Math 5601 Homework 7

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Problem 1.

Let x_0, x_1, x_2 and w_0, w_1, w_2 be the nodes and weights of the the three-point Gaussian quadrature for $\int_{-1}^{1} f(x)\,\mathrm{d}x$. Then the quadrature must be exact for $f(x) = x^n$, $n \in \{0, 1, 2, 3, 4, 5\}$. That is,

$$\int_{-1}^{1} x^n\,\mathrm{d}x = \sum_{j=0}^{2} w_j x_j^n, \qquad n \in \{0, 1, 2, 3, 4, 5\}. \tag{1}$$

Since

$$\int_{-1}^{1} x^n\,\mathrm{d}x = \left.\frac{x^{n+1}}{n+1}\right|_{-1}^{1} = \begin{cases} \frac{2}{n+1} & n \text{ even} \\ 0 & n \text{ odd}, \end{cases} \tag{2}$$

we obtain the following system of six equations in the six unknowns x_0, x_1, x_2 and w_0, w_1, w_2:

$$\begin{aligned} 2 &= w_0 + w_1 + w_2 & 0 &= w_0 x_0 + w_1 x_1 + w_2 x_2 \\ \frac{2}{3} &= w_0 x_0^2 + w_1 x_1^2 + w_2 x_2^2 & 0 &= w_0 x_0^3 + w_1 x_1^3 + w_2 x_2^3 \\ \frac{2}{5} &= w_0 x_0^4 + w_1 x_1^4 + w_2 x_2^4 & 0 &= w_0 x_0^5 + w_1 x_1^5 + w_2 x_2^5. \end{aligned}$$

Using the following `solve` command in MATLAB gives the solution of this nonlinear system of equations. Note that the system is symmetric with respect to permutation of the index $j \in \{0, 1, 2\}$. Therefore, MATLAB returns $o(S_3) = 3! = 6$ solutions. Since the quadrature is also symmetric with respect to permutations of the index j, each solution results in the same quadrature, so we just use the first one returned by MATLAB.

```
>> syms x0 x1 x2 w0 w1 w2
>> result = solve(...
2 == w0 + w1 + w2, 0 == w0*x0 + w1*x1 + w2*x2,...
2/3 == w0*x0^2 + w1*x1^2 + w2*x2^2, 0 == w0*x0^3 + w1*x1^3 + w2*x2^3,...
2/5 == w0*x0^4 + w1*x1^4 + w2*x2^4, 0 == w0*x0^5 + w1*x1^5 + w2*x2^5);
>> [result.x0(1), result.x1(1), result.x2(1), result.w0(1), result.w1(1), result.w2(1)]

ans =

[15^(1/2)/5, -15^(1/2)/5, 0, 5/9, 5/9, 8/9]
```

Thus, we get $x_0 = \frac{\sqrt{15}}{5}$, $x_1 = -\frac{\sqrt{15}}{5}$, $x_2 = 0$, and $w_0 = w_1 = \frac{5}{9}$, $w_2 = \frac{8}{9}$.

Problem 2.

Let x_0, x_1, x_2 and w_0, w_1, w_2 be the same as in the previous problem. Let $u_4(x)$ be a polynomial of degree 3 on $[-1, 1]$ that is orthogonal to $\operatorname{span}\{1, x, x^2\}$. Then x_0, x_1, and x_2 are the roots of u_4. We can find such a polynomial using the Gram-Schmidt process on $\{1, x, x^2, x^3\}$.

Let $u_1(x) = 1$. Note that $(u_1, u_1) = 2$, and for any continuous function f, $(f, u_1) = \int_{-1}^1 f(x)\,\mathrm{d}x$. By the Gram-Schmidt process, we obtain $u_2(x)$ orthogonal to $u_1(x)$ via

$$u_2(x) = x - \frac{(x, u_1)}{(u_1, u_1)} u_1(x) = x \tag{3}$$

because $(x, u_1) = \int_{-1}^1 x\,\mathrm{d}x = 0$. Next, we can find u_2 orthogonal to both u_1 and u_2 via

$$u_3(x) = x^2 - \frac{(x^2, u_2)}{(u_2, u_2)} u_2(x) - \frac{(x^2, u_1)}{(u_1, u_1)} u_1(x). \tag{4}$$

The last term is just the constant function $\frac{1}{3}$. As for the second term, note that

$$(u_2, u_2) = \int_{-1}^1 x^2\,\mathrm{d}x = \frac{2}{3}, \qquad (x^2, u_2) = \int_{-1}^1 x^3\,\mathrm{d}x = 0, \tag{5}$$

so $u_3(x) = x^2 - \frac{1}{3}$. Lastly, to obtain $u_4(x)$ of degree 3 and orthogonal to $\operatorname{span}\{1, x, x^2\}$, we use

$$u_4(x) = x^3 - \frac{(x^3, u_3)}{(u_3, u_3)} u_3(x) - \frac{(x^3, u_2)}{(u_2, u_2)} u_2(x) - \frac{(x^3, u_1)}{(u_1, u_1)} u_1(x). \tag{6}$$

Since x^3 is odd, the last term is 0. Since

$$(x^3, u_2) = \int_{-1}^1 x^4\,\mathrm{d}x = \frac{2}{5}, \tag{7}$$

the second term is $\frac{3}{5}x$ (after dividing by the value of (u_2, u_2) from above). Lastly, since $u_3(x)$ is even, $x^3 u_3(x)$ is odd, so $(x^3, u_3) = 0$. This gives

$$u_4(x) = x^3 - \frac{3}{5}x. \tag{8}$$

The roots of u_4, and the nodes of the Gaussian quadrature with three points on $[-1, 1]$, are clearly $x_0 = \sqrt{\frac{3}{5}} = \frac{\sqrt{15}}{5}$, $x_1 = -\sqrt{\frac{3}{5}} = -\frac{\sqrt{15}}{5}$, and $x_2 = 0$, the same as we got in Problem 1.

To obtain the weights, we can now integrate the Lagrange basis polynomials for interpolation at the points x_0, x_1 and x_2. That is,

$$w_0 = \int_{-1}^1 L_0(x)\,\mathrm{d}x = \int_{-1}^1 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}\,\mathrm{d}x = \frac{5}{6}\left[\frac{x^3}{3} - \frac{(x_1 + x_2)x^2}{2} + x_1 x_2 x\right]_{-1}^1 = \frac{5}{9}, \tag{9}$$

and

$$w_1 = \int_{-1}^1 L_1(x)\,\mathrm{d}x = \int_{-1}^1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}\,\mathrm{d}x = \frac{5}{6}\left[\frac{x^3}{3} - \frac{(x_0 + x_2)x^2}{2} + x_0 x_2 x\right]_{-1}^1 = \frac{5}{9}, \tag{10}$$

and

$$w_2 = \int_{-1}^1 L_2(x)\,\mathrm{d}x = \int_{-1}^1 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\,\mathrm{d}x = -\frac{5}{3}\left[\frac{x^3}{3} - \frac{(x_0 + x_1)x^2}{2} + x_0 x_1 x\right]_{-1}^1 \tag{11}$$

$$= \frac{5}{3} \cdot \left(\frac{6}{5} - \frac{2}{3}\right) = 2 - \frac{10}{9} = \frac{8}{9}. \tag{12}$$

These are the same weights that we obtained in Problem 1.

Problem 3.

Let $I(h)$ be an approximation of $\int_a^b f(x)\,\mathrm{d}x$ depending on a parameter h such that the error satisfies

$$I(h) - \int_a^b f(x)\,\mathrm{d}x = c_1 h + c_2 h^2 + \mathcal{O}(h^3) \tag{13}$$

for some constants c_1 and c_2. If $I(h)$, $I\left(\frac{h}{2}\right)$, and $I\left(\frac{h}{3}\right)$ are known, then we can use a linear combination to obtain a third-order ($\mathcal{O}(h^3)$) approximation of the integral:

$$Q(h) = a_1 I(h) + a_2 I\left(\frac{h}{2}\right) + a_3 I\left(\frac{h}{3}\right). \tag{14}$$

Now we just need to determine what a_1, a_2 and a_3 should be so that

$$Q(h) - \int_a^b f(x)\,\mathrm{d}x = \mathcal{O}(h^3). \tag{15}$$

By (13), we have

$$\begin{aligned}
Q(h) - \int_a^b f(x)\,\mathrm{d}x &= a_1 I(h) + a_2 I\left(\frac{h}{2}\right) + a_3 I\left(\frac{h}{3}\right) - \int_a^b f(x)\,\mathrm{d}x \\
&= a_1\left[I(h) - \int_a^b f(x)\,\mathrm{d}x\right] + a_2\left[I\left(\frac{h}{2}\right) - \int_a^b f(x)\,\mathrm{d}x\right] + a_3\left[I\left(\frac{h}{3}\right) - \int_a^b f(x)\,\mathrm{d}x\right] \\
&\quad - (1 - a_1 - a_2 - a_3)\int_a^b f(x)\,\mathrm{d}x \\
&= a_1(c_1 h + c_2 h^2 + \mathcal{O}(h^3)) + a_2\left(\frac{c_1 h}{2} + \frac{c_2 h^2}{4} + \mathcal{O}(h^3)\right) + a_3\left(\frac{c_1 h}{3} + \frac{c_2 h^2}{9} + \mathcal{O}(h^3)\right) \\
&\quad - (1 - a_1 - a_2 - a_3)\int_a^b f(x)\,\mathrm{d}x \\
&= (a_1 + a_2 + a_3 - 1)\int_a^b f(x)\,\mathrm{d}x + \left(a_1 + \frac{a_2}{2} + \frac{a_3}{3}\right)c_1 h + \left(a_1 + \frac{a_2}{4} + \frac{a_3}{9}\right)c_2 h^2 + \mathcal{O}(h^3).
\end{aligned}$$

Thus, the error between $Q(h)$ and the integral is $\mathcal{O}(h^3)$ as long as a_1, a_2, and a_3 are chosen such that

$$1 = a_1 + a_2 + a_3, \tag{16}$$

$$0 = a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3, \tag{17}$$

$$0 = a_1 + \frac{1}{4}a_2 + \frac{1}{9}a_3. \tag{18}$$

Substituting $a_1 = 1 - a_2 - a_3$ from the first equation into the last two, we get the system of equations

$$1 = \frac{1}{2}a_2 + \frac{2}{3}a_3, \tag{19}$$

$$1 = \frac{3}{4}a_2 + \frac{8}{9}a_3. \tag{20}$$

Therefore, $\frac{1}{4}a_2 = -\frac{2}{9}a_3$, so $a_2 = -\frac{8}{9}a_3$. Then $a_3 = \frac{9}{2}$, and $a_2 = -4$. Finally, this gives $a_1 = 1 - a_2 - a_3 = \frac{1}{2}$. Hence,

$$Q(h) = \frac{1}{2}I(h) - 4I\left(\frac{h}{2}\right) + \frac{9}{2}I\left(\frac{h}{3}\right) \tag{21}$$

is an approximation of $\int_a^b f(x)\,\mathrm{d}x$ with $\mathcal{O}(h^3)$ accuracy.