Math 5601 Homework 1

Rajrani Gupta and Jacob Hauck

January 25, 2024

1. 1.3 (iv)

Let $\mathbb{T} = \{\sqrt{n} \mid n \in \mathbb{N}_0\}$. Then for $t = \sqrt{n} \in \mathbb{T}$,

- the next point to the right of $\sqrt{n}$ is $\sqrt{n+1}$, so $\sigma(t) = \sigma(\sqrt{n}) = \sqrt{n+1} = \sqrt{t^2+1}$,
- the next point to the left of $\sqrt{n}$ is $\sqrt{n-1}$ if $n > 0$. If $n = 0$, then there is no point in $\mathbb{T}$ to the left of $t = 0$, so

$$\begin{aligned}\rho(t) = \rho(\sqrt{n}) &= \begin{cases} \sqrt{n-1} & n > 0 \\ 0 & n = 0 \end{cases} \\ &= \begin{cases} \sqrt{t^2-1} & t > 0 \\ 0 & t = 0. \end{cases}\end{aligned}$$

- $\mu(t) = \sigma(t) - t = \sqrt{t^2+1} - t$.

Every point in $\mathbb{T}$ is right-scattered because $\sigma(t) = \sqrt{t^2+1} > t$. If $t > 0$, then t is left-scattered because $\rho(t) = \sqrt{t^2-1} < t$. The point $0 \in \mathbb{T}$ is not left-scattered because $\rho(0) = 0$, and it is not left-dense either because $0 = \inf \mathbb{T}$.

2. 1.4 (ii)

Let $\mathbb{T} = \{0\} \cup [1, 2]$. Then $\mathbb{T}$ is a time-scale, and $1 \in \mathbb{T}$ does not satisfy $\rho(\sigma(1)) = 1$. Indeed, $\sigma(1) = 1$, and $\rho(1) = 0$, so $\rho(\sigma(1)) = 0 \neq 1$.

Given any time-scale $\mathbb{T}$ and $t \in \mathbb{T}$, then $\rho(\sigma(t)) = t$ if and only if t is not left-scattered or t is right-scattered.

Proof. Suppose that t is left-scattered and not right-scattered. Then $\sigma(t) = t$, so $\rho(\sigma(t)) = \rho(t) \neq t$. Hence, $\rho(\sigma(t))$ implies that t is not left-scattered or t is right-scattered.

Conversely, if t is right-scattered, then $\sigma(t) \in \mathbb{T}$ is left-scattered with $\rho(\sigma(t)) = t$. If t is not right-scattered and not left-scattered, then $\rho(t) = t$ and $\sigma(t) = t$, so $\rho(\sigma(t)) = t$. ☐

3. 1.14 (i)

Define $f : \mathbb{T} \to \mathbf{R}$ by $f(t) = t^2$. Then $f^\Delta(t) = t + \sigma(t)$.

Proof. Let $t \in \mathbb{T}$, and let $\varepsilon > 0$ be given. Set $\delta = \varepsilon$. Then for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$,

$$\begin{aligned}|f(\sigma(t)) - f(s) - (t + \sigma(t))(\sigma(t) - s)| &= |\sigma(t)^2 - s^2 - (t + \sigma(t))(\sigma(t) - s)| \\ &= |ts + \sigma(t)s - s^2 - t\sigma(t)| \\ &= |(s - t)(\sigma(t) - s)| \\ &\leq \varepsilon|\sigma(t) - s|,\end{aligned}$$

so $f^\Delta(t) = t + \sigma(t)$ by definition. $\square$

4. 1.19 (ii)

Let $\mathbb{T} = \{\sqrt{n} \mid n \in \mathbb{N}_0\}$, and define $f : \mathbb{T} \to \mathbf{R}$ by $f(t) = t^2$. Recall from 1.3 (iv) that $\sigma(t) = \sqrt{t^2+1}$, and every point in $\mathbb{T}$ is right-scattered. Note that every point $t \in \mathbb{T}$ is (topologically) isolated, so f is continuous on $\mathbb{T}$. Therefore, by Theorem 1.16, f is differentiable everywhere on $\mathbb{T}$, and for $t \in \mathbb{T}$,

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{\left(\sqrt{t^2+1}\right)^2 - t^2}{\sqrt{t^2+1} - t} = \sqrt{t^2+1} + t.$$

5. 1.19 (iii)

Let $\mathbb{T} = \left\{\frac{n}{2} \mid n \in \mathbb{N}_0\right\}$, and define $f : \mathbb{T} \to \mathbf{R}$ by $f(t) = t^2$. Then for $t = \frac{n}{2} \in \mathbb{T}$, the next point to the right of t is $\frac{n+1}{2} = t + \frac{1}{2}$. Hence, $\sigma(t) = t + \frac{1}{2}$. Moreover, every point in $\mathbb{T}$ is right-scattered, and every point in $\mathbb{T}$ is (topologically) isolated, so f is continuous on $\mathbb{T}$. By Theorem 1.16, for $t \in \mathbb{T}$,

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{\left(t + \frac{1}{2}\right)^2 - t^2}{t + \frac{1}{2} - t} = 2\left(t + \frac{1}{4}\right) = 2t + \frac{1}{2}.$$

6. 1.22

Suppose that $f : \mathbb{T} \to \mathbf{R}$ is differentiable at $t \in \mathbb{T}$, and $f(t)f(\sigma(t)) \neq 0$. Then $\frac{1}{f}$ is differentiable at t, and

$$\left(\frac{1}{f}\right)^\Delta(t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))}.$$

Proof. We know from Theorem 1.16 that f is continuous at t. Since $f(t) \neq 0$ by assumption, it follows that f is bounded away from 0 in a neighborhood of t. That is, there exists $C > 0$ and $\delta_0 > 0$ such that for all $s \in (t - \delta_0, t + \delta_0) \cap \mathbb{T}$, we have $|f(s)| \geq C$.

Let $\varepsilon > 0$ be given, and set

$$\varepsilon^* = \varepsilon\left(\frac{1}{C|f(\sigma(t))|} + \frac{\left|f^\Delta(t)\right|}{C\left|f(t)f(\sigma(t))\right|}\right)^{-1}.$$

Since f is continuous and delta-differentiable at t, we can choose $\delta \in (0, \delta_0]$ such that for all $s \in (t-\delta, t+\delta) \cap \mathbb{T}$,

1. $|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon^*|\sigma(t) - s|$,
2. $|f(t) - f(s)| \leq \varepsilon^*$.

Note also that $|f(s)| \geq C$ for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$ because $\delta < \delta_0$.

Then

$$
\begin{aligned}
&\left|\frac{1}{f(\sigma(t))} - \frac{1}{f(s)} - \left(-\frac{f^\Delta(t)}{f(t)f(\sigma(t))}\right)(\sigma(t)-s)\right| \\
&\qquad = \left|\frac{f(t)f(s) - f(t)f(\sigma(t)) + f(s)f^\Delta(t)(\sigma(t)-s)}{f(t)f(\sigma(t))f(s)}\right| \\
&\qquad = \left|\frac{f(t)\big[f(s) - f(\sigma(t)) + f^\Delta(t)(\sigma(t)-s)\big] + (f(s)-f(t))f^\Delta(t)(\sigma(t)-s)}{f(t)f(\sigma(t))f(s)}\right| \\
&\qquad \le \frac{\varepsilon^*|\sigma(t)-s|}{\big|f(\sigma(t))f(s)\big|} + \frac{\varepsilon^*\left|f^\Delta(t)\right| \cdot |\sigma(t)-s|}{\big|f(t)f(\sigma(t))f(s)\big|} \\
&\qquad \le \left(\frac{1}{C|f(\sigma(t))|} + \frac{\left|f^\Delta(t)\right|}{C\big|f(t)f(\sigma(t))\big|}\right)\varepsilon^*|\sigma(t)-s| \\
&\qquad = \varepsilon|\sigma(t)-s|,
\end{aligned}
$$

so

$$\left(\frac{1}{f}\right)^\Delta(t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))}$$

by definition. □

7. 1.26