

# Math 6417 Homework 1

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## Question 1.

Let  $f$  be continuous on  $[0, 1] \times \mathbf{R}$  and satisfy  $|f(x, u) - f(x, v)| \leq L|u - v|$  for all  $x \in [0, 1]$  and  $u, v \in \mathbf{R}$ , where  $0 \leq L < 8$ .

For  $\alpha, \beta \in \mathbf{R}$ , consider the boundary value problem

$$\begin{aligned} -u''(x) &= f(x, u(x)) \quad \text{if } x \in (0, 1) \\ u(0) &= \alpha \quad u(1) = \beta. \end{aligned} \tag{1}$$

This problem has one and only one solution  $u \in C^2[0, 1]$ .

Indeed, define

$$G(x, \xi) = \begin{cases} \xi(1-x) & 0 \leq \xi \leq x \leq 1 \\ x(1-\xi) & 0 \leq x \leq \xi \leq 1 \end{cases} \tag{2}$$

and also consider the integral equation

$$u(x) = \alpha(1-x) + \beta x + \int_0^1 G(x, \xi) f(\xi, u(\xi)) \, d\xi \quad \text{if } x \in [0, 1]. \tag{3}$$

We show that if  $u \in C^2[0, 1]$ , then  $u$  solves (1) if and only if  $u$  solves (3), and that there is a unique solution  $u \in C^2[0, 1]$  of (3) by the Banach Fixed Point Theorem. Then the claim follows.

(i) If  $u \in C^2[0, 1]$ , then  $u$  is a solution of (1) if and only if  $u$  is a solution of (3).

*Proof.* Suppose that  $u \in C^2[0, 1]$  is a solution of (1). Then, using integration by parts,

$$\begin{aligned} \int_0^1 G(x, \xi) f(\xi, u(\xi)) \, d\xi &= - \int_0^x \xi(1-x) u''(\xi) \, d\xi - \int_x^1 x(1-\xi) u''(\xi) \, d\xi \\ &= -(1-x) \left[ \xi u'(\xi) \Big|_0^x - \int_0^x u'(\xi) \, d\xi \right] - x \left[ (1-\xi) u'(\xi) \Big|_x^1 + \int_x^1 u'(\xi) \, d\xi \right] \\ &= -(1-x) x u'(x) + (1-x)(u(x) - u(0)) \\ &\quad + x(1-x) x u'(x) - x(u(1) - u(x)) \\ &= -\alpha(1-x) - \beta x + u(x) \end{aligned}$$

for any  $x \in [0, 1]$ . Therefore,  $u$  solves (3).

Conversely, suppose that  $u \in C^2[0, 1]$  is a solution of (3). Then differentiating both sides of (3) implies that

$$u'(x) = \beta - \alpha + \frac{d}{dx} \int_0^x \xi(1-x) f(\xi, u(\xi)) \, d\xi + \frac{d}{dx} \int_x^1 x(1-\xi) f(\xi, u(\xi)) \, d\xi \tag{4}$$

for  $x \in (0, 1)$ . Since the integrands in both integrals above are obviously continuous and have a continuous partial derivative with respect to  $x$  on  $[0, 1]^2$ , the action of the derivative on the integrals gives

$$\begin{aligned} u'(x) &= \beta - \alpha + x(1-x)f(x, u(x)) - \int_0^x \xi f(\xi, u(\xi)) \, d\xi - x(1-x)f(x, u(x)) + \int_x^1 (1-\xi)f(\xi, u(\xi)) \, d\xi \\ &= \beta - \alpha - \int_0^x \xi f(\xi, u(\xi)) \, d\xi + \int_x^1 (1-\xi)f(\xi, u(\xi)) \, d\xi \end{aligned} \quad (5)$$

for  $x \in (0, 1)$ . Since  $f$  is continuous, the integrands in the above integrals are continuous, and the Fundamental Theorem of Calculus implies that

$$u''(x) = -xf(x, u(x)) - (1-x)f(x, u(x)) = -f(x, u(x)) \quad (6)$$

for  $x \in (0, 1)$ . Lastly, note that the definition of  $G$  implies that  $G(0, \xi) = 0 = G(1, \xi)$  for all  $\xi \in [0, 1]$ . Thus,  $u(0) = \alpha$ , and  $u(1) = \beta$ , so  $u$  solves (1).  $\square$

(ii) There is one and only one solution  $u \in C^2[0, 1]$  of (3).

*Proof.* First, note that  $G$  is continuous on  $[0, 1]^2$ . Indeed, it is obviously continuous on the regions  $\{x < \xi\}$  and  $\{\xi < x\}$  by definition, and we have

$$\lim_{\substack{(x, \xi) \rightarrow (x_0, x_0) \\ x \leq \xi}} G(x, \xi) = x_0(1-x_0) = \lim_{\substack{(x, \xi) \rightarrow (x_0, x_0) \\ x \geq \xi}} G(x, \xi) \quad (7)$$

for any  $x_0 \in [0, 1]$ . Thus,  $G$  is continuous on  $\{x = \xi\}$  as well, and consequently on all of  $[0, 1]^2$ .

Second, for  $u \in C[0, 1]$ , define

$$Au(x) = \alpha(1-x) + \beta x + \int_0^1 G(x, \xi) f(\xi, u(\xi)) \, d\xi. \quad (8)$$

Since  $f$  is continuous on  $[0, 1]^2$ , it is also bounded on  $[0, 1]^2$ , say, by  $M > 0$ . Then

$$\begin{aligned} \left| \int_0^1 G(x, \xi) f(\xi, u(\xi)) \, d\xi - \int_0^1 G(y, \xi) f(\xi, u(\xi)) \, d\xi \right| &\leq M \int_0^1 |G(x, \xi) - G(y, \xi)| \, d\xi \\ &\leq M \left[ \int_0^x \xi |x-y| \, d\xi + \int_x^1 |x-y|(1-\xi) \, d\xi \right] \\ &\leq M|x-y| \end{aligned}$$

Hence,  $Au$  is the sum of a polynomial and a Lipschitz function, so  $Au \in C[0, 1]$ , and  $A : C[0, 1] \rightarrow C[0, 1]$ .

Third,  $A$  is a contraction on  $C[0, 1]$  in the uniform metric  $\rho$  on  $C[0, 1]$ . Indeed, for  $u, v \in C[0, 1]$ ,

$$\rho(Au, Av) = \max_{x \in [0, 1]} \left| \int_0^1 G(x, \xi) [f(\xi, u(\xi)) - f(\xi, v(\xi))] \, d\xi \right| \quad (9)$$

$$\leq \max_{x \in [0, 1]} L \int_0^1 |G(x, \xi)| \cdot |u(\xi) - v(\xi)| \, d\xi \quad (10)$$

$$\leq L \cdot \left( \max_{x \in [0, 1]} \int_0^1 |G(x, \xi)| \, d\xi \right) \rho(u, v). \quad (11)$$

By the Extreme Value Theorem,

$$\begin{aligned} p(x) &= \int_0^1 |G(x, \xi)| \, d\xi = \int_0^x \xi(1-x) \, d\xi + \int_x^1 x(1-\xi) \, d\xi = \frac{1}{2} [x^2(1-x) + x(1-x)^2] \\ &= \frac{1}{2} x(1-x) \end{aligned} \quad (12)$$

achieves its maximum for  $x \in [0, 1]$  when  $x \in \{0, 1\}$ , which implies  $p(x) = 0$ , or when

$$0 = p'(x) = \frac{1}{2}(1-x-x) \quad (13)$$

that is, when  $x = \frac{1}{2}$ , in which case  $p(x) = \frac{1}{8}$ . Thus,  $p(x) \leq \frac{1}{8}$  for  $x \in [0, 1]$ , and

$$\rho(Au, Av) \leq 8L\rho(u, v). \quad (14)$$

Since  $8L < 1$  by hypothesis, it follows that  $A$  is a contraction on  $C[0, 1]$ .

Fourth, by the Banach Fixed Point Theorem, there is a unique solution  $u \in C[0, 1]$  of (3). Since  $C^2[0, 1] \subseteq C[0, 1]$ , it follows that if  $u \in C^2[0, 1]$ , then (3) has a unique solution in  $C^2[0, 1]$ , namely,  $u$ . Thus, to finish the proof, we need to show that  $u'$  and  $u''$  exist and are continuous.

Since  $u$  is a solution of (3), it follows that

$$u(x) = \alpha(1-x) + \beta x + \int_0^x (1-x)\xi f(\xi, u(\xi)) \, d\xi + \int_x^1 x(1-\xi)f(\xi, u(\xi)) \, d\xi. \quad (15)$$

The calculations in (4, 5, 6) relied only the fact that  $u$  was continuous (so that  $f(\cdot, u(\cdot))$  would be continuous), so they apply to  $u$  here. Thus,  $u'$  and  $u''$  exist, and

$$u''(x) = -f(x, u(x)), \quad (16)$$

which is continuous on  $[0, 1]$ . Therefore  $u \in C^2[0, 1]$ .  $\square$