

# Math 6417 Homework 3

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## Question 1.

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Let  $B(\cdot, \cdot)$  be a continuous, bilinear form on a real Hilbert space  $H$ . Suppose that  $B$  is coercive in the sense that there is some  $\alpha > 0$  such that  $|B(x, x)| \geq \alpha \|x\|^2$  for all  $x \in H$ .

- 1.1) Let  $y \in H$ . Then the map  $f_y : H \rightarrow \mathbf{R}$  defined by  $f_y(x) = B(x, y)$  is a bounded linear functional on  $H$ . Consequently, there exists a unique  $w \in H$  such that  $B(x, y) = f_y(x) = (x, w)$  for all  $x \in H$ .

*Proof.* Firstly, it is clear that  $f_y$  is linear; indeed, given  $a_1, a_2 \in \mathbf{R}$  and  $x_1, x_2 \in H$ ,

$$f_y(a_1x_1 + a_2x_2) = B(a_1x_1 + a_2x_2, y) = a_1B(x_1, y) + a_2B(x_2, y) = a_1f_y(x_1) + a_2f_y(x_2) \quad (1)$$

by the bilinearity of  $B$ .

Secondly,  $B(\cdot, y) = f_y$  must be continuous because  $B$  is continuous. Hence,  $f_y$  is bounded.

Thirdly, by the Riesz representation theorem, there exists a unique  $w \in H$  such that  $B(x, y) = f_y(x) = (x, w)$  for all  $x \in H$ .  $\square$

- 1.2) Given  $y \in H$ , by 1.1), there is a unique  $w \in H$  such that  $B(x, y) = (x, w)$  for all  $x \in H$ ; this defines a function  $A : H \rightarrow H$ , where  $Ay = w$ . Then  $A$  is a bounded, linear operator on  $H$ , that is,  $A \in B(H)$ .

*Proof.* There are two steps to this proof: showing that  $A$  is linear, and showing that  $A$  is bounded.

### Step 1: linearity

Let  $a_1, a_2 \in \mathbf{R}$  and  $y_1, y_2 \in H$ . Then for all  $x \in H$ ,

$$\begin{aligned} (x, A(a_1y_1 + a_2y_2)) &= B(x, a_1y_1 + a_2y_2) = a_1B(x, y_1) + a_2B(x, y_2) = a_1(x, Ay_1) + a_2(x, Ay_2) \\ &= (x, a_1Ay_1 + a_2Ay_2). \end{aligned} \quad (2)$$

Thus,  $w = A(a_1y_1 + a_2y_2)$  and  $w' = a_1Ay_1 + a_2Ay_2$  satisfy the property that  $B(x, a_1y_1 + a_2y_2) = (x, w) = (x, w')$  for all  $x \in H$ . By the Riesz representation theorem, there is only one element of  $H$  that can satisfy this property; therefore,  $w = w'$ , or  $A(a_1y_1 + a_2y_2) = a_1Ay_1 + a_2Ay_2$ . Thus,  $A$  is linear.

### Step 2: boundedness

Note that  $B$  is continuous if and only if (see, e.g., Theorem 8.10 assumption (a) in Arbogast and Bona) there exists some  $M > 0$  such that

$$|B(x, y)| \leq M \|x\| \|y\|, \quad \text{for all } x, y \in H. \quad (3)$$

Let  $y \in H$ . Then

$$\|Ay\| = \left| \left( \frac{Ay}{\|Ay\|}, Ay \right) \right| = \left| B \left( \frac{Ay}{\|Ay\|}, y \right) \right| \leq M \|y\|. \quad (4)$$

Since  $y$  was arbitrary, it follows that  $A$  is bounded, and  $\|A\| \leq M$ .  $\square$

1.3)  $A$  is bounded below in the sense that there exists  $\gamma > 0$  such that  $\|Ay\| \geq \gamma\|y\|$  for all  $y \in H$ .

*Proof.* This follows from the coercivity of  $B$ : for all  $y \in H$ ,

$$\|Ay\|\|y\| \geq |(y, Ay)| = |B(y, y)| \geq \alpha\|y\|^2, \quad (5)$$

so  $\|Ay\| \geq \alpha\|y\|$  for all  $y \in H$ , as claimed.  $\square$

1.4)  $A$  is one-to-one, and the range of  $A$  is closed.

*Proof.* Let  $y_1, y_2 \in H$ , and suppose that  $Ay_1 = Ay_2$ . Then, by the previous part,

$$\|y_1 - y_2\| \leq \frac{1}{\gamma} \|A(y_1 - y_2)\| = \frac{1}{\gamma} \|Ay_1 - Ay_2\| = 0. \quad (6)$$

Therefore,  $y_1 = y_2$ . This shows that  $A$  is one-to-one.

Let  $R(A)$  denote the range of  $A$ , and let  $\{w_n\} \subseteq R(A)$  be a convergent sequence in the range of  $A$ . By the definition of  $R(A)$ , there exists  $y_n \in H$  such that  $w_n = Ay_n$ .

Let  $\varepsilon > 0$  be given. Since  $\{w_n\}$  is convergent, it is also Cauchy, so we can choose  $N$  such that  $n, m > N$  implies that  $\|w_n - w_m\| < \varepsilon$ . By the linearity of  $A$ , we have  $A(y_n - y_m) = w_n - w_m$ ; hence,

$$\begin{aligned} \alpha\|y_n - y_m\|^2 &\leq |B(y_n - y_m, y_n - y_m)| = |(y_n - y_m, w_n - w_m)| \leq \|y_n - y_m\| \cdot \|w_n - w_m\| \\ &\leq \varepsilon\|y_n - y_m\| \end{aligned} \quad (7)$$

$$(8)$$

if  $n, m > N$ . Thus,  $n, m > N$  implies that  $\|y_n - y_m\| < \frac{\varepsilon}{\alpha}$ . This implies that  $\{y_n\}$  is Cauchy.

Since  $H$  is complete, there exists  $y \in H$  such that  $y_n \rightarrow y$  as  $n \rightarrow \infty$ . By the continuity of  $A$ ,

$$Ay = \lim_{n \rightarrow \infty} Ay_n = \lim_{n \rightarrow \infty} w_n = w. \quad (9)$$

This means that  $w \in R(A)$ . Since the convergent sequence  $\{w_n\} \subseteq R(A)$  was arbitrary, and its limit  $w \in R(A)$ , it follows that  $R(A)$  is closed.  $\square$

1.5)  $A$  is onto.

*Proof.* Suppose that  $x \in R(A)^\perp$ , that is,  $(x, w) = 0$  for all  $w \in R(A)$ . This implies that  $(x, Ay) = 0$  for all  $y \in H$ , which is equivalent to saying that  $B(x, y) = 0$  for all  $y \in H$ . In particular, if we choose  $y = x$ , then  $\|x\|^2 \leq \frac{1}{\alpha} |B(x, x)| = 0$ . Therefore,  $x = 0$ . This shows that  $R(A)^\perp = \{0\}$  because  $x$  was arbitrary.

Let  $y \in H$ . Since  $R(A)$  is a closed subspace of  $H$  by (1.4), there exists a best approximation  $w \in R(A)$  of  $y$ , which satisfies the property  $(y - w, x) = 0$  for all  $x \in R(A)$  (Theorem 3.7 and Corollary 3.8 in Arbogast and Bona). That is,  $y - w \in R(A)^\perp$ . Since  $R(A)^\perp = \{0\}$  by the above, it follows that  $y - w = 0$ , and  $y = w \in R(A)$ . Since  $y$  was arbitrary and  $R(A) \subseteq H$ , it follows that  $R(A) = H$ , that is,  $A$  is onto.  $\square$

1.6)  $A$  is invertible.

*Proof.* By the previous two parts,  $A$  is bijective, so it has a set-theoretic inverse function  $A^{-1}$ . By 1.2),  $A$  is bounded and linear, and by 1.5) it is surjective. Therefore, by the open mapping theorem,  $A$  maps open sets to open sets, which means that the preimage of an open set under  $A^{-1}$  is open, that is,  $A^{-1}$  is continuous. Therefore,  $A$  is invertible.  $\square$

- 1.7) Given  $f \in H^*$ , the Riesz representation theorem implies that there exists a unique  $w \in H$  such that  $f(x) = (x, w)$  for all  $x \in H$ , and we can view  $H^*$  and  $H$  as the same under the correspondence  $f \leftrightarrow w$ .
- 1.8) Consider the equation  $B(x, y) = f(x)$  for all  $x \in H$ , where  $f \in H^*$ . By the remark in part 1.7), we can choose  $w \in H$  such that  $f(x) = (x, w)$  for all  $x \in H$ . Then the equation is equivalent to  $B(x, y) = (x, w)$  for all  $x \in H$ . If  $y$  is a solution of this equation, then, by the definition of  $A$ , we must have  $Ay = w$ . Using the invertibility of  $A$ , we obtain  $y = A^{-1}w$  as the unique solution of the equation. Viewing  $f$  and  $w$  as the same under the correspondence in 1.7), we might also write  $y = A^{-1}f$ .

### Question 2.

Let  $e_j \in L^2(-\pi, \pi)$  be defined by  $e_j(x) = \frac{1}{\sqrt{2\pi}} e^{ijx}$  for  $j \in \mathbf{Z}$ . Define

$$H = \left\{ f \in L^2(-\pi, \pi) : f = \bar{f} \text{ and } f = \sum_{j \neq 0} f_j e_j \text{ for some } \{f_j\} \subseteq \mathbf{C} \text{ such that } \sum_{j \neq 0} j^2 |f_j|^2 < \infty \right\}, \quad (10)$$

and

$$H^{-1} = \left\{ f = \sum_{j \neq 0} f_j e_j : \sum_{j \neq 0} j^{-2} |f_j|^2 < \infty \text{ and } f = \bar{f} \right\}. \quad (11)$$

- 2.1)  $H$  and  $H^{-1}$  are real Hilbert spaces when equipped with the inner products

$$(f, g)_H = \sum_{j \neq 0} j^2 f_j \bar{g}_j, \quad (f, g)_{H^{-1}} = \sum_{j \neq 0} j^{-2} f_j \bar{g}_j. \quad (12)$$

*Proof.* Before we can show that  $H$  and  $H^{-1}$  are real Hilbert spaces, we need to study their definitions. For the definition  $H$ , there isn't much trouble – its elements are just elements of  $L^2(-\pi, \pi)$ . For the definition  $H^{-1}$ , the series might not converge to an element of  $L^2(-\pi, \pi)$  (say,  $f_j = 1$  for all  $j$ ), so understanding what the convergence of the infinite sum means is more difficult.

As we will see, we can understand  $H$  in terms of a corresponding space of coefficient sequences, which is isomorphic to  $H$  as a real vector space. By analogy, then, we can understand  $H^{-1}$  in terms of its corresponding space of coefficient sequences, which will suffice for our purposes.

To understand these spaces of coefficient sequences, we introduce the following Lemma showing that  $H$  is a real vector space isomorphic to its space of coefficient sequences.

**Lemma 1.** *Define*

$$S_H = \left\{ \{f_j\}_{j \neq 0} \subseteq \mathbf{C} : \sum_{j \neq 0} j^2 |f_j|^2 < \infty \text{ and } \forall j \in \mathbf{Z} \setminus \{0\}, f_j = \bar{f}_{-j} \right\}. \quad (13)$$

*Then  $S_H$  is a real vector space, and there is an isomorphism (of real vector spaces)  $\varphi : H \rightarrow S_H$  such that if  $\varphi(f) = \{f_j\}$ , then*

$$f = \sum_{j \neq 0} f_j e_j. \quad (14)$$

*Proof.* We need to prove that  $\varphi$  is well-defined, bijective, and, after showing that  $H$  and  $S_H$  are real vector spaces, that  $\varphi$  is a linear mapping between them.

**Step 1: definition of  $\varphi$**

Let  $f \in H$ . Then  $f \in L^2(-\pi, \pi)$ . Recalling from our lecture that  $\{e_j\}_{j \in \mathbf{Z}}$  is an orthonormal basis for  $L^2(-\pi, \pi)$ , it follows that there exists exactly one sequence of coefficients  $\{g_j\} \in \ell^2(\mathbf{Z})$  such that

$$f = \sum_{j \in \mathbf{Z}} g_j e_j. \quad (15)$$

On the other hand, since  $f \in H$ , there must be a sequence of coefficients  $\{f_j\} \subseteq \mathbf{C}$  such that

$$f = \sum_{j \neq 0} f_j e_j, \quad \sum_{j \neq 0} j^2 |f_j|^2 < \infty, \quad (16)$$

It follows by the uniqueness of  $\{g_j\}$  that  $g_0 = 0$ , and  $g_j = f_j$  for all  $j \neq 0$ . Furthermore, since  $f = \bar{f}$ , it follows that

$$\bar{f}(x) = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} \overline{f_j e^{ijx}} = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} \bar{f}_j e^{-ijx} = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} \bar{f}_{-j} e^{ijx} = f(x). \quad (17)$$

That is,  $f = \sum_{j \neq 0} \bar{f}_{-j} e_j$ ; by the uniqueness of  $\{g_j\}$  again, we must have  $\bar{f}_{-j} = g_j = f_j$  for all  $j \neq 0$ .

Hence,  $\{f_j\} \in S_H$ . Since  $\{f_j\}$  is uniquely determined by  $f$  by the uniqueness of  $\{g_j\}$ , we can define a function  $\varphi : H \rightarrow S_H$  by  $\varphi(f) = \{f_j\}$ .

**Step 2:  $\varphi$  is a one-to-one correspondence**

First, suppose that  $\varphi(f) = \{f_j\} = \varphi(g) = \{g_j\}$  for some  $f, g \in H$  and  $\{f_j\} \in S_H$ . Then, by the definition of  $\varphi$ ,

$$f = \sum_{j \neq 0} f_j e_j = g, \quad (18)$$

so  $\varphi$  is one-to-one.

Second, let  $\{f_j\} \in S_H$ . Since  $L^2(-\pi, \pi)$  is a Hilbert space, its Riesz-Fischer map  $F : L^2(-\pi, \pi) \rightarrow \ell^2(\mathbf{Z})$  corresponding to the orthonormal basis  $\mathcal{B}$  is an isomorphism. If we set  $g_j = f_j$  for  $j \neq 0$ , and  $g_0 = 0$ , then we have

$$\sum_{j \in \mathbf{Z}} |g_j|^2 = \sum_{j \neq 0} |f_j|^2 \leq \sum_{j \neq 0} j^2 |f_j|^2 < \infty \quad (19)$$

because  $\{f_j\} \in S_H$ . Therefore,  $\{g_j\} \in \ell^2(\mathbf{Z})$ , and  $f = F^{-1}(\{g_j\}) \in L^2(-\pi, \pi)$ . By the definition of  $F$  and the fact that  $\{e_j\}$  is an orthonormal basis, we have

$$f = F^{-1}(\{g_j\}) = \sum_{j \in \mathbf{Z}} g_j e_j = \sum_{j \neq 0} f_j e_j. \quad (20)$$

Since  $\{f_j\} \in S_H$ , we have  $f_j = \bar{f}_{-j}$ , so

$$\bar{f}(x) = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} \overline{f_j e^{ijx}} = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} \bar{f}_j e^{-ijx} = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} \bar{f}_{-j} e^{ijx} = \frac{1}{\sqrt{2\pi}} \sum_{j \neq 0} f_j e^{ijx} = f(x), \quad (21)$$

so  $f = \bar{f}$ . Since  $\{f_j\} \in S_H$ , we also have  $\sum_{j \neq 0} j^2 |f_j|^2 < \infty$ . Therefore,  $f \in H$  by definition.

Finally,  $\varphi(f) = \{f_j\}$  because  $f = \sum_{j \neq 0} f_j e_j$ , and  $\varphi(f)$  is, by definition, the unique sequence of coefficients in  $S_H$  that make that statement true.

Thus,  $\varphi$  is one-to-one and onto.

**Step 3:  $H$  and  $S_H$  are real vector spaces**

$H$  is a nonempty subset (it contains 0) of the real vector space  $L^2(-\pi, \pi)$ , and  $S_H$  is a nonempty subset (it contains 0) of the real vector space  $\ell^2(\mathbf{Z} \setminus \{0\})$ . Thus, it suffices to show that  $H$  and  $S_H$  are subspaces of these two vector spaces.

Let  $\{f_j\}, \{g_j\} \in S_H$ , and let  $\alpha, \beta \in \mathbf{R}$ . Then  $\{h_j\} = \alpha\{f_j\} + \beta\{g_j\} \in S_H$  because  $\bar{h}_{-j} = \alpha\bar{f}_{-j} + \beta\bar{g}_{-j} = \alpha f_j + \beta g_j = h_j$ , and

$$\sum_{j \neq 0} j^2 |h_j|^2 = \sum_{j \neq 0} j^2 |\alpha f_j + \beta g_j|^2 \leq 2\alpha^2 \sum_{j \neq 0} j^2 |f_j|^2 + 2\beta^2 \sum_{j \neq 0} j^2 |g_j|^2 < \infty \quad (22)$$

since  $|\alpha f_j + \beta g_j|^2 \leq 2(\alpha^2 |f_j|^2 + \beta^2 |g_j|^2)$  for all  $j$ . Thus,  $S_H$  is a real vector subspace of the space of all sequences of complex numbers.

Now let  $f, g \in H$ , and  $\alpha, \beta \in \mathbf{R}$ . Set  $\{f_j\} = \varphi(f)$ , and  $\{g_j\} = \varphi(g)$ . Then  $\{h_j\} = \alpha\{f_j\} + \beta\{g_j\} \in S_H$ , so we can define  $h = \varphi^{-1}(\{h_j\})$ . By the definition of  $h$ , we have

$$\alpha f + \beta g = \alpha \sum_{j \neq 0} f_j e_j + \beta \sum_{j \neq 0} g_j e_j = \sum_{j \neq 0} (\alpha f_j + \beta g_j) e_j = h \in H. \quad (23)$$

Therefore,  $H$  is a (real) vector subspace of  $L^2(-\pi, \pi)$ .

**Step 4:  $\varphi$  is linear**

Let  $f, g \in H$ , and let  $\alpha, \beta \in \mathbf{R}$ . Define  $\{f_j\} = \varphi(f)$ , and  $\{g_j\} = \varphi(g)$ . Then

$$\alpha f + \beta g = \alpha \sum_{j \neq 0} f_j e_j + \beta \sum_{j \neq 0} g_j e_j = \sum_{j \neq 0} (\alpha f_j + \beta g_j) e_j. \quad (24)$$

This implies that  $\alpha\varphi(f) + \beta\varphi(g) = \{\alpha f_j + \beta g_j\} = \varphi(\alpha f + \beta g)$  by the definition of  $\varphi$ . Thus,  $\varphi$  is linear.  $\square$

The Lemma establishes that  $H$  is essentially the same as  $S_H$  as a vector space. If we define by analogy

$$S_{H^{-1}} = \left\{ \{f_j\}_{j \neq 0} \subseteq \mathbf{C} : \sum_{j \neq 0} j^{-2} |f_j|^2 < \infty \text{ and } \forall j \in \mathbf{Z} \setminus \{0\}, f_j = \bar{f}_{-j} \right\}, \quad (25)$$

then we can understand  $H^{-1}$  to be a real vector space isomorphic to  $S_{H^{-1}}$ . Note that  $S_{H^{-1}}$  is indeed a real vector space; like  $S_H$ , it is a nonempty subset (it contains 0) of the vector space of all sequences of complex numbers, and if  $\{f_j\}, \{g_j\} \in S_{H^{-1}}$ , and  $\alpha, \beta \in \mathbf{R}$ , then  $\{h_j\} = \alpha\{f_j\} + \beta\{g_j\} \in S_H$  because  $\bar{h}_{-j} = \alpha\bar{f}_{-j} + \beta\bar{g}_{-j} = \alpha f_j + \beta g_j = h_j$ , and

$$\sum_{j \neq 0} j^{-2} |h_j|^2 = \sum_{j \neq 0} j^{-2} |\alpha f_j + \beta g_j|^2 \leq 2\alpha^2 \sum_{j \neq 0} j^{-2} |f_j|^2 + 2\beta^2 \sum_{j \neq 0} j^{-2} |g_j|^2 < \infty \quad (26)$$

since  $|\alpha f_j + \beta g_j|^2 \leq 2(\alpha^2 |f_j|^2 + \beta^2 |g_j|^2)$  for all  $j$ .

We notice that it is possible for the series in the definition of  $H^{-1}$  to converge for some  $\{f_j\} \in S_{H^{-1}}$ . In particular, if  $\{f_j\} \in \ell^2(\mathbf{Z} \setminus \{0\})$ , then the series converges to a function  $f \in L^2(-\pi, \pi)$ . Moreover, by the same reasoning in Lemma 1, the function  $f$  is uniquely determined by the coefficients  $\{f_j\}$ , and vice versa.

Thus, when  $f \in L^2(-\pi, \pi)$ , and the coefficients  $\{f_j\}$  of  $f$  with respect to the orthonormal basis  $\{e_j\}$  belong to  $\ell^2(\mathbf{Z} \setminus \{0\})$  with  $f_0 = 0$  and  $f_j = \bar{f}_{-j}$ , it makes sense to view  $f$  as an element of  $H^{-1}$ , and we can define a function  $\psi : H^{-1} \rightarrow S_{H^{-1}}$  by  $\psi(f) = \{f_j\}_{j \neq 0}$ . By the exact same reasoning in Lemma 1, we can easily verify that  $\psi$  is one-to-one and linear. This gives us an interpretation of at least some of the elements of  $H^{-1}$ ; from now on, we will simply assume that  $\psi$  can be extended to an isomorphism between  $H^{-1}$  and  $S_{H^{-1}}$ .

Now we turn to the issue of equipping  $H$  and  $H^{-1}$  with inner products. The given inner products are defined in terms of the sequence representations of elements of  $H$  and  $H^{-1}$ ; this is well-defined due to the (actual) isomorphism between  $H$  and  $S_H$  and the (assumed) isomorphism between  $H^{-1}$  and  $S_{H^{-1}}$ . It also allows us to work with  $S_{H^{-1}}$  without needing to worry about how to interpret its elements – we will only need to work with the sequence representations in  $S_{H^{-1}}$ .

We need to show that  $H$  and  $H^{-1}$  are real inner product spaces when equipped with  $(\cdot, \cdot)_H$  and  $(\cdot, \cdot)_{H^{-1}}$  as inner products. We wrap this into the following Lemma.

**Lemma 2.** For  $G \in \{H, H^{-1}\}$ , define

$$\rho = \rho_G = \begin{cases} \varphi & G = H, \\ \psi & G = H^{-1}, \end{cases} \quad s = s_G = \begin{cases} 1 & G = H, \\ -1 & G = H^{-1}. \end{cases} \quad (27)$$

Then  $G$  is a real inner product space with inner product  $(\cdot, \cdot)_G$  defined by

$$(f, g)_G = \sum_{j \neq 0} j^{2s} f_j \bar{g}_j, \quad f, g \in G, \quad (28)$$

where  $\{f_j\} = \rho(f)$ ,  $\{g_j\} = \rho(g) \in S_G$  are the coefficients of  $f$  and  $g$  in  $S_G$ .

*Proof.* As we have already remarked, the uniqueness of  $\{f_j\}$  and  $\{g_j\}$  implies that  $(f, g)_G$  is well-defined. We still need to show that the series converges to a real number, and that  $(\cdot, \cdot)_G$  is symmetric, linear in the first argument, and positive definite.

**Step 1:  $(f, g)_G$  is a real number**

Let  $f, g \in G$  with  $\{f_j\} = \rho(f)$ , and  $\{g_j\} = \rho(g)$ . Then the series for  $(f, g)_G$  converges absolutely because

$$\sum_{j \neq 0} j^{2s} |f_j| \cdot |\bar{g}_j| \leq \left( \sum_{j \neq 0} j^{2s} |f_j|^2 \right)^{\frac{1}{2}} \left( \sum_{j \neq 0} j^{2s} |g_j|^2 \right)^{\frac{1}{2}} < \infty \quad (29)$$

by the Cauchy-Schwarz inequality.

Next,  $(f, g)_G \in \mathbf{R}$  because

$$\overline{(f, g)_G} = \overline{\sum_{j \neq 0} j^{2s} f_j \bar{g}_j} = \sum_{j \neq 0} j^{2s} \bar{f}_j g_j = \sum_{j \neq 0} j^{2s} f_{-j} \bar{g}_{-j} = \sum_{j \neq 0} j^{2s} f_j \bar{g}_j = (f, g)_G, \quad (30)$$

and only real numbers are equal to their own complex conjugate.

**Step 2:  $(\cdot, \cdot)_G$  is symmetric**

Let  $f, g \in G$ , and let  $\{f_j\} = \rho(f)$ , and  $\{g_j\} = \rho(g)$ . Then

$$(f, g)_G = \sum_{j \neq 0} j^{2s} f_j \bar{g}_j = \sum_{j \neq 0} j^{2s} f_{-j} \bar{g}_{-j} = \sum_{j \neq 0} j^{2s} g_j \bar{f}_j = (g, f)_G. \quad (31)$$

Therefore,  $(\cdot, \cdot)_G$  is symmetric.

**Step 3:  $(\cdot, \cdot)_G$  is linear in the first argument**

Let  $f, g, h \in G$ , and let  $\{f_j\} = \rho(f)$ ,  $\{g_j\} = \rho(g)$ , and  $\{h_j\} = \rho(h)$ . Let  $\alpha, \beta \in \mathbf{R}$ . Then  $\rho(\alpha f + \beta g) = \alpha \rho(f) + \beta \rho(g)$  because  $\rho$  is linear, so

$$(\alpha f + \beta g, h)_G = \sum_{j \neq 0} j^{2s} (\alpha f_j + \beta g_j) \bar{h}_j = \alpha \sum_{j \neq 0} j^{2s} f_j \bar{h}_j + \beta \sum_{j \neq 0} j^{2s} g_j \bar{h}_j = \alpha (f, h)_G + \beta (g, h)_G. \quad (32)$$

Therefore,  $(\cdot, \cdot)_G$  is linear in the first argument.

**Step 4:  $(\cdot, \cdot)_G$  is positive definite**

Let  $f \in G$ , and let  $\{f_j\} = \rho(f)$ . Then

$$(f, f)_G = \sum_{j \neq 0} j^{2s} |f_j|^2 \geq 0 \quad (33)$$

because each term of the series is nonnegative. Moreover, if  $f = 0$ , then  $|f_j|^2 = 0$  (by the linearity of  $\rho$ ), so each term is 0, and  $(f, f)_G = 0$ . Conversely, if  $(f, f)_G = 0$ , then, since each term of the series for  $(f, f)_G$  is nonnegative, it must be that each term is 0. This implies that  $f_j = 0$  for all  $j \neq 0$ , that is,  $\{f_j\} = 0$  in  $S_G$ . Therefore  $f = \rho^{-1}(0) = 0$  by the linearity of  $\rho^{-1}$ .

This shows that  $(\cdot, \cdot)_G$  is positive definite.  $\square$

Now we know that  $H$  and  $H^{-1}$  are real inner product spaces, so there is only one more thing to show in order to prove that they are real Hilbert spaces: they need to be complete with respect to the norms  $\|\cdot\|_H$  and  $\|\cdot\|_{H^{-1}}$  induced by their inner products. For the sake of organization, we put this proof inside one last Lemma.

**Lemma 3.** *For  $G \in \{H, H^{-1}\}$  and  $s = s_G$ ,  $\rho = \rho_G$  defined as in Lemma 2, the space  $G$  is complete with respect to the norm  $\|\cdot\|_G$  induced by the inner product  $(\cdot, \cdot)_G$  from Lemma 2.*

*Proof.* Let  $\{f^n\}_{n=1}^\infty$  be a Cauchy sequence in  $G$  with respect to  $\|\cdot\|_G$ . We need to show that there exists  $f \in G$  such that  $f^n \rightarrow f$  as  $n \rightarrow \infty$  in  $\|\cdot\|_G$ . To do this, we first identify a candidate element, then we show that the sequence converges to the candidate.

**Step 1: identifying a candidate limit**

Given  $\varepsilon > 0$ , we can choose  $N$  such that  $n, m > N$  implies that

$$\varepsilon^2 > \|f^n - f^m\|_G^2 = \sum_{j \neq 0} j^{2s} |f_j^n - f_j^m|^2, \quad (34)$$

where  $\{f_j^n\} = \rho(f^n)$ . Since each term in the summation is nonnegative, this means that  $|j^s f_j^n - j^s f_j^m| < \varepsilon$  for all  $n, m > N$ . Then  $\{j^s f_j^n\}_{n=1}^\infty$  is Cauchy for all  $j \neq 0$ . Thus, by the completeness of  $\mathbf{C}$ , the sequence  $\{j^s f_j^n\}$  converges to a limit  $j^s f_j \in \mathbf{C}$  as  $n \rightarrow \infty$ .

Since  $j^s \bar{f}_j^n = j^s f_{-j}^n$  for all  $j$  and all  $n$ , taking the limit as  $n \rightarrow \infty$  and using the continuity of the complex conjugate function, we get  $j^s \bar{f}_j = j^s f_{-j}$  for all  $j \neq 0$ . Since  $j^s \neq 0$ , it follows that  $\bar{f}_j = f_{-j}$  for all  $j \neq 0$ .

Furthermore, by the convergence of  $\{j^s f_j^n\}$  to  $j^s f_j$ , for all  $J > 0$ , we can choose  $n$  large enough that  $|j^s f_j - j^s f_j^n| \leq \frac{1}{2J}$  for all  $0 < |j| \leq J$ . Then

$$\sum_{0 < |j| \leq J} j^{2s} |f_j|^2 = \sum_{0 < |j| \leq J} |j^s f_j - j^s f_j^n + j^s f_j^n|^2 \leq 2 \sum_{0 < |j| \leq J} |j^s f_j - j^s f_j^n|^2 + 2 \sum_{0 < |j| \leq J} j^{2s} |f_j^n|^2 \quad (35)$$

$$\leq 2 + 2\|f^n\|_G^2. \quad (36)$$

Since  $\{f^n\}$  is Cauchy in  $\|\cdot\|_G$ , it must be bounded; that is, there exists  $M > 0$  such that  $\|f^n\|_G \leq M$  for all  $n$ . Then

$$\sum_{0 < |j| \leq J} j^{2s} |f_j|^2 \leq 2 + 2M^2 \quad (37)$$

for all  $J > 0$ . This implies that

$$\sum_{j \neq 0} j^{2s} |f_j|^2 \leq 2 + 2M^2 < \infty. \quad (38)$$

Therefore,  $\{f_j\} \in S_G$ , and we can define our candidate limit as  $f = \rho^{-1}(\{f_j\})$ .

**Step 2: showing that the candidate is the limit**

Let  $\varepsilon > 0$  be given. Since  $\{f^n\}$  is Cauchy, we can choose  $N$  such that  $n, m > N$  implies that  $\|f^n - f^m\|_G^2 < \varepsilon$ . By the convergence of  $\{j^s f_j^n\}$ , for any  $J > 0$ , we can choose  $m_J > N$  such that for all  $0 < |j| \leq J$  we have  $|j^s f_j - j^s f_j^{m_J}|^2 < \frac{\varepsilon}{2J}$ . Then

$$\sum_{0 < |j| \leq J} j^{2s} |f_j - f_j^n|^2 = \sum_{0 < |j| \leq J} |(j^s f_j - j^s f_j^{m_J}) + (j^s f_j^{m_J} - j^s f_j^n)|^2 \quad (39)$$

$$\leq 2 \sum_{0 < |j| \leq J} |j^s f_j - j^s f_j^{m_J}|^2 + 2 \sum_{0 < |j| \leq J} j^{2s} |f_j^{m_J} - f_j^n|^2 \quad (40)$$

$$\leq 2\varepsilon + 2\|f^{m_J} - f^n\|_G^2 \leq 4\varepsilon \quad (41)$$

if  $n > N$ . Since this estimate is independent of  $J$ , it follows that

$$\|f - f^n\|_G^2 = \sum_{j \neq 0} j^{2s} |f_j - f_j^n|^2 \leq 4\varepsilon \quad (42)$$

if  $n > N$ . Therefore,  $f^n \rightarrow f$  as  $n \rightarrow \infty$  in  $\|\cdot\|_G$ .  $\square$

The three Lemmas above establish that  $H$  and  $H^{-1}$  are Hilbert spaces with the given inner products.  $\square$

**2.2)** Define the bilinear form  $B$  on  $H$  by

$$B(f, g) = \sum_{j \neq 0} (ij + j^2) f_j \bar{g}_j, \quad (43)$$

where  $\{f_j\} = \varphi(f)$ , and  $\{g_j\} = \varphi(g)$ . Then  $B$  satisfies the hypotheses of the Lax-Milgram theorem.

*Proof.* As with the inner products, the isomorphism  $\varphi$  ensures that  $B(f, g)$  is well-defined in terms of  $\{f_j\} = \varphi(f)$  and  $\{g_j\} = \varphi(g)$ . We need to show that the series for  $B$  converges and that  $B$  is actually bilinear over  $\mathbf{R}$ . Then, we need to show that  $B$  satisfies the hypotheses of the Lax-Milgram theorem, that is, that  $B$  is continuous and coercive.

**Step 1:  $B$  is well-defined and bilinear**

Let  $f, g \in H$  with  $\{f_j\} = \varphi(f)$  and  $\{g_j\} = \varphi(g)$ . Then the series for  $B(f, g)$  converges absolutely because

$$\sum_{j \neq 0} |(ij + j^2) f_j \bar{g}_j| \leq \left( \sum_{j \neq 0} j \sqrt{1 + j^2} |f_j|^2 \right)^{\frac{1}{2}} \left( \sum_{j \neq 0} j \sqrt{1 + j^2} |g_j|^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \|f\|_H \|g\|_H < \infty \quad (44)$$

by the Cauchy-Schwarz inequality (note that  $|ij + j^2| = j \sqrt{1 + j^2} \leq \sqrt{2} j^2$  for all  $j \neq 0$ ).

Let  $f, g, h \in H$ , and let  $\alpha, \beta \in \mathbf{R}$ . Set  $\{f_j\} = \varphi(f)$ ,  $\{g_j\} = \varphi(g)$ , and  $\{h_j\} = \varphi(h)$ . By the linearity of  $\varphi$ , we have  $\varphi(\alpha f + \beta g) = \alpha \varphi(f) + \beta \varphi(g)$ ; therefore,

$$B(\alpha f + \beta g, h) = \sum_{j \neq 0} (ij + j^2) (\alpha f_j + \beta g_j) \bar{h}_j = \alpha \sum_{j \neq 0} (ij + j^2) f_j \bar{h}_j + \beta \sum_{j \neq 0} (ij + j^2) g_j \bar{h}_j \quad (45)$$

$$= \alpha B(f, h) + \beta B(g, h). \quad (46)$$



Similarly,

$$B(h, \alpha f + \beta g) = \sum_{j \neq 0} (ij + j^2) h_j \overline{(\alpha f_j + \beta g_j)} = \alpha \sum_{j \neq 0} (ij + j^2) h_j \bar{f}_j + \beta \sum_{j \neq 0} (ij + j^2) h_j \bar{g}_j \quad (47)$$

$$= \alpha B(h, f) + \beta B(h, g). \quad (48)$$

Thus,  $B$  is bilinear.

**Step 2:  $B$  is continuous and coercive**

We have practically already shown that  $B$  is continuous: by (44),

$$|B(f, g)| \leq \sqrt{2} \|f\|_H \|g\|_H \quad (49)$$

for all  $f, g \in H$  with  $\{f_j\} = \varphi(f)$  and  $\{g_j\} = \varphi(g)$ . This implies that  $B$  is continuous because  $B$  is bilinear.

For coercivity, observe that

$$|B(f, f)| = \left| \sum_{j \neq 0} (ij + j^2) |f_j|^2 \right| = \left( \left[ \sum_{j \neq 0} j |f_j|^2 \right]^2 + \left[ \sum_{j \neq 0} j^2 |f_j|^2 \right]^2 \right)^{\frac{1}{2}} \geq \|f\|_H^2. \quad (50)$$

Thus,  $B$  is coercive (note that the first summation converges absolutely by a comparison test, by the way:  $|j| |f_j|^2 \leq j^2 |f_j|^2$  for all integers  $j \neq 0$ ).  $\square$

**2.3)**  $H^{-1} \subseteq H^*$  if we assign the following action to  $f \in H^{-1}$ :

$$f(g) = \sum_{j \neq 0} f_j \bar{g}_j, \quad g \in H, \quad (51)$$

where  $\{g_j\} = \varphi(g)$ , and  $\{f_j\} = \psi(f)$ .

*Proof.* Let  $f \in H^{-1}$ . Note that, because  $H$  is a real Hilbert space, an element of  $H^*$  should be a real-valued, bounded linear functional. Thanks to the isomorphisms  $\varphi$  and  $\psi$ , the action of  $f \in H^{-1}$  is well-defined, but we still need to show that the series converges to a real number. Then, we need to show that  $f$  is linear over  $\mathbf{R}$  and bounded. Then we will have  $f \in H^*$ , completing the proof.

**Step 1:  $f(g)$  converges to a real number**

Let  $f \in H^{-1}$  and  $g \in H$ , with  $\{f_j\} = \psi(f)$ , and  $\{g_j\} = \varphi(g)$ . Then the series for  $f(g)$  converges absolutely because

$$\sum_{j \neq 0} |f_j| |\bar{g}_j| \leq \left( \sum_{j \neq 0} j^{-2} |f_j|^2 \right)^{\frac{1}{2}} \left( \sum_{j \neq 0} j^2 |g_j|^2 \right)^{\frac{1}{2}} = \|f\|_{H^{-1}} \cdot \|g\|_H \quad (52)$$

by the Cauchy-Schwarz inequality.

Next,  $f(g) \in \mathbf{R}$  because

$$\overline{f(g)} = \overline{\sum_{j \neq 0} f_j \bar{g}_j} = \sum_{j \neq 0} \bar{f}_j g_j = \sum_{j \neq 0} f_{-j} \bar{g}_{-j} = \sum_{j \neq 0} f_j \bar{g}_j = f(g), \quad (53)$$

and only real numbers equal their own complex conjugate.

**Step 2:  $f$  is linear and bounded**

Let  $f \in H^{-1}$  and  $g, h \in H$  with  $\{f_j\} = \psi(f)$ , and  $\{g_j\} = \varphi(g)$ ,  $\{h_j\} = \varphi(h)$ . Let  $\alpha, \beta \in \mathbf{R}$ . Then, by the linearity of  $\varphi$ , we have  $\varphi(\alpha g + \beta h) = \alpha \varphi(g) + \beta \varphi(h)$ . Hence,

$$f(\alpha g + \beta h) = \sum_{j \neq 0} f_j \overline{(\alpha g_j + \beta h_j)} = \alpha \sum_{j \neq 0} f_j \bar{g}_j + \beta \sum_{j \neq 0} f_j \bar{h}_j = \alpha f(g) + \beta f(h). \quad (54)$$

Therefore,  $f$  is linear.

We have practically already shown that  $f$  is bounded. By (52), we have

$$|f(g)| \leq \|f\|_{H^{-1}} \cdot \|g\|_H \quad (55)$$

for all  $g \in G$ . Therefore,  $f$  is bounded.

Hence,  $f \in H^*$  for all  $f \in H^{-1}$ , using the action we have assigned to  $f$  to view it as a functional on  $H$ . Thus,  $H^{-1} \subseteq H^*$ .  $\square$

**2.4)** For every  $f \in H^{-1}$ , there exists  $u \in H$  such that

$$B(x, u) = f(x) \quad \text{for all } x \in H. \quad (56)$$

*Proof.* After all the work on the previous parts, this is a simple application of the Lax-Milgram theorem. Since  $B$  satisfies the assumptions of the theorem by 2.2, it follows by the Lax-Milgram theorem that for all  $f \in H^*$ , there exists  $u \in H$  such that (56) is true. Since  $H^{-1} \subseteq H^*$  by 2.3, given  $f \in H^{-1}$ , we have  $f \in H^*$ , so we can find  $u \in H$  such that (56) holds.  $\square$

**2.5)** Suppose that  $u$  satisfies (56). Then, formally,  $u$  solves the ODE

$$-(u' + u'') = f. \quad (57)$$

*Proof.* Set  $\{u_j\} = \varphi(u)$ , and  $\{f_j\} = \psi(f)$ . Formally,

$$u'(x) = \sum_{j \neq 0} i j u_j e^{ijx}, \quad u''(x) = - \sum_{j \neq 0} j^2 u_j e^{ijx}. \quad (58)$$

Define  $\{u'_j\} = \{i j u_j\}$ , and  $\{u''_j\} = \{-j^2 u_j\}$ . Then  $\{u'_j\}, \{u''_j\} \in S_{H^{-1}}$  because  $\{u_j\} \in S_H$  implies that

$$\sum_{j \neq 0} j^{-2} |i j u_j|^2 \leq \sum_{j \neq 0} j^2 |u_j|^2 < \infty, \quad \sum_{j \neq 0} j^{-2} |-j^2 u_j|^2 \leq \sum_{j \neq 0} j^2 |u_j|^2 < \infty, \quad (59)$$

and  $\overline{u'_{-j}} = \overline{-i j u_{-j}} = i j u_j = u'_j$ , and  $\overline{u''_{-j}} = \overline{-j^2 u_{-j}} = -j^2 u_j = u''_j$ .

Thus, for all  $x \in H$ , if  $\{x_j\} = \varphi(x)$ , then

$$\sum_{j \neq 0} f_j \bar{x}_j = f(x) = B(x, u) = \sum_{j \neq 0} (i j + j^2) x_j \bar{u}_j = \sum_{j \neq 0} (-i j + j^2) x_{-j} \bar{u}_{-j} = - \sum_{j \neq 0} (u'_j + u''_j) \bar{x}_j. \quad (60)$$

We can choose  $\{x_j\} \in S_H$  such that  $x_j = 0$  if  $j \neq \pm k$  and  $x_{\pm k} = 1$ . Then (60) implies that

$$f_{-k} + f_k = -(u'_{-k} + u'_k + u''_{-k} + u''_k). \quad (61)$$

We can also choose  $\{x_j\} \in S_H$  such that  $x_j = 0$  if  $j \neq \pm k$  and  $x_{\pm k} = \pm i$ . Then (60) implies that

$$-f_{-k} + f_k = -(-u'_{-k} + u'_k - u''_{-k} + u''_k). \quad (62)$$

Together, (61) and (62) imply that  $f_j = -(u'_j + u''_j)$  for all  $j \neq 0$ . This implies that  $f = -(u' + u'')$  by the linearity of  $\psi$ .  $\square$

This can be made into a classical ODE solution by adding assumptions. For example, if  $u \in C^2[-\pi, \pi]$ , and  $u$  is periodic, then the Fourier series of  $u$  and  $u'$  are continuous, and  $u'$  and  $u''$  are piecewise smooth, meaning that the Fourier series of  $u$  and  $u'$  may be differentiated term-by-term.

2.6) Any bounded set in  $H$  is pre-compact in  $L^2(-\pi, \pi)$ .

*Proof.* Let  $A$  be a bounded set in  $H$ ; that is, there is some  $M > 0$  such that  $\|g\|_H^2 \leq M$  for all  $g \in A$ . We recall that  $\{e_j\}$  is an orthonormal basis in  $L^2(-\pi, \pi)$ . Hence, for any  $f \in L^2(-\pi, \pi)$  (see Arbogast and Bona, Theorem 3.18 part (iii)),

$$\|f\|_{L^2(-\pi, \pi)}^2 = \sum_j |f_j|^2, \quad f_j = (f, e_j). \quad (63)$$

If  $f \in H$  as well, then  $\{f_j\} = \varphi(f)$ , so

$$\|f\|_{L^2(-\pi, \pi)}^2 = \sum_j |f_j|^2 \leq \sum_{j \neq 0} j^2 |f_j|^2 = \|f\|_H^2. \quad (64)$$

This implies that  $A$  is bounded in  $L^2(-\pi, \pi)$  in addition to  $H$ . Then  $\overline{A}$  is also bounded in  $L^2(-\pi, \pi)$ , because the closure of a bounded set is bounded. We need to show that  $\overline{A}$  is also compact.

To this end, let  $\{f^n\}$  be a sequence in  $\overline{A}$ . Since  $\overline{A}$  is bounded in  $L^2(-\pi, \pi)$ , and  $L^2(-\pi, \pi)$  is a Hilbert space, there exists a weakly convergent subsequence  $\{f^{n_k}\}$  of  $\{f^n\}$ . That is, there exists  $f \in L^2(-\pi, \pi)$  such that  $f_j^{n_k} \rightarrow f_j$  for all  $j$ , where  $f_j^{n_k} = (f^{n_k}, e_j)$ , and  $f_j = (f, e_j)$ . If we can show that  $f^{n_k} \rightarrow f$  strongly as well, then we are done.

Before that, however, we need the following fact: for all  $g \in A$  and all  $J > 0$ ,

$$\sum_{|j| > J} |g_j|^2 \leq \frac{1}{J^2} \sum_{|j| > J} j^2 |g_j|^2 \leq \frac{M}{J^2}, \quad \{g_j\} = \varphi(g). \quad (65)$$

Therefore, given  $\varepsilon > 0$ , we can choose  $J > 0$  such that for all  $g \in A$ ,

$$\sum_{|j| > J} |g_j|^2 < \varepsilon, \quad \{g_j\} = \varphi(g). \quad (66)$$

Now we show that  $\{f^{n_k}\}$  converges strongly. Since  $L^2(-\pi, \pi)$  is complete, we only need to show  $\{f^{n_k}\}$  is Cauchy in  $\|\cdot\|_{L^2(-\pi, \pi)}$ .

Let  $\varepsilon > 0$  be given. Then we can choose  $J > 0$  such that for all  $g \in A$ ,

$$\sum_{|j| > J} |g_j|^2 < \varepsilon, \quad \{g_j\} = \varphi(g). \quad (67)$$

Since  $\{f_j^{n_k}\}_{k=1}^\infty$  is convergent for all  $j$ , it is also Cauchy for all  $j$ . Thus, we can choose  $K$  large enough that  $k, \ell > K$  implies that  $|f_j^{n_k} - f_j^{n_\ell}|^2 < \frac{\varepsilon}{2J+1}$  for all  $0 \leq |j| \leq J$ . Then

$$\|f^{n_k} - f^{n_\ell}\|_{L^2(-\pi, \pi)}^2 = \sum_{|j| \leq J} |f_j^{n_k} - f_j^{n_\ell}|^2 + \sum_{|j| > J} |f_j^{n_k} - f_j^{n_\ell}|^2 \leq \varepsilon + \sum_{|j| > J} |f_j^{n_k} - f_j^{n_\ell}|^2 \quad (68)$$

if  $k, \ell > K$ . Since  $f^{n_k} \in \overline{A}$ , for all  $k$ , we can choose  $g^k \in A$  such that  $\|f^{n_k} - g^k\|_{L^2(-\pi, \pi)}^2 < \varepsilon$ . Hence,

if  $g_j^k = \varphi(g^k)$ , then

$$\|f^{n_k} - f^{n_\ell}\|_{L^2(-\pi, \pi)}^2 \leq \varepsilon + \sum_{|j| > J} |(f_j^{n_k} - g_j^k) + (g_j^k - g_j^\ell) + (g_j^\ell - f_j^{n_\ell})|^2 \quad (69)$$

$$\leq \varepsilon + 3 \sum_{|j| > J} |f_j^{n_k} - g_j^k|^2 + 3 \sum_{|j| > J} |f_j^{n_\ell} - g_j^\ell|^2 + 3 \sum_{|j| > J} |g_j^k - g_j^\ell|^2 \quad (70)$$

$$\leq \varepsilon + 3\|f^{n_k} - g^k\|_{L^2(-\pi, \pi)}^2 + 3\|f^{n_\ell} - g^\ell\|_{L^2(-\pi, \pi)}^2 \quad (71)$$

$$+ 6 \sum_{|j| > J} |g_j^k|^2 + 6 \sum_{|j| > J} |g_j^\ell|^2 \quad (72)$$

$$\leq 19\varepsilon \quad (73)$$

if  $k, \ell > K$ . This implies that  $\{f^{n_k}\}$  is Cauchy in  $\|\cdot\|_{L^2(-\pi, \pi)}$ . Therefore,  $\{f^{n_k}\}$  converges in  $L^2(-\pi, \pi)$ .

Thus, any sequence in  $\overline{A}$  has a convergent subsequence. This implies that  $\overline{A}$  is compact, and  $A$  is pre-compact.  $\square$