Groundwater and Laplace's Equation

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Abstract

In this report, we present differential equations modeling the movement of groundwater flow in a bounded system similar to that of Tóth. Groundwater flow is the movement of water through soil or rocks within the earth's surface. Groundwater flow can be demonstrated through Darcy's law. In solving these we will use Laplace's Equation to help better recreate the flow of groundwater to a discharge zone. As well we will take a deeper dive into the groundwater flow of a system with blockades or boundary conditions in the middle of it. We present examples of how fluid would flow given these extra conditions.

1 Groundwater flow and the Laplace Equation

1.1 Overview

Groundwater flow refers to the movement of fluids beneath the earth's surface through various materials such as soil, sediment, and rock. These fluids traverse different geological layers until reaching discharge zones, where they emerge onto the Earth's surface. Common discharge zones include wells, streams, and springs.

Groundwater flow occurs gradually over extended periods, often taking years or even decades to reach a discharge zone. Measurement of groundwater flow are typically represented in units such as feet per day, with rates varying depending on the characteristics of the rocks or soil it flows through.

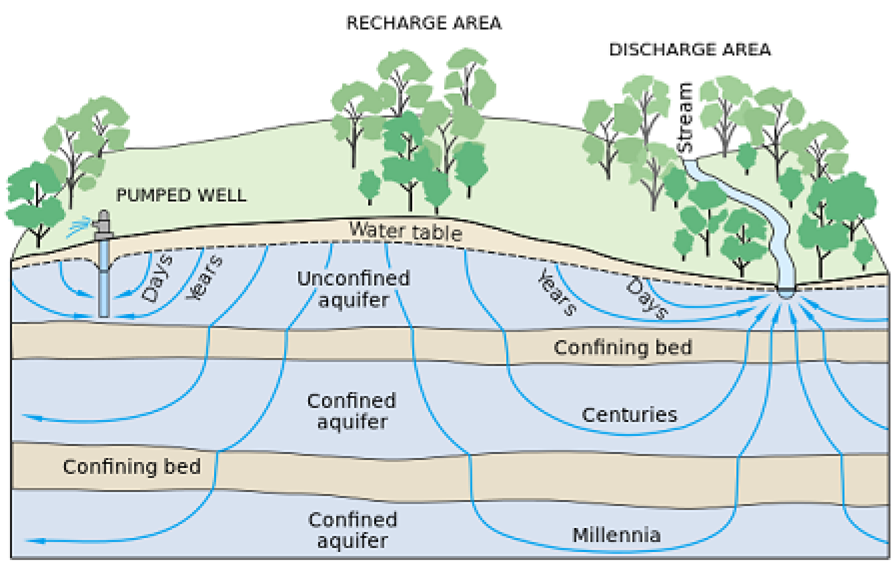


Figure 1: Groundwater flow through aquifers to discharge zones from Friedle [2023].

The materials through which groundwater flows are known as aquifers, which encompass different types of earth formations. There are three primary types of aquifers: unconfined, confined, and perched, each distinguished by its unique hydrogeological properties.

Groundwater flow and aquifers play an important role in the earth's hydrological cycle. Maintaining water balance and supporting ecosystem functions. Groundwater flow with discharge zones such as rivers or lakes influences water quality and aquatic habitats, helping to contribute to increased biodiversity. Groundwater also affects geological processes, having impacts on shaping landscapes through erosion, and sediment transport. Groundwater flow is an important concept to understand when thinking about environmental sustainability and water resource management. (University, Oregon State [2020]).

1.2 Laplace Equation

Laplace's equation is a second-order partial differential equation. The equation states that the sum of the second-order partial derivatives of f (an unknown function) equals zero. The equation is formally known as,

$$\nabla^2 f = 0 \text{ or } \Delta f = 0 \tag{1}$$

but can be written in many different coordinate systems. Three we would like to focus on are 2D Cartesian coordinates (2), 3D Cartesian coordinates (3), and 2D polar coordinates (4).

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{2}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{3}$$

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0 \tag{4}$$

Solutions to Laplace's equation are called harmonic functions. They are all analytical within the domain, or the boundaries, where the equation is satisfied. This is useful when solving because we can take a complicated equation and solve it by summing simpler solutions of Laplace's equation (Wikipedia contributors [2023]).

Laplace's equation can be used to help many different types of problems. Here we will focus on its applications to hydrogeology, in specific connection to understanding groundwater flow. Laplace's equation can provide valuable insights into the behavior of aquifers and the movement of subsurface water resources. Later on in this paper, we'll examine how Laplace's 2D Cartesian coordinate equation can be used to solve and model groundwater flow equations.

2 Understanding Tóth's Paper

Inspiration for our paper came from J. Tóth's paper "A theoretical analysis of groundwater flow in small drainage basins", where Tóth took a deep analysis of the effects of gravity in groundwater circulation. This was a major step forward in groundwater studies as his research had not been done previously. Here we analyze Tóths equations and figures and recreate them ourselves before moving into our adaptations. The key concepts of the groundwater flow investigation is *fluid potential*. Similar to the idea of electron potential, the gradient of fluid potential a steady, imcompressible, and irrotational potential flow (Tóth, J. [1963]).

2.1 Solving Laplace Equation of the Fluid Potential for liquid

To tackle this ground water question, Tóth was trying to solve Laplaces equation to find an expression for the fluid potential, ϕ. Based on Figure 2, the boundary conditions can be treated as the three impermeable planes and the water table. Therefore, the boundary conditions can be written as:

1. $\phi = g(z_0 + cx)$ at $z = z_0$ for $0 \leq x \leq s$
2. $\frac{\partial \phi}{\partial x} = 0$ at $x = 0$ and s for $0 \leq x \leq s$
3. $\frac{\partial \phi}{\partial z} = 0$ at $z = 0$ for $0 \leq x \leq s$

Tóth demonstrates five key equations of (equation 5 - 9) solving the Laplace equation in his paper. Start with these five equations, we provide more detailed steps to show the mathematical strategy of solving a Laplace equation with certain boundary conditions. In Tóth's paper, assuming the flow region is previously filled and the recharge rate is the same as the discharge rate, the Lapace equation of the fluid potential is:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{5}$$

Our first strategy was using *separation of variables* to rewrite the Laplace equation so that we have each variable on each side of the equation by assuming $\phi = f(x)g(z)$.

$$f''(x)g(z) + g''(z)f(x) = 0 \Rightarrow -\frac{f''(x)}{f(x)} = \frac{g''(x)}{g(z)} = \lambda \tag{5.a}$$

Now, the partial differential Laplace equation was converted two ordinary differential equations. Then we used *exponential substitution* to find a solution of both $f(x)$ and $g(z)$, assuming $f(x) = e^{k_1 x}$ and $g(z) = e^{k_2 z}$

$$f''(x) = -f(x) = \lambda \Rightarrow f(x) = A\cos kx + B\sin kx \tag{5.b}$$

$$g''(z) = g(z) = \lambda \Rightarrow g(z) = Ce^{\sqrt{\lambda}z} + De^{-\sqrt{\lambda}z} \tag{5.c}$$

As $\phi = f(x)g(z)$, we reproduce the equation 6 in the Tóth paper. A, B, M, and N are arbitrary coefficient constants.

$$\phi(x, z) = e^{-kz}(A\cos kx + B\sin kx) + e^{kz}(M\cos kx + N\sin kx) \tag{6}$$

When solving the two ordinary differential equations, we arbitrarily assigned $\frac{f''(x)}{f(x)} = -\lambda$ with negative sign due to the boundary condition. As $\frac{\partial\phi}{\partial x} = 0$ at $x = 0$ and s for $0 \le x \le s$, the function of x has harmonic behavior. Therefore, we assigned the negative sign so that the f(x) contains trigonometry terms. Then, we use boundary conditions 2 and 3 to find the constants A, B, M, and N.

$\because \frac{\partial\phi}{\partial x} = 0$ at $x = 0$ for $0 \le x \le s$

$\therefore f'(0) = -A\sqrt{\lambda}\sin(0) + B\sqrt{\lambda}\cos(0) = 0 \Rightarrow B = 0$

$\because \frac{\partial\phi}{\partial x} = 0$ at $x = s$ for $0 \le x \le s$

$\therefore f'(s) = -A\sqrt{\lambda}\sin(\sqrt{\lambda} * s) = 0 \Rightarrow \sqrt{\lambda} = \frac{m\pi}{s}, m = 0, 1, 2, 3...$

$\because \frac{\partial\phi}{\partial z} = 0$ at $z = 0$

$\therefore g'(0) = C\sqrt{\lambda}e^{\sqrt{0}} - D\sqrt{\lambda}e^{-\sqrt{0}} = 0 \Rightarrow C\sqrt{\lambda} - D\sqrt{\lambda} = 0 \Rightarrow C = D$

$\because M = C * A$ and $N = D * B$

$\therefore M = A$ and $N = B$

Therefore, based on $\phi = f(x)g(z)$, the fluid potential function is:

$$\phi(x, z) = \sum_{m=0}^{\infty} E_m \cos\left(\frac{m\pi x}{s}\right)\left(e^{\frac{m\pi}{s}z} + e^{-\frac{m\pi}{s}z}\right) \tag{6.a}$$

To reproduce the equation 7 in Tóth's paper, this equation was substituted with a *hyperbolic function*: $\cosh x = \frac{e^x + e^{-x}}{2}$.

$$\phi = \sum_{m=0}^{\infty} C_m \cos\left(\frac{m\pi x}{s}\right)\cosh\left(\frac{m\pi z_0}{s}\right), \text{ where } C_m = 2E_m \tag{7}$$

This equation 7 is a *Fourier series.* Similar with Taylor polynomials, it is a sum of trigonometric functions. We were able to expanded it to the equation 8, a periodic function, using formulas to find every *Fourier coefficients*, C_m.

Using the first boundary condition $\phi = g(z_0 + cx)$ at $z = z_0$ for $0 \le x \le s$, we are able to rewrite the equation 7 as: $g(z_0 + cx) = \sum_{m=0}^{\infty} C_m \cos\left(\frac{m\pi x}{s}\right) \cosh\left(\frac{m\pi z_0}{s}\right) = c_0 + \sum_{k=1}^{\infty} C_k \cos\left(\frac{k\pi x}{s}\right) \cosh\left(\frac{k\pi z_0}{s}\right)$

When m = 0,

$$\int_0^s C_0 \, dx = \int_0^s f(x) \, dx \Rightarrow C_0 = \frac{1}{s} \int_0^s f(x) \, dx$$

When m = k, k = 1, 2, 3,...,

$$\int_0^s f(x) \cos\left(\frac{m\pi x}{s}\right) dx = \int_0^s C_k \cos^2\left(\frac{m\pi x}{s}\right) \cosh\left(\frac{m\pi z_0}{s}\right) dx$$

$$\frac{\int_0^s f(x) \cos\left(\frac{m\pi x}{s}\right) dx \int_0^s dx}{\cosh\left(\frac{m\pi z_0}{s}\right) \int_0^s cos^2\left(\frac{m\pi x}{s}\right)} = C_k$$

$$\because \int_0^s \cos^2\left(\frac{\pi x}{s}\right) = \frac{s}{2}$$

$$\therefore C_k = 2 \sum_{m=1}^{\infty} \frac{1}{s \cosh\left(\frac{m\pi z_0}{s}\right)} \times \int_0^s f(x) \cos\left(\frac{m\pi x}{s}\right) dx$$

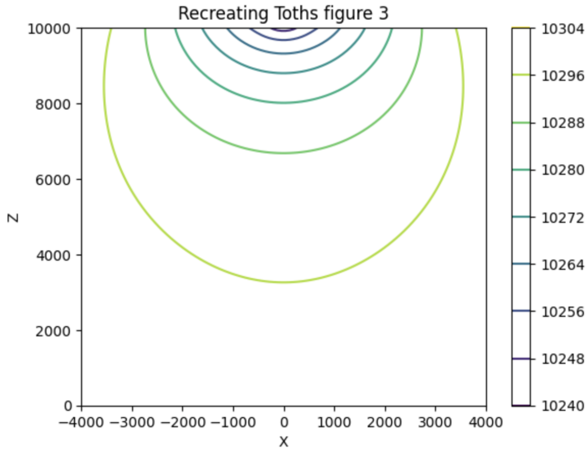
Therefore, by multiplying with C_K with $\cos(\frac{m\pi x}{s})\cosh(\frac{m\pi z_0}{s})$, we have the equation 8. To reproduce the equation 9, we just solve the integrals.

$$\phi(x, z) = \frac{1}{s} \int_0^s f(x) \, dx + 2 \sum_{m=1}^{\infty} \frac{\cos(\frac{m\pi x}{s})\cosh(\frac{m\pi z_0}{s})}{s \cosh(\frac{m\pi z}{s})} \times \int_0^s f(x) \cos\left(\frac{m\pi x}{s}\right) dx \tag{8}$$

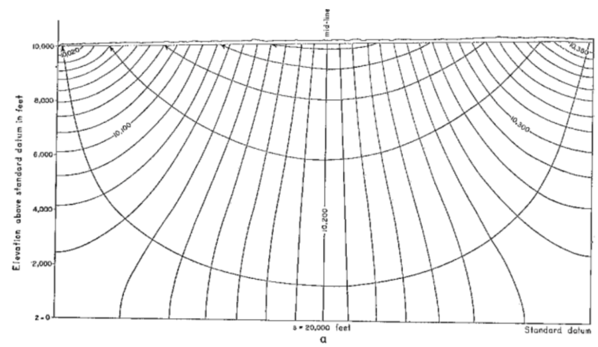
$$\phi(x, z) = g\left(z_0 + \frac{cs}{2}\right) - \frac{4gcs}{\pi^2} \cdot \sum_{m=0}^{\infty} \frac{\cos\left[(2m+1)\pi x/s\right] \cosh\left[(2m+1)\pi z/s\right]}{(2m+1)^2 \cosh\left[(2m+1)\pi z_0/s\right]} \tag{9}$$

2.2 Tóth's Figures

Below we can examine Figure 2b to see Tóth's original creation of demonstrating groundwater flow from equation 8. We reconstructed this equation in Python code and created Figure 2a. Comparing the two figures we can see a lot of similarities although not exactly the same. We see a cyclical movement of groundwater flow on an x axis ranging from -4,000 to 4,000 while Tóth has his x-axis as 0 to 40,000. Looking at the shape of the graphs we see relatively similar shapes on z bounds of 0 to 10,000. As well as similar values occurring on both graphs. Our recreation has values ranging from 10240 to 10304 and Tóth's original has values ranging from 10,000 to 10,300. Considering Tóth's figure it is important to consider the boundary conditions which he chose. Looking at the upper boundary condition we see $z = 10,000$, this was chosen by Tóth as "The general characteristics of the potential and flow pattern are more obvious owing to the large vertical distance." This is important to note as z is measured as the elevation above standard datum in feet, and having a z value of 10,000 does not exactly represent the earth in which Tóth was modeling. Overall we would argue the two Figures represent similar motion of groundwater flow and are indeed similar. And some smaller dissimilarities result from a difference in constants chosen by us.



(a)



(b)

Figure 2: Looking at Figure (b) we can see Tóth's original figure demonstrating a simple example of groundwater flow. Figure (a) is our attempt to recreate Tóth's figure in Python.

3 Our Adaptations

After studying Tóth's paper and reproducing his methods, we began to wonder how we might expand upon them. Tóth's figure three in particular represents a very basic case of groundwater flow. While we are not experts on the geography or hydrology of large groundwater flows we were interested in more specific cases than those depicted in figure 3. Throughout this section we are going to imagine a large earthquake that somehow disrupts an initially harmonious flow. This earthquake will create holes and obstacles, alter boundary conditions, and help us explore solutions to differential equations over non-rectangular domains.

3.1 The Earthquake

At 5:05 PM on April 28th 2024 a hypothetical but mighty earthquake rips through sleepy central Alberta. No one is harmed or even really registers that it happens, but, for our purposes this earthquake will have three principal effects on the groundwater basin shown in our recreation of Tóth's figure 3.

1. A small hole opens along the walls of the basin through which water can flow.
2. A boulder that once clung precariously to the upper impermeable boundary has been cast into our aquifer creating an obstacle around which water must flow.
3. A pipeline carrying toxic chemicals that was previously flowing adjacent to the basin has sprung a small leak and has begun spreading poison into the groundwater.

These three key scenarios are the focus of our modelling in this project.

3.2 The Hole in the Wall

In this scenario we imagine that a new energetically preferable path emerges through which water will travel. In order to accomplish this we needed to first define the boundary conditions of our aquifer. We imagined for simplicity that the 10000 by 20000 foot region that we are plotting is only a portion of the overall aquifer that is itself confined by some larger boundary conditions. This allowed us to create Dirichlet boundary conditions specifying that the amount of water at a location must be zero at those places. We then inserted a circular hole into the aquifer with a nonzero Dirichlet boundary condition. This causes water to flow into the hole from all sides. Figure 3 allows us to visualize this flow from all directions. One physical interpretation of our

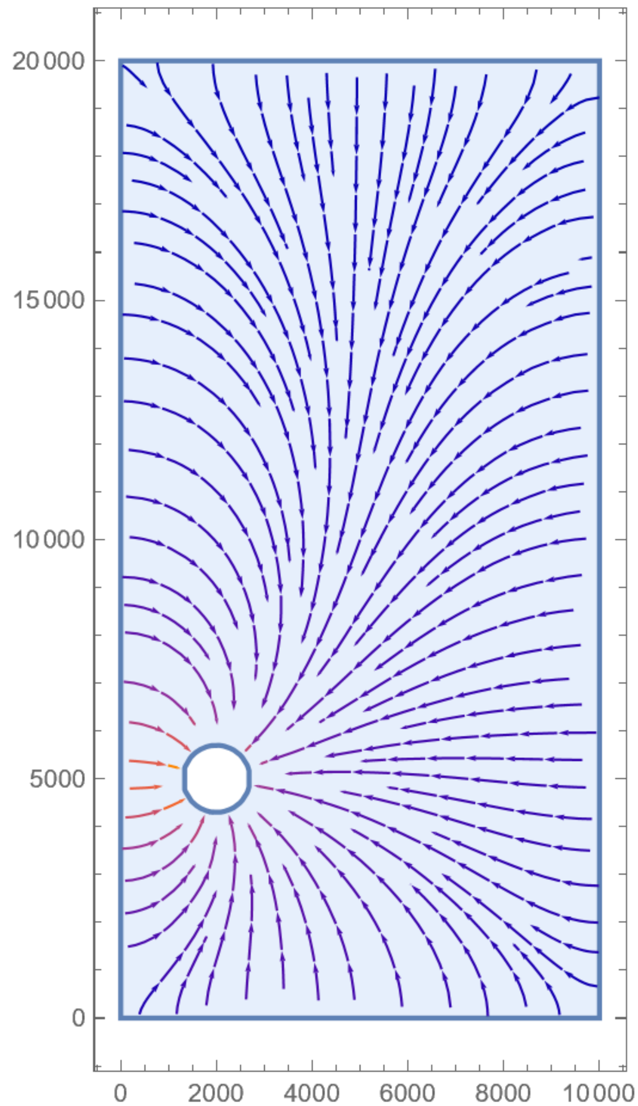


Figure 3: Fluid will flow into the circular hole because of how we defined the boundary conditions.The vector field plotted here shows us that water from all sides will swirl into this newly created hole

2D model is that the hole has formed at the bottom of the aquifer (and is perhaps draining into a deeper one.) This would account for the multi-directional flow into the hole. Figure 4 simply adds contour lines to 3 allowing us to more easily visualize this scenario in 3 dimensions.

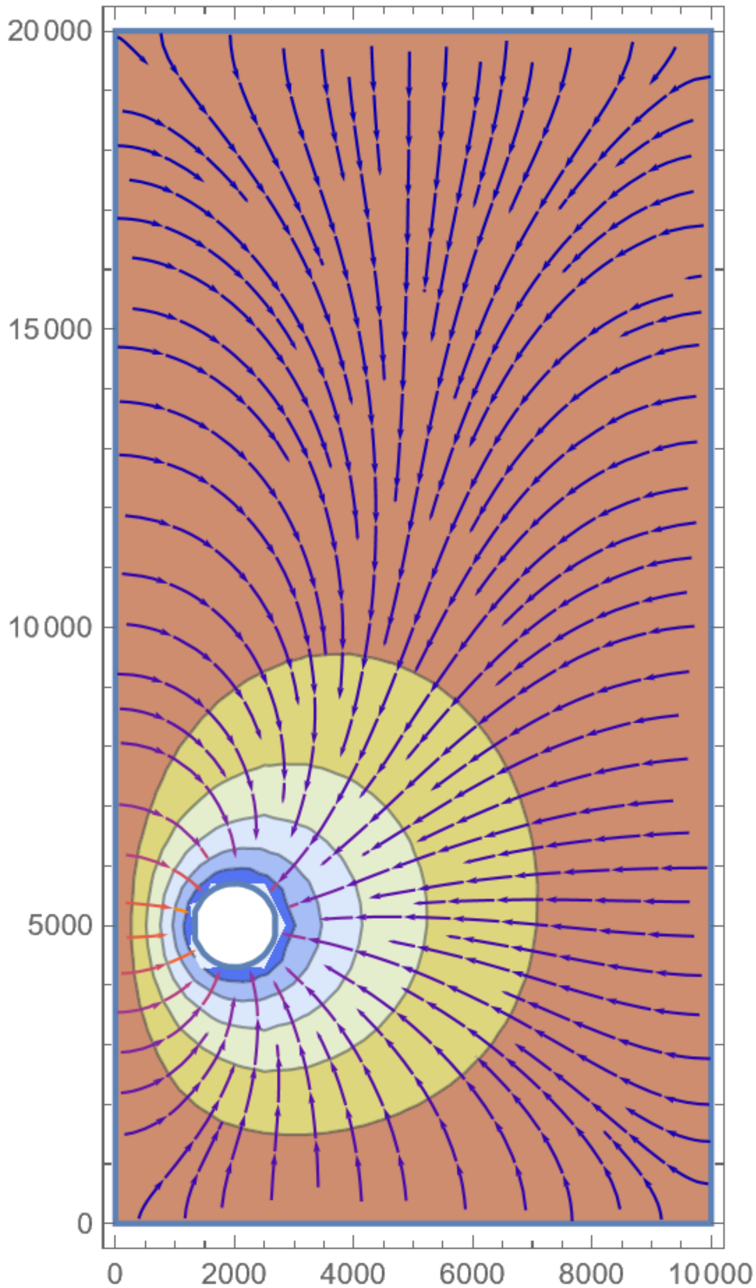


Figure 4: This figure adds contour lines to the previous one. This allows for more effective visualization of water being sucked down into the hole

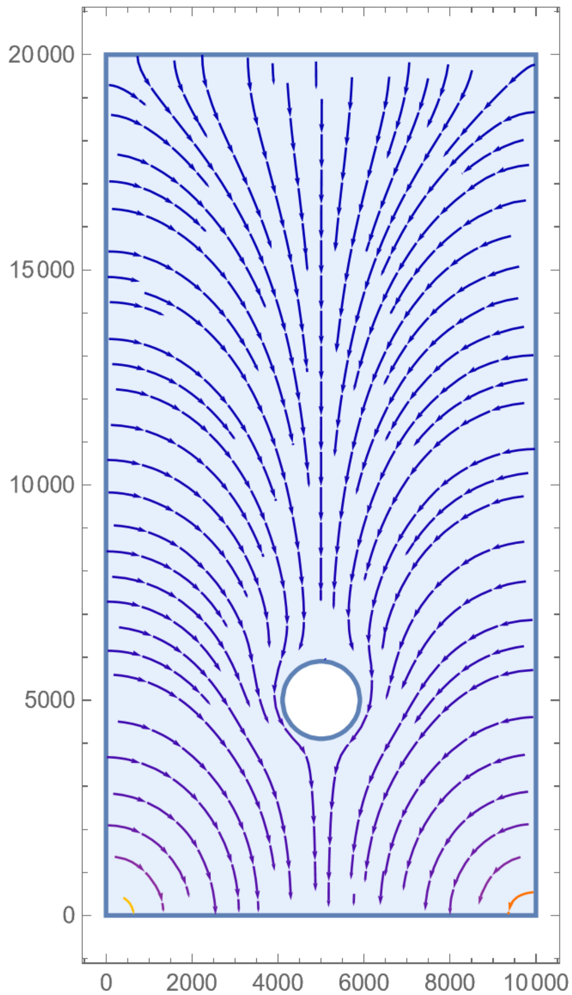


Figure 5: Fluid flows around the circle as it is drawn down a potential hill. Note the point directly above the circle where water must "choose" which side of the circle it will flow around.

3.3 The Obstacle

In this scenario we imagine that an impermeable substance has entered into our aquifer. As a result water must now navigate the non rectangular domain surrounding the obstacle. We accomplished this by setting a nonzero Dirichlet boundary conditions at the bottom of our region of interest which draws fluid down. We then removed the boundary condition from the obstacle (formerly the hole) thereby forcing water to have to move around it. This allowed us to impose an effective Neuman boundary condition along the perimeter of our obstacle, preventing water from flowing into our out of the shape. We did this without actually using the NeumanValue[ ] command in Mathematica. We believe we were able to do this because the RegionDifference[] command used to create our plot simply removes the specified shape from the plotting region. As a result, if no boundary condition is specified along the border of the shape, already existing fluid will flow around and no new fluid will emerge from it. An image of the code used to produce figure 6 is given in figure 8. We attempted this with three different shapes: a circle, triangle, and rectangle (figures 5, 6, and 7.) Fluid exhibits distinct behavior while it flows around each shape. The behavior on the corners of the triangle and rectangle are particularly interesting. Flowing water will try to cut these corners as close as possible. At times it seems like our model breaks down and vector lines run into the shape at these corner locations.

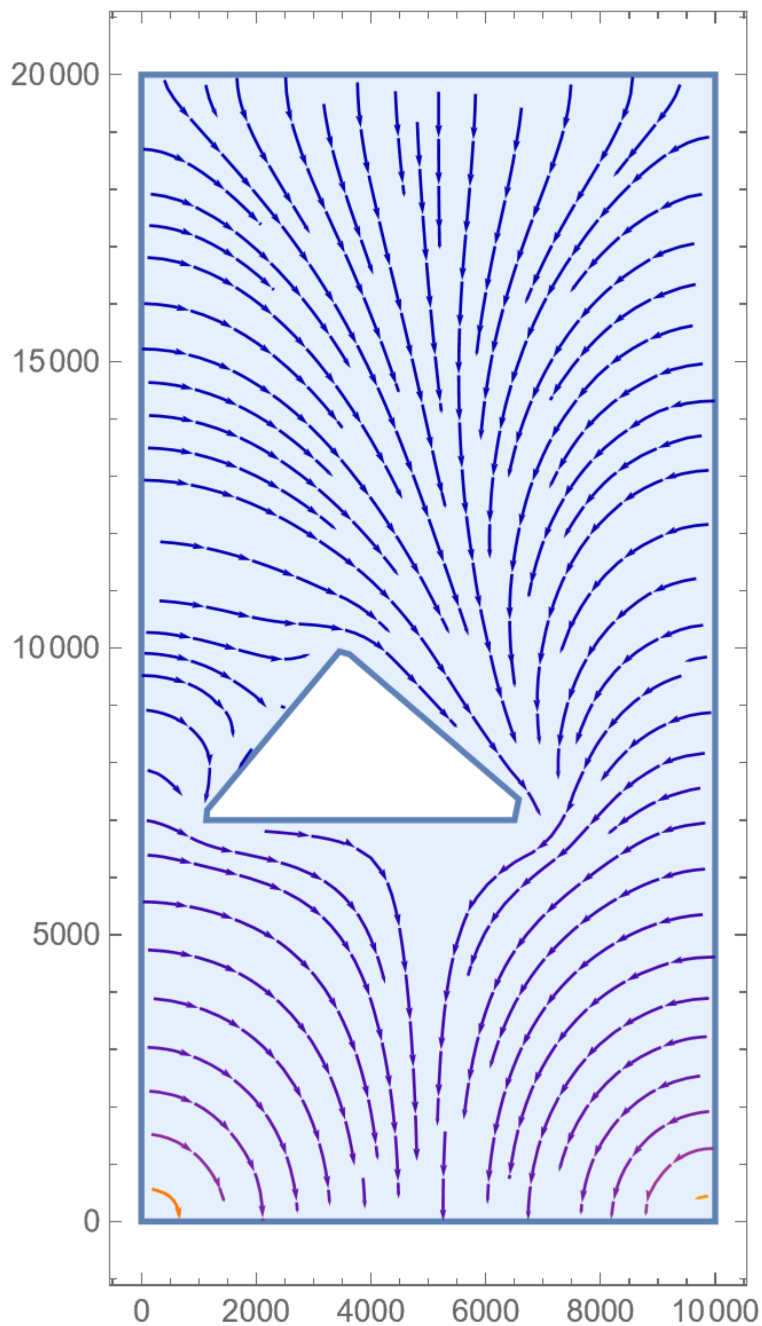


Figure 6: Water flows around a triangle. Particularly on the triangle's left edge, water seems to cling incredibly closely to the surface of the shape.

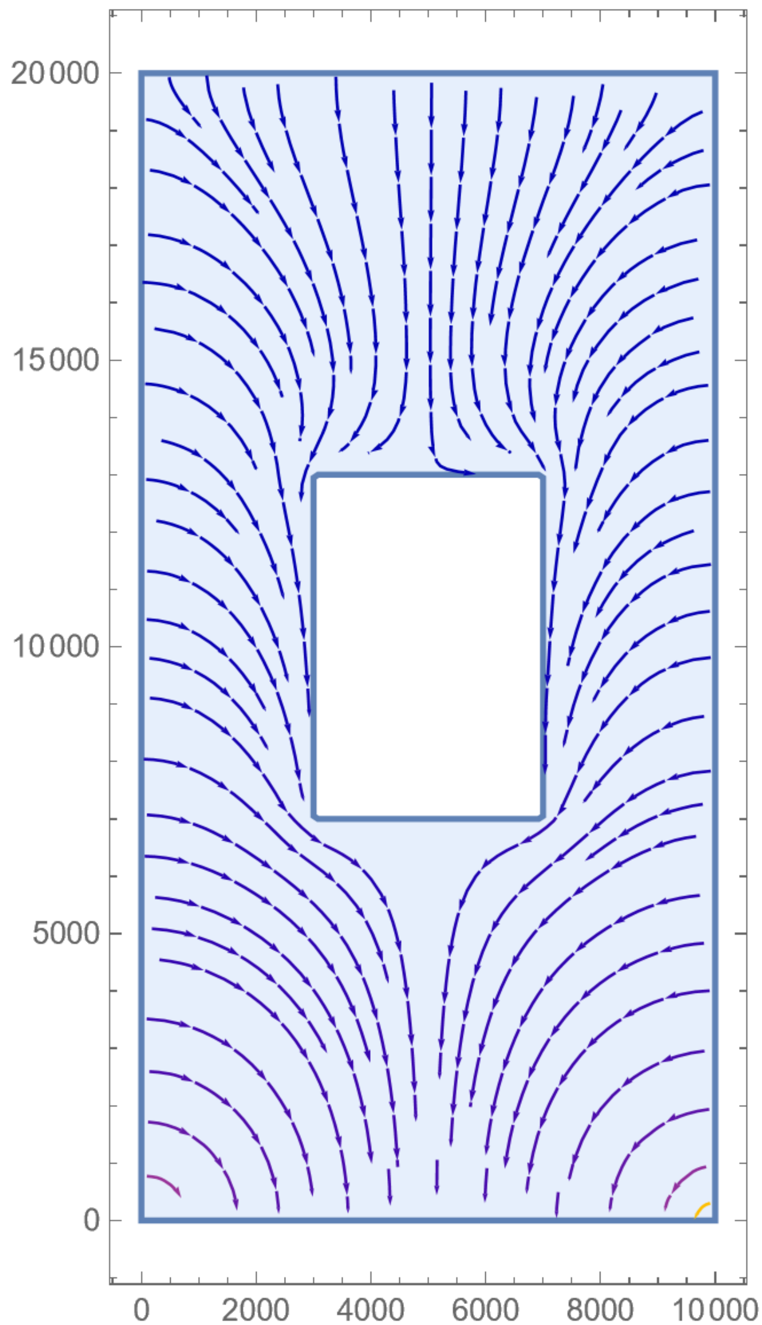


Figure 7: Water flows around a rectangle. As it travels down it starts to approach flowing parallel to the edge of the surface. Some of the contour lines appear to disappear within the shape highlighting a possible area for future improvement in our modelling.

```
triangle = Triangle[{{1500, 7000}, {4000, 10000}, {7000, 7000}}];
Ω = RegionDifference[Rectangle[{0, 0}, {10000, 20000}], triangle];

(*Plot the domain*)
RegionPlot[Ω, AspectRatio → Automatic]

(*Plot the domain*)

(*Define the Laplace operator*)
op = -Laplacian[u[x, y], {x, y}];

(*Define the Dirichlet boundary conditions*)
Subscript[Γ, D] = {DirichletCondition[u[x, y] == 0, x == 0 && 0 ≤ y ≤ 20000],
   DirichletCondition[u[x, y] == 0, x == 10000 && 0 ≤ y ≤ 20000],
   DirichletCondition[u[x, y] == 0, 0 ≤ x ≤ 10000 && y == 20000],
   DirichletCondition[u[x, y] == 1, 0 ≤ x ≤ 10000 && y == 0]};

(*Solve Laplace's equation within the domain with the specified boundary conditions*)
ufun = NDSolveValue[{op == 0, Subscript[Γ, D]}, u, {x, y} ∈ Ω];

(*Generate a contour plot of the solution*)
p1 = ContourPlot[ufun[x, y], {x, 0, 10000}, {y, 0, 20000},
  ColorFunction → "Temperature", AspectRatio → Automatic, PlotRange → All]
gf = {D[ufun[x, y], x], D[ufun[x, y], y]};
gf2[xx_, yy_] := gf /. {x → xx, y → yy}

p2 = StreamPlot[gf2[xx, yy], {xx, yy} ∈ Ω, AspectRatio → Automatic]

Show[p1, p2]
```

```
Out[164]=
{0.0015561, -0.0000119982}
```

Figure 8: This figure shows the code used to generate figure 6. In this code we first define a region Ω, with a triangle sliced out of it. We then define our Laplacian and specify the Dirichlet boundary conditions for the outside of our region. Finally we produce a contour plot, p1, and a stream plot showing the laplacian for our system, p2.

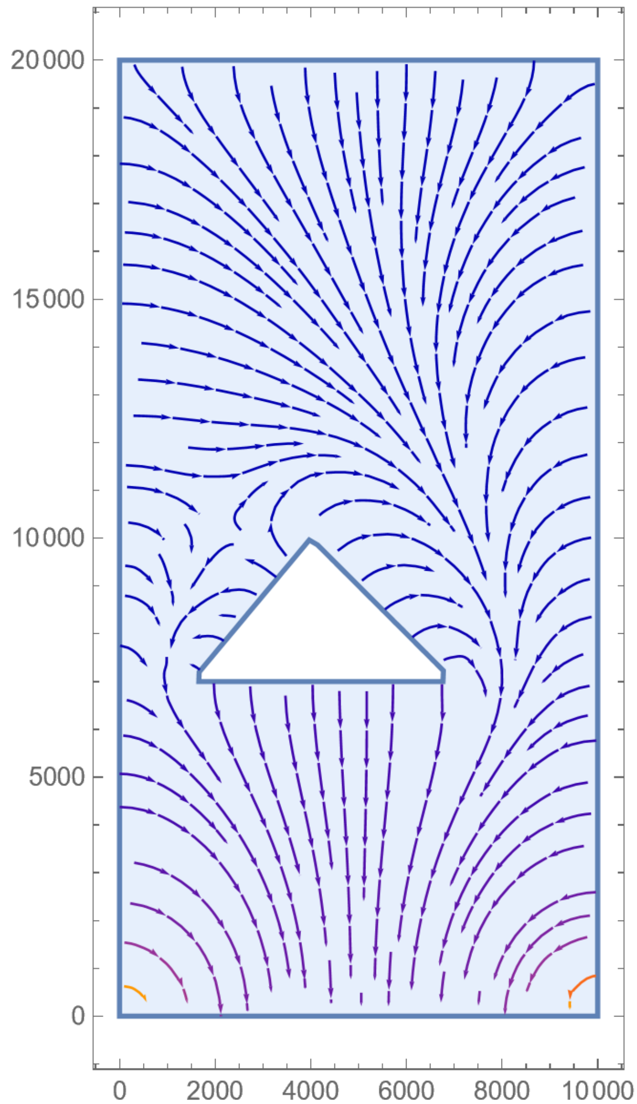


Figure 9: Fluid flows from the outside walls of the triangular hole in the pipeline. Toxic fluid joins the stream produced by the original Dirichlet boundary conditions and is swept into the aquifer.

3.4 The Pipeline

In this scenario we imagined that fluid now flows out of the hole we created before. We accomplished this by redefining our Laplacian operator to include a term for the divergence of our vector field at a given point. We scaled the gradient of our function by this value around the triangular hole. This results in a plot in which fluid around the hole initially flows out in all directions, but is then swept into the natural current of the aquifer and driven towards the nonzero Dirichlet boundary value. Again, we speculate that something similar could have been accomplished using Mathematica's NeumannValue[] command but we were unsuccessful in getting that to work.

4 Conclusion

In summary, Tóth's paper on groundwater flow was a huge advancement in understanding the motion of fluids below the earth's surface. Through reading his paper we have demonstrated knowledge that we understand his processes and can recreate his equations and figures. From basing our understanding in these we were able to create our adaptations trying to understand how the motion of fluid would change with obstacles in the path.

4.1 Further Adaptations

If we were given another month to further develop this paper and our research we would like to expand into understanding Laplace's equation in polar or 3D Cartesian coordinates. Tóth originally presents the representation of groundwater flow in 2D Cartesian coordinates, we believe it would be interesting to see what changes may occur from the problem being represented in a different coordinate system. From a modelling perspective one thing that we might also try is actively implementing Neumann boundary conditions instead of relying on Mathematica's RegionDifference[ ] command. We struggled to get these types of conditions to work in our code and with more time we could see what interesting effects they might have on our three scenarios.

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