Exponential Functions

Jacob Hauck

Math 6010

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$$3 = 3, \qquad \frac{2}{5} = 0.4, \qquad \pi = 3.14159265..., \qquad \sqrt{2} = 1.41421356...$$

Approximate any number x by finitely many of its decimal digits

$$a^{\pi} \approx a^{3.14} = a^{\frac{314}{100}} = \sqrt[100]{a^{314}} \tag{1}$$

Exponential Functions

The more digits we use, the closer we get to the true value of a^π

2^3	$2^{3.1}$	$2^{3.14}$	$2^{3.141}$	$2^{3.1415}$	$\cdots$	2^π
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An **exponential function** is a function f such that

$$f(x) = C \cdot a^x$$

- $C \neq 0$ is the **initial value**
- $a > 0$, $a \neq 1$ is the **growth factor**

Consider the exponential function $f(x) = 5 \cdot 2^x$. Let's make a table of values

x	$f(x)$
-2	1.25
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- $f(0) = 5 = C$.

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$$\frac{f(x+1)}{f(x)} = \frac{C \cdot a^{x+1}}{C \cdot a^x} = \frac{a^{x+1}}{a^x} = a^{x+1-x} = a$$

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x	$f(x)$
-5	0.16125
-4	0.3125
-3	0.625
-2	1.25
-1	2.5
0	5
1	10
2	20
3	40
4	80
5	160

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It seems that $f(x) \to 0$ as $x \to -\infty$, and $f(x) \to \infty$ as $x \to \infty$

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Connect the points continuously and use the asymptotic behavior noted