Math 6330 Homework 10

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7.8(b)

Consider the product system
$$\dot{x}_1 = x_1^2, \qquad \dot{x}_2 = -x_2.$$

Solving each equation, we have $x_2 = x_2(0)e^{-t}$, and $x_1 = \frac{x_1(0)}{x_1(0)-t}$ if $x_1(0) \neq 0$, and $x_1 = 0$ if $x_1(0) = 0$. Therefore, the maximal interval of existence is

$$I = \begin{cases} (-\infty, x_1(0)) & x_1(0) > 0, \\ (-\infty, \infty) & x_1(0) = 0, \\ (x_1(0), \infty) & x_1(0) < 0. \end{cases}$$

From the solution formulas we have
$$\lim_{t\to\infty} x_2 = 0,$$
and
$$\lim_{t\to-\infty} x_2 = \begin{cases} \infty & x_2(0) > 0, \\ 0 & x_2(0) = 0, \\ -\infty & x_2(0) < 0. \end{cases}$$

We also have

$$\lim_{t\to x_1(0)^-} x_1 = \infty \qquad \text{if } x_1(0) > 0,$$
$$\lim_{t\to\infty} x_1 = 0 \qquad \text{if } x_1(0) < 0,$$

and

$$\lim_{t\to-\infty} x_1 = 0 \qquad \text{if } x_1(0) > 0,$$
$$\lim_{t\to x_1(0)^+} x_1 = -\infty \qquad \text{if } x_1(0) < 0.$$

If $x_2 \neq 0$ and $x_1 \neq 0$, then we have

$$\frac{\mathrm{d}x_1}{\mathrm{d}x_2} = -\frac{x_1^2}{x_2} \implies -\frac{\mathrm{d}x_1}{x_1^2} = \frac{\mathrm{d}x_2}{x_2} \implies \frac{1}{x_1} = \ln|x_2| + C,$$

for some constant C, or, equivalently,
$$x_2 = Ae^{\frac{1}{x_1}},$$
for some constant $A \neq 0$.

If $x_1(0) < 0$, then $x_1 \to 0$ from the left as $t \to \infty$, so $x_2 \to 0$, and $(x_1, x_2) \to (0, 0)$. If $x_1(0) > 0$, then $x_1 \to \infty$ as $t \to x_1(0)$, so $x_2 \to x_2(0)e^{-x_1(0)}$ as $t \to x_1(0)$. Additionally, if $x_1(0) < 0$, then $x_1 \to -\infty$ as $t \to x_1(0)$, and if $x_1(0) > 0$, then $x_1 \to 0$ from the right as $t \to -\infty$, so $x_2 \to \infty$ as $t \to -\infty$.

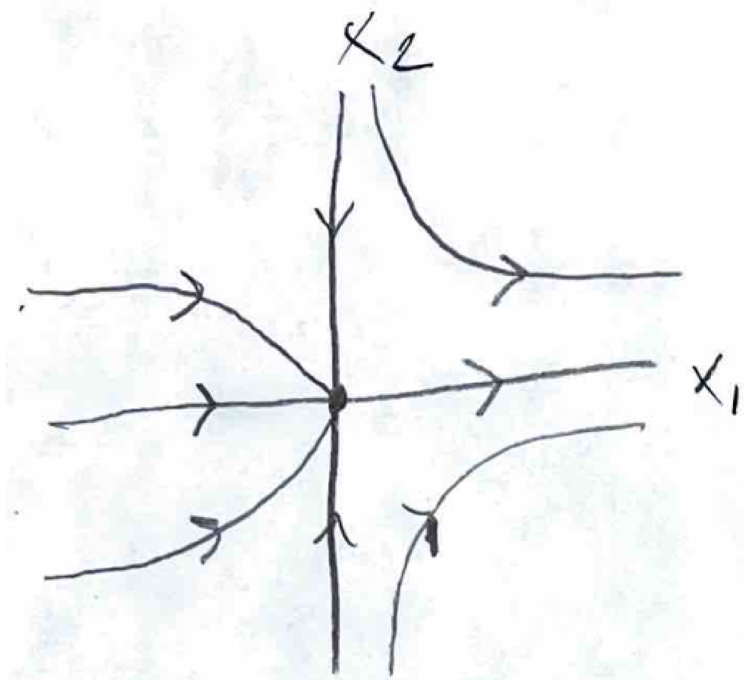

Figure 1: 7.8(b) Phase portrait

These considerations result in the phase plane diagram in Figure 1. Furthermore, the ω-limit and α-limit sets are given by

$$\omega(x_1(0), x_2(0)) = \begin{cases} \{(0,0)\} & x_1(0) \le 0 \\ \varnothing & x_1(0) > 0, \end{cases}$$

$$\alpha(x_1(0), x_2(0)) = \begin{cases} \{(0,0)\} & x_1(0) \ge 0 \text{ and } x_2(0) = 0, \\ \varnothing & \text{otherwise.} \end{cases}$$

7.8(c)

Consider the product system

$$\dot{x}_1 = -x_1, \qquad \dot{x}_2 = x_2 - x_2^3.$$

Then $x_1 = x_1(0)e^{-t}$, and the asymptotic behavior of x_2 is determined by the phase line diagram in Figure 2. Based on these facts, we obtain the phase portrait in Figure 3. Finally, we can determine the α-limit and ω-limit sets from the phase portrait:

$$\omega(x_1(0), x_2(0)) = \begin{cases} (0,0) & x_2(0) = 0, \\ (0,1) & x_2(0) > 0, \\ (0,-1) & x_2(0) < 0 \end{cases}$$

$$\alpha(x_1(0), x_2(0)) = \begin{cases} (0,0) & x_1(0) = 0 \text{ and } |x_2(0)| < 1, \\ \varnothing & \text{otherwise.} \end{cases}$$

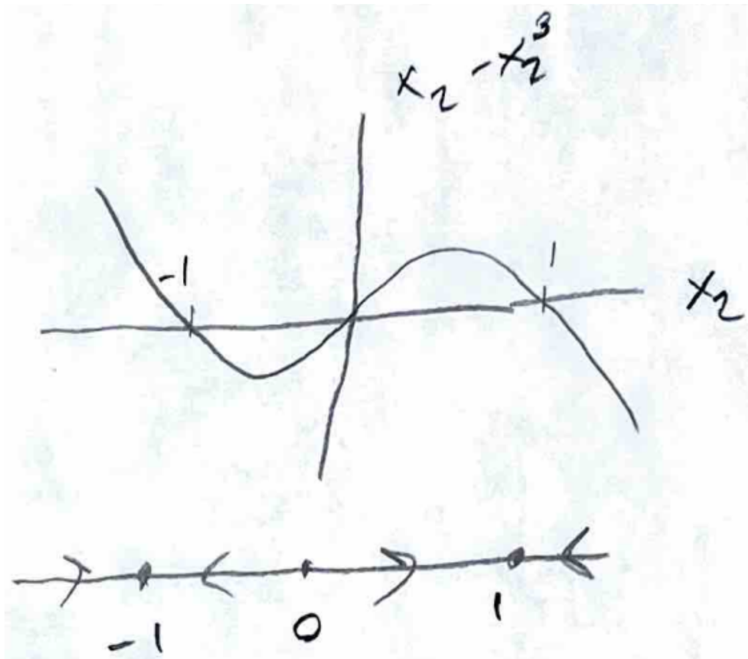

Figure 2: Phase line diagram for x_2 equation

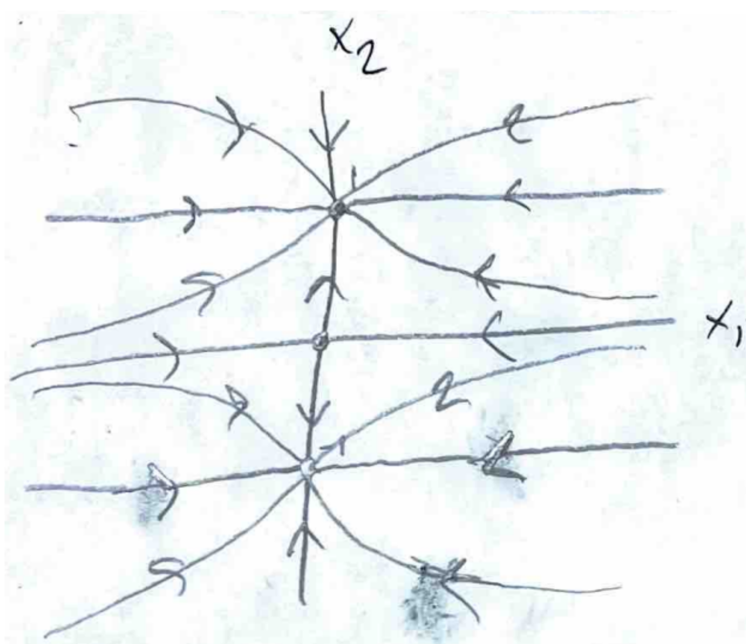

Figure 3: 7.8(c) Phase portrait