Math 5601 Homework 5

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Question 1.

Let $S = \{(t, y) \in \mathbf{R}^2 \mid |t| < 11\}$. Define

$$f(t, y) = \frac{t}{9}\cos(2y) + t^2. \tag{1}$$

Then f is differentiable with respect to y, with

$$\frac{\partial f}{\partial y} = -\frac{2t}{9}\sin(2y) \tag{2}$$

If $|t| < 11$, then $\left|\frac{\partial f}{\partial y}\right| < \frac{22}{9}$ for all $y \in \mathbf{R}$. This implies that f is $\frac{22}{9}$-Lipschitz in y over S, so, by Theorem (I) in the notes, the IVP

$$y' = f(t, y), \qquad y(0) = 1 \tag{3}$$

has a unique solution defined for $|t| < 11$. Then (3) certainly has a unique solution defined for $|t| \leq 10$.

Question 2.

Question 3.

Consider the forward Euler method

$$y_{j+1}^h = y_j^h + f(x_j, y_j^h), \qquad h > 0, \quad j \in \{0, 1, 2, \ldots, N\} \tag{4}$$

for approximating the solution $y(x)$ of $y' = f(x, y)$ with $y(0) = \alpha$. Suppose that

$$y_j^h - y(x_j) = \sum_{m=1}^{\infty} c_m h^m \tag{5}$$

for some $\{c_m\}$ independent of h. To find a third-order approximation z_j^h of $y(x_j)$ using y_j^h, $y_j^{\frac{h}{2}}$, and $y_j^{\frac{h}{3}}$, we take a linear combination of them and attempt to find coefficients that make the combination a third-order approximation. To this end, let

$$z_j^h = a_1 y_j^h + a_2 y_j^{\frac{h}{2}} + a_3 y_j^{\frac{h}{3}}. \tag{6}$$

Consider the difference $z_j - y(x_j)$:

$$z_j^h - y(x_j) = (a_1 + a_2 + a_3 - 1)y(x_j) + \sum_{n=1}^{3} a_n \left(y_j^{\frac{h}{n}} - y(x_j) \right) \tag{7}$$

$$= (a_1 + a_2 + a_3 - 1)y(x_j) + \sum_{n=1}^{3} a_n \sum_{m=1}^{\infty} c_m \left(\frac{h}{n} \right)^m \tag{8}$$

$$= (a_1 + a_2 + a_3 - 1)y(x_j) + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{3} \frac{a_n}{n^m} \right) h^m \tag{9}$$

$$= \left(-1 + \sum_{n=1}^{3} a_n \right) y(x_j) + \left(\sum_{n=1}^{3} \frac{a_n}{n} \right) h + \left(\sum_{n=1}^{3} \frac{a_n}{n^2} \right) h^2 + O(h^3). \tag{10}$$

Evidently, z_j^h will be a third-order approximation if we choose a_1, a_2, and a_3 such that the first three terms in the last line above are all zero. That is, we must choose the coefficients to satisfy

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{4} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{11}$$

This implies that

$$\begin{bmatrix} 18 & 12 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = -36a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{12}$$

which gives

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = -\frac{1}{36} \cdot (-36a_1) \cdot \begin{bmatrix} 4 & -12 \\ -9 & 18 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = a_1 \begin{bmatrix} -8 \\ 9 \end{bmatrix}, \tag{13}$$

or $a_2 = -8a_1$, and $a_3 = 9a_1$. Since we must have $a_1 + a_2 + a_3 = 1$, it follows that $a_1 - 8a_1 + 9a_1 = 1$, so $a_1 = \frac{1}{2}$, $a_2 = -4$, and $a_3 = \frac{9}{2}$. Therefore,

$$z_j^h = \frac{1}{2} y_j^h - 4 y_j^{\frac{h}{2}} + \frac{9}{2} y_j^{\frac{h}{3}} \tag{14}$$

is a third-order approximation of $y(x_j)$.