Math 5604 Homework 2

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Problem 1.

Consider the IVP

$$y' = f(t, y), \qquad y(0) = a. \tag{1}$$

Let $k > 0$ be the time step for a numerical scheme to approximate y'. Assume that f is L-Lipschitz in y for all t.

1. Consider the scheme

$$y^{n+1} = y^n + kf\big(t_{n+1}, y^{n+1}\big). \tag{2}$$

Suppose that $y(t_n) = y^n$. Using the Taylor expansion of y about t_n,

$$y(t_{n+1}) = y(t_n) + ky'(t_n) + R_1(k),$$

where the remainder $R_1(k) = \mathcal{O}(k^2)$ as $k \to 0$. Further expanding y' about t_{n+1} and using the ODE gives

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + k\left[y'(t_{n+1}) + R_2(k)\right] + R_1(k) \\ &= y(t_n) + ky'(t_{n+1}) + kR_2(k) + R_1(k) \\ &= y(t_n) + kf(t_{n+1}, y(t_{n+1})) + kR_2(k) + R_1(k), \end{aligned}$$

where the remainder $R_2(k) = \mathcal{O}(k)$ as $k \to 0$. Using the assumption that $y(t_n) = y^n$ and the definition of the scheme, we have

$$\begin{aligned} y(t_{n+1}) &= y^n + kf\big(t_{n+1}, y^{n+1}\big) + k\left[f(t_{n+1}, y(t_{n+1})) - f\big(t_{n+1}, y^{n+1}\big)\right] + kR_2(k) + R_1(k) \\ &= y^{n+1} + k\left[f(t_{n+1}, y(t_{n+1})) - f\big(t_{n+1}, y^{n+1}\big)\right] + kR_2(k) + R_1(k). \end{aligned}$$

Thus,

$$\text{LTE} = \left|y(t_{n+1}) - y^{n+1}\right| = k\left|f(t_{n+1}, y(t_{n+1})) - f\big(t_{n+1}, y^{n+1}\big) + kR_2(k) + R_1(k)\right|.$$

We can easily show that LTE $\to 0$ as $k \to 0$, that is, that the scheme is consistent.

By the Lipschitz condition on f,

$$\begin{aligned} \text{LTE} = \left|y(t_{n+1}) - y^{n+1}\right| &\le k\left|f(t_{n+1}, y(t_{n+1})) - f\big(t_{n+1}, y^{n+1}\big)\right| + |kR_2(k) + R_1(k)| \\ &\le kL\left|y(t_{n+1}) - y^{n+1}\right| + |kR_2(k) + R_1(k)|. \end{aligned}$$

For all $k < \frac{1}{L}$, we have $1 - kL > 0$, so

$$\text{LTE} \le \frac{|kR_2(k) + R_1(k)|}{1 - kL}, \qquad k < \frac{1}{L}.$$

This implies that

$$0 \le \lim_{k\to 0} \text{LTE} \le \lim_{k\to 0} \frac{|kR_2(k) + R_1(k)|}{1 - kL} = 0$$

because $kR_2(k) + R_1(k) \to 0$ as $k \to 0$, and $1 - kL \to 1$ as $k \to 0$. That is, LTE $\to 0$ as $k \to 0$, and the scheme is consistent.

2. Consider the scheme

$$y^{n+1} = y^{n-1} + 2kf(t_n, y_n). \tag{3}$$

Suppose that $y(t_{n-1}) = y^{n-1}$, and $y(t_n) = y^n$. Using the Taylor expansion of y about t_n to the left and to the right, we have

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + ky'(t_n) + R_1(k) \\ y(t_{n-1}) &= y(t_n) - ky'(t_n) + R_2(k), \end{aligned}$$

where the remainders $R_1(k)$ and $R_2(k)$ satisfy $R_1(k) = \mathcal{O}(k^2)$ and $R_2(k) = \mathcal{O}(k^2)$ as $k \to 0$.

By the ODE and the assumptions that $y(t_{n-1}) = y^{n-1}$ and $y(t_n) = y^n$, this implies that

$$\begin{aligned} y(t_{n+1}) - y^{n-1} &= y(t_{n+1}) - y(t_{n-1}) \\ &= 2ky'(t_n) + R_1(k) - R_2(k) \\ &= 2kf(t_n, y(t_n)) + R_1(k) - R_2(k) \\ &= 2kf(t_n, y^n) + R_1(k) - R_2(k). \end{aligned}$$

Therefore, the LTE is given by

$$\text{LTE} = \left|y^{n+1} - y(t_{n+1})\right| = |R_1(k) - R_2(k)|.$$

Since both $R_1(k) \to 0$ and $R_2(k) \to 0$ as $k \to 0$, it follows that LTE $\to 0$ as $k \to 0$. That is, the scheme is consistent.