Difference Equations and Finance

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Problem

Suppose you would like to retire with 10 million dollars in savings. To keep the difference equations simple, let's say that:

- you invest uniform monthly payments for a fixed number of years,
- that you receive a fixed annual percent yield,
- and that the interest is compounded monthly.

Solve this problem with z-transform techniques for:

- three different interest rates (low, medium, high), and
- three different time intervals (short, average, long).

For each interest rate and investment duration, compute the required (fixed) monthly payments $C_{i,D}$ needed to realize the 10 million dollar retirement goal.

Rates and Interval

Pick three annual percentage rates reflective of different investments:

$$i_{low} = 0.5\,\%$$
$$i_{med} = 6.0\,\%$$
$$i_{high} = 10.5\,\%$$

Investment time intervals:

$$D_{short} = 16\text{ years}$$
$$D_{med} = 32\text{ years}$$
$$D_{long} = 48\text{ years}$$

Difference Equation

Let

- $n = 0$ correspond to the first investment contribution,
- $n = 12D - 1$ be the last investment contribution, and
- $n = 12D$ be the time when the retirement account is to have the desired 10 million dollars in savings.

Our difference equation is then

$$y[n] = \left(1 + \frac{i}{100}\right)^{\frac{1}{12}} y[n-1] + C(u[n] - u[n-12D]), \tag{1}$$

where

- $y[n]$ is the balance at month n,
- $\left(1 + \frac{i}{100}\right)^{\frac{1}{12}} y[n-1]$ is last month's balance plus interest, and
- $C(u[n] - u[n-12D])$ is the payments made each month.

Now putting equation 1 in standard form and simplifying we get,

$$y[n] - \left(1 + \frac{i}{100}\right)^{\frac{1}{12}} y[n-1] = C(u[n] - u[n-12D]) \tag{2}$$

$$\left(1 - \left(1 + \frac{i}{100}\right)^{\frac{1}{12}} E^{-1}\right) y[n] = C(u[n] - u[n-12D]). \tag{3}$$

Note that E^{-1} is used to denote a delay operation. Having put equation 1 in a more standard form we can take the z-transform of both sides,

$$\left(1 - \left(1 + \frac{i}{100}\right)^{\frac{1}{12}} z^{-1}\right) Y(z) = C\left(\frac{z}{z-1} - \frac{z^{-12D+1}}{z-1}\right). \tag{4}$$

Which can be written as

$$Y(z) = C\left(\frac{\frac{z}{z-1} - \frac{z^{-12D+1}}{z-1}}{1-\left(1+\frac{i}{100}\right)^{\frac{1}{12}} z^{-1}}\right) \tag{5}$$

$$= C\left(\frac{z - z^{-12D+1}}{(z-1)(1-pz^{-1})}\right), \tag{6}$$

where

$$p = \left(1+\frac{i}{100}\right)^{\frac{1}{12}}. \tag{7}$$

Note that a delay in the time domain corresponds to a negative power of z in the z-domain by the time shifting property:

$$\text{If } m > 0 : x[n-m]u[n-m] \stackrel{Z_u}{\Longleftrightarrow} z^{-m}X(z) \tag{8}$$

We also used one z-transform pair:

$$\gamma^n u[n] \stackrel{Z_u}{\Longleftrightarrow} \frac{z}{z-\gamma} \quad \text{ROC: } |z| > |\gamma| \tag{9}$$

We'll simplify (6) and pull out a z before performing the expansion

$$\frac{Y(z)}{z} = C\left(\frac{z - z^{-12D+1}}{z(z-1)(1-pz^{-1})}\right) \tag{10}$$

$$= C\left(\frac{z - z^{-12D+1}}{(z-1)(z-p)}\right). \tag{11}$$

Let

$$\frac{Y_1(z)}{z} = \frac{z}{(z-1)(z-p)}, \tag{12}$$

$$\frac{Y_2(z)}{z} = \frac{z^{-12D+1}}{(z-1)(z-p)}, \tag{13}$$

then

$$\frac{Y(z)}{z} = C\left(\frac{Y_1(z)}{z} - \frac{Y_2(z)}{z}\right). \tag{14}$$

We start by solving for $Y_1(z)$,

$$\begin{aligned} \frac{Y_1(z)}{z} &= \frac{z}{(z-1)(z-p)} && (15) \\ &= \frac{A}{z-1} + \frac{B}{z-p}. && (16) \end{aligned}$$

Using the Heaviside cover-up method

$$\begin{aligned} A &= \frac{1}{1-p}, && (17) \\ B &= \frac{p}{p-1}, && (18) \end{aligned}$$

so

$$\frac{Y_1(z)}{z} = \left(\frac{1}{1-p}\right)\left(\frac{1}{z-1}\right) + \left(\frac{p}{p-1}\right)\left(\frac{1}{z-p}\right). \qquad (19)$$

Modified PFE (3/3)

Now in order to use (9) we need a z in the numerator of each component of the sum, so we move the z on the LHS back over,

$$Y_1(z) = \left(\frac{1}{1-p}\right)\left(\frac{z}{z-1}\right) + \left(\frac{p}{p-1}\right)\left(\frac{z}{z-p}\right). \tag{20}$$

Using (9) we can find the inverse z-transform of $Y_1(z)$,

$$\begin{aligned} Z^{-1}\left[Y_1(z)\right] &= \frac{1}{1-p}u[n] + \frac{p}{p-1}p^n u[n] && (21)\\ &= \left(\frac{1}{1-p} + \frac{p}{p-1}p^n\right) u[n] && (22)\\ &= \left(\frac{1}{p-1}p^{n+1} - \frac{1}{p-1}\right) u[n] && (23)\\ &= \frac{1}{p-1}\left(p^{n+1}-1\right) u[n] && (24)\\ &= y_1[n]. && (25)\end{aligned}$$

Having found $y_1[n]$ we can use the time shift property (8) to find $y_2[n]$. First note that we originally wrote $Y_2(z)$ as

$$\frac{Y_2(z)}{z} = \frac{z^{-12D+1}}{(z-1)(z-p)}. \tag{26}$$

Now

$$Y_1(z) = \frac{z^2}{(z-1)(z-p)} \tag{27}$$

$$Y_2(z) = \frac{z^{-12D+2}}{(z-1)(z-p)} \tag{28}$$

$$Y(z) = C\left(Y_1(z) - Y_2(z)\right). \tag{29}$$

Notice

$$\begin{aligned} Y_2(z) &= \frac{z^{-12D+2}}{(z-1)(z-p)} && (30)\\ &= \frac{z^2}{(z-1)(z-p)} z^{-12D} && (31)\\ &= Y_1(z) z^{-12D}. && (32) \end{aligned}$$

We can then use the time shift property (8) to find $y_2[n]$,

$$
\begin{aligned}
Z^{-1}\left[Y_2\right] &= Z^{-1}\left[Y_1(z)z^{-12D}\right] && (33)\\
&= \frac{1}{p-1}\left(p^{n+1-12D}-1\right)u[n-12D] && (34)\\
&= y_2[n]. && (35)
\end{aligned}
$$

Having found both $y_1[n]$ and $y_2[n]$ we can take advantage of linearity to find $y[n]$. The z-transform is a linear transformation, so

$$ax[n]+by[n] \overset{Z_u}{\Longleftrightarrow} aX(z)+bY(z), \tag{36}$$

which means that

$$
\begin{aligned}
C\left(Y_1(z)-Y_2(z)\right) &\overset{Z_u}{\Longleftrightarrow} C\left(y_1[n]-y_2[n]\right), && (37)\\
Y(z) &\overset{Z_u}{\Longleftrightarrow} y[n]. && (38)
\end{aligned}
$$

Inv. Z-Transform (4/4)

Using (36)

$$
\begin{aligned}
y[n] &= C(y_1[n] - y_2[n]) && (39)\\
&= C\Big(\frac{1}{p-1}\left(p^{n+1}-1\right)u[n] && (40)\\
&\qquad - \frac{1}{p-1}\left(p^{n+1-12D}-1\right)u[n-12D]\Big) && (41)\\
&= \frac{C}{p-1}\left(\left(p^{n+1}-1\right)u[n] - \left(p^{n+1-12D}-1\right)u[n-12D]\right). && (42)
\end{aligned}
$$

Equation (42) is useful if we want to see what our projected balance would be on a given month. However we are interested in our monthly payments, given a goal, interest, and duration. So we'll rearrange (42) as

$$C = \frac{y[n](p-1)}{\left(p^{n+1}-1\right)u[n] - \left(p^{n+1-12D}-1\right)u[n-12D]} \tag{43}$$

$$= \frac{y[12D](p-1)}{\left(p^{12D+1}-1\right)u[12D] - \left(p^{12D+1-12D}-1\right)u[12D-12D]} \tag{44}$$

$$= \frac{10\,\mathrm{M}(p-1)}{p^{12D+1}-1-p^{1}+1} \tag{45}$$

$$= \frac{10\,\mathrm{M}(p-1)}{p\left(p^{12D}-1\right)} \tag{46}$$

Monthly Payments (2/2)

Having solved for C we use MATLAB to find $C_{i,D}$ given our three APYs and durations:

D, i	0.5%	6%	10.5%
16 years	\$50,022.43	\$31,447.18	\$21,026.04
32 years	\$24,013.78	\$8,882.5	\$3,539.26
48 years	\$15,362.61	\$3,146.68	\$692.73

Table 1: Monthly contribution for particular APYs and investment durations

Investment Growth