Math 6330 Homework 8

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4.9

Suppose that $b(t)$ is 1-periodic, and let $b_0 = \int_0^1 b(s)\,\mathrm{d}s$. Then

1. If $b_0 = 0$, then all solutions of $\dot{x} = b(t)$ are 1-periodic.
2. If $b_0 \neq 0$, then all solutions of $\dot{x} = b(t)$ are unbounded.

Proof. Note that simply integrating both sides of $\dot{x} = b(t)$ implies that $\varphi(t, 0, x_0) = x_0 + \int_0^t b(s)\,\mathrm{d}s$. Thus, $\Pi(x_0) = \varphi(1, 0, x_0) = x_0 + \int_0^1 b(s)\,\mathrm{d}s = x_0 + b_0$.

If $b_0 = 0$, then x_0 is a fixed point of Π for all x_0, so all solutions of $\dot{x} = b(t)$ are 1-periodic.

If $b_0 \neq 0$, then Π has no fixed points, which implies that $\dot{x} = b(t)$ has no 1-periodic solutions; hence, every solution must be unbounded by the contrapositive of Theorem 4.11. $\square$

4.10

Consider the 1-periodic differential equation $\dot{x} = f(t, x)$. Suppose that $f(t, 0) = 0$, and there exists $r > 0$ such that $|x_0| < r$ implies that $\varphi(t, t_0, x_0) \to 0$ as $t \to \infty$. Then the zero solution $x = 0$ is stable.

Proof. Let $|x_0| < r$. Recalling that $\Pi^k(x_0) = \varphi(k, 0, x_0)$, it follows by the assumption that

$$\lim_{k\to\infty} \Pi^k(x_0) = 0. \tag{$*$}$$

Given $\varepsilon > 0$, choose $\delta < \min\{r, \varepsilon\}$. Then $|x_0| < \delta$ implies that $|\Pi(x_0)| < \varepsilon$ by $(*)$ and the monotonicity of Π. This shows that $x_0 = 0$ is a stable fixed point of Π by definition; hence, $x = 0 = \varphi(t, 0, 0)$ is a stable, 1-periodic solution of $\dot{x} = f(t, x)$. $\square$

4.11

Suppose that $a(t)$ and $b(t)$ are 1-periodic and continuous functions. Let

$$a_0 = \int_0^1 a(s)\,\mathrm{d}s, \qquad c_0 = \int_0^1 e^{\int_s^1 a(u)\,\mathrm{d}u} b(s)\,\mathrm{d}s.$$

By the variation of constants formula,

$$\varphi(t, 0, x_0) = e^{\int_0^t a(s)\,\mathrm{d}s} x_0 + \int_0^1 e^{\int_s^t a(u)\,\mathrm{d}u} b(s)\,\mathrm{d}s,$$

so

$$\Pi(x_0) = \varphi(1, 0, x_0) = e^{a_0} x_0 + c_0.$$

1. If $a_0 \neq 0$, then $\Pi(x_0) = x_0$ if and only if $x_0 = -\frac{c_0}{e^{a_0}-1}$, which implies that there is exactly one 1-periodic solution of $\dot{x} = a(t)x + b(t)$. If $a_0 < 0$, then $|\Pi'(x_0)| = e^{a_0} < 1$, so the fixed point of Π is asymptotically stable, and 1-periodic solution is also asymptotically stable. If $a_0 > 0$, then $|\Pi'(x_0)| = e^{a_0} > 1$, so the fixed point of Π is unstable, and the 1-periodic solution is also unstable.
2. Suppose that $a_0 = 0$. Then $\Pi(x_0) = x_0 + c_0$. Thus, if $c_0 = 0$, then every point is a fixed point of Π, and every solution is 1-periodic. Conversely, if every solution is periodic, then every point is a fixed point of Π, which can clearly only happen if $c_0 = 0$.
3. Suppose that $a_0 = 0$ so that $\Pi(x_0) = x_0 + c_0$. If $c_0 \neq 0$, then Π has no fixed points, so there are no 1-periodic solutions. Then the contrapositive of Theorem 4.11 implies that all solutions are unbounded.
4. One version of Fredholm's Alternative from linear algebra: if A is a matrix and b is a vector, then exactly one of the following is true
   - $Ax = b$ has a unique solution.
   - $A^T y = 0$ has a nonzero solution.

   In our case, we are interested in the fixed points of Π, which we obtain by solving the linear equation $\Pi(x_0) = e^{a_0}x_0 + c_0 = x_0$, so in Fredholm's Alternative we would set $A = e^{a_0} - 1$, and $b = -c_0$ and obtain cases similar to the three above.

4.16

Let $c(t)$ be a continuous, 1-periodic function. Then there is a unique 1-periodic solution of $\dot{x} = -x^5 + c(t)$, and it is asymptotically stable.

Proof. Since $c(t)$ is bounded, there exists $M > 0$ such that $-M \leq c(t) \leq M$ for all t. Hence, $\dot{x} < 0$ if $x > \sqrt[5]{M}$, and $\dot{x} > 0$ if $x < -\sqrt[5]{M}$. This implies that every solution is bounded by similar reasoning that was used for $\dot{x} = -x^3 + c(t)$. By Theorem 4.11, it follows that there is a 1-periodic solution $\Phi(t)$.

Suppose that $x(t)$ is another solution, and define $y = x - \Phi$. Then y satisfies

$$\dot{y} = \dot{x} - \dot{\Phi} = -x^5 + c(t) + \Phi^5 - c(t) = -(y + \Phi)^5 + \Phi^5.$$

Since

$$-(y + \Phi)^5 + \Phi^5 = -yg(y, \Phi),$$

where

$$g(y, \Phi) = y^4 + 5y^3\Phi + 10y^2\Phi^2 + 10y\Phi^3 + 5\Phi^4$$

is a positive-definite function, it follows that $\dot{y} < 0$ if $y > 0$, and $\dot{y} > 0$ if $y < 0$; therefore, $y(t) \to 0$ as $t \to \infty$, so $x(t) \to \Phi(t)$ as $t \to \infty$. That is, Φ is asymptotically stable. Moreover, if x is 1-periodic, the only possibility is that $x = \Phi$, so Φ is unique.

To see that $g(y, \Phi)$ is positive-definite, let $a, t \in \mathbf{R}$ and consider that

$$g(t, at) = (1 + 5a + 10a^2 + 10a^3 + 5a^4)t^4 = h(a)t^4$$

where $h(a) = 5a^4 + 10a^3 + 10a^2 + 5a + 1$. Note that

$$h'(a) = 20a^3 + 30a^2 + 20a + 5 = 5(4a^3 + 6a^2 + 4a + 1) = 5\left((a + 1)^4 - a^4\right).$$

Then $h'(a) = 0$ implies that $(a + 1)^4 = a^4$, which implies that $a + 1 = \pm a$. $a + 1 = a$ is impossible, so $a + 1 = -a$, which implies that $a = -\frac{1}{2}$ is the only critical point of h. Since $h''(a) = 60a^2 + 60a + 20$, it follows that

$$h''\left(-\frac{1}{2}\right) = 15 - 30 + 20 = 5 > 0,$$

so $-\frac{1}{2}$ is a local minimizer of h. In fact, since it is the only critical point of h, it must also be a global minimizer. Hence,

$$h(a) \geq h\left(-\frac{1}{2}\right) = \frac{5}{16} - \frac{10}{8} + \frac{10}{4} - \frac{5}{2} + 1 = \frac{1}{16}$$

for all a. It follows that

$$g(y, \Phi) \geq \frac{y^4}{16} > 0 \quad \text{if} \quad y \neq 0.$$

If $y = 0$, then $g(y, \Phi) = 5\Phi^4 > 0$ if $\Phi \neq 0$, so g is positive definite, as claimed. $\square$