Stat 6841 Homework 4

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October 27, 2024

Problem 1.

We need to show that $E[|M_n|] < \infty$ for $n \geq 1$, that $\{M_n\}$ is adapted to $\{\mathcal{F}_n\}$, and that $E[M_{n+1}|\mathcal{F}_n] = M_n$ for $n \geq 1$.

1. We have
$$\begin{aligned}
E[|M_n|] = E\left[\left|S_n^2 - v_n\right|\right] &= E\left[\left|\left(S_0 + \sum_{i=1}^n \xi_i\right)^2 - \sum_{i=1}^n \sigma_i^2\right|\right] \\
&\leq E\left[\left|S_0 + \sum_{i=1}^n \xi_i^2\right|\right] + \sum_{i=1}^n \sigma_i^2 \\
&\leq S_0^2 + \sum_{i=1}^n E[\xi_i^2] + \sum_{i=1}^n \sigma_i^2 < \infty.
\end{aligned}$$

2. Since M_n is a continuous function of $\xi_1, \ldots, \xi_n$, and ξ_i is $\mathcal{F}_n$-predictable for $i = 1, \ldots, n$, it follows that M_n is $\mathcal{F}_n$-predictable. Thus, $\{M_n\}$ is adapted to $\{\mathcal{F}_n\}$.

3. We have
$$\begin{aligned}
E[M_{n+1}|\mathcal{F}_n] &= E\left[\left(S_0 + \sum_{i=1}^{n+1} \xi_i\right)^2 - \sum_{i=1}^{n+1} \sigma_i^2 \middle| \mathcal{F}_n\right] \\
&= E\left[\left(S_0 + \sum_{i=1}^n \xi_i\right)^2 + 2\left(S_0 + \sum_{i=1}^n \xi_i\right)\xi_{n+1} + \xi_{n+1}^2 \middle| \mathcal{F}_n\right] - \sum_{i=1}^{n+1} \sigma_i^2.
\end{aligned}$$

   Since $\left(S_0 + \sum_{i=1}^n \xi_i\right)^2$ is a continuous function of $\xi_1, \ldots, \xi_n$, it is $\mathcal{F}_n$-predictable because $\xi_1, \ldots, \xi_n$ are all $\mathcal{F}_n$-predictable. Then
$$E[M_{n+1}|\mathcal{F}_n] = \left(S_0 + \sum_{i=1}^n \xi_i\right)^2 - \sum_{i=1}^n \sigma_i^2 + 2\left(S_0 + \sum_{i=1}^n \xi_i\right) E[\xi_{n+1}|\mathcal{F}_n] + E[\xi_{n+1}^2|\mathcal{F}_n] - \sigma_{n+1}^2.$$

   Since ξ_{n+1} is independent of $\mathcal{F}_n$, it follows that ξ_{n+1}^2 is also independent of $\mathcal{F}_n$, and $E[\xi_{n+1}|\mathcal{F}_n] = E[\xi_{n+1}] = 0$, and $E[\xi_{n+1}^2|\mathcal{F}_n] = E[\xi_{n+1}] = \sigma_{n+1}^2$. Therefore,
$$E[M_{n+1}|\mathcal{F}_n] = \left(S_0 + \sum_{i=1}^n \xi_i\right)^2 - \sum_{i=1}^n \sigma_i^2 = M_n.$$

Problem 2.

(a) Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Then $\{M_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$. To show this, we need to show that $E[|M_n|] < \infty$ for $n \geq 1$, that$\{M_n\}$ is adapted to $\{\mathcal{F}_n\}$, and that $E[M_{n+1}|\mathcal{F}_n] = M_n$.

1. We have
$$E[|M_n|] = E[2^n X_n] = 2^n E[U_1 \cdots U_n] = 2^n \prod_{i=1}^{n} E[U_i] = 1 < \infty$$
because $U_1, \ldots, U_n$ are independent with $E[U_i] = \frac{1}{2}$ for $i = 1, 2, \ldots, n$.
2. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Then M_n is a continuous function of $X_1, \ldots, X_n$, each of which is $\mathcal{F}_n$-predictable, so M_n is $\mathcal{F}_n$-predictable.
3. We have
$$E[M_{n+1}|\mathcal{F}_n] = E[2^{n+1} X_{n+1}|\mathcal{F}_n] = 2^{n+1} E[X_n U_n|\mathcal{F}_n].$$
Since U_n is independent of $\mathcal{F}_n$ and X_n is $\mathcal{F}_n$-predictable, we have
$$E[M_{n+1}|\mathcal{F}_n] = 2^{n+1} X_n E[U_n|\mathcal{F}_n] = 2^{n+1} X_n E[U_n] = 2^n X_n = M_n.$$

(b) Note that $-\log(U_i)$ is exponentially-distributed with mean 1 because, by the CDF method,
$$P(-\log(U_i) \leq x) = P(U_i \leq e^{-x}) = e^{-x}, \qquad x \geq 0.$$
Then $E[\log(U_i)] = -E[-\log(U_i)] = -1$, and, by the Strong Law of Large Numbers,
$$\frac{1}{n} \log(X_n) = \frac{1}{n} \sum_{i=1}^{n} \log(U_i) \xrightarrow{\text{a.s.}} -1$$
as $n \to \infty$.

(c) To show that $M_n \to 0$ almost surely, we must show that $P(B) = 1$, where $B = \{M_n \to 0\}$. Since $P(B) \leq 1$ in any case, we only need to show that $P(B) \geq 1$.

To this end, let $A = \{\frac{1}{n} M_n \to \log(2) - 1\}$. Then $P(A) = 1$ because, by **(b)**,
$$\frac{1}{n} \log(M_n) = \log(2) + \frac{1}{n} X_n \xrightarrow{\text{a.s.}} \log(2) - 1.$$
The natural exponent base e is greater than 2, so $\log(2) < 1$, and there exists $\delta > 0$ such that $\log(2) + \delta < 1$. Let $\omega \in A$, and let $\varepsilon > 0$. Since $\omega \in A$, there exists $N_1 > 0$ such that
$$\frac{1}{n} \log(X_n(\omega)) < -\log(2) - \delta \quad \text{for all } n > N_1.$$
Choose $N > \max\left\{N_1, -\frac{\log(\varepsilon)}{\delta}\right\}$. Then for all $n > N$, we have
$$\begin{aligned}\frac{1}{n} \log(X_n(\omega)) < -\log(2) - \delta &\implies \log(X_n(\omega)) + n\log(2) < -n\delta \\ &\implies |M_n(\omega)| < e^{-n\delta} < e^{-N\delta} < \varepsilon.\end{aligned}$$
Thus, $M_n(\omega) \to 0$ as $n \to \infty$, so $\omega \in B$. Since $\omega \in A$ was arbitrary, it follows that $A \subseteq B$, and $1 = P(A) \leq P(B)$. This proves that $M_n \to 0$ almost surely.

Problem 3.

T_1 and $T_1 + 1$ are stopping times, but $T_1 - 1$ is not.

1. T_1 is a stopping time because
$$\{T_1 \leq n\} = \bigcup_{i=1}^{n} \{M_i = 1\} \in \mathcal{F}_n$$
because $\{M_i = 1\} \in \mathcal{F}_i \subseteq \mathcal{F}_n$ for $i = 1, 2, \dots, n$.

2. $T_1 + 1$ is a stopping time because
$$\{T_1 + 1 \leq n\} = \{T_1 \leq n - 1\} = \bigcup_{i=1}^{n-1} \{M_i = 1\} \in \mathcal{F}_n$$
because $\{M_i = 1\} \in \mathcal{F}_i \subseteq \mathcal{F}_n$ for $i = 1, 2 \dots, n - 1$.

3. $T_1 - 1$ is not necessarily a stopping time because
$$\{T_1 - 1 \leq n\} = \{T_1 \leq n + 1\} = \{M_{n+1} = 1\} \cup \bigcup_{i=1}^{n} \{M_i = 1\} = \{M_{n+1} = 1\} \cup \{T_1 \leq n\}.$$
Since $\{M_{n+1} = 1\}$ may not be in $\mathcal{F}_n$, we cannot conclude that $\{T_1 - 1 \leq n\} \in \mathcal{F}_n$.

Problem 4.

Let A_s be the event that $X_t - 1$ for all $t > s$, for $s = 1, 2, \dots$, that is,
$$A_s = \bigcap_{t>s} \{X_t = -1\}.$$
For an outcome $\omega \in A_s$, it is clear that $M_t(\omega) \to -\infty$. Thus, $A_s \subseteq \{M_t \to -\infty\}$ for all $s = 1, 2, \dots$ Then $\{M_t \to -\infty\}^C \subseteq A_s^C$ for all $s = 1, 2, \dots$

By Boole's inequality, it follows that
$$P\left(A_s^C\right) = P\left(\bigcup_{t>s} \{X_t = t^2 - 1\}\right) \leq \sum_{t>s} P(X_t = t^2 - 1) = \sum_{t>s} \frac{1}{t^2}.$$
Then $P\left(A_s^C\right) \to 0$ as $s \to \infty$ because the last expression above is the tail of the convergent sum $\sum_{t=1}^{\infty} \frac{1}{t^2}$. Hence, by the squeeze theorem,
$$0 \leq P(M_t \not\to -\infty) \leq P(A_s^C) \implies P(M_t \not\to -\infty) = 0.$$
This means that $P(M_t \to -\infty) = 1$, that is, $M_t \to -\infty$ almost surely.

Problem 5.

Since $\{X_t\}$ is a sub-martingale, there is a filtration $\{\mathcal{F}_t\}$ to which $\{X_t\}$ is adapted.

Problem 6.

To show that $\{Y_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$, where $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, we need to show that $E[|Y_n|] < \infty$ for all n, that $\{Y_n\}$ is adapted to $\{\mathcal{F}_n\}$, and that $E[Y_{n+1}|\mathcal{F}_n] = Y_n$.

1. We have

$$E[|Y_n|] = E\left[\left|S_n^2 - \sum_{k=1}^n X_k^2\right|\right] \leq E\left[\left(\sum_{k=1}^n X_k\right)^2\right] + \sum_{k=1}^n E[X_k^2] \leq E\left[n^2\sum_{k=1}^n X_k^2\right] + n\sigma^2 \leq (n^3+n)\sigma^2 < \infty.$$

because for any set of real numbers $\{a_i : i = 1, 2 \dots, n\}$, we have

$$\left(\sum_{i=1}^n a_i\right)^2 \leq \left(\sum_{i=1}^n |a_i|\right)^2 \leq \left(n \max_{i=1,2,\dots,n} |a_i|\right)^2 = n^2 \max_{i=1,2,\dots,n} a_i^2 \leq n^2 \sum_{i=1}^n a_i^2.$$

2. It is easy to see that Y_n is a continuous function of $X_1, X_2, \dots, X_n$, so Y_n is adapted to $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ for all n.

3. Since X_{n+1} is independent of $X_1, \dots, X_n$, it follows that X_{n+1} is independent of $\mathcal{F}_n$. Thus,

$$\begin{aligned}
E[Y_{n+1}|\mathcal{F}_n] &= E\left[\left(\sum_{k=1}^{n+1} X_k\right)^2 - \sum_{k=1}^{n+1} X_k^2 \middle| \mathcal{F}_n\right] \\
&= E\left[\left(\sum_{k=1}^{n} X_k\right)^2 + 2X_{n+1}\sum_{k=1}^n X_k + X_{n+1}^2 - \sum_{k=1}^n X_k^2 - X_{n+1}^2 \middle| \mathcal{F}_n\right] \\
&= \left(\sum_{k=1}^n X_k\right)^2 - \sum_{k=1}^n X_k^2 + 2E[X_{n+1}|\mathcal{F}_n]\sum_{k=1}^n X_k \\
&= Y_n + 2E[X_{n+1}]\sum_{k=1}^n X_k \\
&= Y_n.
\end{aligned}$$

We have $S_n = \sum_{k=1}^n X_k^2$. Since $\{X_k\}$ are independent, so, too, are $\{X_k^2\}$. Furthermore, we have $E[X_k^2] = \sigma^2$ for all k. Then, by Wald's Equation,

$$E[S_\tau] = E[X_1^2]E[\tau] = \sigma^2 E[\tau].$$

Problem 7.

Let $Y_n = e^{2b(S_n - bn)}$. Then $\{Y_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$, where $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. To show this, we need to prove that $E[|Y_n|] < \infty$ for all n, that $\{Y_n\}$ is adapted to $\{\mathcal{F}_n\}$, and that $E[Y_{n+1}|\mathcal{F}_n] = Y_n$.

1. Since X_k is standard normal for $k = 1, 2, \dots,$ and $S_n = X_1 + X_2 + \cdots + X_n$, it follows that S_n is also normal with mean 0 and variance n, so that the PDF of S_n is

$$f(s) = \frac{1}{\sqrt{2\pi n}} e^{-\frac{1}{2n}s^2}.$$

Then

$$E[|Y_n|] = E\left[e^{2b(S_n - bn)}\right] = \int_{-\infty}^{\infty} e^{2bs - 2bn}\frac{1}{\sqrt{2\pi n}} e^{-\frac{1}{2n}s^2}\,\mathrm{d}s \lesssim \int_{-\infty}^{\infty} e^{-\frac{1}{2n}(s-2nb)^2}\,\mathrm{d}s < \infty.$$

2. Y_n is a continuous function of $X_1, \dots, X_n$, so Y_n is adapted to $\mathcal{F}_n$ for all n.

3. Since X_{n+1} is independent of $X_1, \dots, X_n$, it is also independent of $\mathcal{F}_n$. Then

$$\begin{aligned}
E[Y_{n+1} \big| \mathcal{F}_n] &= E\left[e^{2b(S_{n+1}-b(n+1))} \big| \mathcal{F}_n\right] = E\left[e^{2b(S_n+X_{n+1}-bn-b)} \big| \mathcal{F}_n\right] \\
&= E\left[e^{2b(S_n-bn)} e^{2b(X_{n+1}-b)} \big| \mathcal{F}_n\right] \\
&= Y_n E\left[e^{2b(X_{n+1}-b)}\right] \\
&= Y_n e^{-2b^2} \int_{-\infty}^{\infty} e^{2bx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, \mathrm{d}x \\
&= Y_n e^{-2b^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2b)^2+2b^2} \, \mathrm{d}x \\
&= Y_n \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, \mathrm{d}x \\
&= Y_n,
\end{aligned}$$

where the second-to-last line follows from the change of variables $u = x - 2b$, and the last line follows from the fact that the integrand is the PDF of a standard normal random variable.

All martingales are submartingales, so $\{Y_n\}$ is also a submartingale. Additionally, $Y_n > 0$ for all n. Hence, by the Doob submartingale inequality,

$$P\left(Y_n > e^{2bc}\right) \le P\left(\max_{1 \le k \le n} Y_k > e^{2bc}\right) \le e^{-2bc} E[Y_n].$$

Since Y_n is a martingale,

$$\begin{aligned}
E[Y_n] = E[Y_1] = E\left[e^{2b(X_1-b)}\right] &= e^{-2b^2} E\left[e^{2bX_1}\right] \\
&= e^{-2b^2} \int_{-\infty}^{\infty} e^{2bx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, \mathrm{d}x \\
&= e^{-2b^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2b)^2+2b^2} \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, \mathrm{d}x \\
&= 1.
\end{aligned}$$

Furthermore,

$$\{Y_n = e^{2bc}\} = \left\{e^{2b(S_n-bn)} > e^{2bc}\right\} = \{2b(S_n - bn) > 2bc\} = \{S_n > bn + c\};$$

therefore,

$$P(S_n > bn + c) \le e^{-2bc}.$$