Math 5601 Homework 3

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Problem 1.

(a) To find the best approximation $p \in P^3[-1,1]$ of $f(t) = \sin(t)$ on $[-1,1]$, we simply use the definition. If p is the best approximation, then

$$E(q) = \|q - f\|_{L^2} \tag{1}$$

must be minimal with respect to $q \in P^3[-1,1]$ if $q = p$. Since every element $q \in P^3[-1,1]$ satisfies $q(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3$ for some $\{q_i\}_{i=0}^3 \in \mathbf{R}^4$, and E is minimal precisely when E^2 is minimal, it follows that the representation $\{p_i\} \in \mathbf{R}^4$ of p is the minimizer of

$$F(\{q_i\}) = E^2(q) = \|q - f\|_{L^2}^2 = \int_{-1}^{1} (q_0 + q_1 t + q_2 t^2 + q_3 t^3 - \sin(t))^2 \, \mathrm{d}t \tag{2}$$

with respect to $\{q_i\} \in \mathbf{R}^4$. Since F is clearly continuously differentiable, the Extreme Value Theorem implies that its gradient is 0 when $\{q_i\} = \{p_i\}$ because $\{p_i\}$ is a minimizer of F. Therefore,

$$\frac{\partial F}{\partial q_i}(\{p_i\}) = \int_{-1}^{1} 2(p_0 + p_1 t + p_2 t^2 + p_3 t^3 - \sin(t)) t^i \, \mathrm{d}t = 0 \tag{3}$$

for $i \in \{0,1,2,3\}$. Then

$$\begin{aligned} 0 &= \int_{-1}^{1} (p_0 t^i + p_1 t^{i+1} + p_2 t^{i+2} + p_3 t^{i+3} - t^i \sin(t)) \, \mathrm{d}t & (4)\\ &= \left[\frac{p_0}{i+1} t^{i+1} + \frac{p_1}{i+2} t^{i+2} + \frac{p_2}{i+3} t^{i+3} + \frac{p_3}{i+4} t^{i+4}\right]_{-1}^{1} - \int_{-1}^{1} t^i \sin(t) \, \mathrm{d}t. & (5)\end{aligned}$$

Note that $t^i \sin(t)$ is odd if i is even, which makes $\int_{-1}^{1} t^i \sin(t) \, \mathrm{d}t = 0$. If i is odd, then $i \in \{1,3\}$, and

$$\int_{-1}^{1} t \sin(t) \, \mathrm{d}t = \big[-t\cos(t) + \sin(t)\big]_{-1}^{1} = 2\sin(1) - 2\cos(1) \tag{6}$$

and

$$\int_{-1}^{1} t^3 \sin(t) \, \mathrm{d}t = \big[-t^3\cos(t) + 3t^2 \sin(t) + 6t\cos(t) - 6\sin(t)\big]_{-1}^{1} = 10\cos(1) - 6\sin(1) \tag{7}$$

Evaluating (2) for $t \in \{0,1,2,3\}$, we obtain a system of four equations

$$\begin{aligned} &(i=0) & 0 &= 2p_0 + \frac{2}{3}p_2, & (8)\\ &(i=1) & 2\sin(1) - 2\cos(1) &= \frac{2}{3}p_1 + \frac{2}{5}p_3, & (9)\\ &(i=2) & 0 &= \frac{2}{3}p_0 + \frac{2}{5}p_2, & (10)\\ &(i=3) & 10\cos(1) - 6\sin(1) &= \frac{2}{5}p_1 + \frac{2}{7}p_3. & (11)\end{aligned}$$

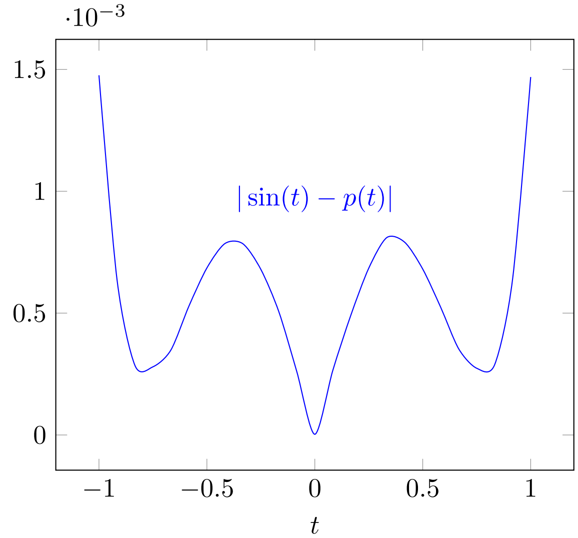

Figure 1: Absolute error between p and sin on $[-1, 1]$. Evidently, the approximation is pretty good.

Using substitution on the first and third equations, we see that $p_0 = -\frac{1}{3}p_2$, so that $\frac{8}{45}p_2 = 0$. Thus, $p_0 = p_2 = 0$ (as expected, since sin is odd). Solving the second pair of equations is less fun; if $x = (p_1, p_3)^T$, then x solves the equation

$$\begin{bmatrix} \frac{2}{3} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{7} \end{bmatrix} x = \begin{bmatrix} -2 \\ 10 \end{bmatrix} \cos(1) + \begin{bmatrix} 2 \\ -6 \end{bmatrix} \sin(1). \tag{12}$$

Using the formula for 2×2 inverse matrices gives

$$x = \frac{1}{\frac{4}{21} - \frac{4}{25}} \begin{bmatrix} \frac{2}{7} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{2}{3} \end{bmatrix} \left(\begin{bmatrix} -2 \\ 10 \end{bmatrix} \cos(1) + \begin{bmatrix} 2 \\ -6 \end{bmatrix} \sin(1) \right) \tag{13}$$

$$= \frac{525}{16} \left(\begin{bmatrix} -\frac{32}{7} \\ \frac{112}{15} \end{bmatrix} \cos(1) + \begin{bmatrix} \frac{104}{35} \\ -\frac{24}{5} \end{bmatrix} \sin(1) \right) \tag{14}$$

$$= \begin{bmatrix} -150\cos(1) + \frac{195}{2}\sin(1) \\ 245\cos(1) - \frac{315}{2}\sin(1) \end{bmatrix}. \tag{15}$$

That is, $p_1 = -150\cos(1) + \frac{195}{2}\sin(1)$ and $p_3 = 245\cos(1) - \frac{315}{2}\sin(1)$, and the best approximation $p \in P^3[-1, 1]$ of f on $[-1, 1]$ in L^2 norm is

$$p(t) = \left(-150\cos(1) + \frac{195}{2}\sin(1) \right) t + \left(245\cos(1) - \frac{315}{2}\sin(1) \right) t^3 \tag{16}$$

$$\approx 0.998075139t - 0.157615170t^3 \tag{17}$$

Figure 1 provides a visualization of the approximation error.

(b) The degree 3 Taylor approximation polynomial $p(t)$ for $f(t) = \sin(t)$ centered at $t = 0$ is defined to be

$$p(t) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{6} \tag{18}$$

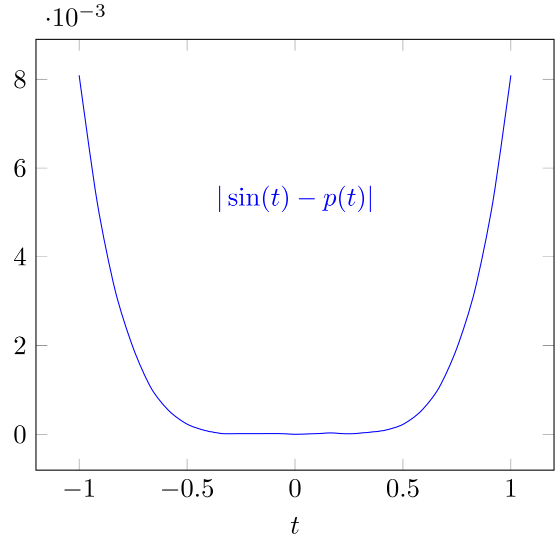

Figure 2: Absolute error between p and sin on $[-1, 1]$. This approximation is also pretty good.

because $f(0) = 0$, $f'(0) = \cos(0) = 1$, $f''(0) = -\sin(0) = 0$, and $f'''(0) = -\cos(0) = -1$. Note how the coefficients are fairly close to those found in the previous part (the best L^2 approximation). Figure 2 gives a visualization of the absolute error between p and $f = \sin$. Note how the error is smaller near $t = 0$ but larger away from $t = 0$ than the best L^2 approximation error.

(c) The degree 3 Lagrange polynomial approximation of $f(t) = \sin(t)$ that interpolates at the points $T = \left\{-1, -\frac{1}{3}, \frac{1}{3}, 1\right\}$ is defined to be the degree 3 polynomial p such that $p(t) = f(t)$ for all $t \in T$. There are numbers $\{p_i\}_{i=0}^3 \in \mathbf{R}^4$ such that $p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$; the definition of p therefore requires that the following equations be true

$$\sin(-1) = p_0 - p_1 + p_2 - p_3, \tag{19}$$

$$\sin\left(-\frac{1}{3}\right) = p_0 - \frac{1}{3}p_1 + \frac{1}{9}p_2 - \frac{1}{27}p_3, \tag{20}$$

$$\sin\left(\frac{1}{3}\right) = p_0 + \frac{1}{3}p_1 + \frac{1}{9}p_2 + \frac{1}{27}p_3, \tag{21}$$

$$\sin(1) = p_0 + p_1 + p_2 + p_3. \tag{22}$$

Adding the first and last equations, we get $2p_0 + 2p_2 = 0$, so $p_0 = -p_2$. Adding the middle two equations, we get $2p_0 + \frac{2}{9}p_2 = 0$, which implies that $\frac{17}{9}p_0 = 0$, so $p_0 = 0 = p_2$ (as expected from the oddness of sin and odd symmetry of T).

Subtracting the first equation from the last, we get $p_1 + p_3 = \sin(1)$. Subracting the third equation from the second, we get $\frac{1}{3}p_1 + \frac{1}{27}p_3 = \sin\left(\frac{1}{3}\right)$, or $9p_1 + p_3 = 27\sin\left(\frac{1}{3}\right)$. Therefore, $\sin(1) - p_1 = 27\sin\left(\frac{1}{3}\right) - 9p_1$, which implies that $p_1 = \frac{1}{8}\left(27\sin\left(\frac{1}{3}\right) - \sin(1)\right)$, and $p_3 = \sin(1) - p_1 = \frac{1}{8}\left(9\sin(1) - 27\sin\left(\frac{1}{3}\right)\right)$. Thus,

$$p(t) = \frac{1}{8}\left(27\sin\left(\frac{1}{3}\right) - \sin(1)\right)t + \frac{1}{8}\left(9\sin(1) - 27\sin\left(\frac{1}{3}\right)\right)t^3 \tag{23}$$

$$\approx 0.999098228t - 0.157627243t^3. \tag{24}$$

Figure 3 shows a visulization of the error between p and f. Note that the error is 0 when $t \in T$, and also when $t = 0$ because of the oddness of both p and f.

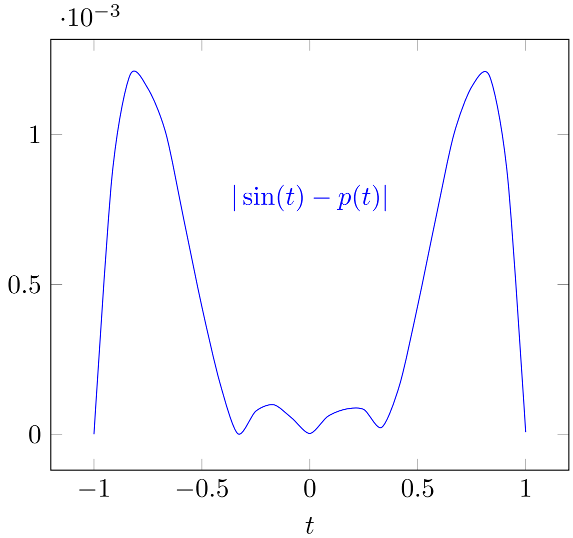

Figure 3: Absolute error between p and sin on $[-1, 1]$. This approximation is also quite good.

Problem 2.

First, we find $w_1 = \frac{u_1}{\|u_1\|}$. Since

$$\|u_1\|^2 = \int_0^1 1^2 \, \mathrm{d}x = 1, \tag{25}$$

we get $\|u_1\| = 1$, and $w_1 = 1$. By the Gram-Schmidt process, $v_2 = u_2 - (u_2, w_1)w_1$ is orthogonal to w_1. Since

$$(u_2, w_1) = \int_0^1 x \, \mathrm{d}x = \frac{1}{2}, \tag{26}$$

it follows that $v_2 = x - \frac{1}{2}$ is orthogonal to w_1. Then $w_2 = \frac{v_2}{\|v_2\|}$ is orthogonal to w_1 and is a unit vector. Since

$$\|v_2\|^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2 \, \mathrm{d}x = \frac{1}{3}\left(x - \frac{1}{2}\right)^3 \Big|_0^1 = \frac{1}{12}, \tag{27}$$

it follows that $w_2 = \sqrt{12}x - \sqrt{3}$ is orthogonal to w_1 and is a unit vector. By the Gram-Schmidt process again, $v_3 = u_3 - (u_3, w_2)w_2 - (u_3, w_1)w_1$ is orthogonal to w_1 and to w_2. Since

$$(u_3, w_2) = \int_0^1 x^2 \left(\sqrt{12}x - \sqrt{3}\right) \, \mathrm{d}x = \frac{\sqrt{12}}{4}x^4 - \frac{\sqrt{3}}{3}x^3 \Big|_0^1 = \frac{\sqrt{12}}{12}, \tag{28}$$

and

$$(u_3, w_1) = \int_0^1 x^2 \, \mathrm{d}x = \frac{1}{3}, \tag{29}$$

it follows that

$$v_3 = x^2 - x + \frac{1}{2} - \frac{1}{3} = x^2 - x + \frac{1}{6}. \tag{30}$$

Then v_3 is orthogonal to w_1 and w_2, and $w_3 = \frac{v_3}{\|v_3\|}$ is orthogonal to w_1 and w_2 and is a unit vector. Since

$$\|v_3\|^2 = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 \, \mathrm{d}x = \left[\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}x\right]_0^1 = \frac{1}{180} \tag{31}$$

it follows that

$$w_3 = \sqrt{180}\left(x^2 - x + \frac{1}{6}\right) \tag{32}$$

is orthogonal to w_1 and w_2 and is a unit vector. Altogether then, the orthonormal basis for $P^2[0,1]$ obtained by the Gram-Schmidt process applied to $\{1, x, x^2\}$ is

$$B = \left\{1, \sqrt{12}x - \sqrt{3}, \sqrt{180}\left(x^2 - x + \frac{1}{6}\right)\right\} \tag{33}$$

The best approximation for $f(x) = \sqrt{x}$ in $P^2[0,1]$ with respect to the L^2 norm is therefore

$$p(x) = p_1 w_1(x) + p_2 w_2(x) + p_3 w_3(x) \tag{34}$$

where $p_i = (\sqrt{x}, w_i)$. Computing these inner products, we get

$$p_1 = (\sqrt{x}, w_1) = \int_0^1 \sqrt{x}\,\mathrm{d}x = \frac{2}{3}, \tag{35}$$

$$p_2 = (\sqrt{x}, w_2) = \int_0^1 \sqrt{x}\left(\sqrt{12}x - \sqrt{3}\right)\,\mathrm{d}x = \frac{2\sqrt{12}}{5} - \frac{2\sqrt{3}}{3} = \frac{\sqrt{12}}{15}, \tag{36}$$

$$p_3 = (\sqrt{x}, w_3) = \int_0^1 \sqrt{x}\cdot\sqrt{180}\left(x^2 - x + \frac{1}{6}\right)\,\mathrm{d}x = \sqrt{180}\left(\frac{2}{7} - \frac{2}{5} + \frac{1}{9}\right) = -\frac{\sqrt{180}}{315} \tag{37}$$

Therefore, the best approximation is

$$p(x) = \frac{2}{3} + \frac{\sqrt{12}}{15}\left(\sqrt{12}x - \sqrt{3}\right) - \frac{180}{315}\left(x^2 - x + \frac{1}{6}\right) \tag{38}$$

$$= \frac{2}{3} + \frac{4}{5}x - \frac{2}{5} - \frac{4}{7}x^2 + \frac{4}{7}x - \frac{2}{21} \tag{39}$$

$$= -\frac{4}{7}x^2 + \frac{48}{35}x + \frac{6}{35} \tag{40}$$

$$\approx -0.571428571x^2 + 1.371428571x + 0.171428571 \tag{41}$$

See figure 4 for a visualization of the error.

Problem 3.

Let p be the quadratic polynomial satisfying $p(0) = f(0)$, $p(2) = f(2)$, and $p'(2) = f'(2)$. There exists $\{p_i\}_{i=0}^2 \in \mathbf{R}^3$ such that $p(x) = p_0 + p_1 x + p_2 x^2$. Since $p(0) = f(0)$, it follows that $p_0 = f(0)$. Since $p'(2) = p_1 + 4p_2 = f'(2)$, and $p(2) = f(0) + 2p_1 + 4p_2 = f(2)$, it follows that $f(2) - f'(2) = f(0) + p_1$, so $p_1 = f(2) - f'(2) - f(0)$, and $p_2 = \frac{1}{4}(f'(2) - p_1) = \frac{1}{4}(2f'(2) + f(0) - f(2))$. Therefore,

$$p(x) = p_0 + p_1 x + p_2 x^2 = f(0) + (f(2) - f'(2) - f(0))x + \frac{1}{4}(2f'(2) + f(0) - f(2))x^2 \tag{42}$$

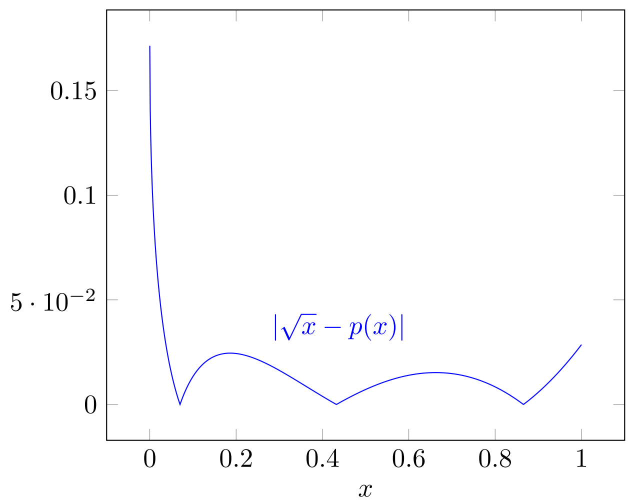

Figure 4: Absolute error between p and $\sqrt{\cdot}$ on $[0, 1]$