HW2 no.1. Due Oct. 19, 2023

A continuous function σ is called sigmoidal if

$$\sigma(t) = \begin{cases} 1 & \text{if } t \geq T \\ 0 & \text{if } t \leq -T \end{cases}$$

1. Let $\sigma_\lambda(x) = \sigma(\lambda(y^T x + \theta) + \phi)$, where λ, θ, and ϕ are parameters, and y is fixed. Show that $\sigma_\lambda(x) \to \gamma(x) = \begin{cases} 1 & \text{for } y^T x + \theta > 0 \\ 0 & \text{for } y^T x + \theta < 0 \\ \sigma(\phi) & \text{for } y^T x + \theta = 0 \end{cases}$
2. Let $\Pi_{y,\theta} = \{x | y^T x + \theta = 0\}$, $H_{y,\theta} = \{x | y^T x + \theta > 0\}$. Show that if for a given finite Borel measure μ on the unit interval, $\int_0^1 \sigma_\lambda(x) d\mu(x) = 0, \forall \lambda, \theta, \phi$, imples $0 = \sigma(\phi)\mu(\Pi_{y,\theta}) + \mu(H_{y,\theta}), \forall \phi, \theta, y$
   (Hint: use Lebesgue dominated convergence theorem)
3. Let μ be as defined in part (2). Define a linear function F as $F(h) = \int_0^1 h(y^T x) d\mu(x), h \in L^\infty(\mathbb{R})$. Show that $\mu = 0$ by showing (i) $F(h) = 0$ for $h = \chi_{[\theta,\infty)}$, and (ii) $F(h) = 0$ for $h = \chi_{[a,b)}$ $\forall\, a < b$.
4. Let σ be a continuous sigmoidal function. Then finite sums of the form

$$G(x) = \sum_{j=1}^{N} \alpha_j \sigma(y_j^T x + \theta_j)$$

   are dense in $C[0,1]$ with the usual norm.
   (Hint: you may assume that the duel space of $C[0,1]$ is the space of Radon measures on $[0,1]$. Use Hahn-Banach theorem in your BWOC proof.)

HW2 no.2. Due Oct. 19, 2023

Define a family of linear functionals on $C([0,1]) = X$ as

$$I_n(f) = \sum_{j=0}^{n} w_j^n f(x_j^n), f \in X,$$

where $0 \leq x_0^n < x_1^n < \cdots < x_n^n \leq 1$ are the chosen nodes, and $w_j^n, 0 \leq j \leq n$ are the chosen weights. These are numerical quadrature formula with weight function $w \in L^1([0,1])$.

1. Show that I_n are bounded linear functionals with the norm given by $\sum_{j=0}^{n} |w_j^n|$ where X is equipped with the standard supremum norm. (Hint: For the equal part, construct a piecewise linear continuous function that equals the sign of the weights at each node.)
2. Suppose that the numerical quadrature formula converges in the sense that $\lim_{n\to\infty} |\int_0^1 f(x)w(x)\,dx - I_n(f)| = 0, \forall f \in C([0,1])$. Show that $\sup_{n\geq 0} \left(\sum_{j=0}^{n} |w_j^n|\right) < \infty$. (Hint: Banach-Steinhaus theorem could be useful.)
3. Suppose that $\sup_{n\geq 0} \left(\sum_{j=0}^{n} |w_j^n|\right) < \infty$, and assume that the quadrature formula works on the polynomials, i.e., $\lim_{n\to\infty} |\int_0^1 p(x)w(x)\,dx - I_n(p)| = 0, \forall p \in \mathcal{P}([0,1])$. Show that the quadrature formula is valid, i.e., $\lim_{n\to\infty} |\int_0^1 f(x)w(x)\,dx - I_n(f)| = 0, \forall f \in C([0,1])$. (Hint: Any continuous function can be approximated well by polynomials thanks to Weierstrass theorem.)