Math 5215 Homework 1

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1. Rudin 6.1

Let α be increasing on $[a, b]$ and continuous at $x_0 \in [a, b]$. Suppose that $f(x_0) = 1$ and $f(x) = 0$ when $x \neq x_0$. Then $f \in \mathscr{R}(\alpha)$ and $\int_a^b f \, d\alpha = 0$.

Proof. Let $\varepsilon > 0$ be given. Since α is continuous at x_0, there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |\alpha(x) - \alpha(x_0)| < \varepsilon$. Let $P = \{y_0, y_1, y_2, y_3\}$ be a partition of $[a, b]$ such that $0 < |y_1 - x_0| < \delta$ and $0 < |y_2 - x_0| < \delta$.

Then we have $\Delta\alpha_2 < 2\varepsilon$, and $M_1 = M_3 = 0$, and $M_2 = 1$. This implies that

$$U(P, f, \alpha) = M_2 \Delta\alpha_2 < 2\varepsilon$$

On the other hand, we have $m_i = 0$ for $i = 1, 2, 3$. This implies that

$$L(P, f, \alpha) = 0$$

Hence,

$$0 = L(P, f, \alpha) \leq \int_a^b f \, d\alpha \leq U(P, f, \alpha) < 2\varepsilon$$

Since ε was arbitrary, we conclude that $f \in \mathscr{R}(\alpha)$ and $\int_a^b f \, d\alpha = 0$. □

2. Rudin 6.2

Suppose $f(x) \geq 0$ for $x \in [a, b]$ and f is continuous on $[a, b]$, and $\int_a^b f(x) \, dx = 0$. Then $f = 0$.

Proof. Suppose on the contrary that $f(x^*) = y > 0$ for some $x^* \in [a, b]$. Using the continuity of f, choose δ such that $|x - x^*| < \delta$ implies that $|f(x) - f(x^*)| = |f(x) - y| < \frac{y}{2}$. If $x^* = a$ or $x^* = b$, then by the continuity of f, there is another point $x^{**} \in (a, b)$ so that $f(x^{**}) > 0$, so we may assume that $x^* \in (a, b)$.

Let $P = \{x_0, x_1 = x^*, x_2, x_3\}$ be a partition of $[a, b]$ such that $\Delta x_2 < \delta$. Then for all $x \in [x_1, x_2]$, we have $|x - x^*| < \delta$ and, consequently, $|f(x) - y| < \frac{y}{2}$, which implies that $f(x) > \frac{y}{2}$. Therefore, $m_2 \geq \frac{y}{2} > 0$.

On the other hand, since $f(x) \geq 0$ for all $x \in [a, b]$, we must have $m_i \geq 0$ for $i = 1, 3$ as well. Thus,

$$L(P, f) = \sum_{i=1}^{3} m_i \Delta x_i \geq m_2 \Delta x_2 > 0$$

This would imply that

$$0 < L(P, f) \leq \int_a^b f(x) \, dx = 0$$

This is a contradiction, so the assumption $f(x^*) > 0$ is false, that is, $f(x^*) \leq 0$. This shows that $f(x^*) = 0$, but $x^* \in [a, b]$ was arbitrary, so $f = 0$ identically. □

3. Rudin 6.3

Define three functions $\beta^1, \beta^2, \beta^3$ as follows: $\beta^j(x) = 0$ if $x < 0$ and $\beta^j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta^1(0) = 0$, $\beta^2(0) = 1$, and $\beta^3(0) = \frac{1}{2}$. Let f be bounded on $[-1, 1]$.

(a) $f \in \mathscr{R}(\beta^1)$ if and only if $f(0+) = f(0)$, and then

$$\int f \, \mathrm{d}\beta^1 = f(0)$$

Proof. **"If"**

Suppose that $f(0+) = f(0)$. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < x < \delta$ implies that $|f(x) - f(0)| < \varepsilon$.

Let $P = \{x_0, x_1 = 0, x_2, x_3\}$ be a partition of $[-1, 1]$ such that $0 < x_2 < \delta$. Then $\Delta\beta^1_2 = 1$, and $\Delta\beta^1_i = 0$ if $i \neq 2$. Therefore,

$$U(P, f) - L(P, f) = M_2 - m_2 \leq 2\varepsilon$$

since

$$\begin{aligned} M_2 - m_2 &= \sup_{x,y \in [x_1, x_2]} |f(x) - f(y)| \\ &\leq \sup_{x,y \in [x_1, x_2]} \Big[|f(x) - f(0)| + |f(y) - f(0)| \Big] \\ &\leq 2\varepsilon \end{aligned}$$

This implies that $f \in \mathscr{R}(\beta^1)$ since ε was arbitrary.

The integral

Note that

$$\begin{aligned} M_2 &= \sup_{x \in [x_1, x_2]} f(x) \leq f(0) + \sup_{x \in [x_1, x_2]} |f(x) - f(0)| \\ &\leq f(0) + \varepsilon \end{aligned}$$

and

$$\begin{aligned} -m_2 &= \sup_{x \in [x_1, x_2]} \Big[-f(x) \Big] \leq -f(0) + \sup_{x \in [x_1, x_2]} |f(x) - f(0)| \\ &\leq -f(0) + \varepsilon \end{aligned}$$

so that $m_2 \geq f(0) - \varepsilon$.

In the above we have $L(P, f) = m_2$ and $U(P, f) = M_2$. Therefore

$$f(0) - \varepsilon \leq m_2 \leq \int f \, \mathrm{d}\beta^1 \leq M_2 \leq f(0) + \varepsilon$$

which implies that

$$\left| \int f \, \mathrm{d}\beta^1 - f(0) \right| \leq \varepsilon$$

This proves that

$$\int f \, \mathrm{d}\beta^1 = f(0)$$

because $\varepsilon > 0$ was arbitrary.

"Only if"

Suppose that $f \in \mathscr{R}(\beta_1)$. Then for $\varepsilon > 0$ there exists a partition P of $[-1, 1]$ such that

$$U(P, f) - L(P, f) < \varepsilon$$

Assume that $0 \in P$ (if not, replace P by a refinement containing 0, which still satisfies the above inequality), say, $x_{i-1} = 0$. Then we have $\Delta\beta_j^1 = 0$ for all $j \neq i$, and $\Delta\beta_i^1 = 1$. Thus, the inequality reduces to

$$M_i - m_i < \varepsilon$$

Choose $\delta = \Delta x_i$; then $0 < x < \delta$ implies that $|f(x) - f(0)| < \varepsilon$, as $f(x)$ and $f(0)$ are between m_i and M_i (because $x \in [x_{i-1}, x_i]$). This means that $f(0+) = f(0)$ by definition. □

(b) Similar to part (a), we have $f \in \mathscr{R}(\beta^2)$ if and only if $f(0-) = f(0)$, and then

$$\int f \, \mathrm{d}\beta^2 = f(0)$$

Proof. **"If"**

Suppose that $f(0-) = f(0)$. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that $-\delta < x < 0$ implies that $|f(x) - f(0)| < \varepsilon$.

Let $P = \{x_0, x_1, x_2 = 0, x_3\}$ be a partition of $[-1, 1]$ such that $-\delta < x_1 < 0$. Then $\Delta\beta_2^2 = 1$, and $\Delta\beta_i^2 = 0$ if $i \neq 2$. Therefore,

$$U(P, f) - L(P, f) = M_2 - m_2 \leq 2\varepsilon$$

by the same reasoning from (a). This implies that $f \in \mathscr{R}(\beta^2)$ since ε was arbitrary.

The integral

In the above we have $L(P, f) = m_2$ and $U(P, f) = M_2$. Therefore (by the same reasoning from (a))

$$f(0) - \varepsilon \leq m_2 \leq \int f \, \mathrm{d}\beta^2 \leq M_2 \leq f(0) + \varepsilon$$

which implies that

$$\int f \, \mathrm{d}\beta^2 = f(0)$$

because $\varepsilon > 0$ was arbitrary.

"Only if"

Suppose that $f \in \mathscr{R}(\beta_2)$. Then for $\varepsilon > 0$ there exists a partition P of $[-1, 1]$ such that

$$U(P, f) - L(P, f) < \varepsilon$$

Assume that $0 \in P$ (if not, replace P by a refinement containing 0, which still satisfies the above inequality), say, $x_i = 0$. Then we have $\Delta\beta_j^2 = 0$ for all $j \neq i$, and $\Delta\beta_i^2 = 1$. Thus, the inequality reduces to

$$M_i - m_i < \varepsilon$$

Choose $\delta = \Delta x_i$; then $-\delta < x < 0$ implies that $|f(x) - f(0)| < \varepsilon$, as $f(x)$ and $f(0)$ are between m_i and M_i (because $x \in [x_{i-1}, x_i]$). This means that $f(0-) = f(0)$ by definition. □

(c) $f \in \mathscr{R}(\beta^3)$ if and only if f is continuous at 0.

Proof. **"If"**
Suppose that f is continuous at 0. Then $f \in \mathscr{R}(\beta^1)$ and $f \in \mathscr{R}(\beta^2)$ by (a) and (b). Therefore $f \in \mathscr{R}(\beta^3)$ because $\beta^3 = \frac{\beta^1+\beta^2}{2}$.
"Only if"
Suppose that $f \in \mathscr{R}(\beta^3)$. Let $\varepsilon > 0$. Then there exists a partition P of $[-1, 1]$ such that

$$U(P, f) - L(P, f) < \varepsilon$$

Let Q be a refinement of P containing 0, say $x_i = 0$. Then

$$U(Q, f) - L(Q, f) \leq U(P, f) - L(P, f) < \varepsilon$$

Since $\Delta\beta^3_i = \Delta\beta^3_{i+1} = \frac{1}{2}$, and $\Delta\beta^3_j = 0$ for all $j \notin \{i, i+1\}$, the inequality reduces to

$$M_i - m_i + M_{i+1} - m_{i+1} < 2\varepsilon$$

$M_j \geq m_j$ for all j, so the above implies that

$$M_i - m_i < 2\varepsilon \qquad M_{i+1} - m_{i+1} < 2\varepsilon$$

If we choose $\delta < \min\{\Delta x_i, \Delta x_{i+1}\}$, then $|x| < \delta$ implies that $x \in [x_{i-1}, x_i]$ or $x \in [x_i, x_{i+1}]$, so that either

$$|f(x) - f(0)| \leq M_i - m_i < 2\varepsilon$$

or

$$|f(x) - f(0)| \leq M_{i+1} - m_{i+1} < 2\varepsilon$$

so that $|f(x) - f(0)| < 2\varepsilon$ in any case. Therefore, f is continuous at 0. □

(d) If f is continuous at 0, then

$$\int f \, \mathrm{d}\beta^1 = \int f \, \mathrm{d}\beta^2 = \int f \, \mathrm{d}\beta^3 = f(0).$$

Proof. By the previous parts, if f is continuous at 0, then

$$\int f \, \mathrm{d}\beta^1 = f(0) = \int f \, \mathrm{d}\beta^2$$

and

$$\int f \, \mathrm{d}\beta^3 = \int f \, \mathrm{d}\left(\frac{\beta^1 + \beta^2}{2}\right) = \frac{1}{2}\int f \, \mathrm{d}\beta^1 + \frac{1}{2}\int f \, \mathrm{d}\beta^2 = f(0)$$

□

4. Rudin 6.4

Let $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x. Then $f \notin \mathscr{R}$ on any interval $[a, b]$.

Proof. Let P be a partition of an interval $[a, b]$, $a < b$. Then $m_i = 0$ and $M_i = 1$ for all i because each subinterval $[x_{i-1}, x_i]$ contains both rational and irrational numbers. Thus,

$$L(P, f) = 0 \qquad U(P, f) = b - a$$

Choose $\varepsilon = \frac{b-a}{2}$. Since P above was arbitrary there is no partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon$$

that is, $f \notin \mathscr{R}$. □

5. Rudin 6.5

Suppose that f is a bounded real function on $[a, b]$ and $f^2 \in \mathscr{R}$. It does not necessarily follow that $f \in \mathscr{R}$.

Consider, for example, a function f defined by $f(x) = 1$ if x is rational and $f(x) = -1$ if x is irrational. Then $f^2 = 1$, which is continuous and therefore integrable on $[a, b]$. On the other hand, if g is the function from 6.4, then $f = 2g - 1$, so $f \in \mathscr{R}$ implies $g \in \mathscr{R}$, but we know that $g \notin \mathscr{R}$, so it follows that $f \notin \mathscr{R}$.

On the other hand, if we know instead that $f^3 \in \mathscr{R}$, then it *does* follow that $f \in \mathscr{R}$. This is because the function $c^{-1}(x) = x^{\frac{1}{3}}$ is continuous everywhere, and it is the inverse of $c(x) = x^3$. This implies that $c^{-1} \circ f^3 = c^{-1} \circ c \circ f = f$, and the continuity of c^{-1} combined with the integrability of f^3 therefore implies that $f \in \mathscr{R}$.

6. Rudin 6.7

Suppose that f is a function on $(0, 1]$ and $f \in \mathscr{R}$ on $[c, 1]$ for all $c > 0$. Define

$$\int_0^1 f(x)\ \mathrm{d}x = \lim_{c \to 0} \int_c^1 f(x)\ \mathrm{d}x$$

if the limit exists and is finite.

(a) Let $f \in \mathscr{R}$ on $[0, 1]$. Then the definition above agrees with the integral of f on $[0, 1]$.

Proof. Let $\varepsilon > 0$ be given. Let $I = \lim_{c\to 0} \int_c^1 f(x)\ \mathrm{d}x$. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that $c < \delta$ implies

$$\left| I - \int_c^1 f(x)\ \mathrm{d}x \right| < \varepsilon$$

Or, equivalently,

$$\left| I - \int_0^1 f(x)\ \mathrm{d}x + \int_0^c f(x)\ \mathrm{d}x \right| < \varepsilon$$

which implies that

$$\left| I - \int_0^1 f(x)\ \mathrm{d}x \right| < \left| \int_0^c f(x)\ \mathrm{d}x \right| + \varepsilon$$

Since f is integrable on $[0, 1]$, it must be bounded. Let $|f(x)| \le M$ for all $x \in [0, 1]$. Then the above inequality becomes

$$\left| I - \int_0^1 f(x)\ \mathrm{d}x \right| < Mc + \varepsilon$$

This inequality holds for any $\varepsilon > 0$ and any sufficiently small $c > 0$. This implies that

$$I = \int_0^1 f(x)\ \mathrm{d}x$$

$\square$

(b) Let $f : (0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \frac{(-1)^{n(x)}}{x}$$

where $n(x)$ is the unique positive integer n such that $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$. Then the imporoper integral $\int_0^1 f$ exists, but the improper integral $\int_0^1 |f|$ does not.

Proof. For a given $0 < c < 1$, let N be the largest positive integer such that $c < \frac{1}{N}$. Then $f \in \mathscr{R}$ on $[c, 1]$ because it is a piecewise continuous function, and

$$\begin{aligned}\int_c^1 f &= (-1)^N \int_c^{\frac{1}{N}} \frac{1}{x}\,\mathrm{d}x + \sum_{n=1}^{N-1} (-1)^n \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{1}{x}\,\mathrm{d}x \\ &= R + \sum_{n=1}^{N-1} (-1)^n a_n\end{aligned}$$

where $R = (-1)^N \int_c^{\frac{1}{N}} \frac{1}{x}\,\mathrm{d}x$ and $a_n = \ln\left(1 + \frac{1}{n}\right)$ (take the sum to be 0 when $N = 1$). Then - $\{a_n\}$ is decreasing and $a_n \to 0$ as $n \to \infty$; - $\sum_{n=1}^{\infty} (-1)^n a_n$ converges by the Alternating Series Test to some number S; - $|R| \leq \int_{\frac{1}{N+1}}^{\frac{1}{N}} \frac{1}{x}\,\mathrm{d}x = a_N$.

Given $\varepsilon > 0$, choose c so that N is large enough to make $a_N < \varepsilon$ and so that

$$\left| S - \sum_{n=1}^{N-1} (-1)^n a_n \right| < \varepsilon$$

Then it follows that

$$\left| S - \int_c^1 f \right| = \left| S - R - \sum_{n=1}^{N-1} (-1)^n a_n \right| < 2\varepsilon$$

Therefore $\int_c^1 f \to S$ as $c \to 0$, that is, $\int_0^1 f = S$.

On the other hand, $|f(x)| = \frac{1}{x}$, which has antiderivative $\ln(x)$ on any interval $[c, 1]$, where $c > 0$. Therefore,

$$\lim_{c\to 0} \int_c^1 |f| = \lim_{c \to 0} \big[-\ln(c) \big] = \infty$$

so $\int_0^1 |f|$ does not exist. □

7. Rudin 6.8

Suppose that $f \in \mathscr{R}$ on $[a, b]$ for all $b > a$, where a is fixed. Define

$$\int_a^{\infty} f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x$$

if this limit exists (and is finite). If $f(x) \geq 0$ and f is decreasing on $[1, \infty)$ then

$$\int_1^{\infty} f(x)\,\mathrm{d}x$$

exists if and only if

$$\sum_{n=1}^{\infty} f(n)$$

converges.

Proof. Since $f(x) \geq 0$, it follows that $g(N) = \int_1^N f(x)\,\mathrm{d}x$ is an increasing function of N.

Integral converges implies series converges

Suppose that the integral $I = \int_1^\infty f(x)\,\mathrm{d}x$ converges. This and the fact that g is increasing imply $g(N) \leq I$ for any $N > 1$.

Let $N > 1$ be an integer, and let $P = \{1, 2, \dots, N\}$ be a partition of $[1, N]$. Then $m_n = f(n+1)$ because f is decreasing, and therefore

$$\sum_{n=2}^{N} f(n) = L(P, f) \leq \int_1^N f(x)\,\mathrm{d}x \leq I$$

Thus, the Nth partial sum of the series is bounded above by $I + f(1)$. On the other hand, the terms of the series are nonnegative, so the partial sums are increasing. Therefore, the series converges.

Series converges implies integral converges

Suppose that $\sum_{n=1}^\infty f(n)$ converges. Then the partial sums are bounded, say by M. Let $N > 1$ be an integer and let $P = \{1, 2, \dots, N\}$ be a partition of $[1, N]$. Then $M_n = f(n)$ because f is decreasing, and therefore

$$g(N) \leq U(P, f) = \sum_{n=1}^{N-1} f(n) \leq M$$

Since g is increasing and bounded above, it must have a limit; that is, the improper integral converges. □

8. Rudin 6.10

Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

Equality holds if and only if $u^p = v^q$.

Proof. **Lemma**

Let $f \in C^1([a, b])$ be strictly increasing. Then f is invertible with strictly increasing inverse $f^{-1} \in C^1([f(a), f(b)])$, and

$$\int_a^b f(x)\,\mathrm{d}x + \int_{f(a)}^{f(b)} f^{-1}(x)\,\mathrm{d}x = bf(b) - af(a)$$

because we can change variables $y = f^{-1}(x)$ in the second integral to write it as

$$\int_{f(a)}^{f(b)} f^{-1}(x)\,\mathrm{d}x = \int_a^b xf'(x)\,\mathrm{d}x$$

and then use integration by parts to obtain

$$\int_{f(a)}^{f(b)} f^{-1}(x)\,\mathrm{d}x = xf(x)\Big|_a^b - \int_a^b f(x)\,\mathrm{d}x$$

from which the claim follows.

Inequality

Define $f(x) = x^{p-1}$. Then $f \in C^1([0,\infty))$ and f is strictly increasing. Thus, f has a strictly increasing C^1 inverse on $[0,\infty)$. In fact, simple algebra shows that $\frac{1}{p-1} = q-1$, so $f^{-1}(x) = x^{\frac{1}{p-1}} = x^{q-1}$.

Assume without loss of generality that $f(u) \leq v$; if not, then $f^{-1}(v) \leq u$ because f is increasing, so, without altering the content of the claim to be proven, we could relabel the variables $p \leftrightarrow q$ and $u \leftrightarrow v$, in which case f would become f^{-1} and we would have $f(u) \leq v$.

Define the function ϕ as follows:

$$\phi(x) = \begin{cases} f^{-1}(x) & 0 \leq x \leq f(u) \\ u & f(u) < x \end{cases}$$

Then

(a) ϕ is continuous on $[0,\infty)$;

(b) $\phi = f^{-1}$ on $[0, f(u)]$;

(c) $\phi(x) \leq f^{-1}(x)$ for all $x \geq 0$ because f^{-1} is increasing.

Applying the Lemma and the above facts gives

$$\begin{aligned} \int_0^v \phi &= \int_0^{f(u)} f^{-1} + \int_{f(u)}^v u \,\mathrm{d}x && \text{(A)} \\ &= uf(u) - \int_0^u f + u(v - f(u)) && (1) \\ &= uv - \int_0^u f && (2) \end{aligned}$$

On the other hand,

$$\int_0^v \phi \leq \int_0^v f^{-1} \tag{B}$$

Applying the FTC gives

$$\int_0^u f = \int_0^u x^{p-1} \,\mathrm{d}x = \frac{u^p}{p} \qquad \int_0^v f^{-1} = \int_0^v x^{q-1} \,\mathrm{d}x = \frac{v^q}{q}$$

Combining all of the preceding results together shows that

$$uv \leq \int_0^u f + \int_0^v f^{-1} = \frac{u^p}{p} + \frac{u^q}{q}$$

Equality

Suppose that $uv = \frac{u^p}{p} + \frac{v^q}{q}$. Then by (A) we have $\int_0^v \phi = \frac{v^q}{q}$, that is, equality holds in (B). If $g = f^{-1} - \phi$, then we have $g(x) \geq 0$ for all $x \in [f(u), v]$ by 3., but equality in (B) and 2. together imply that

$$\int_{f(u)}^v g = 0$$

Then exercise 6.2 implies that $g(x) = 0$, that is, $\phi(x) = f^{-1}(x)$, for all $x \in [f(u), v]$. The only way this can be true given the definition of ϕ and the fact that f^{-1} is *strictly* increasing is if $f(u) = v$, or $u^{p-1} = v$. If this is the case, then substitution for v yields

$$u^p = \frac{u^p}{p} + \frac{v^q}{q}$$

or

$$u^p\left(1-\frac{1}{p}\right)=\frac{1}{q}v^q$$

which implies that $u^p = v^q$.

Conversely, if $u^p = v^q$, then $u = v^{\frac{q}{p}}$, and

$$\begin{aligned} uv &= v^{\frac{q}{p}+1} = v^{q\left(\frac{1}{p}+\frac{1}{q}\right)} = v^q = v^q\left(\frac{1}{p}+\frac{1}{q}\right) \\ &= \frac{u^p}{p}+\frac{v^q}{q} \end{aligned}$$

□

(b) If $f, g \in \mathscr{R}(\alpha)$ and $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p \,\mathrm{d}\alpha = 1 = \int_a^b g^q \,\mathrm{d}\alpha,$$

then

$$\int_a^b fg \,\mathrm{d}\alpha \leq 1.$$

Proof. By part (a), we have $(fg)(x) \leq \frac{f(x)^p}{p}+\frac{g(x)^q}{q}$ for all $x \in [a,b]$, so

$$\int_a^b fg \,\mathrm{d}\alpha \leq \int_a^b \left[\frac{f^p}{p}+\frac{g^q}{q}\right] \,\mathrm{d}\alpha = \frac{1}{p}+\frac{1}{q} = 1.$$

□

(c) If $f, g \in \mathscr{R}(\alpha)$ are functions on $[a,b]$, then fg, $|f|^p$, and $|g|^q$ are in $\mathscr{R}(\alpha)$, and

$$\left|\int_a^b fg \,\mathrm{d}\alpha\right| \leq \left[\int_a^b |f|^p \,\mathrm{d}\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g|^q \,\mathrm{d}\alpha\right]^{\frac{1}{q}}$$

Proof. Define

$$F = \frac{|f|}{\left[\int_a^b |f|^p \,\mathrm{d}\alpha\right]^{\frac{1}{p}}} \qquad G = \frac{|g|}{\left[\int_a^b |g|^q \,\mathrm{d}\alpha\right]^{\frac{1}{q}}}$$

Then $\int_a^b F^p \,\mathrm{d}\alpha = 1$ and $\int_a^b G^q \,\mathrm{d}\alpha = 1$, so by part (b)

$$\begin{aligned} \left|\int_a^b fg \,\mathrm{d}\alpha\right| &\leq \int_a^b |f||g| \,\mathrm{d}\alpha \\ &= \left[\int_a^b |f|^p \,\mathrm{d}\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g|^q \,\mathrm{d}\alpha\right]^{\frac{1}{q}} \cdot \int_a^b FG \,\mathrm{d}\alpha \\ &\leq \left[\int_a^b |f|^p \,\mathrm{d}\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g|^q \,\mathrm{d}\alpha\right]^{\frac{1}{q}} \end{aligned}$$

□

(d) The integral from 6.7

Let f, g be functions on $(0,1]$ such that $f, g \in \mathscr{R}$ on $[c,1]$ for all $0 < c < 1$. Then fg, $|f|^p$, and $|g|^q$ are functions on $(0,1]$ that are integrable on $[c,1]$ for all $0 < c < 1$, and

$$\left| \int_0^1 fg \right| \leq \left[\int_0^1 |f|^p \right]^{\frac{1}{p}} \cdot \left[\int_0^1 |g|^q \right]^{\frac{1}{q}}$$

where the integrals are taken in the manner defined in exercise 6.7, assuming that all of the improper integrals exist and are finite.

Proof. Let $\varepsilon > 0$ be given. Then by the definition of the improper integral and the continuity of the functions $(\cdot)^{\frac{1}{p}}$ and $(\cdot)^{\frac{1}{q}}$ there exists $0 < c < 1$ such that

$$\int_0^1 fg \leq \int_c^1 fg + \varepsilon$$

$$\left[\int_c^1 |f|^p \right]^{\frac{1}{p}} \leq \left[\int_0^1 |f|^p \right]^{\frac{1}{p}} + \varepsilon \qquad \left[\int_c^1 |g|^q \right]^{\frac{1}{q}} \leq \left[\int_0^1 |g|^q \right]^{\frac{1}{q}} + \varepsilon$$

Then by (c)

$$\begin{aligned} \left| \int_0^1 fg \right| &\leq \left| \int_c^1 fg \right| + \varepsilon \leq \left[\int_c^1 |f|^p \right]^{\frac{1}{p}} \cdot \left[\int_c^1 |g|^q \right]^{\frac{1}{q}} + \varepsilon \\ &\leq \left(\left[\int_0^1 |f|^p \right]^{\frac{1}{p}} + \varepsilon \right) \cdot \left(\left[\int_0^1 |g|^q \right]^{\frac{1}{q}} + \varepsilon \right) + \varepsilon \\ &\leq \left[\int_0^1 |f|^p \right]^{\frac{1}{p}} \cdot \left[\int_0^1 |g|^q \right]^{\frac{1}{q}} + M\varepsilon + \varepsilon^2 \end{aligned}$$

where $M > 0$ is number depending on f and g but not ε. The conclusion follows because ε was arbitrary. $\square$

The integral from 6.8

Let f, g be functions on $[a,b]$ for all $b > a$, where a is fixed. Then

$$\left| \int_a^\infty fg \right| \leq \left[\int_a^\infty |f|^p \right]^{\frac{1}{p}} \cdot \left[\int_a^\infty |g|^q \right]^{\frac{1}{q}}$$

where the integrals are taken in the sense defined in 6.8, asssuming that all of the improper integrals exist and are finite.

Proof. Let $\varepsilon > 0$ be given. Then by the definition of the improper integral and the continuity of the functions $(\cdot)^{\frac{1}{p}}$ and $(\cdot)^{\frac{1}{q}}$ there exists $b > a$ such that

$$\int_a^\infty fg \leq \int_a^b fg + \varepsilon$$

$$\left[\int_a^b |f|^p \right]^{\frac{1}{p}} \leq \left[\int_a^\infty |f|^p \right]^{\frac{1}{p}} + \varepsilon \qquad \left[\int_a^b |g|^q \right]^{\frac{1}{q}} \leq \left[\int_a^\infty |g|^q \right]^{\frac{1}{q}} + \varepsilon$$

Then by (c)

$$\begin{aligned}\left|\int_a^\infty fg\right| &\leq \left|\int_a^b fg\right| + \varepsilon \leq \left[\int_a^b |f|^p\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g|^q\right]^{\frac{1}{q}} + \varepsilon \\ &\leq \left(\left[\int_a^\infty |f|^p\right]^{\frac{1}{p}} + \varepsilon\right) \cdot \left(\left[\int_a^\infty |g|^q\right]^{\frac{1}{q}} + \varepsilon\right) + \varepsilon \\ &\leq \left[\int_a^\infty |f|^p\right]^{\frac{1}{p}} \cdot \left[\int_a^\infty |g|^q\right]^{\frac{1}{q}} + M\varepsilon + \varepsilon^2\end{aligned}$$

where $M > 0$ is number depending on f and g but not ε. The conclusion follows because ε was arbitrary. □

9. Rudin 6.11

Let α be an increasing function on $[a, b]$. For $u \in \mathscr{R}(\alpha)$, define

$$\|u\|_2 = \left[\int_a^b |u|^2 \, \mathrm{d}\alpha\right]^{\frac{1}{2}}$$

Then for $f, g, h \in \mathscr{R}(\alpha)$

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

Proof. Let $p = 2 = q$ from 6.10. Then 6.10 (c) imlies that for any $u, v \in \mathscr{R}(\alpha)$

$$\int_a^b uv \, \mathrm{d}\alpha \leq \|u\|_2 \|v\|_2$$

This, and the fact that u and v are real-valued imply that

$$\begin{aligned}\|u + v\|_2^2 = \int_a^b |u + v|^2 \, \mathrm{d}\alpha &= \int_a^b (u^2 + 2uv + v^2) \, \mathrm{d}\alpha \\ &\leq \|u\|_2^2 + 2\|u\|_2\|v\|_2 + \|v\|_2^2 = (\|u\|_2 + \|v\|_2)^2\end{aligned}$$

which furthermore implies that $\|u + v\|_2 \leq \|u\|_2 + \|v\|_2$. Since $f - h = (f - g) + (g - h)$, the above implies that

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

□

10. Rudin 6.15

Suppose that f is a real, continuously differentiable function on $[a, b]$ such that $f(a) = 0 = f(b)$, and

$$\int_a^b f^2(x) \, \mathrm{d}x = 1$$

Then

$$\int_a^b x f(x) f'(x) \, \mathrm{d}x = -\frac{1}{2}$$

and

$$\int_a^b [f'(x)]^2 \, \mathrm{d}x \cdot \int_a^b x^2 f^2(x) \, \mathrm{d}x > \frac{1}{4}$$

Proof. First, we can apply integration by parts to $\int_a^b xf(x)f'(x)\,\mathrm{d}x$ to obtain

$$\begin{aligned}\int_a^b xf(x)f'(x)\,\mathrm{d}x &= xf^2(x)\Big|_a^b - \int_a^b f(x)(f(x)+xf'(x))\,\mathrm{d}x \\ &= -1 - \int_a^b xf(x)f'(x)\,\mathrm{d}x\end{aligned}$$

which implies that $\int_a^b xf(x)f'(x)\,\mathrm{d}x = -\frac{1}{2}$. Second, using 6.10 (c) with $p = 2 = q$ gives

$$\frac{1}{4} = \left|\int_a^b [f'(x)]\cdot[xf(x)]\,\mathrm{d}x\right|^2 \leq \int_a^b [f'(x)]^2\,\mathrm{d}x \cdot \int_a^b x^2f^2(x)\,\mathrm{d}x$$

This inequality implies in particular that $\int_a^b [f']^2 > 0$. Therefore, if the inequality were an equality, we would have

$$\begin{aligned}\int_a^b \left(xf(x) + \frac{f'(x)}{2\int_a^b [f']^2}\right)^2\,\mathrm{d}x &= \int_a^b x^2f^2(x)\,\mathrm{d}x - \frac{1}{4\int_a^b [f']^2} \\ &= \frac{1}{\int_a^b [f']^2}\left(\int_a^b [f'(x)]^2\,\mathrm{d}x \cdot \int_a^b x^2f^2(x)\,\mathrm{d}x - \frac{1}{4}\right) = 0\end{aligned}$$

which implies that $\left(xf(x) + \frac{f'(x)}{2\int_a^b [f']^2}\right)^2 = 0$ for all $x \in [a,b]$ by exercise 6.2. From this we see that f would solve the following linear ODE with continuous coefficients:

$$y'(x) = -2\lambda x y(x)$$

where $\lambda = \int_a^b [f']^2$. By the existence and uniqueness theorem for linear ODEs with continuous coefficients, it follows that

$$f(x) = Ce^{-\lambda x^2}$$

for some constant C. Then $f(a) = 0$ implies that $f = 0$, so that $\int_a^b f^2 = 0$, contradicting one of our assumptions. Therefore, the inequality must be strict. ☐

11. Rudin 6.17

Suppose that α is increasing on $[a,b]$, g is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Then

$$\int_a^b \alpha(x)g(x)\,\mathrm{d}x = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\,\mathrm{d}\alpha \tag{1}$$

Proof. Consider the following hypothesis.

Equation (1) is true when both α and g are nonnegative on $[a,b]$. (H)

Then set $\beta = \alpha + \gamma$, and set $f = g + h$, and $F(x) = G(x) + hx$, where γ and h are large enough that β and f are nonnegative on $[a,b]$. Then $F' = f$, and (H) implies that

$$\int_a^b \beta(x)f(x)\,\mathrm{d}x = F(b)\alpha(b) - F(a)\alpha(a) - \int_a^b F\,\mathrm{d}\beta$$

It is a straightforward computation to verify that this implies (1) if

$$\int_a^b \alpha(x)\ \mathrm{d}x = b\alpha(b) - a\alpha(a) - \int_a^b x\ \mathrm{d}\alpha \tag{2}$$

But the function $p(x) = 1$ and the weight β satisfy the hypotheses of (H), so (H) also implies that

$$\int_a^b \beta(x)\ \mathrm{d}x = b\beta(b) - a\beta(a) - \int_a^b x\ \mathrm{d}\beta$$

from which (2) follows immediately.

Thus, (H) implies our desired result, so we may assume without loss of generality that both α and g are nonnegative on $[a, b]$.

Note that $\alpha \in \mathscr{R}$ because it is increasing and bounded, so $\alpha g \in \mathscr{R}$. Thus, we can find a partition P such that

$$U(P, \alpha g) - \varepsilon \leq \int_a^b \alpha(x)g(x)\ \mathrm{d}x \leq L(P, \alpha g) + \varepsilon$$
$$U(P, g, \alpha) - \varepsilon \leq \int_a^b G\ \mathrm{d}\alpha \leq L(P, g, \alpha) + \varepsilon$$

and Δx_i is small enough that $|g(t) - g(s)| < \varepsilon$ for all $t, s \in [x_{i-1}, x_i]$ for all $i = 1$ to n (by the uniform continuity of g).

By the MVT there are $t_i \in [x_{i-1}, x_i]$ such that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. This implies that

$$\begin{aligned} A = \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i &= \sum_{i=1}^n \alpha(x_i)(G(x_i) - G(x_{i-1})) \\ &= \alpha(b)G(b) - \alpha(a)G(a) - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i \\ &= C - B \end{aligned}$$

where $C = \alpha(b)G(b) - \alpha(a)G(a)$, and $B = \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i$.

It is easy to see that $L(P, G, \alpha) \leq B \leq U(P, G, \alpha)$ becase $G(x_{i-1})$ is between the inf and sup of G on each interval $[x_{i-1}, x_i]$. On the other hand, since α and g are both nonnegative, and α is increasing, we have

$$\inf_{x\in[x_{i-1},x_i]} \alpha(x)g(x) \leq \alpha(t_i)g(t_i) \leq \alpha(x_i)g(t_i)$$

while, using the fact that $t_i \in [x_{i-1}, x_i]$ implies that $g(x_i) \geq g(t_i) - \varepsilon$,

$$\begin{aligned} \sup_{x\in[x_{i-1},x_i]} \alpha(x)g(x) &\geq \alpha(x_i)g(x_i) \\ &\geq \alpha(x_i)g(t_i) - \varepsilon\alpha(x_i) \\ &\geq \alpha(x_i)g(t_i) - \varepsilon\alpha(b) \end{aligned}$$

Thus,

$$L(P, \alpha g) \leq A \leq U(P, \alpha g) + \varepsilon\alpha(b)(b - a)$$

from which we conclude that

$$L(P, \alpha g) + L(P, g, \alpha)\ \leq\ C\ \leq\ U(P, \alpha g) + U(P, g, \alpha) + \varepsilon\alpha(b)(b - a)$$

because $A + B = C$. Then

$$\int_a^b \alpha(x)g(x)\ \mathrm{d}x + \int_a^b G\ \mathrm{d}\alpha - 2\varepsilon$$

$$\leq C \leq$$

$$\int_a^b \alpha(x)g(x)\ \mathrm{d}x + \int_a^b G\ \mathrm{d}\alpha + \varepsilon(2 + \alpha(b)(b-a))$$

from which the claim follows, as ε was arbitrary. □

12. Rudin 6.18

Let $\gamma_1, \gamma_2, \gamma_3$ be curves in the complex plane, defined on $[0, 2\pi]$ by

$$\gamma_1(t) = e^{it} \qquad \gamma_2(t) = e^{2it} \qquad \gamma_3(t) = e^{2\pi i t \sin\left(\frac{1}{t}\right)}$$

Then each curve has the same range, γ_1 and γ_2 are rectifiable with lengths 2π and 4π, and γ_3 is not rectifiable.

Proof. Viewing these curves as curves in $\mathbb{R}^2$, we have

$$\gamma_1(t) = \begin{bmatrix}\cos(t)\\ \sin(t)\end{bmatrix} \qquad \gamma_2(t) = \begin{bmatrix}\cos(2t)\\ \sin(2t)\end{bmatrix} \qquad \gamma_3(t) = \begin{bmatrix}\cos\left(2\pi t\sin\left(\frac{1}{t}\right)\right)\\ \sin\left(2\pi t\sin\left(\frac{1}{t}\right)\right)\end{bmatrix}$$

Clearly $|\gamma_1| = |\gamma_2| = |\gamma_3| = 1$, so the range of each curve is a subset of the unit circle. Each curve has the form $\gamma_j(t) = \begin{bmatrix}\cos(g_j(t))\\ \sin(g_j(t))\end{bmatrix}$ for $j = 1, 2, 3$. Given (x, y) in the unit circle and any $\ell \in \mathbb{R}$, there exists a $\theta \in [\ell, \ell + 2\pi)$ such that $\cos(\theta) = x$ and $\sin(\theta) = y$, so if the range of g_j contains $[\ell, \ell + 2\pi)$, then the range of γ_j must be equal to the unit circle.

The range of $g_1(t) = t$ is $[0, 2\pi] \supset [0, 2\pi)$, and the range of $g_2(t) = 2t$ is $[0, 4\pi] \supset [0, 2\pi)$, so the unit circle is the range of γ_1 and γ_2. Let $t_1 = \frac{2}{3\pi}$ and $t_2 = \frac{16}{\pi}$. Then $\pi > 3$ implies that $t_2 < 6 < 2\pi$, and clearly $0 < t_1 < t_2$, so $[t_1, t_2] \subset [0, 2\pi]$. Also, g_3 is continuous on $[t_1, t_2]$, so the range of g_3 contains $[g_3(t_1), g_3(t_2)]$ by the Intermediate Value Theorem. We have $g_3(t_1) = -\frac{4}{3} < -\frac{\pi}{3}$; $g_3(t_2)$ is a little trickier. Clearly, $25 \cdot 81 < 56 \cdot 137$, so

$$\begin{aligned}
& 25 \cdot 81 + 137 \cdot 81 && < 56 \cdot 137 + 81 \cdot 137 \\
\implies\ & 2 \cdot 81^2 && < 137^2 \\
\implies\ & 2 + \sqrt{2} && < \frac{289}{81} \\
\implies\ & \sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}} && < \frac{17}{18} \\
\implies\ & \frac{1}{2} - \frac{1}{2}\sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}} && > \frac{1}{36} \\
\implies\ & \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}}} && > \frac{1}{6}
\end{aligned}$$

where the last quantity on the left is $\sin\left(\frac{\pi}{16}\right)$ by two applications of the half-angle identity. Therefore, $g_3(t_2) > \frac{16}{3} > \frac{5\pi}{3}$ because $\pi < 3 + \frac{1}{5}$. Thus, the range of g_3 contains $\left[-\frac{\pi}{3}, \frac{5\pi}{3}\right]$, which implies that the range of γ_3 is the unit circle.

Since γ_1 and γ_2 are clearly C^1 curves, they are rectifiable, and we have

$$\Lambda(\gamma_1) = \int_0^{2\pi} \|\gamma_1'\| = \int_0^{2\pi} 1 = 2\pi$$

since $\|\gamma_1'\| = 1$, and

$$\Lambda(\gamma_2) = \int_0^{2\pi} \|\gamma_2'\| = \int_0^{2\pi} 2 = 4\pi$$

since $\|\gamma_2'\| = 2$.

Now let $N \geq 1$ be an integer, and consider the partition

$$P_N = \left\{0, \frac{2}{(2N+1)\pi}, \frac{2}{(2(N-1)+1)\pi}, \ldots, \frac{2}{\pi}, 2\pi\right\}$$

of $[0, 2\pi]$. Then

$$\Lambda(P_N, \gamma_3) \geq \sum_{n=1}^{N} \|\gamma_3(t_n) - \gamma_3(t_{n-1})\|$$

where $t_n = \frac{2}{(2n+1)\pi}$.

For any a, b, the following trigonometric identities hold

$$\begin{aligned} 4\sin^2(b) &= 4\cos^2(a)\sin^2(b) + 4\sin^2(a)\sin^2(b) \\ &= [\cos(\alpha) - \cos(\beta)]^2 + [\sin(\alpha) - \sin(\beta)]^2 \end{aligned}$$

where $2a = \alpha + \beta$ and $2b = \alpha - \beta$.

Taking $\alpha = 2\pi t_n \sin\left(\frac{1}{t_n}\right)$ and $\beta = 2\pi t_{n-1} \sin\left(\frac{1}{t_{n-1}}\right)$ and noting that $\sin\left(\frac{1}{t_n}\right) = (-1)^n$, we can compute

$$\begin{aligned} \|\gamma_3(t_n) - \gamma_3(t_{n-1})\| &= 2\left|\sin\left(\frac{2}{2n+1} + \frac{2}{2n-1}\right)\right| \\ &= 2\left|\sin\left(\frac{4n}{4n^2-1}\right)\right| \end{aligned}$$

By Taylor's Theorem, there exists K large enough so that $n > K$ implies that

$$\sin\left(\frac{4n}{4n^2-1}\right) \geq \frac{4n}{4n^2-1} - M\left(\frac{4n}{4n^2-1}\right)^3$$

for some constant $M > 0$ not depending on n or K. Therefore, if we take $N \geq K$,

$$\Lambda(P_N, \gamma_3) \geq \sum_{n=K}^{N} \frac{4n}{4n^2-1} - M\left(\frac{4n}{4n^2-1}\right)^3$$

Since the series $\sum_{n=K}^{\infty}\left(\frac{4n}{4n^2-1}\right)^3$ converges and the series $\sum_{n=K}^{\infty}\frac{4n}{4n^2-1}$ diverges to infinity, it follows that $\Lambda(P_N, \gamma_3) \to \infty$ as $N \to \infty$. Thus, γ_3 is not rectifiable. $\square$

13. Supplementary 1

Let $f(x) = 0$ for $x \in [0,1]$ and $f(x) = 1$ for $x \in (1,2]$, and let $\alpha = f$. Then $f \in \mathscr{R}$, but $f \notin \mathscr{R}(\alpha)$.

Proof. $f \in \mathscr{R}$

Let $\varepsilon > 0$ be given. Let $P = \{x_0, x_1 = 1, x_2, x_3\}$ be a partition of $[0,2]$ such that $\Delta x_2 < \varepsilon$. Then we have

$$\begin{aligned} m_1 &= 0 \quad M_1 = 0 \\ m_2 &= 0 \quad M_2 = 1 \\ m_3 &= 1 \quad M_3 = 1 \end{aligned}$$

Thus,

$$U(P,f) - L(P,f) = \Delta x_2 < \varepsilon$$

which implies that $f \in \mathscr{R}$.

$f \notin \mathscr{R}(\alpha)$

Let P be a partition of $[0,2]$, and let Q be a refinement of P containing 1, say, $x_{i-1} = 1$. Then $\Delta\alpha_i = 1$ and $\Delta\alpha_j = 0$ for all $j \neq i$. On the other hand, we have $M_i - m_i = 1$. Therefore,

$$U(P,f) - L(P,f) \geq U(Q,f) - L(Q,f) = 1$$

This implies that $f \notin \mathscr{R}(\alpha)$ since P was arbitrary. □

14. Supplementary 2

(a) Let $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational for $x \in [0,1]$. Then $f \notin \mathscr{R}$.

Proof. Let g be the function from Rudin 6.4 above. Then $g = 1 - f$, so $f \in \mathscr{R} \implies g \in \mathscr{R}$. But we showed in Rudin 6.4 that $g \notin \mathscr{R}$. Therefore $f \notin \mathscr{R}$. □

(b) Let α be an increasing function on $[0,1]$. Then $f \in \mathscr{R}(\alpha)$ if and only if α is a constant function.

Proof. Suppose that $f \in \mathscr{R}(\alpha)$. Then given $\varepsilon > 0$ there exists a partition P of $[0,1]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

The density of both rational and irrational numbers implies that both occur in each interval $[x_{i-1}, x_i]$, so $m_i = 0$ and $M_i = 1$. Then

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} \Delta\alpha_i = \alpha(1) - \alpha(0) < \varepsilon$$

Therefore, $\alpha(1) = \alpha(0)$ because $\alpha(1) \geq \alpha(0)$ and ε was arbitrary. Since α is increasing on $[0,1]$, it follows that $\alpha(x) = \alpha(0)$ for all $x \in [0,1]$, that is, α is a constant function.

On the other hand, every bounded function is integrable with respect to constant weight functions, so $f \in \mathscr{R}(\alpha)$ if α is a constant function. □

15. Supplementary 3

Let $f(x) = 1$ if x is rational and $f(x) = -1$ if x is irrational for $x \in [0,1]$. Then $f \notin \mathscr{R}$ because if it were, then $\frac{f+1}{2}$, the function from the previous problem, would be integrable.

On the other hand, it is obvious that $|f| = 1$, which is continuous and therefore integrable.

16. Supplementary 4

Let $I = I^+$ be defined by

$$I(x) = I^+(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

(a) If $a < s \leq b$, f is bounded on $[a,b]$, f is continuous at s, and $\alpha(x) = I^+(x-s)$, then

$$\int_a^b f \,\mathrm{d}\alpha = f(s)$$

Proof. Let $\varepsilon > 0$ be given. Since f is bounded and continuous at s, and α is continuous everywhere but s, we have $f \in \mathscr{R}(\alpha)$. Furthermore, there exists $\delta > 0$ such that $|x-s| < \delta$ implies that $|f(x)-f(s)| < \varepsilon$.

Suppose $s < b$. Let $P = \{x_0, x_1, x_2 = s, x_3\}$ be a partition of $[a,b]$ such that $\Delta x_2 < \delta$. Then $\Delta\alpha_2 = 1$, and $\Delta\alpha_i = 0$ for $i \neq 2$, and

$$L(P,f,\alpha) = m_2 \qquad U(P,f,\alpha) = M_2$$

Since $x \in [x_1, x_2]$ implies that $|x-s| < \delta$, it follows that $f(x) < f(s) + \varepsilon$ and $f(x) > f(s) - \varepsilon$ for all $x \in [x_1, x_2]$, which implies that $M_2 \leq f(s) + \varepsilon$ and $m_2 \geq f(s) - \varepsilon$. Then

$$f(s) - \varepsilon \leq L(P,f,\alpha) \leq \int_a^b f \,\mathrm{d}\alpha \leq U(P,f,\alpha) \leq f(s) + \varepsilon$$

Therefore, $\int_a^b f \,\mathrm{d}\alpha = f(s)$ because ε was arbitrary.

If $s = b$, then consider the partition $P = \{x_0, x_1, x_2\}$ such that $\Delta x_2 < \delta$. Then a virtually identical argument to the one above shows that $\int_a^b f \,\mathrm{d}\alpha = f(s)$. □

(b) Suppose $c_n \geq 0$ for $n = 1, 2, \ldots, N$ and $s_1, \ldots, s_N$ are distinct points in $(a,b]$, and

$$\alpha(x) = \sum_{n=1}^N c_n I^+(x - s_n)$$

Let f be continuous on $[a,b]$. Then

$$\int_a^b f \,\mathrm{d}\alpha = \sum_{n=1}^N c_n f(s_n)$$

Proof. Since each of $I^+(x-s_n)$ is increasing and continuous everywhere but s_n, and f is continuous on $[a,b]$, we have $f \in \mathscr{R}(\alpha_n)$, where $\alpha_n = I^+(x-s_n)$. Then, because $c_n \geq 0$, we also have $f \in \mathscr{R}(c_n\alpha_n)$ and $f \in \mathscr{R}(\alpha)$, and

$$\int_a^b f \,\mathrm{d}\alpha = \sum_{n=1}^N c_n \int_a^b f(x) \,\mathrm{d}I^+(x-s_n) = \sum_{n=1}^N f(s_n)$$

by part (a) of this problem. □

(c) Suppose that $c_n \geq 0$ for $n = 1, 2, \ldots$, $\sum_{n=1}^\infty c_n$ converges, and $s_1, s_2, \ldots$ is a sequence of distinct points in $(a, b]$, and

$$\alpha(x) = \sum_{n=1}^\infty c_n I^+(x - s_n)$$

Let f be continuous on $[a, b]$. Then

$$\int_a^b f \, \mathrm{d}\alpha = \sum_{n=1}^\infty c_n f(s_n)$$

Proof. The convergence of $\sum_{n=1}^\infty c_n$ ensures that α is defined for each $x \in [a, b]$ by the comparison test. It is clear that α is increasing. Since f is continuous, we have $f \in \mathscr{R}(\alpha)$.

Let $\varepsilon > 0$ be given. There is some K such that $N > K$ implies

$$\sum_{n=N+1}^\infty c_n < \varepsilon$$

Define α_1 and α_2 by

$$\alpha_1(x) = \sum_{n=1}^N c_n I^+(x - s_n) \qquad \alpha_2(x) = \sum_{n=N+1}^\infty c_n I^+(x - s_n)$$

Then $\alpha = \alpha_1 + \alpha_2$. Clearly, α_1 and α_2 are also defined for all x and are both increasing, so $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$. Moreover, $\alpha_2(b) - \alpha_2(a) = \sum_{n=N+1}^\infty c_n < \varepsilon$. Therefore,

$$\begin{aligned} \left| \int_a^b f \, \mathrm{d}\alpha - \int_a^b f \, \mathrm{d}\alpha_1 \right| &= \left| \int_a^b f \, \mathrm{d}\alpha_2 \right| \\ &\leq M(\alpha_2(b) - \alpha_2(a)) \leq M\varepsilon \end{aligned}$$

where $M = \sup_{x \in [a,b]} |f(x)|$. By part (b), we have $\int_a^b f \, \mathrm{d}\alpha_1 = \sum_{n=1}^N c_n f(s_n)$, so

$$\left| \int_a^b f \, \mathrm{d}\alpha - \sum_{n=1}^N c_n f(s_n) \right| \leq M\varepsilon$$

for all $N > K$. Therefore, since $\varepsilon > 0$ was arbitrary and the series $\sum_{n=1}^\infty c_n f(s_n)$ is absolutely convergent by comparison with the convergent series $\sum_{n=1}^\infty M c_n$, we can conclude that

$$\int_a^b f \, \mathrm{d}\alpha = \sum_{n=1}^\infty c_n f(s_n)$$

□

17. Supplementary 5

For $x \in \mathbb{R}$ let $[x]$ denote the integer part of x, that is

$$[x] = \sup \{k \in \mathbb{Z} \mid k \leq x\}$$

Then for $n \in \mathbb{N}$, and $\alpha(x) = [x]$, use part (c) of the problem 4 to obtain

$$\int_0^n x \, \mathrm{d}\alpha = \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

because $\alpha(x) = [x] = \sum_{i=1}^n I^+(x-i)$ on $[0,n]$.

18. Supplementary 6

Let $f(x) = x^2$, and define α as follows:

$$\alpha(x) = \begin{cases} 0 & x < 2 \\ \frac{2}{3}x - 1 & 2 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

Since f is continuous, we have $f \in \mathscr{R}(\alpha)$ on $[0,3]$. This means that $f \in \mathscr{R}(\alpha)$ on $[0,2]$ and $[2,3]$ as well, and

$$\int_0^3 f \, \mathrm{d}\alpha = \int_0^2 f \, \mathrm{d}\alpha + \int_2^3 f \, \mathrm{d}\alpha$$

On $[0,2]$, we have $\alpha(x) = \frac{1}{3}I^+(x-2)$, so by 4. (b)

$$\int_0^2 f \, \mathrm{d}\alpha = \frac{f(2)}{3} = \frac{4}{3}$$

On $[2,3]$, we have $\alpha(x) = \frac{2}{3}x - 1$ (since $\frac{2}{3}3 - 1 = 1$, which agrees with the definition of $\alpha(3)$ above). Then $\alpha' = \frac{2}{3}$, so $\alpha' \in \mathscr{R}$ and $f\alpha' \in \mathscr{R}$ on $[2,3]$, which implies that

$$\int_2^3 f \, \mathrm{d}\alpha = \int_2^3 f\alpha' = \int_2^3 \frac{2}{3}x^2 \, \mathrm{d}x = \frac{2}{9}x^3 \Big|_2^3 = \frac{38}{9}$$

by the FTC. Thus,

$$\int_0^3 f \, \mathrm{d}\alpha = \frac{38}{9} + \frac{4}{3} = \frac{46}{9}$$

19. Supplementary 7

(a) (a)

If $\alpha(x) = x^2$, then α is increasing on $[0,1]$, and $\alpha'(x) = 2x$, so $\alpha' \in \mathscr{R}$, and $f\alpha' \in \mathscr{R}$ if $f(x) = x$. Therefore,

$$\int_0^1 f \, \mathrm{d}\alpha = \int_0^1 x \, \mathrm{d}\left(x^2\right) = \int_0^1 2x^2 \, \mathrm{d}x = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$$

by the FTC.

(b) (b)

If $\alpha(x) = \sin(x)$, then α is increasing on $\left[0, \frac{\pi}{2}\right]$, and $\alpha'(x) = \cos(x)$, so $\alpha' \in \mathscr{R}$, and $f\alpha' \in \mathscr{R}$ if $f = \sin$. Therefore,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} f \,\mathrm{d}\alpha &= \int_0^{\frac{\pi}{2}} \sin(x) \,\mathrm{d}\left(\sin(x)\right) = \int_0^{\frac{\pi}{2}} \sin(x)\cos(x) \,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin(2x) \,\mathrm{d}x = -\left.\frac{\cos(2x)}{4}\right|_0^{\frac{\pi}{2}} = \frac{1}{2}\end{aligned}$$

by the FTC.

20. Supplementary 8

I'm not sure if you meant to type o instead of 0 in the lower limit, but I am going to assume that you did, and that $o < 5$. Let $\alpha_1(x) = x$ and $\alpha_2(x) = [x]$. Then $\alpha_1(x) + \alpha_2(x) = x + [x]$, and α_1 and α_2 are both increasing on $[o, 5]$. $f(x) = x^2$ is continuous on $[o, 5]$ and, hence, $f \in \mathscr{R}(\alpha_1)$, and $f \in \mathscr{R}(\alpha_2)$. Therefore,

$$\int_o^5 x^2 \,\mathrm{d}(\alpha_1(x) + \alpha_2(x)) = \int_o^5 x^2 \,\mathrm{d}x + \int_o^5 x^2 \,\mathrm{d}\alpha_2$$

Note that $\alpha_2(x) = [o] + \sum_{i=[o]+1}^5 I^+(x - i)$ on $[o, 5]$. The constant $[o]$ can be ignored, so by Supplementary 4

$$\int_o^5 x^2 \,\mathrm{d}\alpha_2 = \sum_{i=[o]+1}^{5} i^2 = 55 - \frac{[o]([o]+1)(2[o]+1)}{6}$$

On the other hand,

$$\int_o^5 x^2 \,\mathrm{d}x = \left.\frac{x^3}{3}\right|_o^5 = \frac{125}{3} - \frac{o^3}{3}$$

by the FTC. Therefore,

$$\int_o^5 x^2 \,\mathrm{d}(x + [x]) = \frac{290}{3} - \frac{o^3}{3} - \frac{[o]([o]+1)(2[o]+1)}{6}$$

In particular, if $o = 0$, then the integral is $\frac{290}{3}$.

21. Supplementary 9

Let α be strictly increasing on $[a, b]$, and let $f \in \mathscr{R}(\alpha)$ on $[a, b]$. Suppose that $f(x) \geq 0$ for all $x \in [a, b]$, and f is continuous at a point $c \in [a, b]$, and $f(c) > 0$. Then

$$\int_a^b f \,\mathrm{d}\alpha > 0$$

Proof. Let $y = f(c) > 0$. Since f is continuous at c, choose $\delta > 0$ such that $|f(x) - f(c)| < \frac{y}{2}$. Then $|x - c| < \delta$ implies that $f(x) > \frac{y}{2}$.

Let $P = \{x_0, x_1, x_2, x_3\}$ be a partition of $[a, b]$ such that $c \in [x_1, x_2]$, and $\Delta x_2 < \delta$. Then:

- $f(x) \geq 0$ implies that $m_i \geq 0$ for all i, so $m_i \Delta\alpha_i \geq 0$ for all i. - $x \in [x_1, x_2]$ implies that $|x - c| < \delta$, so $m_2 \geq \frac{y}{2}$.

Thus,

$$L(P, f, \alpha) = \sum_{n=1}^{3} m_i \Delta\alpha_i \geq \frac{y}{2} \Delta\alpha_2 > 0$$

since $\Delta\alpha_2 > 0$ as a consequence of α being *strictly* increasing. Therefore,

$$\int_a^b f \, \mathrm{d}\alpha \geq L(P, f, \alpha) > 0$$

$\square$