5.1 $X_1, \ldots, X_4$ are i.i.d., $E[X_i] = 5$, $SD[X_i] = 3$.

(a) $E[X_1 + 2X_2 + X_3 - X_4] = E[X_1] + 2E[X_2] + E[X_3] - E[X_4]$
$= 5 + 10 + 5 - 5 = 15$

(b) $Var[X_1 + 2X_2 + X_3 - X_4] = Var[X_1] + 4Var[X_2] + Var[X_3] + Var[X_4]$
$= 7Var[X_i] = 7\,SD[X_i]^2 = 7 \cdot 9 = 63$

5.5 X, Y continuous with joint pdf

$$f(x,y) = \begin{cases} x+y & 0<x<1 \text{ and } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

(a) $E[X] = \int_{\mathbb{R}^2} x f(x,y)\,dx\,dy = \int_0^1\int_0^1 x(x+y)\,dx\,dy = \int_0^1 \frac{x^3}{3} + \frac{x^2 y}{2}\Big|_0^1 dy$
$= \int_0^1 \left(\frac{1}{3} + \frac{1}{2}y\right)dy = \frac{1}{3}y + \frac{1}{4}y^2\Big|_0^1 = \frac{7}{12}$

(b) $E[X+Y] = E[X] + E[Y]$
$E[Y] = \int_{\mathbb{R}^2} y f(x,y)\,dx\,dy = \int_0^1\int_0^1 y(x+y)\,dx\,dy = \frac{7}{12}$ by symmetry

so $E[X+Y] = \frac{14}{12} = \frac{7}{6}$

(c) $E[XY] = \int_{\mathbb{R}^2} xy f(x,y)\,dx\,dy = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \int_0^1 y\left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_0^1 dy$
$= \int_0^1 \left(\frac{1}{3}y + \frac{1}{2}y^2\right)dy = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

(d) $Cov(X,Y) = E[XY] - E[X]E[Y]$
$= \frac{1}{3} - \left(\frac{7}{12}\right)^2 = -\frac{1}{144}$

So: $Cov(2X, 3Y) = 6\,Cov(X,Y) = -\frac{1}{24}$

(e) $E[Y|X=x] = \int y f_{Y|X=x}(y|x)\,dy$

$f_{Y|X}(y|x) = \frac{f(x,y)}{f(x)}$, for $0<x<1$, $f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (x+y)\,dy = yx + \frac{y^2}{2}\Big|_0^1$
$= x + \frac{1}{2}$, $0<x<1$

$f_{Y|X}(y|x) = \frac{x+y}{x+\frac{1}{2}}$ $0<x<1,\ 0<y<1$, So $E[Y|X=x] = 0$ if not $0<x<1$,

otherwise
$$E[Y|X=x] = \int_0^1 y \frac{x+y}{x+\frac{1}{2}}\,dy = \frac{1}{x+\frac{1}{2}}\left[\frac{y^2}{2}x + \frac{y^3}{3}\right]_0^1$$
$$= \frac{\frac{x}{2}+\frac{1}{3}}{x+\frac{1}{2}}$$

5.6 (a) $\mathrm{Cov}(X \pm Y, Z) = E[(X \pm Y)Z] - E[X \pm Y]E[Z]$
$$= E[XZ] \pm E[YZ] - E[X]E[Z] \mp E[Y]E[Z]$$
$$= E[XZ] - E[X]E[Z] \pm (E[YZ] - E[Y]E[Z])$$
$$= \mathrm{Cov}(X, Z) \pm \mathrm{Cov}(Y, Z)$$

(b) $\mathrm{Cov}(X+Y, Z+W) = \mathrm{Cov}(X, Z+W) + \mathrm{Cov}(Y, Z+W)$

by (a)
$$= \mathrm{Cov}(Z+W, X) + \mathrm{Cov}(Z+W, Y)$$ by symmetry of Cov
$$= \mathrm{Cov}(Z, X) + \mathrm{Cov}(W, X) + \mathrm{Cov}(Z, Y) + \mathrm{Cov}(W, Y)$$

by (a) again
$$= \mathrm{Cov}(X, Z) + \mathrm{Cov}(X, W) + \mathrm{Cov}(Y, Z) + \mathrm{Cov}(Y, W)$$

by symmetry again

(c) $\mathrm{Cov}(X+Y, X-Y) = \mathrm{Cov}(X, X) + \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, -Y)$
$$+ \mathrm{Cov}(Y, -Y)$$

For any A, B
$\mathrm{Cov}(A, -B) = E[A(-B)] - E[A]E[-B] = -E[AB] + E[A]E[B] = -\mathrm{Cov}(A, B)$

So $\mathrm{Cov}(X+Y, X-Y) = \mathrm{Cov}(X, X) - \mathrm{Cov}(Y, Y)$
$$= \mathrm{Var}[X] - \mathrm{Var}[Y]$$

5.12 Let X & Y have joint pdf

$$f(x,y) = \begin{cases} 4x(1-y) & 0<x<1,\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

(a) $$E[X^2Y] = \int_0^1\int_0^1 x^2 y\, 4x(1-y)\,dx\,dy = \int_0^1 x^4(y-y^2)\Big|_0^1 dy$$

$$= \frac{y^2}{2} - \frac{y^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

(b) $$E[X-Y) = \int_0^1\int_0^1 (4x^2(1-y) - 4xy(1-y))\,dx\,dy = \int_0^1 [\tfrac{4}{3}x^3(1-y) - 2x^2(y-y^2$$

$$= \left(\frac{4}{3}\left(y - \frac{y^2}{2}\right) - 2\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\right)\Big|_0^1$$

$$= \frac{2}{3} - 2\left(\frac{1}{6}\right) = \frac{1}{3}$$

(c) $$f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 4x(1-y)\,dy = -4x\left(\frac{(1-y)^2}{2}\right)\Big|_0^1 = 2x, \quad 0<x<1$$

$$f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 4x(1-y)\,dy = 2x^2(1-y)\Big|_0^1 = 2(1-y), \quad 0<y<1$$

So $f(x,y) = (2x)(2(1-y)) = f_X(x)f_Y(y) \Rightarrow$ X & Y are independ
$\quad 0<x<1 \quad 0<y<1$

and thus $\mathrm{Cov}(X,Y) = 0 \Rightarrow \mathrm{Var}[X-Y] = \mathrm{Var}[X] + \mathrm{var}[Y]$

$$E[X] = \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}, \quad E[Y] = \int_0^1 2y(1-y)\,dy = 2\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$

$$E[X^2] = \int_0^1 2x^3\,dx = \frac{1}{2}x^4\Big|_0^1 = \frac{1}{2}, \quad E[Y^2] = \int_0^1 (2y^2 - 2y^3)\,dy = \frac{2}{3}y^3 - \frac{1}{2}y^4\Big|_0^1 = \frac{1}{6}$$

So $\mathrm{Var}[X] = E[X^2] - E[X]^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$

$\mathrm{var}[Y] = E[Y^2] - E[Y]^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$

So $\mathrm{var}[X-Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] = \frac{1}{9}$

(d) Since X & Y are independent,

$\text{corr}(X,Y) = 0$

(e) $E[Y|X=x] = E[Y]$ because X & Y are independent,

So $E[Y|X=x] = \frac{1}{3}$ by (c)

5.17 Let $Y \sim \text{poisson}(X)$, $X \sim \text{Exp}(1)$

(a) $E[Y] = E[E[Y|X]]$

$= E[X] = 1 = 1$

(b) $\text{Var}\{Y\} = \text{Var}[E[Y|X]] + E[\text{Var}[Y|X]]$

$= \text{Var}[X] + E[X]$

expected value of poisson(X)

variance of poisson(X)

$= 1^2 + 1 = 2$

5.19 $N \sim \text{Bin}(6, 0.8)$, $X \sim \text{Bin}(N, 0.3)$

(a) $E[X] = E[E[X|N]] = E[0.3N] = 0.3E[N]$

$= 0.3 \cdot 0.8 \cdot 6 = 1.44$

(b) $\text{Var}[X] = \text{Var}[E[X|N]] + E[\text{Var}[X|N]]$

$= \text{Var}[0.3N] + E[0.3 \cdot 0.7 \cdot N]$

$= 0.3^2 \text{Var}[N] + 0.3 \cdot 0.7 \cdot 0.8 \cdot 6$

$= 0.3^2 \cdot 0.8 \cdot 0.2 \cdot 6 + 0.3 \cdot 0.7 \cdot 0.8 \cdot 6$

$= 1.0944$

(c) $E[X^2] = \text{Var}[X] + E[X]^2 = 1.0944 + 1.44^2 = 3.168$

5.20 $X \sim \text{Poisson}(3)$, $E[Y|X] = \frac{X}{2}$, $Var(Y|X) = \frac{X+1}{3}$

(a) $E[Y] = E[E[Y|X]] = E[\frac{X}{2}] = \frac{1}{2}E(X) = \frac{3}{2}$

(b) $Var[Y] = Var[E[Y|X]] + E[Var(Y|X)]$

$= Var[\frac{X}{2}] + E[\frac{X+1}{3}]$

$= \frac{1}{4} \cdot 3 + \frac{3+1}{3} = \frac{25}{12}$

(c) $E[X \cdot Y] = E[E[XY|X]] = E[X E(Y|X)]$

$= E[X \cdot X/2] = \frac{1}{2} E[X^2]$

$= \frac{1}{2}[Var[X] + E[X]^2]$

$= \frac{1}{2}(3+9) = 6$