5.1 (i) $(1-t^2)x'' - 2tx' + n(n+1)x = 0 \qquad t \in (-1,1)$

$1-t^2 \neq 0$ if $t \in (-1,1)$, so

$$x'' - \frac{2t}{1-t^2}x' + \frac{n(n+1)}{1-t^2}x = 0$$

$$x'' - \frac{2t}{1-t^2}x' = e^{\int \frac{2t}{1-t^2}dt}\left(e^{\int \frac{-2t}{1-t^2}dt}x'\right)'$$

$$= e^{-\ln(1-t^2)}\left(e^{\ln(1-t^2)}x'\right)'$$

$$= \frac{1}{1-t^2}\left((1-t^2)x'\right)'$$

so equation becomes

$$\left((1-t^2)x'\right)' + n(n+1)x = 0$$

(ii) $(1-t^2)x'' - tx' + n^2x = 0 \qquad t \in (-1,1)$

$1-t^2 \neq 0$ if $t \in (-1,1)$, so

$$x'' - \frac{t}{1-t^2}x' + \frac{n^2x}{1-t^2} = 0$$

$$x'' - \frac{t}{1-t^2} = e^{\int \frac{t}{1-t^2}dt}\left(e^{\int -\frac{t}{1-t^2}dt}x'\right)'$$

$$= e^{-\frac{1}{2}\ln(1-t^2)}\left(e^{\frac{1}{2}\ln(1-t^2)}x'\right)'$$

$$= \frac{1}{\sqrt{1-t^2}}\left(\sqrt{1-t^2}\,x'\right)'$$

so equation becomes

$$\left(\sqrt{1-t^2}\,x'\right)' + \frac{n^2}{\sqrt{1-t^2}}x = 0$$

(iii) $tx'' + (1-t)x' + ax = 0 \quad t \in (0,\infty)$

$t \in (0,\infty) \Rightarrow t \neq 0$, so

$$x'' + \left(\frac{1}{t} - 1\right)x' + \frac{a}{t}x = 0$$

$$x'' + \left(\frac{1}{t}-1\right)x' = e^{\int (1 - \frac{1}{t})dt}\left(e^{\int (\frac{1}{t}-1)dt} x'\right)'$$

$$= e^{t - \ln t}\left(e^{\ln t - t}x'\right)'$$

$$= \frac{e^t}{t}\left(\frac{t}{e^t}x'\right)'$$

so

$$\left(\frac{t}{e^t}x'\right)' + \frac{a}{e^t}x = 0$$

(iv) $x'' - 2tx' + 2nx = 0 \quad t \in \mathbb{R}$

$$x'' - 2tx' = e^{\int 2t\,dt}\left(e^{\int -2t\,dt}x'\right)'$$

$$= e^{t^2}\left(e^{-t^2}x'\right)'$$

So $\left(e^{-t^2}x'\right)' + 2ne^{-t^2}x = 0$

(v) $x'' + \frac{2}{3+t}x' + \frac{\lambda}{(3+t)^2}x = 0 \quad t \in (-3,\infty)$

$$x'' + \frac{2}{3+t}x' = e^{-\int \frac{2}{3+t}dt}\left(e^{\int \frac{2}{3+t}dt}x'\right)'$$

$$= e^{-2\ln(t+3)}\left(e^{2\ln(3+t)}x'\right)'$$

$$= \frac{1}{(t+3)^2}\left((t+3)^2x'\right)'$$

So $\left((t+3)^2x'\right)' + \lambda x = 0$

5.19 (i) $(e^{-3t}x')' + 2e^{-3t}x = 0 \quad I = (-\infty, \infty)$

The Cauchy function is a function $x(t,s)$ s.t. $x(\cdot, s)$ solves the equation with $x(s) = 0$, $x'(s) = \frac{1}{p(s)} = e^{3s}$

$$0 = (e^{-3t}x')' + 2e^{-3t}x = e^{-3t}x'' - 3e^{3t}x' + 2e^{-3t}x$$

$$\Rightarrow x'' - 3x' + 2x = 0, \quad t \in I$$

Then the solution with $x(s) = 0$, $x'(s) = e^{3s}$ is

$$x(t,s) = c_1(s)e^{2t} + c_2(s)e^{t}$$ because the characteristic eqn. is $r^2 - 3r + 2 = (r-2)(r-1) = 0$ i.e. $r = 2, r = 1$

$$x(s,s) = 0 = c_1(s)e^{2s} + c_2(s)e^{s} \Rightarrow c_2(s) = -c_1(s)e^{s}$$

$$x'(s,s) = 2c_1(s)e^{2s} + c_2(s)e^{s} = e^{3s}$$

or $\quad c_1(s) = e^{s} \Rightarrow c_2(s) = -e^{2s}$

$$x(t,s) = e^{s}e^{2t} - e^{2s}e^{t}$$ is the Cauchy function.

(iii) $\left(\frac{x'}{t^4}\right)' + \frac{6}{t^6}x = 0 \quad I = (0, \infty)$

$p(t) = t^{-4}$

$$0 = \left(\frac{x'}{t^4}\right)' + \frac{6}{t^6}x = t^{-4}x'' - 4t^{-5}x' + 6t^{-6}x = 0$$

$$t \neq 0 \Rightarrow t^2x'' - 4tx' + 6x = 0$$

Characteristic equation is $r(r-1) - 4r + 6 = 0$

$r^2 - 5r + 6 = 0$ or $(r-3)(r-2) = 0$

$r = 3, r = 2$

So the solution of the equation with $x(s) = 0$, $x'(s) = \frac{1}{p(s)} = s^4$

is $x(t,s) = c_1(s)t^3 + c_2(s)t^2$, and

$$x(s,s) = 0 = c_1(s)s^3 + c_2(s)s^2 \Rightarrow c_1(s) = -c_2(s)s^{-1}$$

$$x'(s,s) = 3c_1(s)s^2 + 2c_2(s)s = -c_2(s)s = s^4 \Rightarrow c_2(s) = -s^3, \quad c_1(s) = s^2$$

$$x(t,s) = s^2t^3 - s^3t^2$$ is the Cauchy function

(v) $x'' + x = 0 \quad I = (-\infty, \infty)$

In self-adjoint form: $(1 \cdot x')' + x = 0$, so $p(t) = 1$

Solution $x(t,s)$ of $x(s) = 0$, $x'(s) = \frac{1}{p(s)} = 1$ is

$$x(t,s) = c_1(s)\cos(t) + c_2(s)\sin(t)$$

because characteristic equation is $r^2 + 1 = 0 \Rightarrow r = \pm i$

$$x(s,s) = 0 = c_1(s)\cos(s) + c_2(s)\sin(s) \Rightarrow c_1(s) = -c_2(s)\tan(s)$$

$$x'(s,s) = -c_1(s)\sin(s) + c_2(s)\cos(s) = 1$$

$$\Rightarrow \frac{\sin^2(s)}{\cos(s)} c_2(s) + c_2(s)\cos(s) = 1$$

$$\Rightarrow c_2(s) = \cos(s), \quad c_1(s) = -\sin(s)$$

$x(t,s) = -\sin(s)\cos(t) + \cos(s)\sin(t)$ is the Cauchy function

5.11 (i) $(e^{-3t}x')' + 2e^{-3t}x = e^{-t} \quad x(0) = 0, \; x'(0) = 0$

From 5.10(i) $x(t,s) = e^{s}e^{2t} - e^{2s}e^{t}$ is the Cauchy function,

So by VOCF

$$x(t) = \int_0^t x(t,s)\, e^{-s}\, ds = \int_0^t (e^{s}e^{2t} - e^{2s}e^{t})\, e^{-s}\, ds$$

$$= te^{2t} - e^{t}(e^{t} - 1) = (t-1)e^{2t} + e^{t}$$

(iii) $\left(\frac{x'}{t^4}\right)' + \frac{6}{t^6}x = e^{-5t} \quad x(1) = 0, \; x'(1) = 0$

From 5.10(iii) $x(t,s) = s^2t^3 - s^3t^2$ is the Cauchy function

So by VOCF

$$x(t) = \int_1^t (s^2t^3 - s^3t^2)\, e^{-5s}\, ds$$

$$= \left[\frac{t^3 e^{-5s}}{-5}\left[s^2 - \frac{2s}{-5} + \frac{2}{25}\right] - \frac{t^2 e^{-5s}}{-5}\left[s^3 - \frac{3s^2}{-5} + \frac{6s}{25} - \frac{6}{-125}\right]\right]_1^t$$

$$= -\frac{1}{5}\Big(t^3 e^{-5t}\left(t^2 + \frac{2t}{5} + \frac{2}{25}\right) - t^3 e^{-5}\left(1 + \frac{2}{5} + \frac{2}{25}\right) - t^2 e^{-5t}\left[t^3 + \frac{3}{5}t^2 + \frac{6}{25}t + \frac{6}{125}\right]$$

$$+ t^2 e^{-5}\left[1 + \frac{3}{5} + \frac{6}{25} + \frac{6}{125}\right]\Big)$$

$$= -\frac{1}{5}\left(e^{-5t}\left(t^5 + \frac{2}{5}t^4 + \frac{2}{25}t^3\right) - t^3 e^{-5}\frac{37}{25} - e^{-5t}\left(t^5 + \frac{3}{5}t^4 + \frac{6}{25}t^3 + \frac{6}{125}t^2\right) + t^2 e^{-5}\frac{236}{125}\right)$$

$$= -\frac{1}{5}\left(e^{-5t}\left(-\frac{1}{5}t^4 - \frac{4}{25}t^3 - \frac{6}{125}t^2\right) + t^2 e^{-5}\frac{236}{125} - t^3 e^{-5}\frac{37}{25}\right)$$

$$= e^{-5t}\left(\frac{t^4}{25} + \frac{4}{125}t^3 + \frac{6}{625}t^2\right) - t^2 e^{-5}\cdot\frac{236}{625} + t^3 e^{-5}\cdot\frac{37}{125}$$

$$= \frac{e^{-5}}{625}\cdot\left[e^{-5(t-1)}\left(25t^4 + 20t^3 + 6t^2\right) - 236t^2 + 185t^3\right]$$

(v) $x'' + x = 4 \quad x(0) = 0, \; x'(0) = 0$

From 5.10(v), the Cauchy function is

$x(t,s) = -\sin(s)\cos(t) + \cos(s)\sin(t)$, so by VoCF

$$x(t) = \int_0^t x(t,s)\cdot 4\,ds = 4\int_0^t [\sin(t)\cos(s) - \cos(t)\sin(s)]\,ds$$

$$= 4\sin(t)\sin(s)\Big|_0^t + 4\cos(t)\cos(s)\Big|_0^t$$

$$= 4\sin^2(t) + 4\cos^2(t) - 4\cos(t)$$

$$= 4(1-\cos(t))$$