Math 6331 Homework 4

Jacob Hauck

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3.55

Let $y(n)$ be the number of ways to paint a strip of length n. Consider a strip of length $n+2$.

If the first square is blue, then the next $n+1$ squares may be painted in any way as long as no two consecutive squares are red. Thus, there are $y(n+1)$ ways to paint $n+2$ squares with the first one blue.

If the first square is red, then the next square must be blue, and the subsequent n squares may be painted in any way as long as no two consecutive squares are red. Thus, there are $y(n)$ ways to paint $n+2$ squares with the first one red.

The first square is either red or blue, so $y(n+2) = y(n+1) + y(n)$. The characteristic equation of this homogeneous difference equation is $0 = \lambda^2 - \lambda - 1$, which has roots $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$, so

$$y(n) = c_1\lambda_1^n + c_2\lambda_2^n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Since there are two ways to paint a strip of length 1, we have $y(1) = 2$. There are 3 ways to paint a strip of length 2 (red-blue, blue-red, blue-blue), so $y(2) = 3$. This implies that

$$\begin{aligned} 2 &= c_1\lambda_1 + c_2\lambda_2 \\ 3 &= c_1\lambda_1^2 + c_2\lambda_2^2 \end{aligned}$$

so

$$\begin{aligned} c_1 &= \frac{1}{\lambda_1\lambda_2^2 - \lambda_2\lambda_1^2}(2\lambda_2^2 - 3\lambda_2) \\ c_2 &= \frac{1}{\lambda_1\lambda_2^2 - \lambda_2\lambda_1^2}(3\lambda_1 - 2\lambda_1^2) \end{aligned}$$

We have $\lambda_1^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2}$ and $\lambda_2^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{3-\sqrt{5}}{2}$, and therefore

$$\begin{aligned} \lambda_1\lambda_2^2 - \lambda_2\lambda_1^2 &= \frac{\big(1+\sqrt{5}\big)\big(3-\sqrt{5}\big) - \big(1-\sqrt{5}\big)\big(3+\sqrt{5}\big)}{4} \\ &= \frac{3+2\sqrt{5}-5-(3-2\sqrt{5}-5)}{4} = \sqrt{5} \end{aligned}$$

so

$$\begin{aligned} c_1 &= \frac{6-2\sqrt{5}-3+3\sqrt{5}}{2\sqrt{5}} = \frac{5+3\sqrt{5}}{10} \\ c_2 &= \frac{3+3\sqrt{5}-6-2\sqrt{5}}{2\sqrt{5}} = \frac{5-3\sqrt{5}}{10} \end{aligned}$$

and

$$y(n) = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

3.56

Let $y(n)$ be the number of ways to tile a hallway of length n. Consider a hallway of length $n+2$. If the first tile is a blue tile, then there are $y(n+1)$ ways to tile the remaining $n+1$ tiles. If the first tile is a red tile, then there are $y(n)$ ways to tile the remaining n tiles. Therefore, $y(n+2) = y(n+1) + y(n)$. This is the same equation as in 3.55, so we get

$$y(n) = c_1\lambda_1^n + c_2\lambda_2^n$$

for the same λ_1 and λ_2 from before but different c_1 and c_2. In particular, since there is one way to tile a hallway of length 1 (with the blue tile) and two ways to tile a hallway of length 2 (with two blue tiles or one red tile), we have $y(1) = 1$, and $y(2) = 2$. This gives

$$\begin{aligned} 1 &= c_1\lambda_1 + c_2\lambda_2 \\ 2 &= c_2\lambda_1^2 + c_2\lambda_2^2 \end{aligned}$$

so

$$\begin{aligned} c_1 &= \frac{1}{\lambda_1\lambda_2^2 - \lambda_2\lambda_1^2}(\lambda_2^2 - 2\lambda_2) \\ c_2 &= \frac{1}{\lambda_1\lambda_2^2 - \lambda_2\lambda_1^2}(2\lambda_1 - \lambda_1^2) \end{aligned}$$

Using the calculations from 3.55, we get

$$\begin{aligned} c_1 &= \frac{3-\sqrt{5}-2+2\sqrt{5}}{2\sqrt{5}} = \frac{5+\sqrt{5}}{10} \\ c_2 &= \frac{2+2\sqrt{5}-3-\sqrt{5}}{2\sqrt{5}} = \frac{5-\sqrt{5}}{10} \end{aligned}$$

and

$$y(n) = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

3.74

(a) Let a y-hallway of length n be a $2 \times n$ hallway on top of a $2 \times (n-1)$ hallway with the extra two squares on top being on the left end, as below (Figure 1). Let $y(n)$ be the number of tilings of a y-hallway of length n.

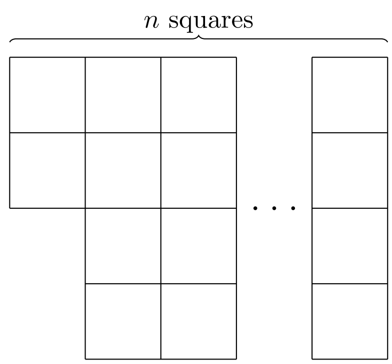


Figure 1: A y-hallway of length n.

Let a z-hallway of length n be a $2 \times n$ hallway with two $1 \times (n-1)$ halways on top and bottom, with the extra two squares in the middle hanging out to the left, as below (Figure 2). Let $z(n)$ be the number of tilings of a z-hallway of length n.

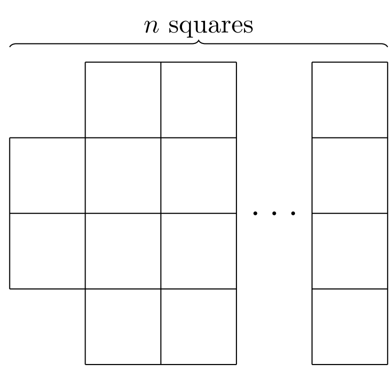
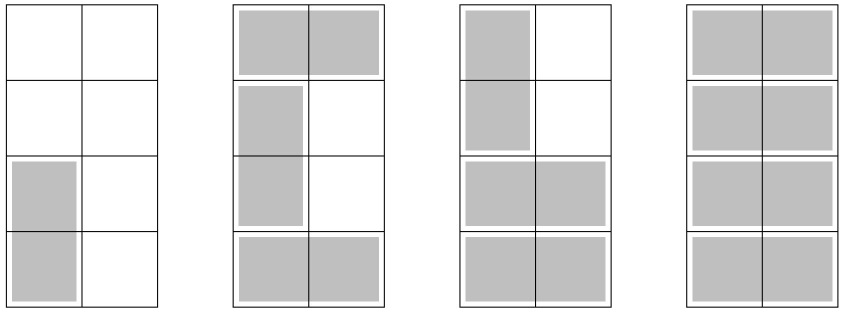


Figure 2: A z-hallway of length n.

Now consider a $4 \times (n+2)$ hallway. The leftmost two columns of every tiling of this hallway must look like one and only one of the four possibilities below, where the shaded areas represent tiles.



(a) (b) (c) (d)

Figure 3: Four possible beginnings for a tiling of a $4 \times (n+2)$ hallway.

The number of tilings with each beginning in Figure 3 are

(a) $y(n+2)$

(b) $z(n+1)$

(c) $y(n+1)$

(d) $x(n)$

It follows that $x(n+2) = x(n) + y(n+2) + y(n+1) + z(n+1)$.

Now consider a y-hallway of length $n+1$. There are two possibilities for the leftmost two columns, shown in Figure 4. There are $x(n)$ tilings for the first possibility, and, by symmetry, $y(n)$ tilings for the second possibility. Thus, $y(n+1) = x(n) + y(n)$.

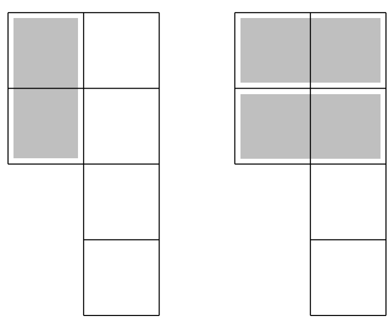


Figure 4: Two possible beginnings for a tiling of a y-hallway of length $n+1$.

Finally, consider a z-hallway of length $n+2$. There are two possibilities for the leftmost three columns, shown in Figure 5. There are $x(n+1)$ tilings for the first possibility and $z(n)$ tilings for the second, so $z(n+2) = x(n+1) + z(n)$.

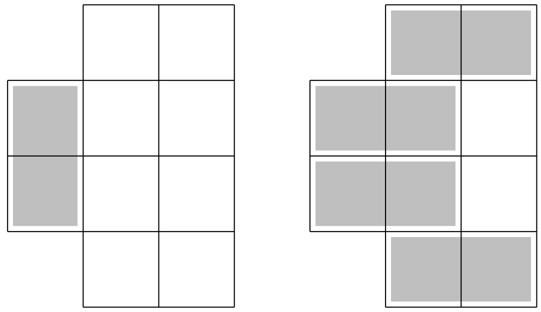


Figure 5: Two possible beginnings for a tiling of a z-hallway of length $n+2$.

This gives the system of three equations

$$\begin{aligned}
x(n+2) &= x(n) + y(n+2) + y(n+1) + z(n+1) \\
y(n+1) &= x(n) + y(n) \\
z(n+2) &= x(n+1) + z(n)
\end{aligned}$$

(b) It is obvious that $x(1) = 1$, $y(1) = 1$, and $z(1) = 1$. Figure 6 shows the possible tilings of a 4×2 hallway, a y-hallway of length 2, and a z-hallway of length 2.

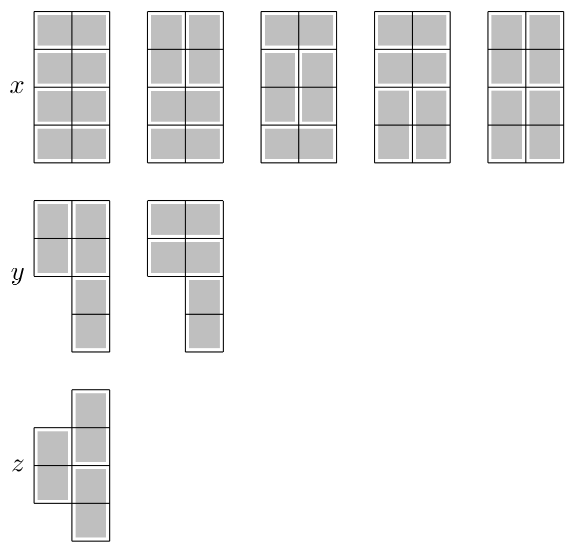


Figure 6: The possible tilings of a hallway, y-hallway, and z-hallway of length 2.

It is easy to see from this that $x(2) = 5$, $y(2) = 2$, and $z(2) = 1$. We can use the system from (a) and these initial conditions to compute $x(10) = 18061$; see the table of values obtained by iteratively applying the system in Table 1.

n	1	2	3	4	5	6	7	8	9	10
$x(n)$	1	5	11	36	95	281	781	2245	6336	18061
$y(n)$	1	2	7	18	54	149	430	1211	3456	9792
$z(n)$	1	1	6	12	42	107	323	888	2568	7224

Table 1: Computed values of $x(n)$, $y(n)$, and $z(n)$ for $n = 1, 2, \ldots, 10$.

3.75

Summing the three equations gives

$$\Delta x(t) + \Delta y(t) + \Delta z(t) = -x(t) + \frac{1}{3}y(t) + \frac{2}{3}z(t) + \frac{1}{3}x(t) - \frac{1}{3}y(t) + \frac{1}{3}z(t) + \frac{2}{3}x(t) - z(t) = 0$$

so $x(t) + y(t) + z(t) = \text{constant}$ because we are working over $t = 0, 1, 2, \dots$ Since, initially, $x(0) = .5$, $y(0) = .3$, and $z(0) = .2$, we must have $x(t) + y(t) + z(t) = 1$ for all t. Substituting $z(t) = 1 - x(t) - y(t)$ into the original equations gives

$$\begin{aligned}
\Delta x(t) &= -x(t) + \frac{1}{3}y(t) + \frac{2}{3}(1 - x(t) - y(t)) = -\frac{5}{3}x(t) - \frac{1}{3}y(t) + \frac{2}{3} \\
\Delta y(t) &= \frac{1}{3}x(t) - \frac{1}{3}y(t) + \frac{1}{3}(1 - x(t) - y(t)) = -\frac{2}{3}y(t) + \frac{1}{3}
\end{aligned}$$

The second equation is the same as $y(t+1) - \frac{1}{3}y(t) = \frac{1}{3}$. Since $u(t) = \left(\frac{1}{3}\right)^t$ is a solution of the homogeneous equation $u(t+1) - \frac{1}{3}u(t) = 0$, and $y_p(t) = \frac{1}{2}$ is a solution of the equation for y, a general solution for $y(t)$ is $y(t) = \frac{1}{2} + c\left(\frac{1}{3}\right)^t$, for some constant c.

Plugging in to the $x(t)$ equation above gives

$$x(t+1) + \frac{2}{3}x(t) = -\frac{1}{3}\left(\frac{1}{2} + c\left(\frac{1}{3}\right)^t\right) + \frac{2}{3} = \frac{1}{2} - c\left(\frac{1}{3}\right)^{t+1}$$

Then $u(t) = \left(-\frac{2}{3}\right)^t$ is a solution of the homogeneous equation $u(t+1) + \frac{2}{3}u(t) = 0$, and

$$\begin{aligned}
x(t) = u(t)\sum \frac{\frac{1}{2} - c\left(\frac{1}{3}\right)^{t+1}}{u(t+1)} &= \left(-\frac{2}{3}\right)^t \sum \frac{\frac{1}{2} - c\left(\frac{1}{3}\right)^{t+1}}{\left(-\frac{2}{3}\right)^{t+1}} \\
&= -\frac{3}{2} \cdot \left(-\frac{2}{3}\right)^t \sum \left[\frac{1}{2} \cdot \left(-\frac{3}{2}\right)^t - \frac{c}{3} \cdot \left(-\frac{1}{2}\right)^t\right] \\
&= -\frac{3}{2} \cdot \left(-\frac{2}{3}\right)^t \left[\frac{1}{2} \cdot \frac{1}{-\frac{3}{2} - 1} \cdot \left(-\frac{3}{2}\right)^t - \frac{c}{3} \cdot \frac{1}{-\frac{1}{2} - 1} \cdot \left(-\frac{1}{2}\right)^t + d'\right] \\
&= \frac{3}{10} - \frac{c}{3} \cdot \left(\frac{1}{3}\right)^t + d\left(-\frac{2}{3}\right)^t
\end{aligned}$$

for some constants d, d'. Applying the initial conditions $x(0) = .5$ and $y(0) = .3$, we get $c = -\frac{1}{5}$ and $d = \frac{2}{15}$.

Therefore, the solution of the original equations is

$$\begin{aligned}
x(t) &= \frac{3}{10} + \frac{1}{15}\left(\frac{1}{3}\right)^t + \frac{2}{15}\left(-\frac{2}{3}\right)^t \\
y(t) &= \frac{1}{2} - \frac{1}{5}\left(\frac{1}{3}\right)^t \\
z(t) &= \frac{1}{5} + \frac{2}{15}\left(\frac{1}{3}\right)^t - \frac{2}{15}\left(-\frac{2}{3}\right)^t
\end{aligned}$$

3.76

(a) Let $\mathcal{A}$ be the event that A wins, let $\mathcal{B}$ be the event that B wins, and let $\mathcal{N}$ be the event that neither wins (the game goes on forever). Let $[n,t]$ be the event that player A has t chips on turn n.

All probabilities conditioned on the state of the game at turn n depend only on how many chips each player has at turn n, rather than the exact history of the game. Assuming a fixed total number of chips, any probability conditioned on the state of the game at turn n actually depends only on the number of chips t that player A has on turn n (since B must have the total minus t chips). Hence, the function $u(t) = P(\mathcal{A} \mid [n,t],[n_1,t_1],\ldots,[t_k,n_k])$ for any $n > n_1 > \cdots > n_k$ is well-defined.

As long as t is not 0 or the total number of chips, then conditioning on the complementary events $[n+1,t-1] \mid [n,t]$ and $[n+1,t+1] \mid [n,t]$ allows us to write

$$\begin{aligned} P(\mathcal{A} \mid [n,t]) &= P(\mathcal{A} \mid [n+1,t-1],[n,t])P([n+1,t-1] \mid [n,t]) \\ &\quad + P(\mathcal{A} \mid [n+1,t+1],[n,t])P([n+1,t+1] \mid [n,t]) \end{aligned}$$

which implies that

$$\begin{aligned} u(t) &= u(t-1)P([n+1,t-1] \mid [n,t]) + u(t+1)P([n+1,t+1] \mid [n,t]) \\ &= (1-p)u(t-1) + pu(t+1) \end{aligned}$$

because by assumption $P([n+1,t-1] \mid [n,t]) = 1-p$, and $P([n+1,t+1] \mid [n,t]) = p$.

(b) Suppose that at the beginning of the game A has a chips and B has b chips. Suppose that A wins on turn n; then we have

$$u(a+b) = P(\mathcal{A} \mid [n,a+b]) = 1$$

Now suppose that A loses on turn n; then we have

$$u(0) = P(\mathcal{A} \mid [n,0]) = 0$$

Assume that $p \neq 0$. Rewriting the difference equation from above gives

$$u(t+2) - \frac{1}{p}u(t+1) + \frac{1-p}{p}u(t) = 0$$

which has characteristic equation $\lambda^2 - \frac{1}{p}\lambda + \frac{1-p}{p} = 0$. This equation has roots $\lambda = 1, \frac{1}{p}-1$, so a general solution of the difference equation is

$$u(t) = c_1 + c_2\left(\frac{1}{p}-1\right)^t$$

for constants c_1 and c_2. Using the condition $u(0) = 0$ gives $c_1 = -c_2$, and $u(t) = c_1\left[1 - \left(\frac{1}{p}-1\right)^t\right]$. Applying the condition $u(a+b) = 1$ gives $c_1 = \frac{1}{1-\left(\frac{1}{p}-1\right)^{a+b}}$, so

$$u(t) = \frac{1-\left(\frac{1}{p}-1\right)^t}{1-\left(\frac{1}{p}-1\right)^{a+b}}$$

Finally, the probability that A wins is just

$$P(\mathcal{A}) = P(\mathcal{A} \mid [1,a]) = u(a) = \frac{1-\left(\frac{1}{p}-1\right)^a}{1-\left(\frac{1}{p}-1\right)^{a+b}}$$

Note: using symmetry to obtain $P(\mathcal{B})$ shows that $P(\mathcal{B}) = 1 - P(\mathcal{A})$, which implies that $P(\mathcal{N}) = 0$, that is, the game must eventually end after finitely many turns.