Math 5601 Homework 2

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Problem 1.

Let $R > 0$, and define $f(x) = 1 - \frac{1}{Rx}$ for $x > 0$. Then clearly $f(x) = 0$ if and only if $x = \frac{1}{R}$, so calculating a zero of f is equivalent to calculating the reciprocal of R.

Let $\{x_k\}$ be the sequence of approximate solutions of $f(x) = 0$ obtained by using Newton's method. Then, by definition,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \left(1 - \frac{1}{Rx_k}\right) \cdot Rx_k^2 = x_k - Rx_k^2 + x_k = x_k(2 - Rx_k) \tag{1}$$

Problem 2.

See `newton.m`; also included here for convenience.

```
function result = newton(f, f_prime, x0, epsilon, epsilon_f, epsilon_f_prime, max_it)
    x_next = x0;

    for k = 0:max_it
        xk = x_next;
        fk = f(xk);
        f_primek = f_prime(xk);

        % check f_prime not zero *before* dividing by it
        if abs(f_primek) <= epsilon_f_prime
            fprintf("Failed. f_prime too small.\n");
            break;
        end

        % now we can update x_next and compute Cauchy error
        x_next = xk - fk / f_primek;
        cauchy_error = abs(x_next - xk);

        fprintf(...
            ['k = %d, x_k = %.5g, Cauchy error = %.5g, '...
            'f(x_k) = %.5g, f''(x_k) = %.5g\n'], ...
            k, xk, cauchy_error, fk, f_primek ...
        );

        if cauchy_error < epsilon || abs(fk) < epsilon_f
            break;
        end
    end

    result = xk;
```

x_0	Converged
0.5	✓
1	✓
1.3	✓
1.4	✗
1.35	✓
1.375	✓
1.3875	✓
1.39375	✗
1.390625	✓
1.3921875	✗

Table 1: Convergence of Newton's method for $f(x) = \tan^{-1}(x)$

The iteration appears to converge (quickly) for some starting values and diverge for others. See Table 1 for a summary of the results. The full outputs from the MATLAB console can be found in `outputs.txt`. In particular, there seems to be a cutoff $c \approx 1.391$ such that the method converges if $|x_0| < c$ and diverges if $|x_0| > c$.

Problem 3.

See `secant.m`; also included here for convenience. Outputs from the MATLAB console can be found in `outputs.txt`.

```
function result = secant(f, x0, x1, epsilon, epsilon_f, max_it)
    xk = x0;
    fk = f(xk);
    x_next = x1;

    for k = 1:max_it
        xkm1 = xk;
        xk = x_next;

        fkm1 = fk;
        fk = f(xk);

        cauchy_error = abs(xk - xkm1);
        x_next = xk - fk * (xk - xkm1) / (fk - fkm1);

        fprintf( ...
            ['k = %d, x_{k-1} = %.05g, xk = %.05g, Cauchy error = %.05g, ' ...
            'f(xk) = %.05g\n'], ...
            k, xkm1, xk, cauchy_error, fk ...
        );

        if cauchy_error < epsilon || abs(fk) < epsilon_f
            break;
        end
    end

    result = xk;
```