3.9 (i) $\begin{pmatrix}x\\y\end{pmatrix}' = \begin{pmatrix}-5 & -1\\ 6 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$. If $A = \begin{pmatrix}-5 & -1\\ 6 & 0\end{pmatrix}$, then

$$\det(A - I\lambda) = \det\begin{pmatrix}-5-\lambda & -1\\ 6 & -\lambda\end{pmatrix} = \lambda(5+\lambda)+6$$

$$= \lambda^2+5\lambda+6 = (\lambda+3)(\lambda+2)=0$$

iff $\lambda=-2$ or $\lambda=-3$

If $\lambda=\lambda_1=-2$, then $Av=\lambda_1 v$ if $-5v_1 - v_2 = -2v_1$

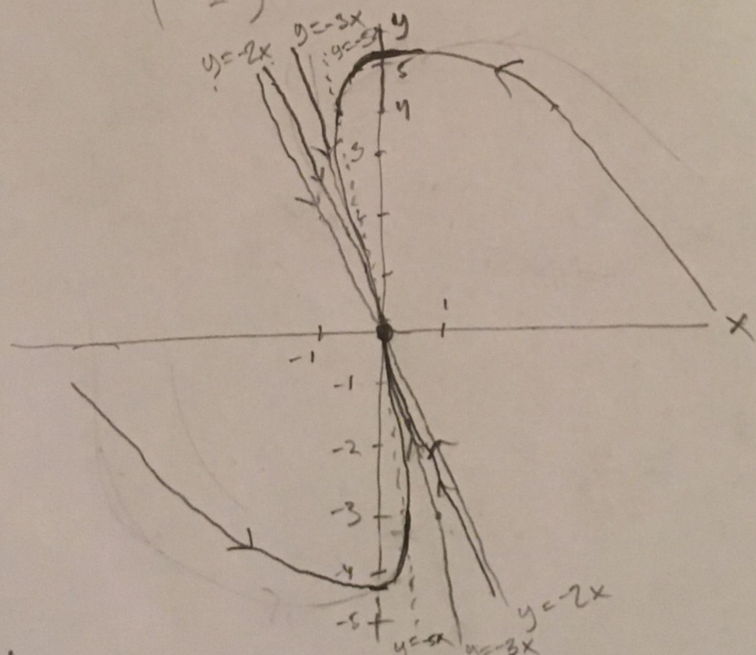
or $v_2=-3v_1$

So $v^{(1)} = \begin{bmatrix}1\\-3\end{bmatrix}$ is an e.v. for $\lambda_1=-2$.

If $\lambda=\lambda_2=-3$, then $Av=\lambda_2 v$ if $6v_1=-3v_2$

or $v_2=-2v_1$

So $v^{(2)}=\begin{pmatrix}1\\-2\end{pmatrix}$ is an e.v for $\lambda_2=-3$.



If $x'=0$, then $5x+y=0 \Rightarrow$ x nullcline is $y=-5x$

if $y'=0$, then $6x=0 \Rightarrow x=0$ is y nullcline

(0,0) is a stable node, so all arrows point toward (0,0)

3.10 (i) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$,

$$\det(A - \lambda I) = \det\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = 1 - 2\lambda + \lambda^2 - 4 = 0$$

if $(\lambda - 3)(\lambda + 1) = 0$

or $\lambda = 3$, $\lambda = -1$

If $\lambda = \lambda_1 = 3$, then $Av = \lambda_1 v \Rightarrow v_1 + 4v_2 = 3v_1$

$\Rightarrow v_1 = 2v_2$

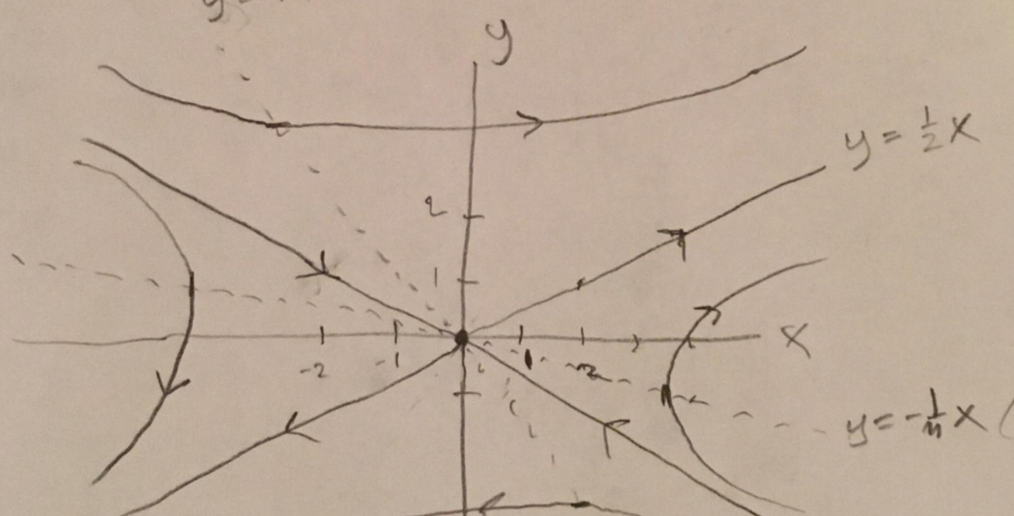
So $v^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an e.v. for $\lambda_1 = 3$.

If $\lambda = \lambda_2 = -1$, then $Av = \lambda_2 v \Rightarrow v_1 + 4v_2 = -v_1$

$\Rightarrow v_1 = -2v_2$

So $v^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an e.v. for $\lambda_2 = -1$.

$y = -x$ (y nullcline)



$x' = 0 \Rightarrow$

$y = -\frac{1}{4}x$

$y' = 0 \Rightarrow$

$y = -x$

$(0,0)$ is a saddle point since $\lambda_2 < 0 < \lambda_1$.

3.11 (i) $\begin{pmatrix}x\\y\end{pmatrix}' = \begin{pmatrix}2 & 5/2\\0 & -3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$, $A = \begin{pmatrix}2 & 5/2\\0 & -3\end{pmatrix}$ has

e.v.s $2, -3$

if $\lambda_1 = 2$, then $Av = \lambda_1 v \Rightarrow 2v_1 + \frac{5}{2}v_2 = 2v_1$

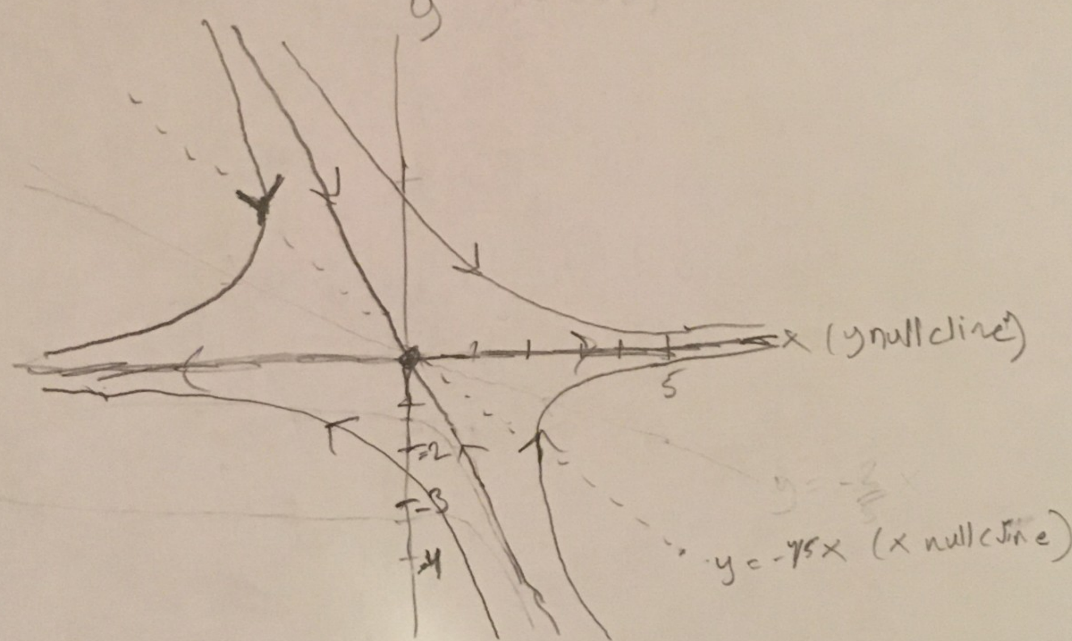
$\Rightarrow v_2 = 0$

So $v^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}$ is an e.v. for $\lambda_1 = 2$

if $\lambda_2 = -3$, then $Av = \lambda_2 v \Rightarrow 2v_1 + \frac{5}{2}v_2 = -3v_1$

$\Rightarrow v_1 = -\frac{1}{2}v_2$

so $v^{(2)} = \begin{pmatrix}1\\-2\end{pmatrix}$ is an ei.v. for $\lambda_2 = -3$.



$x' = 0$ if $4x + 5y = 0$ or $y = -\frac{4}{5}x$

$y' = 0$ if $y = 0$

$(0,0)$ is a saddle point

3.11 (ii) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -\frac{9}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{6}{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} -9/5 & 2/5 \\ 2/5 & -6/5 \end{pmatrix}$.

e.v.'s of $A = \dfrac{\text{e.v.'s of } 5A}{5}$: if $5Av = \lambda v$, then $Av = \frac{\lambda}{5} v$

$\det(5A - I\lambda) = \det \begin{pmatrix} -9-\lambda & 2 \\ 2 & -6-\lambda \end{pmatrix} = \lambda^2 + 15\lambda + 50$

$= (\lambda+5)(\lambda+10) = 0$

iff $\lambda = -5$ or $\lambda = -10$

So e.v.'s of A are $\lambda_1 = -1$, $\lambda_2 = -2$.

If $\lambda_1 = -1$, then $Av = \lambda_1 v \Rightarrow -\frac{9}{5}v_1 + \frac{2}{5}v_2 = -v_1$

$\Rightarrow v_2 = 2v_1$

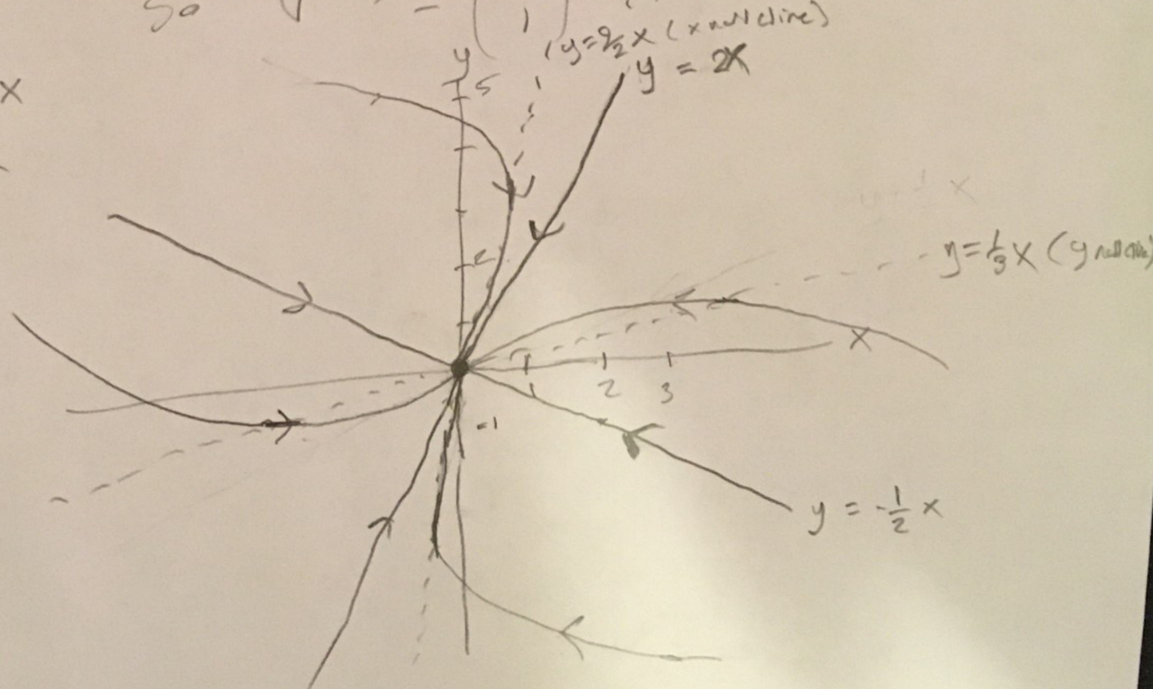
So $v^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an e.v. for $\lambda_1 = -1$

$Av = \lambda_2 v \Rightarrow -\frac{9}{5}v_1 + \frac{2}{5}v_2 = -2v_1 \Rightarrow -2v_2 = v_1$

So $v^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an e.v. for $\lambda_2 = -2$

$x' = 0 \Rightarrow y = \frac{9}{2}x$

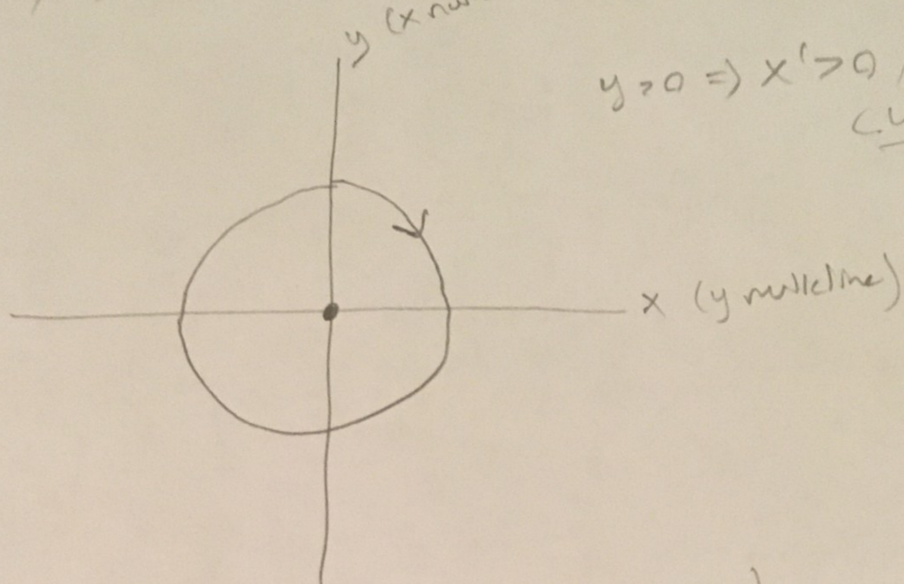
$y' = 0 \Rightarrow y = \frac{1}{3}x$



(0,0) is a stable node

3.12 (i) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eivls are $\lambda = \pm 4i$,

So $(0,0)$ is a center



$y > 0 \Rightarrow x' > 0$, so
C.W. circles

(ii) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 4 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 0 & 4 \\ -1 & 4 \end{pmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 4 \\ -1 & 4-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda - 2)^2 = 0$
$\lambda = 2$

$Av = \lambda v \Rightarrow 4V_2 = 2V_1 \Rightarrow V_1 = 2V_2$
so $V^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is e.v.

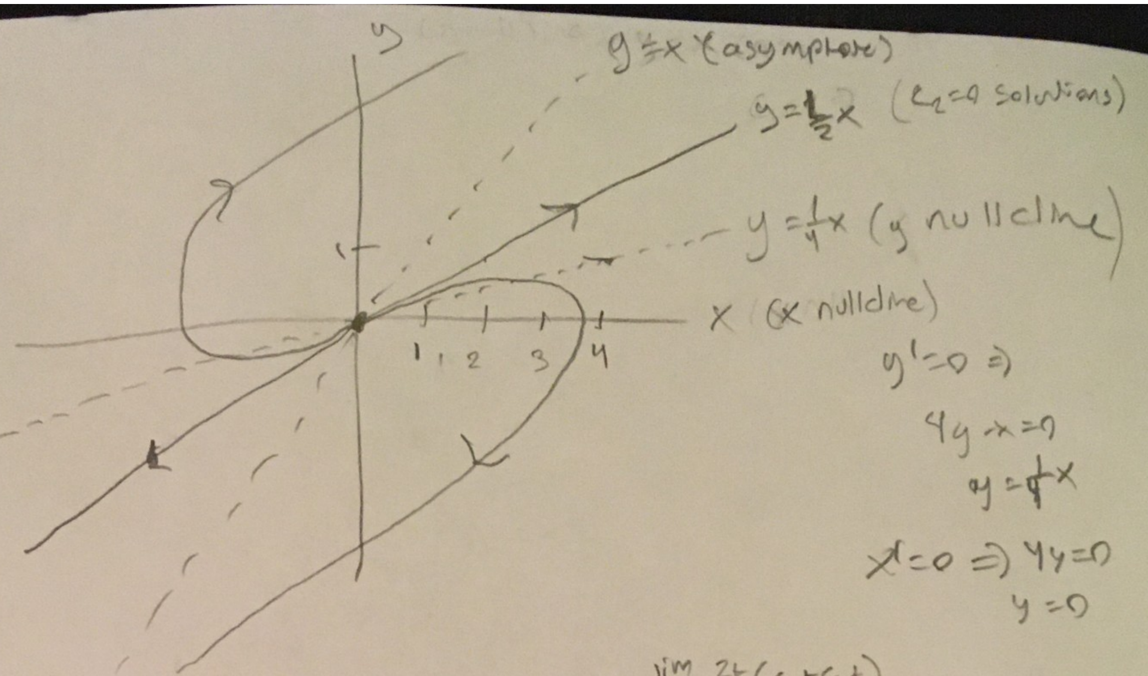
Get $V^{(2)}$ s.t. $(A - I\lambda)V^{(2)} = V^{(1)}$: $-2V_1^{(2)} + 4V_2^{(2)} = 2$
Then choose $V_2^{(2)} = 0$
$\Rightarrow V_1^{(2)} = -1$

so $x^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$ & $x^{(2)} = e^{2t}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + te^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ are lin. indp.

Solutions.

So $x = c_1 2e^{2t} + c_2(2te^{2t} - e^{2t})$
$y = c_1 e^{2t} + c_2 te^{2t}$ is a general soln.

$(0,0)$ is unstable, so
all arrows go away



$y' = 0 \Rightarrow$

$4y - x = 0$

$y = \frac{1}{4}x$

$x' = 0 \Rightarrow 4y = 0$

$y = 0$

$$\lim_{t\to\infty} \frac{y}{x} = \lim \frac{e^{2t}(c_1 + c_2 t)}{2e^{2t}(c_1 + c_2 t) - c_2 e^{2t}}$$

$$= \lim_{t\to\infty} \frac{c_1 + c_2 t}{c_1 + c_2 t - c_2}$$

$$= 1$$

so $y \to x$ as $t \to \infty$

3.12 (iii) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -8 & 10 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} -8 & 10 \\ -4 & 4 \end{pmatrix}$

$\det(A - \lambda I) = \det \begin{pmatrix} -8-\lambda & 10 \\ -4 & 4-\lambda \end{pmatrix} = -(8+\lambda)(4-\lambda) + 40$

$= \lambda^2 + 4\lambda + 8$

So $\lambda = -2 \pm \dfrac{\sqrt{16-32}}{2} = -2 \pm 2i$

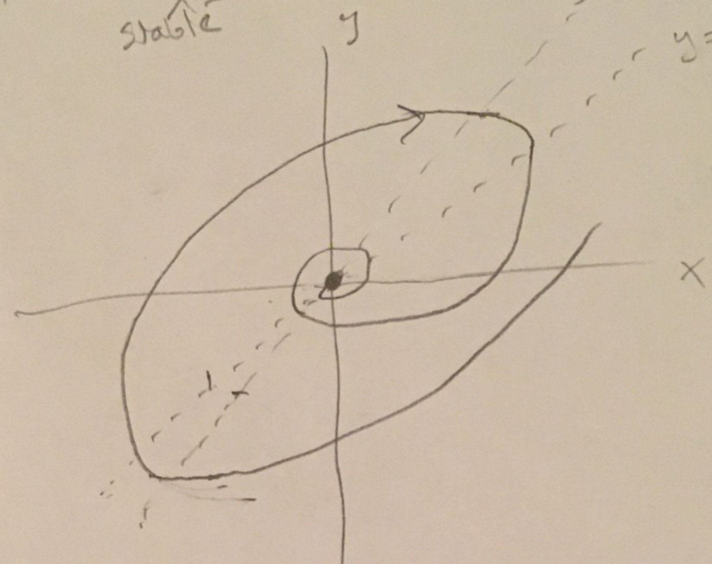
(0,0) is a $\underset{\text{stable}}{\wedge}$ spiral because $\mathrm{Re}\,\lambda < 0$, $\mathrm{Im}\,\lambda \neq 0$

$y = x$ (y nullline)

$x' = 0 \Rightarrow 5y = 4x$

$y = \frac{4}{5}x$ (x nullline) $\quad y = \frac{4}{5}x$

$y' = 0 \Rightarrow y = x$



When $x = 0$, $y > 0$

$x' = 10y > 0$

so spiral

is CW.

(iv) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & -5 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} -1 & -5 \\ 2 & 5 \end{pmatrix}$

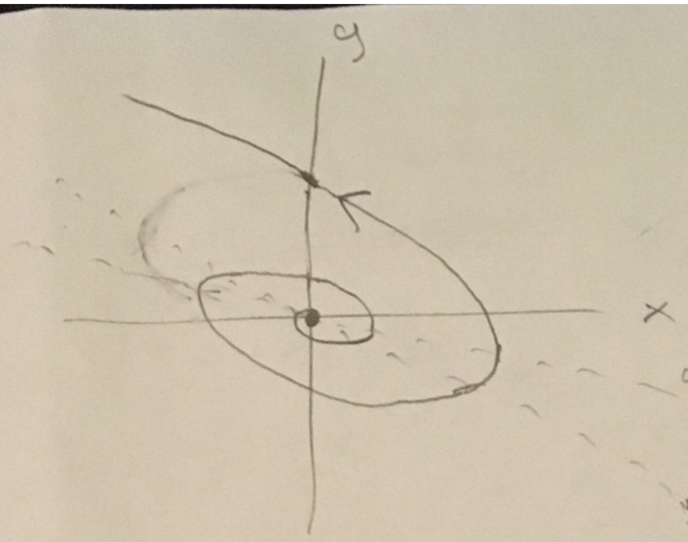
$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & -5 \\ 2 & 5-\lambda \end{pmatrix} = (5-\lambda)(-1-\lambda) + 10$

$= \lambda^2 - 4\lambda + 5$

So $\lambda = 2 \pm \dfrac{\sqrt{16-25}}{2} = 2 \pm \frac{3}{2}i$

and (0,0) is an unstable spiral because $\mathrm{Re}\,\lambda > 0$ and $\mathrm{Im}(\lambda) \neq 0$.

3.12 (iv)



when $x=0$, $y>0$

$x' = -5y < 0$

$x'=0$ if

$y = -\frac{1}{5}x$

$y'=0$ if

$y = -\frac{2}{5}x$

3.14 $x' = -axy + bz$

$y' = axy - cy$

$z' = cy - bz$

(i) $(x+y+z)' = x' + y' + z' = -\cancel{axy} + \cancel{bz} + \cancel{axy} - \cancel{cy} + \cancel{cy} - \cancel{bz}$

$= 0$

$\Rightarrow x + y + z = \text{constant} = D$

~~hat~~ Then $z = D - x - y$

So $x' = -axy + b(D - x - y) = -axy + bD - bx - by$

$y' = axy - cy$

is enough to solve the equations

(ii) $x' = 0 \Rightarrow -axy + bD - bx - by = 0 \Rightarrow y = \frac{bD - bx}{ax + b}$

$y' = 0 \Rightarrow axy - cy = 0 \Rightarrow y(ax - c) = 0$

$\Rightarrow y = 0$ or $ax - c = 0$

So x nullcline is: $y = \frac{b}{a} \frac{D - x}{x + \frac{b}{a}}$

y nullcline is: $y = 0$ or $x = \frac{c}{a}$

Let $\beta = \frac{b}{a}$, $\gamma = \frac{c}{a}$, then

x nullcline: $y = \beta \frac{D - x}{x + \beta} = \beta \frac{D + \beta}{x + \beta} - \beta$

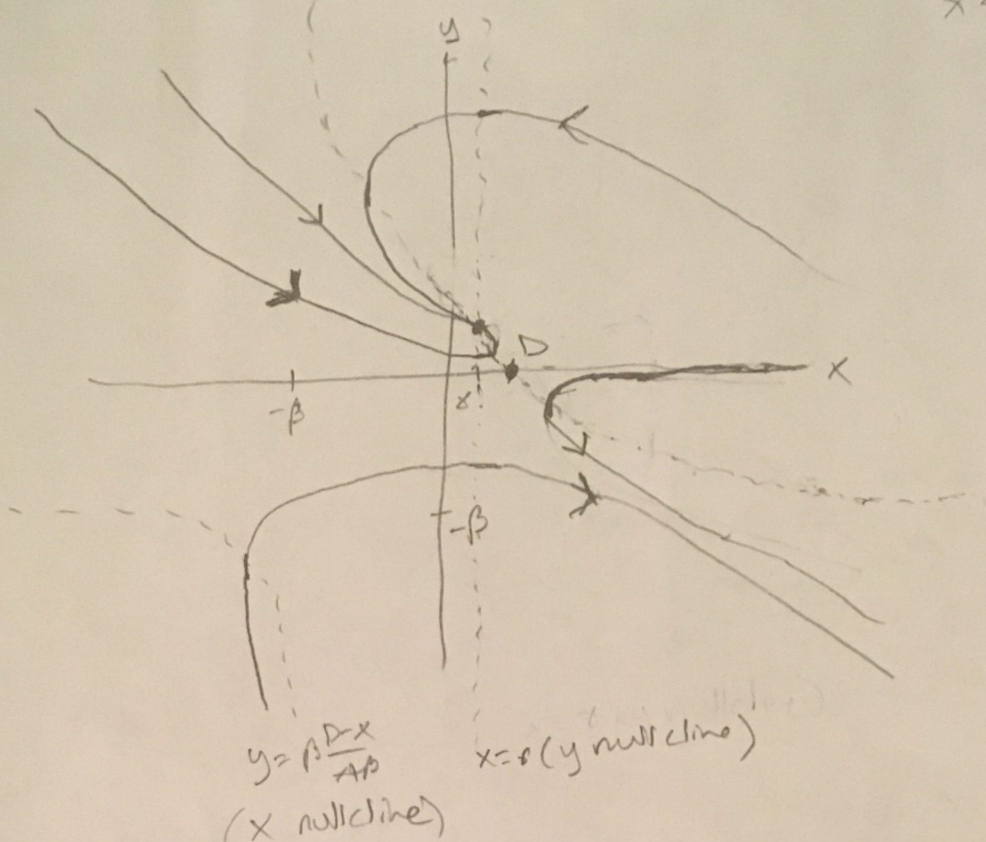
y nullcline: $y = 0$ or $x = \gamma$

c.p. when ($y = 0$ or $x = \gamma$) and $y = \beta \frac{D + \beta}{x + \beta} - \beta$

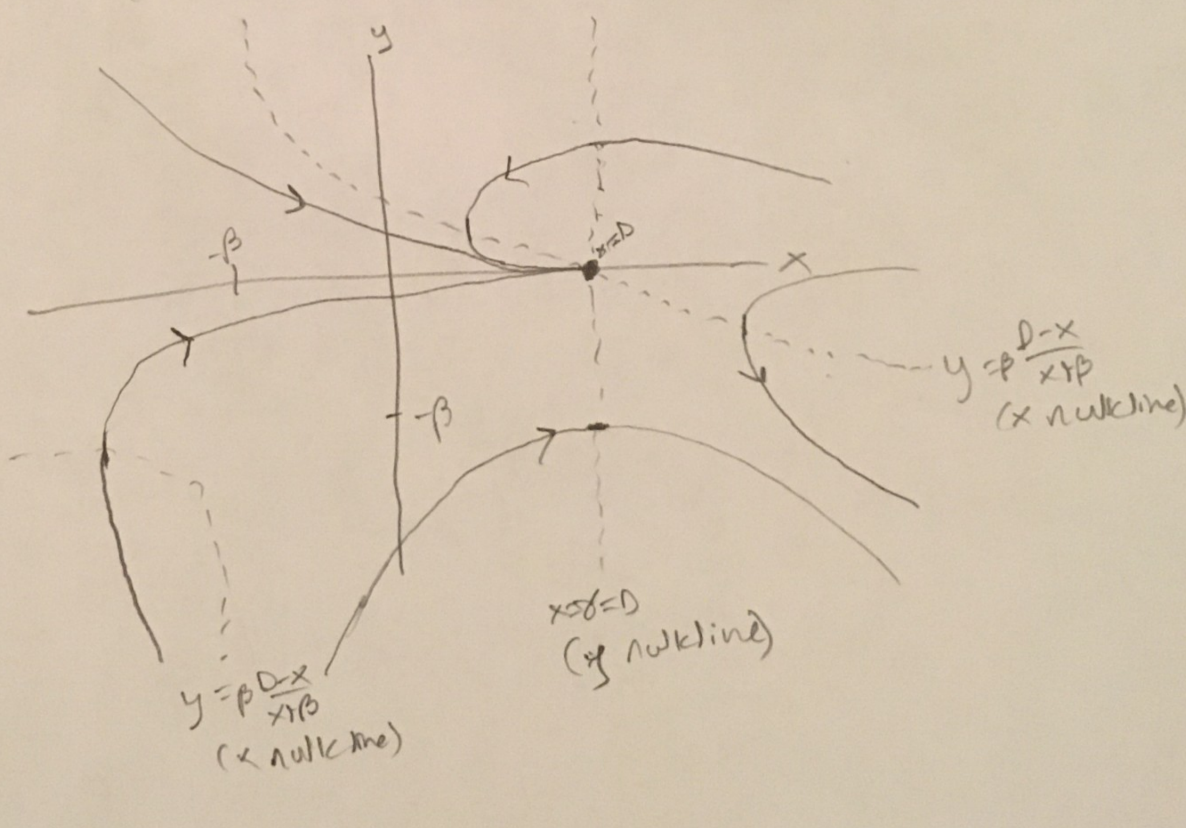
So ($y = 0$ and $\beta + x = \beta + D \Rightarrow x = D$) or ($x = \gamma$ & $y = \beta \frac{D - \gamma}{\beta + \gamma}$)

$\gamma < D$

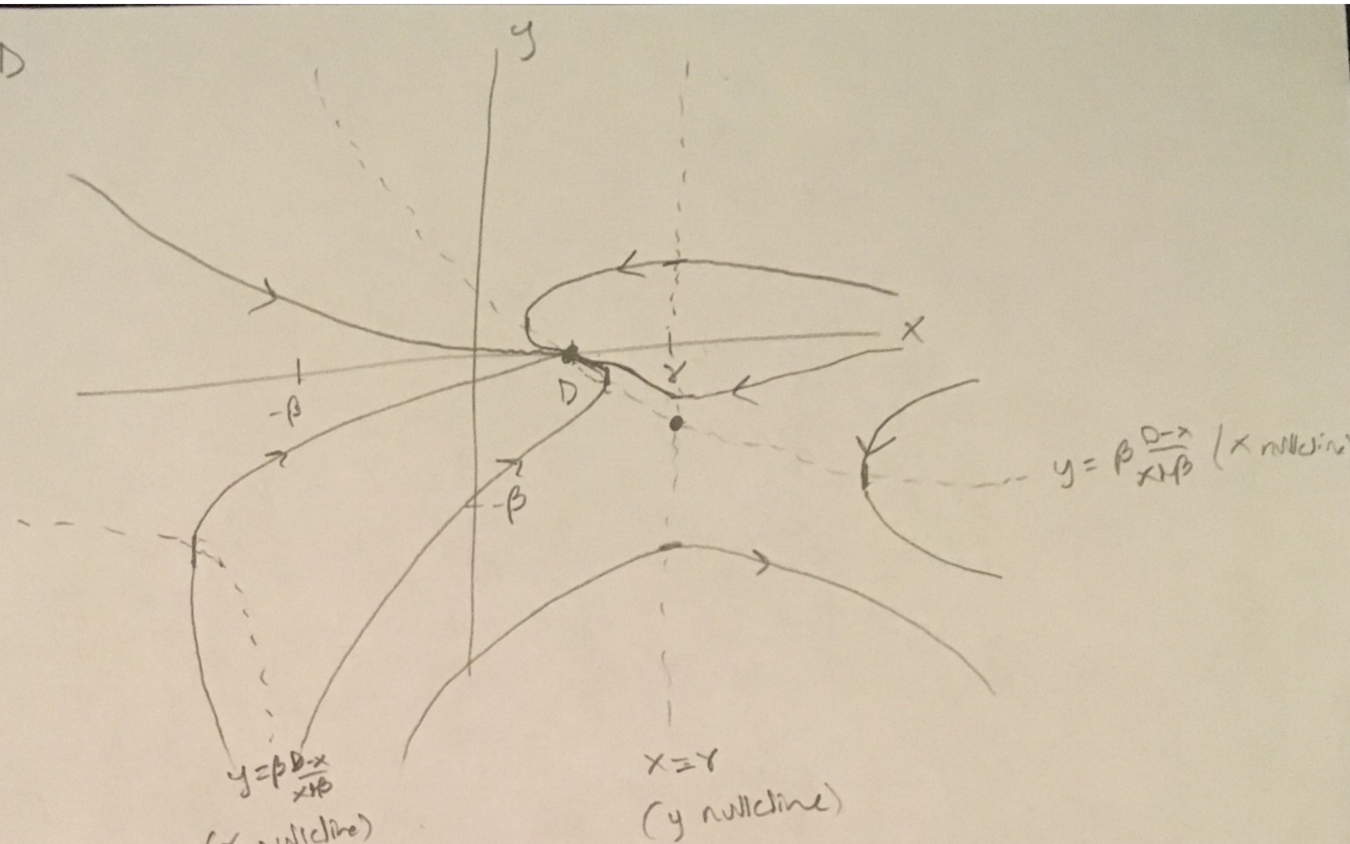
$x < -\beta \Rightarrow x+\beta < 0$
$\Rightarrow x < D \Rightarrow x - D < 0$
$\Rightarrow D - x > 0$
$\Rightarrow \beta \frac{D-x}{x+\beta} < 0$



$\gamma = D$



$\gamma > D$


y
x
$-\beta$
D
γ
$-\beta$
$y = \beta\frac{D-x}{x+\beta}$ (x nullcline)
$y = \beta\frac{D-x}{x+\beta}$
(x nullcline)
$x = \gamma$
(y nullcline)

3.15

$$x' = a\left(\frac{y}{1+y}\right)x - x$$

$$y' = -\left(\frac{y}{1+y}\right)x - y + b \quad , \quad a > 1, \quad b > \frac{1}{a-1} > 0$$

$$x' = 0 \Rightarrow x = a\frac{y}{1+y}x \Rightarrow x\left(1 - \frac{ay}{1+y}\right) = 0$$

$$\Rightarrow x = 0 \text{ or } 1+y = ay \Rightarrow y = \frac{1}{a-1}$$

$$y' = 0 \Rightarrow y = b - \frac{y}{1+y}x \Rightarrow y(1+y) = (1+y)b - xy$$

$$\Rightarrow (y-b)(y+1) = -xy$$

C.P. when $\left(x = 0 \text{ or } y = \frac{1}{a-1}\right)$ and $(y-b)(y+1) = -xy$

$\left(x = 0 \text{ and } (y = b \text{ or } y = -1)\right)$ or $\left(y = \frac{1}{a-1} \text{ and } (y-b)(y+1) = -xy\right.$

$$\Rightarrow x = -\left(\frac{1}{a-1} - b\right)(1 + a - 1)$$
$$= ba - \frac{a}{a-1} \Big)$$

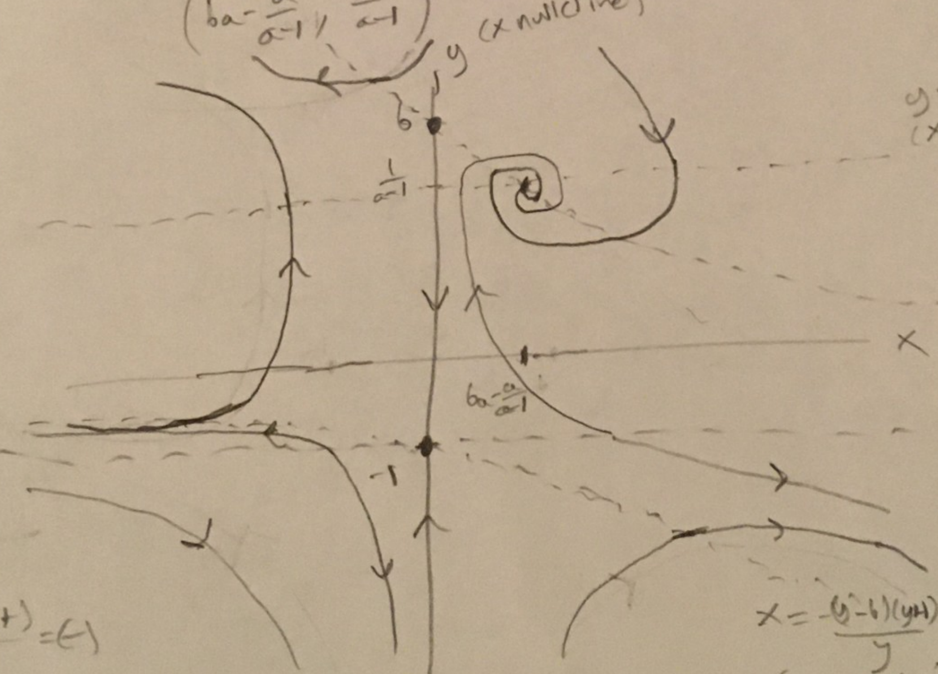
C.P.s: $(0, b)$

$(0, -1)$

$\left(ba - \frac{a}{a-1}, \frac{1}{a-1}\right)$

$(y-b)(y+1) = -xy$

$$\Rightarrow x = -\frac{(y-b)(y+1)}{y}$$



$y = \frac{1}{a-1}$
(x nullcline)

$x = -\frac{(y-b)(y+1)}{y}$
(y nullcline)

$x \to \pm\infty$ as $y \to 0^\pm$

if $0 < y < \varepsilon$
then $y+1 > 0$
and if $\varepsilon < b$
then $y < b$
$\Rightarrow y - b < 0$
so $x > 0$

if $-\varepsilon < y < 0$
then $y+1 > 0$
$y - b < 0$
$x = (-)\frac{(-)(+)}{(-)} = (-)$
so $x < 0$
$\Rightarrow x \to -\infty$ as $y \to 0^-$

$(x \to \infty$ as $y \to \infty$, $x \to -\infty$ as $y \to -\infty)$ and $x \to \infty$ as $y \to 0^+$