Math 5601 Homework 10

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Problem 1.

Let

$$A = \begin{pmatrix} 5 & -2 \\ -4 & 7 \end{pmatrix}. \tag{1}$$

Then the eigenvalues of A are the roots λ of the equation $(5-\lambda)(7-\lambda) - 8 = 0$. That is, $\lambda^2 - 12\lambda + 27 = 0$, or $(\lambda - 3)(\lambda - 9) = 0$. Thus, the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 9$. This implies that the spectral radius of A is $\rho(A) = \max\{3, 9\} = 9$.

By Lemma (I) in the lecture, we can compute $\|A\|_2$ by computing $\sqrt{\rho(A^T A)}$. Since

$$A^T A = \begin{pmatrix} 41 & -38 \\ -38 & 53 \end{pmatrix}, \tag{2}$$

we see that the eigenvalues of $A^T A$ are the roots σ of the equation $(41-\sigma)(53-\sigma) - 38^2 = 0$, or $\sigma^2 - 94\sigma + 41 \cdot 53 - 38^2 = 0$. Then

$$\sigma = 47 \pm \sqrt{47^2 - 41 \cdot 53 + 38^2} = 47 \pm \sqrt{1480}. \tag{3}$$

Therefore, $\rho(A^T A) = 47 + \sqrt{1480}$, and $\|A\|_2 = \sqrt{47 + \sqrt{1480}} \approx 9.254 \geq \rho(A)$.

By Lemma (I) in the lecture, we can compute $\|A\|_1$ by computing the maximum absolute column sum. The absolute column sums of A are $5 + 4 = 9$, and $2 + 7 = 9$. Therefore, $\|A\|_1 = 9 \geq \rho(A)$.

By Lemma (I) in the lecture, we can compute $\|A\|_\infty$ by computing the maximum absolute row sum. The absolute row sums of A are $5 + 2 = 7$, and $4 + 7 = 11$. Therefore, $\|A\|_\infty = 11 \geq \rho(A)$.

Problem 2.

Let $A = \{a_{ij}\}$ be the tridiagonal matrix with entries

$$a_{ij} = \begin{cases} 4 & \text{if } i = j, \\ -1 & \text{if } i = j+1 \text{ or } i = j-1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Suppose that the Jacobi method for solving $Ax = b$ has the form $x^{(k+1)} = Bx^{(k)} + c$. We can compute B and c by the definition of the Jacobi method as $B = D^{-1}(L+U)$, and $c = D^{-1}b$, where L, U, and D are the lower, upper and diagonal parts of A. Clearly, D^{-1} is a diagonal matrix with diagonal elements all equal to $\frac{1}{4}$. Then $B = \{b_{ij}\}$ is the matrix with entries

$$b_{ij} = \begin{cases} -\frac{1}{4} & \text{if } i = j+1 \text{ or } i = j-1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

By Lemma (I) in the lecture, $\|B\|_1$ is the maximum absolute column sum of B. Since each column of B has no more than two nonzero entries, each of which is $-\frac{1}{4}$, it follows that $\|B\|_1 = \frac{1}{2} < 1$. Thus, by condition (c) of the given Theorem, the Jacobi method converges.

Problem 3.

Rewriting the system of equations in matrix-vector form $Ax = b$, we must have

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -3 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6}$$

Suppose that the Jacobi method for $Ax = b$ is given by $x^{(k+1)} = B_J x^{(l)} + c_J$, and the Gauss-Seidel method for $Ax = b$ is given by $x^{(k+1)} = B_G x^{(k)} + c_G$. Recalling computations from Homework 9, we have

$$B_J = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -\frac{1}{3} & -\frac{2}{3} & 0 \end{pmatrix}, \qquad B_G = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}. \tag{7}$$

For B_J, the eigenvalues are the roots λ of the equation $\lambda\left(\lambda^2 + \frac{2}{3}\right) + \frac{2}{3} = 0$. Using MATLAB to find the roots of this equation, we see that the maximum modulus of the eigenvalues is around 0.944, so $\rho(B_J) < 1$. Thus, part (b) of the given Theorem implies that the Jacobi method for this problem converges, which is consistent with the numerical behavior observed in Homework 9.

For B_G, note that our initial point $x_0 = (1, 1, 1)^T$ is an eigenvector with corresponding eigenvalue -1:

$$B_G x_0 = -x_0. \tag{8}$$

Therefore, $\rho(B_G) \geq 1$, and part (b) of the given Theorem implies that the Gauss-Seidel method may not converge for this problem. Indeed, the equation (8) explains why the Gauss-Seidel method simply oscillated back and forth between x_0 and $-x_0$ in Homework 9 (note that $c_G = 0$).