

Math 5604 Homework 1

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Problem 1.

Consider the IVP

$$y' = 3 + e^{-t} - y, \quad t > 0; \quad y(0) = 1. \quad (1)$$

1.1) Multiplying both sides by the integrating factor e^t gives

$$y'e^t + ye^t = 3e^t + 1. \quad (2)$$

The left-hand side is $(ye^t)'$, so integrating on both sides gives

$$ye^t = 3e^t + t + C, \quad (3)$$

for some constant C , so $y(t) = 3 + (t + C)e^{-t}$. The initial condition $y(0) = 1$ implies that $C = -2$, so

$$y(t) = 3 + (t - 2)e^{-t}. \quad (4)$$

1.2) (a) To discretize the IVP on $[0, 2]$ using the forward Euler method, we need to have an evenly-spaced set of time samples $\{t_i\}_{i=0}^n$ defined by

$$t_i = \begin{cases} 0 & i = 0 \\ t_{i-1} + k & i \geq 1, \end{cases}, \quad i = 0, 1, \dots, n. \quad (5)$$

The value k is the step size and is chosen so that $t_n = 2$; that is, $k = \frac{2}{n}$. We will attempt to find an approximation $\{y_i\}_{i=0}^n$ of the values $\{y(t_i)\}_{i=0}^n$. To find $\{y_i\}$, we create and solve a system of equations from the ODE by approximating $y'(t_i)$ by the forward difference $y'(t_i) \approx \frac{y(t_{i+1}) - y(t_i)}{k}$, where $i < n$. Since we know that $y(0) = 1$ from the initial condition, we are led to the scheme

$$\begin{cases} y_0 = 1 \\ \frac{y_{i+1} - y_i}{k} = 3 + e^{-t_i} - y_i \end{cases} \quad 0 \leq i < n, \quad (6)$$

which allows to write an explicit recursive formula for y_i :

$$\begin{cases} y_0 = 1 \\ y_{i+1} = y_i + k(3 + e^{-t_i} - y_i) \end{cases} \quad 0 \leq i < n. \quad (7)$$

(b)

(c)

1.3) (a) To discretize the IVP on $[0, 2]$ using the backward Euler method, we need to have an evenly-spaced set of time samples $\{t_i\}_{i=0}^n$ defined by

$$t_i = \begin{cases} 0 & i = 0 \\ t_{i-1} + k & i \geq 1, \end{cases}, \quad i = 0, 1, \dots, n. \quad (8)$$

The value k is the step size and is chosen so that $t_n = 2$; that is, $k = \frac{2}{n}$. We will attempt to find an approximation $\{y_i\}_{i=0}^n$ of the values $\{y(t_i)\}_{i=0}^n$. To find $\{y_i\}$, we create and solve a system of equations from the ODE by approximating $y'(t_i)$ by the backward difference $y'(t_i) \approx \frac{y(t_i) - y(t_{i-1}))}{k}$, where $i > 0$. Since we know that $y(0) = 1$ from the initial condition, we are led to the scheme

$$\begin{cases} y_0 = 1 \\ \frac{y_i - y_{i-1}}{k} = 3 + e^{-t_i} - y_i \quad 0 < i \leq n, \end{cases} \quad (9)$$

which allows to write an explicit recursive formula for y_i :

$$\begin{cases} y_0 = 1 \\ y_i = \frac{y_{i-1} + k(3 + e^{-t_i})}{1+k} \quad 0 < i \leq n. \end{cases} \quad (10)$$

(b)

(c)

1.4)

Problem 2.

Consider the IVP

$$y' = \frac{3t^2 + 10t + 1}{2(y+1)}, \quad t > 0; \quad y(0) = -2. \quad (11)$$

2.1) Multiplying both sides by $2(y+1)$ gives

$$2(y+1)(y+1)' = 3t^2 + 10t + 1. \quad (12)$$

The left-hand side is $((y+1)^2)'$, so integrating on both sides gives

$$(y+1)^2 = t^3 + 5t^2 + t + C \quad (13)$$

for some constant C . The initial condition $y(0) = -2$ implies that $C = 1$. Therefore,

$$y(t) = -1 \pm \sqrt{t^3 + 5t^2 + t + 1}. \quad (14)$$

The initial condition forces us to choose a negative sign after taking the square root; thus,

$$y(t) = -1 - \sqrt{t^3 + 5t^2 + t + 1}. \quad (15)$$