Math 5604 Homework 8

Jacob Hauck

April 24, 2024

Problem 1.

Consider the BVP

$$\Delta u = -2\pi^2 \sin(\pi(x+y)) = f(x,y), \qquad\qquad 0 < x < 1, \quad 0 < y < 1$$
$$u(0,y) = \sin(\pi y) = g_\ell(y), \quad u(1,y) = \sin(\pi(1+y)) = g_r(y), \qquad 0 \le y \le 1$$
$$u(x,0) = \sin(\pi x) = g_b(x), \quad u(x,1) = \sin(\pi(1+x)) = g_t(x), \qquad 0 \le x \le 1.$$

The exact solution of this equation is given by $u(x,y) = \sin(\pi(x+y))$.

(a) Consider a grid of sample points $\{(x_i, y_j)\}$ on the domain $[0,1]^2$, where $i = 0, 1, \ldots, M$, and $j = 0, 1, \ldots, N$. If the points are evenly spaced horizontally by $h_x = \frac{1}{M}$ and vertically by $h_y = \frac{1}{N}$, then $x_i = ih_x$, and $y_j = jh_y$.

We approximate $u(x_i, y_j)$ by $u_{i,j}$. Using a centered-difference scheme to approximate Δu on the interior and applying the boundary conditions on the boundary points, we are led to the numerical scheme

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} = f(x_i, y_i), \qquad\qquad (i,j)$$
$$i = 1, 2, \ldots, M-1, \ j = 1, 2, \ldots N-1,$$

and

$$u_{0,j} = g_\ell(y_j), \qquad u_{M,j} = g_r(y_j), \qquad j = 0, 1, \ldots, N,$$
$$u_{i,0} = g_b(x_i), \qquad u_{i,N} = g_t(x_i), \qquad i = 0, 1, \ldots M.$$

In order to solve this linear system, we need to reshape the matrix of unknowns $\{u_{i,j}\}_{i=1,j=1}^{M-1,N-1}$ into a vector U and rewrite the corresponding equations (i,j) as a matrix-vector system, substituting the known boundary values where applicable.

We use row-wise ordering to reshape the matrix of unknowns; that is, we define the block vector of rows of the unknown matrix

$$U = \begin{bmatrix} U^{(1)} \\ U^{(2)} \\ \vdots \\ U^{(N-1)} \end{bmatrix}, \qquad U^{(j)} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{M-1,j} \end{bmatrix}, \qquad j = 1, 2, \ldots, N-1.$$

We can rewrite the equations (i,j) into a matrix-vector system, expressing the matrix A and vector b in block form corresponding to the blocks of U:

$$A = \begin{bmatrix} A^{(1,1)} & \cdots & A^{(1,N-1)} \\ \vdots & \ddots & \vdots \\ A^{(N-1,1)} & \cdots & A^{(N-1,N-1)} \end{bmatrix}, \qquad b = \begin{bmatrix} b^{(1)} \\ b^{(2)} \\ \vdots \\ b^{(N-1)} \end{bmatrix}.$$

We remark that the block $A^{(j,j')}$ expresses the dependence of equations $(1,j), (2,j), \ldots, (M-1,j)$ on the unknowns in row j' of the unknown matrix $\{u_{i,j}\}$.

We construct A and b one block row at a time. Consider the blocks $A^{(1,j')}$ for $j' = 1, 2, \ldots, N-1$, the first row of blocks of A. As mentioned, these blocks correspond to equations $(1,1), (2,1), \ldots, (M-1,1)$. Substituting in boundary conditions, we see that

$$(1,1) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,1} + \frac{1}{h_x^2} u_{2,1} + \frac{1}{h_y^2} u_{1,2} = f(x_1, y_1) - \frac{g_\ell(y_1)}{h_x^2} - \frac{g_b(x_1)}{h_y^2}$$

$$\overset{(i,1)}{\underset{i=2,\ldots,M-2}{\implies}} \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,1} + \frac{1}{h_x^2} u_{i-1,1} + \frac{1}{h_x^2} u_{i+1,1} + \frac{1}{h_y^2} u_{i,2} = f(x_i, y_1) - \frac{g_b(x_i)}{h_y^2}$$

$$(M-1,1) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M-1,1} + \frac{1}{h_x^2} u_{M-2,1} + \frac{1}{h_y^2} u_{M-1,2} = f(x_{M-1}, y_1) - \frac{g_r(y_1)}{h_x^2} - \frac{g_b(x_{M-1})}{h_y^2}.$$

Thus, each equation depends only on rows 1 and 2 of the matrix $\{u_{i,j}\}$, so only blocks $A^{(1,1)}$ and $A^{(1,2)}$ are nonzero. Examining these dependencies, we get

$$A^{(1,1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & \\ & & \ddots & \\ & & \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(1,2)} = \begin{bmatrix} \frac{1}{h_y^2} & & \\ & \ddots & \\ & & \frac{1}{h_y^2} \end{bmatrix},$$

where blanks indicate zero entries. The block $b^{(1)}$ corresponding to the right hand sides of equations $(1,1), (2,1), \ldots, (M-1,1)$ we can read off easily:

$$b^{(1)} = \begin{bmatrix} f(x_1, y_1) - \frac{g_\ell(y_1)}{h_x^2} - \frac{g_b(x_1)}{h_y^2} \\ f(x_2, y_1) - \frac{g_b(x_2)}{h_y^2} \\ f(x_3, y_1) - \frac{g_b(x_3)}{h_y^2} \\ \vdots \\ f(x_{M-2}, y_1) - \frac{g_b(x_{M-2})}{h_y^2} \\ f(x_{M-1}, y_1) - \frac{g_r(y_1)}{h_x^2} - \frac{g_b(x_{M-1})}{h_y^2} \end{bmatrix}.$$

Now consider the blocks $A^{(j,j')}$ for $j = 2, 3, \ldots, N-2$, and $j' = 1, 2, \ldots, N-1$. These correspond to equations $(1,j), (2,j), \ldots, (M-1,j)$. Substituting boundary conditions, we have

$$(1,j) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,j} + \frac{1}{h_x^2} u_{2,j} + \frac{1}{h_y^2} u_{1,j-1} + \frac{1}{h_y^2} u_{1,j+1} = f(x_1, y_j) - \frac{g_\ell(y_j)}{h_x^2}$$

$$\overset{(i,j)}{\underset{i=2,\ldots,M-2}{\implies}} \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,j} + \frac{1}{h_x^2} u_{i-1,j} + \frac{1}{h_x^2} u_{i+1,j} + \frac{1}{h_y^2} u_{i,j-1} + \frac{1}{h_y^2} u_{i,j+1} = f(x_i, y_j)$$

$$(M-1,j) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M-1,j} + \frac{1}{h_x^2} u_{M-2,j} + \frac{1}{h_y^2} u_{M-1,j-1} + \frac{1}{h_y^2} u_{M-1,j+1} = f(x_{M-1}, y_j) - \frac{g_r(y_j)}{h_x^2}.$$

Thus, each equation depends only on rows $j-1$, j, and $j+1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(j,j-1)}$, $A^{(j,j)}$, and $A^{(j,j+1)}$ are nonzero. Examining these dependencies gives

$$A^{(j,j)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & \\ & & \ddots & \\ & & \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(j,j-1)} = A^{(j,j+1)} = \begin{bmatrix} \frac{1}{h_y^2} & & \\ & \ddots & \\ & & \frac{1}{h_y^2} \end{bmatrix}.$$

The block $b^{(j)}$ can be read off from the right hand sides easily:

$$b^{(j)} = \begin{bmatrix} f(x_1, y_j) - \frac{g_\ell(y_j)}{h_x^2} \\ f(x_2, y_j) \\ f(x_3, y_j) \\ \vdots \\ f(x_{M-2}, y_j) \\ f(x_{M-1}, y_j) - \frac{g_r(y_j)}{h_x^2} \end{bmatrix}.$$

Finally, consider the blocks $A^{(N-1,j')}$, $j' = 1, 2, \ldots, N-1$. These correspond to equations $(1, N-1), (2, N-1), \ldots, (M-1, N-1)$. Substituting boundary conditions, we have

$$(1, N-1) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,N-1} + \frac{1}{h_x^2} u_{2,N-1} + \frac{1}{h_y^2} u_{1,N-2} = f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2}$$

$$\overset{(i,N-1)}{\underset{i=2,\ldots,M-2}{}} \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,N-1} + \frac{1}{h_x^2} u_{i-1,N-1} + \frac{1}{h_x^2} u_{i+1,N-1} + \frac{1}{h_y^2} u_{i,2} = f(x_i, y_{N-1}) - \frac{g_t(x_i)}{h_y^2}$$

$$(M-1, N-1) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M-1,N-1} + \frac{1}{h_x^2} u_{M-2,N-1} + \frac{1}{h_y^2} u_{M-1,N-2}$$
$$= f(x_{M-1}, y_{N-1}) - \frac{g_r(y_{N-1})}{h_x^2} - \frac{g_t(x_{M-1})}{h_y^2}.$$

Thus, each equation depends only on rows $N-2$ and $N-1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(N-1,N-2)}$ and $A^{(N-1,N-1)}$ are nonzero. Examining these dependencies gives

$$A^{(N-1,N-1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & \\ & & \ddots & \\ & & \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(N-1,N-2)} = \begin{bmatrix} \frac{1}{h_y^2} & & \\ & \ddots & \\ & & \frac{1}{h_y^2} \end{bmatrix}.$$

The block $b^{(N-1)}$ can be read off from the right hand sides of the equations easily:

$$b^{(N-1)} = \begin{bmatrix} f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \\ f(x_2, y_{N-1}) - \frac{g_t(x_2)}{h_y^2} \\ f(x_3, y_{N-1}) - \frac{g_t(x_3)}{h_y^2} \\ \vdots \\ f(x_{M-2}, y_{N-1}) - \frac{g_t(x_{M-2})}{h_y^2} \\ f(x_{M-1}, y_{N-1}) - \frac{g_r(y_{N-1})}{h_x^2} - \frac{g_t(x_{M-1})}{h_y^2} \end{bmatrix}.$$

Therefore, the entire system of equations (i, j) is equivalent to the matrix-vector equation $AU = b$.

(b) See `problem1.m` for the implementation of the scheme in (a).

(c) See Figure 1 for the mesh plots of the numerical solution and the error function.

(d) See Table 1 for the table of numerical errors and convergence rates for $h = 1/8, 1/16, 1/32, 1/64, 1/128, 1/256$.
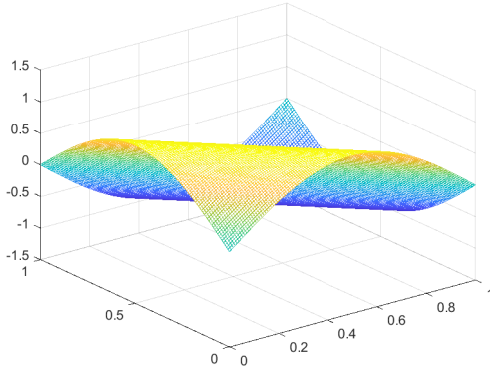
(a) Numerical solution
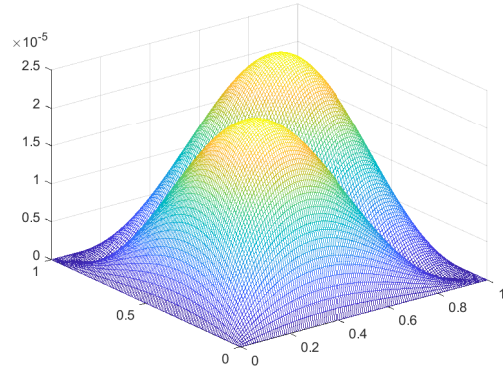
(b) Numerical error function

Figure 1: Numerical solution and error function mesh plots with $h_x = h_y = \frac{1}{128}$

h	∞ error	∞ rate	2 error	2 rate
1/8	5.992487e-03	-	2.193459e-03	-
1/16	1.527086e-03	1.972373	5.495032e-04	1.997008
1/32	3.870626e-04	1.980143	1.373949e-04	1.999799
1/64	9.676442e-05	2.000018	3.434909e-05	1.999985
1/128	2.419921e-05	1.999517	8.587283e-06	1.999998
1/256	6.051066e-06	1.999699	2.146821e-06	2.000000

Table 1: Numerical errors and convergence rates for problem 1

Problem 2.

Consider the BVP

$$\Delta u = -2\pi^2 \sin(\pi x)\sin(\pi y) = f(x,y), \qquad 0 < x < 1, \quad 0 < y < 2$$
$$u(0,y) = 2 = g_\ell(y), \quad u(1,y) = 2 = g_r(y), \qquad 0 \le y \le 2$$
$$u(x,0) = 2 = g_b(x), \quad u(x,1) = 2 = g_t(x), \qquad 0 \le x \le 1.$$

The exact solution of this equation is given by $u(x,y) = 2 + \sin(\pi x)\sin(\pi y)$.

(a) Consider a grid of sample points $\{(x_i, y_j)\}$ on the domain $[0,1] \times [0,2]$, where $i = 0, 1, \dots, M$, and $j = 0, 1, \dots, N$. If the points are evenly spaced horizontally by $h_x = \frac{1}{M}$ and vertically by $h_y = \frac{2}{N}$, then $x_i = ih_x$, and $y_j = jh_y$.

We approximate $u(x_i, y_j)$ by $u_{i,j}$. Using a centered-difference scheme to approximate Δu on the interior and applying the boundary conditions on the boundary points, we are led to the numerical scheme

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} = f(x_i, y_i), \qquad (i,j)$$
$$i = 1, 2, \dots, M-1, \; j = 1, 2, \dots N-1,$$

and

$$u_{0,j} = g_\ell(y_j), \quad u_{M,j} = g_r(y_j), \qquad j = 0, 1, \dots, N,$$
$$u_{i,0} = g_b(x_i), \quad u_{i,N} = g_t(x_i), \qquad i = 0, 1, \dots M.$$

In order to solve this linear system, we need to reshape the matrix of unknowns $\{u_{i,j}\}_{i=1,j=1}^{M-1,N-1}$ into a vector U and rewrite the corresponding equations (i,j) as a matrix-vector system, substituting the known boundary values where applicable.

We use column-wise ordering to reshape the matrix of unknowns; that is, we define the block vector of columns of the unknown matrix

$$
U = \begin{bmatrix} U^{(1)} \\ U^{(2)} \\ \vdots \\ U^{(M-1)} \end{bmatrix}, \qquad U^{(i)} = \begin{bmatrix} u_{i,1} \\ u_{i,2} \\ \vdots \\ u_{i,N-1} \end{bmatrix}, \quad i = 1, 2, \ldots, M-1.
$$

We can rewrite the equations (i,j) into a matrix-vector system, expressing the matrix A and vector b in block form corresponding to the blocks of U:

$$
A = \begin{bmatrix} A^{(1,1)} & \cdots & A^{(1,M-1)} \\ \vdots & \ddots & \vdots \\ A^{(M-1,1)} & \cdots & A^{(M-1,M-1)} \end{bmatrix}, \qquad b = \begin{bmatrix} b^{(1)} \\ b^{(2)} \\ \vdots \\ b^{(M-1)} \end{bmatrix}.
$$

We remark that the block $A^{(i,i')}$ expresses the dependence of equations $(i,1),(i,2),\ldots,(i,N-1)$ on the unknowns in column i' of the unknown matrix $\{u_{i,j}\}$.

We construct A and b one block row at a time. Consider the blocks $A^{(1,i')}$ for $i' = 1,2,\ldots,M-1$, the first row of blocks of A. As mentioned, these blocks correspond to equations $(1,1),(1,2),\ldots,(1,N-1)$. Substituting in boundary conditions, we see that

$$
(1,1) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,1} + \frac{1}{h_x^2} u_{2,1} + \frac{1}{h_y^2} u_{1,2} = f(x_1, y_1) - \frac{g_\ell(y_1)}{h_x^2} - \frac{g_b(x_1)}{h_y^2}
$$

$$
\overset{(1,j)}{\underset{j=2,\ldots,N-2}{}} \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,j} + \frac{1}{h_x^2} u_{2,j} + \frac{1}{h_y^2} u_{1,j-1} + \frac{1}{h_y^2} u_{1,j+1} = f(x_1, y_j) - \frac{g_\ell(y_j)}{h_x^2}
$$

$$
(1, N-1) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,N-1} + \frac{1}{h_x^2} u_{2,N-1} + \frac{1}{h_y^2} u_{1,N-2} = f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2}.
$$

Thus, each equation depends only on columns 1 and 2 of the matrix $\{u_{i,j}\}$, so only blocks $A^{(1,1)}$ and $A^{(1,2)}$ are nonzero. Examining these dependencies, we get

$$
A^{(1,1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & & & \\ \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & & \\ & & \ddots & & \\ & & & \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(1,2)} = \begin{bmatrix} \frac{1}{h_x^2} & & \\ & \ddots & \\ & & \frac{1}{h_x^2} \end{bmatrix},
$$

where blanks indicate zero entries. The block $b^{(1)}$ corresponding to the right hand sides of equations $(1,1),(1,2),\ldots,(1,N-1)$ we can read off easily:

$$
b^{(1)} = \begin{bmatrix} f(x_1, y_1) - \frac{g_\ell(y_1)}{h_x^2} - \frac{g_b(x_1)}{h_y^2} \\ f(x_1, y_2) - \frac{g_\ell(y_2)}{h_x^2} \\ f(x_1, y_3) - \frac{g_\ell(y_3)}{h_x^2} \\ \vdots \\ f(x_1, y_{N-2}) - \frac{g_\ell(y_{N-2})}{h_x^2} \\ f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \end{bmatrix}.
$$

Now consider the blocks $A^{(i,i')}$ for $i = 2, 3, \ldots, N-2$, and $i' = 1, 2, \ldots, N-1$. These correspond to equations $(i, 1), (i, 2), \ldots, (i, N-1)$. Substituting boundary conditions, we have

$$(i, 1) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,1} + \frac{1}{h_x^2} u_{i-1,1} + \frac{1}{h_x^2} u_{i+1,1} + \frac{1}{h_y^2} u_{i,2} = f(x_i, y_1) - \frac{g_b(x_i)}{h_y^2}$$

$$\underset{j=2,\ldots,N-2}{\overset{(i,j)}{}} \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,j} + \frac{1}{h_x^2} u_{i-1,j} + \frac{1}{h_x^2} u_{i+1,j} + \frac{1}{h_y^2} u_{i,j-1} + \frac{1}{h_y^2} u_{i,j+1} = f(x_i, y_j)$$

$$(i, N-1) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,N-1} + \frac{1}{h_x^2} u_{i-1,N-1} + \frac{1}{h_x^2} u_{i+1,N-1} + \frac{1}{h_y^2} u_{i,N-2} = f(x_i, y_{N-1}) - \frac{g_t(x_i)}{h_y^2}.$$

Thus, each equation depends only on columns $i - 1$, i, and $i + 1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(i,i-1)}$, $A^{(i,i)}$, and $A^{(i,i+1)}$ are nonzero. Examining these dependencies gives

$$A^{(i,i)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & & \\ \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & \\ & & \ddots & \\ & & \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(i,i-1)} = A^{(i,i+1)} = \begin{bmatrix} \frac{1}{h_x^2} & & \\ & \ddots & \\ & & \frac{1}{h_x^2} \end{bmatrix}.$$

The block $b^{(i)}$ can be read off from the right hand sides easily:

$$b^{(i)} = \begin{bmatrix} f(x_i, y_1) - \frac{g_b(x_i)}{h_y^2} \\ f(x_i, y_2) \\ f(x_i, y_3) \\ \vdots \\ f(x_i, y_{N-2}) \\ f(x_i, y_{N-1}) - \frac{g_t(x_i)}{h_y^2} \end{bmatrix}.$$

Finally, consider the blocks $A^{(M-1,i')}$, $i' = 1, 2, \ldots, M-1$. These correspond to equations $(M-1, 1), (M-1, 2), \ldots, (M-1, N-1)$. Substituting boundary conditions, we have

$$(M-1, 1) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M-1,1} + \frac{1}{h_x^2} u_{M-2,1} + \frac{1}{h_y^2} u_{M-1,2} = f(x_1, y_{N-1}) - \frac{g_r(y_1)}{h_x^2} - \frac{g_b(x_{M-1})}{h_y^2}$$

$$\underset{j=2,\ldots,N-2}{\overset{(M-1,j)}{}} \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M-1,j} + \frac{1}{h_x^2} u_{M-2,j} + \frac{1}{h_y^2} u_{M-1,j-1} + \frac{1}{h_y^2} u_{M-1,j+1} = f(x_{M-1}, y_j) - \frac{g_r(y_j)}{h_x^2}$$

$$(M-1, N-1) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M-1,N-1} + \frac{1}{h_x^2} u_{M-2,N-1} + \frac{1}{h_y^2} u_{M-1,N-2}$$

$$= f(x_{M-1}, y_{N-1}) - \frac{g_r(y_{N-1})}{h_x^2} - \frac{g_t(x_{M-1})}{h_y^2}.$$

Thus, each equation depends only on columns $M - 2$ and $M - 1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(M-1,M-2)}$ and $A^{(M-1,M-1)}$ are nonzero. Examining these dependencies gives

$$A^{(N-1,N-1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & & \\ \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_y^2} & \\ & & \ddots & \\ & & \frac{1}{h_y^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(N-1,N-2)} = \begin{bmatrix} \frac{1}{h_x^2} & & \\ & \ddots & \\ & & \frac{1}{h_x^2} \end{bmatrix}.$$

The block $b^{(M-1)}$ can be read off from the right hand sides of the equations easily:

$$b^{(M-1)} = \begin{bmatrix} f(x_{M-1}, y_1) - \frac{g_r(y_1)}{h_x^2} - \frac{g_b(x_{M-1})}{h_y^2} \\ f(x_{M-1}, y_2) - \frac{g_r(y_2)}{h_x^2} \\ f(x_{M-1}, y_3) - \frac{g_r(y_3)}{h_x^2} \\ \vdots \\ f(x_{M-1}, y_{N-2}) - \frac{g_r(y_{N-2})}{h_x^2} \\ f(x_{M-1}, y_{N-1}) - \frac{g_r(y_{N-1})}{h_x^2} - \frac{g_t(x_{M-1})}{h_y^2} \end{bmatrix}.$$

Therefore, the entire system of equations (i, j) is equivalent to the matrix-vector equation $AU = b$.

(b) See `problem2.m` for the implementation of the scheme in (a).

(c) See Figure 2 for the mesh plots of the numerical solution and the error function.
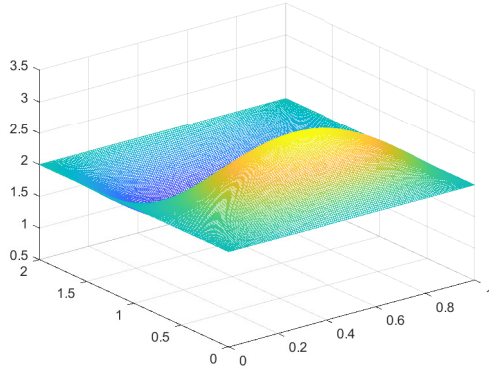
(a) Numerical solution
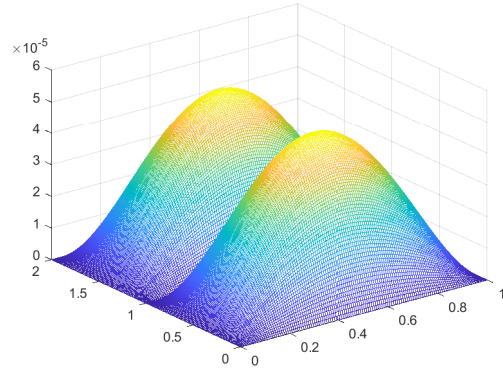
(b) Numerical error function

Figure 2: Numerical solution and error function mesh plots with $h_x = h_y = \frac{1}{128}$

(d) See Table 2 for the table of numerical errors and convergence rates for $h = 1/8, 1/16, 1/32, 1/64, 1/128, 1/256$.

h	∞ error	∞ rate	2 error	2 rate
1/8	1.295075e-02	-	9.157561e-03	-
1/16	3.218964e-03	2.008367	2.276152e-03	2.008367
1/32	8.035777e-04	2.002087	5.682152e-04	2.002087
1/64	2.008218e-04	2.000522	1.420025e-04	2.000522
1/128	5.020092e-05	2.000130	3.549741e-05	2.000130
1/256	1.254995e-05	2.000032	8.874152e-06	2.000033

Table 2: Numerical errors and convergence rates for problem 2

Problem 3.

Consider the BVP

$$\Delta u = -2\pi^2 \sin(\pi(x+y)) = f(x,y), \qquad\qquad 0 < x < 1, \quad 0 < y < 1$$
$$u(0,y) = \sin(\pi y) = g_\ell(y), \quad u_x(1,y) = \pi\cos(\pi(1+y)) = g_r(y), \qquad 0 \le y \le 1$$
$$u_y(x,0) = \pi\cos(\pi x) = g_b(x), \quad u(x,1) = \sin(\pi(1+x)) = g_t(x), \qquad 0 \le x \le 1.$$

The exact solution of this equation is given by $u(x,y) = \sin(\pi(x+y))$.

(a) Consider a grid of sample points $\{(x_i, y_j)\}$ on the domain $[0,1]^2$, where $i = 0, 1, \ldots, M$, and $j = 0, 1, \ldots, N$. If the points are evenly spaced horizontally by $h_x = \frac{1}{M}$ and vertically by $h_y = \frac{1}{N}$, then $x_i = ih_x$, and $y_j = jh_y$.

We approximate $u(x_i, y_j)$ by $u_{i,j}$. For the Neumann boundary conditions on the right and bottom boundaries, we use a centered difference scheme with ghost points to approximate the derivatives, and extend the PDE to the boundaries to eliminate the ghost points, as follows:

$$\frac{u_{M+1,j} - u_{M-1,j}}{2h_x} = g_r(y_j), \quad j = 1, 2, \ldots, N-1, \qquad \frac{u_{i,1} - u_{i,-1}}{2h_y} = g_b(x_i), \quad i = 1, 2, \ldots M - 1.$$

Using a centered-difference scheme to approximate Δu on the interior and the points on the parts of the boundary that have a Neumann condition, we obtain the following scheme:

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} = f(x_i, y_i), \qquad\qquad (i,j)$$
$$i = 1, 2, \ldots, M, \; j = 0, 2, \ldots N - 1.$$

For the left and top boundaries, we have Dirichlet boundary conditions, giving

$$u_{0,j} = g_\ell(y_j), \quad j = 0, 1, \ldots, N, \qquad u_{i,N} = g_t(x_i), \quad i = 0, 1, \ldots, M.$$

In order to solve this linear system, we need to reshape the matrix of unknowns $\{u_{i,j}\}_{i=1,j=0}^{M,N-1}$ into a vector U and rewrite the corresponding equations (i,j) as a matrix-vector system, substituting the known boundary values and ghost point relationships where applicable.

We use row-wise ordering to reshape the matrix of unknowns; that is, we define the block vector of rows of the unknown matrix

$$U = \begin{bmatrix} U^{(0)} \\ U^{(1)} \\ \vdots \\ U^{(N-1)} \end{bmatrix}, \qquad U^{(j)} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{M,j} \end{bmatrix}, \quad j = 0, 1, \ldots, N-1.$$

We can rewrite the equations (i,j) into a matrix-vector system, expressing the matrix A and vector b in block form corresponding to the blocks of U:

$$A = \begin{bmatrix} A^{(0,0)} & \cdots & A^{(0,N-1)} \\ \vdots & \ddots & \vdots \\ A^{(N-1,0)} & \cdots & A^{(N-1,N-1)} \end{bmatrix}, \qquad b = \begin{bmatrix} b^{(0)} \\ b^{(1)} \\ \vdots \\ b^{(N-1)} \end{bmatrix}.$$

We remark that the block $A^{(j,j')}$ expresses the dependence of equations $(1,j), (2,j), \ldots, (M,j)$ on the unknowns in row j' of the unknown matrix $\{u_{i,j}\}$.

We construct A and b one block row at a time. Consider the blocks $A^{(0,j')}$ for $j' = 1, 2, \ldots, N-1$, the first row of blocks of A. As mentioned, these blocks correspond to equations $(1,0), (2,0), \ldots, (M,0)$. Substituting in boundary conditions, we see that

$$(1,0) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,0} + \frac{1}{h_x^2} u_{2,0} + \frac{2}{h_y^2} u_{1,1} = f(x_1, y_0) - \frac{g_\ell(y_0)}{h_x^2} + \frac{2}{h_y} g_b(x_1)$$

$$\underset{i=2,\ldots,M-1}{\overset{(i,0)}{}} \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,0} + \frac{1}{h_x^2} u_{i-1,0} + \frac{1}{h_x^2} u_{i+1,0} + \frac{2}{h_y^2} u_{i,1} = f(x_i, y_0) + \frac{2}{h_y} g_b(x_i)$$

$$(M,0) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M,0} + \frac{2}{h_x^2} u_{M-1,0} + \frac{2}{h_y^2} u_{M-1,1} = f(x_M, y_0) - \frac{2}{h_x} g_r(y_0) + \frac{2}{h_y} g_b(x_M).$$

Thus, each equation depends only on rows 0 and 1 of the matrix $\{u_{i,j}\}$, so only blocks $A^{(0,0)}$ and $A^{(0,1)}$ are nonzero. Examining these dependencies, we get

$$A^{(1,1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ & & \ddots & & \\ & & & \frac{2}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(1,2)} = \begin{bmatrix} \frac{2}{h_y^2} & & \\ & \ddots & \\ & & \frac{2}{h_y^2} \end{bmatrix},$$

where blanks indicate zero entries. The block $b^{(0)}$ corresponding to the right hand sides of equations $(1,0), (2,0), \ldots, (M-1,0)$ we can read off easily:

$$b^{(0)} = \begin{bmatrix} f(x_1, y_0) - \frac{g_\ell(y_0)}{h_x^2} + \frac{2}{h_y} g_b(x_1) \\ f(x_2, y_0) + \frac{2}{h_y} g_b(x_2) \\ f(x_3, y_0) + \frac{2}{h_y} g_b(x_3) \\ \vdots \\ f(x_{M-1}, y_0) + \frac{2}{h_y} g_b(x_{M-1}) \\ f(x_M, y_0) - \frac{2}{h_x} g_r(y_0) + \frac{2}{h_y} g_b(x_M) \end{bmatrix}.$$

Now consider the blocks $A^{(j,j')}$ for $j = 1, 2, 3, \ldots, N-2$, and $j' = 0, 1, \ldots, N-1$. These correspond to equations $(1,j), (2,j), \ldots, (M,j)$. Substituting boundary conditions, we have

$$(1,j) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,j} + \frac{1}{h_x^2} u_{2,j} + \frac{1}{h_y^2} u_{1,j-1} + \frac{1}{h_y^2} u_{1,j+1} = f(x_1, y_j) - \frac{g_\ell(y_j)}{h_x^2}$$

$$\underset{i=2,\ldots,M-1}{\overset{(i,j)}{}} \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,j} + \frac{1}{h_x^2} u_{i-1,j} + \frac{1}{h_x^2} u_{i+1,j} + \frac{1}{h_y^2} u_{i,j-1} + \frac{1}{h_y^2} u_{i,j+1} = f(x_i, y_j)$$

$$(M,j) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M,j} + \frac{2}{h_x^2} u_{M-1,j} + \frac{1}{h_y^2} u_{M,j-1} + \frac{1}{h_y^2} u_{M,j+1} = f(x_M, y_j) - \frac{2}{h_x} g_r(y_j).$$

Thus, each equation depends only on rows $j-1$, j, and $j+1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(j,j-1)}$, $A^{(j,j)}$, and $A^{(j,j+1)}$ are nonzero. Examining these dependencies gives

$$A^{(j,j)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ & & \ddots & & \\ & & & \frac{2}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \qquad A^{(j,j-1)} = A^{(j,j+1)} = \begin{bmatrix} \frac{1}{h_y^2} & & \\ & \ddots & \\ & & \frac{1}{h_y^2} \end{bmatrix}.$$

The block $b^{(j)}$ can be read off from the right hand sides easily:

$$b^{(j)} = \begin{bmatrix} f(x_1, y_j) - \frac{g_\ell(y_j)}{h_x^2} \\ f(x_2, y_j) \\ f(x_3, y_j) \\ \vdots \\ f(x_{M-1}, y_j) \\ f(x_M, y_j) - \frac{2}{h_x} g_r(y_j) \end{bmatrix}.$$

Finally, consider the blocks $A^{(N-1,j')}$, $j' = 0, 1, \ldots, N-1$. These correspond to equations $(1, N-1), (2, N-1), \ldots, (M, N-1)$. Substituting boundary conditions, we have

$$(1, N-1) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{1,N-1} + \frac{1}{h_x^2} u_{2,N-1} + \frac{1}{h_y^2} u_{1,N-2} = f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2}$$

$$\overset{(i,N-1)}{\underset{i=2,\ldots,M-1}{}} \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{i,N-1} + \frac{1}{h_x^2} u_{i-1,N-1} + \frac{1}{h_x^2} u_{i+1,N-1} + \frac{1}{h_y^2} u_{i,2} = f(x_i, y_{N-1}) - \frac{g_t(x_i)}{h_y^2}$$

$$(M, N-1) \implies \left(-\frac{2}{h_x^2} - \frac{2}{h_y^2}\right) u_{M,N-1} + \frac{2}{h_x^2} u_{M-1,N-1} + \frac{1}{h_y^2} u_{M,N-2}$$

$$= f(x_M, y_{N-1}) - \frac{2}{h_x} g_r(y_{N-1}) - \frac{g_t(x_M)}{h_y^2}.$$

Thus, each equation depends only on rows $N-2$ and $N-1$ of the matrix $\{u_{i,j}\}$, so only blocks $A^{(N-1,N-2)}$ and $A^{(N-1,N-1)}$ are nonzero. Examining these dependencies gives

$$A^{(N-1,N-1)} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} & \frac{1}{h_x^2} & & \\ & \ddots & & & \\ & & & \frac{2}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} \end{bmatrix}, \quad A^{(N-1,N-2)} = \begin{bmatrix} \frac{1}{h_y^2} & & \\ & \ddots & \\ & & \frac{1}{h_y^2} \end{bmatrix}.$$

The block $b^{(N-1)}$ can be read off from the right hand sides of the equations easily:

$$b^{(N-1)} = \begin{bmatrix} f(x_1, y_{N-1}) - \frac{g_\ell(y_{N-1})}{h_x^2} - \frac{g_t(x_1)}{h_y^2} \\ f(x_2, y_{N-1}) - \frac{g_t(x_2)}{h_y^2} \\ f(x_3, y_{N-1}) - \frac{g_t(x_3)}{h_y^2} \\ \vdots \\ f(x_{M-1}, y_{N-1}) - \frac{g_t(x_{M-1})}{h_y^2} \\ f(x_M, y_{N-1}) - \frac{2}{h_x} g_r(y_{N-1}) - \frac{g_t(x_M)}{h_y^2} \end{bmatrix}.$$

Therefore, the entire system of equations (i, j) is equivalent to the matrix-vector equation $AU = b$.

(b) See `problem3.m` for the implementation of the scheme in (a).

(c) See Figure 3 for the mesh plots of the numerical solution and the error function.

(d) See Table 3 for the table of numerical errors and convergence rates for $h = 1/8, 1/16, 1/32, 1/64, 1/128, 1/256$.
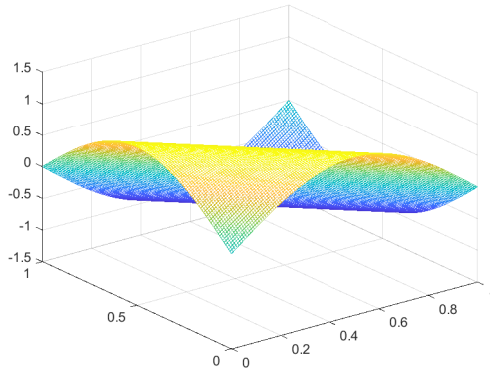
(a) Numerical solution
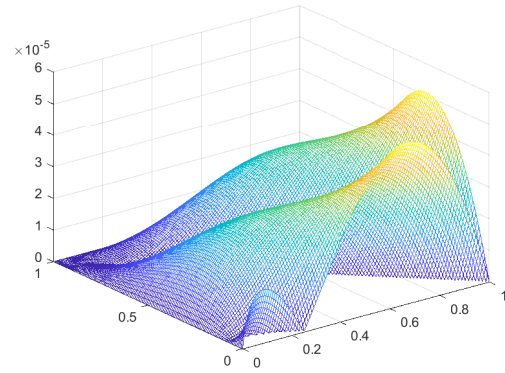
(b) Numerical error function

Figure 3: Numerical solution and error function mesh plots with $h_x = h_y = \frac{1}{128}$

h	∞ error	∞ rate	2 error	2 rate
1/8	1.296423e-02	-	4.368608e-03	-
1/16	3.256417e-03	1.993179	1.007073e-03	2.117006
1/32	8.147140e-04	1.998920	2.419857e-04	2.057174
1/64	2.038673e-04	1.998663	5.933794e-05	2.027895
1/128	5.096342e-05	2.000096	1.469410e-05	2.013717
1/256	1.274146e-05	1.999932	3.656267e-06	2.006794

Table 3: Numerical errors and convergence rates for problem 3