Math 6417 Homework 4

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November 25, 2023

Question 1.

Define the **Fourier transform operator** $\mathscr{F} : L^1(\mathbf{R}) \to L^\infty(\mathbf{R})$ by

$$\mathscr{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f(x) \, \mathrm{d}x. \tag{1}$$

1.1) We note that the function $x \mapsto e^{iyx} f(x)$ is clearly integrable if f is, so the integral in (1) exists for all y. We show that $\mathscr{F}(f) \in L^\infty(\mathbf{R})$ as claimed, and $\|\mathscr{F} f\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$. Indeed, for $y \in \mathbf{R}$,

$$|\mathscr{F}(f)(y)| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} e^{iyx} f(x) \, \mathrm{d}x \right| \tag{2}$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| e^{iyx} f(x) \right| \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}. \tag{3}$$

Therefore, $\|\mathscr{F} f\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$.

1.2) Suppose that $f \in C^2(\mathbf{R})$, and $f, f', f'' \in L^1(\mathbf{R})$, and $f(x), f'(x), f''(x) \to 0$ as $x \to \pm\infty$. Then there exists a constant C such that $|y^2 \mathscr{F}(f)(y)| \leq C$ for all $y \in \mathbf{R}$. Furthermore, $\mathscr{F}(f) \in L^1(\mathbf{R})$.

Proof. Since $f'' \in L^1(\mathbf{R})$, we can take its Fourier transform, which yields

$$\mathscr{F}(f'')(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f''(x) \, \mathrm{d}x. \tag{4}$$

We can integrate by parts because $f', f \in L^1(\mathbf{R})$ and are continuous, and $f(x), f'(x) \to 0$ as $x \to \pm\infty$. This gives

$$\mathscr{F}(f'')(y) = \frac{1}{\sqrt{2\pi}} \left[f'(x) e^{iyx} \Big|_{-\infty}^{\infty} - iy \int_{-\infty}^{\infty} e^{iyx} f'(x) \, \mathrm{d}x \right] \tag{5}$$

$$= \frac{iy}{\sqrt{2\pi}} \left[-f(x) e^{iyx} \Big|_{-\infty}^{\infty} + iy \int_{-\infty}^{\infty} e^{iyx} f(x) \, \mathrm{d}x \right] \tag{6}$$

$$= -y^2 \mathscr{F}(f)(y). \tag{7}$$

By the reasoning in 1.1), it follows that

$$|y^2 \mathscr{F}(f)(y)| = |\mathscr{F}(f'')(y)| \leq \frac{1}{\sqrt{2\pi}} \|f''\|_{L^1} \tag{8}$$

for all $y \in \mathbf{R}$.

Thus, if $C = \frac{1}{\sqrt{2\pi}} \|f''\|_{L^1}$, then $|\mathscr{F}(f)(y)| \leq \frac{C}{y^2}$ for all $y \in \mathbf{R}$. On the other hand, $\mathscr{F}(f) \in L^\infty(\mathbf{R})$ by part 1.1), so $\mathscr{F}(f)$ is dominated by the integrable function

$$\phi(y) = \begin{cases} \|\mathscr{F}(f)\|_{L^\infty} & y \in [-1, 1], \\ \frac{C}{y^2} & \text{otherwise}. \end{cases} \tag{9}$$

By the integral comparison test, $\mathscr{F}(f) \in L^1(\mathbf{R})$. ∎

1.3) Formally, $\mathscr{F}^2(f)(y) = f(-y)$.

Proof. Let $\delta_{x_0} = \delta(x - x_0)$, where δ is the Dirac delta function. Then, formally,

$$\mathscr{F}(\delta_{x_0})(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} \delta(x - x_0)\ \mathrm{d}x = \frac{e^{iyx_0}}{\sqrt{2\pi}}. \tag{10}$$

Next, we note that if $f \in C^1 \cap L^1(\mathbf{R})$, and $f' \in L^1(\mathbf{R})$, and $f(x) \to 0$ as $x \to \pm\infty$, then we can use integration by parts to show that

$$\mathscr{F}(f')(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f'(x)\ \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[e^{iyx} f(x) \Big|_{-\infty}^{\infty} - iy \int_{-\infty}^{\infty} e^{iyx} f(x)\ \mathrm{d}x \right] \tag{11}$$

$$= -iy\mathscr{F}(f)(y). \tag{12}$$

On the other hand, let $f \in L^1(\mathbf{R})$, and define $g(x) = ixf(x)$. If $g \in L^1(\mathbf{R})$ as well, then

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathscr{F}(f)(y) = \frac{\mathrm{d}}{\mathrm{d}y} \int_{-\infty}^{\infty} e^{iyx} f(x)\ \mathrm{d}x = \int_{-\infty}^{\infty} \frac{\partial}{\partial y} \left[e^{iyx} f(x) \right]\ \mathrm{d}x \tag{13}$$

$$= \int_{-\infty}^{\infty} e^{iyx} ixf(x)\ \mathrm{d}x = \mathscr{F}(g)(y). \tag{14}$$

If we take $f(x) = e^{-ax^2}$, then f satisfies the above assumptions. Since $f'(x) = -2axf(x)$,

$$2ai\frac{\mathrm{d}}{\mathrm{d}y}\mathscr{F}(f)(y) = 2ai\mathscr{F}(i(\cdot)f(\cdot))(y) = \mathscr{F}(-2a(\cdot)f(\cdot))(y) = \mathscr{F}(f')(y) = -iy\mathscr{F}(f)(y). \tag{15}$$

Hence, $\mathscr{F}(f)(y)$ is the unique solution of the IVP

$$u' = -\frac{1}{2a}u, \qquad u(0) = \mathscr{F}(f)(0). \tag{16}$$

Since

$$\mathscr{F}(f)(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2}\ \mathrm{d}x = \frac{1}{\sqrt{2a}}, \tag{17}$$

it follows that

$$\mathscr{F}(f)(y) = \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}. \tag{18}$$

Thus, formally, if $\phi_a(x) = e^{-ax^2}$, then

$$\mathscr{F}(1)(y) = \lim_{a \to 0} \mathscr{F}(\phi_a)(y) = \lim_{a \to 0} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}. \tag{19}$$

Note that for an integrable function g,

$$\int_{-\infty}^{\infty} \phi_a(y)g(y)\ \mathrm{d}y = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}\ \mathrm{d}y = \frac{1}{\sqrt{\pi}}, \tag{20}$$

and $\lim_{a \to 0} \phi(x) = 0$ if $x \neq 0$, we can formally interpret $\lim_{a \to 0} \phi_a$ as the Dirac delta function. $\square$