Math 6418 Homework 1

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Define

$$k_\varepsilon(x) = \frac{1}{\varepsilon}\chi_{\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]}(x). \tag{1}$$

1.

For $u \in \mathcal{D}'(\mathbf{R})$, define $u * k_\varepsilon \in \mathcal{D}'(\mathbf{R})$ by

$$\langle u * k_\varepsilon, \varphi\rangle = \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi)(y)\,\mathrm{d}y, \qquad \varphi \in \mathcal{D}(\mathbf{R}). \tag{2}$$

Note that the integral is defined because $u * R\varphi$ is continuous. Furthermore, $u * k_\varepsilon$ is a distribution because it is linear and continuous.

Linearity

We can verify linearity easily using the linearity of convolution with a test function, the linearity of integration, and the linearity of reflection. If $\alpha, \beta \in \mathbf{R}$ and $\varphi, \psi \in \mathcal{D}(\mathbf{R})$, then

$$\begin{aligned}
\langle u * k_\varepsilon, \alpha\varphi + \beta\psi\rangle &= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R(\alpha\varphi + \beta\psi))(y)\,\mathrm{d}y & (3)\\
&= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * (\alpha R\varphi + \beta R\psi))(y)\,\mathrm{d}y & (4)\\
&= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \big[\alpha(u * R\varphi)(y) + \beta(u * R\psi)(y)\big]\,\mathrm{d}y & (5)\\
&= \alpha\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi)(y)\,\mathrm{d}y + \beta\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\psi)(y)\,\mathrm{d}y & (6)\\
&= \alpha\langle u * k_\varepsilon, \varphi\rangle + \beta\langle u * k_\varepsilon, \psi\rangle, & (7)
\end{aligned}$$

so $u * k_\varepsilon$ is linear.

Continuity

Let $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbf{R})$. Then clearly $R\varphi_n \to R\varphi$ in $\mathcal{D}(\mathbf{R})$. Recalling that convolution of a test function with a distribution is continuous on $\mathcal{D}(\mathbf{R})$, it follows that $u * R\varphi_n \to u * R\varphi$ in $\mathcal{D}(\mathbf{R})$. Then $u * R\varphi_n$ also converges to $u * R\varphi$ uniformly on $\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$, so

$$\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi_n)(y)\,\mathrm{d}y \to \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi)(y)\,\mathrm{d}y; \tag{8}$$

that is, $\langle u * k_\varepsilon, \varphi_n\rangle \to \langle u * k_\varepsilon, \varphi\rangle$. Thus, $u * k_\varepsilon$ is continuous.

Extension

Definition (2) is a good definition of convolution at least in the sense that it reduces to convolution with k_ε for regular distributions. Indeed, suppose that $f \in L^1_{\text{loc}}(\mathbf{R})$. Then for any $\varphi \in \mathcal{D}(\mathbf{R})$,

$$\begin{aligned}
\langle f * k_\varepsilon, \varphi \rangle &= \int_{-\infty}^{\infty} (f * k_\varepsilon)(x)\varphi(x)\ \mathrm{d}x && (9)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)k_\varepsilon(x-y)\varphi(x)\ \mathrm{d}y\ \mathrm{d}x && (10)\\
&= \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} f(y)\varphi(x)\ \mathrm{d}y\ \mathrm{d}x. && (11)
\end{aligned}$$

Using the change of variables $y' = y - x$, we get

$$\begin{aligned}
\langle f * k_\varepsilon, \varphi \rangle &= \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} f(y'+x)\varphi(x)\ \mathrm{d}y'\ \mathrm{d}x && (12)\\
&= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}}\int_{-\infty}^{\infty} f(y'+x)\varphi(x)\ \mathrm{d}x\ \mathrm{d}y'. && (13)
\end{aligned}$$

Using the change of variables $x' = y' + x$, we get

$$\begin{aligned}
\langle f * k_\varepsilon, \varphi \rangle &= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}}\int_{-\infty}^{\infty} f(x')\varphi(x'-y')\ \mathrm{d}x'\ \mathrm{d}y' && (14)\\
&= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (f * R\varphi)(y')\ \mathrm{d}y', && (15)
\end{aligned}$$

which agrees with our definition of $f * k_\varepsilon$ in (2) if we view f as a distribution.

2.

Consider $\delta_0 * k_\varepsilon$ using our definition of convolution from (2). Since δ_0 is supposed to be the identity for the convolution operator, we expect that $\delta_0 * k_\varepsilon = k_\varepsilon$ (viewing k_ε as a distribution).

This turns out to be the case. According to the definition in (2), for any $\varphi \in \mathcal{D}(\mathbf{R})$,

$$\begin{aligned}
\langle \delta_0 * k_\varepsilon, \varphi \rangle &= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (\delta_0 * R\varphi)(y)\ \mathrm{d}y && (16)\\
&= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} R\varphi(y)\ \mathrm{d}y && \text{because } \delta_0 \text{ is identity for convolution} \quad (17)\\
&= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \varphi(y)\ \mathrm{d}y && \text{using change of variables } y \mapsto -y \quad (18)\\
&= \langle k_\varepsilon, \varphi \rangle. && (19)
\end{aligned}$$

Thus, $\delta_0 * k_\varepsilon = k_\varepsilon$, viewing k_ε as a distribution.

3.

Since $\int k_\varepsilon = 1$ all ε, and $k_\varepsilon(x) \to 0$ as $\varepsilon \to 0$ if $x \neq 0$, it would seem that k_ε behaves like δ_0 as $\varepsilon \to 0$. Thus, it would make sense that $u * k_\varepsilon$ "$\to u * \delta_0 = u$". That is, it would make sense that $u * k_\varepsilon \to u$ as $\varepsilon \to 0$ in the topology of $\mathcal{D}'(\mathbf{R})$.

In fact, this turns out to be the case. Let $\varphi \in \mathcal{D}(\mathbf{R})$. Since $u * R\varphi$ is a test function and therefore continuous, it has an antiderivative ψ. Then

$$\begin{aligned}
\lim_{\varepsilon \to 0} \langle u * k_\varepsilon, \varphi \rangle &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (u * R\varphi)(y)\, \mathrm{d}y && (20) \\
&= \lim_{\varepsilon \to 0} \frac{\psi\left(\frac{\varepsilon}{2}\right) - \psi\left(-\frac{\varepsilon}{2}\right)}{\varepsilon} = \psi'(0) && (21) \\
&= (u * R\varphi)(0) = \langle u, \tau_0 RR\varphi \rangle && (22) \\
&= \langle u, \varphi \rangle. && (23)
\end{aligned}$$

Hence, $u * k_\varepsilon \to u$ in $\mathcal{D}'(\mathbf{R})$ as $\varepsilon \to 0$.

4.