2.63 (i)

$$x' = \begin{bmatrix} 3 & 0 \\ 0 & \sin^2 t \end{bmatrix} x.$$

To find the Floquet multipliers we need a fundamental matrix. The system can be solved easily, though, by separation of variables for x_1 & x_2:

$$x_1' = 3x_1 \Rightarrow x_1 = c_1 e^{3t}$$

$$x_2' = \sin^2 t\, x_2 \Rightarrow x_2 = c_2 e^{\int \sin^2 t\,dt} = c_2 e^{\int \frac{1-\cos 2t}{2}} = c_2 e^{\frac{1}{2}t - \frac{1}{4}\sin 2t}$$

Therefore, a general solution of the eqn. is (because eqn. is homogeneous linear & $\binom{1}{0}x_1$ & $\binom{0}{1}x_2$ are lin. indep.)

$$x = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\frac{1}{2}t - \frac{1}{4}\sin 2t}$$ and

$$\Phi(t) = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{\frac{1}{2}t - \frac{1}{4}\sin 2t} \end{bmatrix}$$ is a fund. matrix of $x' = A(t)x$.

Then the Floquet multipliers are the e.v.'s of

$$\Phi^{-1}(0)\,\Phi(\omega) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{3\pi} & 0 \\ 0 & e^{\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} e^{3\pi} & 0 \\ 0 & e^{\pi/2} \end{bmatrix}.$$

Clearly, $\mu_1 = e^{3\pi}$ & $\mu_2 = e^{\pi/2}$ are the Floquet multipliers.

2.63 (iii) $x' = \begin{pmatrix} -3+2\sin t & 0 \\ 0 & -1 \end{pmatrix} x$

To find Floquet multipliers we need a fundamental matrix. We can get one easily by solving for x_1 & x_2 using separation of variables:

$$x_1' = (-3+2\sin t)x_1 \Rightarrow x_1 = c_1 e^{-3t-2\cos t}$$

$$x_2' = -x_2 \Rightarrow x_2 = c_2 e^{-t}$$

Thus, a general solution of the equation is

$$x = c_1\begin{pmatrix}1\\0\end{pmatrix}e^{-3t-2\cos t} + c_2\begin{pmatrix}0\\1\end{pmatrix}e^{-t}$$

because the eqn. is homogeneous linear and $\begin{pmatrix}1\\0\end{pmatrix}e^{-3t-2\cos t}$ & $\begin{pmatrix}0\\1\end{pmatrix}e^{-t}$ are lin. indep. solutions. Thus,

$$\Phi(t) = \begin{bmatrix} e^{-3t-2\cos t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$ is a fundamental matrix

for $x' = A(t)x$, and the period ω of $A(t)$ is 2π, so the Floquet multipliers are the e.v.'s of

$$\Phi^{-1}(0)\,\Phi(\omega) = \begin{bmatrix} e^{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^{-6\pi-2} & 0 \\ 0 & e^{-2\pi} \end{bmatrix} = \begin{bmatrix} e^{-6\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix},$$

whose eigenvalues are clearly $\mu_1 = e^{-6\pi}$ & $\mu_2 = e^{-2\pi}$ the Floquet multipliers.

2.64. (i) $x' = (2\sin 3t)x$

$\Rightarrow x = Ce^{-\frac{2}{3}\cos 3t}$

So $[e^{-\frac{2}{3}\cos 3t}] = \Phi(t)$ is a fundamental matrix.

Then, since $\omega = \frac{2\pi}{3}$ is the period of $A(t) = [2\sin 3t]$, the Floquet multiplier is the eigenvalue of

$$\Phi^{-1}(0)\,\Phi(\omega) = [e^{\frac{2}{3}\cos(3\cdot 0)}][e^{-\frac{2}{3}\cos(\frac{2\pi}{3}\cdot 3)}]$$
$$= [e^{\frac{2}{3}}e^{-\frac{2}{3}}] = [1], \text{ so}$$

clearly the multiplier is $\mu = 1$. This implies that the trivial solution is stable on $[0,\infty)$ ($|\mu| = 1$ and μ is simple) by the Floquet Stability Theorem.

(ii) $x' = (\cos^2 t)x \Rightarrow x = Ce^{\frac{1}{2}t + \frac{1}{4}\sin 2t}$, so

$\Phi(t) = [e^{\frac{1}{2}t + \frac{1}{4}\sin 2t}]$ is a fundamental matrix, $\omega = \pi$ is the period of $A(t) = [\cos^2 t]$, so the Floquet multiplier μ is the eigenvalue of

$$\Phi^{-1}(0)\,\Phi(\omega) = [e^0][e^{\frac{\pi}{2}}] = [e^{\frac{\pi}{2}}]$$

and thus $\mu = e^{\frac{\pi}{2}}$, and the trivial solution is unstable on $[0,\infty)$ ($|\mu| > 1$ for the Floquet multiplier $\mu = e^{\frac{\pi}{2}}$)

(iii) $x' = (-1 + \sin 4t)x \Rightarrow x = Ce^{-t - \frac{1}{4}\cos 4t}$, and therefore

$\Phi(t) = [e^{-t - \frac{1}{4}\cos 4t}]$ is a fundamental matrix. The Floquet multiplier is the e.v. of

$$\Phi^{-1}(0)\,\Phi(\omega) = [e^{\frac{1}{4}}][e^{-\frac{1}{4} - \frac{\pi}{2}}] = [e^{-\frac{\pi}{2}}],$$

where $\omega = \frac{\pi}{2}$ is the period of $A(t) = -1 + \sin 4t$. Thus $\mu = e^{-\pi/2}$ is the Floquet multiplier, and by the Floquet Stability Theorem, the trivial solution is g.a.s. on $[0,\infty)$ ($|\mu| = e^{-\pi/2} < 1$).

2.65 (ii) $x' = \begin{bmatrix} \cos(2\pi t)+1 & 0 \\ \cos(2\pi t) & 1 \end{bmatrix} x$

Then $x_1' = (\cos(2\pi t)+1)x_1 \Rightarrow x_1 = c_1 e^{t+\frac{1}{2\pi}\sin(2\pi t)}$,

and $x_2' = c_1 \cos 2\pi t \, e^{t+\frac{1}{2\pi}\sin 2\pi t} + x_2$

$\Rightarrow x_2' - x_2 = c_1 \cos 2\pi t \, e^{t+\frac{1}{2\pi}\sin 2\pi t}$

$\Rightarrow x_2 e^{-t} = c_1 e^{\frac{1}{2\pi}\sin 2\pi t} + c_2$

$\Rightarrow x_2 = c_1 e^{t+\frac{1}{2\pi}\sin 2\pi t} + c_2 e^{t}$

$\Rightarrow x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t+\frac{1}{2\pi}\sin 2\pi t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$

and therefore a fundamental matrix is

$\Phi(t) = \begin{bmatrix} e^{t+\frac{1}{2\pi}\sin 2\pi t} & 0 \\ e^{t+\frac{1}{2\pi}\sin 2\pi t} & e^{t} \end{bmatrix}$, because $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t+\frac{1}{2\pi}\sin 2\pi t}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$ are lin. indep. solutions.

Then the Floquet multipliers are the eigenvalues of

$\Phi^{-1}(0)\Phi(\omega) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^1 & 0 \\ e^1 & e^1 \end{bmatrix}$, $\omega = 1$, the period of $A(t) = \begin{pmatrix} \cos(2\pi t)+1 & 0 \\ \cos(2\pi t) & 1 \end{pmatrix}$

$= \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$

So the Floquet multipliers are $\mu_1 = \mu_2 = e$. Thus, the trivial solution of $x' = A(t)x$ is unstable on $[0,\infty)$ because $|\mu_1| = e > 1$ (by the Floquet Stability Theorem).

2.65 (iii) $x' = \begin{bmatrix} -2 & 0 \\ \sin 2t & -2 \end{bmatrix} x$

Solving using separation of variables gives

$x_1' = -2x_1 \Rightarrow x_1 = c_1 e^{-2t}$, $x_2' = \sin(2t) c_1 e^{-2t} - 2x_2$

$\Rightarrow x_2' + 2x_2 = c_1 \sin(2t) e^{-2t}$

$\Rightarrow x_2 e^{2t} = -\frac{1}{2} c_1 \cos 2t + c_2$

$\Rightarrow x_2 = -\frac{1}{2} c_1 \cos 2t \, e^{-2t} + c_2 e^{-2t}$

$\therefore x = c_1 \begin{pmatrix} 1 \\ -\frac{1}{2}\cos 2t \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$ is a general soln.

because $\begin{pmatrix} 1 \\ -\frac{1}{2}\cos 2t \end{pmatrix} e^{-2t}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$ are lin. indep. soln's.

Thus

$\Phi(t) = \begin{bmatrix} e^{-2t} & 0 \\ -\frac{1}{2} e^{-2t} \cos 2t & e^{-2t} \end{bmatrix}$ is a fundamental matrix.

Since $\omega = \pi =$ period of $A(t) = \begin{bmatrix} -2 & 0 \\ \sin 2t & -2 \end{bmatrix}$, the Floquet multipliers are the eigenvalues of

$$\Phi^{-1}(0)\Phi(\omega) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} e^{-2\pi} & 0 \\ -\frac{1}{2} e^{-2\pi} & e^{-2\pi} \end{bmatrix} = \begin{bmatrix} e^{-2\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix},$$

So the eigenvalues / Floquet multipliers are clearly

$\mu_1 = \mu_2 = e^{-2\pi}$.

Therefore the trivial solution of $x' = A(t)x$ is g.a.s. on $[0,\infty)$ by the Floquet Stability Theorem, because $|\mu_1| = |\mu_2| = e^{-2\pi} < 1$ for both multipliers μ_1 & μ_2.

3.) $x'' = x^3 - x$

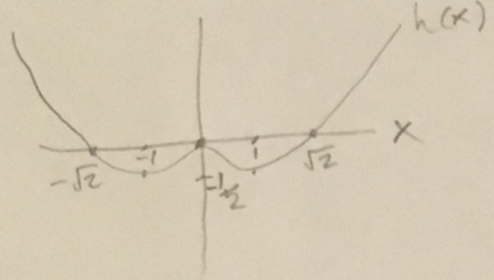
Let $y = x'$, and get the system

$$x' = y \qquad \Rightarrow \quad \frac{y'}{x'} = \frac{dy}{dx} = \frac{x^3 - x}{y}$$
$$y' = x^3 - x$$

$$\Rightarrow \frac{1}{2}y^2 = \frac{1}{4}x^4 - \frac{1}{2}x^2 + C$$

$$\Rightarrow y = \pm\sqrt{\frac{1}{2}x^4 - x^2 + d}$$

Consider $h(x) = \frac{1}{2}x^4 + x^2 = x^2\left(\frac{1}{2}x^2 - 1\right)$; $h(x) = 0$ if $x = 0$ or $x = \pm\sqrt{2}$



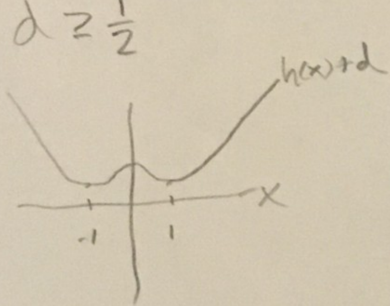
$h'(x) = 2x^3 - 2x = 0$ if $x = 0$, or $x = \pm 1$

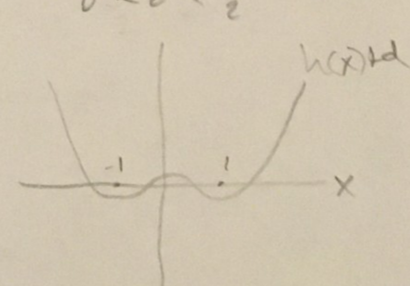
So h has local extrema when $x = 0$, or $x = \pm 1$.

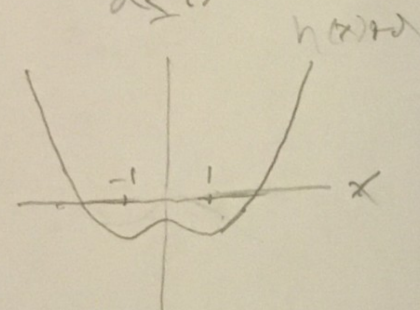
If $x = \pm 1$, then $h(x) = -\frac{1}{2}$

Varying d in $y = \pm\sqrt{h(x) + d}$ shifts the graph of h up and down. Th3 gives 3 types of graphs for $y^2 = h(x) + d$

$d \geq \frac{1}{2}$ $\qquad$ $0 < d < \frac{1}{2}$ $\qquad$ $d \leq 0$

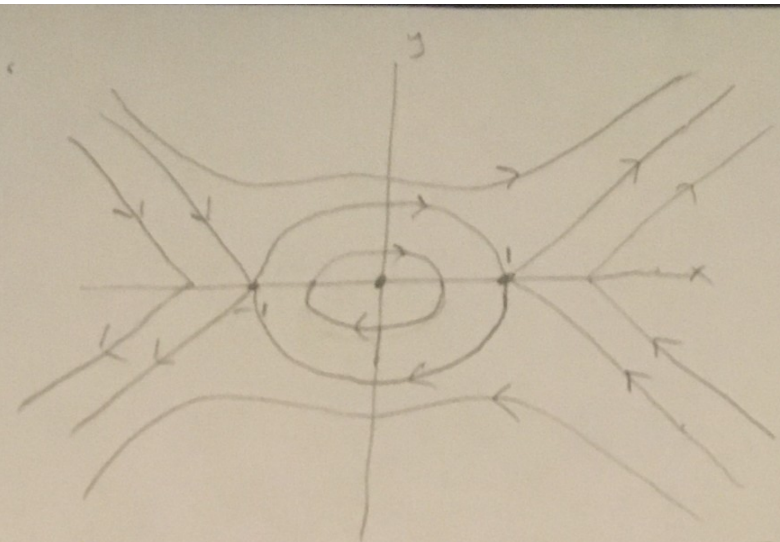






Since we need $y^2 = \pm\sqrt{h(x) + d}$, we can only use the parts of $h(x) + d$ where $h(x) + d \geq 0$. Critical points are when $x' = y = 0$ & $y' = x^3 - x = 0$, so $(0, 0)$ & $(\pm 1, 0)$.

3.1 P.P.D.



When $y>0$, $x'=y>0$

When $y<0$, $x'=y<0$

Separatrix is when $h(x)+d=0$ and touching the x-axis, So when $d=\frac{1}{2}$. Thus $\boxed{y^2=\frac{1}{2}x^4-x^2+\frac{1}{2}}$ is the separatrix equation

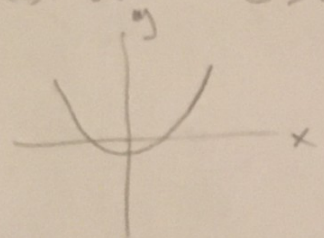
3.2 (c) $x''-2xx'=0$

Let $y=x'$, Then $x'=y$

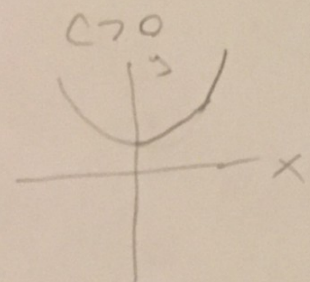
$y'=2xy$ and $\frac{dy}{dx}=\frac{2xy}{y}=2x$

$\Rightarrow y=x^2+C$

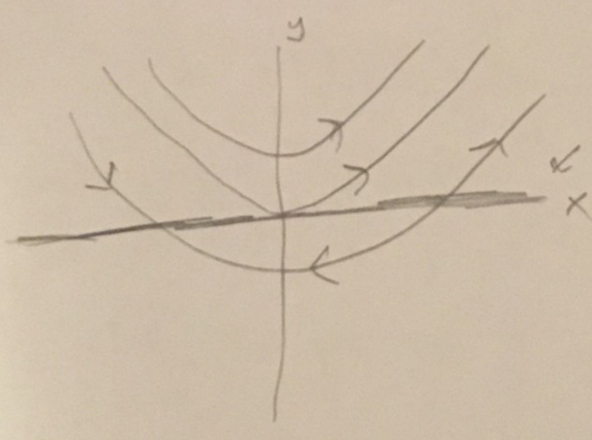
Consider $C\le 0$



$C>0$



C.P.'s when $x'=y=0$ & $y'=2xy=0$. So $x=0$ or $y=0$. Then $(x,0)$ is a c.p. $\forall x$



x-axis is all C.P.'s

$x'=y>0$ when $y>0$

" <0 when $y<0$

3.2 (ii) $x'' + |x| = 0$

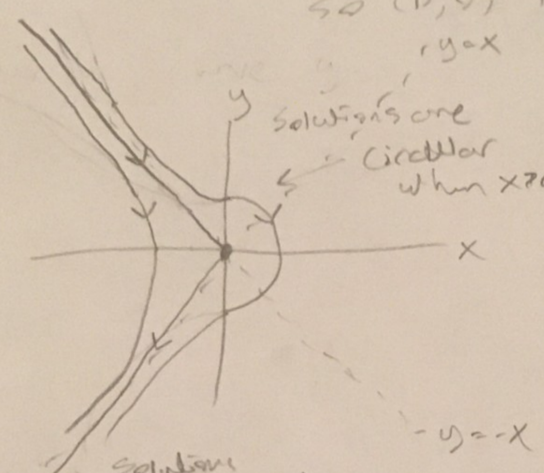
Let $y = x'$, then $x' = y$, $y' = -|x|$ so $\frac{dy}{dx} = \frac{-|x|}{y}$

If $x > 0$, then $-|x| = -x$, and $y^2 = -x^2 + C^2$ or $x^2 + y^2 = C^2$

If $x < 0$, then $-|x| = x$, and $y^2 = x^2 + D$ or $y^2 - x^2 = D$

Only c.p. is when $x' = y = 0$ and $y' = -|x| = 0 \Rightarrow x = 0$

so $(0,0)$ is the only c.p.



$x' = y > 0$ when $y > 0$

$x' = y < 0$ when $y < 0$

Solutions are hyperbolic when $x < 0$

and since the equation is $x^2 - y^2 = D$, either the hyperbola is aligned with the x or y axis depending on sign of D (no xy term in equation means no rotation of the asymptotes of the hyperbolae)

if $D = 0$, then

$x^2 - y^2 = 0 \Rightarrow x = \pm y$

When $D > 0$, the hyperbola opens up/down (along y axis) and when $D < 0$, the hyperbola opens left (along x-axis).

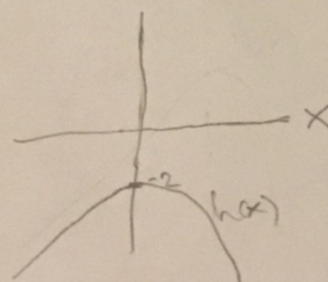
3.2 (iii) $x'' + e^x = 1$

Let $y = x'$, then $x' = y$
$y' = 1 - e^x$ and $\frac{dy}{dx} = \frac{1-e^x}{y}$

$$\Rightarrow \frac{1}{2}y^2 = x - e^x + c$$

$$\text{So } y = \pm\sqrt{2x - 2e^x + d}$$

let $h(x) = 2x - 2e^x$, and consider:



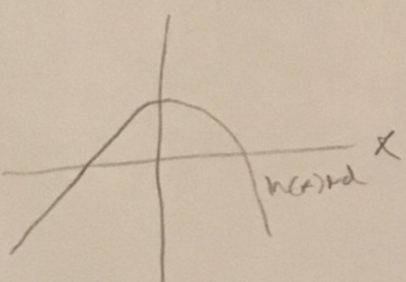
$h(x) = 0$ if $x = e^x$

$h'(x) = 0$ if $1 - e^x = 0$
or $x = 0$

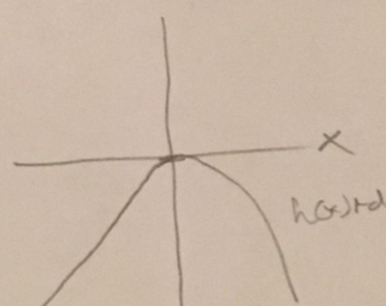
So $h(x)$ has a local max at $x=0$ where $h(0) = -2$ (max because $h''(0) = -2 < 0$)

This gives several cases, since d shifts $h(x)+d$ up and down

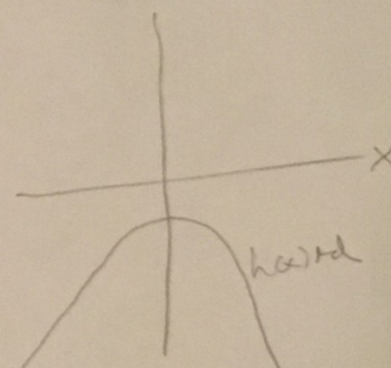
$d > 2$



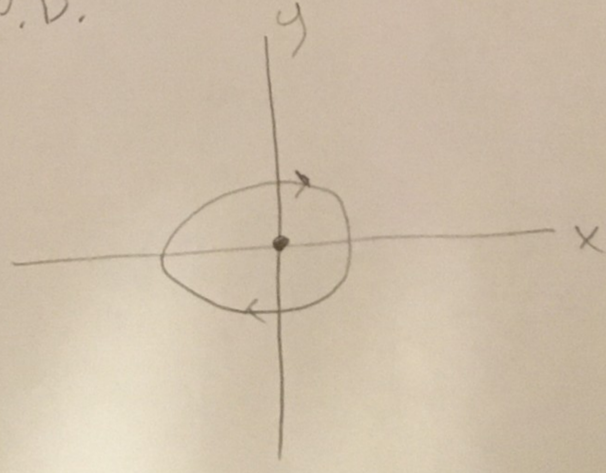
$d = 2$



$d < 2$



P.P.D.



C.P. occurs when $x' = y = 0$
& $y' = 1 - e^x = 0$
$\Rightarrow x = 0$

So $(0,0)$ is only C.P.

$x' > 0$ if $y > 0$
$x' < 0$ if $y < 0$

3.6 $x' = x + 16y$
$y' = -8x - y$

is Hamiltonian because if

$$x' = f(x,y) = x + 16y$$
$$y' = g(x,y) = -8x - y \quad \text{, then}$$

$$\frac{\partial f}{\partial x} = 1 = -\frac{\partial g}{\partial y} = -(-1) = 1,$$

Thus $\exists\, h(x,y)$ s.t. $\frac{\partial h}{\partial y} = f(x,y)$, $\frac{\partial h}{\partial x} = -g(x,y)$

and therefore $h(x,y) = xy + 8y^2 + K(x)$

so $\frac{\partial h}{\partial x} = y + K'(x) = -g(x,y) = +8x + y$

$\Rightarrow K'(x) = 8x$

$\Rightarrow K(x) = 4x^2 + C$

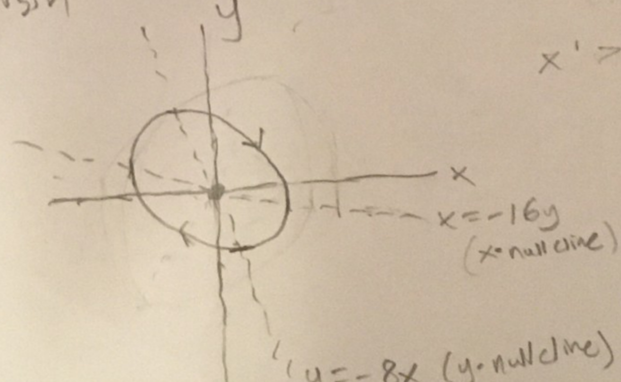
So $h(x,y) = xy + 8y^2 + 4x^2 + C$ is the Hamiltonian function, and the orbits occur when $h(x,y) = 0$ or when $8y^2 + 4x^2 + xy = d$, an equation of an ellipse.

C.P. occurs when $x' = x + 16y = 0$ & $y' = -8x - y = 0$, or $x = -16y$

$\Rightarrow 128y - y = 0 \Rightarrow y = 0$

$\Rightarrow x = 0$

So $(0,0)$ is only C.P. The elliptical orbits are centered at the origin



$x' > 0$ if $x + 16y > 0$
or $x > -16y$
$x' = 0$ when $x = 16y$
$x' < 0$ when $x < -16y$
$y' > 0$ if $-8x - y > 0$
or $y < -8x$
$y' = 0$ if $-8x - y = 0$
or $y = -8x$
$y' < 0$ if $y > -8x$

3.8 (ii)

$x' = x + y$
$y' = -x + y$

$f(x,y) = x + y$
$g(x,y) = y - x$

check for Hamiltonian system

$\frac{\partial f}{\partial x} \stackrel{?}{=} -\frac{\partial g}{\partial y}$

$1 \neq -1$, so

the system is <u>not</u> Hamiltonian.