

# Math 5601 Homework 6

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**Problem 1.**

We already know the nodes  $x_0 = \frac{2a+b}{3}$  and  $x_1 = \frac{a+2b}{3}$ . To compute the weights  $\alpha_0$  and  $\alpha_1$  in the two-point open Newton-Cotes formula, we just compute the following integrals:

$$\alpha_0 = \int_a^b L_0(x) \, dx, \quad \alpha_1 = \int_a^b L_1(x) \, dx, \quad (1)$$

where

$$L_k(x) = \prod_{i=0, i \neq k}^1 \frac{x - x_i}{x_k - x_i}, \quad k \in \{0, 1\} \quad (2)$$

from the definition of Lagrange polynomials. Then  $L_0(x) = \frac{x-x_1}{x_0-x_1}$ , and  $L_1(x) = \frac{x-x_0}{x_1-x_0}$ . Hence

$$\alpha_0 = \int_a^b \frac{x - x_1}{x_0 - x_1} \, dx = \frac{1}{2(x_0 - x_1)} (x - x_1)^2 \Big|_a^b \quad (3)$$

$$= -\frac{3}{2(b-a)} \left[ \frac{(b-a)^2}{9} - \frac{4(b-a)^2}{9} \right] \quad (4)$$

$$= \frac{b-a}{2}, \quad (5)$$

and

$$\alpha_1 = \int_a^b \frac{x - x_0}{x_1 - x_0} \, dx = \frac{1}{2(x_1 - x_0)} (x - x_0)^2 \Big|_a^b \quad (6)$$

$$= \frac{3}{2(b-a)} \left[ \frac{4(b-a)^2}{9} - \frac{(b-a)^2}{9} \right] \quad (7)$$

$$= \frac{b-a}{2}. \quad (8)$$

This results in the Newton-Cotes formula with two-point open rule

$$\int_a^b f(x) \, dx \approx \alpha_0 f(x_0) + \alpha_1 f(x_1) = \frac{b-a}{2} f\left(\frac{2a+b}{3}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\right). \quad (9)$$

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**Problem 2.**

Suppose that we have a numerical quadrature for  $\hat{f}(\hat{x})$  on  $[0, 1]$  defined by

$$\hat{J}(\hat{f}) = \int_a^b \hat{f}(\hat{x}) \, d\hat{x} \approx \hat{Q}(\hat{f}) = \sum_{j=0}^m \hat{\alpha}_j \hat{f}(\hat{x}_j). \quad (10)$$

If we want to compute the integral of  $f(x)$  on  $[a, b]$ , then we can make the change of variables  $x = a + (b-a)\hat{x}$ , which is a one-to-one correspondence between  $x \in [a, b]$  and  $\hat{x} \in [0, 1]$  with  $dx = (b-a) d\hat{x}$ , to obtain

$$J(f) = \int_a^b f(x) dx = \int_0^1 f(a + (b-a)\hat{x})(b-a) d\hat{x}. \quad (11)$$

Define  $\hat{f}(\hat{x}) = f(a + (b-a)\hat{x})$ . Then we obtain a numerical quadrature  $Q(f)$  for  $J(f)$  by using our numerical quadrature  $\hat{Q}(\hat{f})$  to approximate the integral of  $\hat{f}$  on  $[0, 1]$ :

$$J(f) = (b-a) \int_0^1 \hat{f} d\hat{x} = (b-a) \hat{Q}(\hat{f}) = \sum_{j=0}^m (b-a) \hat{\alpha}_j \hat{f}(\hat{x}_j) \quad (12)$$

$$= \sum_{j=0}^m (b-a) \hat{\alpha}_j f(a + (b-a)\hat{x}_j) = \sum_{j=0}^m \alpha_j f(x_j) \quad (13)$$

$$=: Q(f), \quad (14)$$

if we define  $\alpha_j = (b-a)\hat{\alpha}_j$  and  $x_j = a + (b-a)\hat{x}_j$ .

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### Problem 3.

We use the given Newton-Cotes formulas to approximate

$$J = \int_1^2 \frac{\cos\left(\frac{\pi}{4}x\right)}{\sin^2\left(\frac{\pi}{4}x\right)} dx. \quad (15)$$

The following table summarizes the output of the MATLAB code I used to perform these calculations – see `output.txt` and `quad.m`. Note that we use the affine transformation process derived in Problem 2 to transform the quadrature nodes given in part (d) for  $[0, 1]$  to those for  $[1, 2]$ .

| Method | Approximation | Actual | Error |
|--------|---------------|--------|-------|
|--------|---------------|--------|-------|