Study Guide

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1 The How, When, and Why of Mathematics

1. **Polya's method**
    a. Understanding the problem
    b. Devising a plan
    c. Carrying out the plan
    d. Looking back

2 Logically Speaking

2.1 Definitions

Definition a precise and unambiguous description of the meaning of a mathematical term. It characterizes the meaning of a word by giving all the properties and only those properties that must be true.

Theorem a mathematical statement that is proved using rigorous mathematical reasoning. In a mathematical paper, the term theorem is often reserved for the most important results.

Lemma a minor result whose sole purpose is to help in proving a theorem. It is a stepping stone on the path to proving a theorem. Very occasionally lemmas can take on a life of their own (Zorns lemma, Urysohns lemma, Burnsides lemma, Sperners lemma).

Corollary a result in which the (usually short) proof relies heavily on a given theorem (we often say that this is a corollary of Theorem A).

Proposition a proved and often interesting result, but generally less important than a theorem.

Conjecture a statement that is unproved, but is believed to be true (Collatz conjecture, Goldbach conjecture, twin prime conjecture).

Claim an assertion that is then proved. It is often used like an informal lemma.

Axiom/Postulate a statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved (Euclids five postulates, Zermelo-Fraenkel axioms, Peano axioms).

Identity a mathematical expression giving the equality of two (often variable) quantities (trigonometric identities, Eulers identity).

Paradox a statement that can be shown, using a given set of axioms and definitions, to be both true and false. Paradoxes are often used to show the inconsistencies in a flawed theory (Russells paradox). The term paradox is often used informally to describe a surprising or counterintuitive result that follows from a given set of rules (Banach-Tarski paradox, Alabama paradox, Gabriels horn).

Statement - A sentence that is either TRUE or FALSE, but not both.

Statement Form - A statement form is a letter representing an unspecified statement or expression built from such letters using connectives.

Connectives - Operators connecting together statement forms.

Tautology - Two statement forms that are logically equivalent. A statement form for which the final column in the truth table consists of all T's is called a tautology.

Contradiction - A statement form for which the final column is all F's is called a contradiction.

Contrapositive - Switching the hypothesis and conclusion of a conditional statement and negating both. The contrapositive of any true proposition is also true.

Even - An integer x is even if there is an integer n such that $x = 2n$

Odd - An integer is odd if there is an integer n such that $x = 2n + 1$.

Prime - An integer p is prime if $p > 1$ and p cannot be written as a product of two positive integers, both different from p.

Contrapositive - Switching the hypothesis and conclusion of a conditional statement and negating both. The contrapositive of any true proposition is also true.

2.2 Theorems

Theorem 2.7. Two statement forms P and Q are equivalent if and only if they have the same truth table.

Theorem 2.9. Let P and Q denote statement forms. The following are tautologies:

1. **DeMorgan's laws**
   $\neg(P \vee Q) \leftrightarrow (\neg P \wedge \neg Q)$
   $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$
2. **Implication and its negation**
   $(P \to Q) \leftrightarrow (\neg P \vee Q)$
   $\neg(P \to Q) \leftrightarrow (P \vee \neg Q)$
3. **Double negation**
   $\neg(\neg P) \leftrightarrow P$

3 Introducing the Contrapositive and Converse

3.1 Definitions

Definition 3.1. An **integer** x **is odd** if there is an integer n such that $x = 2n + 1$.

Definition 3.2. An **integer** x **is even** if there is an integer n such that $x = 2n$

Definition 3.3. An integer p is **prime** if $p > 1$ and p cannot be written as a product of two positive integers, both different from p.

3.2 Theorems

Theorem 3.1. Let P, Q, and R denote statement forms. Then the following are tautologies:

1. **Distributive property**
   $(P \wedge (Q \vee R)) \leftrightarrow ((P \wedge Q) \vee (P \wedge R))$
   $(P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \vee R))$
2. **Associative property**
   $(P \wedge (Q \wedge R)) \leftrightarrow ((P \wedge Q) \wedge R)$
   $(P \vee (Q \vee R)) \leftrightarrow ((P \vee Q) \vee R)$
3. **Commutative property**
   $(P \wedge Q) \leftrightarrow (Q \wedge P)$
   $(P \vee Q) \leftrightarrow (Q \vee P)$

Theorem 3.3. Let x be an integer. If x^2 is odd, then x is odd.

Theorem (Contrapositive of the statement of Theorem 3.3). Let x be an integer. If x is even, then x^2 is even.

4 Set Notation and Quantifiers

4.1 Common Sets

The natural numbers : $\mathbb{N} = \{0, 1, 2, 3, ...\}$

The integers : $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$

The rational numbers : $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z} \textit{ and } q \neq 0\}$

The real numbers : $\mathbb{R}$

The complex numbers : $\mathbb{C} = \{a + bi : i^2 = -1 \textit{ and } a, b \in \mathbb{R}\}$

If A is one of the sets $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$, then the set of the positive elements is denoted by $A^+ = \{x \in A : x > 0\}$ and the set of the negative elements is denoted by $A^- = \{x \in A : x < 0\}$. Thus we have defined $\mathbb{Z}^+$, $\mathbb{Z}^-$, $\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{R}^+$, and $\mathbb{R}^-$

The plane $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$

For $n \in \mathbb{Z}^+$, Euclidean n-space $\mathbb{R}^n = \{(x_1, x_2, ..., x_n) : x_j \in \mathbb{R}$ for $j = 1, 2, ..., n\}$.

4.2 Symbols

For all $\forall$

There exists $\exists$

5 Proof Techniques

5.1 Definitions

Three methods discussed in this chapter:

- direct proof (just get started and keep going)
- proof by contradiction (show that the negation of the statement you wish to prove implies the impossible)
- proof in cases (which may be used when conditions dictate that different situations occur).

Definition 5.1. A nonzero integer a **divides** an integer b if there is an integer n such that $b = an$. We write this as $a|b$.

Definition 5.2. For a real number x, the **absolute value** of x is defined to be

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

5.2 Theorems

Theorem 5.1. If a, b and c are integers such that a divides b and a divides c, then a divides $b + c$.

Theorem 5.2. The number $\sqrt{2}$ is not rational.

Theorem 5.3. Let x and y be real numbers. Then $|xy| = |x||y|$.

6 Sets

6.1 Definitions

Definition 6.1. The **empty set**, denoted $\emptyset$ is the set with no elements.

Definition 6.2. The set A is a **subset** of the set B or, equivalently, A is **contained** in B, if every element of A is an element of B. We write $A \subseteq B$ to indicate that A is a subset of B.

Definition 6.3. The set A is a **proper subset** of B if $A \subseteq B$ and $A \neq B$, and we write $A \subset B$.

Definition 6.4. The set A is **equal** to B, written $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Definition 6.5. The **union** of the sets A and B is the set $A \cup B = \{x : x \in A \textit{ or } x \in B\}$.

Definition 6.6. The **intersection** of the sets A and B is the set $A \cap B = \{x : x \in A \textit{ and } x \in B\}$.

Definition 6.7. Two sets A and B are **disjoint** if $A \cap B = \emptyset$.

Definition 6.8. The **set difference** of set B in set A is the set $A \setminus B = \{x \in A : x \notin B\}$.

Definition 6.9. If the set X is the universe and A is a subset of X, the **complement** of A is the set $A^c = X \setminus A$.

6.2 Theorem 6.11.

Let A be a set. Then $\emptyset \subseteq A$.

7 Notes

The operator iff requires two statements to prove, one in each direction.

In order to prove two sets equal, you need to prove that each one contains the other.