8.22 $x' = -x - y^3$ $\quad V(x,y) = \alpha x^2 + \beta y^4$

$y' = x - y$

$(0,0)$ is a c.p.

(i) $V(0,0) = 0 \quad \forall \alpha, \beta$

(ii) $V(x,y) > 0$ for $(x,y) \neq 0$ if $\alpha > 0, \beta > 0$

(iii) $\dot{V} = [2\alpha x \;\; 4\beta y^3] \begin{pmatrix} -x - y^3 \\ x - y \end{pmatrix} = -2\alpha x^2 - 2\alpha x y^3 + 4\beta x y^3 - 4\beta y^4$

$\geq 0 \quad \forall x, y$

if $4\beta - 2\alpha = 0$

Choose $\alpha = 2, \beta = 4$. then

$V(x,y) = 2x^2 + 4y^4$ is a Liapunov function for

$x' = -x - y^3$, $y' = x - y$ at $(0,0)$

3.23

$$x' = -x - y^2$$
$$y' = -\frac{1}{2}y + 2xy$$

(0,0) is a c.p.

Let $V(x,y) = 2x^2 + y^2$. Then

(i) $V(0,0) = 0$

(ii) $V(x,y) > 0 \quad \forall (x,y) \neq (0,0)$

(iii) $\dot{V} = [4x \quad 2y] \begin{bmatrix} -x - y^2 \\ -\frac{1}{2}y + 2xy \end{bmatrix} = -4x^2 - 4xy^2 - y^2 + 4xy^2$

$= -4x^2 - y^2 \leq 0 \quad \forall x, y$

3.2T (i)

$x' = -2x - 2xy$ if $x' = 0$, then $x = 0$ or $y = -1$

$y' = 2x^2 - y$ if $y' = 0$, then $2x^2 = y$, $2x^2 \neq -1$, so only c.p. is $(x,y) = (0,0)$.

$V(x,y) = x^2 + y^2$ is a Liapunov function for the system at c.p. $(0,0)$ on $\mathbb{R}^2$

- $V(0,0) = 0$
- $V(x,y) > 0$ if $(x,y) \neq (0,0)$
- $\dot{V} = \begin{pmatrix} 2x & 2y \end{pmatrix} \begin{pmatrix} -2x - 2xy \\ 2x^2 - y \end{pmatrix} = -4x^2 - 4x^2y + 4x^2y - 2y^2$
  $= -4x^2 - 2y^2 < 0$ if $(x,y) \neq (0,0)$.

Let $E = \{(x,y) \mid \dot{V} = 0\}$. Then $E = \{(0,0)\}$ because

$$\dot{V} = -4x^2 - 2y^2 = 0 \Rightarrow x = y = 0.$$

The largest invariant subset of E is $M = \{(0,0)\}$, which is invariant because $x' = y' = 0$ if $x = y = 0$, so any solution starting at $(0,0)$ will stay forever. Thus, by the La Salle Invariance Theorem, any bounded solution $\to (0,0)$ as $t \to \infty$.

3.27 (ii)

$$x' = y$$
$$y' = -x + y^5 - 2y$$

$x' = 0 \Rightarrow y = 0$

$y' = 0 \Rightarrow x = 2y - y^5$

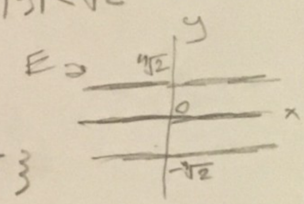
So $(0,0)$ is only c.p.

$V(x,y) = x^2 + y^2$ is a Liapunov function for the system at $(0,0)$ on $U = \{(x,y) \mid |y| < \sqrt[4]{2}\}$

because

- $V(0,0) = 0$
- $V(x,y) > 0$ if $(x,y) \neq (0,0)$
- $\dot{V} = (2x \;\; 2y)\begin{pmatrix} y \\ -x + y^5 - 2y \end{pmatrix} = 2xy - 2xy + 2y^6 - 4y^2$

$$= 2y^6 - 4y^2$$
$$= 2y^2(y^4 - 2) \leq 0 \text{ if } |y| < \sqrt[4]{2}$$



Let $E = \{(x,y) \mid \dot{V} = 0\}$. Then

$$E = \{(x,y) \mid x \in \mathbb{R} \;\&\; y = 0 \text{ or } y = \pm\sqrt[4]{2}\}$$

Let $(x,y) \in E$. If $y \neq 0$, then $y' \neq 0$ if $x \neq y^5 - 2y$ so that a solution through (x,y) would leave E. If $x = y^5 - 2y$, then $x' = y \neq 0$, so a solution through (x,y) would move left or right and then leave E. If $y = 0$, then $y' = -x \neq 0$ if $x \neq 0$, so a solution through (x,y) would leave E. Therefore the largest invariant subset of E is $M = \{(0,0)\}$. Finally, any solution staying in $U = \{(x,y) \mid |y| < \sqrt[4]{2}\} \to (0,0)$ by the LaSalle Invariance Theorem.

3.24 (i) $x' = -x - y + xy^2$
$y' = x - y - x^2 y$

$x' = 0$, $y' = 0$ $\Rightarrow$ $x - x^2 y = 0$
$\Rightarrow x = 0$ or $xy = 1$

If $x = 0$, then $x' = -y = 0$.
If $xy = 1$, then $x' = -x = 0 \Rightarrow y = 0$,
So $(0,0)$ is only c.p.

$V(x,y) = x^2 + y^2$ is a Liapunov function for the system for c.p. $(0,0)$ on $\mathbb{R}^2$

because

- $V(0,0) = 0$
- $V(x,y) > 0$ if $(x,y) \neq (0,0)$
- $\dot{V} = \begin{pmatrix} 2x & 2y \end{pmatrix} \begin{pmatrix} -x - y + xy^2 \\ x - y - x^2 y \end{pmatrix} = -2x^2 - 2xy + 2x^2y^2 + 2xy - 2y^2 - 2x^2y^2$
$= -2x^2 - 2y^2 \leq 0 \quad \forall (x,y) \in \mathbb{R}^2$

Let $E = \{(x,y) \mid \dot{V} = 0\} = \{(x,y) \mid x^2 + y^2 = 0\} = \{(0,0)\}$.

Since $x' = y' = 0$ when $x = y = 0$, E is an invariant set.

Therefore, by La Salle's Invariance Theorem, any bounded solution of the system $\to 0$ as $t \to \infty$.

3.17 (i) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x+x^2-2xy \\ -y+xy \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}' = 0$ if $x+x^2-2xy=0$ and $-y+xy=0$ $\Rightarrow$ $y=0$ or $x=1$,

- If $y=0$, then $x+x^2=0$ $\Rightarrow$ $x=-1$ or $x=0$
- If $x=1$, then $y=1$

So c.p.'s are $(-1,0)$ & $(1,1)$ & $(0,0)$

Jacobian is $J = \begin{pmatrix} 1+2x-2y & -2x \\ y & x-1 \end{pmatrix}$

$J(-1,0) = \begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix}$ which has e.v.'s -1 & -2.

All e.v.'s have real part < 0, so the equilibrium $(-1,0)$ is asymptotically stable.

$J(1,1) = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}$ and e.v.'s satisfy $(1-\lambda)(-\lambda)+2=0$

$\Rightarrow \lambda^2-\lambda+2=0$

$\Rightarrow \lambda = \frac{1}{2} \pm \frac{\sqrt{1-8}}{2}$

$= \frac{1}{2} \pm \frac{i\sqrt{7}}{2}$

There is an e.v. with real part > 0,

so the equilibrium $(1,1)$ is unstable (spiral)

$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which has e.v.'s $1, -1$

since $1 > 0$, the equilibrium $(0,0)$ is unstable

3.17 (ii) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -x+y+y^2-2xy \\ x \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}' = 0$ if $x = 0$ & $y + y^2 = 0$

$\Rightarrow y = -1$ or $y = 0$

so c.p.s are $(0,0)$ & $(0,-1)$

Jacobian is $J = \begin{pmatrix} -1-2y & 1+2y-2x \\ 1 & 0 \end{pmatrix}$

$J(0,0) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ so e.v.s satisfy

$(-\lambda-1)(-\lambda) - 1 = 0$

or $\lambda^2 + \lambda - 1 = 0$

or $\lambda = -\frac{1}{2} \pm \frac{\sqrt{1+4}}{2}$

So, since $\frac{\sqrt{5}}{2} > \frac{\sqrt{4}}{2} = \frac{2}{2} > \frac{1}{2}$

$\Rightarrow -\frac{1}{2} + \frac{\sqrt{5}}{2} > 0 \Rightarrow$ one eigenvalue has real part > 0. Thus the equilibrium $(0,0)$ is unstable.

$J(0,-1) = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$, so e.v.s satisfy $(1-\lambda)(-\lambda) - 3 = 0$ or $\lambda^2 - \lambda - 3 = 0$ or $\lambda = \frac{1}{2} \pm \ldots$ So 1 e.v. has real part > 0, and the equilibrium $(0,-1)$ is unstable.

3.17 (iii) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x-y \\ x^2-1 \end{pmatrix} = 0$ if $x=y$, $x^2-1=0$
$\Rightarrow x = \pm 1$

So c.p.s are $(1,1)$ & $(-1,-1)$.

Jacobian is $J = \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}$ so

$J(1,1) = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$ and e.v.'s satisfy

$(1-\lambda)(-\lambda)+2=0 \Rightarrow \lambda^2-\lambda+2=0 \Rightarrow \lambda = \frac{1}{2} \pm \frac{\sqrt{1-8}}{2}$
$= \frac{1}{2} \pm i\frac{\sqrt{7}}{2}$

So one e.v. has real part >0

& the equilibrium $(1,1)$ is unstable

$J(-1,-1) = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$ and e.v.'s satisfy

$(1-\lambda)(-\lambda) - 2 = 0 \Rightarrow \lambda^2-\lambda-2=0 \Rightarrow (\lambda-2)(\lambda+1)=0$
so $\lambda = 2$ or $\lambda = -1$

$\lambda = 2 > 0$, so the equilibrium $(-1,-1)$

is unstable.

3.19 (i) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x^5 - 2x + y \\ -x \end{pmatrix} = 0$ if $x = 0$ and $y = 0$

So only e.p. is $(0,0)$

Jacobian at $(0,0)$ is $\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$, so e.v.'s satisfy

$$(-2-\lambda)(-\lambda) + 1 = 0 \Rightarrow \lambda^2 + 2\lambda + 1 = 0$$
$$\Rightarrow (\lambda+1)^2 = 0$$

So $\lambda = -1$, all e.v.'s have real part < 0, so the equilibrium $(0,0)$ is A.S.

3.19 (ii) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4-4x^2-y^2 \\ 3xy \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}' = 0$ if ($x=0$ or $y=0$) and

($y^2 = 4-4x^2$)

If $x=0$, then $y^2 = 4$, so $y = \pm 2$

If $y=0$, then $x^2 = 1 \Rightarrow x = \pm 1$, so

Only c.p.'s are $(0, \pm 2)$, $(\pm 1, 0)$

Jacobian is $J = \begin{pmatrix} -8x & -2y \\ 3y & 3x \end{pmatrix}$

$J(0,2) = \begin{pmatrix} 0 & -4 \\ 6 & 0 \end{pmatrix}$, so e.v.'s satisfy $\lambda^2 + 24 = 0$

$\lambda = \pm i\sqrt{24}$

$Re\,\lambda = 0$ for all e.v.'s so No conclusions can be drawn from linearization.

$J(0,-2) = \begin{pmatrix} 0 & 4 \\ -12 & 0 \end{pmatrix}$, so e.v.'s satisfy $\lambda^2 + 48 = 0$

$\Rightarrow \lambda = \pm i\sqrt{48}$, $Re\,\lambda = 0$ for all e.v.'s so no conclusions can be made from linearization

$J(1,0) = \begin{pmatrix} -8 & 0 \\ 0 & 3 \end{pmatrix}$. Has eigenvalue $3 > 0$, so equilibrium $(1,0)$ is unstable

$J(-1,0) = \begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix}$ has eigenvalue $8 > 0$, so the equilibrium $(-1,0)$ is unstable