Bifurcation Analysis of a Discrete-Time Prey-Predator Model

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Presented by Jacob Hauck

Outline

- Description and interpretation of a discrete-time predator-prey model

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- Numerical investigations
  - Bifurcation diagram of period-doubling bifurcation
  - Phase portrait changes at Neimark-Sacker bifurcation

Model Description

$$x_p(n+1) = x_p(n)\left[1 + r\left(1 - \frac{x_p(n)}{k}\right) - a y_p(n)\right]$$
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- $b > 0$: death rate of predators
- $c > 0$: conversion rate (of prey into predators)

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Ecologically, an important fixed points occur when $x_p > 0$, and $y_p > 0$, when predator and prey are in equilibrium.

There is one such fixed point:

$$\mathcal{P}_* = \left(\frac{rkb}{ack + br}, \frac{crk}{ack + br} \right).$$

Period-doubling Bifurcations

On the time scale $\mathbb{Z}$, fixed points are also 1-periodic solutions. In general, if $x(n)$ is a solution of

$$x(n + 1) = f(x(n))$$

such that $x(n + p) = x(n)$, where p is the smallest integer that makes this true, then $x_0 = x(0)$ is called a **periodic point of minimal period** p.

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See Section 3.4 of *Dynamics and Bifurcations*.

Period-doubling Bifurcation in the Predator-Prey Model

Using the predation rate a as a bifurcation parameter, there is a period-doubling bifurcation at the parameter value

$$a_{\mathrm{PD}} = -\frac{br(br-2b-2r+4)}{ck(br-2b+4)}.$$

Furthermore, the bifurcation is supercritical (subcritical) if $\widehat{\beta^{pp}_{\mathrm{PD}}} > 0$ (< 0), where

$$\widehat{\beta^{pp}_{\mathrm{PD}}} = \frac{16r(b-2)^3(r+2)}{(br-2b+4)^2k^2c^2(br-4)}.$$

Recall: supercritical $\iff$ stable $\to$ unstable, subcritical $\iff$ unstable $\to$ stable.

Period-Doubling Bifurcation – Method

One-dimensional case (from *Dynamics and Bifurcations*):

Let $f \in C^3$ with

$$f(0) = 0, \quad f'(0) = -1, \quad (f^2)'''(0) \neq 0.$$

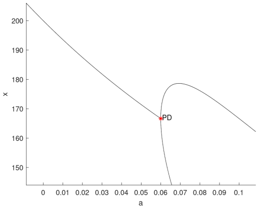
If $F(\lambda, x)$ is a perturbation of f such that

$$F(0,x) = f(x), \quad F(\lambda, 0) = 0, \quad \frac{\partial F}{\partial \lambda}(\lambda, 0) = -(1+\lambda),$$

then the discrete equation $x_{n+1} = F(\lambda, x_n)$ undergoes a period-doubling bifurcation at $\lambda = 0$.

Apply a similar result to higher-dimensional equations – this involves Jacobian matrix and third-order partial derivatives.

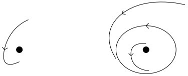
Period-Doubling Bifurcation Diagram



A subcritical period-doubling bifurcation

Neimark-Sacker Bifurcations

In a **Neimark-Sacker bifurcation** the fixed point changes stability type and a closed invariant curve containing the fixed point emerges with opposite stability.



Before Bifurcation   After Bifurcation

Neimark-Sacker Bifurcation in the Predator-Prey Model

A Neimark-Sacker bifurcation occurs with respect to a when

$$a = a_{\text{NS}} = \frac{-r(br - b - r)}{ck(r-1)}.$$

The bifurcation is supercritical (subcritical) if $\widehat{\sigma_{\text{NS}}^{pp}} < 0$ (> 0).

What is $\widehat{\sigma_{\text{NS}}^{pp}}$? This is a value that depends on the parameters and is related to the following result...

Neimark-Sacker Bifurcation – Method

From *Elements of Applied Bifurcation Theory* by Y.A. Kuznetsov:

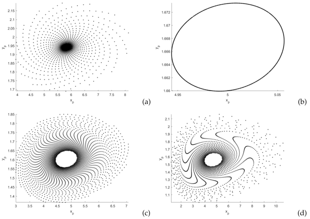
In the two-dimensional discrete system $x_{n+1} = f(\lambda, x_n)$, let $\mu_\pm(\lambda) = r(\lambda)e^{\pm i\theta(\lambda)}$ be the eigenvalues of the Jacobian near $\lambda = 0$. If

$$r(0) = 1, \quad r'(0) \neq 0, \quad e^{ik\theta(0)} \neq 1 \text{ for } k = 1, 2, 3, 4,$$

then the system undergoes a Neimark-Sacker bifurcation, which is supercritical (subcritical) if $\sigma = \Re\left(e^{-i\theta(0)}c_1(0)\right) < 0$ (> 0).

Here, $c_1(0)$ is a complicated function of the first, second, and third derivatives of f at $\lambda = 0$ and at the critical point.

Phase Portraits Near the Neimark-Sacker Bifurcation



(a) $a < a_{\mathrm{NS}}$, (b) $a = a_{\mathrm{NS}}$, (c) $a > a_{\mathrm{NS}}$, (d) $a > a_{\mathrm{NS}}$