

Math 6417 Homework 4

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Question 1.

Define the **Fourier transform operator** $\mathcal{F} : L^1(\mathbf{R}) \rightarrow L^\infty(\mathbf{R})$ by

$$\mathcal{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f(x) \, dx. \quad (1)$$

- 1.1) We note that the function $x \mapsto e^{iyx} f(x)$ is clearly integrable if f is, so the integral in (1) exists for all y . We show that $\mathcal{F}(f) \in L^\infty(\mathbf{R})$ as claimed, and $\|\mathcal{F}f\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$. Indeed, for $y \in \mathbf{R}$,

$$|\mathcal{F}(f)(y)| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} e^{iyx} f(x) \, dx \right| \quad (2)$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{iyx} f(x)| \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \, dx = \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}. \quad (3)$$

Therefore, $\|\mathcal{F}f\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$.

- 1.2) Suppose that $f \in C^2(\mathbf{R})$, and $f, f', f'' \in L^1(\mathbf{R})$, and $f(x), f'(x), f''(x) \rightarrow 0$ as $x \rightarrow \pm\infty$. Then there exists a constant C such that $|y^2 \mathcal{F}(f)(y)| \leq C$ for all $y \in \mathbf{R}$. Furthermore, $\mathcal{F}(f) \in L^1(\mathbf{R})$.

Proof. Since $f'' \in L^1(\mathbf{R})$, we can take its Fourier transform, which yields

$$\mathcal{F}(f'')(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f''(x) \, dx. \quad (4)$$

We can integrate by parts because $f', f \in L^1(\mathbf{R})$ and are continuous, and $f(x), f'(x) \rightarrow 0$ as $x \rightarrow \pm\infty$. This gives

$$\mathcal{F}(f'')(y) = \frac{1}{\sqrt{2\pi}} \left[f'(x) e^{iyx} \Big|_{-\infty}^{\infty} - iy \int_{-\infty}^{\infty} e^{iyx} f'(x) \, dx \right] \quad (5)$$

$$= \frac{iy}{\sqrt{2\pi}} \left[-f(x) e^{iyx} \Big|_{-\infty}^{\infty} + iy \int_{-\infty}^{\infty} e^{iyx} f(x) \, dx \right] \quad (6)$$

$$= -y^2 \mathcal{F}(f)(y). \quad (7)$$

By the reasoning in 1.1), it follows that

$$|y^2 \mathcal{F}(f)(y)| = |\mathcal{F}(f'')(y)| \leq \frac{1}{\sqrt{2\pi}} \|f''\|_{L^1} \quad (8)$$

for all $y \in \mathbf{R}$.

Thus, if $C = \frac{1}{\sqrt{2\pi}} \|f''\|_{L^1}$, then $|\mathcal{F}(f)(y)| \leq \frac{C}{y^2}$ for all $y \in \mathbf{R}$. On the other hand, $\mathcal{F}(f) \in L^\infty(\mathbf{R})$ by part 1.1), so $\mathcal{F}(f)$ is dominated by the integrable function

$$\phi(y) = \begin{cases} \|\mathcal{F}(f)\|_{L^\infty} & y \in [-1, 1], \\ \frac{C}{y^2} & \text{otherwise.} \end{cases} \quad (9)$$

By the integral comparison test, $\mathcal{F}(f) \in L^1(\mathbf{R})$. □

1.3) Formally, $\mathcal{F}^2(f)(y) = f(-y)$.

Proof. We note that if $f \in C^1 \cap L^1(\mathbf{R})$, and $f' \in L^1(\mathbf{R})$, and $f(x) \rightarrow 0$ as $x \rightarrow \pm\infty$, then we can use integration by parts to show that

$$\mathcal{F}(f')(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f'(x) dx = \frac{1}{\sqrt{2\pi}} \left[e^{iyx} f(x) \Big|_{-\infty}^{\infty} - iy \int_{-\infty}^{\infty} e^{iyx} f(x) dx \right] \quad (10)$$

$$= -iy \mathcal{F}(f)(y). \quad (11)$$

On the other hand, let $f \in L^1(\mathbf{R})$, and define $g(x) = ix f(x)$. If $g \in L^1(\mathbf{R})$ as well, then

$$\frac{d}{dy} \frac{1}{\sqrt{2\pi}} \mathcal{F}(f)(y) = \frac{d}{dy} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial y} [e^{iyx} f(x)] dx \quad (12)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} ix f(x) dx = \mathcal{F}(g)(y). \quad (13)$$

If we take $f(x) = e^{-ax^2}$, then f satisfies the above assumptions. Since $f'(x) = -2ax f(x)$,

$$2ai \frac{d}{dy} \mathcal{F}(f)(y) = 2ai \mathcal{F}(i(\cdot) f(\cdot))(y) = \mathcal{F}(-2a(\cdot) f(\cdot))(y) = \mathcal{F}(f')(y) = -iy \mathcal{F}(f)(y). \quad (14)$$

Hence, $\mathcal{F}(f)(y)$ is the unique solution of the IVP

$$u' = -\frac{y}{2a} u, \quad u(0) = \mathcal{F}(f)(0). \quad (15)$$

The general solution of the differential equation is

$$u(y) = u(0) e^{-\frac{y^2}{4a}}. \quad (16)$$

Since

$$\mathcal{F}(f)(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{1}{\sqrt{2a}}, \quad (17)$$

it follows that

$$\mathcal{F}(f)(y) = \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}. \quad (18)$$

Thus, if $\phi_a(x) = e^{-ax^2}$, then, formally,

$$\mathcal{F}(1)(y) = \mathcal{F} \left(\lim_{a \rightarrow 0^+} \phi_a \right) (y) = \lim_{a \rightarrow 0^+} \mathcal{F}(\phi_a)(y) = \lim_{a \rightarrow 0^+} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}. \quad (19)$$

We would like to interpret the last limit formally as a constant multiple of the Dirac delta function. Clearly,

$$\lim_{a \rightarrow 0^+} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}} = \begin{cases} 0 & y \neq 0, \\ \infty & y = 0. \end{cases} \quad (20)$$

At the same time, for any $a > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}} dy = \frac{1}{\sqrt{2a}} \sqrt{4a\pi} = \sqrt{2\pi}, \quad (21)$$

so it makes sense formally that we should have $\mathcal{F}(1)(y) = \sqrt{2\pi} \delta(y)$.

Now, if we consider applying the Fourier transform twice to a function f , we get

$$\mathcal{F}\mathcal{F}(f)(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ixy} e^{izx} f(z) \, dz \, dx \quad (22)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix(y+z)} \, dx \, dz \quad (23)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) \mathcal{F}(1)(y+z) \, dz \quad (24)$$

$$= \int_{-\infty}^{\infty} f(z) \delta(y+z) \, dz \quad (25)$$

$$= \int_{-\infty}^{\infty} f(z-y) \delta(z) \, dz \quad (26)$$

$$= f(-y). \quad (27)$$

□

- 1.4) Define $g(y) = f(-y)$ for some function f . Based on the formal result from part 1.3), we see immediately that

$$\mathcal{F}^4(f)(y) = \mathcal{F}^2(\mathcal{F}^2(f))(y) = \mathcal{F}^2(g)(y) = g(-y) = f(y). \quad (28)$$

Since f was arbitrary, it follows formally that $\mathcal{F}^4 = I$, the identity operator.

- 1.5) Let $p(x) = x^4$. By the Spectral Mapping Theorem,

$$p(\sigma(\mathcal{F})) = \sigma(p(\mathcal{F})). \quad (29)$$

Since $p(\mathcal{F}) = \mathcal{F}^4 = I$, the spectrum of $p(\mathcal{F})$ is just $\sigma(I) = \{1\}$, as the operator $I - \lambda I = (1 - \lambda)I$ is invertible, with inverse $\frac{1}{1-\lambda}I$, if and only if $\lambda \neq 1$. Therefore, if $\lambda \in \sigma(\mathcal{F})$, then $p(\lambda) = 1$, that is, $\lambda^4 = 1$. The possible solutions of this equation are $1, -1, i, -i$, so $\sigma(\mathcal{F}) \subseteq \{1, -1, i, -i\}$.

- 1.6) If we reuse the result in equation (18) with $a = \frac{1}{2}$, we see that if $f(x) = e^{-\frac{1}{2}x^2}$, then

$$\mathcal{F}(f)(y) = e^{-\frac{1}{2}y^2} \quad (30)$$

as well. Thus, $\mathcal{F}f = f$, so f is an eigenfunction of \mathcal{F} with corresponding eigenvalue 1.

Question 2.

On this question, we will reuse the notation from Question 2 of Homework 3.

Let $\dot{L}^2(-\pi, \pi) = \{f \in L^2(-\pi, \pi) : f = \bar{f} \text{ and } \text{mean}(f) = 0\}$, where $\text{mean}(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f$. Consider the following problem.

$$\text{Let } f \in \dot{L}^2(-\pi, \pi). \text{ Find } u \in H \text{ such that } -u'' = f, \quad (31)$$

where H is the space defined in Homework 3.

- 2.1) Let $f \in \dot{L}^2(-\pi, \pi)$. Then $f \in L^2(-\pi, \pi)$, and, recalling from Homework 3, there exists $\{f_j\} \subset \mathbf{C}$ such that

$$f = \sum_j f_j e_j, \quad f_j = (f, e_j). \quad (32)$$

Since $e_0 = \text{constant}$, we have $f_0 = (f, e_0) \propto \text{mean}(f) = 0$, so $f_0 = 0$. Furthermore, by an argument we used several times in Homework 3, the fact that $f = \bar{f}$ implies that $f_{-j} = \bar{f}_j$. Lastly, by Parseval's identity,

$$\sum_{j \neq 0} j^{-2} |f_j|^2 \leq \sum_{j \neq 0} |f_j|^2 = \|f\|_2^2 < \infty, \quad (33)$$

so $f \in H^{-1}$ from Homework 3 because $\{f_j\}_{j \neq 0} \in S_{H^{-1}}$. Therefore, $\dot{L}^2(-\pi, \pi) \subseteq H^{-1}$.

We claim that for $f \in \dot{L}^2(-\pi, \pi)$ and $u \in H$,

$$-u'' = f \iff B(u, v) = f(v) \quad \forall v \in H, \quad (34)$$

where we define the action of f on v in the same way as in Homework 3, and

$$B(u, v) = \sum_{j \neq 0} j^2 u_j \bar{v}_j, \quad \{u_j\} = \varphi(u), \quad \{v_j\} = \varphi(v), \quad (35)$$

where $\varphi : H \rightarrow S_H$ is defined as in Homework 3. We use essentially the same formal argument that we used on 2.5) in Homework 3.

Suppose that $-u'' = f$, and let $\{f_j\} = \psi(f)$, $\{u_j\} = \varphi(u)$. Formally differentiating the Fourier series for u , we have

$$-\sum_{j \neq 0} f_j e_j = -f = u'' = \sum_{j \neq 0} -j^2 u_j e_j. \quad (36)$$

Therefore, $f_j = j^2 u_j$ for all j , and for any $v \in H$,

$$B(u, v) = \sum_{j \neq 0} j^2 u_j \bar{v}_j = \sum_{j \neq 0} f_j \bar{v}_j = f(v). \quad (37)$$

On the other hand, suppose that $B(u, v) = f(v)$ for all $v \in H$. Clearly, $e_j + e_{-j} \in H$, and $e_j - e_{-j} \in H$, so

$$j^2 u_j + j^2 u_{-j} = f_j + f_{-j}, \quad j^2 u_j - j^2 u_{-j} = f_j - f_{-j}, \quad (38)$$

which implies that $j^2 u_j = f_j$ for all j . By the same formal differentiation reasoning, it follows that $-u'' = f$. Additionally, we have $u \in H$ because

$$\bar{u}_{-j} = \frac{\bar{f}_{-j}}{j^2} = \frac{f_j}{j^2} = u_j, \quad \sum_{j \neq 0} j^2 |u_j|^2 = \sum_{j \neq 0} j^2 \left| \frac{f_j}{j^2} \right|^2 \leq \sum_{j \neq 0} |f_j|^2 < \infty, \quad (39)$$

which implies that $\{u_j\} \in S_H$.

The function B is bilinear because for any $\alpha, \beta \in \mathbf{R}$, and any $u, v, w \in H$,

$$B(\alpha u + \beta v, w) = \sum_{j \neq 0} j^2 (\alpha u_j + \beta v_j) \bar{w}_j = \alpha \sum_{j \neq 0} j^2 u_j \bar{w}_j + \beta \sum_{j \neq 0} j^2 v_j \bar{w}_j = \alpha B(u, w) + \beta B(v, w), \quad (40)$$

and

$$B(w, \alpha u + \beta v) = \sum_{j \neq 0} j^2 w_j \overline{\alpha u_j + \beta v_j} = \alpha \sum_{j \neq 0} j^2 w_j \bar{u}_j + \beta \sum_{j \neq 0} j^2 w_j \bar{v}_j = \alpha B(w, u) + \beta B(w, v) \quad (41)$$

because $\varphi(\alpha u + \beta v) = \alpha \varphi(u) + \beta \varphi(v)$.

The function B is also continuous because for any $u, v \in H$,

$$|B(u, v)| = \left| \sum_{j \neq 0} j^2 u_j \bar{v}_j \right| \leq \|u\|_H \|v\|_H \quad (42)$$

by the Cauchy-Schwarz inequality.

Lastly, B is coercive because for any $u \in H$,

$$B(u, u) = \sum_{j \neq 0} j^2 |u_j|^2 = \|u\|_H^2. \quad (43)$$

Hence, the Lax-Milgram Theorem implies that, given $f \in \dot{L}^2(-\pi, \pi) \subseteq H^{-1} \subseteq H^*$, there exists a unique $u \in H$ such that $B(u, v) = f(v)$ for all $v \in H$. That is, there exists a unique $u \in H$ such that $-u'' = f$.

2.2) Let $T : \dot{L}^2(-\pi, \pi) \rightarrow H$ denote the solution operator of (31), which exists by 2.1). Then T is compact as an operator on $\dot{L}^2(-\pi, \pi)$.

Proof. Given $f \in \dot{L}^2(-\pi, \pi)$, there exists $u \in H$ such that

$$\|Tf\|_H = \|u\|_H, \quad (44)$$

and $B(u, v) = f(v)$ for all $v \in H$. In particular, if we take $v = u$, we obtain

$$\|u\|_H^2 = f(u) = \sum_{j \neq 0} f_j \bar{u}_j \leq \|u\|_H \|f\|_{L^2}, \quad (45)$$

which implies that $\|Tf\|_H \leq \|f\|_{L^2}$, so T is bounded.

As we showed in 2.1), $Tf = u$ if and only if $f_j = j^2 u_j$, where $\{u_j\} = \varphi(u)$, and $\{f_j\} = \psi(f)$. Thus, $u_j = \frac{f_j}{j^2}$.

Let A be a bounded set in $\dot{L}^2(-\pi, \pi)$. Then $T(A)$ is bounded in H because T is a bounded operator from $\dot{L}^2(-\pi, \pi)$ to H . Then there exists $M > 0$ such that $\|u\|_H \leq M$ for all $u \in T(A)$.

Thus, for any $J > 0$ and any $u \in T(A)$,

$$\sum_{|j| > J} |u_j|^2 \leq \frac{1}{J^2} \sum_{|j| > J} j^2 |u_j|^2 \leq \frac{M}{J^2}, \quad (46)$$

where $\{u_j\} = \varphi(u)$. We recall from Homework 3 that

$$\|u\|_{L^2}^2 = \sum_{j \neq 0} |u_j|^2, \quad (47)$$

so the tail of the series form of the L^2 norm of u is uniformly small for $u \in T(A)$, which is what allows us to establish the compactness of T .

Let $\{u^n\}$ be a sequence in $T(A)$. The space H is a Hilbert space by Homework 3, and $\{u^n\}$ is bounded because $T(A)$ is bounded, so there exists a weakly convergent subsequence of $\{u^n\}$, call it $\{u^{n_k}\}$. Then $\{u_j^{n_k}\}_k$ converges for all j , where $\{u_j^{n_k}\}$ are the Fourier coefficients of u^{n_k} with respect to $\{e_j\}$. In particular, each such sequence is Cauchy.

Let $\varepsilon > 0$ be given. By (46), we can choose J large enough that

$$\sum_{|j| > J} |u_j^{n_k}|^2 < \varepsilon^2 \quad (48)$$

for all k . We can also choose N large enough that $k, \ell > N$ implies that $|u_j^{n_k} - u_j^{n_\ell}|^2 < \frac{\varepsilon^2}{2J}$ for all $|j| \leq J$. Then

$$\|u^{n_k} - u^{n_\ell}\|_{L^2}^2 = \sum_{|j| \leq J} |u_j^{n_k} - u_j^{n_\ell}|^2 + \sum_{|j| > J} |u_j^{n_k} - u_j^{n_\ell}|^2 \quad (49)$$

$$\leq \varepsilon^2 + 2 \sum_{|j| > J} |u_j^{n_k}|^2 + 2 \sum_{|j| > J} |u_j^{n_\ell}|^2 \quad (50)$$

$$\leq 5\varepsilon^2, \quad (51)$$

so $k, \ell > N$ implies that $\|u^{n_k} - u^{n_\ell}\|_H < \sqrt{5}\varepsilon$.

This implies that $\{u^{n_k}\}_k$ is Cauchy in $L^2(-\pi, \pi)$. Since $L^2(-\pi, \pi)$ is a Hilbert space, it follows that $\{u^{n_k}\}$ converges to some $u \in L^2(-\pi, \pi)$. It remains to show that $u \in \dot{L}^2(-\pi, \pi)$.

We recall from Homework 3 that $\{u_j^{n_k}\}_{j \neq 0} = \varphi(u^{n_k})$, so $\overline{u_j^{n_k}} = u_{-j}^{n_k}$ for all k and all $j \neq 0$, and $u_0^{n_k} = 0$ for all k . Taking the limit as $k \rightarrow \infty$, we get $\bar{u}_j = \bar{u}_{-j}$ for all $j \neq 0$, and $u_0 = 0$. Thus, $\bar{u} = u$ by the same reasoning used several times in Homework 3, and $\text{mean}(u) \propto (u, e_0) = u_0 = 0$; therefore, $u \in \dot{L}^2(-\pi, \pi)$ by definition.

Thus, every sequence in the bounded set $T(A)$ has a convergent subsequence, so $T(A)$ is pre-compact in H . This implies that T is compact by definition. \square

2.3) The operator T is self-adjoint as an operator on $\dot{L}^2(-\pi, \pi)$.

Proof. Let $f, g \in \dot{L}^2(-\pi, \pi)$, and define $u = Tf$, and $v = Tg$. Then, by the same reasoning as in 2.1), if $\{u_j\} = \psi(u)$, $\{v_j\} = \psi(v)$, $\{f_j\} = \psi(f)$, and $\{g_j\} = \psi(g)$, then

$$u_j = \frac{f_j}{j^2}, \quad v_j = \frac{g_j}{j^2}. \quad (52)$$

Thus,

$$(Tf, g) = (u, g) = \sum_{j \neq 0} u_j \bar{g}_j = \sum_{j \neq 0} f_j \bar{v}_j = (f, v) = (f, Tg), \quad (53)$$

so T is self-adjoint. \square

2.4) Suppose that $Tf = \lambda f$ for $f \in H$, with $f \neq 0$. Note that since T is self-adjoint, we must have $\lambda \in \mathbf{R}$. By the reasoning in 2.1) giving the formula for T in terms of Fourier coefficients, we must have $j^{-2}f_j = \lambda f_j$ for all $j \neq 0$, where $\{f_j\} = \psi(f)$. Then either $f_j = 0$ or $\lambda = j^{-2}$ for all $j \neq 0$. We cannot have $f_j = 0$ for all j , because then $f = 0$ by the linearity of ψ^{-1} .

Thus, there exists some $k > 0$ such that $f_k \neq 0$, which implies that $\lambda = k^{-2}$. The equation $\lambda f_j = j^{-2}f_j$ for all j implies that $f_j = 0$ for all $j \neq \pm k$. Since $f_{-k} = \bar{f}_k$, it follows that $f = f_k e_k + \bar{f}_k e_{-k}$. Supposing that $f_j = a + ib$, we must have

$$f(x) = \frac{1}{\sqrt{2\pi}} ((a + ib)e^{ikx} + (a - ib)e^{-ikx}) \quad (54)$$

$$= \frac{1}{\sqrt{2\pi}} (a \cos(kx) - b \sin(kx) + ib \cos(kx) + ia \sin(kx)) \quad (55)$$

$$+ a \cos(kx) - b \sin(kx) - ib \cos(kx) - ia \sin(kx)) \quad (56)$$

$$= \frac{1}{\sqrt{2\pi}} (2a \cos(kx) - 2b \sin(kx)). \quad (57)$$

Thus, if λ is an eigenvalue of T , then $\lambda = k^{-2}$ for some integer $k > 0$, and the corresponding eigenvectors must be linear combinations of $\cos(kx)$ and $\sin(kx)$; that is, the corresponding eigenspace is

$\text{span}\{\cos(kx), \sin(kx)\}$. We note that $\cos(kx)$ and $\sin(kx)$ are indeed in $\dot{L}^2(-\pi, \pi)$ because they have mean zero on $(-\pi, \pi)$.

It is not hard to see by reversing the above logic that the converse is true, namely, that k^{-2} is an eigenvalue of T for all integers $k \neq 0$, and its corresponding eigenspace is $\text{span}\{\cos(kx), \sin(kx)\}$.

Thus, the eigenvalues of T are $\{\lambda_k = k^{-2}\}_{k>0}$, with corresponding eigenspaces $\{\text{span}\{\cos(kx), \sin(kx)\}\}_{k>0}$.

- 2.5)** Let $c \in \mathbf{C}$ be given. Then for any $j > 0$, there exists $a, b \in \mathbf{R}$ such that $ce_j + \bar{c}e_{-j} = a\cos(jx) + b\sin(jx)$; indeed, by the calculation in 2.4), we just need to choose $a = \sqrt{\frac{2}{\pi}}\Re(c)$, and $b = -\sqrt{\frac{2}{\pi}}\Im(c)$. Therefore, given $u \in \dot{L}^2(-\pi, \pi)$, the partial sum of the Fourier series for u can be written as a linear combination of elements of the set $\mathcal{B} = \{\cos(jx), \sin(jx)\}_{j>0} \subseteq \dot{L}^2(-\pi, \pi)$. Since $L^2(-\pi, \pi)$ is a Hilbert space, the Fourier series of u converges to u , so u is the limit of a sequence of elements of $\text{span}(\mathcal{B})$.

In other words, \mathcal{B} is a basis for $\dot{L}^2(-\pi, \pi)$. In fact, it is an orthogonal basis, as we show now. Let $\{c_j^k\}_j = \psi(\cos(kx))$, and let $\{s_j^k\}_j = \psi(\sin(kx))$. Then, by the Euler formula relating e^{ijx} to $\sin(x)$ and $\cos(x)$,

$$c_j^k = \begin{cases} \sqrt{\frac{\pi}{2}} & j = |k| \\ 0 & \text{otherwise,} \end{cases} \quad s_j^k = \begin{cases} -i\sqrt{\frac{\pi}{2}}\text{sgn}(j) & j = |k| \\ 0 & \text{otherwise,} \end{cases} \quad (58)$$

where $\text{sgn}(j)$ is the sign of j . Hence, for $k, \ell > 0$,

$$(\cos(kx), \cos(\ell x)) = \sum_{j \neq 0} c_j^k \bar{c}_j^\ell = \begin{cases} 0 & k \neq \ell \\ \frac{\pi}{2} & k = \ell, \end{cases} \quad (59)$$

$$(\cos(kx), \sin(\ell x)) = \sum_{j \neq 0} c_j^k \bar{s}_j^\ell = \begin{cases} 0 & k \neq \ell \\ \frac{\pi}{2}(-i + i) = 0 & k = \ell \end{cases} = 0, \quad (60)$$

$$(\sin(kx), \sin(\ell x)) = \sum_{j \neq 0} s_j^k \bar{s}_j^\ell = \begin{cases} 0 & k \neq \ell \\ \frac{\pi}{2} & k = \ell. \end{cases} \quad (61)$$

Therefore, \mathcal{B} is orthogonal in H . Moreover, we can clearly modify the elements of \mathcal{B} so that they are orthonormal; if we set $\mathcal{B}' = \left\{ \sqrt{\frac{2}{\pi}}\cos(jx), \sqrt{\frac{2}{\pi}}\sin(jx) \right\}$, then \mathcal{B}' is an orthonormal basis for $\dot{L}^2(-\pi, \pi)$.

Finally, we note that \mathcal{B}' is the orthonormal basis that diagonalizes T in the sense of the spectral theorem for self-adjoint, compact operators. Let $c^k = \sqrt{\frac{2}{\pi}}\cos(kx)$, and let $s^k = \sqrt{\frac{2}{\pi}}\sin(kx)$. Define $u^k = T(c^k)$, and $v^k = T(s^k)$, and define $\{u_j^k\} = \psi(u^k)$, and $\{v_j^k\} = \psi(v^k)$. By our formula for T in terms of Fourier coefficients and by (58),

$$v_{\pm k}^k = k^{-2}s_{\pm k}^k \implies T(s^k) = \lambda_k s^k, \quad u_{\pm k}^k = k^{-2}c_{\pm k}^k \implies T(c^k) = \lambda_k c^k. \quad (62)$$

Furthermore, given $f \in \dot{L}^2(-\pi, \pi)$, if $u = Tf$, $\{f_j\} = \psi(f)$, and $\{u_j\} = \varphi(u)$, then, on the one hand,

$$f(x) = \sum_{j \neq 0} f_j e_j(x) = \sum_{j > 0} f_j e_j(x) + \bar{f}_j e_{-j}(x) = \sum_{j > 0} \Re(f_j) \sqrt{\frac{2}{\pi}} \cos(jx) - \Im(f_j) \sqrt{\frac{2}{\pi}} \sin(jx) \quad (63)$$

$$= \sum_{j > 0} \Re(f_j) c^j(x) - \Im(f_j) s^j(x), \quad (64)$$

so that $(f, c^j) = \Re(f_j)$, and $(f, s^j) = -\Im(f_j)$ by the orthonormality of \mathcal{B}' . On the other hand,

$$Tf(x) = \sum_{j \neq 0} \frac{f_j}{j^2} e_j(x) = \sum_{j > 0} j^{-2} (f_j e_j(x) + \bar{f}_j e_{-j}(x)) \quad (65)$$

$$= \sum_{j > 0} \Re(f_j) j^{-2} \sqrt{\frac{2}{\pi}} \cos(jx) - \Im(f_j) j^{-2} \sqrt{\frac{2}{\pi}} \sin(jx) \quad (66)$$

$$= \sum_{j > 0} \lambda_j(f, c^j) c^j(x) + \lambda_j(f, s^j) s^j(x), \quad (67)$$

which matches the diagonalization formula guaranteed by the spectral theorem for self-adjoint, compact operators.