Math 6418 Homework 2

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Let

$$f(x) = \begin{cases} 1 & x \in [-1, 1], \\ 0 & |x| > 1. \end{cases} \tag{1}$$

1.

Since $f \in H^s(\mathbf{R})$ if and only if $g_s \in L^2(\mathbf{R})$, where $g_s(\xi) = \widehat{f}(\xi)(1 + |\xi|^2)^{\frac{s}{2}}$, we should start by computing $\widehat{f}$:

$$\begin{aligned}
\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ix\xi}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-ix\xi}}{-i\xi}\right]_{-1}^{1} \\
&= \frac{e^{i\xi} - e^{-i\xi}}{i\xi\sqrt{2\pi}} \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin(\xi)}{\xi}.
\end{aligned}$$

We note that $\widehat{f}(0) = \sqrt{\frac{2}{\pi}}$, so $\widehat{f}$ is continuous. Thus,

$$g_s(\xi) = \sqrt{\frac{2}{\pi}} \frac{\sin(\xi)}{\xi} (1 + |\xi|^2)^{\frac{s}{2}}$$

is also continuous.

Therefore, for $a > 0$,

$$\int_{-\infty}^{\infty} |g_s(\xi)|\,\mathrm{d}\xi = \int_{-a}^{a} |g_s(\xi)|^2\,\mathrm{d}\xi + 2\int_{a}^{\infty} |g_s(\xi)|^2\,\mathrm{d}\xi$$

because g_s is even. The first term is always finite because g_s is continuous, so $g_s \in L^2(\mathbf{R})$ if and only if the second term is finite.

Since $\frac{(1+|\xi|^2)^s}{|\xi|^{2s}} = \left(\frac{1}{|\xi|^2} + 1\right)^s \to 1$ as $\xi \to \infty$, we can choose a large enough that $(1 + |\xi|^2)^s \leq 2|\xi|^{2s}$ for all $\xi > a$. Then $|g_s(\xi)|^2 \leq \frac{4}{\pi}|\xi|^{2(s-1)}$ for all $\xi > a$, meaning that the second term is finite if $2(s-1) < -1$, that is, if $s < \frac{1}{2}$.

Similarly, since $\frac{|\xi|^{2s}}{(1+|\xi|^2)^s} \to 1$ as $\xi \to \infty$, we can choose a large enough that $|\xi|^{2s} \leq 2(1 + |\xi|^2)^s$ for all $\xi > a$.