Lax-Milgram Theorem, HW3 no.1, Due date: TBA

Lax-Milgram theorem

(HW3, no.1) Let $B(\cdot, \cdot)$ be a continuous bilinear form on a Hilbert space H. Suppose that B is coercive in the sense that $\exists \alpha > 0$, s.t. $B(\mathbf{x}, \mathbf{x}) \geq \alpha \|\mathbf{x}\|^2, \forall \mathbf{x} \in H$.

1. Fix a $\mathbf{y} \in H$, show that the map $\mathbf{x} \mapsto B(\mathbf{x}, \mathbf{y})$ is a bounded linear functional on H. Therefore, $\exists \mathbf{w} \in H$, s.t. $B(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{w}), \forall \mathbf{x} \in H$.
2. Define $A : H \to H$ by $A\mathbf{y} = \mathbf{w}$. Show that $A \in B(H)$.
3. Show that A is bounded below in the sense that $\exists \gamma > 0$, s.t., $\|A\mathbf{x}\| \geq \gamma \|\mathbf{x}\| \ \forall \mathbf{x} \in H$.
4. Show that A is one-to-one, and the range of A is closed.
5. Show that A is onto by showing that $R(A)^{\perp} = \{0\}$
6. Show that A is invertible
7. Given $f \in H^*$, find $\mathbf{w} \in H$ s.t. $f(\mathbf{x}) = (\mathbf{x}, \mathbf{w}) \ \forall \mathbf{x} \in H$
8. Solve the equation $B(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}), \forall \mathbf{x} \in H$ by converting this to the equation $A\mathbf{y} = \mathbf{w}$

Application of Lax-Milgram Theorem, HW3 no.2, Due date: TBA

Let

$$H = \dot{H}^1_{per} := \{f \in L^2(-\pi, \pi), f(x) = \sum_{j \neq 0} f_j e^{ijx}, \sum_{j \neq 0} j^2 |f_j|^2 < \infty, f = \bar{f}\}, (f, g)_H = \sum_{j \neq o} j^2 f_j \overline{g_j}$$

$$B(f, g) = \sum_{j \neq 0} (ij + j^2) f_j \overline{g_j}, \;\; f, g \in H.$$

$$\dot{H}^{-1}_{per} := \{f(x) = \sum_{j=-\infty, j \neq 0}^{\infty} f_j e^{ijx}, \quad \sum_{j \neq 0} j^{-2} |f_j|^2 < \infty, f = \bar{f}\}, (f, g)_{H^{-1}} = \sum_{j \neq o} j^{-2} f_j \overline{g_j}$$

1. Show that H, H^{-1} are Hilbert spaces.
2. Show that the bilinear form B satisfies the conditions of the Lax-Milgram theorem.
3. Show that $\dot{H}^{-1}_{per} \subset (\dot{H}^1_{per})^*$ with the action given by $f(g) = \sum_{j \neq 0} f_j \overline{g_j}$ for $f \in \dot{H}^{-1}_{per}, g \in H$.
4. Show that for each $f \in \dot{H}^{-1}_{per}$, there exists $\mathbf{u} \in \dot{H}^1_{per}$ s.t. $B(\mathbf{x}, \mathbf{u}) = f(\mathbf{x}), \forall \mathbf{x} \in \dot{H}^1_{per}$
5. Does **u** solve a differential equation? If yes, which one?
   (Hint: you may wish to formally differentiate the Fourier series once and twice to gain insight.)
6. (optional) Show that any bounded set in H is pre-compact in L^2, i.e., for the space of doubly infinite sequence with $\sum_{j \neq 0} j^2 |f_j|^2 < \infty$ is pre-compact in the space of double infinite sequence with $\sum_{j \neq 0} |f_j|^2 < \infty$.
   This is a special case of the so-called Sobolev imbedding of H^1 into L^2.