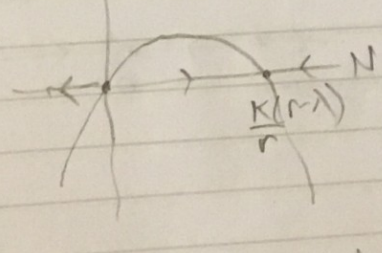
1.30 (i) The DE

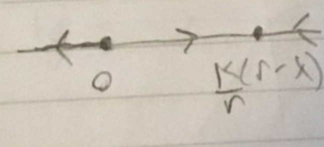
$$N' = rN\left(1-\frac{N}{K}\right) - \lambda N$$

could model logistic growth with harvesting for $\lambda > 0$ because in a situation of logistic growth with harvesting, the population would be growing at a rate proportional to its size (rN) and mediated by a limited resources $\left(1-\frac{N}{K}\right)$, while at the same time decreasing (in this model of harvesting) at a rate proportional to the size due to harvesting ($-\lambda N$; this is NOT mediated by limited resources).

(ii) If $\lambda < r$, then $N' = f(N,\lambda) = \left((r-\lambda) - \frac{rN}{K}\right)N = 0$ if

$N = 0$ or $N = \frac{K(r-\lambda)}{r} > 0$ (c.p.'s)



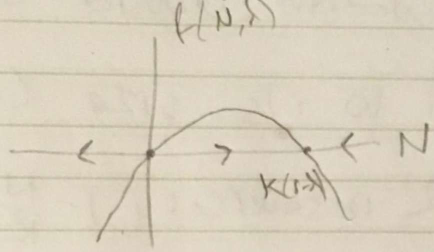


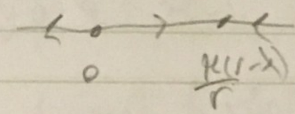
$N = 0$ is unstable

$N = \frac{K(r-\lambda)}{r}$ is a.s. stable

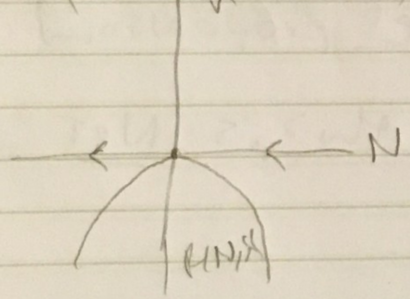
1.30 (iii) $f(N,\lambda) = N'$ is a downward parabola with roots $N = 0$ and $N = K(r-\lambda)$. Thus, the form of the graph depends on whether $\frac{K}{r}(r-\lambda) >, <,$ or $= 0$
$\iff r >, <, = \lambda$

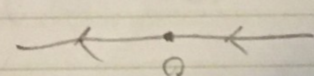
$r > \lambda$ $\left(\frac{K}{r}(r-\lambda) > 0\right)$ is like in part (ii)



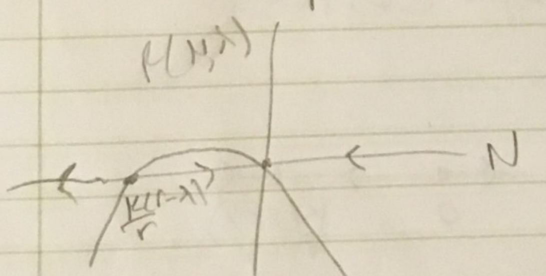


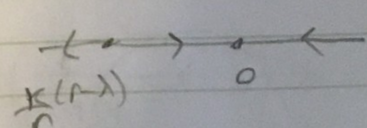
$r = \lambda$ $\left(\frac{K}{r}(r-\lambda) = 0\right.$, so the roots of $f(N,\lambda)$ are both zero$\left.\right)$





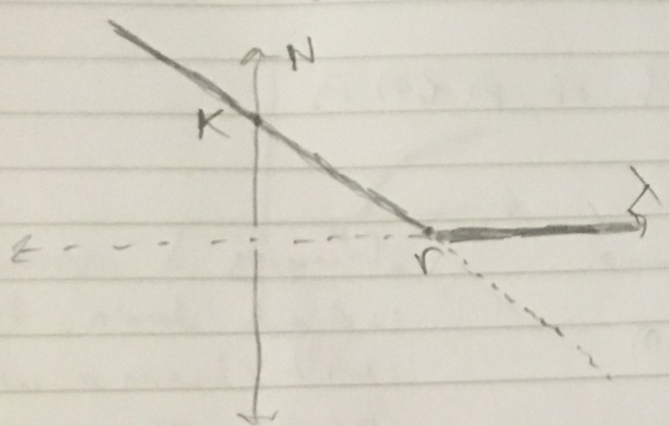
$r < \lambda$ $\left(\frac{K}{r}(r-\lambda) < 0\right)$





So a bifurcation happens at $\lambda = r$.

1.30 (iii) Bifurcation diagram



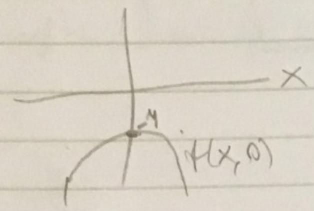
$N = 0$ or

$N = \frac{K}{r}(r-\lambda)$

$= K - \frac{K}{r}\lambda$

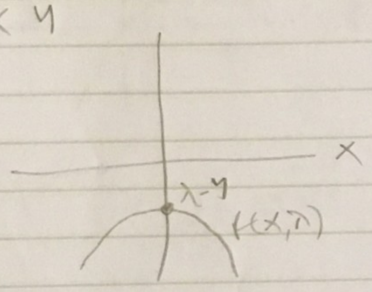
1.33 (i) $x' = f(x,\lambda) = \lambda - 4 - x^2$

When $\lambda = 0$, graph of $f(x,0)$ is

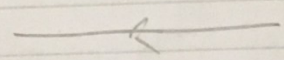


changing λ moves $f(x)$ up and down; the form will change when $\lambda = 4$ (graph crosses x-axis)

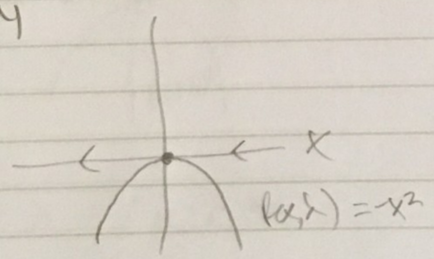
$\lambda < 4$



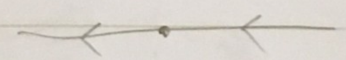
No c.p.'s



$\lambda = 4$



$x = 0$ is only c.p. (semistable/unstable)

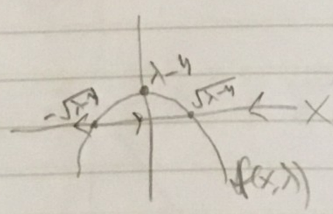


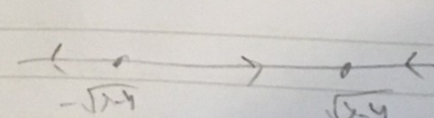
c.p.'s

$\sqrt{\lambda-4}$: as. stable

$-\sqrt{\lambda-4}$: unstable

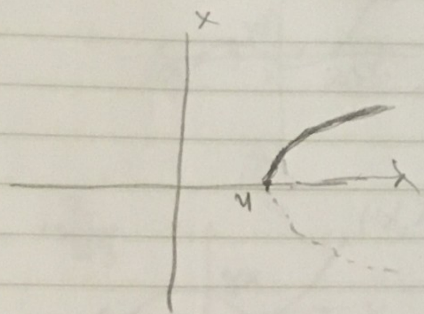
$\lambda > 4$





bifurcation point at $\lambda = 4$

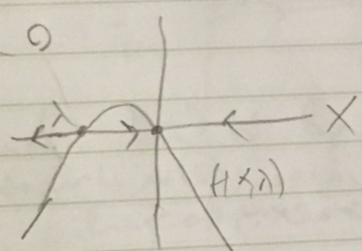
1.33 (i) Bifurcation diagram



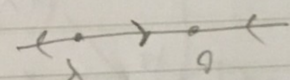
(ii) $x' = f(x,\lambda) = x^3(\lambda - x)$

$f(x,\lambda)=0$ if $x=0$ or $x=\lambda$, so the graph will change form when $\lambda=0$ and these roots coincide

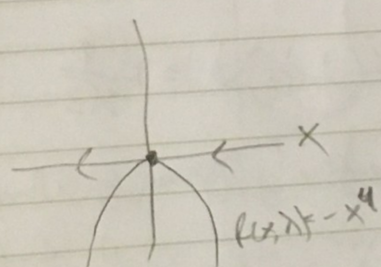
$\lambda < 0$



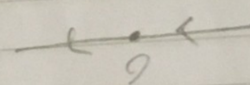
C.P.'s are λ and 0



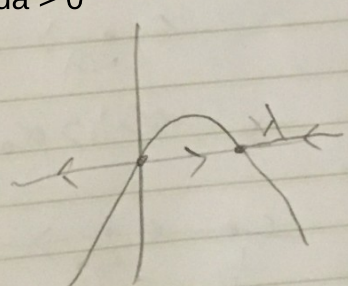
$\lambda = 0$



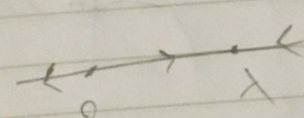
C.P.'s are λ and 0, but $\lambda=0$, so just 0



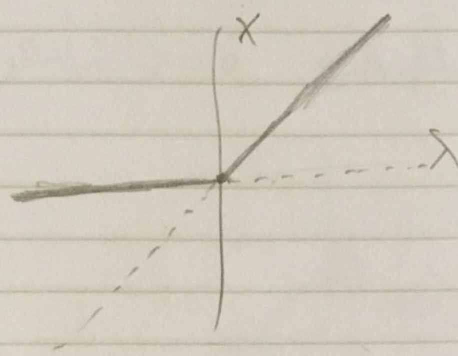
lambda > 0



C.P.'s are λ and 0

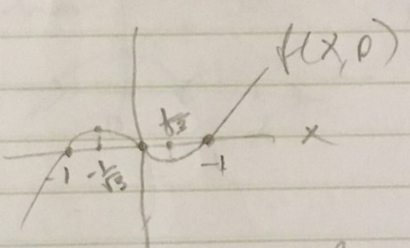


1.33 (ii) Bifurcation diagram



(iii) $x' = f(x,\lambda) = x^3 - x + \lambda$
$= x(x^2-1) + \lambda$

changing λ shifts graph up and down:



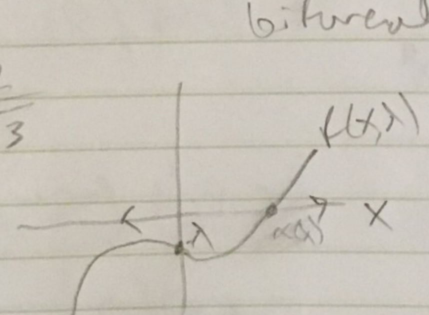
Need to find local min/max, because form changes when they cross x-axis:

$\frac{\partial f}{\partial x} = 0 \quad 3x^2 - 1 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$

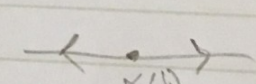
and $f(\pm\frac{1}{\sqrt{3}}, 0) = \mp \frac{2}{3\sqrt{3}}$, so

bifurcation happens when $\lambda = \mp \frac{2}{3\sqrt{3}}$

$\lambda < -\frac{2}{3\sqrt{3}}$

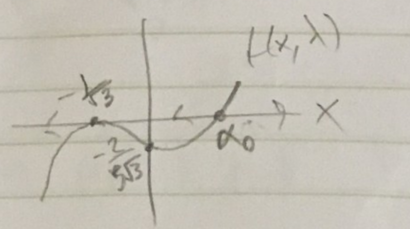


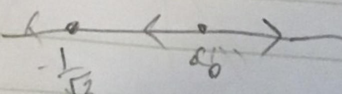
Call c.p. where $f(x,\lambda)=0$ $x = \alpha$



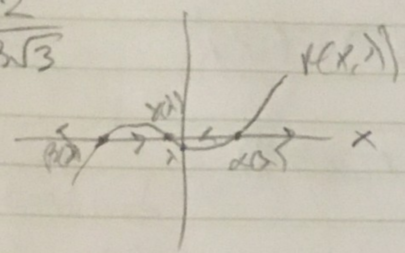
$\alpha(\lambda) > \alpha_0$

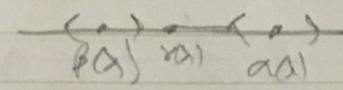
$\lambda = -\frac{2}{3\sqrt{3}}$





$-\frac{2}{3\sqrt{3}} < \lambda < \frac{2}{3\sqrt{3}}$



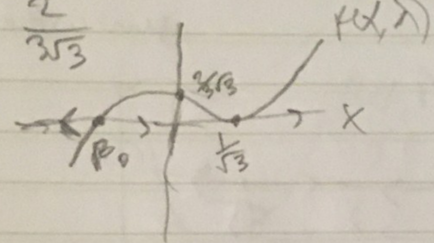


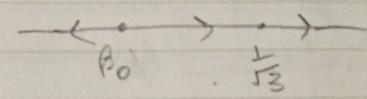
$\beta_0 < \beta(\lambda) < -\frac{1}{\sqrt{3}}$

$-\frac{1}{\sqrt{3}} < \gamma(\lambda) < \frac{1}{\sqrt{3}}$

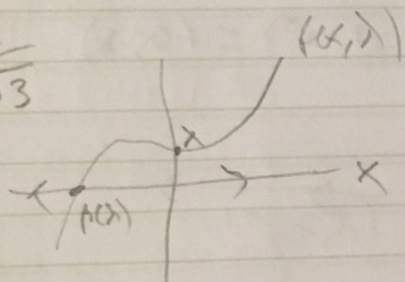
$\alpha_0 > \alpha(\lambda) > \frac{1}{\sqrt{3}}$

$\lambda = \frac{2}{3\sqrt{3}}$

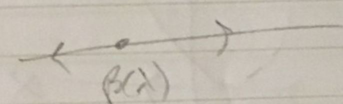




$\lambda > \frac{2}{3\sqrt{3}}$

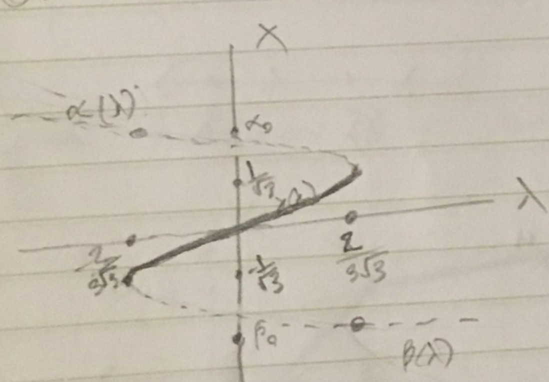


$\beta(\lambda) < \beta_0$



$\beta_0 = -\alpha_0$ by symmetry

Two bifurcations occur at $\lambda = \pm\frac{2}{3\sqrt{3}}$

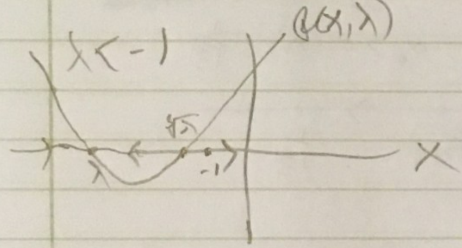


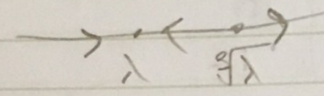
1.36 (ii) $x' = (\lambda - x)(\lambda - x^3) = f(x, \lambda)$

$f(x,\lambda) = 0$ if $x = \lambda$ or $x = \sqrt[3]{\lambda}$

This gives two distinct roots if $\lambda \neq \sqrt[3]{\lambda}$ i.e., if $\lambda \neq \pm 1, 0$; one otherwise

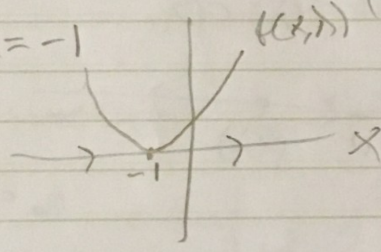
$\lambda < -1$

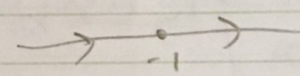




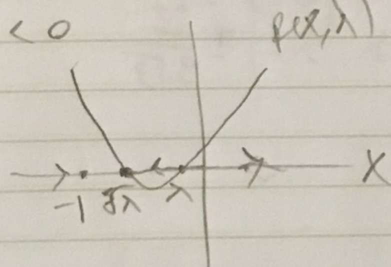
$(-1-x)(-1-x^3) = f(x,\lambda)$

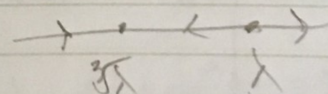
$\lambda = -1$



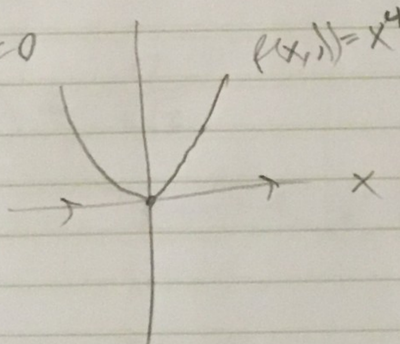


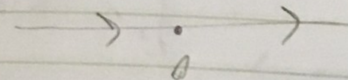
$-1 < \lambda < 0$



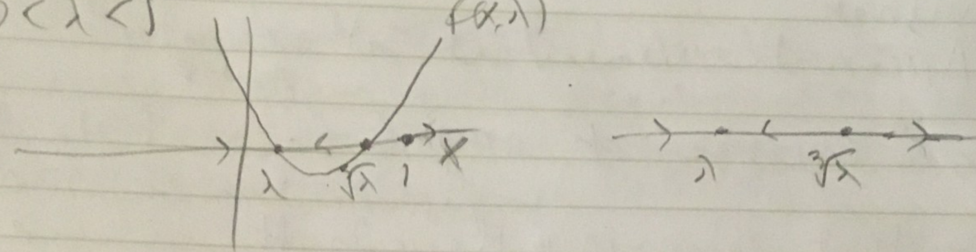


$\lambda = 0$

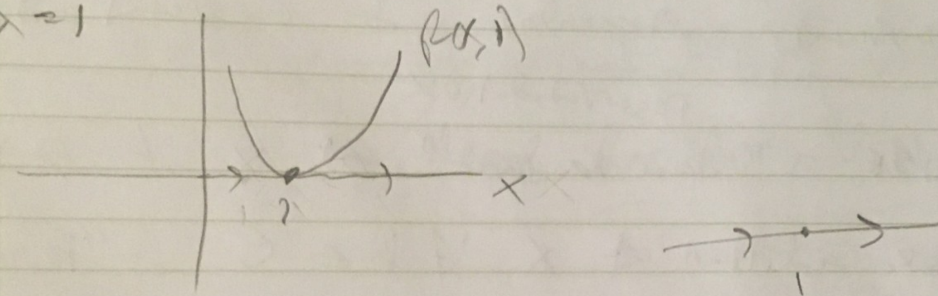




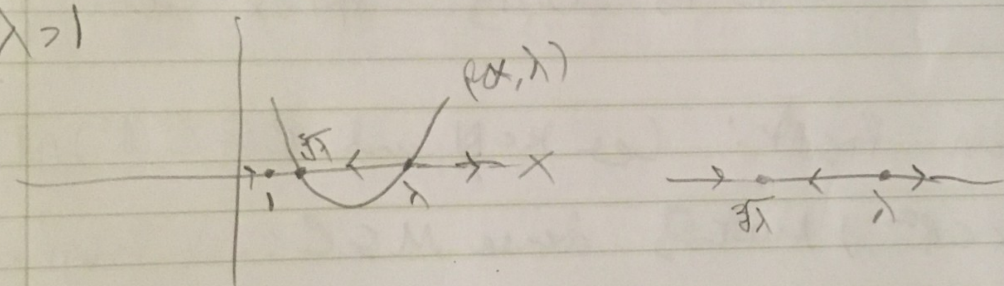
$0 < \lambda < 1$



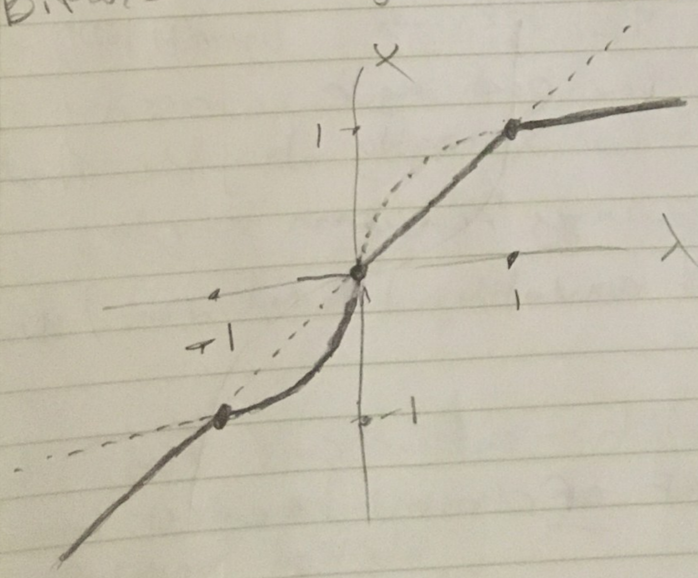
$\lambda = 1$



$\lambda > 1$



Bifurcation diagram - bifurcation points at $\lambda = 1, 0, -1$



137 (i) Let $x' = f(\lambda,x) = (\lambda - x)(\lambda - x^2)$.

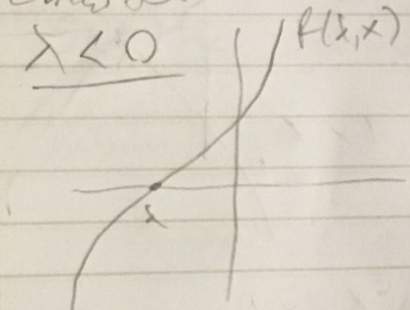
$f(\lambda,x) = 0$ if $x = \lambda$ or $x = \pm\sqrt{\lambda}$

So, if $\lambda < 0$, $f(\lambda,x)$ has only 1 real root.

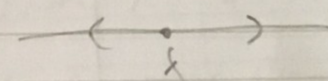
If $\lambda \neq \sqrt{\lambda} \iff \lambda \neq 0$ and $\lambda \neq 1$, then $f(\lambda,x)$ has 3 distinct roots. If $\lambda = 0$, then $f(\lambda,x) = x^3$ has 1 root, and if $x = 1$, then $f(\lambda,x) = (1-x)(1-x^2)$
$= (1-x)^2(1+x)$
has 2 roots.

Consider:

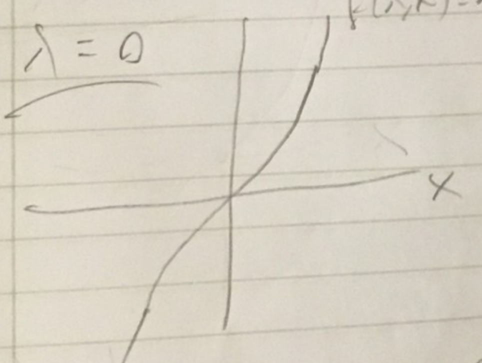
$\underline{\lambda < 0}$



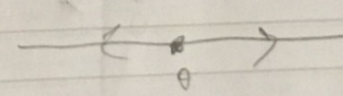
PLD:



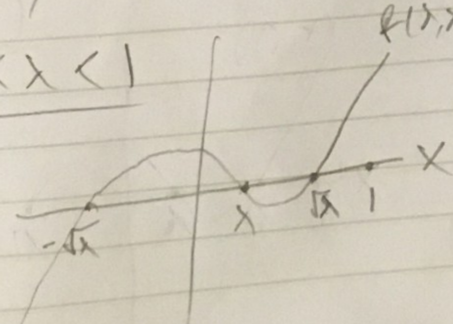
$\underline{\lambda = 0}$



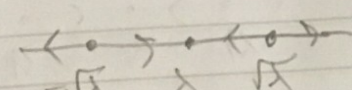
PLD:



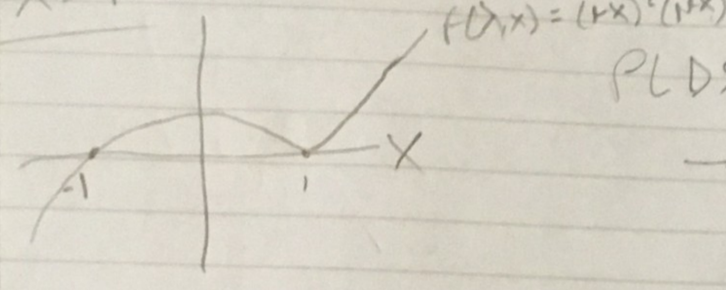
$\underline{0 < \lambda < 1}$



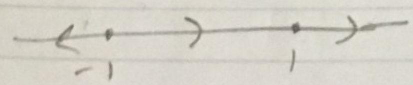
PLD:



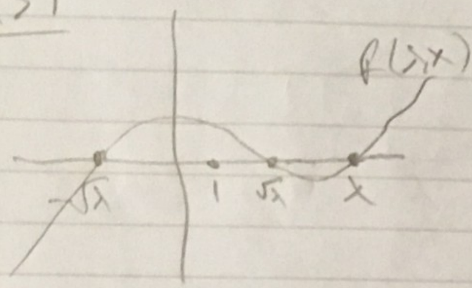
$\lambda = 1$



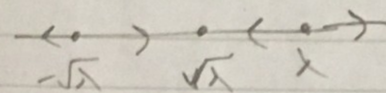
PLD:



$\lambda > 1$

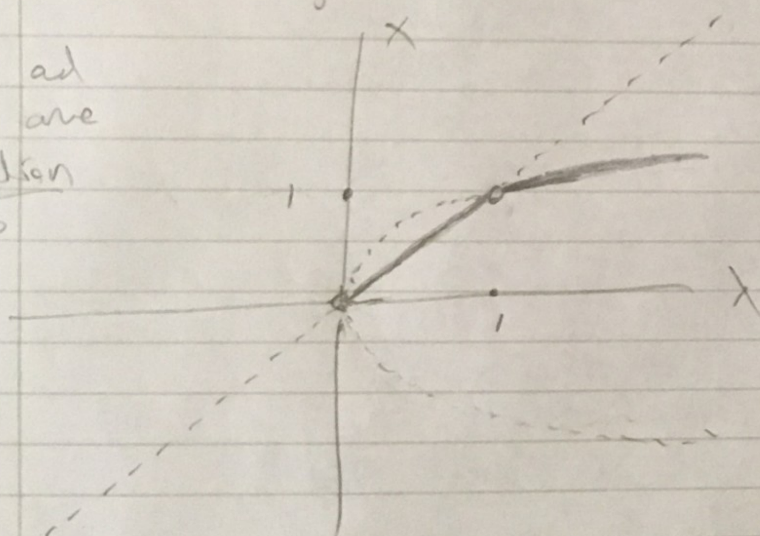


PLD:



Bifurcation diagram

$\lambda_1 = 0$ and $\lambda_2 = 1$ are bifurcation points

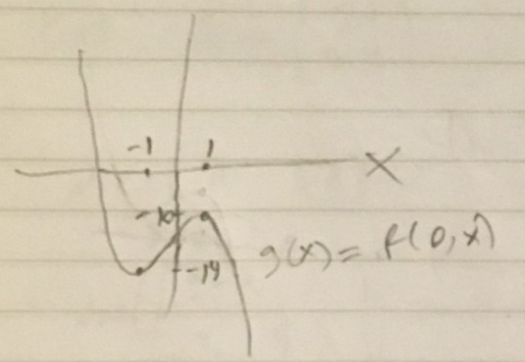


1.37 (ii) Let $x' = f(\lambda, x) = \lambda - 12 + 3x - x^3$

Observe that λ shifts up and down the graph of $-x^3 + 3x - 12 = g(x)$.

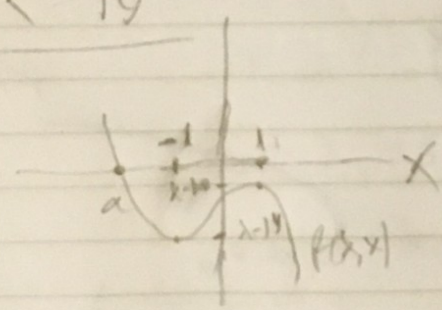
$g'(x) = -3x^2 + 3$
$= 3(1-x^2) = 3(1-x)(1+x)$
$= 0$ if $x = \pm 1$

$g(1) = -10$
$g(-1) = -14$

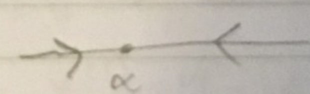


So the roots of $f(\lambda, x)$ undergo a drastic change when $\lambda = \lambda_1 = 10$ and when $\lambda = \lambda_2 = 14$, as this is when a piece of $f(\lambda, x)$ moves from positive to negative, and roots are gained / lost.

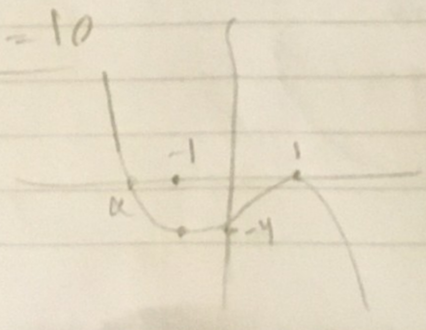
$\lambda < 19$



PLD



$\lambda = 10$



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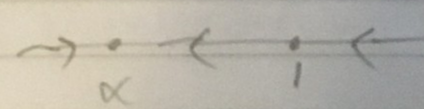
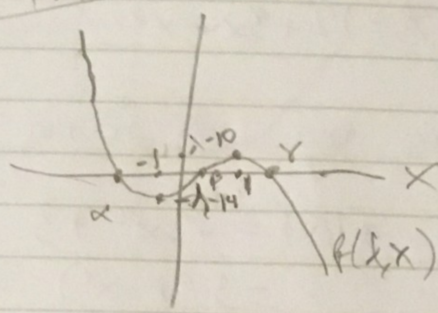


$10 < \lambda < 14$



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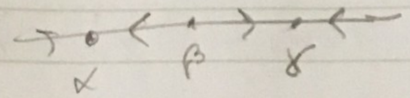
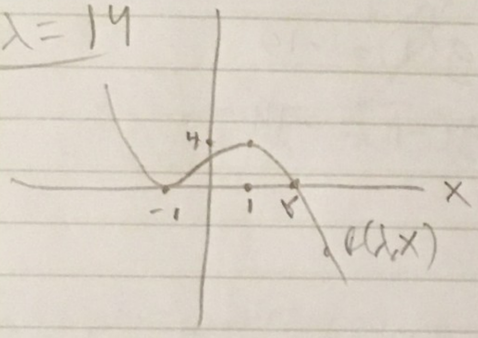
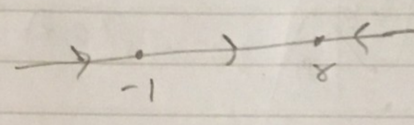


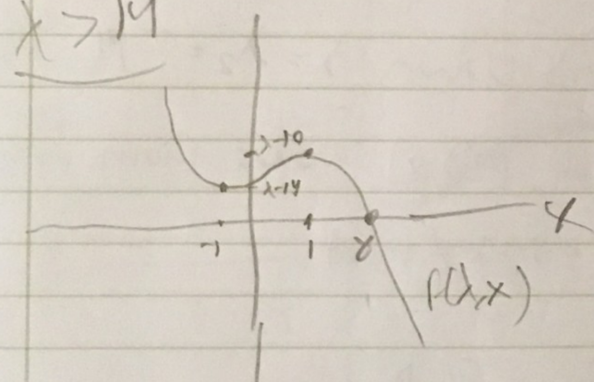
$\lambda = 14$



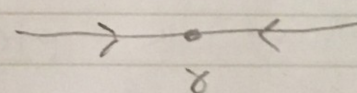
PLD



$\lambda > 14$



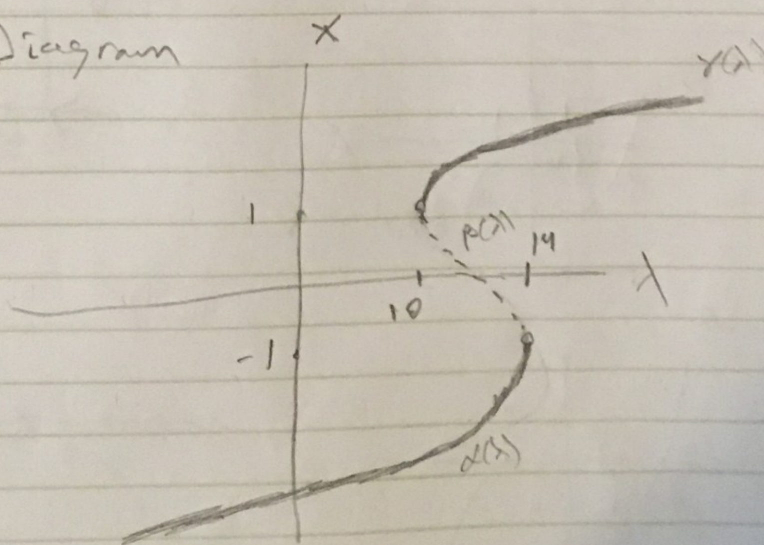
PLD



Bifurcation Diagram

$\lambda_1 = 10$, $\lambda_2 = 14$

are bifurcation points



2.20 (i) using Theorem 2.14 means by finding eigen-values/-vectors of matrix

$$x' = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} x$$

Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

$\det(A - I\lambda) = 0$ if λ is eigenvalue of A;

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{bmatrix}\right) = (2-\lambda)(4-\lambda) - 3 = 0$$

if $8 - 6\lambda + \lambda^2 - 3 = 0$

or $(\lambda - 5)(\lambda - 1) = 0$

or $\lambda = 5$ or $\lambda = 1$

If $\lambda = \lambda_1 = 5$

$Av = \lambda_1 v$ if $2v_1 + v_2 = 5v_1$ or $v_2 = 3v_1$

so $v^{(1)} = \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is eigenvector for $\lambda_1 = 5$. Thus $x^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{5t}$ is a soln. $(= v^{(1)} e^{\lambda_1 t})$ of the diff equation.

If $\lambda = \lambda_2 = 1$, $Av = \lambda_2 v$ if $2v_1 + v_2 = v_1$ or $v_2 = -v_1$

So $v^{(2)} = \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is eigenvector for $\lambda_2 = 1$.

Thus $x^{(2)} = v^{(2)} e^{\lambda_2 t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$ is a soln. of diff eqn.

$x^{(1)}, x^{(2)}$ are lin. indep., so $x = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$ is a general solution.

2.2a (ii) $x' = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} x.$ Let $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$

Again $\det(A - I\lambda) = 0$ if λ is e.v. of A, so

$\det\left(\begin{pmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix}\right) = 0$ if $(2-\lambda)(-1-\lambda) - 4 = 0$

$\Leftrightarrow \lambda^2 - \lambda - 6 = 0$

$\Leftrightarrow (\lambda - 3)(\lambda + 2) = 0$

$\Leftrightarrow \lambda = -2$ or $\lambda = 3$

So $\lambda = \lambda_1 = -2$ and $\lambda = \lambda_2 = 3$ are e.v.'s of $\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$.

If $\lambda = \lambda_1$, then $Av = \lambda_1 v$ if $2v_1 + 2v_2 = -2v_1$
or $v_2 = -2v_1$

So $v^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is e.vector for λ_1. Then

$x^{(1)} = v^{(1)} e^{\lambda_1 t} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$ is a solution of d.e.

If $\lambda = \lambda_2$, then $Av = \lambda_2 v$ if $2v_1 + 2v_2 = 3v_1$ or
$v_2 = \frac{1}{2} v_1$, so

$v^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is e.vector for λ_2. Then

$x^{(2)} = v^{(2)} e^{\lambda_2 t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t}$ is a solution of d.e.

Thus, since $x^{(1)}, x^{(2)}$ are lin. indep.,

$x = c_1 x^{(1)} + c_2 x^{(2)} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t}$

is a general solution of the diff. eqn

2.23 (i) $x' = \begin{bmatrix} 0 & 9 \\ -9 & 0 \end{bmatrix} x$. Let $A = \begin{bmatrix} 0 & 9 \\ -9 & 0 \end{bmatrix}$.

Then eigen value λ of A satisfies $\det(A - I\lambda) = 0$,

or $\det\left(\begin{pmatrix} -\lambda & 9 \\ -9 & -\lambda \end{pmatrix}\right) = 0$ or $\lambda^2 + 9 = 0$

or $\lambda = \pm 3i$.

Take $\lambda_1 = 3i$. Then $Av = \lambda_1 v$ iff $9v_2 = 3iv_1$

or $v_2 = \frac{i}{3} v_1$

Thus $v^{(1)} = \begin{pmatrix} 3 \\ i \end{pmatrix}$ is e. vector for λ_1, and

$\phi = v^{(1)} e^{\lambda_1 t} = \begin{pmatrix} 3 \\ i \end{pmatrix} e^{3it}$ is a complex-valued

solution of diff. eqn. Thus

$x^{(1)} = \mathrm{Re}\,\phi$ and $x^{(2)} = \mathrm{Im}\,\phi$ are real-valued solutions.

$\phi = \begin{pmatrix} 3 \\ i \end{pmatrix} (\cos 3t + i \sin 3t) = \begin{pmatrix} 3\cos 3t \\ -\sin 3t \end{pmatrix} + i \begin{pmatrix} 3\sin 3t \\ \cos 3t \end{pmatrix}$.

So $x^{(1)} = \begin{pmatrix} 3\cos 3t \\ -\sin 3t \end{pmatrix}$ and $x^{(2)} = \begin{pmatrix} 3\sin 3t \\ \cos 3t \end{pmatrix}$ are

lin. indep. solutions of diff eqn. and $\therefore$

$x = c_1 x^{(1)} + c_2 x^{(2)} = c_1 \begin{pmatrix} 3\cos 3t \\ -\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 3\sin 3t \\ \cos 3t \end{pmatrix}$ is

a general solution.

2.2.3 (ii) $x' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} x$. Let $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

λ is e.v. of A if $\det(A - I\lambda) = 0$, or

$\det\left(\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix}\right) = 0$ or $(1-\lambda)^2 + 1 = 0$

or $1-\lambda = \pm i$

or $\lambda = 1 \pm i$

Take $\lambda_1 = 1+i$. Then $Av = \lambda_1 v$ if $v_1 + v_2 = (1+i)v_1$

or $v_2 = i v_1$

Then $v^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is e.vector for λ_1, and

$\phi = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+i)t}$ is a complex-valued solution

Then $x^{(1)} = \mathrm{Re}\,\phi$ and $x^{(2)} = \mathrm{Im}\,\phi$ are real-valued solutions.

$\phi = \begin{pmatrix} 1 \\ i \end{pmatrix} e^t(\cos t + i \sin t) = e^t\left(\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right)$

So $x^{(1)} = e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ and $x^{(2)} = e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$, and are lin. indep., $\therefore$

$x = c_1 x^{(1)} + c_2 x^{(2)} = c_1 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ is

a general solution of diff eqn.