Math 5601 Homework 5

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October 3, 2023

Question 1.

Let $S = \{(t, y) \in \mathbf{R}^2 \mid |t| < 11\}$. Define

$$f(t, y) = \frac{t}{9}\cos(2y) + t^2. \tag{1}$$

Then f is differentiable with respect to y, with

$$\frac{\partial f}{\partial y} = -\frac{2t}{9}\sin(2y) \tag{2}$$

If $|t| < 11$, then $\left|\frac{\partial f}{\partial y}\right| < \frac{22}{9}$ for all $y \in \mathbf{R}$. This implies that f is $\frac{22}{9}$-Lipschitz in y over S, so, by Theorem (I) in the notes, the IVP

$$y' = f(t, y), \qquad y(0) = 1 \tag{3}$$

has a unique solution defined for $|t| < 11$. Then (3) certainly has a unique solution defined for $|t| \leq 10$.

Question 2.

(a) The following MATLAB code (copied from `forward_euler.m`) implements the forward Euler method

$$y_{j+1} = y_j + hf(t_j, y_j), \qquad y_0 = y_a \tag{4}$$

for the IVP $y' = f(t, y)$ on $[a, b]$ with $y(a) = y_a$, where $t_j = a + jh$:

```
function result = forward_euler(f, a, b, ya, h)

% num_steps will get us as close to b as possible
% using steps of size h without going past b
num_steps = floor((b - a) / h);

y = zeros(1, num_steps);
y(1) = ya;

t_jm1 = a;
for j = 2:num_steps
    y(j) = y(j - 1) + h * f(t_jm1, y(j - 1));
    t_jm1 = t_jm1 + h;
end

result = y;
```

(b) The following code can be entered in the Command Window in MATLAB to use the above script to solve the IVP $y' = y^{\frac{1}{3}}$ on $[0, 2]$ with $y(0) = 0$:

```
forward_euler(@(t, y) y^(1/3), 0, 2, 0, h)
```

Running this command with various values of the parameter `h` (I tried .1, .01, .001, and .0001) gives the same output, so I will just describe it rather than copy it: $y_j = 0$ for all j. This is, indeed, the exact values $y(t_j)$ for the solution $y(t) = y_1(t) = 0$ of the IVP. There is, however, *another* solution of the IVP:

$$y(t) = y_2(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}, \tag{5}$$

which is in fact a solution because $y_2(0) = 0$, and $y_2'(t) = \frac{3}{2}\left(\frac{2}{3}\right)^{\frac{3}{2}} t^{\frac{1}{2}} = \left(\frac{2}{3}t\right)^{\frac{1}{2}} = [y_2(t)]^{\frac{1}{3}}$ for $t \in [0, 2]$.

It is evident from the definition of the forward Euler method that it will never be able to approximate this solution no matter what value of h is chosen. Indeed, we have $y_0 = y_a = 0$, and if $y_j = 0$, then

$$y_{j+1} = y_j + hf(t_j, y_j) = 0 + h \cdot 0^{\frac{1}{3}} = 0, \tag{6}$$

so by mathematical induction it follows that $y_j = 0$ for all j, which is precisely what we observed in the numerical experiments (of course, one could argue that floating-point round-off error *might* cause the stored values $\{\tilde{y}_j\}$ to differ slightly from 0, ruining the above explanation; however, in standard floating-point representations, 0 can be represented exactly, and floating-point arithmetic standards ensure that multiplication by exactly 0 is exactly 0, and addition of exactly 0 is exactly the same as before, so the above argument still works).

Question 3.

Consider the forward Euler method

$$y_{j+1}^h = y_j^h + f(x_j, y_j^h), \qquad h > 0, \quad j \in \{0, 1, 2, \ldots, N\} \tag{7}$$

for approximating the solution $y(x)$ of $y' = f(x, y)$ with $y(0) = \alpha$. Suppose that

$$y_j^h - y(x_j) = \sum_{m=1}^{\infty} c_m h^m \tag{8}$$

for some $\{c_m\}$ independent of h. To find a third-order approximation z_j^h of $y(x_j)$ using y_j^h, $y_j^{\frac{h}{2}}$, and $y_j^{\frac{h}{3}}$, we take a linear combination of them and attempt to find coefficients that make the combination a third-order approximation. To this end, let

$$z_j^h = a_1 y_j^h + a_2 y_j^{\frac{h}{2}} + a_3 y_j^{\frac{h}{3}}. \tag{9}$$

Consider the difference $z_j - y(x_j)$:

$$z_j^h - y(x_j) = (a_1 + a_2 + a_3 - 1)y(x_j) + \sum_{n=1}^{3} a_n \left(y_j^{\frac{h}{n}} - y(x_j)\right) \tag{10}$$

$$= (a_1 + a_2 + a_3 - 1)y(x_j) + \sum_{n=1}^{3} a_n \sum_{m=1}^{\infty} c_m \left(\frac{h}{n}\right)^m \tag{11}$$

$$= (a_1 + a_2 + a_3 - 1)y(x_j) + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{3} \frac{a_n}{n^m}\right) h^m \tag{12}$$

$$= \left(-1 + \sum_{n=1}^{3} a_n\right) y(x_j) + \left(\sum_{n=1}^{3} \frac{a_n}{n}\right) h + \left(\sum_{n=1}^{3} \frac{a_n}{n^2}\right) h^2 + O(h^3). \tag{13}$$

Evidently, z_j^h will be a third-order approximation if we choose a_1, a_2, and a_3 such that the first three terms in the last line above are all zero. That is, we must choose the coefficients to satisfy

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{4} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{14}$$

This implies that

$$\begin{bmatrix} 18 & 12 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = -36a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{15}$$

which gives

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = -\frac{1}{36} \cdot (-36a_1) \cdot \begin{bmatrix} 4 & -12 \\ -9 & 18 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = a_1 \begin{bmatrix} -8 \\ 9 \end{bmatrix}, \tag{16}$$

or $a_2 = -8a_1$, and $a_3 = 9a_1$. Since we must have $a_1 + a_2 + a_3 = 1$, it follows that $a_1 - 8a_1 + 9a_1 = 1$, so $a_1 = \frac{1}{2}$, $a_2 = -4$, and $a_3 = \frac{9}{2}$. Therefore,

$$z_j^h = \frac{1}{2} y_j^h - 4 y_j^{\frac{h}{2}} + \frac{9}{2} y_j^{\frac{h}{3}} \tag{17}$$

is a third-order approximation of $y(x_j)$.