$ is an orthonormal basis for $L^2(-\pi, \pi)$, it follows that there exists exactly one sequence of coefficients $\{g_j\} \in \ell^2(\mathbf{Z})$ such that

$$f = \sum_{j \in \mathbf{Z}} g_j e_j. \quad (15)$$

On the other hand, since $f \in H$, there must be a sequence of coefficients $\{f_j\} \subseteq \mathbf{C}$ such that

$$f = \sum_{j \neq 0} f_j e_j, \quad \sum_{j \neq 0} j^2 |f_j|^2 < \infty, \quad (16)$$

It follows by the uniqueness of $\{g_j\}$ that $g_0 = 0$, and $g_j = f_j$ for all $j \neq 0$. Furthermore, since $f = \bar{f}$, it follows that

$$\bar{f}(x) = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} \overline{f_j e^{ijx}} = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} \bar{f}_j e^{-ijx} = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} \bar{f}_{-j} e^{ijx} = f(x). \quad (17)$$

That is, $f = \sum_{j \neq 0} \bar{f}_{-j} e_j$; by the uniqueness of $\{g_j\}$ again, we must have $\bar{f}_{-j} = g_j = f_j$ for all $j \neq 0$.

Hence, $\{f_j\} \in S_H$. Since $\{f_j\}$ is uniquely determined by f by the uniqueness of $\{g_j\}$, we can define a function $\varphi : H \rightarrow S_H$ by $\varphi(f) = \{f_j\}$.

Step 2: φ is a one-to-one correspondence

First, suppose that $\varphi(f) = \{f_j\} = \varphi(g)$ for some $f, g \in H$ and $\{f_j\} \in S_H$. Then, by the definition of φ ,

$$f = \sum_{j \neq 0} f_j e_j = g, \quad (18)$$

so φ is one-to-one.

Second, let $\{f_j\} \in S_H$. Since $L^2(-\pi, \pi)$ is a Hilbert space, its Riesz-Fischer map $F : L^2(-\pi, \pi) \rightarrow \ell^2(\mathbf{Z})$ corresponding to the orthonormal basis \mathcal{B} is an isomorphism. If we set $g_j = f_j$ for $j \neq 0$, and $g_0 = 0$, then we have

$$\sum_{j \in \mathbf{Z}} |g_j|^2 = \sum_{j \neq 0} |f_j|^2 \leq \sum_{j \neq 0} j^2 |f_j|^2 < \infty \quad (19)$$

because $\{f_j\} \in S_H$. Therefore, $\{g_j\} \in \ell^2(\mathbf{Z})$, and $f = F^{-1}(\{g_j\}) \in L^2(-\pi, \pi)$. By the definition of F and the fact that $\{e_j\}$ is an orthonormal basis, we have

$$f = F^{-1}(\{g_j\}) = \sum_{j \in \mathbf{Z}} g_j e_j = \sum_{j \neq 0} f_j e_j. \quad (20)$$

Since $\{f_j\} \in S_H$, we have $f_j = \bar{f}_{-j}$, so

$$\bar{f}(x) = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} \overline{f_j e^{ijx}} = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} \bar{f}_j e^{-ijx} = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} \bar{f}_{-j} e^{ijx} = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} f_j e^{ijx} = f(x), \quad (21)$$

so $f = \bar{f}$. Since $\{f_j\} \in S_H$, we also have $\sum_{j \neq 0} j^2 |f_j|^2 < \infty$. Therefore, $f \in H$ by definition.

Finally, $\varphi(f) = \{f_j\}$ because $f = \sum_{j \neq 0} f_j e_j$, and $\varphi(f)$ is, by definition, the unique sequence of coefficients in S_H that make that statement true.

Thus, φ is one-to-one and onto.

Step 3: H and S_H are real vector spaces

H is a nonempty subset (it contains 0) of the real vector space $L^2(-\pi, \pi)$, and S_H is a nonempty subset (it contains 0) of the real vector space $\ell^2(\mathbf{Z} \setminus \{0\})$. Thus, it suffices to show that H and S_H are subspaces of these two vector spaces.

Let $\{f_j\}, \{g_j\} \in S_H$, and let $\alpha, \beta \in \mathbf{R}$. Then $\{h_j\} = \alpha\{f_j\} + \beta\{g_j\} \in S_H$ because $\bar{h}_{-j} = \alpha\bar{f}_{-j} + \beta\bar{g}_{-j} = \alpha f_j + \beta g_j = h_j$, and

$$\sum_{j \neq 0} j^2 |h_j|^2 = \sum_{j \neq 0} j^2 |\alpha f_j + \beta g_j|^2 \leq 2\alpha^2 \sum_{j \neq 0} j^2 |f_j|^2 + 2\beta^2 \sum_{j \neq 0} j^2 |g_j|^2 < \infty \quad (22)$$

since $|\alpha f_j + \beta g_j|^2 \leq 2(\alpha^2 |f_j|^2 + \beta^2 |g_j|^2)$ for all j . Thus, S_H is a real vector subspace of the space of all sequences of complex numbers.

Now let $f, g \in H$, and $\alpha, \beta \in \mathbf{R}$. Set $\{f_j\} = \varphi(f)$, and $\{g_j\} = \varphi(g)$. Then $\{h_j\} = \alpha\{f_j\} + \beta\{g_j\} \in S_H$, so we can define $h = \varphi^{-1}(\{h_j\})$. By the definition of h , we have

$$\alpha f + \beta g = \alpha \sum_{j \neq 0} f_j e_j + \beta \sum_{j \neq 0} g_j e_j = \sum_{j \neq 0} (\alpha f_j + \beta g_j) e_j = h \in H. \quad (23)$$

Therefore, H is a (real) vector subspace of $L^2(-\pi, \pi)$.

Step 4: φ is linear

Let $f, g \in H$, and let $\alpha, \beta \in \mathbf{R}$. Define $\{f_j\} = \varphi(f)$, and $\{g_j\} = \varphi(g)$. Then

$$\alpha f + \beta g = \alpha \sum_{j \neq 0} f_j e_j + \beta \sum_{j \neq 0} g_j e_j = \sum_{j \neq 0} (\alpha f_j + \beta g_j) e_j. \quad (24)$$

This implies that $\alpha\varphi(f) + \beta\varphi(g) = \{\alpha f_j + \beta g_j\} = \varphi(\alpha f + \beta g)$ by the definition of φ . Thus, φ is linear. \square

The Lemma establishes that H is essentially the same as S_H as a vector space. If we define by analogy

$$S_{H^{-1}} = \left\{ \{f_j\}_{j \neq 0} \subseteq \mathbf{C} : \sum_{j \neq 0} j^{-2} |f_j|^2 < \infty \text{ and } \forall j \in \mathbf{Z} \setminus \{0\}, f_j = \bar{f}_{-j} \right\}, \quad (25)$$

then we can understand H^{-1} to be a real vector space isomorphic to $S_{H^{-1}}$. Note that $S_{H^{-1}}$ is indeed a real vector space; like S_H , it is a nonempty subset (it contains 0) of the vector space of all sequences of complex numbers, and if $\{f_j\}, \{g_j\} \in S_{H^{-1}}$, and $\alpha, \beta \in \mathbf{R}$, then $\{h_j\} = \alpha\{f_j\} + \beta\{g_j\} \in S_H$ because $\bar{h}_{-j} = \alpha\bar{f}_{-j} + \beta\bar{g}_{-j} = \alpha f_j + \beta g_j = h_j$, and

$$\sum_{j \neq 0} j^{-2} |h_j|^2 = \sum_{j \neq 0} j^{-2} |\alpha f_j + \beta g_j|^2 \leq 2\alpha^2 \sum_{j \neq 0} j^{-2} |f_j|^2 + 2\beta^2 \sum_{j \neq 0} j^{-2} |g_j|^2 < \infty \quad (26)$$

since $|\alpha f_j + \beta g_j|^2 \leq 2(\alpha^2 |f_j|^2 + \beta^2 |g_j|^2)$ for all j .

We notice that it is possible for the series in the definition of H^{-1} to converge for some $\{f_j\} \in S_{H^{-1}}$. In particular, if $\{f_j\} \in \ell^2(\mathbf{Z} \setminus \{0\})$, then the series converges to a function $f \in L^2(-\pi, \pi)$. Moreover, by the same reasoning in Lemma 1, the function f is uniquely determined by the coefficients $\{f_j\}$, and vice versa.

Thus, when $f \in L^2(-\pi, \pi)$, and the coefficients $\{f_j\}$ of f with respect to the orthonormal basis $\{e_j\}$ belong to $\ell^2(\mathbf{Z} \setminus \{0\})$ with $f_0 = 0$ and $f_j = \bar{f}_{-j}$, it makes sense to view f as an element of H^{-1} , and we can define a function $\psi : H^{-1} \rightarrow S_{H^{-1}}$ by $\psi(f) = \{f_j\}_{j \neq 0}$. By the exact same reasoning in Lemma 1,

we can easily verify that ψ is one-to-one and linear. This gives us an interpretation of at least some of the elements of H^{-1} ; from now on, we will simply assume that ψ can be extended to an isomorphism between H^{-1} and $S_{H^{-1}}$.

Now we turn to the issue of equipping H and H^{-1} with inner products. The given inner products are defined in terms of the sequence representations of elements of H and H^{-1} ; this is well-defined due to the (actual) isomorphism between H and S_H and the (assumed) isomorphism between H^{-1} and $S_{H^{-1}}$. It also allows us to work with $S_{H^{-1}}$ without needing to worry about how to interpret its elements – we will only need to work with the sequence representations in $S_{H^{-1}}$.

We need to show that H and H^{-1} are real inner product spaces when equipped with $(\cdot, \cdot)_H$ and $(\cdot, \cdot)_{H^{-1}}$ as inner products. We wrap this into the following Lemma.

Lemma 2. For $G \in \{H, H^{-1}\}$, define

$$\rho = \rho_G = \begin{cases} \varphi & G = H, \\ \psi & G = H^{-1}, \end{cases} \quad s = s_G = \begin{cases} 1 & G = H, \\ -1 & G = H^{-1}. \end{cases} \quad (27)$$

Then G is a real inner product space with inner product $(\cdot, \cdot)_G$ defined by

$$(f, g)_G = \sum_{j \neq 0} j^{2s} f_j \bar{g}_j, \quad f, g \in G, \quad (28)$$

where $\{f_j\} = \rho(f)$, $\{g_j\} = \rho(g) \in S_G$ are the coefficients of f and g in S_G .

Proof. As we have already remarked, the uniqueness of $\{f_j\}$ and $\{g_j\}$ implies that $(f, g)_G$ is well-defined. We still need to show that the series converges to a real number, and that $(\cdot, \cdot)_G$ is symmetric, linear in the first argument, and positive definite.

Step 1: $(f, g)_G$ is a real number

Let $f, g \in G$ with $\{f_j\} = \rho(f)$, and $\{g_j\} = \rho(g)$. Then the series for $(f, g)_G$ converges absolutely because

$$\sum_{j \neq 0} j^{2s} |f_j| \cdot |\bar{g}_j| \leq \left(\sum_{j \neq 0} j^{2s} |f_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j \neq 0} j^{2s} |g_j|^2 \right)^{\frac{1}{2}} < \infty \quad (29)$$

by the Cauchy-Schwarz inequality.

Next, $(f, g)_G \in \mathbf{R}$ because

$$\overline{(f, g)_G} = \overline{\sum_{j \neq 0} j^{2s} f_j \bar{g}_j} = \sum_{j \neq 0} j^{2s} \bar{f}_j g_j = \sum_{j \neq 0} j^{2s} f_{-j} \bar{g}_{-j} = \sum_{j \neq 0} j^{2s} f_j \bar{g}_j = (f, g)_G, \quad (30)$$

and only real numbers are equal to their own complex conjugate.

Step 2: $(\cdot, \cdot)_G$ is symmetric

Let $f, g \in G$, and let $\{f_j\} = \rho(f)$, and $\{g_j\} = \rho(g)$. Then

$$(f, g)_G = \sum_{j \neq 0} j^{2s} f_j \bar{g}_j = \sum_{j \neq 0} j^{2s} f_{-j} \bar{g}_{-j} = \sum_{j \neq 0} j^{2s} g_j \bar{f}_j = (g, f)_G. \quad (31)$$

Therefore, $(\cdot, \cdot)_G$ is symmetric.

Step 3: $(\cdot, \cdot)_G$ is linear in the first argument

Let $f, g, h \in G$, and let $\{f_j\} = \rho(f)$, $\{g_j\} = \rho(g)$, and $\{h_j\} = \rho(h)$. Let $\alpha, \beta \in \mathbf{R}$. Then $\rho(\alpha f + \beta g) = \alpha \rho(f) + \beta \rho(g)$ because ρ is linear, so

$$(\alpha f + \beta g, h)_G = \sum_{j \neq 0} j^{2s} (\alpha f_j + \beta g_j) \bar{h}_j = \alpha \sum_{j \neq 0} j^{2s} f_j \bar{h}_j + \beta \sum_{j \neq 0} j^{2s} g_j \bar{h}_j = \alpha (f, h)_G + \beta (g, h)_G. \quad (32)$$

Therefore, $(\cdot, \cdot)_G$ is linear in the first argument.

Step 4: $(\cdot, \cdot)_G$ is positive definite

Let $f \in G$, and let $\{f_j\} = \rho(f)$. Then

$$(f, f)_G = \sum_{j \neq 0} j^{2s} |f_j|^2 \geq 0 \quad (33)$$

because each term of the series is nonnegative. Moreover, if $f = 0$, then clearly $|f_j|^2 = 0$, so each term is 0, and $(f, f)_G = 0$. Conversely, if $(f, f)_G = 0$, then, since each term of the series for $(f, f)_G$ is nonnegative, it must be that each term is 0. This implies that $f_j = 0$ for all $j \neq 0$, that is, $\{f_j\} = 0$ in S_G . Therefore $f = \rho^{-1}(0) = 0$ by the linearity of ρ^{-1} .

This shows that $(\cdot, \cdot)_G$ is positive definite. \square

Now we know that H and H^{-1} are real inner product spaces, so there is only one more thing to show in order to prove that they are real Hilbert spaces: they need to be complete with respect to the norms $\|\cdot\|_H$ and $\|\cdot\|_{H^{-1}}$ induced by their inner products. For the sake of organization, we put this proof inside one last Lemma.

Lemma 3. *For $G \in \{H, H^{-1}\}$ and $s = s_G$, $\rho = \rho_G$ defined as in Lemma 2, the space G is complete with respect to the norm $\|\cdot\|_G$ induced by the inner product $(\cdot, \cdot)_G$ from Lemma 2.*

Proof. Let $\{f^n\}_{n=1}^\infty$ be a Cauchy sequence in G with respect to $\|\cdot\|_G$. We need to show that there exists $f \in G$ such that $f^n \rightarrow f$ as $n \rightarrow \infty$ in $\|\cdot\|_G$. To do this, we first identify a candidate element, then we show that the sequence converges to the candidate.

Step 1: identifying a candidate limit

Given $\varepsilon > 0$, we can choose N such that $n, m > N$ implies that

$$\varepsilon^2 > \|f^n - f^m\|_G^2 = \sum_{j \neq 0} j^{2s} |f_j^n - f_j^m|^2, \quad (34)$$

where $\{f_j^n\} = \rho(f^n)$. Since each term in the summation is nonnegative, this means that $|j^s f_j^n - j^s f_j^m| < \varepsilon$ for all $n, m > N$. Then $\{j^s f_j^n\}_{n=1}^\infty$ is Cauchy for all $j \neq 0$. Thus, by the completeness of \mathbf{C} , the sequence $\{j^s f_j^n\}$ converges to a limit $j^s f_j$ in \mathbf{C} as $n \rightarrow \infty$.

Since $j^s \bar{f}_j^n = j^s f_{-j}^n$ for all j and all n , taking the limit as $n \rightarrow \infty$ and using the continuity of the complex conjugate function, we get $j^s \bar{f}_j = j^s f_{-j}$ for all $j \neq 0$. Since $j^s \neq 0$, it follows that $\bar{f}_j = f_{-j}$ for all $j \neq 0$.

Furthermore, by the convergence of $\{j^s f_j^n\}$ to $j^s f_j$, for all $J > 0$, we can choose n large enough that $|j^s f_j - j^s f_j^n| \leq \frac{1}{2J}$ for all $0 < |j| \leq J$. Then

$$\sum_{0 < |j| \leq J} j^{2s} |f_j|^2 = \sum_{0 < |j| \leq J} |j^s f_j - j^s f_j^n + j^s f_j^n|^2 \leq 2 \sum_{0 < |j| \leq J} |j^s f_j - j^s f_j^n|^2 + 2 \sum_{0 < |j| \leq J} j^{2s} |f_j^n|^2 \quad (35)$$

$$\leq 4 + 2\|f^n\|_G^2. \quad (36)$$

Since $\{f^n\}$ is Cauchy in $\|\cdot\|_G$, it must be bounded; that is, there exists $M > 0$ such that $\|f^n\|_G \leq M$ for all n . Then

$$\sum_{0 < |j| \leq J} j^{2s} |f_j|^2 \leq 4 + 2M^2 \quad (37)$$

for all $J > 0$. This implies that

$$\sum_{j \neq 0} j^{2s} |f_j|^2 \leq 4 + 2M^2 < \infty. \quad (38)$$

Therefore, $\{f_j\} \in S_G$, and we can define our candidate limit as $f = \rho^{-1}(\{f_j\})$.

Step 2: showing that the candidate is the limit

Let $\varepsilon > 0$ be given. Since $\{f^n\}$ is Cauchy, we can choose N such that $n, m > N$ implies that $\|f^n - f^m\|_G^2 < \varepsilon$. By the convergence of $\{j^s f_j^n\}$, for any $J > 0$, we can choose $m_J > N$ such that for all $0 < |j| \leq J$ we have $|j^s f_j - j^s f_j^{m_J}|^2 < \frac{\varepsilon}{2J}$. Then

$$\sum_{0 < |j| \leq J} j^{2s} |f_j - f_j^n|^2 = \sum_{0 < |j| \leq J} |(j^s f_j - j^s f_j^{m_J}) + (j^s f_j^{m_J} - j^s f_j^n)|^2 \quad (39)$$

$$\leq 2 \sum_{0 < |j| \leq J} |j^s f_j - j^s f_j^{m_J}|^2 + 2 \sum_{0 < |j| \leq J} j^{2s} |f_j^{m_J} - f_j^n|^2 \quad (40)$$

$$\leq 2\varepsilon + 2\|f^{m_J} - f^n\|_G^2 \leq 4\varepsilon \quad (41)$$

if $n > N$. Since this estimate is independent of J , it follows that

$$\|f - f^n\|_G^2 = \sum_{j \neq 0} j^{2s} |f_j - f_j^n|^2 < 4\varepsilon \quad (42)$$

if $n > N$. Therefore, $f^n \rightarrow f$ as $n \rightarrow \infty$ in $\|\cdot\|_G$. \square

The three Lemmas above establish that H and H^{-1} are Hilbert spaces with the given inner products. \square

2.2) Define the bilinear form B on H by

$$B(f, g) = \sum_{j \neq 0} (ij + j^2) f_j \bar{g}_j, \quad (43)$$

where $\{f_j\} = \varphi(f)$, and $\{g_j\} = \varphi(g)$. Then B satisfies the hypotheses of the Lax-Milgram theorem.

Proof. As with the inner products, the isomorphism φ ensures that $B(f, g)$ is well-defined in terms of $\{f_j\} = \varphi(f)$ and $\{g_j\} = \varphi(g)$. We need to show that the series for B converges and that B is actually bilinear over \mathbf{R} . Then, we need to show that B satisfies the hypotheses of the Lax-Milgram theorem, that is, that B is continuous and coercive.

Step 1: B is well-defined and bilinear

Let $f, g \in H$ with $\{f_j\} = \varphi(f)$ and $\{g_j\} = \varphi(g)$. Then the series for $B(f, g)$ converges absolutely because

$$\sum_{j \neq 0} |(ij + j^2) f_j \bar{g}_j| \leq \left(\sum_{j \neq 0} j \sqrt{1 + j^2} |f_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j \neq 0} j \sqrt{1 + j^2} |g_j|^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \|f\|_H \|g\|_H < \infty \quad (44)$$

by the Cauchy-Schwarz inequality (note that $|ij + j^2| = j \sqrt{1 + j^2} \leq \sqrt{2} j^2$ for all $j \neq 0$).

Let $f, g, h \in H$, and let $\alpha, \beta \in \mathbf{R}$. Set $\{f_j\} = \varphi(f)$, $\{g_j\} = \varphi(g)$, and $\{h_j\} = \varphi(h)$. By the linearity of φ , we have $\varphi(\alpha f + \beta g) = \alpha \varphi(f) + \beta \varphi(g)$; therefore,

$$B(\alpha f + \beta g, h) = \sum_{j \neq 0} (ij + j^2) (\alpha f_j + \beta g_j) \bar{h}_j = \alpha \sum_{j \neq 0} (ij + j^2) f_j \bar{h}_j + \beta \sum_{j \neq 0} (ij + j^2) g_j \bar{h}_j \quad (45)$$

$$= \alpha B(f, h) + \beta B(g, h). \quad (46)$$

Similarly,

$$B(h, \alpha f + \beta g) = \sum_{j \neq 0} (ij + j^2) h_j \overline{(\alpha f_j + \beta g_j)} = \alpha \sum_{j \neq 0} (ij + j^2) h_j \bar{f}_j + \beta \sum_{j \neq 0} (ij + j^2) h_j \bar{g}_j \quad (47)$$

$$= \alpha B(h, f) + \beta B(h, g). \quad (48)$$

Thus, B is bilinear.

Step 2: B is continuous and coercive

We have practically already shown that B is continuous: by (44),

$$|B(f, g)| \leq \sqrt{2} \|f\|_H \|g\|_H \quad (49)$$

for all $f, g \in H$ with $\{f_j\} = \varphi(f)$ and $\{g_j\} = \varphi(g)$. This implies that B is continuous because B is bilinear.

For coercivity, observe that

$$|B(f, f)| = \left| \sum_{j \neq 0} (ij + j^2) |f_j|^2 \right| = \left(\left[\sum_{j \neq 0} j |f_j|^2 \right]^2 + \left[\sum_{j \neq 0} j^2 |f_j|^2 \right]^2 \right)^{\frac{1}{2}} \geq \|f\|_H. \quad (50)$$

Thus, B is coercive (note that the first summation converges absolutely by a comparison test, by the way: $|j| |f_j|^2 \leq j^2 |f_j|^2$ for all integers $j \neq 0$). \square

2.3) $H^{-1} \subseteq H^*$ if we assign the following action to $f \in H^{-1}$:

$$f(g) = \sum_{j \neq 0} f_j \bar{g}_j, \quad g \in H, \quad (51)$$

where $\{g_j\} = \varphi(g)$, and $\{f_j\} = \psi(f)$.

Proof. Let $f \in H^{-1}$. Note that, because H is a real Hilbert space, an element of H^* should be a real-valued, bounded linear functional. Thanks to the isomorphisms φ and ψ , the action of $f \in H^{-1}$ is well-defined, but we still need to show that the series converges to a real number. Then, we need to show that f is linear over \mathbf{R} and bounded. Then we will have $f \in H^*$, completing the proof.

Step 1: $f(g)$ converges to a real number

Let $f \in H^{-1}$ and $g \in H$, with $\{f_j\} = \psi(f)$, and $\{g_j\} = \varphi(g)$. Then the series for $f(g)$ converges absolutely because

$$\sum_{j \neq 0} |f_j| |\bar{g}_j| \leq \left(\sum_{j \neq 0} j^{-2} |f_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j \neq 0} j^2 |g_j|^2 \right)^{\frac{1}{2}} = \|f\|_{H^{-1}} \cdot \|g\|_H \quad (52)$$

by the Cauchy-Schwarz inequality.

Next, $f(g) \in \mathbf{R}$ because

$$\overline{f(g)} = \overline{\sum_{j \neq 0} f_j \bar{g}_j} = \sum_{j \neq 0} \bar{f}_j g_j = \sum_{j \neq 0} f_{-j} \bar{g}_{-j} = \sum_{j \neq 0} f_j \bar{g}_j = f(g), \quad (53)$$

and only real numbers equal their own complex conjugate.

Step 2: f is linear and bounded

Let $f \in H^{-1}$ and $g, h \in H$ with $\{f_j\} = \psi(f)$, and $\{g_j\} = \varphi(g)$, $\{h_j\} = \varphi(h)$. Let $\alpha, \beta \in \mathbf{R}$. Then, by the linearity of φ , we have $\varphi(\alpha g + \beta h) = \alpha \varphi(g) + \beta \varphi(h)$. Hence,

$$f(\alpha g + \beta h) = \sum_{j \neq 0} f_j \overline{(\alpha g_j + \beta h_j)} = \alpha \sum_{j \neq 0} f_j \bar{g}_j + \beta \sum_{j \neq 0} f_j \bar{h}_j = \alpha f(g) + \beta f(h). \quad (54)$$

Therefore, f is linear.

We have practically already shown that f is bounded. By (52), we have

$$|f(g)| \leq \|f\|_{H^{-1}} \cdot \|g\|_H \quad (55)$$

for all $g \in G$. Therefore, f is bounded.

Hence, $f \in H^*$ for all $f \in H^{-1}$, using the action we have assigned to f to view it as a functional on H . Thus, $H^{-1} \subseteq H^*$. \square

2.4) For every $f \in H^{-1}$, there exists $u \in H$ such that

$$B(x, u) = f(x) \quad \text{for all } x \in H. \quad (56)$$

Proof. After all the work on the previous parts, this is a simple application of the Lax-Milgram theorem. Since B satisfies the assumptions of the theorem by 2.2, it follows by the Lax-Milgram theorem that for all $f \in H^*$, there exists $u \in H$ such that (56) is true. Since $H^{-1} \subseteq H^*$ by 2.3, given $f \in H^{-1}$, we have $f \in H^*$, so we can find $u \in H$ such that (56) holds. \square

2.5) Suppose that u satisfies (56). Then, formally, u solves the ODE

$$-(u' + u'') = f. \quad (57)$$

Rigorously, if $f \in L^2(-\pi, \pi) \cap H^{-1}$, then u is twice differentiable, and

$$-(u'(x) + u''(x)) = f(x). \quad (58)$$

Proof. We start by showing that u solves the ODE formally. Then, for the subclass of elements of H^{-1} that we made sense of in the definition of ψ , we show that u is indeed twice differentiable and solves the ODE in a rigorous sense.

Formal analysis

Set $\{u_j\} = \varphi(u)$, and $\{f_j\} = \psi(f)$. Formally,

$$u'(x) = \sum_{j \neq 0} i j u_j e^{ijx}, \quad u''(x) = - \sum_{j \neq 0} j^2 u_j e^{ijx}. \quad (59)$$

Thus, for all $x \in H$, if $\{x_j\} = \varphi(x)$ and $\{u'_j\} = \varphi(u')$, and $\{u''_j\} = \varphi(u'')$, then

$$\sum_{j \neq 0} f_j \bar{x}_j = \sum_{j \neq 0} (ij + j^2) x_j \bar{u}_j = \sum_{j \neq 0} (-ij + j^2) x_{-j} \bar{u}_{-j} = - \sum_{j \neq 0} (u'_j + u''_j) \bar{x}_j. \quad (60)$$

Choosing $x = e_j$ gives $f_j = -(u'_j + u''_j)$ for all j , so $f = -(u' + u'')$.

Rigorous analysis

Being more rigorous, suppose that $f \in L^2(-\pi, \pi) \cap H^{-1}$. Set $\{u_j\} = \varphi(u)$, and $\{f_j\} = \psi(f)$. Define the coefficient sequences $\{u'_j\} = \{i j u_j\}$ and $\{u''_j\} = \{-j^2 u_j\}$. Then

$$\sum_{j \neq 0} j^{-2} |u'_j|^2 \leq \sum_{j \neq 0} j^2 |u_j|^2 < \infty, \quad \sum_{j \neq 0} j^{-2} |u''_j|^2 = \sum_{j \neq 0} j^2 |u_j|^2 < \infty. \quad (61)$$

Therefore, $\{u'_j\}, \{u''_j\} \in H^{-1}$. Let $u' = \psi^{-1}(\{u'_j\})$, and $u'' = \psi^{-1}(\{u''_j\})$. If $x \in H$ with $\{x_j\} = \varphi(x)$, the same calculation as in (60) gives

$$\sum_{j \neq 0} f_j \bar{x}_j = - \sum_{j \neq 0} (u'_j + u''_j) \bar{x}_j. \quad (62)$$

We can choose $\{x_j\} \in S_H$ such that $x_j = 0$ if $j \neq \pm k$ and $x_{\pm k} = 1$. Then (62) implies that

$$f_{-k} + f_k = -(u'_{-k} + u'_k + u''_{-k} + u''_k). \quad (63)$$

We can also choose $\{x_j\} \in S_H$ such that $x_j = 0$ if $j \neq \pm k$ and $x_{\pm k} = \pm i$. Then (62) implies that

$$-f_{-k} + f_k = -(-u'_{-k} + u'_k - u''_{-k} + u''_k). \quad (64)$$

Together, (63) and (64) imply that $f_j = -(u'_j + u''_j)$ for all $j \neq 0$. This implies that $f = -(u' + u'')$ by the linearity of ψ .

We observe that $u' \in L^2(-\pi, \pi)$ because

$$\sum_{j \neq 0} |u'_j|^2 = \sum_{j \neq 0} j^2 |u_j|^2 < \infty. \quad (65)$$

Then $u'' = -f - u' \in L^2(-\pi, \pi)$ because $f \in L^2(-\pi, \pi)$. This implies that $\sum_{j \neq 0} j^4 |u_j|^2 < \infty$ because $\{u''_j\} = \{-j^2 u_j\}$ are the Fourier coefficients of u'' with respect to the orthonormal basis $\{e_j\}$ if $u'' \in L^2(-\pi, \pi)$. \square

2.6) Any bounded set in H is pre-compact in $L^2(-\pi, \pi)$.

Proof. Let A be a bounded set in H ; that is, there is some $M > 0$ such that $\|g\|_H^2 \leq M$ for all $g \in A$. We recall that $\{e_j\}$ is an orthonormal basis in $L^2(-\pi, \pi)$. Hence, for any $f \in L^2(-\pi, \pi)$ (see Arbogast and Bona, Theorem 3.18 part (iii)),

$$\|f\|_{L^2(-\pi, \pi)}^2 = \sum_j |f_j|^2, \quad f_j = (f, e_j). \quad (66)$$

If $f \in H$ as well, then $\{f_j\} = \varphi(f)$, so

$$\|f\|_{L^2(-\pi, \pi)}^2 = \sum_j |f_j|^2 \leq \sum_{j \neq 0} j^2 |f_j|^2 = \|f\|_H^2. \quad (67)$$

This implies that A is bounded in $L^2(-\pi, \pi)$ in addition to H . Then \bar{A} is also bounded in $L^2(-\pi, \pi)$, because the closure of a bounded set is bounded. We need to show that \bar{A} is also compact.

To this end, let $\{f^n\}$ be a sequence in \bar{A} . Since \bar{A} is bounded in $L^2(-\pi, \pi)$, and $L^2(-\pi, \pi)$ is a Hilbert space, there exists a weakly convergent subsequence $\{f^{n_k}\}$ of $\{f^n\}$. That is, there exists $f \in L^2(-\pi, \pi)$ such that $f_j^{n_k} \rightarrow f_j$ for all j , where $f_j^{n_k} = (f^{n_k}, e_j)$, and $f_j = (f, e_j)$. If we can show that $f^{n_k} \rightarrow f$ strongly as well, then we are done.

Before that, however, we need the following fact: for all $g \in A$ and all $J > 0$,

$$\sum_{|j| > J} |g_j|^2 \leq \frac{1}{J^2} \sum_{|j| > J} j^2 |g_j|^2 \leq \frac{M}{J^2}, \quad \{g_j\} = \varphi(g). \quad (68)$$

Therefore, given $\varepsilon > 0$, we can choose $J > 0$ such that for all $g \in A$,

$$\sum_{|j| > J} |g_j|^2 < \varepsilon, \quad \{g_j\} = \varphi(g). \quad (69)$$

Now we show that $\{f^{n_k}\}$ converges strongly. Since $L^2(-\pi, \pi)$ is complete, we only need to show $\{f^{n_k}\}$ is Cauchy in $\|\cdot\|_{L^2(-\pi, \pi)}$.

Let $\varepsilon > 0$ be given. Then we can choose $J > 0$ such that for all $g \in A$,

$$\sum_{|j|>J} |g_j|^2 < \varepsilon, \quad \{g_j\} = \varphi(g). \quad (70)$$

Since $\{f_j^{n_k}\}_{k=1}^\infty$ is convergent for all j , it is also Cauchy for all j . Thus, we can choose K large enough that $k, \ell > K$ implies that $|f_j^{n_k} - f_j^{n_\ell}|^2 < \frac{\varepsilon}{2J+1}$ for all $0 \leq |j| \leq J$. Then

$$\|f^{n_k} - f^{n_\ell}\|_{L^2(-\pi, \pi)}^2 = \sum_{|j| \leq J} |f_j^{n_k} - f_j^{n_\ell}|^2 + \sum_{|j| > J} |f_j^{n_k} - f_j^{n_\ell}|^2 \leq \varepsilon + \sum_{|j| > J} |f_j^{n_k} - f_j^{n_\ell}|^2 \quad (71)$$

if $k, \ell > K$. Since $f^{n_k} \in \overline{A}$, for all k , we can choose $g^k \in A$ such that $\|f^{n_k} - g^k\|_{L^2(-\pi, \pi)}^2 < \varepsilon$. Hence, if $g_j^k = \varphi(g^k)$, then

$$\|f^{n_k} - f^{n_\ell}\|_{L^2(-\pi, \pi)}^2 \leq \varepsilon + \sum_{|j| > J} |(f_j^{n_k} - g_j^k) + (g_j^k - g_j^\ell) + (g_j^\ell - f_j^{n_\ell})|^2 \quad (72)$$

$$\leq \varepsilon + 3 \sum_{|j| > J} |f_j^{n_k} - g_j^k|^2 + 3 \sum_{|j| > J} |f_j^{n_\ell} - g_j^\ell|^2 + 3 \sum_{|j| > J} |g_j^k - g_j^\ell|^2 \quad (73)$$

$$\leq \varepsilon + 3\|f^{n_k} - g^k\|_{L^2(-\pi, \pi)}^2 + 3\|f^{n_\ell} - g^\ell\|_{L^2(-\pi, \pi)}^2 \quad (74)$$

$$+ 6 \sum_{|j| > J} |g_j^k|^2 + 6 \sum_{|j| > J} |g_j^\ell|^2 \quad (75)$$

$$\leq 19\varepsilon \quad (76)$$

if $k, \ell > K$. This implies that $\{f^{n_k}\}$ is Cauchy in $\|\cdot\|_{L^2(-\pi, \pi)}$. Therefore, $\{f^{n_k}\}$ converges in $L^2(-\pi, \pi)$.

Thus, any sequence in \overline{A} has a convergent subsequence. This implies that \overline{A} is compact, and A is pre-compact. \square