Math 6417 Homework 4

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Question 1.

Define the **Fourier transform operator** $\mathscr{F} : L^1(\mathbf{R}) \to L^\infty(\mathbf{R})$ by

$$\mathscr{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f(x)\, \mathrm{d}x. \tag{1}$$

1.1) We note that the function $x \mapsto e^{iyx} f(x)$ is clearly integrable if f is, so the integral in (1) exists for all y. We show that $\mathscr{F}(f) \in L^\infty(\mathbf{R})$ as claimed, and $\|\mathscr{F} f\|_{L^\infty} \le \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$. Indeed, for $y \in \mathbf{R}$,

$$\begin{aligned} |\mathscr{F}(f)(y)| &= \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} e^{iyx} f(x)\, \mathrm{d}x \right| && (2) \\ &\le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| e^{iyx} f(x) \right|\, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}. && (3) \end{aligned}$$

Therefore, $\|\mathscr{F} f\|_{L^\infty} \le \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$.

1.2) Suppose that $f \in C^2(\mathbf{R})$, and $f, f', f'' \in L^1(\mathbf{R})$, and $f(x), f'(x), f''(x) \to 0$ as $x \to \pm\infty$. Then there exists a constant C such that $|y^2 \mathscr{F}(f)(y)| \le C$ for all $y \in \mathbf{R}$. Furthermore, $\mathscr{F}(f) \in L^1(\mathbf{R})$.

Proof. Since $f'' \in L^1(\mathbf{R})$, we can take its Fourier transform, which yields

$$\mathscr{F}(f'')(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f''(x)\, \mathrm{d}x. \tag{4}$$

We can integrate by parts because $f', f \in L^1(\mathbf{R})$ and are continuous, and $f(x), f'(x) \to 0$ as $x \to \pm\infty$. This gives

$$\begin{aligned} \mathscr{F}(f'')(y) &= \frac{1}{\sqrt{2\pi}} \left[f'(x) e^{iyx} \Big|_{-\infty}^{\infty} - iy \int_{-\infty}^{\infty} e^{iyx} f'(x)\, \mathrm{d}x \right] && (5) \\ &= \frac{iy}{\sqrt{2\pi}} \left[-f(x) e^{iyx} \Big|_{-\infty}^{\infty} + iy \int_{-\infty}^{\infty} e^{iyx} f(x)\, \mathrm{d}x \right] && (6) \\ &= -y^2 \mathscr{F}(f)(y). && (7) \end{aligned}$$

By the reasoning in 1.1), it follows that

$$|y^2 \mathscr{F}(f)(y)| = |\mathscr{F}(f'')(y)| \le \frac{1}{\sqrt{2\pi}} \|f''\|_{L^1} \tag{8}$$

for all $y \in \mathbf{R}$.

Thus, if $C = \frac{1}{\sqrt{2\pi}} \|f''\|_{L^1}$, then $|\mathscr{F}(f)(y)| \le \frac{C}{y^2}$ for all $y \in \mathbf{R}$. On the other hand, $\mathscr{F}(f) \in L^\infty(\mathbf{R})$ by part 1.1), so $\mathscr{F}(f)$ is dominated by the integrable function

$$\phi(y) = \begin{cases} \|\mathscr{F}(f)\|_{L^\infty} & y \in [-1, 1], \\ \frac{C}{y^2} & \text{otherwise}. \end{cases} \tag{9}$$

By the integral comparison test, $\mathscr{F}(f) \in L^1(\mathbf{R})$. $\square$

1.3) Formally, $\mathscr{F}^2(f)(y) = f(-y)$.

Proof. We note that if $f \in C^1 \cap L^1(\mathbf{R})$, and $f' \in L^1(\mathbf{R})$, and $f(x) \to 0$ as $x \to \pm\infty$, then we can use integration by parts to show that

$$\mathscr{F}(f')(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f'(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[e^{iyx} f(x) \Big|_{-\infty}^{\infty} - iy \int_{-\infty}^{\infty} e^{iyx} f(x)\,\mathrm{d}x \right] \tag{10}$$

$$= -iy\mathscr{F}(f)(y). \tag{11}$$

On the other hand, let $f \in L^1(\mathbf{R})$, and define $g(x) = ixf(x)$. If $g \in L^1(\mathbf{R})$ as well, then

$$\frac{\mathrm{d}}{\mathrm{d}y} \frac{1}{\sqrt{2\pi}} \mathscr{F}(f)(y) = \frac{\mathrm{d}}{\mathrm{d}y} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial y} \left[e^{iyx} f(x) \right]\,\mathrm{d}x \tag{12}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} ixf(x)\,\mathrm{d}x = \mathscr{F}(g)(y). \tag{13}$$

If we take $f(x) = e^{-ax^2}$, then f satisfies the above assumptions. Since $f'(x) = -2axf(x)$,

$$2ai \frac{\mathrm{d}}{\mathrm{d}y} \mathscr{F}(f)(y) = 2ai\mathscr{F}(i(\cdot)f(\cdot))(y) = \mathscr{F}(-2a(\cdot)f(\cdot))(y) = \mathscr{F}(f')(y) = -iy\mathscr{F}(f)(y). \tag{14}$$

Hence, $\mathscr{F}(f)(y)$ is the unique solution of the IVP

$$u' = -\frac{y}{2a} u, \qquad u(0) = \mathscr{F}(f)(0). \tag{15}$$

The general solution of the differential equation is

$$u(y) = u(0) e^{-\frac{y^2}{4a}}. \tag{16}$$

Since

$$\mathscr{F}(f)(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2}\,\mathrm{d}x = \frac{1}{\sqrt{2a}}, \tag{17}$$

it follows that

$$\mathscr{F}(f)(y) = \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}. \tag{18}$$

Thus, if $\phi_a(x) = e^{-ax^2}$, then, formally,

$$\mathscr{F}(1)(y) = \mathscr{F}\left(\lim_{a \to 0^+} \phi_a \right)(y) = \lim_{a \to 0^+} \mathscr{F}(\phi_a)(y) = \lim_{a \to 0^+} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}. \tag{19}$$

We would like to interpret the last limit formally as a constant multiple of the Dirac delta function. Clearly,

$$\lim_{a \to 0^+} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}} = \begin{cases} 0 & y \neq 0, \\ \infty & y = 0. \end{cases} \tag{20}$$

At the same time, for any $a > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}\,\mathrm{d}y = \frac{1}{\sqrt{2a}} \sqrt{4a\pi} = \sqrt{2\pi}, \tag{21}$$

so it makes sense formally that we should have $\mathscr{F}(1)(y) = \sqrt{2\pi}\delta(y)$.

Now, if we consider applying the Fourier transform twice to a function f, we get

$$\begin{aligned}
\mathscr{F}\mathscr{F}(f)(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixy}e^{izx}f(z)\,\mathrm{d}z\,\mathrm{d}x && (22)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix(y+z)}\,\mathrm{d}x\,\mathrm{d}z && (23)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)\mathscr{F}(1)(y+z)\,\mathrm{d}z && (24)\\
&= \int_{-\infty}^{\infty} f(z)\delta(y+z)\,\mathrm{d}z && (25)\\
&= \int_{-\infty}^{\infty} f(z-y)\delta(z)\,\mathrm{d}z && (26)\\
&= f(-y). && (27)
\end{aligned}$$

□

1.4) Define $g(y) = f(-y)$ for some function f. Based on the formal result from part 1.3), we see immediately that

$$\mathscr{F}^4(f)(y) = \mathscr{F}^2(\mathscr{F}^2(f))(y) = \mathscr{F}^2(g)(y) = g(-y) = f(y). \tag{28}$$

Since f was arbitrary, it follows formally that $\mathscr{F}^4 = I$, the identity operator.

1.5) Let $p(x) = x^4$. By the Spectral Mapping Theorem,

$$p(\sigma(\mathscr{F})) = \sigma(p(\mathscr{F})). \tag{29}$$

Since $p(\mathscr{F}) = \mathscr{F}^4 = I$, the spectrum of $p(\mathscr{F})$ is just $\sigma(I) = \{1\}$, as the operator $I - \lambda I = (1-\lambda)I$ is invertible, with inverse $\frac{1}{1-\lambda}I$, if and only if $\lambda \neq 1$. Therfore, if $\lambda \in \sigma(\mathscr{F})$, then $p(\lambda) = 1$, that is, $\lambda^4 = 1$. The possible solutions of this equation are $1, -1, i, -i$, so $\sigma(\mathscr{F}) \subseteq \{1, -1, i, -i\}$.

1.6) If we reuse the result in equation (18) with $a = \frac{1}{2}$, we see that if $f(x) = e^{-\frac{1}{2}x^2}$, then

$$\mathscr{F}(f)(y) = e^{-\frac{1}{2}y^2} \tag{30}$$

as well. Thus, $\mathscr{F}f = f$, so f is an eigenfunction of $\mathscr{F}$ with corresponding eigenvalue 1.

Question 2.

On this question, we will reuse the notation from Question 2 of Homework 3.

Let $\dot{L}^2(-\pi,\pi) = \{f \in L^2(-\pi,\pi) : f = \bar{f} \text{ and } \mathrm{mean}(f) = 0\}$, where $\mathrm{mean}(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f$. Consider the following problem.

$$\text{Let } f \in \dot{L}^2(-\pi,\pi). \text{ Find } u \in H \text{ such that } -u'' = f, \tag{31}$$

where H is the space defined in Homework 3.

2.1) Let $f \in \dot{L}^2(-\pi,\pi)$. Then $f \in L^2(-\pi,\pi)$, and, recalling from Homework 3, there exists $\{f_j\} \subset \mathbf{C}$ such that

$$f = \sum_j f_j e_j, \qquad f_j = (f, e_j). \tag{32}$$

Since $e_0 =$ constant, we have $f_j = (f, e_j) \propto \mathrm{mean}(f) = 0$, so $f_0 = 0$. Furthermore, by an argument we used several times in Homework 3, the fact that $f = \bar{f}$ implies that $f_{-j} = \bar{f}_j$. Lastly, by Parseval's identity,

$$\sum_{j\neq 0} j^{-2}|f_j|^2 \leq \sum_{j\neq 0} |f_j|^2 = \|f\|_2^2 < \infty, \tag{33}$$

so $f \in H^{-1}$ from Homework 3 by definition. Therefore, $\dot{L}^2(-\pi,\pi) \subseteq H^{-1}$.

Now suppose that $u \in H$ is twice differentiable, and $-u'' = f$ for some $f \in \dot{L}^2(-\pi,\pi)$. Then

$$u'(\pi) - u'(-\pi) = -\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = 0 \tag{34}$$

because $\mathrm{mean}(f) = 0$. Let $\{u_j\}$ be the coefficients of u, and let u_j'' be the coefficients of u'' with respect to $\{e_j\}$. The