Math 6108 Homework 2

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Problem 1.

1. Let $S = \{(x, y) \in \mathbf{R}^2 \mid x \geq 0, y \geq 0\}$. Then S is not a subspace of $\mathbf{R}^2$ because $(1, 0) \in S$, but $-(1, 0) = (-1, 0) \notin S$; that is, S is not closed under scalar multiplication in $\mathbf{R}^2$.
2. Let $S = \{f \in \mathcal{P}^3 \mid f(4) = 0\}$. Then S is a subspace of $\mathcal{P}^3$. To see this, let $f, g \in S$, and let $a \in \mathbf{R}$. Then $(f + g)(4) = f(4) + g(4) = 0$, so $f + g \in S$, and $(af)(4) = af(4) = 0$, so $af \in S$. Finally, $\mathbf{0} \in S$ because $\mathbf{0}(4) = 0$, so S is nonempty. This shows that S is a subspace of $\mathcal{P}^3$.
3. Let $S = \{f \in \mathcal{P}^4 \mid f(4) = 2\}$. Then S is not a subspace of $\mathcal{P}^4$ because $\mathbf{0} \notin S$, as $\mathbf{0}(4) = 0 \neq 2$. A subspace must contain the zero element of the larger space.
4. Let $S = \{f \in \mathcal{F} \mid f'(x) + f(x) = 2\}$. Then S is not a subspace of $\mathcal{F}$ because $\mathbf{0} \notin S$, as $\mathbf{0}'(x) + \mathbf{0}(x) = 0 \neq 2$.
5. Let $S = \{f \in \mathcal{F} \mid f''(x) - 2f(x) = 0\}$. Then S is a subspace of $\mathcal{F}$. To see this, let $f, g \in S$, and let $a \in \mathbf{R}$. Then $(f + g)''(x) - 2(f + g)(x) = f''(x) - 2f(x) + g''(x) - 2g(x) = 0$, so $f + g \in S$, and $(af)''(x) - 2(af)(x) = a(f''(x) - 2f(x)) = 0$, so $af \in S$. This shows that S is a subspace of $\mathcal{F}$.

Problem 2.

Let $\mathcal{C}$ and $\mathcal{D}$ denote the sets of all continuous and differentiable functions. Then $\mathcal{C}$ and $\mathcal{D}$ are subspaces of $\mathcal{F}$. This is because continuity and differentiability are preserved under addition and scalar multiplication. That is, if f, g are continuous functions, and $a \in \mathbf{R}$, then $f + g$ and af are continuous functions. Similarly, if f, g are differentiable, and $a \in \mathbf{R}$, then $f + g$ and af are differentiable. Since $\mathbf{0} \in \mathcal{C}$, and $\mathbf{0} \in \mathcal{D}$, it follows that $\mathcal{C}$ and $\mathcal{D}$ are subspaces of $\mathcal{F}$.

Problem 3.

Let V be a vector space over a field $\mathbb{F}$, and let $S \subseteq V$ be nonempty. Then S is subspace of V if and only if S is closed under linear combinations.

Proof. Let S be a subspace of V, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v_n} \in S$, let $c_1, c_2, \ldots, c_n \in \mathbf{R}$, and let $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v}_n$. If $\mathbf{u}_i = c_i\mathbf{v}_i$, then $\mathbf{u}_i \in S$ for $i = 1$ to n because subspaces are closed under scalar multiplication. Furthermore, $\mathbf{v} = \mathbf{u}_1 + \mathbf{u_2} + \cdots + \mathbf{u}_n$; therefore, $\mathbf{v} \in S$ because subspaces are closed under addition (we have to apply this inductively). Thus, S is closed under linear combinations.

Suppose that S is closed under linear combinations. If $\mathbf{v}_1, \mathbf{v}_2 \in S$, then $\mathbf{v}_1 + \mathbf{v}_2$ is a linear combination of elements of S and therefore also in S. Similarly, if $a \in \mathbb{F}$, then $a\mathbf{v}_1$ is a liner combination of elements of S and therefore also in S. Hence, S is a subspace of V. $\square$

Problem 4.

Let S_1 and S_2 be subspaces of a vector space V over a field $\mathbb{F}$. Then $S_1 \cap S_2$ is a subspace of V.

Proof. Since S_1 and S_2 are subspaces they both contain $\mathbf{0} \in S$. Therefore, $\mathbf{0} \in S_1 \cap S_2$, so $S_1 \cap S_2$ is nonempty.

Let $\mathbf{v}_1, \mathbf{v}_2 \in S_1 \cap S_2$, and let $a \in \mathbb{F}$. Then for $i \in \{1, 2\}$, $\mathbf{v}_1, \mathbf{v}_2 \in S_i$, so $\mathbf{v}_1 + \mathbf{v}_2 \in S_i$, and $a\mathbf{v}_1 \in S_i$ because S_i is subspace of V. Therefore, $\mathbf{v}_1 + \mathbf{v_2} \in S_1 \cap S_2$, and $a\mathbf{v}_1 \in S_1 \cap S_2$.

This shows that $S_1 \cap S_2$ is a subspace of V. □

Problem 5.

1. Let $S = \{1 + x^2, 1 + x, x^2 - x\} \subseteq \mathcal{P}^2$. Then S is linearly dependent because
$$c_1(x^2 - x) + c_2(1 + x^2) + c_3(1 + x) = 0$$
if we choose $c_1 = 1$, $c_2 = -1$, and $c_3 = 1$. Furthermore, $x^2 - x = 1 + x^2 + (-1)(1 + x)$.

2. Let $S = \{\cos(x), \sin(x), 1\} \subseteq \mathcal{F}$. Then S is linearly independent. To see this, let $c_1, c_2, c_3 \in \mathbf{R}$, and suppose that
$$c_1 \cos(x) + c_2 \sin(x) + c_3 = 0.$$
This must be true for all $x \in \mathbf{R}$ by the definition of scalar multiplication and addition in $\mathcal{F}$; in particular, if $x = 0$, then we obtain $c_1 = -c_3$, and if $x = \pi$, then we obtain $c_1 = c_3$. Thus, $c_1 = c_3 = 0$. This means that $c_2 \sin(x) = 0$. Taking $x = \frac{\pi}{2}$, we get $c_2 = 0$. Thus, $c_1 = c_2 = c_3 = 0$. This means that S is linearly independent.

Problem 6.

Problem 7.

Problem 8.

Let S be the set of all symmetric matrices in $\mathbf{R}^{n\times n}$.

1. S is a subspace of $\mathbf{R}^{n\times n}$.

   Proof. Clearly, S is nonempty (it contains, for example, the zero matrix). Let $A, B \in S$, and let $a \in \mathbf{R}$. Then $(A + B)^T = A^T + B^T = A + B$, so $A + B \in S$. Furthermore, $(aA)^T = aA^T$, so $aA \in S$. This shows that S is a subspace of $\mathbf{R}^{n\times n}$. □

2. Let $[A]_{k\ell}$ mean taking the element in the kth row and ℓth column of a matrix A. Define $B = \{A^{ij} : 1 \le j \le i \le n\}$, where $A^{ij} \in \mathbf{R}^{n\times n}$ is defined by
$$A^{ij}_{k\ell} = \begin{cases} 1 & (k, \ell) = (i, j) \text{ or } (k, \ell) = (j, i) \\ 0 & \text{otherwise,} \end{cases}$$
for $k, \ell = 1, 2, \ldots, n$. Then B is a basis for S.

   Proof. We need to show that B is linearly independent and that $\operatorname{span}(B) = S$.

1. Suppose that
$$\sum_{A^{ij}\in B} c_{ij}A^{ij} = \mathbf{0}$$
for some $\{c_{ij} : 1 \leq j \leq i \leq n\} \subseteq \mathbf{R}$. For $\ell \leq k$, we have
$$\sum_{1\leq j\leq i\leq n} c_{ij}a^{ij}_{k\ell} = 0.$$
Since $\ell \leq k$, and $j \leq i$, the definition of $A^{ij}_{k\ell}$ implies that only the term $c_{k\ell}A^{k\ell}_{k\ell}$ is nonzero. Then we get $c_{k\ell} = 0$. Since $\ell \leq k$ were arbitrary, it follows that $c_{ij} = 0$ for all $1 \leq j \leq i \leq n$, so B is linearly independent.
2. If $(i,j) \neq (k,\ell)$ and $(i,j) \neq (\ell,k)$, then $A^{ij}_{k\ell} = 0 = A^{ij}_{\ell k}$. If $(i,j) = (k,\ell)$, then $A^{ij}_{k\ell} = 1 = A^{ij}\ell k$. Hence, $A^{ij} = (A^{ij})^T$ for all $1 \leq i \leq j \leq n$. Thus $B \subseteq S$, which implies that every element of $\mathrm{span}(B)$ is a linear combination of elements of S. Since S is a subspace by part 1. and subspaces are closed under linear combination by Problem 3., it follows that $\mathrm{span}(B) \subseteq S$.

   Conversely, let $C \in S$ be a symmetric matrix. Since
$$\begin{aligned}\left[\sum_{A^{ij}\in B} C_{ij}A^{ij}\right]_{k\ell} &= \sum_{1\leq j\leq i\leq n} C_{ij}A^{ij}_{k\ell} \\ &= \begin{cases} C_{k\ell} & \ell \leq k \\ C_{\ell k}\ (= C_{k\ell}) & k \leq \ell \end{cases} \\ &= C_{k\ell}\end{aligned}$$
by the symmetry of C, we must have
$$C = \sum_{A^{ij}\in B} C_{ij}A^{ij}.$$
Thus, $C \in \mathrm{span}(B)$. This shows that $S \subseteq \mathrm{span}(B)$. Since $S \subseteq \mathrm{span}(B)$, and $\mathrm{span}(B) \subseteq S$, it follows that $\mathrm{span}(B) = S$.

This completes the proof that B is a basis for S. □