Math 6418 Homework 1

Jacob Hauck

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Define

$$k_\varepsilon(x) = \frac{1}{\varepsilon}\chi_{\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]}(x). \tag{1}$$

1.

For $u \in \mathcal{D}'(\mathbf{R})$, define $u * k_\varepsilon \in \mathcal{D}'(\mathbf{R})$ by

$$\langle u * k_\varepsilon, \varphi\rangle = \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi)(y)\ \mathrm{d}y, \qquad \varphi \in \mathcal{D}(\mathbf{R}). \tag{2}$$

Note that the integral is defined because $u * R\varphi$ is continuous. Furthermore, $u * k_\varepsilon$ is a distribution because it is linear and continuous.

Linearity

We can verify linearity easily using the linearity of convolution with a test function, the linearity of integration, and the linearity of reflection. If $\alpha, \beta \in \mathbf{R}$ and $\varphi, \psi \in \mathcal{D}(\mathbf{R})$, then

$$\begin{aligned}
\langle u * k_\varepsilon, \alpha\varphi + \beta\psi\rangle &= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R(\alpha\varphi + \beta\psi))(y)\ \mathrm{d}y && (3)\\
&= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * (\alpha R\varphi + \beta R\psi))(y)\ \mathrm{d}y && (4)\\
&= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \big[\alpha(u * R\varphi)(y) + \beta(u * R\psi)(y)\big]\ \mathrm{d}y && (5)\\
&= \alpha\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi)(y)\ \mathrm{d}y + \beta\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\psi)(y)\ \mathrm{d}y && (6)\\
&= \alpha\langle u * k_\varepsilon, \varphi\rangle + \beta\langle u * k_\varepsilon, \psi\rangle, && (7)
\end{aligned}$$

so $u * k_\varepsilon$ is linear.

Continuity

Let $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbf{R})$. Then clearly $R\varphi_n \to R\varphi$ in $\mathcal{D}(\mathbf{R})$, and by the continuity of u, $u * R\varphi_n \to u * R\varphi$ pointwise. If $\{u * R\varphi_n\}_{n=1}^\infty$ is uniformly bounded on $\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$, then the bounded convergence theorem implies that

$$\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi_n)(y)\ \mathrm{d}y \to \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi)(y)\ \mathrm{d}y; \tag{8}$$

that is, $\langle u * k_\varepsilon, \varphi_n\rangle \to \langle u * k_\varepsilon, \varphi\rangle$. Thus, $u * k_\varepsilon$ is continuous.

To show that $\{u * R\varphi_n\}$ is uniformly bounded on $\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$, suppose for the sake of contradiction that it is not. Then for all $m > 0$, there exists φ_{n_m} and $x_m \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$ such that

$$m \le |(u * \varphi_{n_m})(x_m)| = |\langle u, \tau_{x_m} R\varphi_{n_m}\rangle|.$$

There is a convergent subsequence $\{x_{m_k}\}_{k=1}^{\infty}$ such that $x_{m_k} \to x \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$. The convergence of $\{x_{m_k}\}$ and the convergence of $\{\varphi_{n_{m_k}}\}$ in $\mathcal{D}(\mathbf{R})$ ensure that $\tau_{x_{m_k}} R\varphi_{n_{m_k}} \to \tau_x R\varphi$ in $\mathcal{D}(\mathbf{R})$ as $k \to \infty$. Then, by the continuity of u,

$$\infty = \lim_{k\to\infty} |\langle u, \tau_{x_{m_k}} R\varphi_{n_{m_k}}\rangle| = |\langle u, \tau_x R\varphi\rangle| < \infty,$$

which is a contradiction.

Extension

Definition (2) is a good definition of convolution at least in the sense that it reduces to convolution with k_ε for regular distributions. Indeed, suppose that $f \in L^1_{\text{loc}}(\mathbf{R})$. Then for any $\varphi \in \mathcal{D}(\mathbf{R})$,

$$\begin{aligned}
\langle f * k_\varepsilon, \varphi\rangle &= \int_{-\infty}^{\infty} (f * k_\varepsilon)(x)\varphi(x)\,\mathrm{d}x && (9)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)k_\varepsilon(x-y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x && (10)\\
&= \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} f(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x. && (11)
\end{aligned}$$

Using the change of variables $y' = y - x$, we get

$$\begin{aligned}
\langle f * k_\varepsilon, \varphi\rangle &= \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} f(y'+x)\varphi(x)\,\mathrm{d}y'\,\mathrm{d}x && (12)\\
&= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}}\int_{-\infty}^{\infty} f(y'+x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y'. && (13)
\end{aligned}$$

Using the change of variables $x' = y' + x$, we get

$$\begin{aligned}
\langle f * k_\varepsilon, \varphi\rangle &= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}}\int_{-\infty}^{\infty} f(x')\varphi(x'-y')\,\mathrm{d}x'\,\mathrm{d}y' && (14)\\
&= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (f * R\varphi)(y')\,\mathrm{d}y', && (15)
\end{aligned}$$

which agrees with our definition of $f * k_\varepsilon$ in (2) if we view f as a distribution.

2.

Consider $\delta_0 * k_\varepsilon$ using our definition of convolution from (2). Since δ_0 is supposed to be the identity for the convolution operator, we expect that $\delta_0 * k_\varepsilon = k_\varepsilon$ (viewing k_ε as a distribution).

This turns out to be the case. According to the definition in (2), for any $\varphi \in \mathcal{D}(\mathbf{R})$,

$$\begin{aligned}
\langle \delta_0 * k_\varepsilon, \varphi\rangle &= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (\delta_0 * R\varphi)(y)\,\mathrm{d}y && (16)\\
&= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} R\varphi(y)\,\mathrm{d}y && \text{because } \delta_0 \text{ is identity for convolution} \quad (17)\\
&= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \varphi(y)\,\mathrm{d}y && \text{using change of variables } y \mapsto -y \quad (18)\\
&= \langle k_\varepsilon, \varphi\rangle. && (19)
\end{aligned}$$

Thus, $\delta_0 * k_\varepsilon = k_\varepsilon$, viewing k_ε as a distribution.

3.

Since $\int k_\varepsilon = 1$ all ε, and $k_\varepsilon(x) \to 0$ as $\varepsilon \to 0$ if $x \neq 0$, it would seem that k_ε behaves like δ_0 as $\varepsilon \to 0$. Thus, it would make sense that $u * k_\varepsilon$ "$\to u * \delta_0 = u$". That is, it would make sense that $u * k_\varepsilon \to u$ as $\varepsilon \to 0$ in the topology of $\mathcal{D}'(\mathbf{R})$.

In fact, this turns out to be the case. Let $\varphi \in \mathcal{D}(\mathbf{R})$. Since $u * R\varphi$ is C^∞ and therefore continuous, it has an antiderivative ψ. Then

$$\begin{aligned}
\lim_{\varepsilon\to 0}\langle u * k_\varepsilon, \varphi\rangle &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi)(y)\,\mathrm{d}y && (20)\\
&= \lim_{\varepsilon\to 0}\frac{\psi\left(\frac{\varepsilon}{2}\right) - \psi\left(-\frac{\varepsilon}{2}\right)}{\varepsilon} = \psi'(0) && (21)\\
&= (u * R\varphi)(0) = \langle u, \tau_0 RR\varphi\rangle && (22)\\
&= \langle u, \varphi\rangle. && (23)
\end{aligned}$$

Hence, $u * k_\varepsilon \to u$ in $\mathcal{D}'(\mathbf{R})$ as $\varepsilon \to 0$.

4.

For $u, v \in \mathcal{D}'(\mathbf{R})$ we are tempted to define $u * v \in \mathcal{D}'(\mathbf{R})$ by

$$\langle u * v, \varphi\rangle = \langle u, Rv * \varphi\rangle, \qquad \varphi \in \mathcal{D}(\mathbf{R}). \tag{24}$$

Unfortunately, although $Rv * \varphi$ is C^∞, it may not be compactly supported. For example, if $Rv = 1$, then $Rv * \varphi = \int \varphi$, which is not compactly supported as long as $\int \varphi \neq 0$. Thus, it may not make sense to take the action of u on $Rv * \varphi$.

Furthermore, this definition would obviously be linear in φ by the linearity of convolution and of u, but to prove continuity, we would need to prove that $Rv * \varphi_n \to Rv * \varphi$ in $\mathcal{D}(\mathbf{R})$ whenever $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbf{R})$. So, not only would we need that $Rv * \varphi_n$ is compactly supported, but we would need that each $Rv * \varphi_n$ is supported in the same compact set.

If we could provide some restriction on v so that Rv could allay these concerns, then there would be some hope of defining $u * v$ as above. Alternatively, we might be able to restrict u so that it's action could be extended to a function that is not compactly supported. This is essentially what we did in part 1. with k_ε. Using the fact the $Rk_\varepsilon = k_\varepsilon$, a simple change of variables shows that the proposed definition above and the definition given for convolution with k_ε are one and the same, assuming that the action of k_ε is defined by integration (in which case we must use the compact support of k_ε to deal with the fact that $Rv * \varphi$ might not be compactly supported).

On the bright side, if $f, g \in L^1(\mathbf{R})$ are viewed as distributions, then $f * g$ as defined above is the same as the usual convolution of f with g, which I will denote $f \star g$ to avoid confusion. Indeed, for any $\varphi \in \mathcal{D}(\mathbf{R})$,

$$\begin{aligned}
\langle f * g, \varphi\rangle &= \langle f, Rg * \varphi\rangle && (25)\\
&= \int_{-\infty}^{\infty} f(x)(Rg \star \varphi)(x)\,\mathrm{d}x \qquad (* \text{ reduces to } \star \text{ for } Rg \text{ and } \varphi) && (26)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)Rg(y)\varphi(x-y)\,\mathrm{d}y\,\mathrm{d}x && (27)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(-y)\varphi(x-y)\,\mathrm{d}y\,\mathrm{d}x. && (28)
\end{aligned}$$

If we let $y' = x - y$, then we get

$$\langle f * g, \varphi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) g(y' - x) \varphi(y') \, \mathrm{d}y' \, \mathrm{d}x \tag{29}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) g(y' - x) \varphi(y') \, \mathrm{d}x \, \mathrm{d}y' \tag{30}$$

$$= \int_{-\infty}^{\infty} (f \star g)(y') \varphi(y') \, \mathrm{d}y' \tag{31}$$

$$= \langle f \star g, \varphi \rangle. \tag{32}$$

Thus, as distributions, we have $f * g = f \star g$, as claimed.