Math 5604 Homework 5 and 6

Jacob Hauck

March 31, 2024

Problem 1.

Consider the IVP

$$y'' + x^2 y = (x^2 - 4)\sin(2x), \qquad x > 0$$
$$y(0) = 0, \quad y'(0) = 2.$$

In order to solve this IVP numerically, we rewrite it as a system of ODEs by defining $z = y'$. Then we can equivalently solve

$$\begin{aligned} y' &= z \\ z' &= -x^2 y + (x^2 - 4)\sin(2x) \\ y(0) = 0, &\qquad z(0) = 2. \end{aligned}$$

For all numerical solutions, we approximation $y(x_n)$ and $z(x_n)$ by y^n and z^n at the points $\{x_n\}_{n=0}^N$, which are evenly spaced on $[0, 1]$ by $k = \frac{1}{N}$.

(a) As the BDF2 method is a two-step method, we need to obtain y^1 and z^1 before we can start the main iteration. For this we can use the backward Euler method, which has second-order local truncation error to match the second-order global truncation error of the BDF2 method. This leads to the following implicit scheme

$$\begin{aligned} y^{n+1} &= \frac{1}{3}\left[4y^n - y^{n-1} + 2kz^{n+1}\right] && n = 1, 2, \dots, N-1 \\ z^{n+1} &= \frac{1}{3}\left[4z^n - z^{n-1} + 2k(-x_{n+1}^2 y^{n+1} + (x_{n+1}^2 - 4)\sin(2x_{n+1}))\right] && n = 1, 2, \dots, N-1 \\ y^1 &= y^0 + kz^1 \\ z^1 &= z^0 + k(-x_1^2 y^1 + (x_1^2 - 4)\sin(2x_1)) \\ y^0 &= 0 \\ z^0 &= 2. \end{aligned}$$

Since the original equation is linear, we can easily solve the implicit equations above to obtain the following equivalent, explicit scheme

$$\begin{aligned} z^{n+1} &= \frac{\frac{1}{3}\left[4z^n - z^{n-1} + 2k\left[-\frac{1}{3}x_{n+1}^2(4y^n - y^{n-1}) + (x_{n+1}^2 - 4)\sin(2x_{n+1})\right]\right]}{1 + \frac{4k^2}{9}x_{n+1}^2} && n = 1, 2, \dots, N-1 \\ y^{n+1} &= \frac{1}{3}\left[4y^n - y^{n-1} + 2kz^{n+1}\right] && n = 1, 2, \dots, N-1 \\ z^1 &= \frac{z^0 + k(-x_1^2 y^0 + (x_1^2 - 4)\sin(2x_1))}{1 + k^2 x_1^2} \\ y^1 &= y^0 + kz^1 \\ y^0 &= 0 \\ z^0 &= 2. \end{aligned}$$

(b) In Table 1 are the errors and convergences rates of the method from part (a). The table shows that the method empirically has convergence rate of 2, as expected theoretically.

h	Error	Rate
1/8	8.457332e-02	-
1/16	2.133647e-02	1.986881
1/32	5.319670e-03	2.003913
1/64	1.325627e-03	2.004662
1/128	3.307156e-04	2.003012
1/256	8.258288e-05	2.001677

Table 1: Errors at $t = 1$ with convergence rates using the BDF2 method in (a)

(c) Since the TR-BDF2 method is a one-step method, we can apply the method immediately to obtain the following implicit scheme

$$\begin{aligned}
y_*^{n+1} &= y^n + \frac{k}{4}\left[z^n + z_*^{n+1}\right] \\
z_*^{n+1} &= z^n + \frac{k}{4}\left[-x_n^2 y^n + (x_n^2 - 4)\sin(2x_n) - x_{n+1/2}^2 y_*^{n+1} + (x_{n+1/2}^2 - 4)\sin(2x_{n+1/2})\right] \\
y^{n+1} &= \frac{1}{3}\left[4y_*^{n+1} - y^n + kz^{n+1}\right] \\
z^{n+1} &= \frac{1}{3}\left[4z_*^{n+1} - z^n + k\left[-x_{n+1}^2 y^{n+1} + (x_{n+1}^2 - 4)\sin(2x_{n+1})\right]\right] \\
&\quad \text{for } n = 0, 1, \dots, N-1, \text{ and} \\
y^0 &= 0 \\
z^0 &= 2,
\end{aligned}$$

where $x_{n+1/2} = x_n + \frac{k}{2}$. As in part (a), we can solve this scheme to obtain an equivalent explicit scheme

$$\begin{aligned}
z_*^{n+1} &= \frac{z^n + \frac{k}{4}\left[-x_n^2 y^n + (x_n^2 - 4)\sin(2x_n) - x_{n+1/2}^2\left(y^n + \frac{k}{4}z^n\right) + (x_{n+1/2}^2 - 4)\sin(2x_{n+1/2})\right]}{1 + \frac{k^2}{16}x_{n+1/2}^2} \\
y_*^{n+1} &= y^n + \frac{k}{4}\left[z^n + z_*^{n+1}\right] \\
z^{n+1} &= \frac{\frac{1}{3}\left[4z_*^{n+1} - z^n + k\left[-\frac{x_{n+1}^2}{3}\left[4y_*^{n+1} - y^n\right] + (x_{n+1}^2 - 4)\sin(2x_{n+1})\right]\right]}{1 + \frac{k^2}{9}x_{n+1}^2} \\
y^{n+1} &= \frac{1}{3}\left[4y_*^{n+1} - y^n + kz^{n+1}\right] \\
&\quad \text{for } n = 0, 1, \dots, N-1, \text{ and} \\
y^0 &= 0 \\
z^0 &= 2.
\end{aligned}$$

(d) In Table 2 are the errors and convergences rates of the method from part (c). The table shows that the method empirically has convergence rate of 2, as expected theoretically.

Problem 2.

h	Error	Rate
1/8	5.161865e-04	-
1/16	1.369409e-04	1.914339
1/32	3.531733e-05	1.955105
1/64	8.970516e-06	1.977114
1/128	2.260645e-06	1.988456
1/256	5.674363e-07	1.994204

Table 2: Errors at $t = 1$ with convergence rates using the TR-BDF2 method in (c)

Consider the BVP

$$\begin{gathered} y'' + x^2 y = (x^2 - 4)\sin(2x), \qquad 0 < x < \pi \\ y(0) = 0, \qquad y'(\pi) + 2y(\pi) = 2. \end{gathered}$$

For all numerical solutions, we approximation $y(x_n)$ by y_n at the points $\{x_n\}_{n=0}^N$, which are evenly spaced on $[0, 1]$ by $h = \frac{1}{N}$.

(a) Using the centered difference method to approximate y'' on the interior of the domain, we get the following scheme for the interior points $y_1, y_2, \dots y_{N-1}$

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + x_n^2 y_n = (x_n^2 - 4)\sin(2x_n), \qquad n = 1, 2, \dots, N-1.$$

The left boundary condition gives the discrete condition $y_0 = 0$, but the right boundary condition involves the first order derivative y'; to approximate this with a centered difference, we would need a point $x_{N+1} = x_N + h$ outside of the domain (assuming that y' can be continuously extended, giving us the approximation $y_{N+1} \approx y(x_{N+1})$). By enforcing the differential equation at the point x_N, we can obtain another equation involving the point x_{N+1}, which we can combine with the boundary condition to eliminate the need for information at x_{N+1}, as follows:

$$\begin{aligned} \frac{y_{N+1} - y_{N-1}}{2h} + 2y_N &= 2 \qquad \text{(right boundary condition)} \\ \frac{y_{N+1} - 2y_N + y_{N-1}}{h^2} + x_N^2 y_N &= (x_N^2 - 4)\sin(2x_N) \qquad \text{(equation at } x_N\text{)} \end{aligned}$$

Eliminating y_{N+1} gives

$$\frac{2y_N - 2y_{N-1} + h^2\left[-x_N^2 y_N + (x_N^2 - 4)\sin(2x_N)\right]}{2h} + 2y_N = 2.$$

Substituting the explicit condition $y_0 = 0$ into the $n = 1$ equation and collecting all our equations together, we obtain the scheme

$$\begin{aligned} \left(x_1^2 - \frac{2}{h^2}\right) y_1 + \frac{1}{h^2} y_2 &= (x_1^2 - 4)\sin(2x_1) \\ \frac{1}{h^2} y_{n-1} + \left(x_n^2 - \frac{2}{h^2}\right) y_n + \frac{1}{h^2} y_{n+1} &= (x_n^2 - 4)\sin(2x_n), \qquad n = 2, 3, \dots, N-1 \\ -\frac{1}{h} y_{N-1} + \left(\frac{1}{h} - \frac{h x_N^2}{2} + 2\right) y_N &= 2 - \frac{h}{2}(x_N^2 - 4)\sin(2x_N). \end{aligned}$$

We can write this system of equations in matrix-vector form $Ay = b$, where

$$A = \begin{bmatrix} x_1^2 - \frac{2}{h^2} & \frac{1}{h^2} & & & & \\ \frac{1}{h^2} & x_2^2 - \frac{2}{h^2} & \frac{1}{h^2} & & & \\ & \frac{1}{h^2} & x_3^3 - \frac{2}{h^2} & \frac{1}{h^2} & & \\ & & \ddots & & & \\ & & \frac{1}{h^2} & x_{N-1}^2 - \frac{2}{h^2} & \frac{1}{h^2} & \\ & & & -\frac{1}{h} & \frac{1}{h} - \frac{hx_N^2}{2} + 2 & \end{bmatrix},$$

where empty entries are assumed to be 0, and

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, \qquad b = \begin{bmatrix} (x_1^2 - 4)\sin(2x_1) \\ (x_2^2 - 4)\sin(2x_2) \\ \vdots \\ (x_{N-1}^2 - 4)\sin(2x_{N-1}) \\ 2 - \frac{h}{2}(x_N^2 - 4)\sin(2x_N) \end{bmatrix}.$$

(b) In Table 3 are the ℓ^2 and ℓ^∞ errors of the scheme in part (a). Based on the table, the ℓ^∞ convergence rate is 2, and the ℓ^2 convergence rate is 1.5, which is not surprising given the equivalence of the ℓ^2 and ℓ^∞ norms:

$$\|u\|_{\ell^\infty} \le \|u\|_{\ell^2} \le \sqrt{N}\|u\|_{\ell^\infty} \qquad \text{for all } u \in \mathbf{R}^N.$$

Indeed, let $e \in \mathbf{R}^{N+1}$ be a vector of errors defined by $e_n = |y_n - y(x_n)|$. If $\|e\|_{\ell^\infty} \approx h^2$, then the right inequality implies that $\|e\|_{\ell^2} \lesssim \left(\sqrt{N+1}\right) h^2 \approx h^{1.5}$ because $N \approx h^{-1}$.

h	ℓ^2 error	ℓ^2 rate	ℓ^∞ error	ℓ^∞ rate
$\pi/8$	2.219190e-01	-	1.280149e-01	-
$\pi/16$	6.790030e-02	1.708543	2.905806e-02	2.139301
$\pi/32$	2.293204e-02	1.566053	6.987220e-03	2.056148
$\pi/64$	7.996325e-03	1.519956	1.727129e-03	2.016342
$\pi/128$	2.815873e-03	1.505755	4.305031e-04	2.004281
$\pi/256$	9.943666e-04	1.501732	1.075450e-04	2.001083

Table 3: Centered difference – ℓ^2 and ℓ^∞ errors with convergence rates

(c) Using the centered difference method to approximate y'' on the interior of the domain, we get the following scheme for the interior points $y_1, y_2, \dots y_{N-1}$

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + x_n^2 y_n = (x_n^2 - 4)\sin(2x_n), \qquad n = 1, 2, \dots, N-1.$$

The left boundary condition gives the discrete condition $y_0 = 0$, but the right boundary condition involves the first order derivative y'; to approximate this with a second-order, one-sided method, we recall from class that, for a function $u(t)$,

$$u'(t) = \frac{-3u(t) + 4u(t+k) - u(t+2k)}{2k} + \mathcal{O}(h^2).$$

Taking $u = y$, $k = -h$, and $t = \pi$, this implies that

$$y'(\pi) = \frac{3y(\pi) - 4y(\pi - h) + y(\pi - 2h)}{2h} + \mathcal{O}(h^2).$$

This leads to the second-order, one-sided discretization of the right boundary condition

$$\frac{3y_N - 4y_{N-1} + y_{N-2}}{2h} + 2y_N = 2.$$

Combining the left boundary condition with the first interior equation, we have the scheme

$$\begin{aligned}
\left(x_1^2 - \frac{2}{h^2}\right) y_1 + \frac{1}{h^2} y_2 &= (x_1^2 - 4)\sin(2x_1) \\
\frac{1}{h^2} y_{n-1} + \left(x_n^2 - \frac{2}{h^2}\right) y_n + \frac{1}{h^2} y_{n+1} &= (x_n^2 - 4)\sin(2x_n), \qquad n = 2, 3, \dots, N-1 \\
\frac{1}{2h} y_{N-2} - \frac{2}{h} y_{N-1} + \left(2 + \frac{3}{2h}\right) y_N &= 2.
\end{aligned}$$

This system of equations can be written in matrix-vector form $Ay = b$, where

$$A = \begin{bmatrix} x_1^2 - \frac{2}{h^2} & \frac{1}{h^2} & & & \\ \frac{1}{h^2} & x_2^2 - \frac{2}{h^2} & \frac{1}{h^2} & & \\ & \frac{1}{h^2} & x_3^2 - \frac{2}{h^2} & \frac{1}{h^2} & \\ & & \ddots & & \\ & & \frac{1}{h^2} & x_{N-1}^2 - \frac{2}{h^2} & \frac{1}{h^2} \\ & & \frac{1}{2h} & -\frac{2}{h} & 2 + \frac{3}{2h} \end{bmatrix},$$

where blank entries are assumed to be 0, and

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, \qquad b = \begin{bmatrix} (x_1^2 - 4)\sin(2x_1) \\ (x_2^2 - 4)\sin(2x_2) \\ \vdots \\ (x_{N-1}^2 - 4)\sin(2x_{N-1}) \\ 2 \end{bmatrix}.$$

(d) In Table 4 are the ℓ^2 and ℓ^∞ errors of the scheme in part (a). Based on the table, it seems that the ℓ^∞ convergence rate is 2, and the ℓ^2 convergence rate is 1.5 (using a few smaller step sizes showed that this is really the case).

h	ℓ^2 error	ℓ^2 rate	ℓ^∞ error	ℓ^∞ rate
$\pi/8$	3.827128e+00	-	1.973108e+00	-
$\pi/16$	1.390236e+00	1.460932	5.126718e-01	1.944362
$\pi/32$	6.531083e-01	1.089936	1.730366e-01	1.566958
$\pi/64$	2.674238e-01	1.288195	5.028926e-02	1.782755
$\pi/128$	9.931652e-02	1.429022	1.322982e-02	1.926457
$\pi/256$	3.559417e-02	1.480393	3.355599e-03	1.979151

Table 4: One-sided difference – ℓ^2 and ℓ^∞ errors with convergence rates

Problem 3.

Consider the boundary-value problem

$$\begin{aligned} &\varepsilon y'' - x^2 y' - y = 0, \qquad 0 < x < 1 \\ &y(0) = 1, \qquad y(1) = 1, \end{aligned}$$

where $\varepsilon > 0$.

(a) We approximation $y(x_n)$ by y_n at the points $\{x_n\}_{n=0}^N$, which are evenly spaced on $[0,1]$ by $h=\frac{1}{N}$. To handle the boundary conditions, we simply set $y_0=1$ and $y_N=1$. At the interior points, we can use central difference approximations of the derivatives to obtain the equations

$$\varepsilon\frac{y_{n+1}-2y_n+y_{n-1}}{h^2}-x_n^2\frac{y_{n+1}-y_{n-1}}{2h}-y_n=0,\qquad n=1,2,\dots,N-1.$$

Combining the boundary conditions with the first and last of these equations, we obtain the scheme

$$\begin{aligned}
\left(\frac{\varepsilon}{h^2}-\frac{x_1^2}{2h}\right)y_2-\left(\frac{2\varepsilon}{h^2}+1\right)y_1&=-\left(\frac{\varepsilon}{h^2}+\frac{x_1^2}{2h}\right) &&\text{(left BC)}\\
-\left(\frac{2\varepsilon}{h^2}+1\right)y_{N-1}+\left(\frac{\varepsilon}{h^2}+\frac{x_{N-1}^2}{2h}\right)y_{N-2}&=-\left(\frac{\varepsilon}{h^2}-\frac{x_{N-1}^2}{2h}\right) &&\text{(right BC)}\\
\left(\frac{\varepsilon}{h^2}-\frac{x_n^2}{2h}\right)y_{n+1}-\left(\frac{2\varepsilon}{h^2}+1\right)y_n+\left(\frac{\varepsilon}{h^2}+\frac{x_n^2}{2h}\right)y_{n-1}&=0, &&n=2,3,\dots,N-2.
\end{aligned}$$

We can write these equations in matrix-vector form $Ay=b$, where

$$A=\begin{bmatrix}
-\frac{2\varepsilon}{h^2}-1 & \frac{\varepsilon}{h^2}-\frac{x_1^2}{2h} & & & \\
\frac{\varepsilon}{h^2}+\frac{x_2^2}{2h} & -\frac{2\varepsilon}{h^2}-1 & \frac{\varepsilon}{h^2}-\frac{x_2^2}{2h} & & \\
 & \frac{\varepsilon}{h^2}+\frac{x_3^2}{2h} & -\frac{2\varepsilon}{h^2}-1 & \frac{\varepsilon}{h^2}-\frac{x_3^2}{2h} & \\
 & & \ddots & & \\
 & & \frac{\varepsilon}{h^2}+\frac{x_{N-2}^2}{2h} & -\frac{2\varepsilon}{h^2}-1 & \frac{\varepsilon}{h^2}-\frac{x_{N-2}^2}{2h}\\
 & & & \frac{\varepsilon}{h^2}+\frac{x_{N-1}^2}{2h} & -\frac{2\varepsilon}{h^2}-1
\end{bmatrix},$$

where blank entries are assumed to be 0, and

$$y=\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_{N-1}\end{bmatrix},\qquad b=\begin{bmatrix}-\left(\frac{\varepsilon}{h^2}+\frac{x_1^2}{2h}\right)\\ 0\\ \vdots\\ 0\\ -\left(\frac{\varepsilon}{h^2}-\frac{x_{N-1}^2}{2h}\right)\end{bmatrix}.$$

(b) In Figure 1 is the "exact" solution for $\varepsilon=.05$ and $h=1/2048$.

(c) In Table 5 are the ℓ^2 and ℓ^∞ errors (computed using the reference solution from (b)) with $\varepsilon=.05$. From the table, it appears that the centered difference method has a convergence rate of 2 in ℓ^∞ and 1.5 in ℓ^2, just as it did on Problem 2.

h	ℓ^2 error	ℓ^2 rate	ℓ^∞ error	ℓ^∞ rate
1/8	8.952792e-02	-	8.748680e-02	-
1/16	3.516391e-02	1.348242	2.969501e-02	1.558845
1/32	1.217098e-02	1.530651	6.755169e-03	2.136157
1/64	4.237348e-03	1.522211	1.719463e-03	1.974034
1/128	1.486917e-03	1.510838	4.260417e-04	2.012891
1/256	5.188923e-04	1.518817	1.051005e-04	2.019226

Table 5: Centered difference method with $\varepsilon=.05$ – ℓ^2 and ℓ^∞ errors with convergence rates

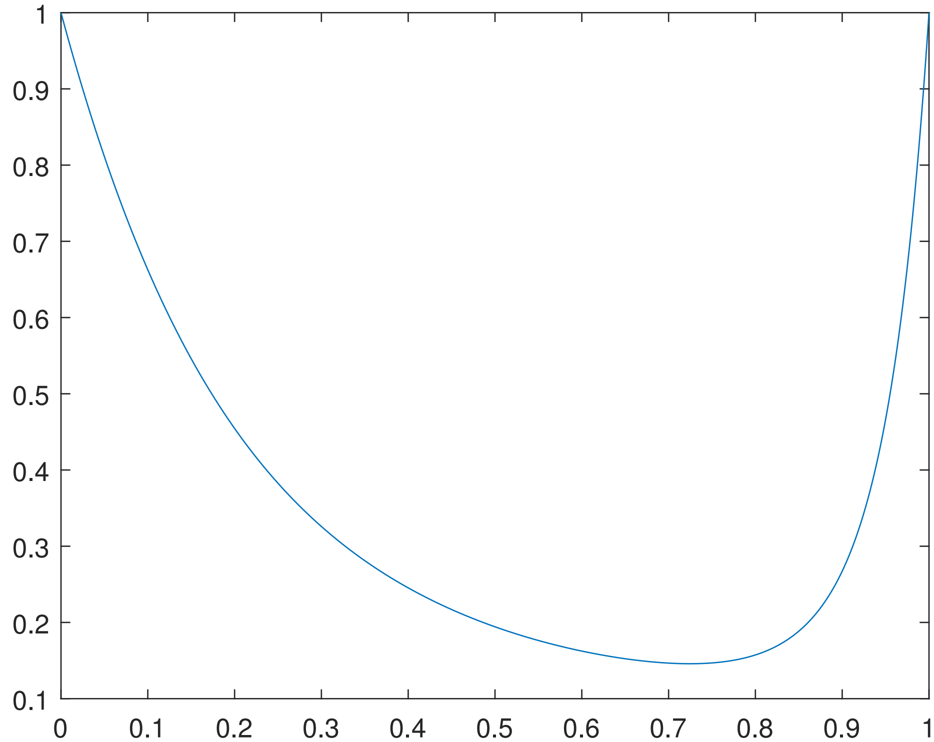

Figure 1: “Exact” solution when $\varepsilon = 0.05$, computed using step size $h = 1/2048$

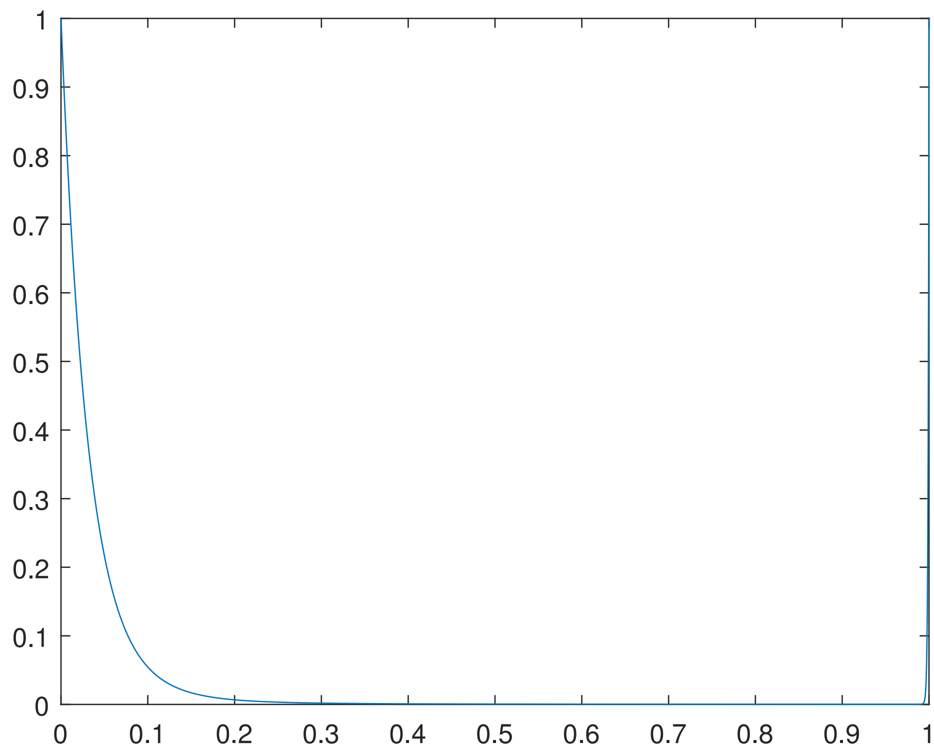

Figure 2: “Exact” solution when $\varepsilon = 0.001$, computed using step size $h = 1/2048$

(d) In Figure 2 is the "exact" solution for $\varepsilon = .05$ and $h = 1/2048$.

(e) In Table 6 are the ℓ^2 and ℓ^∞ errors (computed using the reference solution from (b)) with $\varepsilon = .05$. From the table, it appears that the centered difference method has a convergence rate of 2 in ℓ^∞ and 1.5 in ℓ^2, just as it did for $\varepsilon = 0.05$. The errors, however, are generally greater in this case than they were in the case of $\varepsilon = 0.05$, and the convergence rate doesn't settle down until the step size is already fairly small. This is likely due to the rapid change in the solution near $x = 1$ when ε is small (compare Figures 1 and 2, which show the $\varepsilon = 0.05$ and $\varepsilon = 0.001$ solutions).

h	ℓ^2 error	ℓ^2 rate	ℓ^∞ error	ℓ^∞ rate
1/8	8.839999e-01	-	7.051292e-01	-
1/16	8.168405e-01	0.113992	6.540523e-01	0.108482
1/32	5.408442e-01	0.594841	4.827786e-01	0.438044
1/64	2.517577e-01	1.103177	2.490745e-01	0.954784
1/128	8.763561e-02	1.522447	8.247853e-02	1.594487
1/256	2.856069e-02	1.617486	1.853421e-02	2.153828

Table 6: Centered difference method with $\varepsilon = 0.005$ – ℓ^2 and ℓ^∞ errors with convergence rates