# Math 5601 Homework 6

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**Problem 1.**

We already know the nodes $x_0 = \frac{2a+b}{3}$ and $x_1 = \frac{a+2b}{3}$. To compute the weights $\alpha_0$ and $\alpha_1$ in the two-point open Newton-Cotes formula, we just compute the following integrals:

$$\alpha_0 = \int_a^b L_0(x)\,\mathrm{d}x, \qquad \alpha_1 = \int_a^b L_1(x)\,\mathrm{d}x, \tag{1}$$

where

$$L_k(x) = \prod_{i=0, i\neq k}^{1} \frac{x - x_i}{x_k - x_i}, \qquad k \in \{0, 1\} \tag{2}$$

from the definition of Lagrange polynomials. Then $L_0(x) = \frac{x-x_1}{x_0-x_1}$, and $L_1(x) = \frac{x-x_0}{x_1-x_0}$. Hence

$$\begin{aligned}
\alpha_0 = \int_a^b \frac{x - x_1}{x_0 - x_1}\,\mathrm{d}x &= \frac{1}{2(x_0 - x_1)}(x - x_1)^2\Big|_a^b && (3)\\
&= -\frac{3}{2(b-a)}\left[\frac{(b-a)^2}{9} - \frac{4(b-a)^2}{9}\right] && (4)\\
&= \frac{b-a}{2}, && (5)
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1 = \int_a^b \frac{x - x_0}{x_1 - x_0}\,\mathrm{d}x &= \frac{1}{2(x_1 - x_0)}(x - x_0)^2\Big|_a^b && (6)\\
&= \frac{3}{2(b-a)}\left[\frac{4(b-a)^2}{9} - \frac{(b-a)^2}{9}\right] && (7)\\
&= \frac{b-a}{2}. && (8)
\end{aligned}$$

This results in the Newton-Cotes formula with two-point open rule

$$\int_a^b f(x)\,\mathrm{d}x \approx \alpha_0 f(x_0) + \alpha_1 f(x_1) = \frac{b-a}{2} f\left(\frac{2a+b}{3}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\right). \tag{9}$$

**Problem 2.**

Suppose that we have a numerical quadrature for $\hat{f}(\hat{x})$ on $[0, 1]$ defined by

$$\hat{J}(\hat{f}) = \int_a^b \hat{f}(\hat{x})\,\mathrm{d}\hat{x} \approx \hat{Q}(\hat{f}) = \sum_{j=0}^{m} \hat{\alpha}_j \hat{f}(\hat{x}_j). \tag{10}$$

If we want to compute the integral of $f(x)$ on $[a, b]$, then we can make the change of variables $x = a+(b-a)\hat{x}$, which is a one-to-one correspondence between $x \in [a, b]$ and $\hat{x} \in [0, 1]$ with $\mathrm{d}x = (b-a)\,\mathrm{d}\hat{x}$, to obtain

$$J(f) = \int_a^b f(x)\,\mathrm{d}x = \int_0^1 f(a + (b-a)\hat{x})(b-a)\,\mathrm{d}\hat{x}. \tag{11}$$

Define $\hat{f}(\hat{x}) = f(a+(b-a)\hat{x})$. Then we obtain a numerical quadrature $Q(f)$ for $J(f)$ by using our numerical quadrature $\hat{Q}(\hat{f})$ to approximate the integral of $\hat{f}$ on $[0, 1]$:

$$\begin{aligned} J(f) = (b-a)\int_0^1 \hat{f}\,\mathrm{d}\hat{x} = (b-a)\hat{Q}(\hat{f}) &= \sum_{j=0}^{m}(b-a)\hat{\alpha}_j\hat{f}(\hat{x}_j) && (12)\\ = \sum_{j=0}^{m}(b-a)\hat{\alpha}_j f(a+(b-a)\hat{x}_j) &= \sum_{j=0}^{m}\alpha_j f(x_j) && (13)\\ =: Q(f), & && (14) \end{aligned}$$

if we define $\alpha_j = (b-a)\hat{\alpha}_j$ and $x_j = a + (b-a)\hat{x}_j$.

**Problem 3.**

We use the given Newton-Cotes formulas to approximate

$$J = \int_1^2 \frac{\cos\left(\frac{\pi}{4}x\right)}{\sin^2\left(\frac{\pi}{4}x\right)}\,\mathrm{d}x. \tag{15}$$

The following table summarizes the output of the MATLAB code I used to perform these calculations – see `output.txt` and `quad.m`. Note that we use the affine transformation process derived in Problem 2 to transform the quadrature nodes given in part (d) for $[0, 1]$ to those for $[1, 2]$.

| Part | Method | Approximation | Actual | Error |
|---|---|---|---|---|
| (a) | Midpoint | 4.4834e-01 | 5.2739e-01 | 7.9052e-02 |
| (b) | Two-Point Open | 4.7203e-01 | 5.2739e-01 | 5.5359e-02 |
| (c) | Simpson | 5.3460e-01 | 5.2739e-01 | 7.2035e-03 |
| (d) | – | 5.3080e-01 | 5.2739e-01 | 3.4091e-03 |

As shown in the table, the two-point open rule has lower error than the midpoint rule (one-point open). Simpson's rule (3-point closed) has even less error, and the rule given in part (d) (four-point closed?) has the smallest error.

Here is the code for `quad.m` (just does numerical quadrature given a function, weights and nodes).

```
function result = quad(f, nodes, weights)

% nodes and weights are 1 x (m + 1) ROW vectors
result = f(nodes) * weights';
```