Math 6417 Homework 3

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Question 1.

Let $B(\cdot,\cdot)$ be a continuous, bilinear form on a real Hilbert space H. Suppose that B is coercive in the sense that there is some $\alpha > 0$ such that $B(x,x) \geq \alpha\|x\|^2$ for all $x \in H$.

1.1) Let $y \in H$. Then the map $f_y : H \to \mathbf{R}$ defined by $f_y(x) = B(x,y)$ is a bounded linear functional on H. Consequently, there exists a unique $w \in H$ such that $B(x,y) = f_y(x) = (x,w)$ for all $x \in H$.

Proof. Firstly, it is clear that f_y is linear; indeed, given $a_1, a_2 \in \mathbf{R}$ and $x_1, x_2 \in H$,

$$f_y(a_1x_1 + a_2x_2) = B(a_1x_1 + a_2x_2, y) = a_1B(x_1,y) + a_2B(x_2,y) = a_1f_y(x_1) + a_2f_y(x_2) \tag{1}$$

by the bilinearity of B.

Secondly, $B(\cdot,y) = f_y$ must be continuous because B is continuous. Hence, f_y is bounded.

Thirdly, by the Riesz representation theorem, there exists a unique $w \in H$ such that $B(x,y) = f_y(x) = (x,w)$ for all $x \in H$. □

1.2) Given $y \in H$, by 1.1), there is a unique w such that $B(x,y) = (x,w)$ for all $x \in H$; this defines a function $A : H \to H$, where $Ay = w$. Then A is a bounded, linear operator on H, that is, $A \in B(H)$.

Proof. Let $a_1, a_2 \in \mathbf{R}$ and $y_1, y_2 \in H$. Then for all $x \in H$,

$$\begin{aligned}(x, A(a_1y_1 + a_2y_2)) &= B(x, a_1y_1 + a_2y_2) = a_1B(x,y_1) + a_2B(x,y_2) = a_1(x,Ay_1) + a_2(x,Ay_2)\\ &= (x, a_1Ay_1 + a_2Ay_2).\end{aligned} \tag{2}$$

Thus, $w = A(a_1y_1 + a_2y_2)$ and $w' = a_1Ay_1 + a_2Ay_2$ satisfy the property that $B(x, a_1y_1 + a_2y_2) = (x,w) = (x,w')$ for all $x \in H$. Since there is only one element of H that can satisfy this property by the Riesz representation theorem, it follows that $w = w'$, that is, $A(a_1y_1 + a_2y_2) = a_1Ay_1 + a_2Ay_2$. Therefore, A is linear.

Note that B is continuous if and only if (see, e.g., Theorem 8.10 assumption (a) in Arbogast and Bona) there exists some $M > 0$ such that

$$|B(x,y)| \leq M\|x\|\|y\|, \quad \text{for all } x, y \in H. \tag{3}$$

Let $y \in H$. Then

$$\|Ay\| = \left|\left(\frac{Ay}{\|Ay\|}, Ay\right)\right| = \left|B\left(\frac{Ay}{\|Ay\|}, y\right)\right| \leq M\|y\|. \tag{4}$$

Since y was arbitrary, it follows that A is bounded, and $\|A\| \leq M$. Thus, A is a bounded, linear operator on H. □

1.3) A is bounded below in the sense that there exists $\gamma > 0$ such that $\|Ay\| \geq \gamma\|y\|$ for all $y \in H$.

Proof. This follows from the coercivity of B: for all $y \in H$,

$$\|Ay\|\|y\| \geq |(y, Ay)| = |B(y, y)| \geq \alpha\|y\|^2, \tag{5}$$

so $\|Ay\| \geq \alpha\|y\|$ for all $y \in H$, as claimed. ☐

1.4) A is one-to-one, and the range of A is closed.

Proof. Let $y_1, y_2 \in H$, and suppose that $Ay_1 = Ay_2$. Then, by the previous part,

$$\|y_1 - y_2\| \leq \frac{1}{\gamma}\|A(y_1 - y_2)\| = \frac{1}{\gamma}\|Ay_1 - Ay_2\| = 0. \tag{6}$$

Therefore, $y_1 = y_2$. This shows that A is one-to-one.

Let $R(A)$ denote the range of A. We show that $H \setminus R(A)$ is open. Indeed, let $w \in R(A)$. ☐

1.5) A is onto.

Proof. Suppose that $x \in R(A)^\perp$, that is, $(x, w) = 0$ for all $w \in R(A)$. This implies that $(x, Ay) = 0$ for all $y \in H$, which is equivalent to saying that $B(x, y) = 0$ for all $y \in H$. In particular, if we choose $y = x$, then $\|x\|^2 \leq \frac{1}{\alpha}|B(x, x)| = 0$. Therefore, $x = 0$. This shows that $R(A)^\perp = \{0\}$ because x was arbitrary.

Let $y \in H$. Since $R(A)$ is a closed subspace of H by (1.4), there exists a best approximation $w \in R(A)$ of y, which satisfies the property $(y - w, x) = 0$ for all $x \in R(A)$ (Theorem 3.7 and Corollary 3.8 in Arbogast and Bona). That is, $y - w \in R(A)^\perp$. Since $R(A)^\perp = \{0\}$ by the above, it follows that $y - w = 0$, and $y = w \in R(A)$. Since y was arbitrary and $R(A) \subseteq H$, it follows that $R(A) = H$, that is, A is onto. ☐

1.6) A is invertible.

Proof. By the previous two parts, A is bijective, so it has a set-theoretic inverse function A^{-1}. By 1.2), A is bounded. Therefore, by the open mapping theorem, A maps open sets to open sets, which means that the preimage of an open set under A^{-1} is open, that is, A^{-1} is continuous. Therefore, A is invertible. ☐

1.7) Given $f \in H^*$, the Riesz representation theorem implies that there exists a unique $w \in H$ such that $f(x) = (x, w)$ for all $x \in H$, and we can view H^* and H as the same under the correspondence $f \leftrightarrow w$.

1.8) Consider the equation $B(x, y) = f(x)$ for all $x \in H$, where $f \in H^*$. By the remark in part 1.7), we can choose $w \in H$ such that $f(x) = (x, w)$ for all $x \in H$. Then the equation is equivalent to $B(x, y) = (x, w)$ for all $x \in H$. If y is a solution of this equation, then, by the definition of A, we must have $Ay = w$. Using the invertibility of A, we obtain $y = A^{-1}w$ as the unique solution of the equation. Viewing f and w as the same under the correspondence in 1.7), we might also write $y = A^{-1}f$.

Question 2.

Define

$$H = \left\{ f \in L^2([-\pi, \pi]) : f(x) = \sum_{j \neq 0} f_j e^{ijx} \text{ some } \{f_j\} \text{ such that } \sum_{j \neq 0} j^2 |f_j|^2 < \infty \right\}, \tag{7}$$

and define

$$H^{-1} = \left\{ f(x) = \sum_{j\neq 0} f_j e^{ijx} : \sum_{j\neq 0} j^{-2}|f_j|^2 < \infty \right\}. \tag{8}$$

Before working the problems using these spaces, we make a few general remarks.

- For our purposes, the sum for $f(x)$ in H^{-1} is really a formal interpretation, but we can still rigorously interpret H^{-1} as the set of all sequences of complex numbers satisfying the summability condition. To facilitate this interpretation, define

$$S_{H^{-1}} = \left\{ \{f_j\}_{j\neq 0} \subseteq \mathbf{C} : \sum_{j\neq 0} j^{-2}|f_j|^2 < \infty \right\}. \tag{9}$$

- As for H, we recall that $\{e^{ijx}\}$ is an orthogonal basis for $L^2([-\pi,\pi])$ (using the L^2 inner product), and every element $f \in L^2([-\pi,\pi])$ has a unique sequence of coefficients $\{f_j\}$ such that

$$f(x) = \sum_j f_j e^{ijx}, \tag{10}$$

where the limit of the sum is taken in the $L^2([-\pi,\pi])$ sense, and, conversely, given any sequence $\{f_j\}$ such that $\sum_j |f_j|^2 < \infty$, there is a function $f \in L^2([-\pi,\pi])$ such that $\{f_j\}$ are the coefficients of f in the sense of (10).

- Define

$$S_H = \left\{ \{f_j\}_{j\neq 0} \subseteq \mathbf{C} : \sum_{j\neq 0} j^{2}|f_j|^2 < \infty \right\} \tag{11}$$

By the previous remark, we see that H is in one-to-one correspondence with S_H.

- We can equip S_H and $S_{H^{-1}}$ with element-wise addition and scalar multiplication operators, which make them into vector spaces. Indeed, if $\{f_j\} \in S_H$, and $\{g_j\} \in S_H$, then, by the Cauchy-Schwartz inequality,

$$\sum_{j\neq 0} j^2|f_j + g_j|^2 \leq \sum_{j\neq 0} j^2|f_j|^2 + 2\left(\sum_{j\neq 0} j^2|f_j|^2\right)^{\frac{1}{2}}\left(\sum_{j\neq 0} j^2|g_j|^2\right)^{\frac{1}{2}} + \sum_{j\neq 0} j^2|g_j|^2 < \infty, \tag{12}$$

and if $\alpha \in \mathbf{C}$,

$$\sum_{j\neq 0} j^2|\alpha f_j|^2 = |\alpha|^2 \sum_{j\neq 0} j^2|f_j|^2 < \infty. \tag{13}$$

Similar reasoning proves that $S_{H^{-1}}$ is closed under element-wise addition and scalar multiplication. Thus, S_H and $S_{H^{-1}}$ are vector spaces, since they are nonempty (contain the zero sequence) and are closed under the vector space operations of the vector space of all sequences of complex numbers.

2.1) H and H^{-1} are Hilbert spaces under the inner products

$$(f,g)_H = \sum_{j\neq 0} j^2 f_j \bar{g}_j, \qquad (f,g)_{H^{-1}} = \sum_{j\neq 0} j^{-2} f_j \bar{g}_j. \tag{14}$$

Proof. We can break this proof into 4 parts. For each space $G \in \{H, H^{-1}\}$, we need to show that

(a) G is a vector space;
(b) $(\cdot,\cdot)_G$ is well-defined;
(c) $(\cdot,\cdot)_G$ is an inner product on G;
(d) and G is complete with respect to the norm $\|\cdot\|_G = \sqrt{(\cdot,\cdot)_G}$, that is, the norm induced by $(\cdot,\cdot)_G$.

Proof of (a)

For $G = H^{-1}$, the result is trivial; we are interpreting H^{-1} as $S_{H^{-1}}$, which we already showed was a vector space in the preliminary remarks.

For $G = H$, we show that H is a subspace of $L^2([-\pi,\pi])$. Let $f, g \in H$, and let $\alpha, \beta \in \mathbf{C}$. By the preliminary remarks, we can find $\{f_j\}, \{g_j\} \in S_H$ such that

$$f(x) = \sum_{j\neq 0} f_j e^{ijx}, \qquad g(x) = \sum_{j\neq 0} g_j e^{ijx}. \tag{15}$$

Since $\{\alpha f_j + \beta g_j\} \in S_H$ by the preliminary remarks, and

$$(\alpha f + \beta g)(x) = \sum_{j\neq 0} (\alpha f_j + \beta g_j) e^{ijx}, \tag{16}$$

it follows that $\alpha f + \beta g \in H$. Thus, H is a subspace of $L^2([-\pi,\pi])$.

Proof of (b)

Let $G \in \{H, H^{-1}\}$, and let $f, g \in G$ with corresponding sequences of coefficients $\{f_j\}, \{g_j\} \in S_G$. Define $\sigma(H) = 1$ and $\sigma(H^{-1}) = -1$. Then the inner product

$$(f,g)_G = \sum_{j\neq 0} j^{2\sigma(G)} f_j \bar{g}_j \tag{17}$$

converges by the Cauchy-Schwarz inequality; indeed, it converges absolutely because

$$\sum_{j\neq 0} j^{2\sigma(G)} |f_j||\bar{g}_j| \leq \left(\sum_{j\neq 0} j^{2\sigma(G)} |f_j|^2\right)^{\frac{1}{2}} \left(\sum_{j\neq 0} j^{2\sigma(G)} |g_j|^2\right)^{\frac{1}{2}} < \infty. \tag{18}$$

Lastly, the value of the inner-product is well-defined because the sequences $\{f_j\}$ and $\{g_j\}$ are uniquely determined by f and g by the preliminary remarks.

Proof of (c)

For $G \in \{H, H^{-1}\}$, in order to show that $(\cdot,\cdot)_G$ is an inner product, we need to show that $(\cdot,\cdot)_G$ is

- conjugate symmetric,
- linear in the first argument,
- and positive definite.

Let $f, g \in G$ with corresponding coefficients $\{f_j\}, \{g_j\} \in S_G$. Then

$$(g,f)_G = \sum_{j\neq 0} j^{2\sigma(G)} g_j \bar{f}_j = \overline{\sum_{j\neq 0} j^{2\sigma(G)} \bar{g}_j f_j} = \overline{(f,g)_G}, \tag{19}$$

so $(\cdot,\cdot)_G$ is conjugate symmetric. If $\alpha, \beta \in \mathbf{C}$, and $\tilde{f} \in G$ with corresponding coefficients $\{\tilde{f}_j\} \in S_G$, then the coefficients of $\alpha f + \beta \tilde{f}$ in S_G are clearly $\{\alpha f_j + \beta \tilde{f}_j\}$. Therefore,

$$(\alpha f + \beta \tilde{f}, g)_G = \sum_{j\neq 0} j^{2\sigma(G)} (\alpha f_j + \beta \tilde{f}_j) \bar{g}_j \tag{20}$$

$$= \alpha \sum_{j\neq 0} j^{2\sigma(G)} f_j \bar{g}_j + \beta \sum_{j\neq 0} j^{2\sigma(G)} \tilde{f}_j \bar{g}_j = \alpha (f,g)_G + \beta (\tilde{f}, g)_G. \tag{21}$$

That is, $(\cdot,\cdot)_G$ is linear in the first argument. Finally, observe that

$$(f,f)_G = \sum_{j\neq 0} j^{2\sigma(G)} f_j \bar{f}_j = \sum_{j\neq 0} j^{2\sigma(G)} |f_j|^2 \geq 0. \tag{22}$$

If $(f,f)_G = 0$, then, since each term of the series for $(f,f)_G$ is nonnegative, it follows that each term must be zero, that is, $j^{2\sigma(G)}|f_j|^2 = 0$. This implies that $f_j = 0$ for all j because $j^{2\sigma(G)} \neq 0$. Therefore, $f = 0$. This shows that $(\cdot,\cdot)_G$ is positive definite.

Proof of (d)

Let $G \in \{H, H^{-1}\}$, and let $\{f^n\}_{n=1}^\infty$ be a Cauchy sequence in G with respect to the norm $\|\cdot\|_G$ induced by the inner product $(\cdot,\cdot)_G$. Let $\{f_j^n\}$ be the corresponding coefficients of f^n in S_G.

Then, given $\varepsilon > 0$, we can choose N such that $n, m > N$ implies that

$$\varepsilon > \|f^n - f^m\|_G^2 = (f^n - f^m, f^n - f^m)_G = \sum_{j\neq 0} j^{2\sigma(G)} |f_j^n - f_j^m|^2. \tag{23}$$

Since each term of the above series is nonnegative, it follows that for all j and all $n, m > N$,

$$j^{2\sigma(G)} |f_j^n - f_j^m|^2 < \varepsilon. \tag{24}$$

Thus, given $\varepsilon' > 0$, we can set $\varepsilon = \frac{\sqrt{\varepsilon'}}{j^{\sigma(G)}}$, for which we may choose N_j such that $n, m > N_j$ implies that $|f_j^n - f_j^m| < \varepsilon'$. Thus, $\{f_j^n\}_{n=1}^\infty$ is a Cauchy sequence for all j. By the completeness of $\mathbf{C}$, each of these sequences has a limit, say $f_j \in \mathbf{C}$.

Then $\{f_j\} \in S_G$. Indeed, let $J > 0$ be an integer, and define $\mathcal{J}_J = \{j \in \mathbf{Z} \setminus \{0\} : |j| \leq J\}$. Since $\mathcal{J}_J$ is finite, by the convergence of the sequences $\{f_j^n\}_{n=1}^\infty$, we can choose n such that $|f_j - f_j^n| < \frac{1}{J^2}$ for all $j \in \mathcal{J}_J$. Then

$$\begin{aligned}
\sum_{j\in\mathcal{J}_J} j^{2\sigma(G)} |f_j|^2 &= \sum_{j\in\mathcal{J}_J} j^{2\sigma(G)} \left(|f_j - f_j^n|^2 + (f_j - f_j^n)\bar{f}_j^n + \overline{(f_j - f_j^n)} f_j^n + |f_j^n|^2 \right) \\
&\leq \sum_{j\in\mathcal{J}_J} \frac{j^{2\sigma(G)}}{J^4} + 2\left(\sum_{j\in\mathcal{J}_J} j^{2\sigma(G)} |f_j - f_j^n|^2 \right)^{\frac{1}{2}} \left(\sum_{j\in\mathcal{J}_J} j^{2\sigma(G)} |f_j^n|^2 \right)^{\frac{1}{2}} \\
&\quad + \sum_{j\in\mathcal{J}_J} j^{2\sigma(G)} |f_j^n|^2.
\end{aligned} \tag{25}$$

Since $2\sigma(G) \leq 2$, and $j \in \mathcal{J}_J$ implies that $|j| \leq J$, it follows that $\frac{j^{2\sigma(g)}}{J^4} \leq \frac{1}{j^2}$ for $j \in \mathcal{J}_J$. Also, it is well-known that

$$\lim_{J\to\infty} \sum_{j\in\mathcal{J}} \frac{1}{j^2} = \lim_{J\to\infty} 2\sum_{j=1}^{J} \frac{1}{j^2} = \frac{\pi^2}{3}. \tag{26}$$

Therefore,

$$\sum_{j\in\mathcal{J}_J} \frac{j^{2\sigma(G)}}{J^4} \leq \frac{\pi^2}{3}. \tag{27}$$

Furthermore, by the definition of $\|\cdot\|_G$,

$$\|f^n\|_G^2 = \lim_{J\to\infty} \sum_{j\in\mathcal{J}_J} j^{2\sigma(G)} |f_j^n|^2. \tag{28}$$

Hence,

$$\sum_{j\in\mathcal{J}_J} j^{2\sigma(G)} |f_j^n|^2 \leq \|f^n\|_G^2. \tag{29}$$

The sequence $\{f^n\}$ is Cauchy with respect to $\|\cdot\|_G$, so it must be bounded with respect to $\|\cdot\|_G$, that is, there exists $K > 0$ such that $\|f^n\|_G \leq K$ for all n.

Combining these observations with (25), we get

$$\sum_{j\in\mathcal{J}_J} j^{2\sigma(G)}|f_j|^2 \leq \frac{\pi^2}{3} + \frac{2K\pi}{\sqrt{3}} + K^2 \tag{30}$$

for all $J > 0$. This implies that

$$\sum_{j\neq 0} j^{2\sigma(G)}|f_j|^2 < \infty, \tag{31}$$

so $\{f_j\} \in S_G$.

Thus, there is a function $f \in G$ whose coefficients in S_G are $\{f_j\}$. If we can show that $f^n \to f$ in $\|\cdot\|_G$, then we will have shown that G is complete with respect to $\|\cdot\|_G$, that is, G is a Hilbert space.

Let $\varepsilon > 0$ be given. Then we can choose N such that $n, m > N$ implies that $\|f^n - f^m\| < \varepsilon$. Let $n > N$. For $J > 0$, we can choose $m > n$ such that $|f_j - f_j^m| < \frac{\varepsilon}{J^4}$ for all $j \in \mathcal{J}_J$. Then, by a similar computation to (25),

$$\begin{aligned}\sum_{j\in\mathcal{J}_J} j^{2\sigma(G)}|f_j - f_j^n|^2 &\leq \sum_{j\in\mathcal{J}_J} \frac{\varepsilon j^{2\sigma(G)}}{J^4} + 2\left(\sum_{j\in\mathcal{J}_J} j^{2\sigma(G)}|f_j - f_j^n|^2\right)^{\frac{1}{2}}\left(\sum_{j\in\mathcal{J}_J} j^{2\sigma(G)}|f_j^n - f_j^m|^2\right)^{\frac{1}{2}} \\ &\quad + \sum_{j\in\mathcal{J}_J} j^{2\sigma(G)}|f_j^n - f_j^m|^2 \\ &\leq \varepsilon\frac{\pi^2}{3} + 2\|f - f^n\|_G \cdot \|f^n - f^m\|_G + \|f^n - f^m\|_G^2 \\ &\leq \left(\frac{\pi^2}{3} + \|f - f^n\|_G + \varepsilon\right)\varepsilon\end{aligned} \tag{32}$$

Since $\{f^n\}$ is Cauchy, $\{f - f^n\}$ is also Cauchy, and therefore also bounded; that is, there exists $L > 0$ such that $\|f - f^n\| \leq L$ for all n.

Hence, for all $\varepsilon > 0$ and all $J > 0$, we have $n > N$ implies

$$\sum_{j\in\mathcal{J}_J} j^{2\sigma(G)}|f_j - f_j^n|^2 \leq \left(\frac{\pi^2}{3} + L + \varepsilon\right)\varepsilon. \tag{33}$$

Therefore, $n > N$ implies that

$$\|f - f^n\|_G^2 \leq \left(\frac{\pi^2}{3} + L + \varepsilon\right)\varepsilon. \tag{34}$$

Hence, for any $\varepsilon' > 0$, we can choose N' such that $n > N'$ implies that

$$\|f - f^n\|_G \leq \varepsilon'. \tag{35}$$

That is, $f \to f^n$ in $\|\cdot\|_G$. Therefore, G is complete.

This shows that G is a Hilbert space for $G \in \{H, H^{-1}\}$. □

2.2) For $f, g \in H$, define

$$B(f,g) = \sum_{j\neq 0}(ij + j^2)f_j\bar{g}_j. \tag{36}$$

Then B is a continuous, coercive, bilinear form; that is, B satisfies the assumptions of the Lax-Milgram theorem.

Proof. Like $(\cdot,\cdot)_H$, the function B is well-defined because the sequences $\{f_j\}$ and g_j in its definition are uniquely determined by f and g, and the series converges absolutely because, by the Cauchy-Schwarz inequality,

$$\sum_{j\neq 0} |ij+j^2|\cdot|f_j|\cdot|\bar{g}_j| \leq \left(\sum_{j\neq 0}|ij+j^2||f_j|^2\right)^{\frac{1}{2}}\left(\sum_{j\neq 0}|ij+j^2||g_j|^2\right)^{\frac{1}{2}} \leq \sqrt{2}\|f\|_H\|G\|_H \tag{37}$$

because $|ij+j^2| = \sqrt{j^2+j^4} \leq \sqrt{2j^4} \leq \sqrt{2}j^2$ for all integers $j \neq 0$. This also shows that B is continuous because

$$|B(f,g)| \leq \sum_{j\neq 0}|ij+j^2|\cdot|f_j|\cdot|\bar{g}_j| \leq \sqrt{2}\|f\|_H\|g\|_H. \tag{38}$$

The function B is bilinear because, if $f,\tilde{f},g,\tilde{g}\in H$ with corresponding coefficients $\{f_j\},\{\tilde{f}_j\},\{g_j\},\{\tilde{g}_j\}\in S_H$, and $\alpha,\beta,\gamma,\delta\in\mathbf{R}$, then

$$\begin{aligned}
B(\alpha f+\beta\tilde{f},\gamma g+\delta\tilde{g}) &= \sum_{j\neq 0}(ij+j^2)(\alpha f_j+\beta\tilde{f}_j)(\gamma\bar{g}_j+\delta\bar{\tilde{g}}_j) &(39)\\
&= \alpha\gamma\sum_{j\neq 0}(ij+j^2)f_j\bar{g}_j + \beta\gamma\sum_{j\neq 0}(ij+j^2)\tilde{f}_j\bar{g}_j &(40)\\
&\quad + \alpha\delta\sum_{j\neq 0}(ij+j^2)f_j\bar{\tilde{g}}_j + \beta\delta\sum_{j\neq 0}(ij+j^2)\tilde{f}_j\bar{\tilde{g}}_j &(41)\\
&= \alpha\gamma B(f,g)+\beta\gamma B(\tilde{f},g)+\alpha\delta B(f,\tilde{g})+\beta\delta B(\tilde{f},\tilde{g}), &(42)
\end{aligned}$$

as $\{\alpha f_j+\beta\tilde{f}_j\}\in S_H$ are the coefficients of $\alpha f+\beta\tilde{f}$. Finally, B is coercive because

$$|B(f,f)| = \left|\sum_{j\neq 0}(ij+j^2)|f_j|^2\right| = \left[\left(\sum_{j\neq 0} j|f_j|^2\right)^2 + \left(\sum_{j\neq 0} j^2|f_j|^2\right)^2\right]^{\frac{1}{2}} \geq \sum_{j\neq 0} j^2|f_j|^2 = \|f\|_H^2. \tag{43}$$

□

2.3) We can view $f\in H^{-1}$ as an element of H^* under the action

$$f(g) = \sum_{j\neq 0} g_j\bar{f}_j \tag{44}$$

for $g\in H$, where $\{f_j\}\in S_{H^{-1}}$ and $\{g_j\}\in S_H$ are the coefficients of f and g.

Proof. As with $(\cdot,\cdot)_H$ and $B(\cdot,\cdot)$, the functional $f(g)$ is well-defined because the sequences $\{f_j\}$ and $\{g_j\}$ are uniquely determined by f and g, and the series converges absolutely because

$$\left|\sum_{j\neq 0} g_j\bar{f}_j\right| \leq \sum_{j\neq 0} j^{-1}|f_j|\cdot j|g_j| \leq \left(\sum_{j\neq 0} j^{-2}|f_j|^2\right)^{\frac{1}{2}}\left(\sum_{j\neq 0} j^2|g_j|^2\right)^{\frac{1}{2}} \leq \|f\|_{H^{-1}}\|g\|_H. \tag{45}$$

The functional f is also linear because, given $\alpha,\beta\in\mathbf{C}$, and $g,\tilde{g}\in H$ with coefficients $\{g_j\},\{\tilde{g}_j\}\in S_H$,

$$f(\alpha g+\beta\tilde{g}) = \sum_{j\neq 0}\bar{f}_j(\alpha g_j+\tilde{g}_j) = \alpha\sum_{j\neq 0} g_j\bar{f}_j + \beta\sum_{j\neq 0}\tilde{g}_j\bar{f}_j = \alpha f(g)+\beta f(\tilde{g}) \tag{46}$$

□