(3.3.7) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Proof: By defn.

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x} dx$$

Let $u = x^{\frac{1}{2}}$. Then $2u\,du = dx$

$x \to 0 \Leftrightarrow u \to 0$

$x \to \infty \Leftrightarrow u \to \infty$

So

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty u^{-1} \cdot 2u\, e^{-u^2} du = 2\int_0^\infty e^{-u^2} du.$$

Let $I = \int_0^\infty e^{-u^2} du$. Then

$$I^2 = \int_0^\infty e^{-u^2} du \int_0^\infty e^{-v^2} dv = \int_0^\infty \int_0^\infty e^{-(u^2+v^2)} du\, dv.$$

Let $r^2 = u^2 + v^2$, $u = r\cos\theta$, $v = r\sin\theta$ be a polar coordinate transformation. The region $[0,\infty) \times [0,\infty)$ in (u,v) goes to $(r,\theta) \in [0,\infty, 0, \frac{\pi}{2})$, so

$$I^2 = \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r\, d\theta\, dr = \frac{\pi}{2}\int_0^\infty e^{-r^2} r\, dr.$$

Let $W = r^2$. Then $dW = 2r\,dr$

$r \to 0 \Leftrightarrow w \to 0$, $r \to \infty \Leftrightarrow w \to \infty$,

So $I^2 = \frac{\pi}{4}\int_0^\infty e^{-w} dw = \frac{\pi}{4}(-e^{-w})_0^\infty = \frac{\pi}{4}$

$\Rightarrow I = \frac{\sqrt{\pi}}{2}$.

But $\Gamma(\frac{1}{2}) = 2I$, so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. $\square$

3.32 $X \sim U(50,75)$

(a) PDF of X is $f(x) = \begin{cases} \frac{1}{25} & x \in (50,75) \\ 0 & \text{otherwise} \end{cases}$

by definition. Then the CDF $F(x)$ is

$$F(x) = P(X \leq x) = \begin{cases} \int_{-\infty}^{x} f(x)dx = 0 & \text{if } x \leq 50 \\ \int_{50}^{x} \frac{1}{25} dx = \frac{x-50}{25} & \text{if } 50 < x < 75 \\ \int_{50}^{75} \frac{1}{25} dx = 1 & \text{if } x > 75 \end{cases}$$

$$= \begin{cases} 0 & x \leq 50 \\ \frac{x-50}{25} & 50 < x < 75 \\ 1 & x \geq 75 \end{cases}$$

(b) $P(60 < X < 70) = F(70) - F(60)$ (because X is cont.)

$$= \frac{70-50}{25} - \frac{60-50}{25}$$

$$= \frac{10}{25}$$

$$= 0.4$$

(c) $E[X] = \frac{75+50}{2} = 62.5$ by formula in book.

(d) $Var[X] = \frac{(75-50)^2}{12} = \frac{625}{12} = 52.08$ by formula in book.

3.40 $X \sim Exp(100)$

(a) $P(X > 15) = \int_{15}^{\infty} \frac{1}{100} e^{-\frac{1}{100}x} dx = -e^{-\frac{1}{100}x} \Big|_{15}^{\infty}$

$= e^{-0.15x}$

(b) $P(X > 110) = \int_{110}^{\infty} \frac{1}{100} e^{-\frac{1}{100}x} dx = -e^{-\frac{1}{100}x} \Big|_{110}^{\infty}$

$= e^{-1.1x}$

(c) $P(X > 110 \mid X > 95) = \dfrac{P(X > 110 \cap X > 95)}{P(X > 95)}$

$= \dfrac{P(X > 110)}{P(X > 95)}$

$= \dfrac{e^{-1.1x}}{\frac{1}{100}\int_{95}^{\infty} e^{-\frac{1}{100}x} dx} = \dfrac{e^{-1.1x}}{-e^{-\frac{1}{100}x}\Big|_{95}^{\infty}}$

$= \dfrac{e^{-1.1x}}{e^{-0.95x}}$

$= e^{-0.15x}$

The same as $P(X > 15)$. This is because Exp distribution is <u>Memoryless</u>:

$P(X > x + \Delta x \mid X > x) = P(X > \Delta x)$ for any $x \geq 0$, $\Delta x > 0$.

(d) $Var[X] = \left(\frac{1}{100}\right)^2 = 0.0001$ by example 3.9.2 in book.

3.51) $Z \sim N(0,1)$

(a) $P(Z \leq 1.53) = \Phi(1.53) = 0.937$ by calculator

(b) $P(Z > -0.49) = 1 - P(Z \leq -0.49) = 1 - \Phi(-0.49) = 0.688$ by calculator

(d) $P(|Z| > 1.28) = P(Z < -1.28) + P(Z > 1.28)$

$$= \Phi(-1.28) + 1 - \Phi(1.28)$$

$$= 0.201 \text{ by calculator.}$$

(c) $P(0.35 < Z < 2.01) = P(Z < 2.01) - P(Z < 0.35)$

$$= \Phi(2.01) - \Phi(0.35)$$

$$= 0.341 \text{ by calculator}$$

(e) $P(Z \leq a) = 0.648$ if $a = \Phi^{-1}(0.648)$

$$= 0.380 \text{ by calculator}$$

(f) $P(|Z| < b) = 0.95$. Since Z is symmetric about 0,

$P(Z < K) = P(Z > -K)$ for any K, i.e.

$\Phi(K) = 1 - \Phi(-K)$.

Then $P(|Z| < b) = P(-b < Z < b)$

$$= \Phi(b) - \Phi(-b)$$

$$= 2\Phi(b) - 1 = 0.95$$

if $\Phi(b) = 0.975$

or $b = \Phi^{-1}(0.975) = 1.96$ by calculator.

3.52 Let $X \sim N(3, 0.16)$, $Z \sim N(0,1)$.
$\mu = 3$, $\sigma = 0.4$

(a) $P(X > 3) = P\left(\frac{X-3}{0.4} > 0\right) = P(Z > 0) = 1 - \Phi(0)$
$= 0.5$ by calculator.

(b) $P(X > 3.3) = P\left(\frac{X-3}{0.4} > \frac{0.3}{0.4}\right) = P(Z > \frac{3}{4})$
$= 1 - \Phi(\frac{3}{4}) = 0.2266$ by calculator.

(c) $P(2.8 \le X \le 3.1) = P\left(-\frac{0.2}{0.4} \le Z \le \frac{0.1}{0.4}\right) = \Phi(\frac{1}{4}) - \Phi(-\frac{1}{2})$
$= 0.2902$ by calculator.

(d) 98-th percentile of X is x s.t.

$P(X \le x) = 0.98$, i.e.

$P\left(Z \le \frac{x-3}{0.4}\right) = 0.98$

$\Phi\left(\frac{x-3}{0.4}\right) = 0.98$

$\Rightarrow x = 3 + 0.4\,\Phi^{-1}(.98)$
$= 3.8215$ by calculator.

(e) $P(3-c < X < 3+c) = P\left(\frac{-c}{0.4} < Z < \frac{c}{0.4}\right)$

$= \Phi\left(\frac{c}{0.4}\right) - \Phi\left(\frac{-c}{0.4}\right)$

$= 2\Phi\left(\frac{c}{0.4}\right) - 1 = 0.90$

if $c = 0.4\,\Phi^{-1}(0.95)$
$= 0.658$ by calculator