Math 6418 Homework 2

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Let
$$f(x) = \begin{cases} 1 & x \in [-1,1], \\ 0 & |x| > 1. \end{cases} \tag{1}$$

1.

Since $f \in H^s(\mathbf{R})$ if and only if $g_s \in L^2(\mathbf{R})$, where $g_s(\xi) = \widehat{f}(\xi)(1+|\xi|^2)^{\frac{s}{2}}$, we should start by computing $\widehat{f}$:
$$\begin{aligned}\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ix\xi}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-ix\xi}}{-i\xi}\right]_{-1}^{1} \\ &= \frac{e^{i\xi} - e^{-i\xi}}{i\xi\sqrt{2\pi}} \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin(\xi)}{\xi}.\end{aligned}$$

We note that $\widehat{f}(0) = \sqrt{\frac{2}{\pi}}$, so $\widehat{f}$ is continuous. Thus,
$$g_s(\xi) = \sqrt{\frac{2}{\pi}} \frac{\sin(\xi)}{\xi} (1+|\xi|^2)^{\frac{s}{2}}$$
is also continuous.

Therefore, for $a > 0$,
$$\int_{-\infty}^{\infty} |g_s(\xi)|\,\mathrm{d}\xi = \int_{-a}^{a} |g_s(\xi)|^2\,\mathrm{d}\xi + 2\int_{a}^{\infty} |g_s(\xi)|^2\,\mathrm{d}\xi$$
because g_s is even. The first term is always finite because g_s is continuous, so $g_s \in L^2(\mathbf{R})$ if and only if the second term is finite.

Since $\frac{(1+|\xi|^2)^s}{|\xi|^{2s}} = \left(\frac{1}{|\xi|^2} + 1\right)^s \to 1$ as $\xi \to \infty$, we can choose a large enough that $(1+|\xi|^2)^s \leq 2|\xi|^{2s}$ for all $\xi > a$. Then $|g_s(\xi)|^2 \leq \frac{4}{\pi}|\xi|^{2(s-1)}$ for all $\xi > a$, meaning that the second term is finite if $2(s-1) < -1$, that is, if $s < \frac{1}{2}$. Hence, if $s < \frac{1}{2}$, then $f \in H^s$.

Conversely, suppose that $s \geq \frac{1}{2}$. Since $(1+|\xi|^2)^s \geq |\xi|^{2s}$, it follows that
$$\int_{\frac{\pi}{4}}^{\infty} |g_s(\xi)|^2\,\mathrm{d}\xi \geq \frac{2}{\pi} \int_{\frac{\pi}{4}}^{\infty} \sin^2(\xi)|\xi|^{2s-2}\,\mathrm{d}\xi.$$

If $|\xi|^{2s-2}$ is increasing, then it is clear that the integral on the right hand side is infinite, which implies that $f \notin H^s$. Suppose that $|\xi|^{2s-2}$ is decreasing, and set $I_1 = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right] \cup \dots$ and set $I_2 = \left[\frac{\pi}{4}, \infty\right) \setminus I_1$.

The sets I_1 and I_2 consist of consecutive, interleaved intervals of length $\frac{\pi}{2}$; since $|\xi|^{2s-2}$ is decreasing and each interval of I_1 precedes an interval of I_2, it follows that

$$\int_{I_1} |\xi|^{2s-2}\,\mathrm{d}\xi \geq \int_{I_2} |\xi|^{2s-2}\,\mathrm{d}\xi,$$

so

$$\int_{\frac{\pi}{4}}^{\infty} |\xi|^{2s-2} = \int_{I_1} |\xi|^{2s-2}\,\mathrm{d}\xi + \int_{I_2} |\xi|^{2s-2}\,\mathrm{d}\xi \leq 2\int_{I_1} |\xi|^{2s-2}\,\mathrm{d}\xi,$$

which implies that

$$\int_{I_1} |\xi|^{2s-2}\,\mathrm{d}\xi \geq \frac{1}{2}\int_{\frac{\pi}{4}}^{\infty} |\xi|^{2s-2}\,\mathrm{d}\xi = \infty$$

because $s \geq \frac{1}{2}$. Therefore,

$$\int_{\frac{\pi}{4}}^{\infty} \sin^2(\xi)|\xi|^{2s-2}\,\mathrm{d}\xi \geq \int_{I_1} \sin^2(\xi)|\xi|^{2s-2}\,\mathrm{d}\xi \geq \frac{1}{2}\int_{I_1} |\xi|^{2s-2}\,\mathrm{d}\xi = \infty$$

because $\sin^2(\xi) \geq \frac{1}{2}$ for $\xi \in I_1$. This implies that $f \notin H^s$.

Thus, $f \in H^s$ if and only if $s < \frac{1}{2}$.

2.

Let $g \in L^1(\mathbf{R})$ be a positive function such that $g(1+|\cdot|^2)^{\frac{1}{s}} \in L^2(\mathbf{R})$ for all $s < 1$. For example, $g(x) = e^{-x^2}$ works. Define

$$v(\xi,\eta) = \frac{1}{\|g\|_{L^1}} \frac{2\sin(\xi)}{\xi\sqrt{1+\xi^2}} \cdot g\left(\frac{\eta}{\sqrt{1+\xi^2}}\right).$$

Then

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} v(\xi,\eta)\,\mathrm{d}\eta = \frac{1}{\|g\|_{L^1}}\sqrt{\frac{2}{\pi}}\frac{\sin(\xi)}{\xi\sqrt{1+\xi^2}}\int_{-\infty}^{\infty} g\left(\frac{\eta}{\sqrt{1+\xi^2}}\right)\,\mathrm{d}\eta = \sqrt{\frac{2}{\pi}}\frac{\sin(\xi)}{\xi} = \widehat{f}(\xi),$$

upon making the substitution $w = \frac{\eta}{\sqrt{1+\xi^2}}$. Furthermore, for $s \in \mathbf{R}$,

$$\|v(1+|\cdot|^2)^{\frac{s}{2}}\|_{L^2}^2 = \frac{4}{\|g\|_{L^1}^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin^2(\xi)}{\xi^2(1+\xi^2)} g^2\left(\frac{\eta}{\sqrt{1+\xi^2}}\right)(1+\xi^2+\eta^2)^s\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

Substituting $w = \frac{\eta}{\sqrt{1+\xi^2}}$ in the inner integral over η, we have

$$\begin{aligned}
\|v(1+|\cdot|^2)^{\frac{s}{2}}\|_{L^2}^2 &= \frac{4}{\|g\|_{L^1}^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin^2(\xi)(1+\xi^2)^s}{\xi^2\sqrt{1+\xi^2}} g^2(w)(1+w^2)^s\,\mathrm{d}w\,\mathrm{d}\xi \\
&= \frac{4\|g(1+(\cdot)^2)^{\frac{s}{2}}\|_{L^2}^2}{\|g\|_{L^1}^2}\int_{-\infty}^{\infty} \frac{\sin^2(\xi)(1+\xi^2)^{s-\frac{1}{2}}}{\xi^2}\,\mathrm{d}\xi.
\end{aligned}$$

The remaining integral is finite if $2s-1-2 < -1$, that is, if $s < 1$. Choosing $s = 0$, we see that $v \in L^2(\mathbf{R}^2)$, so there exists $u \in L^2(\mathbf{R}^2)$ such that $\widehat{u} = v$ by Fourier inversion; therefore, $u \in H^s(\mathbf{R}^2)$ for all $s < 1$. Restricting to the upper half plane, we can also say that $u \in H^s(\mathbf{R}^2_+)$ for all $s < 1$.

Finally, $u(x,0) = f(x)$ because

$$u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\xi x}\widehat{u}(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi = \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} v(\xi,\eta)\,\mathrm{d}\eta\right)(x) = \mathcal{F}^{-1}(\hat{f})(x) = f(x).$$

3.

We note that the same u in part 2 works for all $s < 1$.

4.

If we choose $g(x) = e^{-x^2}$, then u from part 2 is C^∞ on $\mathbf{R}^2_+$. Indeed, for $y > 0$, we have

$$u(x,y) = \frac{1}{\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\xi x} e^{i\eta y} \frac{\sin(\xi)}{\xi\sqrt{1+\xi^2}} e^{-\left(\frac{\eta}{\sqrt{1+\xi^2}}\right)^2} \, \mathrm{d}\eta \, \mathrm{d}\xi.$$

Doing the η integral first, we have

$$\begin{aligned} u(x,y) &= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{ix\xi} \frac{\sin(\xi)}{\xi\sqrt{1+\xi^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\eta y} e^{-\left(\frac{\eta}{\sqrt{1+\xi^2}}\right)} \, \mathrm{d}\eta \, \mathrm{d}\xi \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{ix\xi} \frac{\sin(\xi)}{\xi\sqrt{1+\xi^2}} \mathscr{F}^{-1} \left[e^{-\frac{(\cdot)^2}{1+\xi^2}} \right] (y) \, \mathrm{d}\xi. \end{aligned}$$

Consulting a table of Fourier transforms, we find that

$$u(x,y) = \frac{1}{\pi\sqrt{2}} \int_{-\infty}^{\infty} e^{ix\xi} \frac{\sin(\xi)}{\xi} e^{-\frac{y^2(1+\xi^2)}{4}} \, \mathrm{d}\xi.$$

Let α be a multi-index. Then, formally,

$$D^\alpha u(x,y) = \frac{1}{\pi\sqrt{2}} \int_{-\infty}^{\infty} e^{ix\xi} \frac{\sin(\xi)}{\xi} P_\alpha(x,y,\xi) e^{-\frac{y^2(1+\xi^2)}{4}} \, \mathrm{d}\xi,$$

where $P_\alpha(x,y,\xi)$ is a polynomial function in x, y, ξ. Taking the partial derivatives inside the integral is valid, in fact, because the integrand is C^∞ and has exponential decay in ξ, *provided that* $y > 0$, which we are assuming. Since this is true of the integrand no matter how many partial derivatives are taken, it follows that $D^\alpha u$ exists and is continuous for all α, if $y > 0$. That is, $u \in C^\infty(\mathbf{R}^2_+)$.