1. Let $f = \chi_{[0,1]}$.

$$f * f(x) = \int_{-\infty}^{\infty} f(y-x) f(y)\, dy.$$

For any $y$, $f(y-x) f(y) = \begin{cases} 1 & y-x \in [0,1] \text{ and } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

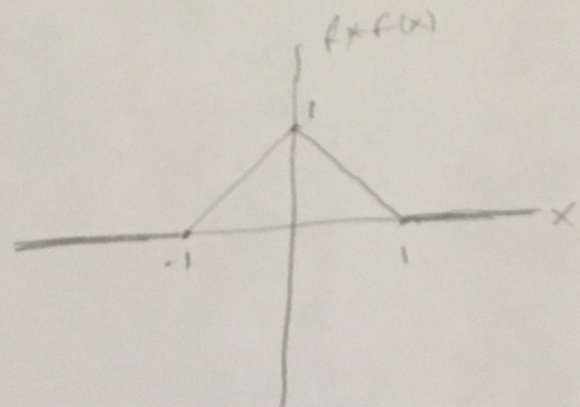
$$= \begin{cases} 1 & y \in [x, x+1] \cap [0,1] \\ 0 & \text{otherwise}, \end{cases}$$

and $[x, x+1] \cap [0,1] = \begin{cases} [x,1] & 0 \le x \le 1 \\ [0, x+1] & -1 \le x \le 0 \\ \emptyset & \text{otherwise}. \end{cases}$

Then $f(y-x) f(y) = \begin{cases} \chi_{[x,1]}(y) & 0 \le x \le 1 \\ \chi_{[0,x+1]}(y) & -1 \le x \le 0 \\ 0 & \text{otherwise} \end{cases}$

and $f * f(x) = \begin{cases} \int_x^1 dy & 0 \le x \le 1 \\ \int_0^{x+1} dy & -1 \le x \le 0 \\ 0 & \text{otherwise} \end{cases}$

$$= \begin{cases} 1-x & 0 \le x \le 1 \\ x+1 & -1 \le x \le 0 \\ 0 & \text{otherwise} \end{cases}$$

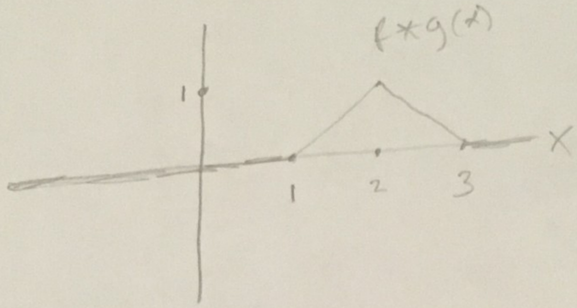


2. Let $f = \chi_{[0,1]}$ and $g = \chi_{[2,3]}$. Then

$g(x-2) = f(x)$. Since $*$ is shift-invariant,

$$f * g(x) = f * f(x-2) = \begin{cases} 1-(x-2) & 0 \le x-2 \le 1 \\ 1+(x-2) & -1 \le x-2 \le 0 \\ 0 & \text{otherwise} \end{cases}$$

by 1., or $f * g(x) = \begin{cases} 3-x & 2 \le x \le 3 \\ x-1 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$




3. Define $T$ by

$$Tf(x) = \int_{-\infty}^{x} f(y)\,dy.$$

(i) $T$ is shift-invariant:

$$T\tau_c f(x) = \int_{-\infty}^{x} \tau_c f(y)\,dy = \int_{-\infty}^{x} f(y-c)\,dy$$

$$= \int_{-\infty}^{x-c} f(y)\,dy \quad (\text{substitute } u = y-c),$$

$$= Tf(x-c) = \tau_c Tf(x),$$

where $\tau_c$ is the shift operator defined by $\tau_c f(x) = f(x-c)$, $c \in \mathbb{R}$.

Therefore $T$ is shift-invariant.

(ii) The kernel of $T$ is $\chi_{[-\infty,0]}$ because

$$\chi_{(-\infty,0]} * f(x) = \int_{-\infty}^{\infty} \chi_{(-\infty,0]}(y-x) f(y)\,dy,$$ and $\chi_{(-\infty,0]}(y-x) = 1$ iff $y - x \le 0 \Leftrightarrow y \le x$

so $\chi_{(-\infty,0]} * f(x) = \int_{-\infty}^{x} f(y)\,dy = Tf(x).$

3. (iii) Observe that $(Tf)'(x) = f(x)$. Then by a previous homework $\widehat{(Tf)'}(\xi) = i\xi \widehat{Tf}(\xi)$. Thus, if $\xi \neq 0$, then

$$\widehat{Tf}(\xi) = \frac{1}{i\xi} \hat{f}(\xi),$$

so the Fourier multiplier of $T$ is $\frac{1}{i\xi}$ (almost everywhere, but this identifies the multiplier as an element of a Lebesgue space).