# Stat 5643 Homework 2

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## 1.22

Let $R_1$ be the event that the track star wins the first race, $R_2$, the event that he wins the second. Then $P(R_1) = 0.7$, $P(R_2) = 0.6$, and $P(R_1 \cap R_2) = 0.5$.

**(a)** The event that he wins at least one race is the same as the event that he wins *either* the first *or* the second, or $R_1 \cup R_2$.

$$\begin{aligned} P(R_1 \cup R_2) &= P(R_1) + P(R_2) - P(R_1 \cap R_2) && \text{by Theorem 1.4.3} \\ &= 0.7 + 0.6 - 0.5 = 0.8. \end{aligned}$$

**(b)** The event that he wins exactly one race is the same as the event that he wins race one and not race two *or* wins race two and not race one, or $(R_1 \cap R_2{}') \cup (R_1{}' \cap R_2)$. Notice that the events $R_1 \cap R_2{}'$ and $R_1{}' \cap R_2$ are mutually exclusive:

$$\left(R_1 \cap R_2{}'\right) \cap \left(R_1{}' \cap R_2\right) = R_1 \cap R_1{}' \cap R_2 \cap R_2{}' = \emptyset \cap \emptyset = \emptyset$$

because the intersection of any event with its complement is empty. Therefore we can apply the third axiom of probability to conclude that

$$\begin{aligned} P\left(\left(R_1 \cap R_2{}'\right) \cup \left(R_1{}' \cap R_2\right)\right) &= P\left(R_1 \cap R_2{}'\right) + P\left(R_1{}' \cap R_2\right) \\ &= P(R_1) - P(R_1 \cap R_2) + P(R_2) - P(R_1 \cap R_2) && \text{by 1.18(a)} \\ &= 0.7 - 0.5 + 0.6 - 0.5 = 0.3. \end{aligned}$$

**(c)** The event that he wins neither race is the same as the event that he doesn't win the first race *and* he doesn't win the second race, or $R_1{}' \cap R_2{}'$.

$$\begin{aligned} P\left(R_1{}' \cap R_2{}'\right) &= P\left((R_1 \cup R_2)'\right) && \text{by De Morgan's laws} \\ &= 1 - P(R_1 \cup R_2) = 1 - 0.8 = 0.2. \end{aligned}$$

**1.35**

Let $M_1$, $M_2$, and $M_3$ be the events that the randomly selected bolt was produced by machines 1, 2, and 3. Let $D$ be the event that the randomly selected bolt is defective. Then, assuming each bolt is equally likely to be selected, $P(M_1) = 0.2$, $P(M_2) = 0.3$, and $P(M_3) = 0.5$. Furthermore, $P(D \mid M_1) = 0.05$, $P(D \mid M_2) = 0.03$, and $P(D \mid M_3) = 0.02$.

**(a)** A bolt is presumably produced by exactly one of machines 1, 2, and 3. This means, firstly, that $M_1$, $M_2$, and $M_3$ are mutually exclusive and, secondly, that every outcome belongs to one of $M_1$, $M_2$, or $M_3$: that is, $S \subseteq M_1 \cup M_2 \cup M_3$. On the other hand, $M_1 \cup M_2 \cup M_3 \subseteq S$. Therefore, $M_1 \cup M_2 \cup M_3 = S$, and

$$\begin{aligned} D &= D \cap S \\ &= D \cap (M_1 \cup M_2 \cup M_3) \\ &= (D \cap M_1) \cup (D \cap M_2) \cup (D \cap M_3). \end{aligned} \tag{1}$$

Since the events $M_1$, $M_2$, and $M_3$ are mutually exclusive, so, too, are the events $D \cap M_1$, $D \cap M_2$, and $D \cap M_3$. Applying the third axiom of probability to (1), then, we get

$$\begin{aligned} P(D) &= P((D \cap M_1) \cup (D \cap M_2) \cup (D \cap M_3)) \\ &= P(D \cap M_1) + P(D \cap M_2) + P(D \cap M_3) \\ &= P(D \mid M_1) P(M_1) + P(D \mid M_2) P(M_2) + P(D \mid M_3) P(M_3) && \text{by definition of conditional probability} \\ &= 0.05 \cdot 0.2 + 0.03 \cdot 0.3 + 0.02 \cdot 0.2 = 0.023. \end{aligned}$$

**(b)** The probability being asked for here is the conditional probability $P(M_1 \mid D)$. By Bayes' law

$$P(M_1 \mid D) = \frac{P(D \mid M_1) P(M_1)}{P(D)}.$$

Substituting $P(D)$ from part (a), we get

$$P(M_1 \mid D) = \frac{0.05 \cdot 0.2}{0.023} \approx 0.435.$$

## 1.37

Let $P(A) = 0.4$ and $P(A \cup B) = 0.6$.

**(a)** If $A$ and $B$ are mutually exclusive, then by definition $P(A \cap B) = 0$, and we can find $P(B)$ by Theorem 1.4.3:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ 0.6 &= 0.4 + P(B) - 0 \\ P(B) &= 0.2. \end{aligned}$$

**(b)** If $A$ and $B$ are independent, then by definition $P(A \cap B) = P(A)P(B)$, and we can find $P(B)$ by Theorem 1.4.3:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ 0.6 &= 0.4 + P(B) - 0.4P(B) \\ 0.2 &= P(B)(1 - 0.4) \\ P(B) &= \frac{1}{3}. \end{aligned}$$

## 1.41

Let $L_1$ and $L_2$ be the events that the parallel components in the left side of the diagram, from top to bottom, fail, and let $R_1$, $R_2$, and $R_3$ be the events that the parallel components in the right side of the diagram, from top to bottom, fail. Then $P(L_1) = 0.1$, $P(L_2) = 0.2$, $P(R_1) = 0.1$, $P(R_2) = 0.2$, and $P(R_3) = 0.3$. Reading the diagram, the event that the system malfunctions, $M$, is given by

$$M = L \cup R, \qquad L = L_1 \cap L_2 \text{ and } R = R_1 \cap R_2 \cap R_3.$$

Since all malfunctions occur independently, we can easily compute $P(L)$, $P(R)$, and $P(L \cap R)$

$$P(L) = P(L_1 \cap L_2) = P(L_1) P(L_2) = 0.1 \cdot 0.2 = 0.02$$

$$\begin{aligned} P(R) &= P(R_1 \cap R_2 \cap R_3) = P(R_1) P(R_2) P(R_3) \\ &= 0.1 \cdot 0.2 \cdot 0.3 = 0.006 \end{aligned}$$

$$\begin{aligned} P(L \cap R) &= P(L_1 \cap L_2 \cap R_1 \cap R_2 \cap R_3) \\ &= P(L_1) P(L_2) P(R_1) P(R_2) P(R_3) \\ &= 0.1 \cdot 0.2 \cdot 0.1 \cdot 0.2 \cdot 0.3 = 0.00012. \end{aligned}$$

By Theorem 1.4.3, we can compute

$$\begin{aligned} P(M) &= P(L \cup R) = P(L) + P(R) - P(L \cap R) \\ &= 0.02 + 0.006 - 0.00012 = 0.02588. \end{aligned}$$

Therefore, the probability that the system does not malfunction is

$$P\left(M'\right) = 1 - P(M) = 1 - 0.02588 = 0.97412.$$

**1.54**

A club consists of 17 men and 13 women, and a committee of five members must be chosen.

**(a)** There are $30 = 17 + 13$ choices of members. Assuming that a committee is uniquely determined by its members, we want to choose 5 from among the 30 *without* repitition, and order *does not* matter. Thus, we see that combinations are the correct way to count the number of committees

$$\begin{aligned}\text{Number of committees } &= \binom{30}{5} = \frac{30!}{25! \cdot 5!}\\ &= \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26}{5 \cdot 4 \cdot 3 \cdot 2} = 142,506.\end{aligned}$$

**(b)** There are two separate tasks: choosing the 3 men, and choosing the 2 women. By the product rule, the total number of committees will be equal to the product of the number of sets of 3 men, $M$, and the number of sets of 2 women, $W$. Selecting a set of 3 men or of 2 women means choosing *without* repitition, and order *does not* matter, so, again, combinations is the correct way to count $M$ and $W$. For $M$ we want to choose 3 out of 17, and for $W$ we want to choose 2 out of 13. Therefore,

$$\begin{aligned}\text{Number of committees } &= MW = \binom{17}{3}\binom{13}{2}\\ &= \frac{17!}{14! \cdot 3!} \cdot \frac{13!}{11! \cdot 2!} = \frac{17 \cdot 16 \cdot 15 \cdot 13 \cdot 12}{3 \cdot 2 \cdot 2}\\ &= 53,040.\end{aligned}$$

**(c)** The above reasoning applies to this case as well, except that if one man must be included, the number of ways to choose the 3 men, $M$, will be equal to the number of ways to choose 2 men out of the 16 men other than the one that must be included, instead of the number of ways to choose 3 out of 17. Again, this choice is *without* repitition, and order *does not* matter, so combinations are still appropriate. Therefore,

$$\begin{aligned}\text{Number of committees } &= MW = \binom{16}{2}\binom{13}{2}\\ &= \frac{16!}{14! \cdot 2!} \cdot \frac{13!}{11! \cdot 2!} = \frac{16 \cdot 15 \cdot 13 \cdot 12}{2 \cdot 2}\\ &= 9,360.\end{aligned}$$

**1.69**

Note that the order in which slips are drawn *does not* matter here because the digits will always be reordered to go from lowest to highest.

**(a)** The task of creating a lottery ticket number is equivalent to choosing 4 items from a set of 9 *without* replacement, where order *does not* matter (as noted above). Therefore, combinations are the appropriate way to count the number of lottery ticket numbers.

$$\text{Number of lottery ticket numbers } = \binom{9}{4} = \frac{9!}{5! \cdot 4!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2} = 126.$$

**(b)** Let $O$ be the number of ways to choose a lottery ticket number with only odd digits, and let $\mathcal{O}$ be the event that a lottery ticket number with only odd digits is chosen. If each lottery ticket number is equally likely to be chosen, then $P(\mathcal{O}) = \frac{O}{126}$. The number of ways to choose a lottery ticket number with only odd digits is equal to the number of ways to choose the 4 ticket number digits out of the 5 potential odd digits *without repitition*, where order *does not* matter, so combinations are the correct way to count $O$. Specifically,

$$O = \binom{5}{4} = \frac{5!}{4! \cdot 1!} = \frac{5}{1} = 5.$$

Therefore, $P(\mathcal{O}) = \frac{O}{126} = \frac{5}{126}$.

**(c)** If order *does* matter, then we need to use permutations to count the total number of ticket numbers instead of combinations. In other words, we want to count the number of permutations of 4 elements taken from 9, or

$$\text{Number of lottery ticket numbers } = {}_9P_4 = \frac{9!}{5!} = 9 \cdot 8 \cdot 7 \cdot 6 = 3,024.$$