Math 5601: Introduction to Numerical Analysis
Homework assignment 4

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Show all relevant work in detail to justify your conclusions. Partial credit depends upon the work you show. For each numerical experiment, all the .m files of your Matlab code should be electronically submitted to hex@mst.edu together with a .txt file which copies all the information in the Matlab command window when you run the code to obtain the numerical results.

Problem #1: Consider the finite difference formula

$$f'(t_j) = \frac{1}{12h}\left[f(t_j - 2h) - 8f(t_j - h) + 8f(t_j + h) - f(t_j + 2h)\right] + O(h^4).$$

(a) Derive this formula by using Taylor's theorem.

(b) Let $\tilde{f}_j$ be the number stored in the computer memory for $f_j = f(t_j)$. Assume $f_j = \tilde{f}_j + e(t_j)$ where $e(t_j)$ is the roundoff error. In general $|e(x)| \leq \varepsilon |f(x)|$ for some relative error ε. Estimate the total error

$$\left| f'(t_j) - \frac{\tilde{f}_{j-2} - 8\tilde{f}_{j-1} + 8\tilde{f}_{j+1} - \tilde{f}_{j+2}}{12h} \right|.$$

(c) Consider the above finite difference formula for $f(x) = ln(x)$:

$$ln'(x) \approx \frac{ln(x - 2h) - 8ln(x - h) + 8ln(x + h) - ln(x + 2h)}{12h}.$$

Compute the numerical differentiation for $x = 3$ and $h = 10^{-n}$ $(n = 1, 2, \cdots, 20)$. Use $ln'(x) = \frac{1}{x}$ to compute the errors $\left| ln'(3) - \frac{ln(3-2h)-8ln(3-h)+8ln(3+h)-ln(3+2h)}{12h} \right|$ of the numerical solutions. What do you observe from the numerical results? Explain your observation according to the conclusion in part (c).

Problem #2: Find the constants α_i $(i = 1, 2, 3, 4, 5)$ for the following finite difference formula with fourth order accuracy:

$$f'(t_j) = \frac{1}{h}\left[\alpha_1 f(t_j) + \alpha_2 f(t_j + h) + \alpha_3 f(t_j + 2h) + \alpha_4 f(t_j + 3h) + \alpha_5 f(t_j + 4h)\right] + O(h^4).$$

Problem #3: Let $f \in C^\infty(-\infty, \infty)$ and let $x \in \mathbb{R}$ be given. Consider the extrapolation method.

(a) Prove that

$$S_h \triangleq \frac{f(x + h) - f(x - h)}{2h} = f'(x) + \sum_{i=1}^{\infty} c_i h^{2i}$$

where c_i $(i = 1, 2, \cdots)$ are independent of h.

(b) Suppose that S_h and $S_{\frac{h}{2}}$ have been calculated. Use the conclusion in (a) to find constants α_1 and α_2 so that

$$\alpha_1 S_h + \alpha_2 S_{\frac{h}{2}} = f'(x) + O(h^4).$$

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