# Math 6108 Homework 1

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**Problem 1.**

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, unit vectors, then $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal.

*Proof.* Since

$$\begin{aligned}(\mathbf{u}+\mathbf{v},\mathbf{u}-\mathbf{v}) &= (\mathbf{u},\mathbf{u}) - (\mathbf{u},\mathbf{v}) + (\mathbf{v},\mathbf{u}) - (\mathbf{v},\mathbf{v}) \\ &= \|\mathbf{u}\|^2 - (\mathbf{u},\mathbf{v}) + (\mathbf{u},\mathbf{v}) - \|\mathbf{v}\|^2 = 1 - 1 = 0,\end{aligned}$$

it follows that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal. $\square$

**Problem 2.**

**Problem 3.**

Let $U$ and $V$ be $n \times n$ unitary matrices. Then $UV$ is an $n \times n$ unitary matrix.

*Proof.* It suffices to show that $(UV)^*(UV) = I$. This is the case because

$$(UV)^*(UV) = V^*U^*UV = V^*IV = V^*V = I$$

by the unitarity of $U$ and $V$. $\square$

**Problem 4.**

Let $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ be $n \times n$ matrices. Suppose that $AB = \{c_{ij}\}$, and $BA = \{d_{ij}\}$. Then, by definition,

$$c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}, \qquad d_{ij} = \sum_{k=1}^n b_{ik}a_{kj}.$$

Hence, by the definition of trace,

$$\mathrm{Tr}(AB) = \sum_{i=1}^n c_{ii} = \sum_{i=1}^n\sum_{k=1}^n a_{ik}b_{ki} = \sum_{k=1}^n\sum_{i=1}^n b_{ki}a_{ik} = \sum_{k=1}^n d_{kk} = \mathrm{Tr}(BA).$$

**Problem 5.**

Let $A \in \mathbb{C}^{n\times n}$ be a matrix whose columns $\{\mathbf{a}_i\}_{i=1}^n$ form an orthogonal set.

1. $A^*A$ is a diagonal matrix.

*Proof.* Since

$$A^*A = \begin{bmatrix} \mathbf{a}_1^* \\ \mathbf{a}_2^* \\ \vdots \\ \mathbf{a}_n^* \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix},$$

if $b_{ij}$ is the entry of $A^*A$ in the $i$th row and $j$th column, then $b_{ij} = \mathbf{a}_i^* \mathbf{a}_j = (\mathbf{a}_i, \mathbf{a}_j) = 0$ if $i \neq j$. Thus, $A^*A$ is diagonal. □

2. $AA^*$ is not necessarily diagonal.

*Proof.* Suppose that

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}.$$

Then $(\mathbf{a}_1, \mathbf{a}_2) = 1 \cdot 2 - 1 \cdot 2 = 0$, so the columns of $A$ form an orthogonal set, but

$$AA^* = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix},$$

which is clearly not diagonal. □