111. Find an equation of a circle with center $(-3, 5)$ and radius 7. $(x+3)^2 + (y-5)^2 = 49$

112. Find an equation of the line that contains the point $(-4, 1)$ and is perpendicular to the line $3x - 6y = 5$. Write the equation in slope-intercept form. $y = -2x - 7$

113. Is the function $f(x) = \dfrac{3x}{5x^3 + 7x}$ even, odd, or neither? Even

*114. Solve for D: $2x + 2yD = xD + y$

115. Find the average rate of change of $f(x) = -3x^2 + 2x + 1$ from 2 to 4. -16

*116. Find the difference quotient of f: $f(x) = \sqrt{2x+3}$

'Are You Prepared?' Answers

1. x
2. Increasing on $[0, \infty)$; decreasing on $(-\infty, 0]$
3. $\{x \mid x \neq -6, x \neq 3\}$
4. $\dfrac{x}{1-x}, x \neq 0, x \neq -1$

6.3 Exponential Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Exponents (Chapter R, Section R.2, pp. 21–23, and Section R.8, pp. 76–77)
- Graphing Techniques: Transformations (Section 3.5, pp. 254–263)
- Solving Equations (Section 1.1, pp. 82–87 and Section 1.2, pp. 92–99)
- Average Rate of Change (Section 3.3, pp. 235–237)
- Quadratic Functions (Section 4.3, pp. 299–308)
- Linear Functions (Section 4.1, pp. 281–284)
- Horizontal Asymptotes (Section 5.3, pp. 359–361)

Now Work the 'Are You Prepared?' problems on page 446.

OBJECTIVES
1 Evaluate Exponential Functions (p. 435)
2 Graph Exponential Functions (p. 439)
3 Define the Number *e* (p. 442)
4 Solve Exponential Equations (p. 444)

1 Evaluate Exponential Functions

Chapter R, Section R.8 gives a definition for raising a real number a to a rational power. That discussion provides meaning to expressions of the form

$$a^r$$

where the base a is a positive real number and the exponent r is a rational number.

But what is the meaning of a^x, where the base a is a positive real number and the exponent x is an irrational number? Although a rigorous definition requires methods discussed in calculus, the basis for the definition is easy to follow: Select a rational number r that is formed by truncating (removing) all but a finite number of digits from the irrational number x. Then it is reasonable to expect that

$$a^x \approx a^r$$

For example, take the irrational number $\pi = 3.14159\ldots$. Then an approximation to a^π is

$$a^\pi \approx a^{3.14}$$

where the digits of π after the hundredths position are truncated. A better approximation is

$$a^\pi \approx a^{3.14159}$$

where the digits after the hundred-thousandths position are truncated. Continuing in this way, we can obtain approximations to a^π to any desired degree of accuracy.

Most calculators have an $\boxed{x^y}$ key or a caret key $\boxed{\wedge}$ for working with exponents. To evaluate expressions of the form a^x, enter the base a, then press the $\boxed{x^y}$ key (or the $\boxed{\wedge}$ key), enter the exponent x, and press $\boxed{=}$ (or $\boxed{\text{ENTER}}$).

EXAMPLE 1 **Using a Calculator to Evaluate Powers of 2**

Use a calculator to evaluate:

(a) $2^{1.4}$ (b) $2^{1.41}$ (c) $2^{1.414}$ (d) $2^{1.4142}$ (e) $2^{\sqrt{2}}$

Solution (a) $2^{1.4} \approx 2.639015822$ (b) $2^{1.41} \approx 2.657371628$

(c) $2^{1.414} \approx 2.66474965$ (d) $2^{1.4142} \approx 2.665119089$

(e) $2^{\sqrt{2}} \approx 2.665144143$

Now Work PROBLEM 17

It can be shown that the laws for rational exponents hold for real exponents.

THEOREM Laws of Exponents

If s, t, a, and b are real numbers with $a > 0$ and $b > 0$, then

- $a^s \cdot a^t = a^{s+t}$
- $(a^s)^t = a^{st}$
- $(ab)^s = a^s \cdot b^s$
- $1^s = 1$
- $a^{-s} = \dfrac{1}{a^s} = \left(\dfrac{1}{a}\right)^s$
- $a^0 = 1$

(1)

Introduction to Exponential Growth

Suppose a function f has the following two properties:

- The value of f doubles with every 1-unit increase in the independent variable x.
- The value of f at $x = 0$ is 5, so $f(0) = 5$.

Table 1 shows values of the function f for $x = 0, 1, 2, 3$, and 4.

Table 1

x	$f(x)$
0	5
1	10
2	20
3	40
4	80

Let's find an equation $y = f(x)$ that describes this function f. The key fact is that the value of f doubles for every 1-unit increase in x.

$f(0) = 5$

$f(1) = 2f(0) = 2 \cdot 5 = 5 \cdot 2^1$ Double the value of f at 0 to get the value at 1.

$f(2) = 2f(1) = 2(5 \cdot 2) = 5 \cdot 2^2$ Double the value of f at 1 to get the value at 2.

$f(3) = 2f(2) = 2(5 \cdot 2^2) = 5 \cdot 2^3$

$f(4) = 2f(3) = 2(5 \cdot 2^3) = 5 \cdot 2^4$

The pattern leads to

$$f(x) = 2f(x-1) = 2(5 \cdot 2^{x-1}) = 5 \cdot 2^x$$

DEFINITION Exponential Function

An **exponential function** is a function of the form

$$\boxed{f(x) = Ca^x}$$

where a is a positive real number $(a > 0)$, $a \neq 1$, and $C \neq 0$ is a real number. The domain of f is the set of all real numbers. The base a is the **growth factor**, and, because $f(0) = Ca^0 = C$, C is called the **initial value**.

In the definition of an exponential function, the base $a = 1$ is excluded because this function is simply the constant function $f(x) = C \cdot 1^x = C$. Bases that are negative are also excluded; otherwise, many values of x would have to be excluded from the domain, such as $x = \frac{1}{2}$ and $x = \frac{3}{4}$. [Recall that $(-2)^{1/2} = \sqrt{-2}$, $(-3)^{3/4} = \sqrt[4]{(-3)^3} = \sqrt[4]{-27}$, and so on, are not defined in the set of real numbers.]

WARNING It is important to distinguish a power function, $g(x) = ax^n$, $n \geq 2$ an integer, from an exponential function, $f(x) = C \cdot a^x$, $a \neq 1$, $a > 0$. In a power function, the base is a variable and the exponent is a constant. In an exponential function, the base is a constant and the exponent is a variable. ■

Transformations (vertical shifts, horizontal shifts, reflections, and so on) of a function of the form $f(x) = Ca^x$ are also exponential functions. Examples of such exponential functions are

$$f(x) = 2^x \qquad F(x) = \left(\frac{1}{3}\right)^x + 5 \qquad G(x) = 2 \cdot 3^{x-3}$$

For each function, note that the base of the exponential expression is a constant and the exponent contains a variable.

In the function $f(x) = 5 \cdot 2^x$, notice that the ratio of consecutive outputs is constant for 1-unit increases in the input. This ratio equals the constant 2, the base of the exponential function. In other words,

$$\frac{f(1)}{f(0)} = \frac{5 \cdot 2^1}{5} = 2 \quad \frac{f(2)}{f(1)} = \frac{5 \cdot 2^2}{5 \cdot 2^1} = 2 \quad \frac{f(3)}{f(2)} = \frac{5 \cdot 2^3}{5 \cdot 2^2} = 2 \quad \text{and so on}$$

This leads to the following result.

THEOREM

For an exponential function $f(x) = Ca^x$, $a > 0$, $a \neq 1$, and $C \neq 0$, if x is any real number, then

$$\boxed{\frac{f(x+1)}{f(x)} = a \quad \text{or} \quad f(x+1) = af(x)}$$

In Words
For 1-unit changes in the input x of an exponential function $f(x) = C \cdot a^x$, the ratio of consecutive outputs is the constant a.

Proof

$$\frac{f(x+1)}{f(x)} = \frac{Ca^{x+1}}{Ca^x} = a^{x+1-x} = a^1 = a$$ ■

EXAMPLE 2 **Identifying Linear or Exponential Functions**

Determine whether the given function is linear, exponential, or neither. For those that are linear, find a linear function that models the data. For those that are exponential, find an exponential function that models the data.

(a)

x	y
−1	5
0	2
1	−1
2	−4
3	−7

(b)

x	y
−1	32
0	16
1	8
2	4
3	2

(c)

x	y
−1	2
0	4
1	7
2	11
3	16

Solution For each function, compute the average rate of change of y with respect to x and the ratio of consecutive outputs. If the average rate of change is constant, then the function is linear. If the ratio of consecutive outputs is constant, then the function is exponential.

(continued)

Table 2 (a)

x	y	Average Rate of Change	Ratio of Consecutive Outputs
−1	5		
		$\frac{\Delta y}{\Delta x} = \frac{2-5}{0-(-1)} = -3$	$\frac{2}{5}$
0	2		
		$\frac{-1-2}{1-0} = -3$	$\frac{-1}{2} = -\frac{1}{2}$
1	−1		
		$\frac{-4-(-1)}{2-1} = -3$	$\frac{-4}{-1} = 4$
2	−4		
		$\frac{-7-(-4)}{3-2} = -3$	$\frac{-7}{-4} = \frac{7}{4}$
3	−7		

(b)

x	y	Average Rate of Change	Ratio of Consecutive Outputs
−1	32		
		$\frac{\Delta y}{\Delta x} = \frac{16-32}{0-(-1)} = -16$	$\frac{16}{32} = \frac{1}{2}$
0	16		
		−8	$\frac{8}{16} = \frac{1}{2}$
1	8		
		−4	$\frac{4}{8} = \frac{1}{2}$
2	4		
		−2	$\frac{2}{4} = \frac{1}{2}$
3	2		

(c)

x	y	Average Rate of Change	Ratio of Consecutive Outputs
−1	2		
		$\frac{\Delta y}{\Delta x} = \frac{4-2}{0-(-1)} = 2$	2
0	4		
		3	$\frac{7}{4}$
1	7		
		4	$\frac{11}{7}$
2	11		
		5	$\frac{16}{11}$
3	16		

(a) See Table 2(a). The average rate of change for every 1-unit increase in x is -3. Therefore, the function is a linear function. In a linear function the average rate of change is the slope m, so $m = -3$. The y-intercept b is the value of the function at $x = 0$, so $b = 2$. The linear function that models the data is $f(x) = mx + b = -3x + 2$.

(b) See Table 2(b). For this function, the average rate of change is not constant. So the function is not a linear function. The ratio of consecutive outputs for a 1-unit increase in the inputs is a constant, $\frac{1}{2}$. Because the ratio of consecutive outputs is constant, the function is an exponential function with growth factor $a = \frac{1}{2}$. The initial value C of the exponential function is $C = 16$, the value of the function at 0. Therefore, the exponential function that models the data is $g(x) = Ca^x = 16 \cdot \left(\frac{1}{2}\right)^x$.

(c) See Table 2(c). For this function, neither the average rate of change nor the ratio of two consecutive outputs is constant. Because the average rate of change is not constant, the function is not a linear function. Because the ratio of consecutive outputs is not a constant, the function is not an exponential function.

Now Work PROBLEM 29

2 Graph Exponential Functions

If we know how to graph an exponential function of the form $f(x) = a^x$, then we can use transformations (shifting, stretching, and so on) to obtain the graph of any exponential function.

First, let's graph the exponential function $f(x) = 2^x$.

EXAMPLE 3 Graphing an Exponential Function

Graph the exponential function: $f(x) = 2^x$

Solution The domain of $f(x) = 2^x$ is the set of all real numbers. Begin by locating some points on the graph of $f(x) = 2^x$, as listed in Table 3.

Because $2^x > 0$ for all x, the graph has no x-intercepts and lies above the x-axis for all x. The y-intercept is 1.

Table 3 suggests that as $x \to -\infty$, the value of f approaches 0. Therefore, the x-axis ($y = 0$) is a horizontal asymptote of the graph of f as $x \to -\infty$. This provides the end behavior for x large and negative.

To determine the end behavior for x large and positive, look again at Table 3. As $x \to \infty$, $f(x) = 2^x$ grows very quickly, causing the graph of $f(x) = 2^x$ to rise very rapidly.

Using all this information, plot some of the points from Table 3 and connect them with a smooth, continuous curve, as shown in Figure 18. From the graph, we conclude that the range of f is $(0, \infty)$. We also conclude that f is an increasing function, and so f is one-to-one.

Table 3

x	$f(x) = 2^x$
-10	$2^{-10} \approx 0.00098$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
10	$2^{10} = 1024$

NOTE Recall $a^{-n} = \frac{1}{a^n}$.

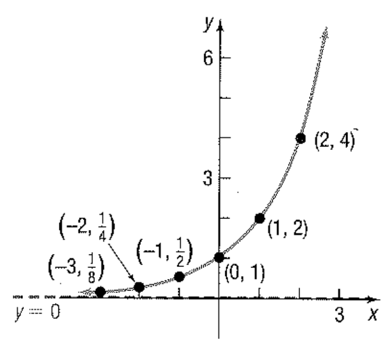
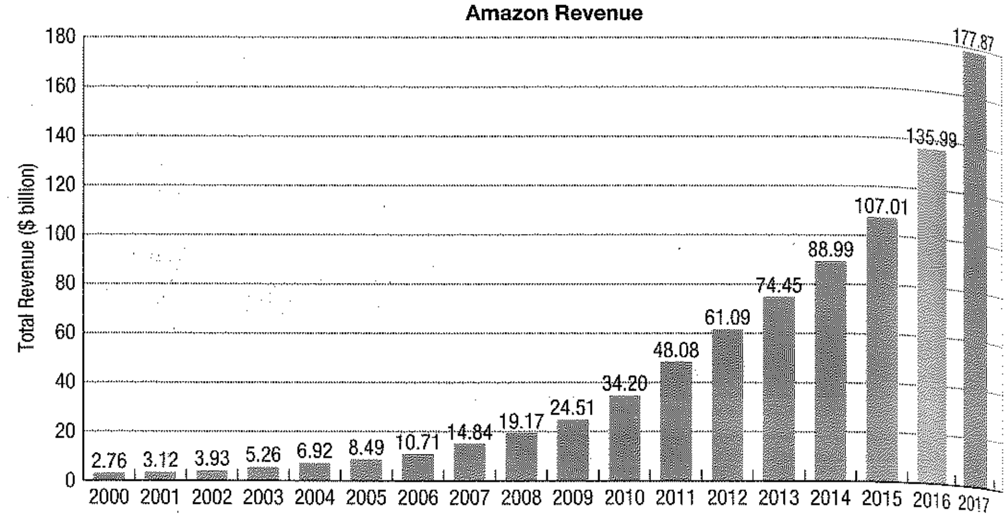


Figure 18 $f(x) = 2^x$

Graphs that look like the one in Figure 18 occur very frequently in a variety of situations. For example, the graph in Figure 19 on the next page shows the annual revenue of Amazon, Inc. from 2000 to 2017. One might conclude from this graph that Amazon's annual revenue is growing *exponentially*.

Later in this chapter, we discuss other situations that exhibit exponential growth. For now, we continue to explore properties of exponential functions.



Source: Amazon, Inc.

Figure 19

The graph of $f(x) = 2^x$ in Figure 18 is typical of all exponential functions of the form $f(x) = a^x$ with $a > 1$. Such functions are increasing functions and so are one-to-one. Their graphs lie above the x-axis, contain the point $(0, 1)$, and rise rapidly as $x \to \infty$. As $x \to -\infty$, the x-axis $(y = 0)$ is a horizontal asymptote. There are no vertical asymptotes. Finally, the graphs are smooth and continuous with no corners or gaps.

Figure 20 illustrates the graphs of two other exponential functions whose bases are larger than 1. Notice that the larger the base, the steeper the graph is when $x > 0$, and when $x < 0$, the larger the base, the closer the graph is to the x-axis.

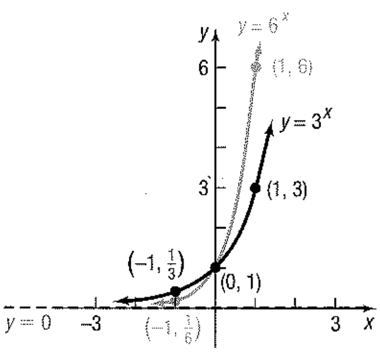


Figure 20

Seeing the Concept

Graph $Y_1 = 2^x$ and compare what you see to Figure 18. Clear the screen, graph $Y_1 = 3^x$ and $Y_2 = 6^x$, and compare what you see to Figure 20. Clear the screen and graph $Y_1 = 10^x$ and $Y_2 = 100^x$.

Properties of the Exponential Function $f(x) = a^x$, $a > 1$

- The domain is the set of all real numbers, or $(-\infty, \infty)$ using interval notation; the range is the set of positive real numbers, or $(0, \infty)$ using interval notation.
- There are no x-intercepts; the y-intercept is 1.
- The x-axis $(y = 0)$ is a horizontal asymptote of the graph of f as $x \to -\infty$.
- $f(x) = a^x$, $a > 1$, is an increasing function and is one-to-one.
- The graph of f contains the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$ and $(1, a)$.
- The graph of f is smooth and continuous, with no corners or gaps. See Figure 21.

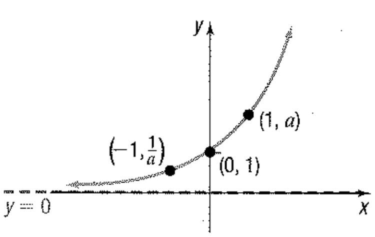


Figure 21 $f(x) = a^x, a > 1$

Now consider $f(x) = a^x$ when $0 < a < 1$.

EXAMPLE 4 Graphing an Exponential Function

Graph the exponential function: $f(x) = \left(\frac{1}{2}\right)^x$

Solution The domain of $f(x) = \left(\frac{1}{2}\right)^x$ is all real numbers. As before, locate some points on the graph as shown in Table 4. Because $\left(\frac{1}{2}\right)^x > 0$ for all x, the range of f is the interval $(0, \infty)$. The graph lies above the x-axis and has no x-intercepts. The y-intercept is 1. As $x \to -\infty$, $f(x) = \left(\frac{1}{2}\right)^x$ grows very quickly. As $x \to \infty$, the values of $f(x)$ approach 0. The x-axis ($y = 0$) is a horizontal asymptote of the graph of f as $x \to \infty$. The function f is a decreasing function and so is one-to-one. Figure 22 shows the graph of f.

Table 4

x	$f(x) = \left(\frac{1}{2}\right)^x$
-10	$\left(\frac{1}{2}\right)^{-10} = 2^{10} = 1024$
-3	$\left(\frac{1}{2}\right)^{-3} = 2^3 = 8$
-2	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$
-1	$\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$
0	$\left(\frac{1}{2}\right)^{0} = 1$
1	$\left(\frac{1}{2}\right)^{1} = \frac{1}{2}$
2	$\left(\frac{1}{2}\right)^{2} = \frac{1}{4}$
3	$\left(\frac{1}{2}\right)^{3} = \frac{1}{8}$
10	$\left(\frac{1}{2}\right)^{10} \approx 0.00098$

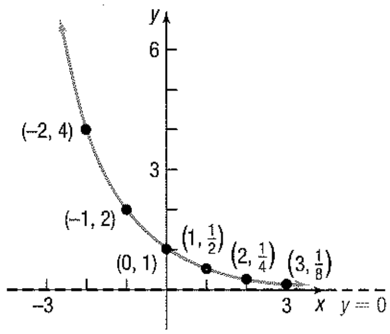


Figure 22 $f(x) = \left(\frac{1}{2}\right)^x$

The graph of $y = \left(\frac{1}{2}\right)^x$ also can be obtained from the graph of $y = 2^x$ using transformations. The graph of $y = \left(\frac{1}{2}\right)^x = 2^{-x}$ is a reflection about the y-axis of the graph of $y = 2^x$ (replace x by $-x$). See Figures 23(a) and (b).

Need to Review?
Reflections about the y-axis are discussed in Section 3.5, p. 261.

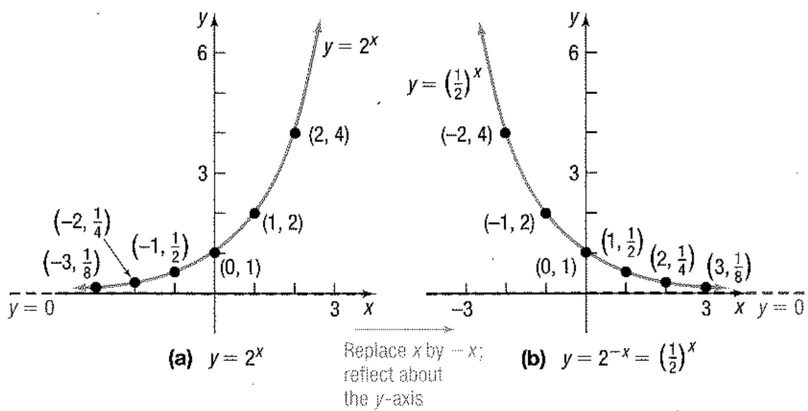


Figure 23

Seeing the Concept
Using a graphing utility, simultaneously graph:

(a) $Y_1 = 3^x$, $Y_2 = \left(\frac{1}{3}\right)^x$

(b) $Y_1 = 6^x$, $Y_2 = \left(\frac{1}{6}\right)^x$

Conclude that the graph of $Y_2 = \left(\frac{1}{a}\right)^x$, for $a > 0$, is the reflection about the y-axis of the graph of $Y_1 = a^x$.

The graph of $f(x) = \left(\frac{1}{2}\right)^x$ in Figure 22 is typical of all exponential functions of the form $f(x) = a^x$ with $0 < a < 1$. Such functions are decreasing and one-to-one. Their graphs lie above the x-axis and contain the point $(0, 1)$. The graphs rise rapidly as $x \to -\infty$. As $x \to \infty$, the x-axis ($y = 0$) is a horizontal asymptote. There are no vertical asymptotes. Finally, the graphs are smooth and continuous, with no corners or gaps.

Figure 24 illustrates the graphs of two other exponential functions whose bases are between 0 and 1. Notice that the smaller base results in a graph that is steeper when $x < 0$. When $x > 0$, the graph of the equation with the smaller base is closer to the x-axis.

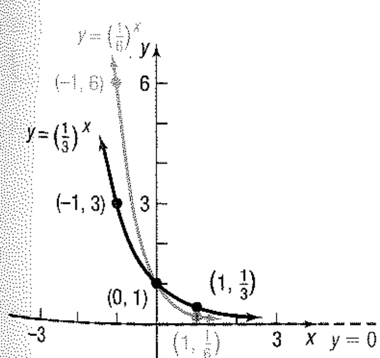


Figure 24

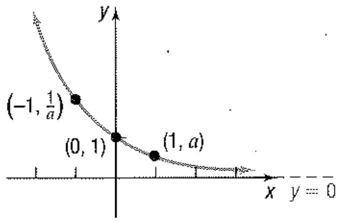


Figure 25 $f(x) = a^x, 0 < a < 1$

Properties of the Exponential Function $f(x) = a^x, 0 < a < 1$

- The domain is the set of all real numbers, or $(-\infty, \infty)$ using interval notation; the range is the set of positive real numbers, or $(0, \infty)$ using interval notation.
- There are no x-intercepts; the y-intercept is 1.
- The x-axis $(y = 0)$ is a horizontal asymptote of the graph of f as $x \to \infty$.
- $f(x) = a^x, 0 < a < 1$, is a decreasing function and is one-to-one.
- The graph of f contains the points $\left(-1, \dfrac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.
- The graph of f is smooth and continuous, with no corners or gaps. See Figure 25.

EXAMPLE 5 **Graphing an Exponential Function Using Transformations**

Graph $f(x) = 2^{-x} - 3$ and determine the domain, range, horizontal asymptote, and y-intercept of f.

Solution Begin with the graph of $y = 2^x$. Figure 26 shows the steps.

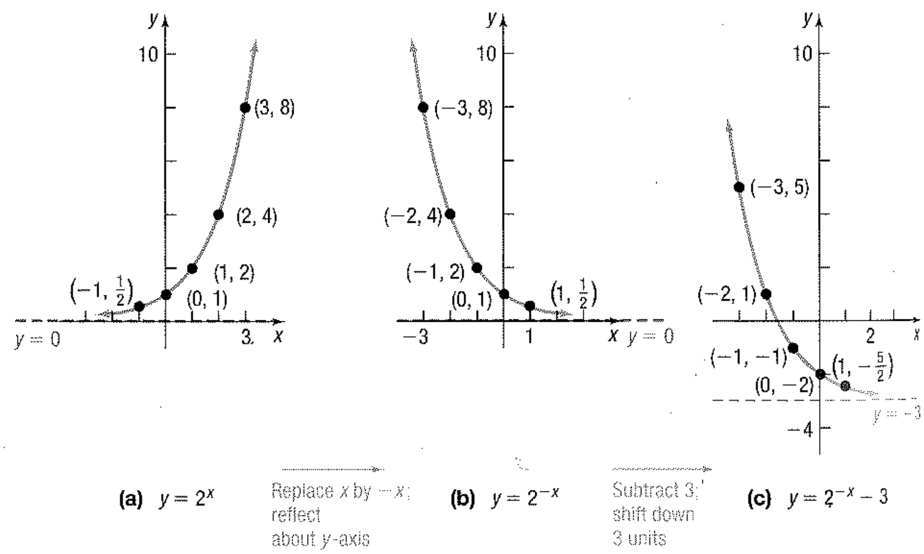


Figure 26

As Figure 26(c) illustrates, the domain of $f(x) = 2^{-x} - 3$ is the interval $(-\infty, \infty)$ and the range is the interval $(-3, \infty)$. The horizontal asymptote of the graph of f is the line $y = -3$. The y-intercept is $f(0) = 2^0 - 3 = 1 - 3 = -2$.

Now Work PROBLEM 45

3 Define the Number e

Many problems that occur in nature require the use of an exponential function whose base is a certain irrational number, symbolized by the letter e.

One way of arriving at this important number e is given next.

Historical Feature

The number e is called *Euler's* number in honor of the Swiss mathematician Leonard Euler (1707–1783). ■

DEFINITION Number e

The **number *e*** is defined as the number that the expression

$$\left(1+\frac{1}{n}\right)^n \qquad (2)$$

approaches as $n \to \infty$. In calculus, this is expressed, using limit notation, as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Table 5 illustrates what happens to the defining expression (2) as n takes on increasingly large values. The last number in the right column in the table approximates e correct to nine decimal places. That is, $e = 2.718281828\ldots$. Remember, the three dots indicate that the decimal places continue. Because these decimal places continue but do not repeat, e is an irrational number. The number e is often expressed as a decimal rounded to a specific number of places. For example, $e \approx 2.71828$ is rounded to five decimal places.

Table 5

n	$\frac{1}{n}$	$1+\frac{1}{n}$	$\left(1+\frac{1}{n}\right)^n$
1	1	2	2
2	0.5	1.5	2.25
5	0.2	1.2	2.48832
10	0.1	1.1	2.59374246
100	0.01	1.01	2.704813829
1,000	0.001	1.001	2.716923932
10,000	0.0001	1.0001	2.718145927
100,000	0.00001	1.00001	2.718268237
1,000,000	0.000001	1.000001	2.718280469
10,000,000,000	10^{-10}	$1+10^{-10}$	2.718281828

Table 6

x	e^x
−2	$e^{-2} \approx 0.14$
−1	$e^{-1} \approx 0.37$
0	$e^0 = 1$
1	$e^1 \approx 2.72$
2	$e^2 \approx 7.39$

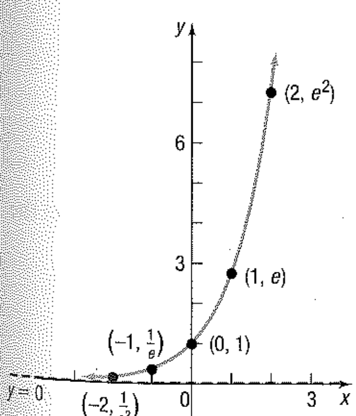


Figure 27 $y = e^x$

The exponential function $f(x) = e^x$, whose base is the number e, occurs with such frequency in applications that it is usually referred to as *the* exponential function. Most calculators have the key $\boxed{e^x}$ or $\boxed{\exp(x)}$, which may be used to approximate the exponential function for a given value of x.*

Use your calculator to approximate e^x for $x = -2, x = -1, x = 0, x = 1$, and $x = 2$. See Table 6. The graph of the exponential function $f(x) = e^x$ is given in Figure 27. Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. Do you see why? (Refer to Figures 18 and 20.)

Seeing the Concept

Graph $Y_1 = e^x$ and compare what you see to Figure 27. Use eVALUEate or TABLE to verify the points on the graph shown in Figure 27. Now graph $Y_2 = 2^x$ and $Y_3 = 3^x$ on the same screen as $Y_1 = e^x$. Notice that the graph of $Y_1 = e^x$ lies between these two graphs.

EXAMPLE 6

Graphing an Exponential Function Using Transformations

Graph $f(x) = -e^{x-3}$ and determine the domain, range, horizontal asymptote, and y-intercept of f.

*If your calculator does not have one of these keys, refer to your owner's manual.

Solution Begin with the graph of $y = e^x$. Figure 28 shows the steps.

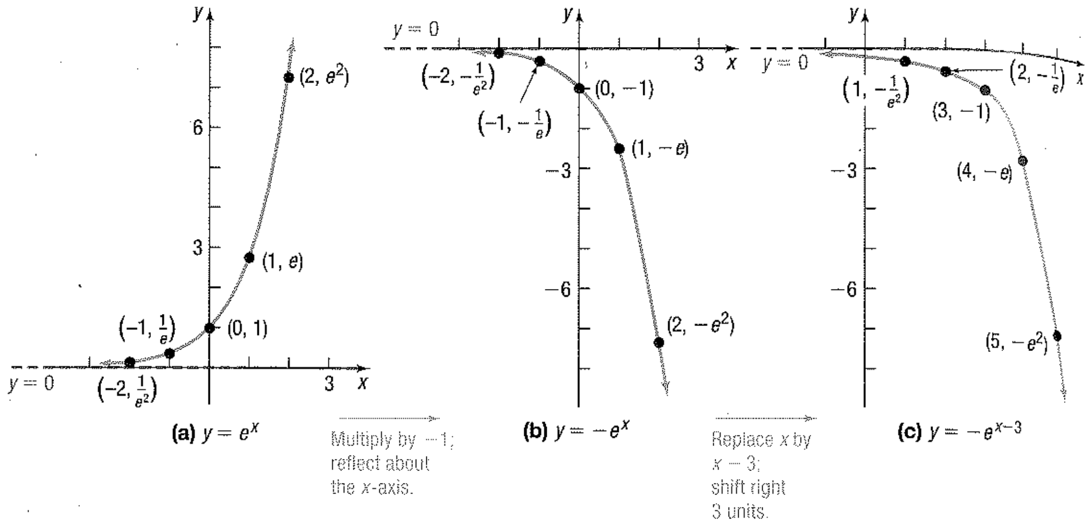


Figure 28

As Figure 28(c) illustrates, the domain of $f(x) = -e^{x-3}$ is the interval $(-\infty, \infty)$, and the range is the interval $(-\infty, 0)$. The horizontal asymptote is the line $y = 0$. The y-intercept is $f(0) = -e^{0-3} = -e^{-3} \approx -0.05$.

Now Work PROBLEM 57

4 Solve Exponential Equations

Equations that involve terms of the form a^x, where $a > 0$ and $a \neq 1$, are referred to as **exponential equations**. Such equations can sometimes be solved by using the Laws of Exponents and property (3):

In Words
When two exponential expressions with the same base are equal, then their exponents are equal.

$$\text{If} \quad a^u = a^v, \quad \text{then} \quad u = v. \tag{3}$$

Property (3) is a consequence of the fact that exponential functions are one-to-one. To use property (3), each side of the equality must be written with the same base.

EXAMPLE 7 **Solving Exponential Equations**

Solve each exponential equation.

(a) $3^{x+1} = 81$ (b) $4^{2x-1} = 8^{x+3}$

Solution (a) Since $81 = 3^4$, write the equation as

$$3^{x+1} = 3^4$$

Now the expressions on both sides of the equation have the same base, 3. Set the exponents equal to each other to obtain

$$x + 1 = 4$$
$$x = 3$$

The solution set is $\{3\}$.

(b)
$$\begin{aligned} 4^{2x-1} &= 8^{x+3} \\ (2^2)^{(2x-1)} &= (2^3)^{(x+3)} && 4 = 2^2;\ 8 = 2^3 \\ 2^{2(2x-1)} &= 2^{3(x+3)} && (a^r)^s = a^{rs} \\ 2(2x-1) &= 3(x+3) && \text{If } a^u = a^v \text{, then } u = v \text{ (property (3)).} \\ 4x - 2 &= 3x + 9 \\ x &= 11 \end{aligned}$$

The solution set is $\{11\}$.

Now Work PROBLEMS 67 AND 77

EXAMPLE 8 Solving an Exponential Equation

Solve: $e^{-x^2} = (e^x)^2 \cdot \frac{1}{e^3}$

Solution Use the Laws of Exponents first to get a single expression with the base e on the right side.

$$(e^x)^2 \cdot \frac{1}{e^3} = e^{2x} \cdot e^{-3} = e^{2x-3}$$

Then,

$$
\begin{aligned}
e^{-x^2} &= e^{2x-3} \\
-x^2 &= 2x - 3 && \text{Use property (3).} \\
x^2 + 2x - 3 &= 0 && \text{Place the quadratic equation in standard form.} \\
(x+3)(x-1) &= 0 && \text{Factor.} \\
x = -3 \quad \text{or} \quad x &= 1 && \text{Use the Zero-Product Property.}
\end{aligned}
$$

The solution set is $\{-3, 1\}$.

Now Work PROBLEM 83

EXAMPLE 9 Exponential Probability

Between 9:00 PM and 10:00 PM, cars arrive at Burger King's drive-thru at the rate of 12 cars per hour (0.2 car per minute). The following formula from probability theory can be used to determine the probability that a car will arrive within t minutes of 9:00 PM.

$$F(t) = 1 - e^{-0.2t}$$

(a) Determine the probability that a car will arrive within 5 minutes of 9 PM (that is, before 9:05 PM).

(b) Determine the probability that a car will arrive within 30 minutes of 9 PM (before 9:30 PM).

(c) Graph F.

(d) What does F approach as t increases without bound in the positive direction?

Solution (a) The probability that a car will arrive within 5 minutes is found by evaluating $F(t)$ at $t = 5$.

$$F(5) = 1 - e^{-0.2(5)} \approx 0.63212$$

Use a calculator.

There is a 63% probability that a car will arrive within 5 minutes.

(b) The probability that a car will arrive within 30 minutes is found by evaluating $F(t)$ at $t = 30$.

$$F(30) = 1 - e^{-0.2(30)} \approx 0.9975$$

Use a calculator.

There is a 99.75% probability that a car will arrive within 30 minutes.

(c) See Figure 29 for the graph of F.

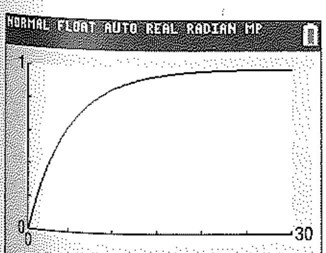


Figure 29 $F(t) = 1 - e^{-0.2t}$

(d) As time passes, the probability that a car will arrive increases. The value that F approaches can be found by letting $t \to \infty$. Since $e^{-0.2t} = \frac{1}{e^{0.2t}}$, it follows that $e^{-0.2t} \to 0$ as $t \to \infty$. Therefore, $F(t) = 1 - e^{-0.2t} \to 1$ as $t \to \infty$. The algebraic analysis is supported by Figure 29.

Now Work PROBLEM 115

SUMMARY

Properties of the Exponential Function

$f(x) = a^x, \quad a > 1$

- Domain: the interval $(-\infty, \infty)$; range: the interval $(0, \infty)$
- x-intercepts: none; y-intercept: 1
- Horizontal asymptote: x-axis $(y = 0)$ as $x \to -\infty$
- Increasing; one-to-one; smooth; continuous
- See Figure 21 for a typical graph.

$f(x) = a^x, \quad 0 < a < 1$

- Domain: the interval $(-\infty, \infty)$; range: the interval $(0, \infty)$
- x-intercepts: none; y-intercept: 1
- Horizontal asymptote: x-axis $(y = 0)$ as $x \to \infty$
- Decreasing; one-to-one; smooth; continuous
- See Figure 25 for a typical graph.

If $a^u = a^v$, then $u = v$.

6.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. $4^3 =$ ___64___; $8^{2/3} =$ ___4___; $3^{-2} =$ ___$\frac{1}{9}$___. (pp. 21–23 and pp. 76–77)

2. Solve: $x^2 + 3x = 4$ (pp. 93–99) $\{-4, 1\}$

3. ***True or False*** To graph $y = (x - 2)^3$, shift the graph of $y = x^3$ to the left 2 units. (pp. 254–263) False

4. Find the average rate of change of $f(x) = 3x - 5$ from $x = 0$ to $x = 4$. (pp. 235–267) 3

5. ***True or False*** The graph of the function $f(x) = \dfrac{2x}{x-3}$ has $y = 2$ as a horizontal asymptote. (pp. 359–361) True

6. If $f(x) = -3x + 10$, then the graph of f is a ___line___ with slope ___-3___ and y-intercept ___10___. (pp. 281–284)

7. Where is the function $f(x) = x^2 - 4x + 3$ increasing? Where is it decreasing? (pp. 302–305) $[2, \infty)$; $(-\infty, 2]$

Concepts and Vocabulary

8. A(n) ___exponential function___ is a function of the form $f(x) = Ca^x$, where $a > 0$, $a \neq 1$, and $C \neq 0$ are real numbers. The base a is the ___growth factor___ and C is the ___initial value___.

9. For an exponential function $f(x) = Ca^x$, $\dfrac{f(x+1)}{f(x)} =$ ___a___.

10. ***True or False*** The domain of the function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is the set of all real numbers. True

11. ***True or False*** The function $f(x) = e^x$ is increasing and is one-to-one. True

*__12.__ The graph of every exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, contains the three points: ______, ______, and ______.

13. If $3^x = 3^4$, then $x =$ ___4___.

14. ***True or False*** The graphs of $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$ are identical. False

15. ***Multiple Choice*** Which exponential function is increasing?

(a) $f(x) = 0.5^x$ **(b)** $f(x) = \left(\frac{5}{2}\right)^x$ **(c)** $f(x) = \left(\frac{2}{3}\right)^x$ **(d)** $f(x) = 0.9^x$ b

16. ***Multiple Choice*** The range of the function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is the interval

(a) $(-\infty, \infty)$ **(b)** $(-\infty, 0)$ **(c)** $(0, \infty)$ **(d)** $[0, \infty)$ c

Skill Building

In Problems 17–28, approximate each number using a calculator. Express your answer rounded to three decimal places.

*__17.__ **(a)** $2^{3.14}$ **(b)** $2^{3.141}$ **(c)** $2^{3.1415}$ **(d)** 2^{π}

*__18.__ **(a)** $2^{2.7}$ **(b)** $2^{2.71}$ **(c)** $2^{2.718}$ **(d)** 2^{e}

*__19.__ **(a)** $3.1^{2.7}$ **(b)** $3.14^{2.71}$ **(c)** $3.141^{2.718}$ **(d)** π^{e}

*__20.__ **(a)** $2.7^{3.1}$ **(b)** $2.71^{3.14}$ **(c)** $2.718^{3.141}$ **(d)** e^{π}

21. $(1 + 0.04)^6$ 1.265

22. $\left(1 + \dfrac{0.09}{12}\right)^{24}$ 1.196

23. $8.4\left(\dfrac{1}{3}\right)^{2.9}$ 0.347

24. $158\left(\dfrac{5}{6}\right)^{8.63}$ 32.758

25. $e^{1.2}$ 3.320

26. $e^{-1.3}$ 0.273

27. $125e^{0.026(7)}$ 149.952

28. $83.6e^{-0.157(9.5)}$ 18.813

*Due to space restrictions, answers to these exercises may be found in the Answers in the back of the text.

In Problems 29–36, determine whether the given function is linear, exponential, or neither. For those that are linear functions, find a linear function that models the data; for those that are exponential, find an exponential function that models the data.

29.

x	$f(x)$
−1	3
0	6
1	12
2	18
3	30

Neither

30.

x	$g(x)$
−1	2
0	5
1	8
2	11
3	14

Linear; $g(x) = 3x + 5$

31.

x	$H(x)$
−1	$\frac{1}{4}$
0	1
1	4
2	16
3	64

Exponential; $H(x) = 4^x$

32.

x	$F(x)$
−1	$\frac{2}{3}$
0	1
1	$\frac{3}{2}$
2	$\frac{9}{4}$
3	$\frac{27}{8}$

33.

x	$f(x)$
−1	$\frac{3}{2}$
0	3
1	6
2	12
3	24

Exponential; $f(x) = 3 \cdot 2^x$

34.

x	$g(x)$
−1	6
0	1
1	0
2	3
3	10

Neither

35.

x	$H(x)$
−1	2
0	4
1	6
2	8
3	10

Linear; $H(x) = 2x + 4$

36.

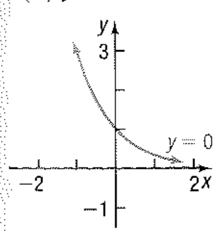
x	$F(x)$
−1	$\frac{1}{2}$
0	$\frac{1}{4}$
1	$\frac{1}{8}$
2	$\frac{1}{16}$
3	$\frac{1}{32}$

In Problems 37–44, the graph of an exponential function is given. Match each graph to one of the following functions.

(A) $y = 3^x$ (B) $y = 3^{-x}$ (C) $y = -3^x$ (D) $y = -3^{-x}$

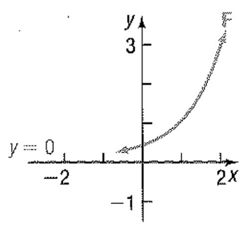
(E) $y = 3^x - 1$ (F) $y = 3^{x-1}$ (G) $y = 3^{1-x}$ (H) $y = 1 - 3^x$

37.



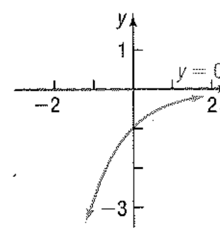
B

38.



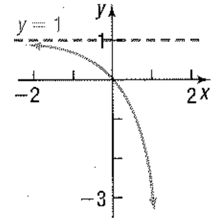
F

39.



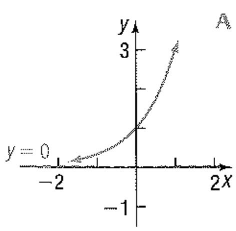
D

40.



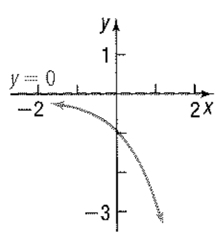
H

41.



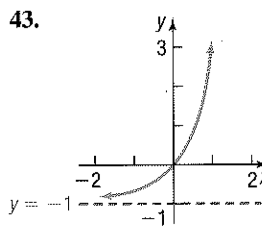
A

42.



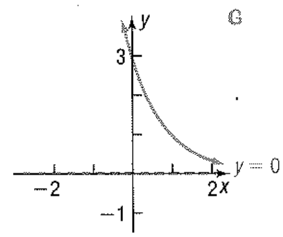
C

43.



E

44.



G

In Problems 45–56, use transformations to graph each function. Determine the domain, range, horizontal asymptote, and y-intercept of each function.

45. $f(x) = 2^x + 1$ **46.** $f(x) = 3^x - 2$ **47.** $f(x) = 3^{x-1}$ **48.** $f(x) = 2^{x+2}$

49. $f(x) = 3 \cdot \left(\frac{1}{2}\right)^x$ **50.** $f(x) = 4 \cdot \left(\frac{1}{3}\right)^x$ **51.** $f(x) = 3^{-x} - 2$ **52.** $f(x) = -3^x + 1$

53. $f(x) = 2 + 4^{x-1}$ **54.** $f(x) = 1 - 2^{x+3}$ **55.** $f(x) = 2 + 3^{x/2}$ **56.** $f(x) = 1 - 2^{-x/3}$

In Problems 57–64, begin with the graph of $y = e^x$ (Figure 27) and use transformations to graph each function. Determine the domain, range, horizontal asymptote, and y-intercept of each function.

57. $f(x) = e^{-x}$ **58.** $f(x) = -e^x$ **59.** $f(x) = e^{x+2}$ **60.** $f(x) = e^x - 1$

61. $f(x) = 5 - e^{-x}$ **62.** $f(x) = 9 - 3e^{-x}$ **63.** $f(x) = 2 - e^{-x/2}$ **64.** $f(x) = 7 - 3e^{2x}$

In Problems 65–84, solve each equation.

65. $6^x = 6^5$ $\{5\}$

66. $5^x = 5^{-6}$ $\{-6\}$

67. $2^{-x} = 16$ $\{-4\}$

68. $3^{-x} = 81$ $\{-4\}$

69. $\left(\frac{1}{5}\right)^x = \frac{1}{25}$ $\{2\}$

70. $\left(\frac{1}{4}\right)^x = \frac{1}{64}$ $\{3\}$

71. $3^{2x-5} = 9$ $\left\{\frac{7}{2}\right\}$

72. $5^{x+3} = \frac{1}{5}$ $\{-4\}$

73. $3^{x^3} = 9^x$ $\{-\sqrt{2}, 0, \sqrt{2}\}$

74. $4^{x^2} = 2^x$ $\left\{0, \frac{1}{2}\right\}$

75. $8^{-x+11} = 16^{2x}$ $\{3\}$

76. $9^{-x+15} = 27^x$ $\{6\}$

77. $3^{x^2-7} = 27^{2x}$ $\{-1, 7\}$

78. $5^{x^2+8} = 125^{2x}$ $\{2, 4\}$

79. $4^x \cdot 2^{x^2} = 16^2$ $\{-4, 2\}$

80. $9^{2x} \cdot 27^{x^2} = 3^{-1}$ $\left\{-1, -\frac{1}{3}\right\}$

81. $e^{2x} = e^{5x+12}$ $\{-4\}$

82. $e^{3x} = e^{2-x}$ $\left\{\frac{1}{2}\right\}$

83. $e^{x^2} = e^{3x} \cdot \frac{1}{e^2}$ $\{1, 2\}$

84. $(e^4)^x \cdot e^{x^2} = e^{12}$ $\{-6, 2\}$

85. If $4^x = 7$, what does 4^{-2x} equal? $\frac{1}{49}$

86. If $2^x = 3$, what does 4^{-x} equal? $\frac{1}{9}$

87. If $3^{-x} = 2$, what does 3^{2x} equal? $\frac{1}{4}$

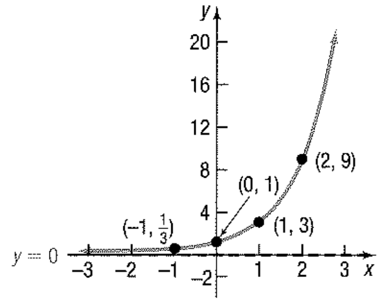
88. If $5^{-x} = 3$, what does 5^{3x} equal? $\frac{1}{27}$

89. If $9^x = 25$, what does 3^x equal? 5

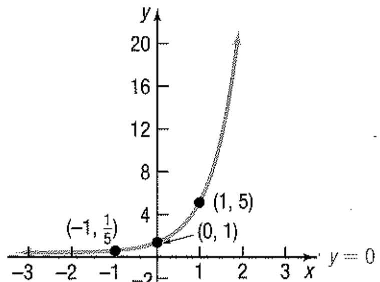
90. If $2^{-3x} = \frac{1}{1000}$, what does 2^x equal? 10

In Problems 91–94, determine the exponential function whose graph is given.

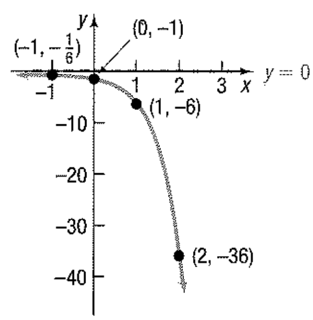
91. $f(x) = 3^x$



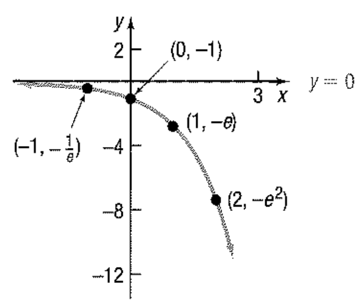
92. $f(x) = 5^x$



93. $f(x) = -6^x$



94. $f(x) = -e^x$



95. Find an exponential function whose graph has the horizontal asymptote $y = 2$ and contains the points $(0, 3)$ and $(1, 5)$.

96. Find an exponential function whose graph has the horizontal asymptote $y = -3$ and contains the points $(0, -2)$ and $(-2, 1)$.

97. Suppose that $f(x) = 2^x$.
(a) What is $f(4)$? What point is on the graph of f?
(b) If $f(x) = \frac{1}{16}$, what is x? What point is on the graph of f?

98. Suppose that $f(x) = 3^x$.
(a) What is $f(4)$? What point is on the graph of f?
(b) If $f(x) = \frac{1}{9}$, what is x? What point is on the graph of f?

99. Suppose that $g(x) = 4^x + 2$.
(a) What is $g(-1)$? What point is on the graph of g?
(b) If $g(x) = 66$, what is x? What point is on the graph of g? $3; (3, 66)$

100. Suppose that $g(x) = 5^x - 3$.
(a) What is $g(-1)$? What point is on the graph of g?
(b) If $g(x) = 122$, what is x? What point is on the graph of g? $3; (3, 122)$

101. Suppose that $H(x) = \left(\frac{1}{2}\right)^x - 4$.
(a) What is $H(-6)$? What point is on the graph of H?
(b) If $H(x) = 12$, what is x? What point is on the graph of H? $-4; (-4, 12)$
(c) Find the zero of H. -2

102. Suppose that $F(x) = \left(\frac{1}{3}\right)^x - 3$.
(a) What is $F(-5)$? What point is on the graph of F?
(b) If $F(x) = 24$, what is x? What point is on the graph of F? $-3; (-3, 24)$
(c) Find the zero of F. -1

Mixed Practice *In Problems 103–106, graph each function. Based on the graph, state the domain and the range, and find any intercepts.*

103. $f(x) = \begin{cases} e^{-x} & \text{if } x < 0 \\ e^x & \text{if } x \geq 0 \end{cases}$

104. $f(x) = \begin{cases} e^{x} & \text{if } x < 0 \\ e^{-x} & \text{if } x \geq 0 \end{cases}$

105. $f(x) = \begin{cases} -e^{x} & \text{if } x < 0 \\ -e^{-x} & \text{if } x \geq 0 \end{cases}$

106. $f(x) = \begin{cases} -e^{-x} & \text{if } x < 0 \\ -e^{x} & \text{if } x \geq 0 \end{cases}$