Math 6108 Homework 5

Jacob Hauck

September 25, 2024

Problem 1.

Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation such that $\operatorname{rank}(T) = \operatorname{rank}(T^2)$. Then $\operatorname{Null}(T) \cap T(V) = \{\mathbf{0}\}$.

Proof. We observe that $\operatorname{Null}(T) \subseteq \operatorname{Null}\left(T^2\right)$ because $T(\mathbf{v}) = \mathbf{0} \implies T^2(\mathbf{v}) = T(T(\mathbf{v})) = \mathbf{0}$. By the Dimension Theorem, we must also have

$$\operatorname{Nullity}(T) + \operatorname{rank}(T) = \dim(V) = \operatorname{Nullity}\left(T^2\right) + \operatorname{rank}\left(T^2\right),$$

which implies that $\operatorname{Null}(T) = \operatorname{Null}\left(T^2\right)$.

Suppose that $\mathbf{v} \in \operatorname{Null}(T) \cap T(V)$. Then $\mathbf{v} \in T(V)$, so there exists $\mathbf{w} \in V$ such that $\mathbf{v} = T(\mathbf{w})$. On the other hand, $\mathbf{v} \in \operatorname{Null}(T)$, so $\mathbf{0} = T(\mathbf{v}) = T^2(\mathbf{w})$. This implies that $\mathbf{w} \in \operatorname{Null}\left(T^2\right) = \operatorname{Null}(T)$. Hence, $\mathbf{v} = T(\mathbf{w}) = \mathbf{0}$. Since $\mathbf{v} \in \operatorname{Null}(T) \cap T(V)$ was arbitrary and $\mathbf{0}$ belongs to any subspace, it follows that $\operatorname{Null}(T) \cap T(V) = \{\mathbf{0}\}$. ☐

Problem 2.

Let $T : V \to W$ and $L : W \to U$ be linear transformations. Then

$$\operatorname{rank}(L \circ T) \leq \min\{\operatorname{rank}(L), \operatorname{rank}(T)\}.$$

Proof. Let $\mathbf{u} \in \operatorname{range}(L \circ T)$. Then there exists $\mathbf{v} \in V$ such that $\mathbf{u} = L(T(\mathbf{v}))$. Thus, $T(\mathbf{v}) \in W$ is a vector whose image under L is $\mathbf{u}$. This implies that $\mathbf{u} \in \operatorname{range}(L)$. Hence, $\operatorname{range}(L \circ T) \subseteq \operatorname{range}(L)$, and it follows that $\operatorname{rank}(L \circ T) \leq \operatorname{rank}(L)$.

Now suppose that $\dim((L \circ T)(V)) = \dim(L(T(V))) > \dim(T(V)) =: n$. Then there exist $n + 1$ vectors $\mathbf{w}_1, \dots, \mathbf{w}_n, \mathbf{w}_{n+1} \in L(T(V))$ that are linearly independent. There also must be $\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}_{n+1} \in T(V)$ such that $L(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, 2, \dots, n + 1$. If

$$c_1\mathbf{v}_1 + \cdots + c_{n+1}\mathbf{v}_{n+1} = \mathbf{0}$$

for some $c_1, \dots, c_{n+1}$ in the underlying field, then applying L on both sides implies

$$\begin{aligned}
\mathbf{0} = L(\mathbf{0}) &= L(c_1\mathbf{v}_1 + \cdots + c_{n+1}\mathbf{v}_{n+1}) \\
&= c_1 L(\mathbf{v}_1) + \cdots + c_{n+1} L(\mathbf{v}_{n+1}) \\
&= c_1\mathbf{w}_1 + \cdots + c_{n+1}\mathbf{w}_{n+1},
\end{aligned}$$

which implies that $c_1 = c_2 = \cdots = c_{n+1} = 0$ by the linear independence of $\mathbf{w}_1, \dots, \mathbf{w}_{n+1}$. This means that $\mathbf{v}_1, \dots, \mathbf{v}_{n+1}$ are linearly independent, contradicting the fact that $\dim(T(V)) = n$. Thus, $\dim(L(T(V))) \leq \dim(T(V))$.

Finally, we obtain $\operatorname{rank}(L \circ T) = \dim((L \circ T)(V)) = \dim(L(T(V))) \leq \dim(T(V)) = \operatorname{rank}(T)$.

This proves that $\operatorname{rank}(L \circ T) \leq \min\{\operatorname{rank}(L), \operatorname{rank}(T)\}$. ☐

Problem 3.

Let A be an $m \times n$ matrix with a left inverse. Then A^* has a right inverse.

Proof. Let B be a left inverse of A; that is, B is an $n \times m$ matrix such that $BA = I$, the $n \times n$ identity matrix. Then

$$I = I^* = (BA)^* = A^* B^*,$$

so B^* is a right inverse for A^*. □

Problem 4.

Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite, and let $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^* A \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Then $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{C}^n$.

Proof. We show that $\langle \cdot, \cdot \rangle$ is positive definite, conjugate symmetric, and linear in the second argument.

1. **Positive definiteness.** For $\mathbf{x} \in \mathbb{C}^n$, we have

$$\langle \mathbf{x}, \mathbf{x}\rangle = \mathbf{x}^* A \mathbf{x} \geq \mathbf{0}$$

because A is positive definite. Furthermore, if $\mathbf{x} \neq \mathbf{0}$, then

$$\langle \mathbf{x}, \mathbf{x}\rangle = \mathbf{x}^* A \mathbf{x} > \mathbf{0},$$

because A is positive definite, so $\langle \mathbf{x}, \mathbf{x}\rangle = \mathbf{0}$ only if $\mathbf{x} = \mathbf{0}$.

2. **Conjugate symmetry.** For $\mathbf{x}, \mathbf{y} \in \mathbb{C}$, we have

$$\langle \mathbf{y}, \mathbf{x}\rangle = \mathbf{y}^* A \mathbf{x} = \overline{(\mathbf{y}^* A \mathbf{x})^*} = \overline{\mathbf{x}^* A^* \mathbf{y}} = \overline{\mathbf{x}^* A \mathbf{y}} = \overline{\langle \mathbf{x}, \mathbf{y}\rangle}$$

because A is Hermitian.

3. **Linearity in second argument.** For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}$ and $a \in \mathbb{C}$, we have

$$\langle \mathbf{x}, \mathbf{y} + a\mathbf{z}\rangle = \mathbf{x}^* A(\mathbf{y} + a\mathbf{z}) = \mathbf{x}^* A \mathbf{y} + a(\mathbf{x}^* A \mathbf{z}) = \langle \mathbf{x}, \mathbf{y}\rangle + a\langle \mathbf{x}, \mathbf{z}\rangle.$$

Problem 5.

1. $\langle \cdot, \cdot \rangle$ is not an inner product because it is not conjugate symmetric. For $\mathbf{v} = (1, 0)^T$ and $\mathbf{w} = (0, 1)^T$, we have

$$\langle \mathbf{v}, \mathbf{w}\rangle = 1 + 1 + 1 = 3,$$

but

$$\langle \mathbf{w}, \mathbf{v}\rangle = 1 + 0 + 1 = 2 \neq 3 = \overline{\langle \mathbf{v}, \mathbf{w}\rangle}$$

2. $\langle \cdot, \cdot \rangle$ is not an inner product because it is not positive definite. Let $\mathbf{v} = (1, -1)^T$. Then

$$\langle \mathbf{v}, \mathbf{v}\rangle = 1 - 1 - 1 + 1 = 0,$$

but $\mathbf{v} \neq 0$.

3. $\langle\cdot,\cdot\rangle$ is not an inner product because it is not positive definite. Let $A \in \mathbb{M}_n$ be the matrix with zero diagonal and all other components equal to 1. Then

$$\langle A, A\rangle = \operatorname{tr}(A^* + A) = 0,$$

but $A \neq 0$.

4. Assuming that elements of $C([-1,1])$ are real-valued, then $\langle\cdot,\cdot\rangle$ is an inner product. We observe that $\langle f, g\rangle$ is an improper integral, but it can be shown to exist for all $f, g \in C([-1,1])$ by making the trigonometric substitution $x = \cos(\theta)$ for $\theta \in [0, \pi]$:

$$\begin{aligned}\langle f, g\rangle &= \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_0^\pi f(\cos(\theta))g(\cos(\theta))\frac{\sin(\theta)}{\sqrt{1-\cos^2(\theta)}}\,\mathrm{d}\theta \\ &= \int_0^\pi f(\cos(\theta))g(\cos(\theta))\,\mathrm{d}\theta.\end{aligned}$$

The latter-most integrand is integrable because $f \circ \cos$ and $g \circ \cos$ are continuous. We now show that $\langle\cdot,\cdot\rangle$ satisfies the requirements for being an inner product.

(a) **Positive definiteness.** Let $f \in C([-1,1])$. Then

$$\langle f, f\rangle = \int_{-1}^{1} \frac{(f(x))^2}{\sqrt{1-x^2}}\,\mathrm{d}x \geq 0$$

because the integrand is pointwise nonnegative. Suppose that $\langle f, f\rangle = 0$. Since the integral is improper, we have to be a bit careful. For any $a \in (0,1)$, the nonnegativity of the integrand implies that

$$0 \leq \int_{-a}^{a} \frac{(f(x))^2}{\sqrt{1-x^2}}\,\mathrm{d}x \leq \int_{-1}^{1} \frac{(f(x))^2}{\sqrt{1-x^2}}\,\mathrm{d}x = 0.$$

Since a was arbitrary, and the integral of a continuous, nonnegative function is zero only if the function is zero, we obtain

$$\frac{(f(x))^2}{\sqrt{1-x^2}} = 0, \qquad x \in (-1,1).$$

Since $\sqrt{1-x^2} \neq 0$ for $x \in (-1,1)$, it follows that $f(x) = 0$ for $x \in (-1,1)$. Since f is continuous on $[-1,1]$, it follows that $f(-1) = f(1) = 0$ as well. Thus, $f = 0$.

(b) **Conjugate symmetry.** For any $f, g \in C([-1,1])$, we have

$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \overline{\int_{-1}^{1} \frac{g(x)f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x} = \overline{\langle g, f\rangle}$$

because f and g are real-valued.

(c) **Linearity in second argument.** Let $f, g, h \in C([-1,1])$, and let $a \in \mathbb{R}$. Then

$$\langle f, g+ah\rangle = \int_{-1}^{1} \frac{f(x)(g(x)+ah(x))}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\,\mathrm{d}x + a\int_{-1}^{1} \frac{f(x)h(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \langle f, g\rangle + a\langle f, h\rangle.$$

Problem 6.

Let V be an inner product space. Then $\langle \mathbf{v}, \mathbf{0}\rangle = 0$ for all $\mathbf{v} \in V$.

Proof. Let $\mathbf{v} \in V$. Since $\mathbf{0} = 0 \cdot \mathbf{0}$, it follows from the linearity of the inner product in the second argument that

$$\langle \mathbf{v}, \mathbf{0} \rangle = \langle \mathbf{v}, 0 \cdot \mathbf{0} \rangle = 0 \cdot \langle \mathbf{v}, \mathbf{0} \rangle = 0.$$

$\square$

$\square$