Math 6418 Homework 2

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Let

$$f(x) = \begin{cases} 1 & x \in [-1, 1], \\ 0 & |x| > 1. \end{cases} \tag{1}$$

1.

Since $f \in H^s(\mathbf{R})$ if and only if $g_s \in L^2(\mathbf{R})$, where $g_s(\xi) = \widehat{f}(\xi)(1 + |\xi|^2)^{\frac{s}{2}}$, we should start by computing $\widehat{f}$:

$$\begin{aligned} \widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ix\xi} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-ix\xi}}{-i\xi} \right]_{-1}^{1} \\ &= \frac{e^{i\xi} - e^{-i\xi}}{i\xi\sqrt{2\pi}} \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin(\xi)}{\xi}. \end{aligned}$$

We note that $\widehat{f}(0) = \sqrt{\frac{2}{\pi}}$, so $\widehat{f}$ is continuous. Thus,

$$g_s(\xi) = \sqrt{\frac{2}{\pi}} \frac{\sin(\xi)}{\xi} (1 + |\xi|^2)^{\frac{s}{2}}$$

is also continuous.

Therefore, for $a > 0$,

$$\int_{-\infty}^{\infty} |g_s(\xi)| \, \mathrm{d}\xi = \int_{-a}^{a} |g_s(\xi)|^2 \, \mathrm{d}\xi + 2 \int_{a}^{\infty} |g_s(\xi)|^2 \, \mathrm{d}\xi$$

because g_s is even. The first term is always finite because g_s is continuous, so $g_s \in L^2(\mathbf{R})$ if and only if the second term is finite.

Since $\frac{(1+|\xi|^2)^s}{|\xi|^{2s}} = \left(\frac{1}{|\xi|^2} + 1\right)^s \to 1$ as $\xi \to \infty$, we can choose a large enough that $(1 + |\xi|^2)^s \leq 2|\xi|^{2s}$ for all $\xi > a$. Then $|g_s(\xi)|^2 \leq \frac{4}{\pi}|\xi|^{2(s-1)}$ for all $\xi > a$, meaning that the second term is finite if $2(s-1) < -1$, that is, if $s < \frac{1}{2}$. Hence, if $s < \frac{1}{2}$, then $f \in H^s$.

Conversely, suppose that $s \geq \frac{1}{2}$. Since $(1 + |\xi|^2)^s \geq |\xi|^{2s}$, it follows that

$$\int_{\frac{\pi}{4}}^{\infty} |g_s(\xi)|^2 \, \mathrm{d}\xi \geq \frac{2}{\pi} \int_{\frac{\pi}{4}}^{\infty} \sin^2(\xi)|\xi|^{2s-2} \, \mathrm{d}\xi.$$

If $|\xi|^{2s-2}$ is increasing, then it is clear that the integral on the right hand side is infinite, which implies that $f \notin H^s$. Suppose that $|\xi|^{2s-2}$ is decreasing, and set $I_1 = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right] \cup \dots$ and set $I_2 = \left[\frac{\pi}{4}, \infty\right) \setminus I_1$.

The sets I_1 and I_2 consist of consecutive, interleaved intervals of length $\frac{\pi}{2}$; since $|\xi|^{2s-2}$ is decreasing and each interval of I_1 precedes an interval of I_2, it follows that

$$\int_{I_1} |\xi|^{2s-2}\,\mathrm{d}\xi \geq \int_{I_2} |\xi|^{2s-2}\,\mathrm{d}\xi,$$

so

$$\int_{\frac{\pi}{4}}^{\infty} |\xi|^{2s-2} = \int_{I_1} |\xi|^{2s-2}\,\mathrm{d}\xi + \int_{I_2} |\xi|^{2s-2}\,\mathrm{d}\xi \leq 2\int_{I_1} |\xi|^{2s-2}\,\mathrm{d}\xi,$$

which implies that

$$\int_{I_1} |\xi|^{2s-2}\,\mathrm{d}\xi \geq \frac{1}{2}\int_{\frac{\pi}{4}} |\xi|^{2s-2}\,\mathrm{d}\xi = \infty$$

because $s \geq \frac{1}{2}$. Therefore,

$$\int_{\frac{\pi}{4}}^{\infty} \sin^2(\xi)|\xi|^{2s-2}\,\mathrm{d}\xi \geq \int_{I_1} \sin^2(\xi)|\xi|^{2s-2}\,\mathrm{d}\xi \geq \frac{1}{2}\int_{I_1} |\xi|^{2s-2}\,\mathrm{d}\xi = \infty$$

because $\sin^2(\xi) \geq \frac{1}{2}$ for $\xi \in I_1$. This implies that $f \notin H^s$.

Thus, $f \in H^s$ if and only if $s < \frac{1}{2}$.