Math 6417 Homework 1

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Question 1.

Let f be continuous on $[0,1] \times \mathbf{R}$ and satisfy $|f(x,u) - f(x,v)| \le L|u - v|$ for all $x \in [0,1]$ and $u, v \in \mathbf{R}$, where $0 \le L < 8$.

For $\alpha, \beta \in \mathbf{R}$, consider the boundary value problem

$$\begin{gathered} -u''(x) = f(x, u(x)) \quad \text{if } x \in (0,1) \\ u(0) = \alpha \qquad u(1) = \beta. \end{gathered} \tag{1}$$

This problem has one and only one solution $u \in C^2[0,1]$.

Indeed, define

$$G(x,\xi) = \begin{cases} \xi(1-x) & 0 \le \xi \le x \le 1 \\ x(1-\xi) & 0 \le x \le \xi \le 1 \end{cases} \tag{2}$$

and also consider the integral equation

$$u(x) = \alpha(1-x) + \beta x + \int_0^1 G(x,\xi) f(\xi, u(\xi))\, \mathrm{d}\xi \quad \text{if } x \in [0,1]. \tag{3}$$

We show that if $u \in C^2[0,1]$, then u solves (1) if and only if u solves (3), and that there is a unique solution $u \in C^2[0,1]$ of (3) by the Banach Fixed Point Theorem. Then the claim follows.

(i) If $u \in C^2[0,1]$, then u is a solution of (1) if and only if u is a solution of (3).

Proof. Suppose that $u \in C^2[0,1]$ is a solution of (1). Then, using integration by parts,

$$\begin{aligned} \int_0^1 G(x,\xi) f(\xi, u(\xi))\, \mathrm{d}\xi &= -\int_0^x \xi(1-x) u''(\xi)\, \mathrm{d}\xi - \int_x^1 x(1-\xi) u''(\xi)\, \mathrm{d}\xi \\ &= -(1-x)\left[\xi u'(\xi)\Big|_0^x - \int_0^x u'(\xi)\, \mathrm{d}\xi\right] - x\left[(1-\xi)u'(\xi)\Big|_x^1 + \int_x^1 u'(\xi)\, \mathrm{d}\xi\right] \\ &= -(1-x)xu'(x) + (1-x)(u(x) - u(0)) \\ &\quad + x(1-x)xu'(x) - x(u(1) - u(x)) \\ &= -\alpha(1-x) - \beta x + u(x) \end{aligned}$$

for any $x \in [0,1]$. Therefore, u solves (3).

Conversely, suppose that $u \in C^2[0,1]$ is a solution of (3). Then differentiating both sides of (3) implies that

$$u'(x) = \beta - \alpha + \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \xi(1-x) f(\xi, u(\xi))\, \mathrm{d}\xi + \frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 x(1-\xi) f(\xi, u(\xi))\, \mathrm{d}\xi \tag{4}$$

for $x \in (0,1)$. Since the integrands in both integrals above are obviously continuous and have a continuous partial derivative with respect to x on $[0,1]^2$, the action of the derivative on the integrals gives

$$\begin{aligned} u'(x) &= \beta - \alpha + x(1-x)f(x,u(x)) - \int_0^x \xi f(\xi,u(\xi))\,\mathrm{d}\xi - x(1-x)f(x,u(x)) + \int_x^1 (1-\xi)f(\xi,u(\xi))\,\mathrm{d}\xi \\ &= \beta - \alpha - \int_0^x \xi f(\xi,u(\xi))\,\mathrm{d}\xi + \int_x^1 (1-\xi)f(\xi,u(\xi))\,\mathrm{d}\xi \qquad (5) \end{aligned}$$

for $x \in (0,1)$. Since f is continuous, the integrands in the above integrals are continuous, and the Fundamental Theorem of Calculus implies that

$$u''(x) = -xf(x,u(x)) - (1-x)f(x,u(x)) = -f(x,u(x)) \qquad (6)$$

for $x \in (0,1)$. Lastly, note that the definition of G implies that $G(0,\xi) = 0 = G(1,\xi)$ for all $\xi \in [0,1]$. Thus, $u(0) = \alpha$, and $u(1) = \beta$, so u solves (1). $\square$

(ii) There is one and only one solution $u \in C^2[0,1]$ of (3).

Proof. First, note that G is continuous on $[0,1]^2$. Indeed, it is obviously continuous on the regions $\{x < \xi\}$ and $\{\xi < x\}$ by definition, and we have

$$\lim_{\substack{(x,\xi)\to(x_0,x_0) \\ x\le\xi}} G(x,\xi) = x_0(1-x_0) = \lim_{\substack{(x,\xi)\to(x_0,x_0) \\ x\ge\xi}} G(x,\xi) \qquad (7)$$

for any $x_0 \in [0,1]$. Thus, G is continuous on $\{x = \xi\}$ as well, and consequently on all of $[0,1]^2$.

Second, for $u \in C[0,1]$, define

$$Au(x) = \alpha(1-x) + \beta x + \int_0^1 G(x,\xi)f(\xi,u(\xi))\,\mathrm{d}\xi. \qquad (8)$$

Since f is continuous on $[0,1]^2$, it is also bounded on $[0,1]^2$, say, by $M > 0$. Then

$$\begin{aligned} \left|\int_0^1 G(x,\xi)f(\xi,u(\xi))\,\mathrm{d}\xi - \int_0^1 G(y,\xi)f(\xi,u(\xi))\,\mathrm{d}\xi\right| &\le M\int_0^1 |G(x,\xi) - G(y,\xi)|\,\mathrm{d}\xi \\ &\le M\left[\int_0^x \xi|x-y|\,\mathrm{d}\xi + \int_x^1 |x-y|(1-\xi)\,\mathrm{d}\xi\right] \\ &\le M|x-y| \end{aligned}$$

Hence, Au is the sum of a polynomial and a Lipschitz function, so $Au \in C[0,1]$, and $A : C[0,1] \to C[0,1]$.

Third, A is a contraction on $C[0,1]$ in the uniform metric ρ on $C[0,1]$. Indeed, for $u, v \in C[0,1]$,

$$\begin{aligned} \rho(Au,Av) &= \max_{x\in[0,1]}\left|\int_0^1 G(x,\xi)\big[f(\xi,u(\xi)) - f(\xi,v(\xi))\big]\,\mathrm{d}\xi\right| &(9) \\ &\le \max_{x\in[0,1]} L\int_0^1 |G(x,\xi)|\cdot|u(\xi)-v(\xi)|\,\mathrm{d}\xi &(10) \\ &\le L\cdot\left(\max_{x\in[0,1]}\int_0^1 |G(x,\xi)|\,\mathrm{d}\xi\right)\rho(u,v). &(11) \end{aligned}$$

By the Extreme Value Theorem,

$$\begin{aligned} p(x) = \int_0^1 |G(x,\xi)|\ \mathrm{d}\xi &= \int_0^x \xi(1-x)\ \mathrm{d}\xi + \int_x^1 x(1-\xi)\ \mathrm{d}\xi = \frac{1}{2}\left[x^2(1-x) + x(1-x)^2\right] \\ &= \frac{1}{2}x(1-x) \end{aligned} \tag{12}$$

achieves its maximum for $x \in [0,1]$ when $x \in \{0,1\}$, which implies $p(x) = 0$, or when

$$0 = p'(x) = \frac{1}{2}(1-x-x) \tag{13}$$

that is, when $x = \frac{1}{2}$, in which case $p(x) = \frac{1}{8}$. Thus, $p(x) \leq \frac{1}{8}$ for $x \in [0,1]$, and

$$\rho(Au, Av) \leq 8L\rho(u,v). \tag{14}$$

Since $8L < 1$ by hypothesis, it follows that A is a contraction on $C[0,1]$.

Fourth, by the Banach Fixed Point Theorem, there is a unique solution $u \in C[0,1]$ of (3). Since $C^2[0,1] \subseteq C[0,1]$, it follows that if $u \in C^2[0,1]$, then (3) has a unique solution in $C^2[0,1]$, namely, u. Thus, to finish the proof, we need to show that u' and u'' exist and are continuous.

Since u is a solution of (3), it follows that

$$u(x) = \alpha(1-x) + \beta x + \int_0^x (1-x)\xi f(\xi, u(\xi))\ \mathrm{d}\xi + \int_x^1 x(1-\xi)f(\xi,u(\xi))\ \mathrm{d}\xi. \tag{15}$$

The calculations in (4, 5, 6) relied only the fact that u was continuous (so that $f(\cdot, u(\cdot))$ would be continuous), so they apply to u here. Thus, u' and u'' exist, and

$$u''(x) = -f(x, u(x)), \tag{16}$$

which is continuous on $[0,1]$. Therefore $u \in C^2[0,1]$. $\square$