Math 5604 Homework 9

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Consider the 1D heat equation

$$u_t = \frac{1}{2}u_{xx} + f(x,t), \qquad 0 < x < 1,\ t > 0,$$

with source term

$$f(x,t) = \left(\frac{\pi^2}{2} - 1\right)\sin\left(\pi\left(x + \frac{1}{2}\right)\right)$$

and Dirichlet boundary conditions given by

$$u(0,t) = e^{-t}, \qquad u(1,t) = -e^{-t}, \qquad t > 0$$

and initial condition

$$u(x,0) = \sin\left(\pi\left(x + \frac{1}{2}\right)\right), \qquad 0 \le x \le 1.$$

The exact solution is $u(x,t) = e^{-t}\sin\left(\pi\left(x + \frac{1}{2}\right)\right)$.

Problem 1.

(a) Discretizing this equation on the time interval $[0,1]$ using a central difference method in space and the forward Euler method in time with the space sample points $x_i = ih$ for $i = 0, 1, \ldots, M$ and time sample points $t_n = nk$ for $n = 0, 1, \ldots N$, where $h = \frac{1}{M}$, and $k = \frac{1}{N}$, we obtain

$$\begin{aligned}
\frac{u_{i+1}^n - u_i^n}{k} &= \frac{1}{2}\cdot\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2} + f(x_i, t_n), \qquad i = 1, 2, \ldots, M-1,\ n = 0, 1, \ldots, N-1,\\
u_0^n &= e^{-t_n}, \quad u_M^n = -e^{-t_n}, \qquad n = 0, 1, \ldots, N,\\
u_i^0 &= \sin\left(\pi\left(x + \frac{1}{2}\right)\right), \qquad i = 0, 1, \ldots, M.
\end{aligned}$$

where $u_i^n \approx u(x_i, t_n)$. We can rewrite this system as

$$U^{n+1} = U^n + k(AU^n + b^n),$$

where

$$U^n = \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{M-1}^n \end{bmatrix}, \qquad A = \frac{1}{2h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}, \qquad b^n = \begin{bmatrix} f(x_1, t_n) + \frac{e^{-t_n}}{2h^2} \\ f(x_2, t_n) \\ \vdots \\ f(x_{M-2}, t_n) \\ f(x_{M-1}, t_n) - \frac{e^{-t_n}}{2h^2} \end{bmatrix}.$$

This linear recurrence is implemented in `problem1.m` using a simple for loop.

k	L^∞ error	L^∞ rate	L^2 error	L^2 rate
1/1024	2.508190e-04	-	1.824251e-04	-
1/4096	6.352731e-05	0.990599	4.552322e-05	1.001315
1/16384	1.587421e-05	1.000346	1.137542e-05	1.000342
1/65536	3.968078e-06	1.000086	2.843515e-06	1.000086
1/262144	9.920215e-07	0.999998	7.108574e-07	1.000022

Table 1: First-order convergence in time of the forward Euler method. Note that $h^2 = 4k$. Error is computed at $t = 1$.

h	L^∞ error	L^∞ rate	L^2 error	L^2 rate
1/16	2.586792e-04	-	1.881419e-04	-
1/32	6.542069e-05	1.983345	4.688000e-05	2.004777
1/64	1.625258e-05	2.009078	1.164655e-05	2.009070
1/128	3.968078e-06	2.034156	2.843515e-06	2.034156
1/256	8.974522e-07	2.144533	6.430914e-07	2.144580

Table 2: Second-order convergence in space of the forward Euler method. Note that $k = \frac{1}{16384}$. Error is computed at $t = 1$.

(b) The forward Euler method is unstable if $k > ch^2$ for some constant c. We determine this constant empirically to be roughly 1 by using the bisection method in `stability_test.m`. By setting h^2 a little bit larger than k, we can still see the first-order convergence in k; see Table 1.

Observing second-order convergence in space is easier given the constraint $k < h^2$. We simply calculate the error for various values of h with k fixed and much smaller than the smallest value of h^2; see Table 2. These tables are generated by running `problem1_calculations.m`.

Problem 2.

(a) Discretizing this equation on the time interval $[0, 1]$ using a central difference method in space and the Crank-Nicolson method in time with the space sample points $x_i = ih$ for $i = 0, 1, \ldots, M$ and time sample points $t_n = nk$ for $n = 0, 1, \ldots N$, where $h = \frac{1}{M}$, and $k = \frac{1}{N}$, we obtain

$$\frac{u_{i+1}^n - u_i^n}{k} = \frac{1}{2}\left[\frac{1}{2} \cdot \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2} + f(x_i, t_n) + \frac{1}{2} \cdot \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{h^2} + f(x_i, t_{n+1})\right],$$

$$i = 1, 2, \ldots, M-1,\ n = 0, 1, \ldots, N-1,$$

$$u_0^n = e^{-t_n}, \quad u_M^n = -e^{-t_n}, \qquad n = 0, 1, \ldots, N,$$

$$u_i^0 = \sin\left(\pi\left(x + \frac{1}{2}\right)\right), \qquad i = 0, 1, \ldots, M.$$

where $u_i^n \approx u(x_i, t_n)$. We can rewrite this system as

$$U^{n+1} = U^n + \frac{k}{2}\left(AU^n + b^n + AU^{n+1} + b^{n+1}\right),$$

where A and b^n are defined in the same way as in Problem 1. This linear recurrence is implemented in `problem2.m` using a simple for loop; note that we can solve the following, equivalent linear system for U^{n+1} in terms of U^n by using a linear solver like the \ operator in MATLAB:

$$\left(I - \frac{k}{2}A\right)U^{n+1} = U^n + \frac{k}{2}\left(AU^n + b^n + b^{n+1}\right)$$

(b) To observe second-order convergence in time of the Crank-Nicolson method, we calculate the error at various time step values k while keeping the spatial step size h much smaller than the smallest value of k; see Table 3.

To observe second-order convergence in space of the Crank-Nicolson method, we calculate the error for various values of h with k fixed; see Table 2. These tables are generated by running `problem2_calculations.m`. See `p2_outputs.txt` for the raw outputs.

k	L^∞ error	L^∞ rate	L^2 error	L^2 rate
1/8	2.173901e-05	-	1.536419e-05	-
1/16	5.358565e-06	2.020368	3.841918e-06	1.999673
1/32	1.328935e-06	2.011576	9.522130e-07	2.012471
1/64	3.202598e-07	2.052956	2.294772e-07	2.052933
1/128	6.808230e-08	2.233891	4.877780e-08	2.234055

Table 3: Second-order convergence in time of the Crank-Nicolson method. Note that $h = \frac{1}{2048}$. Error is computed at $t = 1$.

h	L^∞ error	L^∞ rate	L^2 error	L^2 rate
1/16	2.587832e-04	-	1.882175e-04	-
1/32	6.552588e-05	1.981607	4.695538e-05	2.003039
1/64	1.635768e-05	2.002097	1.172187e-05	2.002088
1/128	4.073157e-06	2.005749	2.918814e-06	2.005748
1/256	1.002529e-06	2.022503	7.183869e-07	2.022549

Table 4: Second-order convergence in space of the Crank-Nicolson method. Note that $k = \frac{1}{256}$. Error is computed at $t = 1$.