Exponential Functions

Jacob Hauck

Math 6010

9-20-2023

Objectives

- Define exponential functions ($f(x) = C \cdot a^x$)

Objectives

- Define exponential functions ($f(x) = C \cdot a^x$)
- Graph exponential functions

Objectives

- Define exponential functions ($f(x) = C \cdot a^x$)
- Graph exponential functions
- Understand basic properties of exponential functions

Extended Exponents

Want a function like

$$f(x) = C \cdot a^x$$

Extended Exponents

Want a function like

$$f(x) = C \cdot a^x$$

Recall: if $x = \frac{m}{n}$, then $a^x = a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Extended Exponents

Want a function like

$$f(x) = C \cdot a^x$$

Recall: if $x = \frac{m}{n}$, then $a^x = a^{\frac{m}{n}} = \sqrt[n]{a^m}$

But x might not be $\frac{m}{n}$, for integers m, n...

Extended Exponents

Want a function like

$$f(x) = C \cdot a^x$$

Recall: if $x = \frac{m}{n}$, then $a^x = a^{\frac{m}{n}} = \sqrt[n]{a^m}$

But x might not be $\frac{m}{n}$, for integers m, n...

- $x = \pi, \sqrt{2}$, and so on

Extended Exponents

Want a function like

$$f(x) = C \cdot a^x$$

Recall: if $x = \frac{m}{n}$, then $a^x = a^{\frac{m}{n}} = \sqrt[n]{a^m}$

But x might not be $\frac{m}{n}$, for integers m, n...

- $x = \pi, \sqrt{2}$, and so on

...but *every* number has a decimal expansion

$$3 = 3, \qquad \frac{2}{5} = 0.4, \qquad \pi = 3.14159265..., \qquad \sqrt{2} = 1.41421356...$$

Extended Exponents

Want a function like

$$f(x) = C \cdot a^x$$

Recall: if $x = \frac{m}{n}$, then $a^x = a^{\frac{m}{n}} = \sqrt[n]{a^m}$

But x might not be $\frac{m}{n}$, for integers m, n...

- $x = \pi, \sqrt{2}$, and so on

...but *every* number has a decimal expansion

$$3 = 3, \qquad \frac{2}{5} = 0.4, \qquad \pi = 3.14159265..., \qquad \sqrt{2} = 1.41421356...$$

Approximate any number x by *finitely many* of its decimal digits.

$$a^\pi \approx a^{3.14} = a^{\frac{314}{100}} = \sqrt[100]{a^{314}}$$

Extended Exponents

The more digits we use, the closer we get to the true value of a^π

2^3	$2^{3.1}$	$2^{3.14}$	$2^{3.141}$	$2^{3.1415}$	$\cdots$	2^π
8.0000	8.5742	8.8152	8.8213	8.8244	$\cdots$	8.8250

Extended Exponents

The more digits we use, the closer we get to the true value of a^π

2^3	$2^{3.1}$	$2^{3.14}$	$2^{3.141}$	$2^{3.1415}$	$\cdots$	2^π
8.0000	8.5742	8.8152	8.8213	8.8244	$\cdots$	8.8250

So we can use any number x in the exponent. Hooray!

Extended Exponents

The more digits we use, the closer we get to the true value of a^π

2^3	$2^{3.1}$	$2^{3.14}$	$2^{3.141}$	$2^{3.1415}$	$\cdots$	2^π
8.0000	8.5742	8.8152	8.8213	8.8244	$\cdots$	8.8250

So we can use any number x in the exponent. Hooray!

Better still, we get to keep all the properties of exponents that we already know:

Exponent Properties

If x, y are real numbers, and $a, b > 0$, then

- $a^x \cdot a^y = a^{x+y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x \cdot b^x$
- $1^x = 1$
- $a^{-x} = \dfrac{1}{a^x} = \left(\dfrac{1}{a}\right)^x$
- $a^0 = 1$

Exponential Functions

Now that we know what a^x means for *any* x, we can define an exponential function.

Exponential Function

An **exponential function** is a functiton f such that

$$f(x) = C \cdot a^x,$$

where

- $C \neq 0$ is called the **initial value**, and
- $a > 0$, $a \neq 1$ is called the **growth factor**.

Exponential Functions

Consider the exponential function $f(x) = 5 \cdot 2^x$. Let's make a table of values

x	$f(x)$
-2	1.25
-1	2.5
0	5
1	10
2	20

Exponential Functions

Consider the exponential function $f(x) = 5 \cdot 2^x$. Let's make a table of values

x	$f(x)$
-2	1.25
-1	2.5
0	5
1	10
2	20

- $f(x)$ goes up by $\times 2 = a$ each time x increases

Exponential Functions

Consider the exponential function $f(x) = 5 \cdot 2^x$. Let's make a table of values

x	$f(x)$
-2	1.25
-1	2.5
0	5
1	10
2	20

- $f(x)$ goes up by $\times 2 = a$ each time x increases
- $f(0) = 5 = C$.

Initial Value and Growth Factor

$$f(x) = C \cdot a^x$$

- $f(0) = C \cdot a^0 = C$, the initial value

Initial Value and Growth Factor

$$f(x) = C \cdot a^x$$

- $f(0) = C \cdot a^0 = C$, the initial value
- The growth factor: how much the function grows (or shrinks) every time x goes up by 1

$$f(x+1) = a \cdot f(x)$$

Initial Value and Growth Factor

$$f(x) = C \cdot a^x$$

- $f(0) = C \cdot a^0 = C$, the initial value
- The growth factor: how much the function grows (or shrinks) every time x goes up by 1

$$f(x+1) = a \cdot f(x)$$

because

$$\frac{f(x+1)}{f(x)} = \frac{C \cdot a^{x+1}}{C \cdot a^x}$$

Initial Value and Growth Factor

$$f(x) = C \cdot a^x$$

- $f(0) = C \cdot a^0 = C$, the initial value
- The growth factor: how much the function grows (or shrinks) every time x goes up by 1

$$f(x+1) = a \cdot f(x)$$

because

$$\frac{f(x+1)}{f(x)} = \frac{C \cdot a^{x+1}}{C \cdot a^x} = \frac{a^{x+1}}{a^x}$$

Initial Value and Growth Factor

$$f(x) = C \cdot a^x$$

- $f(0) = C \cdot a^0 = C$, the initial value
- The growth factor: how much the function grows (or shrinks) every time x goes up by 1

$$f(x+1) = a \cdot f(x)$$

because

$$\frac{f(x+1)}{f(x)} = \frac{C \cdot a^{x+1}}{C \cdot a^x} = \frac{a^{x+1}}{a^x} = a^{x+1-x} = a$$

Graphing Exponential Functions

Let's try to graph the function $f(x) = 5 \cdot 2^x$ from before. First, extend the table of values

Graphing Exponential Functions

Let's try to graph the function $f(x) = 5 \cdot 2^x$ from before. First, extend the table of values

x	$f(x)$
-5	0.16125
-4	0.3125
-3	0.625
-2	1.25
-1	2.5
0	5
1	10
2	20
3	40
4	80
5	160

Graphing Exponential Functions

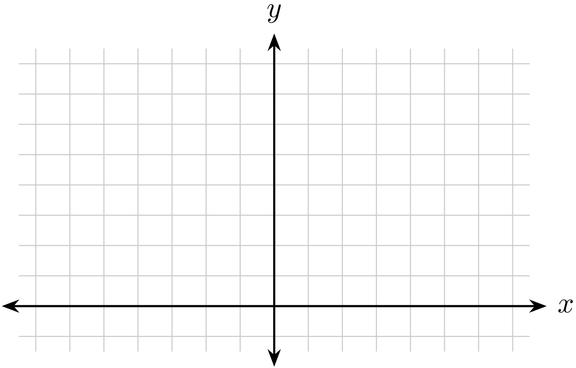
Let's try to graph the function $f(x) = 5 \cdot 2^x$ from before. First, extend the table of values

x	$f(x)$
-5	0.16125
-4	0.3125
-3	0.625
-2	1.25
-1	2.5
0	5
1	10
2	20
3	40
4	80
5	160

It seems that $f(x) \to 0$ as $x \to -\infty$, and $f(x) \to \infty$ as $x \to \infty$

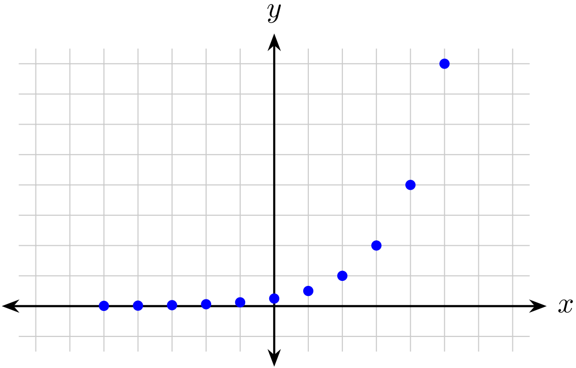
Graphing Exponential Functions

Connect the points continuously and use the asymptotic behavior noted



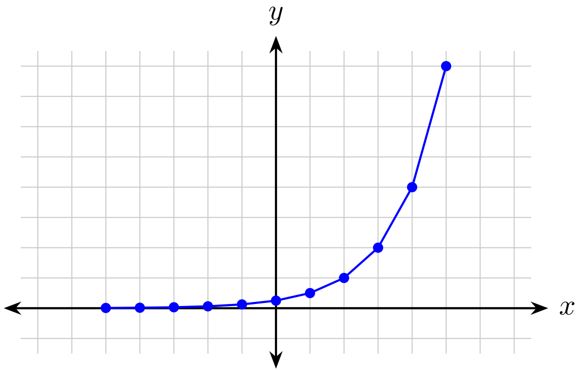
Graphing Exponential Functions

Connect the points continuously and use the asymptotic behavior noted



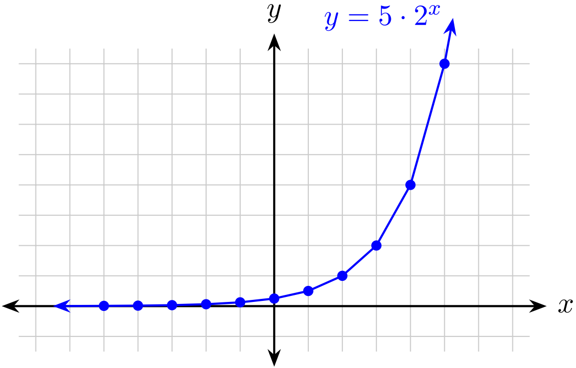
Graphing Exponential Functions

Connect the points continuously and use the asymptotic behavior noted



Graphing Exponential Functions

Connect the points continuously and use the asymptotic behavior noted



Properties

Basic properties:

- Domain is $(-\infty, \infty)$; range is $(0, \infty)$ if $C > 0$, and $(-\infty, 0)$ if $C < 0$

Properties

Basic properties:

- Domain is $(-\infty, \infty)$; range is $(0, \infty)$ if $C > 0$, and $(-\infty, 0)$ if $C < 0$
- Increasing or decreasing, one-to-one, continuous

Properties

Basic properties:

- Domain is $(-\infty, \infty)$; range is $(0, \infty)$ if $C > 0$, and $(-\infty, 0)$ if $C < 0$
- Increasing or decreasing, one-to-one, continuous
- No x-intercept; y-intercept is C

Properties

Basic properties:

- Domain is $(-\infty, \infty)$; range is $(0, \infty)$ if $C > 0$, and $(-\infty, 0)$ if $C < 0$
- Increasing or decreasing, one-to-one, continuous
- No x-intercept; y-intercept is C
- Horizontal asymptote at $x = \infty$ if $a < 1$ and at $x = -\infty$ if $a > 1$

Properties

Basic properties:

- Domain is $(-\infty, \infty)$; range is $(0, \infty)$ if $C > 0$, and $(-\infty, 0)$ if $C < 0$
- Increasing or decreasing, one-to-one, continuous
- No x-intercept; y-intercept is C
- Horizontal asymptote at $x = \infty$ if $a < 1$ and at $x = -\infty$ if $a > 1$

Due to our inherited exponent properties, if $f(x) = C \cdot a^x$, and $g(x) = C \cdot \left(\frac{1}{a}\right)^x$, then

Properties

Basic properties:

- Domain is $(-\infty, \infty)$; range is $(0, \infty)$ if $C > 0$, and $(-\infty, 0)$ if $C < 0$
- Increasing or decreasing, one-to-one, continuous
- No x-intercept; y-intercept is C
- Horizontal asymptote at $x = \infty$ if $a < 1$ and at $x = -\infty$ if $a > 1$

Due to our inherited exponent properties, if $f(x) = C \cdot a^x$, and $g(x) = C \cdot \left(\frac{1}{a}\right)^x$, then

$$f(x) = C \cdot a^x = C \cdot \left(\frac{1}{a}\right)^{-x} = g(-x)$$

That is, g is the reflection of f over the y-axis.