Math 5001 Homework 2

Jacob Hauck

January 14, 2024

Question 1.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then f has an antiderivative F. The Mean Value Theorem applied to F implies that for $x_0, \varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$,

$$\frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon} f(y)\,\mathrm{d}y = \frac{F(x_0+\varepsilon) - F(x_0-\varepsilon)}{(x_0+\varepsilon)-(x_0-\varepsilon)} = f(x_\varepsilon)$$

for some $x_\varepsilon \in (x_0 - |\varepsilon|, x_0 + |\varepsilon|)$.

Let $e > 0$; since f is continuous, there exists some $d > 0$ such that $|x - x_0| < d \implies |f(x) - f(x_0)| < e$. If $|\varepsilon| < d$, then $|x_\varepsilon - x_0| < d$, which implies that $|f(x_\varepsilon) - f(x_0)| < e$. Therefore

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon} f(y)\,\mathrm{d}y = f(x_0).$$

Question 2.

Let $f \in L^1$ be differentiable, and let $f' \in L^1$. Let $u = e^{-ix\xi}$ and $v' = f'$. Then $u' = -i\xi e^{-ix\xi}$, and $v = f$. Using integration by parts,

$$\begin{aligned}
\int_{\mathbb{R}} f'(x)e^{-ix\xi}\,\mathrm{d}x &= uv|_{-\infty}^{\infty} - \int_{\mathbb{R}} u'v\,\mathrm{d}x \\
&= e^{-ix\xi}f(x)\Big|_{-\infty}^{\infty} + i\xi\int_{\mathbb{R}} e^{-ix\xi}f(x)\,\mathrm{d}x \\
&= i\xi\int_{\mathbb{R}} e^{-ix\xi}f(x)\,\mathrm{d}x.
\end{aligned}$$

The last equation follows because $f(x) \to 0$ as $x \to \pm\infty$ (since $f \in L^1$), and $\left|e^{-ix\xi}\right| = 1$. Thus, as $x \to \pm\infty$, we have $\left|e^{-ix\xi}f(x)\right| \to 0$, which means that $e^{-ix\xi}f(x) \to 0$.

Question 3.

Let $a > 0$ and $I = \int_{\mathbb{R}} e^{-ax^2}\,\mathrm{d}x$. Then (by Fubini's Theorem)

$$I^2 = \int_{\mathbb{R}} e^{-ax^2}\,\mathrm{d}x \int_{\mathbb{R}} e^{-ay^2}\,\mathrm{d}y = \int_{\mathbb{R}^2} e^{-a(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y.$$

Convert to polar coordinates, and get

$$\begin{aligned}
I^2 &= \int_{\mathbb{R}^2} e^{-ar^2} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-ar^2} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\pi\int_0^{\infty} e^{-ar^2} r\,\mathrm{d}r.
\end{aligned}$$

Using the substitution $u = ar^2$, the limits of integration are the same, and we find

$$\begin{aligned} I^2 &= 2\pi \frac{1}{2a} \int_0^\infty e^{-u} \, \mathrm{d}u = \frac{\pi}{a} \left[-e^{-u} \right]_0^\infty = \frac{\pi}{a} \\ \implies I &= \sqrt{\frac{\pi}{a}}. \end{aligned}$$