HW4 no.1. eigenvalue and eigenfunctions of the Fourier transform operator, Due December 12

- Define the Fourier transform operator from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$ as

$$(\mathcal{F}f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} f(x)\, dx$$

- Show that $\|\mathcal{F}f\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$
- Show that if $f \in C^2$ and $f', f'' \in L^1$, $f, f', f'' \to 0$ as $x \to \pm\infty$, then $|y^2(\mathcal{F}f)(y)| \leq$ constant $\forall y$. Therefore, $\mathcal{F}f \in L^1$.
- Show formally that $(\mathcal{F}^2 f)(x) = f(-x)$ for "nice" f
  (Compute the formal Fourier transform of the Dirac delta function and the constant function. You may wish to use an appropriate auxiliary function and utilize Fubini's theorem.)
- Show formally that $\mathcal{F}^4 = I$
- Find all possible eigenvalues of $\mathcal{F}$
- Find at least one eigenfunction of $\mathcal{F}$. Bonus points for additional linearly independent eigenfunctions.

HW4 no.2, explicit example of a compact SA operator, Due Decmber 12

Let T be the solution operator associated with the problem of finding $\mathbf{u} \in \dot{H}^1_{per}(-\pi, \pi)$ for given $f \in \dot{L}^2(-\pi, \pi)$[6] such that $-\mathbf{u}'' = f$.

1. Show that the problem can be solved uniquely by invoking the Lax-Milgram Theorem.
   (Hint: HW3 no.2 could be useful.)
2. Show that the solution operator $T : f \to \mathbf{u}$ is a compact operator on $\dot{L}^2(-\pi, \pi)$
3. Show that the solution operator is self-adjoint
4. Find all the eigenvalues and the corresponding eigenspaces of T
5. Find an ONB so that T is diagonalized as specified by the spectral theorem for self-adjoint cpt operators
6. (optional) / Suppose you are looking for an approximate solution that belongs to the linear span of the first $2n$ elements of the ONB. Which equation the approximate solution should satisfy if you try to utilize the bilinear form B? (This is a case of the so-called spectral method of solving differential equations.)

[6] $\dot{L}^2$ is the mean zero subspace of L^2