3.41

$$x' = x + y - x(x^2+y^2)$$
$$y' = -x + y - y(x^2+y^2)$$

Let $\alpha \in \mathbb{R}$, and $x(t) := \cos(t+\alpha)$, $y(t) := -\sin(t+\alpha)$. Then $(x(t), y(t))$ is clearly periodic with period 2π. Moreover

$$\begin{aligned} x'(t) = -\sin(t+\alpha) &= y(t) + x(t) - x(t) \\ &= x(t) + y(t) - x(t)(\cos^2(t+\alpha) + \sin^2(t+\alpha)) \\ &= x(t) + y(t) - x(t)(x^2(t) + y^2(t)) \end{aligned}$$

$$\begin{aligned} y'(t) = -\cos(t+\alpha) &= -x(t) + y(t) - y(t) \\ &= -x(t) + y(t) - y(t)(\cos^2(t+\alpha) + \sin^2(t+\alpha)) \\ &= -x(t) + y(t) - y(t)(x^2(t) + y^2(t)), \end{aligned}$$

so $(x(t), y(t))$ is a periodic solution of the system.

3.45 $x' = x - y - x^3$

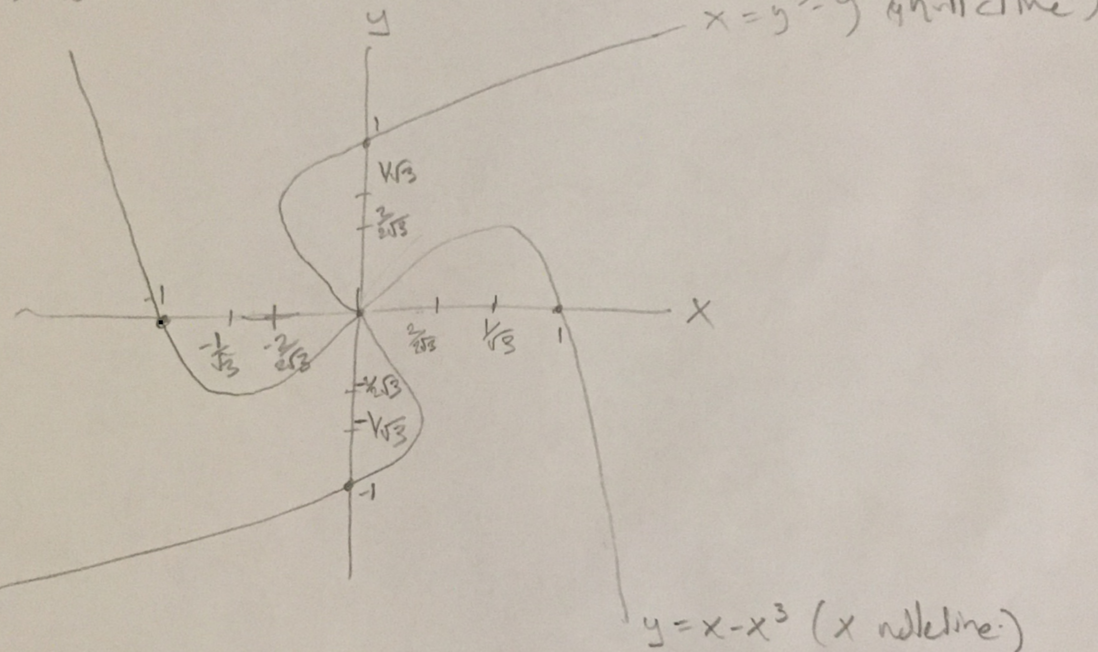
$y' = x + y - y^3$

If $x' = y' = 0$, then $y = x - x^3$ (x nullcline)

$x = y^3 - y$ (y nullcline)

Note that we can plot both nullclines by observing that the function $f(z) = z^3 - z$ has zeroes at $z = \pm 1$, $f(z) \to \pm\infty$ as $z \to \pm\infty$, and f has local extrema when $f'(z) = 3z^2 - 1 = 0 \Rightarrow z = \pm\frac{1}{\sqrt{3}}$, at which points $f(\pm\frac{1}{\sqrt{3}}) = \pm\frac{2}{2\sqrt{3}}$.

Drawing the nullclines shows that $(0,0)$ is the only C.P. (intersection of the nullclines):



Let $S = [-2,2] \times [-2,2]$. If $x = 2$, $y \in [-2,2)$, then

$x' = -6 - y \leq -4 < 0$, and if $y = -2$, then also $y' = -4 + 8 = 4 > 0$,

So for any $(x,y) \in \{2\} \times [-2,2)$, the right side of S, trajectories are going into S.

Let $(\xi, \eta) = (-y, x)$ be $90°$ rotated coordinates. Then

$\xi' = -y' = -x - y + y^3 = -\eta + \xi - \xi^3$

$\eta' = x' = x - y - x^3 = \eta + \xi - \eta^3$, the same system as before. Since the system and the region S are both invariant under $90°$ rotations, the conclusions above apply when rotated by $90°$ arbitrarily many times; that is, trajectories

3.45 starting on any of the sides of S go into S, so S is positively invariant.

On the other hand, the Jacobian of the system is

$$J = \begin{pmatrix} 1-3x^2 & -1 \\ 1 & 1-3y^2 \end{pmatrix}, \text{ and } J(0,0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

has e.v.s satisfying $(1-\lambda)^2+1=0 \Rightarrow \lambda = 1 \pm i$, so $Re\lambda > 0$ for all e.v.s, and therefore no nontrivial solutions $\to (0,0)$.

Let $P \in S \setminus \{(0,0)\}$, and let W = ω-limit set of $\phi(t,P)$, the solution starting at P. Then $\phi(t,P) \not\to (0,0) \Rightarrow (0,0) \notin W$. By the Poincaré-Bendixson theorem either W is a cycle, or $\forall Q \in W$, the ω-limit set of $\phi(t,Q)$ is a nonempty set of C.P.'s. The latter is impossible because $Q \in W \Rightarrow Q \neq (0,0)$, so the ω-limit set of $\phi(t,Q)$ does not contain $(0,0)$, which is the only C.P. of the system. Therefore W is a cycle of the system.

3.52 $x' = 3x - y - xe^{x^2+y^2}$ if $x'=y'=0$, then $x=0 \Rightarrow y=0$, and

$y' = x + 3y - ye^{x^2+y^2}$ if $x \neq 0$, $e^{x^2+y^2} = \frac{3x-y}{x} = 3 - \frac{y}{x}$

$\Rightarrow x + 3y - y(3 - \frac{y}{x}) = 0$

$\Rightarrow x^2 + 3xy - 3xy + y^2 = 0$

$\Rightarrow x^2+y^2 = 0 \Rightarrow x = y = 0$, a contradiction.

therefore $(0,0)$ is the only c.p.

Convert to polar coordinates:

$rr' = xx' + yy' = 3x^2 - x^2e^{x^2+y^2} + 3y^2 - y^2e^{x^2+y^2}$

$= 3r^2 - r^2e^{r^2}$

$\Rightarrow r' = 3r(1 - \frac{e^{r^2}}{3})$

Choose $r = \frac{1}{2}$, then $r'\big|_{r=\frac{1}{2}} = \frac{3}{2}(1 - \frac{1}{3}e^{1/4}) > 0$,

as $e^{1/4} < e < 3 \Rightarrow 1 - \frac{1}{3}e^{1/4} > 0$

Choose $r = 2$, then $r'\big|_{r=2} = 6(1 - \frac{1}{3}e^{4}) < 0$,

as $e^4 > 2^4 = 16 \Rightarrow 1 - \frac{1}{3}e^4 < 0$

Then the annular region $\frac{1}{2} \leq r \leq 2$ is positively invariant and contains no c.p.'s, so it must contain a cycle by the Poincaré-Bendixson Theorem

3.53 (i) $x' = x - y + xy^2$
$y' = x$

Let $F(x,y) = \begin{pmatrix} x - y + xy^2 \\ x \end{pmatrix}$. Set $\alpha(x,y) = 1$. Then

$\operatorname{div}(\alpha F) = 1 + y^2 > 0 \ \forall x,y$ $\therefore$ the system has no cycles by the Bendixson-Dulac Theorem.

(ii) $x' = 1 + xy$ $\qquad x' = 0 \Rightarrow x^2(2 + y^2) = 0 \Rightarrow x^2 = 0$
$y' = 2x^2 + x^2y^2$ $\qquad y' = 0 \qquad\qquad\qquad \Rightarrow x = 0$
$\Rightarrow 1 = 0$. a contradiction

Therefore $\begin{pmatrix} x' \\ y' \end{pmatrix} \neq 0 \ \forall x,y$, i.e., the system has no c.p.s. This implies that it also has no cycles.

(iii) $x' = -x - y + 2x^2 + y^2$
$y' = x$

Let $F(x,y) = \begin{pmatrix} -x - y + 2x^2 + y^2 \\ x \end{pmatrix}$.

$\operatorname{div}(\alpha F) = \alpha_x(-x - y + 2x^2 + y^2) + \alpha(4x - 1) + \alpha_y x$

Try $\alpha_x = 0$, $4x\alpha + \alpha_y x = 0 \Rightarrow \alpha_y = -4\alpha \Rightarrow \alpha = e^{-4y}$

Then $\operatorname{div}(\alpha F) = -e^{-4y} < 0 \ \forall x,y$, so the system has no cycles by the Bendixson-Dulac Theorem.

3.54 (i) $x' = y + x^2 y$, $y' = xy + 2$, $x' = 0 \Rightarrow y(1+x^2) = 0 \Rightarrow y = 0$
$y' = 0 \Rightarrow 2 = 0$, so the system cannot have any c.p.s.

Therefore it also has no cycles.

(ii) $x' = y + x^3$
$y' = x + y + 2y^3$

Let $F(x,y) = \begin{pmatrix} y + x^3 \\ x + y + 2y^3 \end{pmatrix}$. Then for $\alpha(x,y) = 1$

$\operatorname{div}(\alpha F) = 3x^2 + 1 + 6y^2 > 0 \ \forall x, y$, $\therefore$ by the Bendixson-Dulac Theorem the system has no cycles.

(iii) $x' = -y$
$y' = x + y - x^2 - y^2$

Let $F(x,y) = \begin{pmatrix} -y \\ x + y - x^2 - y^2 \end{pmatrix}$, For $\alpha(x,y)$,

$\operatorname{div}(\alpha F) = -y\alpha_x + \alpha_y (x + y - x^2 - y^2) + \alpha(1 - 2y)$

Let $\alpha_y = 0$, $-2y\alpha - y\alpha_x = 0 \Leftarrow 2\alpha + \alpha_x = 0$
or if $\alpha = e^{-2x}$

$\operatorname{div}(\alpha F) = e^{-2x} > 0 \ \forall x, y$, so by the Bendixson-Dulac Theorem the system has no cycles.

3.57 $x'' + 2x^2(x^2-2)x' + x^3 = 0$

Let $f(x) = 2x^2(x^2-2)$, $g(x) = x^3$, Then $x'' + f(x)x' + g(x) = 0$, and

- f is even, and g is odd
- $g(0) = 0$, $g(x) > 0$ if $x > 0$
- $F(x) = \int_0^x f(t)dt = \int_0^x (2t^4 - 4t^2)dt = \frac{2x^5}{5} - \frac{4x^3}{3}$ has zeroes when

$\frac{2}{5}x^5 - \frac{4}{3}x^3 = 0$ or $x^3\left(\frac{2}{5}x^2 - \frac{4}{3}\right) = 0$, or $x = 0$, $x = \pm\sqrt{\frac{10}{3}}$,

So F has 1 positive zero $\sqrt{\frac{10}{3}}$

- $F(x) < 0$ if $x \in (0, \sqrt{\frac{10}{3}})$ and $F(x) > 0$ if $x \in (\sqrt{\frac{10}{3}}, \infty)$
- $F'(x) = f(x) = 2x^2(x^2-2) > 0$ if $x > \sqrt{\frac{10}{3}}$: $2x^2 > 0$, and $x^2 = \frac{10}{3} > 2$

$\Rightarrow x^2 - 2 > 0$. Therefore $F(x)$ is nondecreasing on $(\sqrt{\frac{10}{3}}, \infty)$

- $\lim_{x\to\infty} F(x) = \frac{2}{5}x^5 - \frac{4}{3}x^3 = \infty$

Therefore, by Liénard's Theorem, there exists a unique cycle surrounding $(0,0)$

3.62 $x' = -x + 0.05y + x^2 y$

$y' = \lambda - 0.05y - x^2 y$

If $x' = y' = 0$, then $x' + y' = \lambda - x = 0 \Rightarrow x = \lambda$, and

$y' = 0 \Rightarrow \lambda = y(0.05 + x^2) \Rightarrow y = \frac{\lambda}{0.05 + x^2}$. So the

C.P. as a function of λ is $x_0(\lambda) = \lambda$, $y_0(\lambda) = \frac{\lambda}{0.05 + \lambda^2}$

Then the Jacobian at $(x_0(\lambda), y_0(\lambda))$ is

$$J = \begin{pmatrix} -1 + 2x_0 y_0 & 0.05 + x_0^2 \\ -2x_0 y_0 & -0.05 - x_0^2 \end{pmatrix} \quad \text{so}$$

$$|J - \mu I| = (-1 + 2x_0 y_0 - \mu)(-0.05 - x_0^2 - \mu) + 2x_0 y_0 (0.05 + x_0^2)$$
$$= 0$$

$$\Rightarrow \mu^2 - \mu(-1 + 2x_0 y_0 - 0.05 - x_0^2) - (0.05 + x_0^2)(2x_0 y_0 - 1)$$
$$+ 2x_0 y_0 (0.05 + x_0^2) = 0$$

$$\Rightarrow 0 = \mu^2 + \mu(x_0^2 + 0.05 + 1 - 2x_0 y_0) + 0.05 + x_0^2$$

$$\Rightarrow \mu = \frac{-(x_0^2 + 1.05 - 2x_0 y_0)}{2} \pm \frac{1}{2}\sqrt{(x_0^2 + 1.05 - 2x_0 y_0)^2 - 4(0.05 + x_0^2)}$$

Let $\alpha(\lambda) = \frac{-(x_0^2 + 1.05 - 2x_0 y_0)}{2}$, $f(\lambda) = (x_0^2 + 1.05 - 2x_0 y_0)^2 - 4(0.05 + x_0^2)$
$= \alpha^2 - 4(0.05 + x_0^2)$

$\beta(\lambda) = \frac{1}{2}\sqrt{-f(\lambda)}$ so that $\mu = \alpha(\lambda) + i\beta(\lambda)$. We will see that $\alpha = \mathrm{Re}\,\mu$, $\beta = \mathrm{Im}\,\mu$ when $\alpha(\lambda) = 0$.

$$\alpha(\lambda) = \frac{-\left(\lambda^2 + 1.05 - \frac{2\lambda^2}{0.05 + \lambda^2}\right)}{2}$$

$$f(\lambda) = \left(\lambda^2 + 1.05 - \frac{2\lambda^2}{0.05 + \lambda^2}\right)^2 - 4(0.05 + \lambda^2)$$

If $\alpha(\lambda)=0$, then $\lambda^2+1.05-\frac{2\lambda^2}{0.05+\lambda^2}=0$

or $(0.05+\lambda^2)(\lambda^2+1.05)-2\lambda^2=0$ or $\lambda^4+0.05\lambda^2+1.05\lambda^2-2\lambda^2+1.05\cdot 0.05=0$

or $\lambda^4-0.9\lambda^2+\frac{21}{400}=0 \Rightarrow \lambda^2=\frac{0.9}{2}\pm\frac{1}{2}\sqrt{0.81-0.21}$

$= 0.45\pm\frac{1}{2}\sqrt{.6}$

$\Rightarrow \lambda=\pm\sqrt{0.45\pm\frac{1}{2}\sqrt{0.6}}$

We need $\lambda\in\mathbb{R}$, $\lambda>0$, so we take $\lambda=\sqrt{0.45\pm\frac{1}{2}\sqrt{0.6}}$, which is real because $\frac{1}{2}\sqrt{0.6}<\frac{1}{2}\sqrt{0.64}=\frac{1}{2}0.8=0.4<0.45$

Let $\lambda_0=\sqrt{0.45+\frac{1}{2}\sqrt{0.6}}\approx 0.915$. Then

$\lambda_0^2=0.45+\frac{1}{2}\sqrt{0.6} \Rightarrow \lambda_0^2+0.05=\frac{1+\sqrt{0.6}}{2}$

$\frac{\lambda_0}{\lambda_0^2+0.05}=\frac{2\sqrt{0.45+\frac{1}{2}\sqrt{0.6}}}{1+\sqrt{0.6}}=y_0(\lambda_0)$, $x_0(\lambda_0)=\lambda_0=\sqrt{0.45+\frac{1}{2}\sqrt{0.6}}$

Then $\beta(\lambda_0)=\alpha(\lambda_0)^2-4\cdot(0.05+\lambda_0^2)$
$=-4\cdot(0.05+\lambda_0^2)<0 \Rightarrow \beta(\lambda_0)=0.05+\lambda_0^2=\frac{1+\sqrt{0.6}}{2}\in\mathbb{R}$

So $\operatorname{Re}\mu=\alpha(\lambda_0)=0$, and $\operatorname{Im}\mu=\beta(\lambda_0)\neq 0$

Now, $\alpha'(\lambda_0)=-\frac{1}{2}\left(2\lambda_0+\frac{-2\lambda_0(0.05+\lambda_0^2)+\lambda_0^2\cdot 2\lambda_0}{(\lambda_0^2+0.05)^2}\right)$

$\approx -0.857<0$, so $\alpha(\lambda_0)=0$, $\beta(\lambda_0)\neq 0$, $\alpha'(\lambda_0)\neq 0$.

To apply Hopf bifurcation theorem, we only need to show that $(x_0(\lambda_0), y_0(\lambda_0))$ is a.s. (How to do this?)

Also, in exercise 3.5.1 the existence of a periodic solution with $\lambda=0.5<\lambda_0$ is shown, so the value $\lambda_0=\sqrt{0.45+\frac{1}{2}\sqrt{0.6}}$ is a Hopf Bifurcation.