Math 6417 Homework 1

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September 14, 2023

Question 1.

Let f be continuous on $[0,1]\times\mathbf{R}$ and satisfy $|f(x,u)-f(x,v)|\le L|u-v|$ for all $x\in[0,1]$ and $u,v\in\mathbf{R}$, where $0\le L<8$.

For $\alpha,\beta\in\mathbf{R}$, consider the boundary value problem

$$\begin{aligned} -u''(x) = f(x,u(x)) \quad &\text{if } x\in(0,1) \\ u(0)=\alpha \qquad & u(1)=\beta. \end{aligned} \tag{1}$$

This problem has one and only one solution $u\in C^2[0,1]$.

Indeed, define

$$G(x,\xi)=\begin{cases}\xi(1-x) & 0\le\xi\le x\le 1\\ x(1-\xi) & 0\le x\le\xi\le 1\end{cases} \tag{2}$$

and also consider the integral equation

$$u(x)=\alpha(1-x)+\beta x+\int_0^1 G(x,\xi)f(\xi,u(\xi))\,\mathrm{d}\xi \quad \text{if } x\in[0,1]. \tag{3}$$

We show that if $u\in C^2[0,1]$, then u solves (1) if and only if u solves (3), and that there is a unique solution $u\in C^2[0,1]$ of (3) by the Banach Fixed Point Theorem. Then the claim follows.

(i) If $u\in C^2[0,1]$, then u is a solution of (1) if and only if u is a solution of (3).

Proof. Suppose that $u\in C^2[0,1]$ is a solution of (1). Then, using integration by parts,

$$\begin{aligned}\int_0^1 G(x,\xi)f(\xi,u(\xi))\,\mathrm{d}\xi &= -\int_0^x \xi(1-x)u''(\xi)\,\mathrm{d}\xi-\int_x^1 x(1-\xi)u''(\xi)\,\mathrm{d}\xi\\ &= -(1-x)\left[\xi u'(\xi)\Big|_0^x-\int_0^x u'(\xi)\,\mathrm{d}\xi\right]-x\left[(1-\xi)u'(\xi)\Big|_x^1+\int_x^1 u'(\xi)\,\mathrm{d}\xi\right]\\ &= -(1-x)xu'(x)+(1-x)(u(x)-u(0))\\ &\quad +x(1-x)xu'(x)-x(u(1)-u(x))\\ &= -\alpha(1-x)-\beta x+u(x)\end{aligned}$$

for any $x\in[0,1]$. Therefore, u solves (3).

Conversely, suppose that $u\in C^2[0,1]$ is a solution of (3). Then differentiating both sides of (3) implies that

$$u'(x)=\beta-\alpha+\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \xi(1-x)f(\xi,u(\xi))\,\mathrm{d}\xi+\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 x(1-\xi)f(\xi,u(\xi))\,\mathrm{d}\xi \tag{4}$$

for $x \in (0,1)$. Since the integrands in both integrals above are obviously continuous and have a continuous partial derivative with respect to x on $[0,1]^2$, the action of the derivative on the integrals gives

$$\begin{aligned} u'(x) &= \beta - \alpha + x(1-x)f(x,u(x)) - \int_0^x \xi f(\xi,u(\xi))\,\mathrm{d}\xi - x(1-x)f(x,u(x)) + \int_x^1 (1-\xi)f(\xi,u(\xi))\,\mathrm{d}\xi \\ &= \beta - \alpha - \int_0^x \xi f(\xi,u(\xi))\,\mathrm{d}\xi + \int_x^1 (1-\xi)f(\xi,u(\xi))\,\mathrm{d}\xi \end{aligned} \tag{5}$$

for $x \in (0,1)$. Since f is continuous, the integrands in the above integrals are continuous, and, upon differentiating both sides again, the Fundamental Theorem of Calculus implies that

$$u''(x) = -xf(x,u(x)) - (1-x)f(x,u(x)) = -f(x,u(x)) \tag{6}$$

for $x \in (0,1)$. Lastly, note that the definition of G implies that $G(0,\xi) = 0 = G(1,\xi)$ for all $\xi \in [0,1]$. Thus, $u(0) = \alpha$, and $u(1) = \beta$, so u solves (1). □

(ii) There is one and only one solution $u \in C^2[0,1]$ of (3).

Proof. First, note that G is continuous on $[0,1]^2$. Indeed, it is obviously continuous on the regions $\{x < \xi\}$ and $\{\xi < x\}$ by definition, and we have

$$\lim_{\substack{(x,\xi)\to(x_0,x_0) \\ x\le\xi}} G(x,\xi) = x_0(1-x_0) = \lim_{\substack{(x,\xi)\to(x_0,x_0) \\ x\ge\xi}} G(x,\xi) \tag{7}$$

for any $x_0 \in [0,1]$. Thus, G is continuous on $\{x = \xi\}$ as well, and, consequently, on all of $[0,1]^2$.

Second, for $u \in C[0,1]$, define

$$Au(x) = \alpha(1-x) + \beta x + \int_0^1 G(x,\xi)f(\xi,u(\xi))\,\mathrm{d}\xi. \tag{8}$$

Since f and u are both continuous, it follows that $f(\cdot,u(\cdot))$ is continuous and therefore bounded on $[0,1]$ by, say, $M > 0$. Then

$$\begin{aligned} \left|\int_0^1 G(x,\xi)f(\xi,u(\xi))\,\mathrm{d}\xi - \int_0^1 G(y,\xi)f(\xi,u(\xi))\,\mathrm{d}\xi\right| &\le M\int_0^1 |G(x,\xi)-G(y,\xi)|\,\mathrm{d}\xi \\ &\le M\left[\int_0^x \xi|x-y|\,\mathrm{d}\xi + \int_x^1 |x-y|(1-\xi)\,\mathrm{d}\xi\right] \\ &\le 2M|x-y| \end{aligned}$$

Hence, Au is the sum of a polynomial and a Lipschitz function, so $Au \in C[0,1]$, and $A : C[0,1] \to C[0,1]$.

Third, A is a contraction on $C[0,1]$ in the uniform metric ρ on $C[0,1]$. Indeed, for $u, v \in C[0,1]$,

$$\begin{aligned} \rho(Au,Av) &= \max_{x\in[0,1]}\left|\int_0^1 G(x,\xi)\big[f(\xi,u(\xi)) - f(\xi,v(\xi))\big]\,\mathrm{d}\xi\right| && (9) \\ &\le \max_{x\in[0,1]} L\int_0^1 |G(x,\xi)|\cdot|u(\xi)-v(\xi)|\,\mathrm{d}\xi && (10) \\ &\le L\cdot\left(\max_{x\in[0,1]}\int_0^1 |G(x,\xi)|\,\mathrm{d}\xi\right)\rho(u,v). && (11) \end{aligned}$$

By the Extreme Value Theorem,

$$\begin{aligned} p(x) = \int_0^1 |G(x,\xi)| \,\mathrm{d}\xi = \int_0^x \xi(1-x) \,\mathrm{d}\xi + \int_x^1 x(1-\xi) \,\mathrm{d}\xi = \frac{1}{2}\left[x^2(1-x) + x(1-x)^2\right] \\ = \frac{1}{2}x(1-x) \end{aligned} \tag{12}$$

achieves its maximum for $x \in [0,1]$ either when $x \in \{0,1\}$, which implies $p(x) = 0$, or else when

$$0 = p'(x) = \frac{1}{2}(1-x-x) \tag{13}$$

that is, when $x = \frac{1}{2}$, in which case $p(x) = \frac{1}{8}$. Thus, $p(x) \leq \frac{1}{8}$ for $x \in [0,1]$, and

$$\rho(Au, Av) \leq 8L\rho(u,v). \tag{14}$$

Since $8L < 1$ by hypothesis, it follows that A is a contraction on $C[0,1]$.

Fourth, by the Banach Fixed Point Theorem, there is a unique solution $u \in C[0,1]$ of (3). Since $C^2[0,1] \subseteq C[0,1]$, it follows that if $u \in C^2[0,1]$, then (3) has a unique solution in $C^2[0,1]$, namely, u. Thus, to finish the proof, we need to show that u' and u'' exist and are continuous.

The calculations on the right-hand sides of (4, 5, 6) relied only the fact that u was continuous (so that $f(\cdot, u(\cdot))$ would be continuous) and solved (3), so they apply to u here as well. Thus, u' and u'' exist, and

$$u''(x) = -f(x, u(x)), \tag{15}$$

which is continuous on $[0,1]$. Therefore $u \in C^2[0,1]$. □

Question 2.

Let $u(x,t)$ be a smooth solution of the generalized heat equation

$$\begin{aligned} u_t - \nabla \cdot (A(x)\nabla u) = 0, \qquad (x,t) \in \Omega \times (0,\infty) \\ u\big|_{t=0} = u_0 \end{aligned} \tag{16}$$

where $\Omega \subset \mathbf{R}^n$ is a smooth bounded domain, and $A : \mathbf{R}^n \to \mathbf{R}^{n\times n}$ is a positive definite matrix function.

(i) If $u\big|_{\partial\Omega} = 0$, then

$$\|u(\cdot,t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} \tag{17}$$

Proof. Applying the vector calculus identity $\nabla \cdot (\phi B) = \nabla\phi^T B + \phi \nabla \cdot B$, where $\phi : \mathbf{R}^n \to \mathbf{R}$ is a differentiable scalar function, and $B : \mathbf{R}^n \to \mathbf{R}^n$ is a differentiable vector function, to the quantity $u^{2k-1}A(x)\nabla u$, where $k \geq 1$ is an integer, we obtain the identity

$$\nabla \cdot (u^{2k-1}A(x)\nabla u) = (2k-1)u^{2(k-1)}\nabla u^T A(x)\nabla u + u^{2k-1}\nabla \cdot (A(x)\nabla u) \tag{18}$$

Multiplying both sides of (16) by u^{2k-1} and integrating both sides over Ω gives

$$\int_\Omega u^{2k-1}u_t - \nabla \cdot (A(x)\nabla u) \,\mathrm{d}x = 0 \tag{19}$$

for $t > 0$. Using (18) and the fact that $\frac{\partial(u^{2k})}{\partial t} = 2ku^{2k-1}u_t$, we get

$$\int_\Omega \frac{\partial(u^{2k})}{\partial t} \,\mathrm{d}x = 2k\int_\Omega \nabla \cdot (u^{2k-1}A(x)\nabla u) \,\mathrm{d}x - 2k(2k-1)\int_\Omega u^{2(k-1)}\nabla u^T A(x)\nabla u \,\mathrm{d}x. \tag{20}$$

Since $A(x)$ is positive definite by hypothesis, the integrand of the second term on RHS(20) is pointwise nonnegative; hence, the entire second term is nonpositive because $k \geq 1$. Applying the Divergence Theorem to the first term, we obtain the inequality

$$\int_\Omega \frac{\partial(u^{2k})}{\partial t}\,\mathrm{d}x \leq 2k \int_{\partial\Omega} u^{2k-1} A(x)\nabla u \cdot \mathbf{n}\,\mathrm{d}S. \tag{21}$$

where S is the surface measure on $\partial\Omega$, and $\mathbf{n}$ is the outward unit normal vector to Ω. Using the assumptions $u\big|_{\partial\Omega} = 0$ and $k \geq 1$, we see that the integrand in RHS(21) is equal to 0 over the domain of integration $\partial\Omega$. Hence, RHS(21) = 0. Since u is smooth, the time derivative commutes with the integral on LHS(21), so we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|_{L^{2k}}^{2k} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{2k}\,\mathrm{d}x = \int_\Omega \frac{\partial(u^{2k})}{\partial t}\,\mathrm{d}x \leq 0 \tag{22}$$

for $t > 0$. This implies that

$$\|u(\cdot,t)\|_{L^{2k}}^{2k} \leq \|u(\cdot,0)\|_{L^{2k}}^{2k} \iff \|u(\cdot,t)\|_{L^{2k}} \leq \|u_0\|_{L^{2k}} \tag{23}$$

for $t > 0$ and $k \geq 1$ an integer. Taking the limit as $k \to \infty$ on both sides and applying Proposition 2.18 from Arbogast and Bona, we obtain the desired result. □

(ii) Suppose that $u\big|_{\partial\Omega} = g$, a nonzero, smooth function on $\partial\Omega$. Let v be a smooth solution of the equation

$$\begin{aligned} \nabla\cdot(A(x)\nabla v) &= 0, \qquad x \in \Omega \\ v\big|_{\partial\Omega} &= g. \end{aligned} \tag{24}$$

Then $u - v$ is a smooth solution of (16) such that $u - v\big|_{\partial\Omega} = 0$. Hence, by the previous problem,

$$\|u(\cdot,t) - v\|_{L^\infty} \leq \|u(\cdot,0) - v\|_{L^\infty} = \|u_0 - v\|_{L^\infty} \tag{25}$$

for $t > 0$. Interpreting this inequality, we might say that u does not deviate from 0 by no more than the initial value u_0 in L^∞ norm (the previous situation) but, rather, that u deviates from the *equilibrium* v by no more than than the initial value u_0 in L^∞ norm.