

# Math 6417 Homework 4

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## Question 1.

Define the **Fourier transform operator**  $\mathcal{F} : L^1(\mathbf{R}) \rightarrow L^\infty(\mathbf{R})$  by

$$\mathcal{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f(x) \, dx. \quad (1)$$

1.1) We note that the function  $x \mapsto e^{iyx} f(x)$  is clearly integrable if  $f$  is, so the integral in (1) exists for all  $y$ . We show that  $\mathcal{F}(f) \in L^\infty(\mathbf{R})$  as claimed, and  $\|\mathcal{F}f\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$ . Indeed, for  $y \in \mathbf{R}$ ,

$$|\mathcal{F}(f)(y)| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} e^{iyx} f(x) \, dx \right| \quad (2)$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{iyx} f(x)| \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \, dx = \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}. \quad (3)$$

Therefore,  $\|\mathcal{F}f\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}$ .

1.2) Suppose that  $f \in C^2(\mathbf{R})$ , and  $f, f', f'' \in L^1(\mathbf{R})$ , and  $f(x), f'(x), f''(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ . Then there exists a constant  $C$  such that  $|y^2 \mathcal{F}(f)(y)| \leq C$  for all  $y \in \mathbf{R}$ . Furthermore,  $\mathcal{F}(f) \in L^1(\mathbf{R})$ .

*Proof.* Since  $f'' \in L^1(\mathbf{R})$ , we can take its Fourier transform, which yields

$$\mathcal{F}(f'')(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f''(x) \, dx. \quad (4)$$

We can integrate by parts because  $f', f \in L^1(\mathbf{R})$  and are continuous, and  $f(x), f'(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ . This gives

$$\mathcal{F}(f'')(y) = \frac{1}{\sqrt{2\pi}} \left[ f'(x) e^{iyx} \Big|_{-\infty}^{\infty} - iy \int_{-\infty}^{\infty} e^{iyx} f'(x) \, dx \right] \quad (5)$$

$$= \frac{iy}{\sqrt{2\pi}} \left[ -f(x) e^{iyx} \Big|_{-\infty}^{\infty} + iy \int_{-\infty}^{\infty} e^{iyx} f(x) \, dx \right] \quad (6)$$

$$= -y^2 \mathcal{F}(f)(y). \quad (7)$$

By the reasoning in 1.1), it follows that

$$|y^2 \mathcal{F}(f)(y)| = |\mathcal{F}(f'')(y)| \leq \frac{1}{\sqrt{2\pi}} \|f''\|_{L^1} \quad (8)$$

for all  $y \in \mathbf{R}$ .

Thus, if  $C = \frac{1}{\sqrt{2\pi}} \|f''\|_{L^1}$ , then  $|\mathcal{F}(f)(y)| \leq \frac{C}{y^2}$  for all  $y \in \mathbf{R}$ . On the other hand,  $\mathcal{F}(f) \in L^\infty(\mathbf{R})$  by part 1.1), so  $\mathcal{F}(f)$  is dominated by the integrable function

$$\phi(y) = \begin{cases} \|\mathcal{F}(f)\|_{L^\infty} & y \in [-1, 1], \\ \frac{C}{y^2} & \text{otherwise.} \end{cases} \quad (9)$$

By the integral comparison test,  $\mathcal{F}(f) \in L^1(\mathbf{R})$ . □

1.3) Formally,  $\mathcal{F}^2(f)(y) = f(-y)$ .

*Proof.* We note that if  $f \in C^1 \cap L^1(\mathbf{R})$ , and  $f' \in L^1(\mathbf{R})$ , and  $f(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ , then we can use integration by parts to show that

$$\mathcal{F}(f')(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f'(x) dx = \frac{1}{\sqrt{2\pi}} \left[ e^{iyx} f(x) \Big|_{-\infty}^{\infty} - iy \int_{-\infty}^{\infty} e^{iyx} f(x) dx \right] \quad (10)$$

$$= -iy \mathcal{F}(f)(y). \quad (11)$$

On the other hand, let  $f \in L^1(\mathbf{R})$ , and define  $g(x) = ix f(x)$ . If  $g \in L^1(\mathbf{R})$  as well, then

$$\frac{d}{dy} \frac{1}{\sqrt{2\pi}} \mathcal{F}(f)(y) = \frac{d}{dy} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial y} [e^{iyx} f(x)] dx \quad (12)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} ix f(x) dx = \mathcal{F}(g)(y). \quad (13)$$

If we take  $f(x) = e^{-ax^2}$ , then  $f$  satisfies the above assumptions. Since  $f'(x) = -2ax f(x)$ ,

$$2ai \frac{d}{dy} \mathcal{F}(f)(y) = 2ai \mathcal{F}(i(\cdot) f(\cdot))(y) = \mathcal{F}(-2a(\cdot) f(\cdot))(y) = \mathcal{F}(f')(y) = -iy \mathcal{F}(f)(y). \quad (14)$$

Hence,  $\mathcal{F}(f)(y)$  is the unique solution of the IVP

$$u' = -\frac{y}{2a} u, \quad u(0) = \mathcal{F}(f)(0). \quad (15)$$

The general solution of the differential equation is

$$u(y) = u(0) e^{-\frac{y^2}{4a}}. \quad (16)$$

Since

$$\mathcal{F}(f)(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{1}{\sqrt{2a}}, \quad (17)$$

it follows that

$$\mathcal{F}(f)(y) = \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}. \quad (18)$$

Thus, if  $\phi_a(x) = e^{-ax^2}$ , then, formally,

$$\mathcal{F}(1)(y) = \mathcal{F} \left( \lim_{a \rightarrow 0^+} \phi_a \right) (y) = \lim_{a \rightarrow 0^+} \mathcal{F}(\phi_a)(y) = \lim_{a \rightarrow 0^+} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}}. \quad (19)$$

We would like to interpret the last limit formally as a constant multiple of the Dirac delta function. Clearly,

$$\lim_{a \rightarrow 0^+} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}} = \begin{cases} 0 & y \neq 0, \\ \infty & y = 0. \end{cases} \quad (20)$$

At the same time, for any  $a > 0$ ,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2a}} e^{-\frac{y^2}{4a}} dy = \frac{1}{\sqrt{2a}} \sqrt{4a\pi} = \sqrt{2\pi}, \quad (21)$$

so it makes sense formally that we should have  $\mathcal{F}(1)(y) = \sqrt{2\pi} \delta(y)$ .

Now, if we consider applying the Fourier transform twice to a function  $f$ , we get

$$\mathcal{F}\mathcal{F}(f)(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ixy} e^{izx} f(z) \, dz \, dx \quad (22)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix(y+z)} \, dx \, dz \quad (23)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) \mathcal{F}(1)(y+z) \, dz \quad (24)$$

$$= \int_{-\infty}^{\infty} f(z) \delta(y+z) \, dz \quad (25)$$

$$= \int_{-\infty}^{\infty} f(z-y) \delta(z) \, dz \quad (26)$$

$$= f(-y). \quad (27)$$

□

- 1.4) Define  $g(y) = f(-y)$  for some function  $f$ . Based on the formal result from part 1.3), we see immediately that

$$\mathcal{F}^4(f)(y) = \mathcal{F}^2(\mathcal{F}^2(f))(y) = \mathcal{F}^2(g)(y) = g(-y) = f(y). \quad (28)$$

Since  $f$  was arbitrary, it follows formally that  $\mathcal{F}^4 = I$ , the identity operator.

- 1.5) Let  $p(x) = x^4$ . By the Spectral Mapping Theorem,

$$p(\sigma(\mathcal{F})) = \sigma(p(\mathcal{F})). \quad (29)$$

Since  $p(\mathcal{F}) = \mathcal{F}^4 = I$ , the spectrum of  $p(\mathcal{F})$  is just  $\sigma(I) = \{1\}$ , as the operator  $I - \lambda I = (1 - \lambda)I$  is invertible, with inverse  $\frac{1}{1-\lambda}I$ , if and only if  $\lambda \neq 1$ . Therefore, if  $\lambda \in \sigma(\mathcal{F})$ , then  $p(\lambda) = 1$ , that is,  $\lambda^4 = 1$ . The possible solutions of this equation are  $1, -1, i, -i$ , so  $\sigma(\mathcal{F}) \subseteq \{1, -1, i, -i\}$ .

- 1.6) If we reuse the result in equation (18) with  $a = \frac{1}{2}$ , we see that if  $f(x) = e^{-\frac{1}{2}x^2}$ , then

$$\mathcal{F}(f)(y) = e^{-\frac{1}{2}y^2} \quad (30)$$

as well. Thus,  $\mathcal{F}f = f$ , so  $f$  is an eigenfunction of  $\mathcal{F}$  with corresponding eigenvalue 1.

## Question 2.

On this question, we will reuse the notation from Question 2 of Homework 3.

Let  $\dot{L}^2(-\pi, \pi) = \{f \in L^2(-\pi, \pi) : f = \bar{f} \text{ and } \text{mean}(f) = 0\}$ , where  $\text{mean}(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f$ . Consider the following problem.

$$\text{Let } f \in \dot{L}^2(-\pi, \pi). \text{ Find } u \in H \text{ such that } -u'' = f, \quad (31)$$

where  $H$  is the space defined in Homework 3.

- 2.1) Let  $f \in \dot{L}^2(-\pi, \pi)$ . Then  $f \in L^2(-\pi, \pi)$ , and, recalling from Homework 3, there exists  $\{f_j\} \subset \mathbf{C}$  such that

$$f = \sum_j f_j e_j, \quad f_j = (f, e_j). \quad (32)$$

Since  $e_0 = \text{constant}$ , we have  $f_j = (f, e_j) \propto \text{mean}(f) = 0$ , so  $f_0 = 0$ . Furthermore, by an argument we used several times in Homework 3, the fact that  $f = \bar{f}$  implies that  $f_{-j} = \bar{f}_j$ . Lastly, by Parseval's identity,

$$\sum_{j \neq 0} j^{-2} |f_j|^2 \leq \sum_{j \neq 0} |f_j|^2 = \|f\|_2^2 < \infty, \quad (33)$$

so  $f \in H^{-1}$  from Homework 3 by definition. Therefore,  $\dot{L}^2(-\pi, \pi) \subseteq H^{-1}$ .

We claim that for  $f \in \dot{L}^2(-\pi, \pi)$  and  $u \in H$ ,

$$-u'' = f \iff B(u, v) = f(v) \quad \forall v \in H, \quad (34)$$

where we define the action of  $f$  on  $v$  in the same way as in Homework 3, and

$$B(u, v) = \sum_{j \neq 0} j^2 u_j \bar{v}_j, \quad \{u_j\} = \varphi(u), \quad \{v_j\} = \varphi(v), \quad (35)$$

where  $\varphi : H \rightarrow S_H$  is defined as in Homework 3. We use essentially the same formal argument that we used on 2.5) in Homework 3.

Suppose that  $-u'' = f$ , and let  $\{f_j\} = \psi(f)$ ,  $\{u_j\} = \varphi(u)$ . Formally differentiating the Fourier series for  $u$ , we have

$$-\sum_{j \neq 0} f_j e_j = -f = u'' = \sum_{j \neq 0} -j^2 u_j e_j. \quad (36)$$

Therefore,  $f_j = j^2 u_j$  for all  $j$ , and for any  $v \in H$ ,

$$B(u, v) = \sum_{j \neq 0} j^2 u_j \bar{v}_j = \sum_{j \neq 0} f_j \bar{v}_j = f(v). \quad (37)$$

On the other hand, suppose that  $B(u, v) = f(v)$  for all  $v \in H$ . Clearly,  $e_j + e_{-j} \in H$ , and  $e_j - e_{-j} \in H$ , so

$$j^2 u_j + j^2 u_{-j} = f_j + f_{-j}, \quad j^2 u_j - j^2 u_{-j} = f_j - f_{-j}, \quad (38)$$

which implies that  $j^2 u_j = f_j$  for all  $j$ . By the same formal differentiation reasoning, it follows that  $-u'' = f$ .

The function  $B$  is bilinear because for any  $\alpha, \beta \in \mathbf{R}$ , and any  $u, v, w \in H$ ,

$$B(\alpha u + \beta v, w) = \sum_{j \neq 0} j^2 (\alpha u_j + \beta v_j) \bar{w}_j = \alpha \sum_{j \neq 0} j^2 u_j \bar{w}_j + \beta \sum_{j \neq 0} j^2 v_j \bar{w}_j = \alpha B(u, w) + \beta B(v, w), \quad (39)$$

and

$$B(w, \alpha u + \beta v) = \sum_{j \neq 0} j^2 w_j \overline{\alpha u_j + \beta v_j} = \alpha \sum_{j \neq 0} j^2 w_j \bar{u}_j + \beta \sum_{j \neq 0} j^2 w_j \bar{v}_j = \alpha B(w, u) + \beta B(w, v) \quad (40)$$

because  $\varphi(\alpha u + \beta v) = \alpha \varphi(u) + \beta \varphi(v)$ .

The function  $B$  is also continuous because for any  $u, v \in H$ ,

$$|B(u, v)| = \left| \sum_{j \neq 0} j^2 u_j \bar{v}_j \right| \leq \|u\|_H \|v\|_H \quad (41)$$

by the Cauchy-Schwarz inequality.

Lastly,  $B$  is coercive because for any  $u \in H$ ,

$$B(u, u) = \sum_{j \neq 0} j^2 |u_j|^2 = \|u\|_H^2. \quad (42)$$

Hence, the Lax-Milgram Theorem implies that, given  $f \in \dot{L}^2(-\pi, \pi) \subseteq H^{-1}$ , there exists a unique  $u \in H$  such that  $B(u, v) = f(v)$  for all  $v \in H$ . That is, there exists a unique  $u \in H$  such that  $-u'' = f$ .

**2.2)** Let  $T : \dot{L}^2(-\pi, \pi) \rightarrow H$  denote the solution operator of (31), which exists by 2.1). Then  $T$  is compact.

*Proof.* Given  $f \in \dot{L}^2(-\pi, \pi)$ , there exists  $u \in H$  such that

$$\|Tf\|_H = \|u\|_H, \quad (43)$$

and  $B(u, v) = f(v)$  for all  $v \in H$ . In particular, if we take  $v = u$ , we obtain

$$\|u\|_H^2 = f(u) = \sum_{j \neq 0} f_j \bar{u}_j \leq \|u\|_H \|f\|_{L^2}, \quad (44)$$

which implies that  $\|Tf\|_H \leq \|f\|_{L^2}$ , so  $T$  is bounded.

As we showed in 2.1),  $Tf = u$  if and only if  $f_j = j^2 u_j$ , where  $\{u_j\} = \varphi(u)$ , and  $\{f_j\} = \psi(f)$ . Thus,  $u_j = \frac{f_j}{j^2}$ .

Let  $A$  be a bounded set in  $\dot{L}^2(-\pi, \pi)$ , so that there exists  $M > 0$  such that  $\|f\|_{L^2} \leq M$  for all  $f \in A$ . Suppose that  $\{u^n\}$  is a sequence in  $\overline{T(A)}$ . Then for any  $n$ , we have  $u^n = T(f^n)$  for some  $f^n \in A$ , and  $u_j^n = \frac{f_j^n}{j^2}$ , where  $u_j^n = \varphi(u^n)$ , and  $f_j^n = \psi(f^n)$ .

For any  $J > 0$  and any  $n$ ,

$$\sum_{|j| > J} j^2 |u_j^n|^2 = \sum_{|j| > J} j^{-2} |f_j^n|^2 \leq \frac{1}{J^2} \sum_{|j| > J} |f_j^n|^2 \leq \frac{M}{J^2}. \quad (45)$$

Since  $T$  is bounded, and  $A$  is bounded in  $\dot{L}^2(-\pi, \pi)$ , the set  $T(A)$  is bounded in  $H$ . The space  $H$  is a Hilbert space by Homework 3, so there exists a weakly convergent subsequence of  $\{u^n\}$ , call it  $\{u^{n_k}\}$ . Then  $\{u_j^{n_k}\}_k$  converges for all  $j$ . In particular, each such sequence is Cauchy.

Let  $\varepsilon > 0$  be given. By (45), we can choose  $J$  large enough that

$$\sum_{|j| > J} j^2 |u_j^{n_k}|^2 < \varepsilon^2 \quad (46)$$

for all  $k$ . We can also choose  $N$  large enough that  $k, \ell > N$  implies that  $|u_j^{n_k} - u_j^{n_\ell}|^2 < \frac{\varepsilon^2}{2J^3}$  for all  $|j| \leq J$ . Then

$$\|u^{n_k} - u^{n_\ell}\|_H^2 = \sum_{|j| \leq J} j^2 |u_j^{n_k} - u_j^{n_\ell}|^2 + \sum_{|j| > J} j^2 |u_j^{n_k} - u_j^{n_\ell}|^2 \quad (47)$$

$$\leq \varepsilon^2 + \sum_{|j| > J} j^2 |u_j^{n_k}|^2 + \sum_{|j| > J} j^2 |u_j^{n_\ell}|^2 \quad (48)$$

$$\leq 3\varepsilon^2, \quad (49)$$

so  $k, \ell > N$  implies that  $\|u^{n_k} - u^{n_\ell}\|_H < \sqrt{3}\varepsilon$ .

This implies that  $\{u^{n_k}\}_k$  is Cauchy in  $H$ . Since  $H$  is a Hilbert space, it follows that  $\{u^{n_k}\}$  converges in  $H$ . Thus, every sequence in the bounded set  $T(A)$  has a convergent subsequence, so  $T(A)$  is pre-compact in  $H$ .

This implies that  $T$  is compact by definition.  $\square$

**2.3)** The operator  $T$  is self-adjoint.

*Proof.* Let  $f, g \in H$ , and define  $u = Tf$ , and  $v = Tg$ . Then, by the same reasoning as in 2.1), if  $\{u_j\} = \varphi(u)$ ,  $\{v_j\} = \varphi(v)$ ,  $\{f_j\} = \varphi(f)$ , and  $\{g_j\} = \varphi(g)$ , then

$$u_j = \frac{f_j}{j^2}, \quad v_j = \frac{g_j}{j^2}. \quad (50)$$

Thus,

$$(Tf, g)_H = (u, g)_H = \sum_{j \neq 0} j^2 u_j \bar{g}_j = \sum_{j \neq 0} j^2 f_j \bar{v}_j = (f, v)_H = (f, Tg)_H, \quad (51)$$

so  $T$  is self-adjoint.  $\square$

- 2.4)** Suppose that  $Tf = \lambda f$  for  $f \in H$ , with  $f \neq 0$ . Note that since  $T$  is self-adjoint, we must have  $\lambda \in \mathbf{R}$ . By the reasoning in 2.1), we must have  $j^{-2}f_j = \lambda f_j$  for all  $j \neq 0$ , where  $\{f_j\} = \varphi(f)$ . Then either  $f_j = 0$  or  $\lambda = j^{-2}$  for all  $j \neq 0$ . We cannot have  $f_j = 0$  for all  $j$ , because then  $f = 0$  by the linearity of  $\varphi^{-1}$ .

Thus, there exists some  $k > 0$  such that  $f_k \neq 0$ , which implies that  $\lambda = k^{-2}$ . The equation  $\lambda f_j = j^{-2}f_j$  for all  $j$  implies that  $f_j = 0$  for all  $j \neq \pm k$ . Since  $f_{-k} = \bar{f}_k$ , it follows that  $f = f_k e_k + \bar{f}_k e_{-k}$ . Supposing that  $f_j = a + ib$ , we must have

$$f(x) = \frac{1}{\sqrt{2\pi}} ((a + ib)e^{ikx} + (a - ib)e^{-ikx}) \quad (52)$$

$$= \frac{1}{\sqrt{2\pi}} (a \cos(kx) - b \sin(kx) + ib \cos(kx) + ia \sin(kx)) \quad (53)$$

$$+ a \cos(kx) - b \sin(kx) - ib \cos(kx) - ia \sin(kx)) \quad (54)$$

$$= \frac{1}{\sqrt{2\pi}} (2a \cos(kx) - 2b \sin(kx)). \quad (55)$$

Thus, if  $\lambda$  is an eigenvalue of  $T$ , then  $\lambda = k^{-2}$  for some integer  $k \neq 0$ , and the corresponding eigenvectors must be linear combinations of  $\cos(kx)$  and  $\sin(kx)$ ; that is, the corresponding eigenspace is  $\text{span}\{\cos(kx), \sin(kx)\}$ .

It is not hard to see by reversing the above logic that the converse is true, namely, that  $k^{-2}$  is an eigenvalue of  $T$  for all integers  $k \neq 0$ , and its corresponding eigenspace is  $\text{span}\{\cos(kx), \sin(kx)\}$ .

- 2.5)** Let  $c \in \mathbf{C}$  be given. Then for any  $j > 0$ , there exists  $a, b \in \mathbf{R}$  such that  $ce_j + \bar{c}e_{-j} = a \cos(jx) + b \sin(jx)$ ; indeed, by the calculation in 2.4), we just need to choose  $a = \sqrt{\frac{2}{\pi}} \Re(c)$ , and  $b = -\sqrt{\frac{2}{\pi}} \Im(c)$ . Therefore, given  $u \in H$ , the partial sum of the Fourier series for  $u$  can be written as a linear combination of elements of the set  $\mathcal{B} = \{\cos(jx), \sin(jx)\}_{j>0}$ . Since  $H$  is a Hilbert space, the Fourier series of  $u$  converges to  $u$ , so  $u$  is the limit of a sequence of elements of  $\text{span}(\mathcal{B})$ .

In other words,  $\mathcal{B}$  is a basis for  $H$ . In fact, it is an orthogonal basis, as we show now. Let  $\{c_j^k\}_j = \varphi(\cos(kx))$ , and let  $\{s_j^k\}_j = \varphi(\sin(kx))$ . Then, by the Euler formula relating  $e^{ijx}$  to  $\sin(x)$  and  $\cos(x)$ ,

$$c_j^k = \begin{cases} \sqrt{\frac{\pi}{2}} & j = |k| \\ 0 & \text{otherwise,} \end{cases} \quad s_j^k = \begin{cases} -i\sqrt{\frac{\pi}{2}} \text{sgn}(j) & j = |k| \\ 0 & \text{otherwise,} \end{cases} \quad (56)$$

where  $\text{sgn}(j)$  is the sign of  $j$ . Hence, for  $k, \ell > 0$ ,

$$(\cos(kx), \cos(\ell x))_H = \sum_{j \neq 0} j^2 c_j^k \overline{c_j^\ell} = \begin{cases} 0 & k \neq \ell \\ \frac{\pi}{2} k^2 & k = \ell, \end{cases} \quad (57)$$

$$(\cos(kx), \sin(\ell x))_H = \sum_{j \neq 0} j^2 c_j^k \overline{s_j^\ell} = \begin{cases} 0 & k \neq \ell \\ \frac{\pi}{2} (-i + i) k^2 = 0 & k = \ell \end{cases} = 0, \quad (58)$$

$$(\sin(kx), \sin(\ell x))_H = \sum_{j \neq 0} j^2 s_j^k \overline{s_j^\ell} = \begin{cases} 0 & k \neq \ell \\ \frac{\pi}{2} k^2 & k = \ell. \end{cases} \quad (59)$$

Therefore,  $\mathcal{B}$  is orthogonal in  $H$ . Moreover, we can clearly modify the elements of  $\mathcal{B}$  so that they are orthonormal; if we set  $\mathcal{B}' = \left\{ \sqrt{\frac{2}{\pi}} j^{-1} \cos(jx), \sqrt{\frac{2}{\pi}} j^{-1} \sin(jx) \right\}$ , then  $\mathcal{B}'$  is an orthonormal basis for  $H$ .

Finally, we note that  $\mathcal{B}'$  is the orthonormal basis that diagonalizes  $T$  in the sense of the spectral theorem for self-adjoint, compact operators. Let  $c^k = \cos(kx)$ , and let  $s^k = \sin(kx)$ . Define  $u^k = T(c^k)$ , and  $v^k = T(s^k)$ , and define  $\{u_j^k\} = \varphi(u^k)$ , and  $\{v_j^k\} = \varphi(v^k)$ . By the reasoning in 2.1), and by (56),

$$u_j^k = j^{-2} c_j^k \implies u^k = k^{-2} c^k, \quad v_j^k = j^{-2} s_j^k \implies v^k = k^{-2} s^k; \quad (60)$$

that is,  $Tf = \lambda f$  for any  $f \in \mathcal{B}'$ , where  $\lambda$  is the eigenvalue corresponding to  $f$ .