2.25 (i) Use Theorem 2.14 to solve

$$x' = Ax, \quad \text{where } A = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix}.$$

By Theorem 2.14, we need to find eigenvalues/vectors of A:

λ is e.v $\Leftrightarrow$ $\det(A - I\lambda) = 0 \Rightarrow \det\begin{pmatrix} -2-\lambda & 5 \\ -2 & 4-\lambda \end{pmatrix} = 0$

$$\Rightarrow (-2-\lambda)(4-\lambda) + 10 = 0$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\Rightarrow \lambda = 1 \pm i$$

Consider $\lambda = 1+i$.

$Av = \lambda v$ if $\quad -2v_1 + 5v_2 = (1+i)v_1$

or $v_2 = \frac{3+i}{5} v_1$

so $v = \begin{pmatrix} 5 \\ 3+i \end{pmatrix}$ is e.vector for $\lambda = 1+i$.

By Theorem 2.14,

$$x^c = ve^{\lambda t} = \begin{pmatrix} 5 \\ 3+i \end{pmatrix} e^{(1+i)t} \quad \text{solves the eqn.}$$

Then $\operatorname{Re} x^c$ and $\operatorname{Im} x^c$ also solve the eqn.

$$x^c = \begin{pmatrix} 5 \\ i+3 \end{pmatrix} e^t(\cos t + i\sin t) = \begin{pmatrix} 5e^t\cos t \\ 3e^t\cos t - e^t\sin t \end{pmatrix} + i\begin{pmatrix} 5e^t\sin t \\ e^t\cos t + 3e^t\sin t \end{pmatrix}$$

So $\operatorname{Re} x^c = x^{(1)} = \begin{pmatrix} 5\cos t \\ 3\cos t - \sin t \end{pmatrix} e^t$ & $\operatorname{Im} x^c = x^{(2)} = \begin{pmatrix} 5\sin t \\ \cos t + 3\sin t \end{pmatrix} e^t$

solve the eqn. Then since $x^{(1)}, x^{(2)}$ are lin. indep.

$x = c_1 x^{(1)} + c_2 x^{(2)} = c_1 e^t \begin{pmatrix} 5\cos t \\ 3\cos t - \sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} 5\sin t \\ \cos t + 3\sin t \end{pmatrix}$ is a general solution

2.31 (i)

$$A = \begin{pmatrix} 2 & 1 \\ -9 & -4 \end{pmatrix}$$

$$\det \begin{pmatrix} 2-\lambda & 1 \\ -9 & -4-\lambda \end{pmatrix} = (2-\lambda)(-4-\lambda)+9$$

$$= \lambda^2+2\lambda+1=0$$

$$\text{if } \lambda_1 = \lambda_2 = \lambda = -1$$

$M_0 = I$

$$M_1 = A - I\lambda_1 = A + I = \begin{pmatrix} 3 & 1 \\ -9 & -3 \end{pmatrix}$$

$$\rho' = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \rho, \quad \rho(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so $\rho_1 = e^{-t}$

$$\rho_2' = \rho_1 - \rho_2 = e^{-t} - \rho_2 \Rightarrow \rho_2' + \rho_2 = e^{-t}$$

$$\Rightarrow \rho_2 e^t = t + c$$

$$\rho_2(0) = 0 \Rightarrow c = 0$$

$$\text{so } \rho_2 = te^{-t}$$

$$e^{At} = \sum_{k=0}^{1} \rho_{k+1}(t) M_k = e^{-t} \cdot I + te^{-t} \begin{pmatrix} 3 & 1 \\ -9 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} + 3te^{-t} & te^{-t} \\ -9te^{-t} & e^{-t} - 3te^{-t} \end{pmatrix}$$

2.31 (ii)

$$A = \begin{pmatrix} 10 & 4 \\ -9 & -2 \end{pmatrix}$$

$$\det(A - I\lambda) = 0 \Rightarrow \det\begin{pmatrix} 10-\lambda & 4 \\ -9 & -2-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (10-\lambda)(-2-\lambda) + 36 = 0$$

$$\Rightarrow \lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)^2 = 0$$

$$\lambda = 4 = \lambda_1 = \lambda_2$$

$$M_0 = I \quad M_1 = A - I\lambda_1 = \begin{pmatrix} 6 & 4 \\ -9 & -6 \end{pmatrix}$$

$$P' = \begin{pmatrix} 4 & 0 \\ 1 & 4 \end{pmatrix} P, \quad P(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow P_1 = e^{4t}, \quad P_2' = P_1 + 4P_2 = e^{4t} + 4P_2$$

If $P_2 = te^{4t}$, then $P_2(0) = 0$ and

$$P_2' = e^{4t} + 4te^{4t} = P_1 + 4P_2.$$

$$e^{At} = \sum_{k=0}^{1} P_{k+1}(t) M_k = e^{4t} I + te^{4t}\begin{pmatrix} 6 & 4 \\ -9 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} e^{4t} + 6te^{4t} & 4te^{4t} \\ -9te^{4t} & e^{4t} - 6te^{4t} \end{pmatrix}$$

2.33 (ii)

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

$$\det(A - I\lambda) = \det\begin{pmatrix} 2-\lambda & 3 \\ -3 & 2-\lambda \end{pmatrix} = 0$$

$$\therefore\ (2-\lambda)^2 + 9 = 0$$

$$\text{or}\quad \lambda^2 - 4\lambda + 13 = 0$$

$$\lambda = 2 \pm 3i$$

$$\lambda_1 = 2+3i,\ \lambda_2 = 2-3i$$

$$M_0 = I,\quad M_1 = A - I\lambda_1 = \begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix}$$

$$p' = \begin{pmatrix} 2+3i & 0 \\ 1 & 2-3i \end{pmatrix} p,\quad p(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow p_1 = e^{(2+3i)t},\quad p_2' = e^{(2+3i)t} + (2-3i)p_2$$

$$p_2' + (-2+3i)p_2 = e^{(2+3i)t}$$

$$p_2 e^{(-2+3i)t} = \frac{1}{6i}e^{6it} + C$$

$$p_2(0) = 0 \Rightarrow C = -\frac{1}{6i}$$

$$\Rightarrow p_2 = \frac{1}{6i}\left(e^{(2+3i)t} - e^{(2-3i)t}\right)$$

$$e^{At} = \sum_{k=0}^{1} p_{k+1} M_k = e^{(2+3i)t} I + \frac{e^{2t}}{6i}\left(e^{3it} - e^{-3it}\right)\begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix}$$

$$= \begin{pmatrix} e^{(2+3i)t} - \frac{1}{2}e^{2t}(e^{3it} - e^{-3it}) & \frac{e^{2t}}{2i}(e^{3it} - e^{-3it}) \\ -\frac{e^{2t}}{2i}(e^{3it} - e^{-3it}) & e^{(2t+3i)t} - \frac{1}{2}e^{2t}(e^{3it} - e^{-3it}) \end{pmatrix}$$

2.33 (ii)

$$e^{At} = \begin{pmatrix} e^{2t}\cos 3t & e^{2t}\sin 3t \\ -e^{2t}\sin 3t & e^{2t}\cos 3t \end{pmatrix}$$

2.33 (iii) $A = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}$ $\quad \det(A - I\lambda) = \det\begin{pmatrix} -\lambda & 2 \\ -1 & 2-\lambda \end{pmatrix} = 0$

iff $\lambda^2 - 2\lambda + 2 = 0$

or $\lambda = 1 \pm i$

$\lambda_1 = 1+i, \ \lambda_2 = 1-i$

$M_0 = I, \ M_1 = A - I\lambda_1 = \begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix}$

$p' = \begin{pmatrix} 1+i & 0 \\ 1 & 1-i \end{pmatrix} p, \quad p(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow p_1 = e^{(1+i)t}$

$p_2' = e^{(1+i)t} + (1-i)p_2 \Rightarrow p_2' + (-1+i)p_2 = e^{(1+i)t}$

$\Rightarrow p_2 e^{(-1+i)t} = \frac{1}{2i}e^{2it} + C$

$p_2(0) = 0 \Rightarrow C = -\frac{1}{2i}$

$\Rightarrow p_2 = \frac{1}{2i}\left(e^{(1+i)t} - e^{(1-i)t}\right)$

$$e^{At} = \sum_{k=0}^{1} p_{k+1}(t) M_k = e^{(1+i)t} I + \frac{1}{2i}\left(e^{(1+i)t} - e^{(1-i)t}\right)\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix}$$

$$= \begin{pmatrix} e^{(1+i)t} - \frac{1}{2i}\left(e^{(1+i)t} - e^{(1-i)t}\right) - \frac{1}{2}\left(e^{(1+i)t} - e^{(1-i)t}\right) & \frac{1}{i}\left(e^{(1+i)t} - e^{(1-i)t}\right) \\ -\frac{1}{2i}\left(e^{(1+i)t} - e^{(1-i)t}\right) & e^{(1+i)t} + \frac{1}{2i}\left(e^{(1+i)t} - e^{(1-i)t}\right) - \frac{1}{2}\left(e^{(1+i)t} - e^{(1-i)t}\right) \end{pmatrix}$$

$$= \begin{pmatrix} e^t(\cos t - \sin t) & 2e^t \sin t \\ -e^t \sin t & e^t(\cos t + \sin t) \end{pmatrix}$$

2.34 (i) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

using Putzer Algorithm

$\det(A - I\lambda) = \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = 0 \Rightarrow (1-\lambda)^2 - 1 = 0$

$\Rightarrow |\lambda| = \pm 1$

$\lambda = 1 \pm 1 \Rightarrow \lambda = 0$ or $\lambda = 2$

$\lambda_1 = 0, \ \lambda_2 = 2$

$M_0 = I, \quad M_1 = A - I\lambda_1 = A$

$p' = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} p \quad p(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow p_1 = 1, \quad p_2' = 1 + 2p_2 \Rightarrow p_2 = \frac{-1}{2} + \frac{1}{2}e^{2t}$

$(p_2(0) = 0, \quad p_2' = e^{2t} = 1 + 2\left(-\frac{1}{2} + \frac{1}{2}e^{2t}\right)$

$= 1 + 2p_2$

$\Rightarrow e^{At} = I + A \cdot \frac{1}{2}(e^{2t} - 1)$

$= \begin{pmatrix} \frac{1}{2}(e^{2t}+1) & \frac{1}{2}(e^{2t}-1) \\ \frac{1}{2}(e^{2t}-1) & \frac{1}{2}(e^{2t}+1) \end{pmatrix}$

$\therefore \ x = c_1 \begin{pmatrix} e^{2t}+1 \\ e^{2t}-1 \end{pmatrix} + c_2 \begin{pmatrix} e^{2t}-1 \\ e^{2t}+1 \end{pmatrix}$ is a general solution

of $x' = Ax$.

2.54 (iii) $A = \begin{pmatrix} -1 & -6 \\ 1 & 4 \end{pmatrix}$

Using Putzer Algorithm

$\det(A - I\lambda) = \det\begin{pmatrix} -1-\lambda & -6 \\ 1 & 4-\lambda \end{pmatrix} = 0$ if $(-1-\lambda)(4-\lambda) + 6 = 0$

or $\lambda^2 - 3\lambda + 2 = 0$

or $\lambda = 1$ or $\lambda = 2$

$\lambda_1 = 1, \ \lambda_2 = 2$

$M_0 = I, \ M_1 = A - I\lambda_1 = \begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix}$

$p' = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} p, \ \ p(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow p_1(t) = e^t, \ p_2' = e^t + 2p_2$

$p_2' - 2p_2 = e^t \Rightarrow e^{-2t} p_2 = -e^{-t} + C$

$p_2(0) = 0 \Rightarrow C = 1$

$\Rightarrow p_2 = -e^t + e^{2t}$

So $e^{At} = e^t I + (-e^t + e^{2t}) \begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix}$

$= \begin{pmatrix} e^t - 2(-e^t + e^{2t}) & -6(-e^t + e^{2t}) \\ -e^t + e^{2t} & e^t + 3(-e^t + e^{2t}) \end{pmatrix}$

$= \begin{pmatrix} 3e^t + e^{2t} & 6e^t - 6e^{2t} \\ e^{2t} - e^t & e^{2t} - 2e^t \end{pmatrix}$

and $x = c_1 \begin{pmatrix} 3e^t + e^{2t} \\ e^{2t} - e^t \end{pmatrix} + c_2 \begin{pmatrix} 6e^t - 6e^{2t} \\ e^{2t} - 2e^t \end{pmatrix}$ is a general solution of $x' = Ax$.

1.42(iii) To solve

$$x' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} e^{2t} \\ te^{2t} \end{pmatrix}, \quad x(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

with VOCF, we need a fundamental matrix for

$x' = Ax$, $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Eigenvalues of A are $\lambda_1 = \lambda_2 = 2$,

Using Putzer algorithm: $M_0 = I$, $M_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

and $p' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} p$, $p(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow p_1 = e^{2t}$

and $p_2' - 2p_2 = e^{2t} \Rightarrow p_2 e^{-2t} = t + C$

$p_2(0) = 0 \Rightarrow C = 0$

$p_2 = te^{2t}$

so e^{At} is a f.M. for $x' = Ax$, and

$$e^{At} = e^{2t} I + te^{2t} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}$$

Then VOCF gives (for the original IVP.)

$$x = e^{At} e^{-A0} x(0) + e^{At} \int_0^t e^{-As} \begin{pmatrix} e^{2s} \\ se^{2s} \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{-2s} & -se^{-2s} \\ 0 & e^{-2s} \end{pmatrix} \begin{pmatrix} e^{2s} \\ se^{2s} \end{pmatrix} ds$$

$$= \begin{pmatrix} te^{2t} - e^{2t} \\ e^{2t} \end{pmatrix} + \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \int_0^t \begin{pmatrix} 1 - s^2 \\ s \end{pmatrix} ds$$

$$= \begin{pmatrix} te^{2t} - e^{2t} \\ e^{2t} \end{pmatrix} + \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} t - t^3/3 \\ t^2/2 \end{pmatrix}$$

2.42(iii)

$$x = \begin{pmatrix} te^{2t} - e^{2t} \\ e^{2t} \end{pmatrix} + \begin{pmatrix} (t - \frac{t^3}{3}) e^{2t} + \frac{t^3}{2} e^{2t} \\ \frac{t^2}{2} e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2te^{2t} + \frac{t^3}{6} e^{2t} - e^{2t} \\ e^{2t} + \frac{t^2}{2} e^{2t} \end{pmatrix}$$ is the solution of the original IVP.

2.44

$$x' = \begin{pmatrix} 2t & 0 \\ 0 & 3 \end{pmatrix} x + \begin{pmatrix} t \\ 1 \end{pmatrix} \qquad x(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Since $A(t) = \begin{pmatrix} 2t & 0 \\ 0 & 3 \end{pmatrix}$ is diagonal, $A(t)A(s) = A(s)A(t)$ $\forall t, s \in I$, the interval on which we solve the eqn.

Then $\Phi(t) = e^{\int_0^t A(s)ds}$ is a fund. matrix for

$x' = A(t)x$ by Theorem 2.42. Then

$\Phi(t) = e^{\int_0^t \begin{pmatrix} 2s & 0 \\ 0 & 3 \end{pmatrix} ds} = e^{\begin{pmatrix} t^2 & 0 \\ 0 & 3t \end{pmatrix}}$. using the power

series definition of e^{At}, we see that because

$\int_0^t A(s)ds$ is diagonal, $e^{\begin{pmatrix} t^2 & 0 \\ 0 & 3t \end{pmatrix}} = \begin{pmatrix} e^{t^2} & 0 \\ 0 & e^{3t} \end{pmatrix} = \Phi(t)$.

Using VoCF (Theorem 2.40) gives the solution x of the original eqn.

$$x = \Phi(t)\Phi^{-1}(0)\begin{pmatrix} 0 \\ 2 \end{pmatrix} + \Phi(t)\int_0^t \Phi^{-1}(s)\begin{pmatrix} s \\ 1 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{t^2} & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} e^{t^2} & 0 \\ 0 & e^{3t} \end{pmatrix}\int_0^t \begin{pmatrix} se^{-s^2} \\ e^{-3s} \end{pmatrix} ds$$

$$= \begin{pmatrix} 0 \\ 2e^{3t} \end{pmatrix} + \begin{pmatrix} e^{t^2} & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} -\frac{1}{2}e^{-t^2} + \frac{1}{2} \\ -\frac{1}{3}e^{-3t} + \frac{1}{3} \end{pmatrix}$$

2.14) $x = \begin{pmatrix} 0 \\ 2e^{3t} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(e^{t^2}-1) \\ \frac{1}{3}(e^{3t}-1) \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2}(e^{t^2}-1) \\ \frac{7}{3}e^{3t} - \frac{1}{3} \end{pmatrix}$ is the solution of the original IVP.

2.48 (i) $x' = \begin{bmatrix} -1 & -6 \\ 0 & 4 \end{bmatrix} x$

Let $A = \begin{bmatrix} -1 & -6 \\ 0 & 4 \end{bmatrix}$

Then $\det(A - I\lambda) = \det\begin{pmatrix} -1-\lambda & -6 \\ 0 & 4-\lambda \end{pmatrix} = -(\lambda+1)(4-\lambda) = 0$

if $\lambda = -1$ or $\lambda = 4$

So $\lambda_1 = -1$ & $\lambda_2 = 4$ are e.v.'s of A. Then by the Stability Theorem, the trivial solution of $x' = Ax$ is <u>unstable</u> on $[0,\infty)$ because one of the eigenvalues, $\lambda_2 = 4$, has positive real part.

(ii) $x' = \begin{bmatrix} 0 & -9 \\ 4 & 0 \end{bmatrix} x$, Let $A = \begin{bmatrix} 0 & -9 \\ 4 & 0 \end{bmatrix}$. Then

$\det(A - I\lambda) = \det\begin{pmatrix} -\lambda & -9 \\ 4 & -\lambda \end{pmatrix} = \lambda^2 + 36 = 0$ if $\lambda = \pm 6i$.

Thus $\lambda = \pm 6i$ are the eigenvalues of A. $\lambda_1 = 6i$ and $\lambda_2 = -6i$ are both simple roots with real part 0, so by the Stability Theorem, the trivial solution of $x' = Ax$ is <u>stable</u> on $[0,\infty)$

(iii) $x' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x$. Let $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Then $\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} = \lambda(3+\lambda) + 2 = (\lambda+1)(\lambda+2) = 0$
if $\lambda = -1$ or $\lambda = -2$, so $\lambda_1 = -1$ and $\lambda_2 = -2$ are the e.v.'s of A. Thus by the Stability Theorem, the trivial solution of $x' = Ax$ is g.a.s. on $[0,\infty)$ because all eigenvalues have negative real parts.