5.27 Choose $x'' + x = 0$ on $[0, \pi)$. Every solution has the form $x = A\cos(t + \varphi)$ for arbitrary A, φ. There is exactly 1 $t \in [0, \pi)$ s.t. $x(t) = 0$ given this form because the zeroes of cos occur every π. Thus, the equation is disconjugate on $[0, \pi)$, but every solution has a zero, so there are no positive solutions.

5.29 $t^2 x'' + 2tx' + bx = 0 \quad t \in [1, \infty) = I$

$r(r-1) + 2r + b = 0 \Rightarrow r^2 + r + b = 0$ or

$r = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4b}$

(i) $b > \frac{1}{4}$. Then $1 - 4b < 0$ so general soln. is

$x = t^{-1/2}\left(c_1 \cos(\alpha t) + c_2 \sin(\alpha t)\right)$. Clearly every nontrivial soln. has infinitely many zeroes, so the equation is oscillatory when $b > \frac{1}{4}$

(ii) $b = \frac{1}{4}$. Then a solution is $x = t^{-1/2} > 0$ if $t \in [1, \infty)$, so the equation is not oscillatory when $b = \frac{1}{4}$

(iii) $b < \frac{1}{4}$. Then a solution is $x = t^{-1/2 + \frac{1}{2}\sqrt{1 - 4b}} > 0$ if $t \in [1, \infty)$ because $1 - 4b > 0$. Then the equation is not oscillatory when $b < \frac{1}{4}$.

5.35 (i) $x'' - 5x' + 6x = 0 \quad J = [0, \infty) \iff (e^{-5t}x')' + e^{-5t}6x = 0$

So $p(t) = e^{-5t}$

$r^2 - 5r + 6 = 0$ or $(r-3)(r-2) = 0$,

I think $u(t) = e^{2t}$ is recessive

$v(t) = e^{3t}$ is dominant.

(a) $\lim_{t\to\infty} \frac{u(t)}{v(t)} = \lim_{t\to\infty} e^{-t} = 0$ ✓

(b) $\int_0^\infty \frac{1}{p(t)u^2(t)} dt = \int_0^\infty \frac{1}{e^{-5t}e^{4t}} dt = \lim_{s\to\infty} \int_0^s e^{t} dt = \lim_{s\to\infty} [e^{s} - 1]$

$= \infty$ ✓

(c) $\int_0^\infty \frac{1}{p(t)v^2(t)} dt = \int_0^\infty \frac{1}{e^{-5t}e^{6t}} dt = \lim_{s\to\infty} \int_0^s e^{-t} dt = \lim_{s\to\infty} [1 - e^{-s}]$

$= 1 < \infty$ ✓

(d) $\frac{p(t)u'(t)}{u(t)} = 2e^{-5t} < 3e^{-5t} = \frac{p(t)v'(t)}{v(t)} \quad \forall t$

(ii) $t^2x'' - 5tx' + 9x = 0 \quad J = [1, \infty) \iff (t^{-5}x')' + 9t^{-5}x = 0$

So $p(t) = t^{-5}$

$r(r-1) - 5r + 9 = 0$

$r^2 - 6r + 9 = 0$ or $(r-3)^2 = 0$, $r = 3, 3$

I think $u(t) = t^3$ is recessive

$v(t) = t^3\ln(t)$ is dominant

(a) $\lim_{t\to\infty} \frac{u(t)}{v(t)} = \lim_{t\to\infty} \frac{1}{\ln(t)} = 0$ ✓

(b) $\int_1^\infty \frac{1}{p(t)u^2(t)} dt = \int_1^\infty \frac{1}{t} dt = \lim_{s\to\infty} \int_1^s \frac{1}{t} dt = \lim_{s\to\infty} \ln(s) = \infty$ ✓

(c) $\int_2^\infty \frac{1}{p(t)v^2(t)} dt = \int_2^\infty \frac{1}{t\ln^2(t)} dt = \lim_{s\to\infty} \int_2^s \frac{1}{t\ln^2(t)} dt = \lim_{s\to\infty} \left[-\frac{1}{\ln(t)}\right]_2^s$

$= \lim_{s\to\infty} \left[\frac{1}{\ln(2)} - \frac{1}{\ln(s)}\right] = \frac{1}{\ln(2)} < \infty$ ✓

(d) $\frac{p(t)u'(t)}{u(t)} = \frac{t^{-5}\, 3t^2}{t^3} = 3t^{-6}$

$\frac{p(t)v'(t)}{v(t)} = t^{-5}\left(\frac{3t^2\ln(t) + t^2}{t^3\ln(t)}\right) = t^{-6}\left(3 + \frac{1}{\ln(t)}\right) > 3t^{-6} = \frac{pu'}{u}$

for $t > 1$

(iii) $x'' - 4x' + 4x = 0 \quad J = [0, \infty) \iff (e^{-4t}x')' + e^{-4t}x = 0$

so $p(t) = e^{-4t}$

$r^2 - 4r + 4 = 0$, $(r-2)^2 = 0$, $r = 2, 2$

I think $u(t) = e^{2t}$ is recessive

$v(t) = te^{2t}$ is dominant

(a) $\lim_{t\to\infty} \frac{u(t)}{v(t)} = \lim_{t\to\infty} \frac{1}{t} = 0$ ✓

(b) $\int_0^\infty \frac{1}{p(t)u^2(t)} = \int_0^\infty 1\,dt = \lim_{s\to\infty} \int_0^s dt = \lim_{s\to\infty} s = \infty$ ✓

(c) $\int_1^\infty \frac{1}{p(t)v^2(t)} = \int_1^\infty \frac{1}{t^2}\,dt = \lim_{s\to\infty} \int_1^s \frac{1}{t^2}\,dt = \lim_{s\to\infty} \left[-t^{-1}\right]_1^s$

$= \lim_{s\to\infty} \left[1 - 1/s\right] = 1 < \infty$ ✓

(d) $\frac{p(t)u'(t)}{u(t)} = 2e^{-4t}$

$\frac{p(t)v'(t)}{v(t)} = e^{-4t}\frac{(e^{2t} + 2te^{2t})}{te^{2t}} = \left(2 + \frac{1}{t}\right)e^{-4t} > 2e^{-4t} = \frac{pu'}{u}$

for $t > 0$

5.41 (i) $z' - 3e^{2t} + e^{-2t} z^2 = 0$

$$q(t) = -3e^{2t}, \quad p(t) = e^{2t}$$

corresponding self-adjoint is

$$(e^{2t}x')' - 3e^{2t}x = 0 \Rightarrow x'' + 2x' - 3x = 0$$

$$r^2 + 2r - 3 = 0$$

$$(r+3)(r-1) = 0, \quad r = -3,\ r = 1$$

So solution of S.A. is

$$x = c_1 e^{-3t} + c_2 e^{t}$$

Riccati subs. gives soln. of Riccati Eqn.

$$z = \frac{px'}{x} = \frac{e^{2t}\left(-3c_1 e^{-3t} + c_2 e^{t}\right)}{c_1 e^{-3t} + c_2 e^{t}}$$

So

$$z = \begin{cases} e^{2t} & (\text{setting } c_1 = 0) \\ \dfrac{e^{2t}(ce^{t} - 3e^{-3t})}{ce^{t} + e^{-3t}} & \left(\text{setting } c = \dfrac{c_2}{c_1}\right) \end{cases}$$

(iii) $z' - \frac{1}{t} + \frac{1}{t} z^2 = 0$

$q(t) = -\frac{1}{t}$, $p(t) = t$, S.A. corresponding to eqn is

$$(tx')' - \frac{1}{t}x = 0 \Rightarrow tx'' + x' - \frac{1}{t}x = 0 \text{ or } t^2x'' + tx' - x = 0$$

$r(r-1) + r - 1 = 0$ or $r^2 - 1 = 0$, $r = \pm 1$, and a general

Solution of S.A. is

$$x = c_1 t + c_2 t^{-1}$$

Riccati subs. gives soln. of Riccati Eqn.

$$z = \frac{px'}{x} = \frac{t\left(c_1 - c_2 t^{-2}\right)}{c_1 t + c_2 t^{-1}} = \frac{c_1 t^2 - c_2}{c_1 t^2 + c_2} = \begin{cases} 1 & (\text{setting } c_2 = 0) \\ \dfrac{ct^2 - 1}{ct^2 + 1} & \left(\text{setting } c = \dfrac{c_1}{c_2}\right) \end{cases}$$

(V) $z' = -3e^{-4t} - e^{4t} z^2$

$$q(t) = 3e^{-4t}, \quad p(t) = e^{-4t}$$

Corresponding S.A. is

$$(e^{-4t}x')' + 3e^{-4t}x = 0 \Rightarrow x'' - 4x' + 3x = 0$$

$$r^2 - 4r + 3 = 0 \text{ or } (r-3)(r-1) = 0, \quad r = 1, 3$$

So soln of S.A. is

$$x = c_1 e^t + c_2 e^{3t}$$

Riccati soln. gives soln. of Riccati eqn.

$$z = \frac{px'}{x} = \frac{e^{-4t}(c_1 e^t + 3c_2 e^{3t})}{c_1 e^t + c_2 e^{3t}}$$

$$= \begin{cases} e^{-4t} & (\text{setting } c_2 = 0) \\ \dfrac{e^{-4t}(ce^t + 3e^{3t})}{ce^t + e^{3t}} & \left(\text{setting } c = \dfrac{c_1}{c_2}\right) \end{cases}$$

5.43 (i) $\left(t \log(t) x'\right)' + \dfrac{1}{t \log t} x = 0$

$$p(t) = t\log(t), \quad q(t) = \frac{1}{t \log(t)}$$

- $\displaystyle\int_e^\infty \frac{1}{p(t)}\,dt = \int_e^\infty \frac{1}{t\log(t)}\,dt = \lim_{s\to\infty} \int_e^s \frac{1}{t\log(t)}\,dt = \lim_{s\to\infty} \Big[\log(\log(t))\Big]_e^s$

$$= \lim_{s\to\infty} \log(\log(s)) = \infty$$

- $\displaystyle\int_e^\infty q(t)\,dt = \int_e^\infty \frac{1}{t\log(t)}\,dt = \infty$ by above, so

the eqn is oscillatory on $[e, \infty)$ by the Fite-Wintner Theorem.