Stat 6841 Homework 1

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Problem 1.

If dividends are paid at times $t = 0, 1, 2, \ldots$, then the payment and suspension of dividends can be modeled by a Markov chain $\{X_t : t \geq 0\}$, where $X_t = 1$ means that a dividend is paid at time t, and $X_t = 2$ means that a dividend is suspended at time t. Based on the information given, the transition matrix for this chain must be

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{bmatrix}.$$

Let $\pi = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}$ be the proportion of the time the chain spends in states 1 and 2 in the long run. Then

$$\begin{aligned} \pi P = \pi &\implies \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \\ &\implies 0.9\pi_1 + 0.4\pi_2 = \pi_1 \\ &\implies \pi_1 = 4\pi_2. \end{aligned}$$

Since we also have $\pi_1 + \pi_2 = 1$, it follows that $\pi_2 = \frac{1}{5}$, and $\pi_1 = \frac{4}{5}$. Thus, the chain spends $\pi_1 = \frac{4}{5}$ of its time in state 1 in the long run; that is, in the long run dividends are paid $\frac{4}{5}$ of the time.

Problem 2.

Let $\pi = \begin{bmatrix} \pi_L & \pi_M & \pi_H \end{bmatrix}$ be the limiting fraction of the population in each income class. Then we must have

$$\begin{aligned} \pi P = \pi &\implies \begin{bmatrix} \pi_L & \pi_M & \pi_H \end{bmatrix} \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} = \begin{bmatrix} \pi_L & \pi_M & \pi_H \end{bmatrix} \\ &\implies 0.3\pi_L + 0.7\pi_M + 0.3\pi_H = \pi_M, \\ &\qquad 0.1\pi_L + 0.1\pi_M + 0.6\pi_H = \pi_H \\ &\implies \pi_L + \pi_H = \pi_M, \\ &\qquad \pi_L + \pi_M = 4\pi_H. \end{aligned}$$

Since we also have $\pi_L + \pi_M + \pi_H = 1$, it follows that $2\pi_M = 1$, and $5\pi_H = 1$, so $\pi_M = \frac{1}{2}$, $\pi_H = \frac{1}{5}$, and $\pi_L = \frac{3}{10}$.

Problem 3.

(a) Let $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 & \pi_6 \end{bmatrix}$ be the fraction of time the chain spends in states 1 through 6 in the long run. The machine is working if and only it the chain is in state 1, 2 or 3, so $\pi_1 + \pi_2 + \pi_3$ represents the fraction of the time the machine is working in the long run. Noting that the transition matrix with the additional repair states is given by

$$P = \begin{bmatrix} 0.95 & 0.05 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.875 & 0.125 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we have

$$\begin{aligned}
\pi P = \pi \implies \begin{bmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 & \pi_6\end{bmatrix} &\begin{bmatrix} 0.95 & 0.05 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.875 & 0.125 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 & \pi_6\end{bmatrix} \\
\implies &\ 0.125\pi_3 = \pi_4 = \pi_5 = \pi_6, \\
&\ 0.95\pi_1 + \pi_6 = \pi_1, \\
&\ 0.05\pi_1 + 0.9\pi_2 = \pi_2 \\
\implies &\ \pi_1 = 20\pi_6, \quad \pi_2 = 0.5\pi_1 = 10\pi_6, \\
&\ \pi_3 = 8\pi_6, \quad \pi_4 = \pi_5 = \pi_6.
\end{aligned}$$

Since we also have $\pi_1+\pi_2+\pi_3+\pi_4+\pi_5+\pi_6 = 1$, it follows that $\pi_6 = \frac{1}{41}$, so $\pi_1+\pi_2+\pi_3 = 38\pi_6 = \frac{38}{41}$; that is, the machine is working a fraction $\frac{38}{41}$ of the time in the long run.

(b) Once again, we search for a solution π of $\pi P = \pi$ such that $\pi_1 + \pi_2 + \pi_3 = 1$. In this case, states 1 and 2 are working states, and state 3 is not working. Thus, we need to calculate $\pi_1 + \pi_2$. We obtain this from

$$\begin{aligned}
\pi P = \pi \implies \begin{bmatrix}\pi_1 & \pi_2 & \pi_3\end{bmatrix} &\begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix}\pi_1 & \pi_2 & \pi_3\end{bmatrix} \\
\implies &\ 0.95\pi_1 + \pi_3 = \pi_1, \qquad 0.1\pi_2 = \pi_3 \\
\implies &\ \pi_1 = 20\pi_3 = 2\pi_2.
\end{aligned}$$

Since we must also have $\pi_1 + \pi_2 + \pi_3 = 1$, it follows that $\pi_3 = \frac{1}{31}$, so $\pi_1 + \pi_2 = \frac{30}{31}$. Thus, the machine is working a fraction $\frac{30}{31}$ of the time.

Problem 4.

(a) Let $I_n = 1$ if Xavier has not been caught at time n, and let $I_n = 0$ if Xavier has been caught at time n. Then $\{I_n \mid n \geq 0\}$ is a Markov chain, with the following transition matrix

$$P = \begin{bmatrix} 1 & 0 \\ a & 1-a \end{bmatrix},$$

where a is the probability that Xavier will be caught at the next location given that he has not been caught at the current one. Based on the rules of the game, this means that $a = P(X_{n+1} = Y_{n+1} \mid X_n \neq Y_n)$. We can calculate a from the transition matrices P_X and P_Y. We note that

$$T = \sum_{n=0} I_n,$$

so we can compute $E[T]$ by computing $E\left[\sum_{n=0}^{\infty} I_n\right] = \sum_{n=0}^{\infty} E[I_n]$.

Using the law of total probability and the independence of $\{X_n \mid n \geq 0\}$ and $\{Y_n \mid n \geq 0\}$, we have

$$\begin{aligned}
a &= P(X_{n+1} = Y_{n+1} \mid X_n \neq Y_n) \\
&= \sum_{i \neq j} P(X_{n+1} = Y_{n+1} \mid X_n = i, Y_n = j) \frac{P(X_n = i, Y_n = j)}{P(X_n \neq Y_n)} \\
&= \sum_{i \neq j} \sum_{k=1}^{3} P(X_{n+1} = k, Y_{n+1} = k \mid X_n = i, Y_n = j) \frac{P(X_n = i, Y_n = j)}{P(X_n \neq Y_n)} \\
&= \sum_{i \neq j} \sum_{k=1}^{3} P(X_{n+1} = k \mid X_n = i) P(Y_{n+1} = k \mid Y_n = j) \frac{P(X_n = i, Y_n = j)}{P(X_n \neq Y_n)} \\
&= \sum_{i \neq j} \frac{P(X_n = i, Y_n = j)}{P(X_n \neq Y_n)} \sum_{k=1}^{3} P_{X,ik} P_{Y,jk}. \qquad (*)
\end{aligned}$$

At this point, the symmetry of the problem helps us. In particular, the values of $P_{X,ik}$ and $P_{Y,jk}$ depend only on whether $i = k$ and $j = k$, with

$$P_{X,ij} = \begin{cases} 0.6 & i = j \\ 0.2 & i \neq j, \end{cases} \qquad P_{Y,ij} = \begin{cases} 0 & i = j \\ 0.5 & i \neq j \end{cases}$$

Thus, the inner summation over k always has the same value, with $k = i \neq j$ in one term, $k = j \neq i$, in another term, and $k \neq i$, $k \neq j$ in the remaining term. Hence,

$$\begin{aligned}
a &= \sum_{i \neq j} \frac{P(X_n = i, Y_n = j)}{P(X_n \neq Y_n)} [0.6 \cdot 0.5 + 0.2 \cdot 0 + 0.2 \cdot 0.5] \\
&= \frac{0.4}{P(X_n \neq Y_n)} \sum_{i \neq j} P(X_n = i, Y_n = j) = 0.4 \cdot \frac{P(X_n \neq Y_n)}{P(X_n \neq Y_n)} \\
&= 0.4.
\end{aligned}$$

Now we can compute $E[T]$. Since $I_0 = 1$, we have

$$E[T] = \sum_{n=0}^{\infty} E[I_n] = \sum_{n=0}^{\infty} P(I_n = 1) = \sum_{n=0}^{\infty} P_{11}^n.$$

By the Chapman-Kolmogorov equation, $P_{11}^n = P_{10}^{n-1} P_{01} + P_{11}^{n-1} P_{11}$. Since $P_{01} = 0$, it follows that $P_{11}^n = (P_{11})^n = (1-a)^n$. Hence,

$$E[T] = \sum_{n=0}^{\infty} (1-a)^n = \frac{1}{a} = \frac{1}{0.4} = 2.5.$$

(b) We perform essentially the same calculation as in (a) to find $E[T]$ in terms of p and q. In fact, everything is the same up to equation $(*)$ from part (a). At that point, we note that under our new assumptions,

$$P_{X,ij} = \begin{cases} 1 - 2p & i = j \\ p & i \neq j, \end{cases} \qquad P_{Y,ij} = \begin{cases} 1 - 2q & i = j \\ q & i \neq j \end{cases},$$

which gives

$$\begin{aligned}
a &= \sum_{i \neq j} \frac{P(X_n = i, Y_n = i)}{P(X_n \neq Y_n)} [(1-2p)q + (1-2q)p + pq] \\
&= p + q - 3pq.
\end{aligned}$$

Replacing this value for a in the calculation of $E[T]$ from part (a) we get

$$E[T] = \frac{1}{a} = \frac{1}{p + q - 3pq}.$$

Suppose $q = 0.5$, then $E[T] = g(p) = \frac{2}{1-p}$. The derivative of g is $g'(p) = \frac{2}{(1-p)^2}$. Hence, $g' > 0$ for all $p \in [0, 0.5]$, which is the range of possible p values that Xavier could use. It follows that $E[T] = g(p)$ is maximal for $p = 0$ or $p = 0.5$. Since $g(0) = 1$, and $g(0.5) = 4$, Xavier should use $p = 0.5$ to maximize the expected amount of time before he gets caught.

If $q = \frac{1}{3}$, then $E[T] = \frac{1}{p+\frac{1}{3}-p} = 3$ independent of p, so any value of p that Xavier uses will trivially maximize $E[T]$.

Problem 5.

```python
import numpy as np

# Define state space, and construct index
s = [1, 2, 3, 4]
s_index = {s_i: i for i, s_i in enumerate(s)}

# Define transition matrix
p = np.array([
    [0.2, 0.3, 0, 0.5],
    [0.1, 0.6, 0.1, 0.2],
    [0.45, 0.55, 0, 0],
    [0, 0, 0.9, 0.1]
])

# Define initial distribution
pi_0 = np.array([0.2, 0.25, 0.25, 0.3])

# Set end time
end_time = 5


# Define simulation increment function
def step(x):
    """Simulate next value X_{n+1} given current X_n"""
    return np.random.choice(s, p=p[s_index[x]])


# Simulate Markov chain
x_n = np.random.choice(s, p=pi_0)
for n in range(end_time + 1):
    print(f'X_{n} = {x_n}')
    x_n = step(x_n)
```

6. Textbook: 14, p. 285

(a) For P_1, there is only one class consisting of all states: $\{0, 1, 2\}$. This class must be recurrent.

(b) For P_2, we have $0 \to 1 \to 2$, and $2 \to 0$, and $2 \to 1$. Thus, $\{0, 1, 2\}$ is a class. Since 3 is not accessible from any state other than 3, the other class is $\{3\}$. The state 3 will always be left with probability 1, so the class $\{3\}$ is transient. This means that $\{0, 1, 2\}$ must be recurrent.

(c) For P_3, we have $0 \to 1 \to 2 \to 0$, but none of $0, 1, 2$ are accessible from 3 or 4, so $\{0, 1, 2\}$ is a class. Additionally, we have $3 \to 4$, and $4 \to 3$, so $\{3, 4\}$ is the other class. Since neither class is accessible from the other, they must both be recurrent.

(d) For P_4, we have $0 \to 1$, and $1 \to 0$, but no other state is accessible from 0 or 1, so $\{0, 1\}$ is a class. In addition, states 3 and 4 are not accessible from any different states, so they must be in classes of their own, which leaves 2 in a class of its own. Thus, the classes are $\{0, 1\}$, $\{2\}$, $\{3\}$, and $\{4\}$. Since there is a nonzero probability of the chain leaving states 3 and 4, the classes $\{3\}$ and $\{4\}$ must be transient. The probability of staying in state 2 given that the chain starts in state 2 is 1, so $\{2\}$ is recurrent. Lastly, the class $\{0, 1\}$ is recurrent because the probability that the chain leaves the class is 0.

7. Textbook: 21, p. 286

(a) Since the chain is symmetric with respect to permutation of the states, it follows that the transition probability P_{ij}^n depends only on whether i and j are the same. In other words,

$$P_{ij}^n = \begin{cases} d_n & i = j \\ f_n & i \neq j. \end{cases}$$

On the other hand, for any fixed i, we must have $\sum_{j=1}^4 P_{ij}^n = 1$. It follows that $f_n = \frac{1-d_n}{3}$. Note that $d_1 = P_{11} = 1 - 3\alpha$, and $f_1 = P_{12} = \alpha$. By the Chapman-Kolmogorov equation, for $n \geq 0$,

$$\begin{aligned} d_{n+1} &= P_{11}^{n+1} \\ &= \sum_{k=1}^4 P_{1k}^n P_{k1} \\ &= P_{11}^n(1 - 3\alpha) + \alpha \sum_{k=2}^4 P_{k1}^n \\ &= d_n(1 - 3\alpha) + 3\alpha f_n = d_n(1 - 3\alpha) + \alpha(1 - d_n) \\ &= \alpha + (1 - 4\alpha)d_n. \end{aligned}$$

A particular solution of this nonhomogeneous, linear recurrence relation is $d_n = \frac{1}{4}$, as $\frac{1}{4} = \alpha + (1-4\alpha)\frac{1}{4}$. The general solution of the homogeneous equation $d_{n+1} = (1 - 4\alpha)d_n$ is $d_n = c(1 - 4\alpha)^n$, where c is a constant. Thus, the solution of the nonhomogeneous equation is

$$d_n = \frac{1}{4} + c(1 - 4\alpha)^n.$$

Since $d_0 = P_{11}^0 = 1$, we must have $c = \frac{3}{4}$, giving

$$d_n = \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^n.$$

(b) Let $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix}$ be the long-run proportions of time the chain is in each state. Then we must have

$$\pi P = \pi \implies \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix} \begin{bmatrix} 1 - 3\alpha & \alpha & \alpha & \alpha \\ \alpha & 1 - 3\alpha & \alpha & \alpha \\ \alpha & \alpha & 1 - 3\alpha & \alpha \\ \alpha & \alpha & \alpha & 1 - 3\alpha \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix}.$$

From this we obtain

$$1 = \pi_1 + \pi_2 + \pi_3 + \pi_4 = 4\pi_j, \qquad j = 1, 2, 3, 4.$$

Thus, $\pi_1 = \pi_2 = \pi_3 = \pi_4 = \frac{1}{4}$; that is, in the long-run, the chain is in each state an equal proportion of the time.

8. Textbook: 23, p. 287

Let X_n be a random variable indicating whether year n was good or bad, with $X_n = 1$ meaning good weather, and $X_n = 2$ meaning bad weather. Then $\{X_n : n \geq 0\}$ is a Markov chain because of the assumption that the weather condition depends only on the previous year's condition. Based on the description of the problem, the chain has the following transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix},$$

with the first row and column corresponding to good weather (state 1), and the second row and column to bad weather (state 2). Let S_n denote the number of storms in year n.

(a) We want to compute $E[S_1 + S_2]$. By the law of iterated expectation,

$$E[S_n] = E[E[S_n \mid X_n]] = E[S_n \mid X_n = 1]P(X_n = 1) + E[S_n \mid X_n = 2]P(X_n = 2).$$

We are starting in state 1, so $P(X_n = 1) = P_{11}^n$, and $P(X_n = 2) = P_{12}^n$. We see immediately that $P_{11}^1 = \frac{1}{2}$, and $P_{12}^1 = \frac{1}{2}$. We use the Chapman-Kolmogorov equation to compute P_{11}^2 and P_{12}^2:

$$\begin{aligned} P_{11}^2 &= P_{11}^1 P_{11}^1 + P_{12}^1 P_{21}^1 = \frac{5}{12} \\ P_{12}^2 &= P_{11}^1 P_{12}^1 + P_{12}^1 P_{22}^1 = \frac{7}{12}. \end{aligned}$$

Thus,

$$\begin{aligned} E[S_1 + S_2] &= E[S_1 \mid X_1 = 1]P_{11}^1 + E[S_1 \mid X_1 = 2]P_{12}^1 + E[S_2 \mid X_2 = 1]P_{11}^2 + E[S_2 \mid X_2 = 2]P_{12}^2 \\ &= \frac{1}{2} + \frac{3}{2} + \frac{5}{12} + \frac{21}{12} = \frac{25}{6}. \end{aligned}$$

(b) Using the law of total probability, the p.m.f. of the Poisson distribution, and the fact that the chain starts in state 1, we have

$$\begin{aligned} P(S_3 = 0) &= P(S_3 = 0 \mid X_3 = 1)P(X_3 = 1) + P(S_3 = 0 \mid X_3 = 2)P(X_3 = 2) \\ &= e^{-1}P(X_3 = 1) + e^{-3}P(X_3 = 2) \\ &= e^{-1}P_{11}^3 + e^{-3}P_{12}^3. \end{aligned}$$

A simple computation shows that

$$P^3 = \frac{1}{216}\begin{bmatrix} 87 & 129 \\ 86 & 130 \end{bmatrix}.$$

Thus,

$$P(S_3 = 0) = e^{-1}\frac{87}{216} + e^{-3}\frac{129}{216} \approx 0.178.$$

(c) To determine the long-run average number of storms per year, we need the probabilities π_1 and π_2 that the chain is in states 1 and 2 in the long-run. Then the long-run average number of storms S is given by

$$S = E[S_n \mid X_n = 1]\pi_1 + E[S_n \mid X_n = 2]\pi_2.$$

To find $\pi = \begin{bmatrix}\pi_1 & \pi_2\end{bmatrix}$, we must solve

$$\pi P = \pi \quad \text{or} \quad \begin{bmatrix}\pi_1 & \pi_2\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3}\end{bmatrix} = \begin{bmatrix}\pi_1 & \pi_2\end{bmatrix}.$$

This gives us the equation $3\pi_1 + 2\pi_2 = 6\pi_1$. Together with the fact that $\pi_1 + \pi_2 = 1$, this implies that $\pi_1 = \frac{2}{5}$, and $\pi_2 = \frac{3}{5}$. Thus,

$$S = \frac{2}{5} + 3 \cdot \frac{3}{5} = \frac{11}{5} = 2.2$$

is the long-run average number of storms per year.

(d) The proportion of years that have no storms is the probability of $S_n = 0$ in the long-run. That is, the proportion of years without storms p is

$$p = P(S_n = 0 \mid X_n = 1)\pi_1 + P(S_n = 0 \mid X_n)\pi_2 = e^{-1}\frac{2}{5} + e^{-3}\frac{3}{5} \approx 0.177.$$

9. Textbook: 31, p. 288

The weather can be described by a Markov chain $\{X_n : n \geq 0\}$ with state space $\{1, 2, 3\}$, where $X_n = 1$ means that day n is sunny, $X_n = 2$ means that day n is cloudy, and $X_n = 3$ means that day n is rainy. The transition matrix is then

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}.$$

To find the proportion of days with given weather in the long run, we need to find $\pi = \begin{bmatrix}\pi_1 & \pi_2 & \pi_3\end{bmatrix}$ such that $\pi P = \pi$. Since

$$\begin{aligned}\pi P = \pi &\implies \begin{bmatrix}\pi_1 & \pi_2 & \pi_3\end{bmatrix}\begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}\begin{bmatrix}\pi_1 & \pi_2 & \pi_3\end{bmatrix} \\ &\implies \frac{1}{4}\pi_2 + \frac{1}{4}\pi_3 = \pi_1, \qquad \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 = \pi_2.\end{aligned}$$

Since we also have $\pi_1 + \pi_2 + \pi_3 = 1$, it follows that $5\pi_1 = \pi_1 + \pi_2 + \pi_3$, so $\pi_1 = \frac{1}{5}$. Furthermore, $2\pi_1 + 2\pi_2 + \pi_3 = 4\pi_2$, so $2 - \pi_3 = 4\pi_2$, meaning $\pi_3 = 2 - 4\pi_2$. Hence,

$$1 = \frac{1}{5} + \pi_2 + 2 - 4\pi_2 \implies -\frac{6}{5} = -3\pi_2 \implies \pi_2 = \frac{2}{5}.$$

This leaves $\pi_3 = \frac{2}{5}$. Therefore, in the long run $\frac{1}{5}$ of days are sunny, and $\frac{2}{5}$ are cloudy (or $\frac{4}{5}$, if cloudy means cloudy with or without rain).

10. Textbook: 49, p. 292

(a) We need to find P_{12}^2. Using the Chapman-Kolmogorov equation, we have

$$P_{12}^2 = P_{11}P_{12} + P_{12}P_{22} + P_{13}P_{32} = 0.15 + 0.12 + 0 = 0.27.$$

(b) Let $\pi = \begin{bmatrix}\pi_1 & \pi_2 & \pi_3\end{bmatrix}$ be the long-run proportion of the time the chain spends in each state. Then the long-run average reward per unit time R is

$$R = \pi_1 \cdot 1^2 + \pi_2 \cdot 2^2 + \pi_3 \cdot 3^2.$$

To find π, we solve $\pi P = \pi$:

$$\begin{aligned}
\pi P = \pi &\implies \begin{bmatrix}\pi_1 & \pi_2 & \pi_3\end{bmatrix}\begin{bmatrix}0.5 & 0.3 & 0.2\\ 0 & 0.4 & 0.6\\ 0.8 & 0 & 0.2\end{bmatrix} = \begin{bmatrix}\pi_1 & \pi_2 & \pi_3\end{bmatrix}\\
&\implies 0.5\pi_1 + 0.8\pi_3 = \pi_1, \quad 0.3\pi_1 + 0.4\pi_2 = \pi_2\\
&\implies \pi_3 = \frac{5}{8}\pi_1, \quad \pi_2 = \frac{1}{2}\pi_1.
\end{aligned}$$

Since we must also have $\pi_1 + \pi_2 + \pi_3 = 1$, it follows that $8\pi_1 + 4\pi_1 + 5\pi_1 = 8$, so $\pi_1 = \frac{8}{17}$. Then $\pi_2 = \frac{4}{17}$, and $\pi_3 = \frac{5}{17}$. This allows us to compute R:

$$R = \frac{8}{17} + \frac{16}{17} + \frac{45}{17} = \frac{69}{17}.$$

(c) Due to the Markov property, the number of transitions to reach state 3 after the first one given that the chain does not go to state 3 on the first step is distributed the same as the number of transitions to reach state 3 given that the chain starts in state 3. Thus,

$$E[N_1 \mid X_1 = 1] = 1 + E[N_1], \quad E[N_1 \mid X_1 = 2] = 1 + E[N_2], \quad E[N_1 \mid X_1 = 3] = 1.$$

Similarly,

$$E[N_2 \mid X_1 = 1] = 1 + E[N_1], \quad E[N_2 \mid X_2 = 2] = 1 + E[N_2], \quad E[N_2 \mid X_1 = 3] = 1.$$

By the law of iterated expectation, we must have

$$\begin{aligned}
E[N_1] = E[N_1 \mid X_1 = 1]P(X_1 = 1 \mid X_0 = 1) + E[N_1 \mid X_1 = 2]P(X_1 = 1 \mid X_0 = 1)\\
+ E[N_1 \mid X_1 = 3]P(X_1 = 3 \mid X_0 = 1),\\
E[N_2] = E[N_2 \mid X_1 = 1]P(X_1 = 1 \mid X_0 = 2) + E[N_2 \mid X_1 = 2]P(X_1 = 1 \mid X_0 = 2)\\
+ E[N_2 \mid X_1 = 3]P(X_1 = 3 \mid X_0 = 2),
\end{aligned}$$

so

$$\begin{aligned}
E[N_1] = (1 + E[N_1])P_{11} + (1 + E[N_2])P_{12} + P_{13},\\
E[N_2] = (1 + E[N_1])P_{21} + (1 + E[N_2])P_{22} + P_{23}.
\end{aligned}$$

This implies that

$$E[N_2] = \frac{(1 + E[N_1])P_{21} + P_{22} + P_{23}}{1 - P_{22}},$$

so

$$E[N_1]\left(1 - P_{11} - \frac{P_{21}P_{12}}{1 - P_{22}}\right) = P_{11} + P_{12} + \frac{P_{12}(P_{21} + P_{22} + P_{23})}{1 - P_{22}} + P_{13}.$$

Solving for $E[N_1]$ gives

$$E[N_1] = \frac{1 - P_{22} + P_{12}}{(1 - P_{11})(1 - P_{22}) - P_{21}P_{12}}.$$

Substituting the known values from the transition matrix P gives

$$E[N_1] = \frac{1 - 0.4 + 0.3}{(1 - 0.5)(1 - 0.4) - 0 \cdot 0.3} = \frac{0.9}{0.3} = 3.$$

(d) Let $a_n = P(N_1 \le n)$, and let $b_n = P(N_2 \le n)$. By the same logic we used in (c), it follows that for $n \ge 1$,

$$P(N_1 \le n+1 \mid X_1 = 1) = P(N_1 \le n), \qquad P(N_1 \le n+1 \mid X_1 = 2) = P(N_2 \le n),$$

and similarly that

$$P(N_2 \le n+1 \mid X_1 = 1) = P(N_1 \le n), \qquad P(N_2 \le n+1 \mid X_1 = 2) = P(N_2 \le n).$$

Moreover, $P(N_1 \le n+1 \mid X_1 = 3) = 1$, and $P(N_2 \le n+1 \mid X_1 = 3) = 1$ for $n \ge 1$. Hence,

$$\begin{aligned} a_{n+1} = P(N_1 \le n+1) &= \sum_{i=1}^{3} P(N_1 \le n+1 \mid X_1 = i) P(X_1 = i \mid X_0 = 1) \\ &= P(N_1 \le n) P_{11} + P(N_2 \le n) P_{12} + P_{13} \\ &= P_{11} a_n + P_{12} b_n + P_{13}, \end{aligned}$$

and similarly we can obtain

$$\begin{aligned} b_{n+1} = P(N_2 \le n+1) &= \sum_{i=1}^{3} P(N_2 \le n+1 \mid X_1 = i) P(X_1 = i \mid X_0 = 2) \\ &= P(N_1 \le n) P_{21} + P(N_2 \le n) P_{22} + P_{23} \\ &= P_{21} a_n + P_{22} b_n + P_{23}. \end{aligned}$$

In addition, we have $a_1 = P(N_1 \le 1) = P(N_1 = 1) = P(X_1 = 3 \mid X_0 = 1) = P_{13}$, and similarly $b_1 = P_{23}$. Substituting the required values from the transition matrix, we need to solve the linear recurrence relation

$$\begin{cases} a_{n+1} = 0.5a_n + 0.3b_n + 0.2, & \\ b_{n+1} = 0.4b_n + 0.6, & n \ge 1; \\ a_1 = 0.2, \quad b_1 = 0.6. & \end{cases}$$

We start by finding b_n, as it is easier. A general solution of the homogeneous equation is $b_n^h = C(0.4)^n$, where C is a constant. On the other hand, a particular solution of the equation is $b_n^p = 1$, so a general solution of the equation is $b_n = b_n^p + b_n^h = 1 + C(0.4)^n$. To satisfy the initial condition $b_1 = 0.6$, we need $C = -1$, so $b_n = 1 - (0.4)^n$.

Thus, we obtain a modified equation for a_n:

$$\begin{cases} a_{n+1} = 0.5a_n - 0.3 \cdot (0.4)^n + 0.5, \quad n \ge 1; \\ a_1 = 0.2. \end{cases}$$

We note that a general solution of the homogeneous equation is $a_n^h = C(0.5)^n$, where C is a constant. We hypothesize that $a_n^p = A(0.4)^n + B$ is a particular solution of the equation for some constants A and B. In order to make this work, we need

$$a_{n+1}^p = 0.5a_n^p - 0.3 \cdot (0.4)^n + 0.5, \quad n \ge 1,$$

or

$$0.4A(0.4)^n + B = 0.5A(0.4)^n + 0.5B - 0.3 \cdot (0.4)^n + 0.5, \quad n \ge 1.$$

Thus, we see that a_n^p is a solution only if we choose $0.4A = 0.5A - 0.3$ and $B = 0.5B + 0.5$, so that $A = 3$, and $B = 1$. That is, $a_n^p = 3(0.4)^n + 1$ is a particular solution. Then $a_n = a_n^p + a_n^h = 1 + 3(0.4)^n + C(0.5)^n$, for some constant C. To find C, we apply the initial condition, which requires that

$$0.2 = 1 + 1.2 + 0.5C \implies C = -4,$$

so $a_n = 3(0.4)^n - 4 \cdot (0.5)^n + 1$. This gives the value of $P(N_1 \le 4)$ as

$$P(N_1 \le 4) = a_4 = 3 \cdot (0.4)^4 - 4 \cdot (0.5)^4 + 1 = 0.8268.$$

(e) Using the previous part, we can easily calculate $P(N_1 = 4)$ by noting that

$$P(N_1 = 4) = P(N_1 \le 4) - P(N_1 \le 3) = a_4 - a_3.$$

Thus,

$$P(N_1 = 4) = a_4 - a_3 = 0.8268 - 3 \cdot (0.4)^3 + 4 \cdot (0.5)^3 - 1 = 0.1348.$$

11. Textbook: 60, p. 292

(a) Let A denote the event that state 3 is entered before state 4. We want to calculate $P(A \mid X_0 = 1)$. Conditioning further on the first transition, we have

$$\begin{aligned} P(A \mid X_0 = 1) &= \sum_{i=1}^{4} P(A \mid X_0 = 1,\, X_1 = i) P(X_1 = i \mid X_0 = 1) \\ &= \sum_{i=1}^{2} P(A \mid X_0 = 1,\, X_1 = i) P_{1i} + 1P_{13} + 0P_{14} \\ &= P(A \mid X_1 = 1)P_{11} + P(A \mid X_1 = 2)P_{12} + P_{13}, \end{aligned}$$

where the last equation follows from the Markov property. The Markov property also implies that $P(A \mid X_0 = 1) = P(A \mid X_1 = 1)$, and $P(A \mid X_0 = 2) = P(A \mid X_1 = 2)$. On the other hand, we can use similar reasoning to obtain

$$P(A \mid X_0 = 2) = P(A \mid X_1 = 1)P_{21} + P(A \mid X_1 = 2)P_{22} + P_{23}.$$

Thus, we obtain the system of equations

$$\begin{aligned} p &= P_{11}p + P_{12}q + P_{13}, \\ q &= P_{21}p + P_{22}q + P_{23}, \end{aligned}$$

where $p = P(A \mid X_0 = 1)$, and $q = P(A \mid X_0 = 2)$. Substituting the values from the transition matrix, we have

$$\begin{aligned} p &= 0.4p + 0.3q + 0.2, \\ q &= 0.2p + 0.2q + 0.2. \end{aligned}$$

The first equation implies that $q = 2p - \frac{2}{3}$. Thus,

$$2p - \frac{2}{3} = 0.2p + 0.4p - \frac{0.4}{3} + 0.2,$$

which gives

$$4.2p = 2.2 \implies p = \frac{11}{21}.$$

Thus, $P(A \mid X_0 = 1) = p = \frac{11}{21}$.

(b) Let N be the number of transitions before state 3 or state 4 is entered. We want to find $E[N \mid X_0 = 1]$. Conditioning on the first transition, we have

$$\begin{aligned} E[N \mid X_0 = j] &= \sum_{i=1}^{4} E[N \mid X_0 = j,\, X_1 = i] P(X_1 = i \mid X_0 = j) \\ &= \sum_{i=1}^{2} E[N \mid X_0 = j,\, X_1 = i] P_{ji} + 1 \cdot P_{j3} + 1 \cdot P_{j4} \\ &= E[N \mid X_1 = 1]P_{j1} + E[N \mid X_1 = 2]P_{j2} + P_{j3} + P_{j4} \end{aligned}$$

for $j = 1, 2$. Let $\mu = E[N \mid X_0 = 1]$, and let $\nu = E[N \mid X_0 = 2]$. The Markov property implies that $E[N \mid X_1 = j] = E[N \mid X_0 = j] + 1$, for $j = 1, 2$. Then from the above we obtain the system of equations

$$\begin{aligned}\mu &= P_{11}(\mu + 1) + P_{12}(\nu + 1) + P_{13} + P_{14},\\ \nu &= P_{21}(\mu + 1) + P_{22}(\nu + 1) + P_{23} + P_{24}.\end{aligned}$$

Substituting from the transition matrix, we have

$$\begin{aligned}\mu &= 0.4\mu + 0.3\nu + 1\\ \nu &= 0.2\mu + 0.2\nu + 1.\end{aligned}$$

The first equation implies that $\nu = 2\mu - \frac{10}{3}$. Substituting into the second gives

$$2\mu - \frac{10}{3} = 0.2\mu + 0.4\mu - \frac{2}{3} + 1,$$

so

$$4.2\mu = 11 \implies \mu = \frac{110}{42} = \frac{55}{21}.$$

Thus, $E[N \mid X_0 = 1] = \mu = \frac{55}{21}$.