Abstract Mathematics Homework 4

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1 Book Problems

1.1 Theorem 7.4.9

Let X denote a set and A,B, and C denote subsets of X. Then

$$(A \cup B) \cup C = A \cup (B \cup C) \tag{1}$$

Proof.

For sets X and Y, $X \cup Y$ is defined a follows.

$$\text{If } X \cup Y \text{, then } x \in X \text{ or } x \in Y \tag{2}$$

So $A \cup B$ is the set where $x \in X$ or $x \in Y$.

The union of $(A \cup B)$ with C is then the set where $x \in A$, $x \in B$, or $x \in C$.

Now if $B \cup C$, then $x \in B$ or $x \in C$.

The union of $A \cup (B \cup C)$ is then the set where $x \in A$, $x \in B$, or $x \in C$.

Since both the set $(A \cup B) \cup C$ and $A \cup (B \cup C)$ share the same definition, we can conclude that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$, conversely that $A \cup (B \cup C) \subseteq (A \cup B) \cup C$, and as a result of the last two conclusions, that $(A \cup B) \cup C = A \cup (B \cup C)$.

$\square$

1.2 Theorem 7.4.15

Let X denote a set and A,B, and C denote subsets of X. Then

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \tag{1}$$

Proof.

Observe the following:

1. If $x \in X \setminus (A \cup B)$, then $x \in X$ and $x \notin (A \cup B)$.
2. If $x \in (X \setminus A) \cap (X \setminus B)$, then $x \in X$, $x \notin A$, and $x \notin B$.

We must first prove that $X \setminus (A \cup B) \subseteq (X \setminus A) \cap (X \setminus B)$.

Notice for the set where $x \in X \setminus (A \cup B)$, that $x \in X$, therefore what remains to show is that $x \notin A$ and $x \notin B$.

The set where $x \notin A \cup B$ is defined as the set $\{x \mid \neg(x \in A \vee x \in B)\}$.

By DeMorgan's Laws of Logic we can say that this is equivalent to the set $\{x \mid x \notin A \wedge x \notin B\}$

Hence $x \notin A$, $x \notin B$, and $X \setminus (A \cup B) \subseteq (X \setminus A) \cap (X \setminus B)$.

We now must show that $(X \setminus A) \cap (X \setminus B) \subseteq X \setminus (A \cup B)$.

The set $(X \setminus A) \cap (X \setminus B) \subseteq X \setminus (A \cup B)$ can be defined as the set where $\{x \mid x \in X \wedge x \notin A \wedge x \notin B\}$.

By DeMorgan's Laws of Logic we can say this is equivalent to the set $\{x \mid x \in X \wedge \neg(x \in A \vee x \in B)\}$, which is the set $x \in X \setminus (A \cup B)$.

We conclude that $(X \setminus A) \cap (X \setminus B) \subseteq X \setminus (A \cup B)$ and therefore:

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \tag{2}$$

$\square$

1.3 Theorem 7.4.17

Let X denote a set and A,B, and C denote subsets of X. Then

$$A \setminus B = A \cap B^c \tag{1}$$

Proof.

Observe the following:

1. $A^c = \{x \mid x \notin B\}$
2. $A \cap B = \{x \mid x \in A \land x \in B\}$

Following the definitions observed above we can describe $A \cap B^c$ as follows:

$$A \cap B^c = \{x \mid x \in A \land x \notin B\} \tag{2}$$

Well that's the definition of $A \setminus B$, so we conclude $A \setminus B = A \cap B^c$.

$\square$

1.4 Theorem 7.4.19

Let X denote a set and A,B, and C denote subsets of X. Then

$$(A \subseteq C) \wedge (B \subseteq C) \iff (A \cup B) \subseteq C \tag{1}$$

Proof.

Observe the following:

1. $A \subseteq B$ indicates that every element of A is in B.
2. $A \cup B = \{x \mid x \in A \vee x \in B\}$

We start by proving $(A \subseteq C) \wedge (B \subseteq C) \rightarrow (A \cup B) \subseteq C$.

The statement form $(A \cup B) \subseteq C$ states that every element in the sets A,B is contained in C.

So the set A must be contained in C and the set B must be contained in C.

We have shown $(A \subseteq C) \wedge (B \subseteq C) \rightarrow (A \cup B) \subseteq C$.

Now we must prove $(A \cup B) \subseteq C \rightarrow (A \subseteq C) \wedge (B \subseteq C)$.

Let $A \subseteq C$, $B \subseteq C$, and $x \in (A \cup B)$.

Then $x \in A$ or $x \in B$, which are both subsets of C.

This makes $(A \cup B) = \{x \mid x \in A \vee x \in B\}$ a subset of C.

We have shown $(A \cup B) \subseteq C \rightarrow (A \subseteq C) \wedge (B \subseteq C)$.

We conclude that $(A \subseteq C) \wedge (B \subseteq C) \iff (A \cup B) \subseteq C$

$\square$

1.5 Problem 7.15

Prove that $A^c \cup B^c = X$ if and only if A and B are disjoint.

Proof.

Observe that A and B are disjoint if $A \cap B = \emptyset$.

To prove that "$A^c \cup B^c = X$ if and only if A and B are disjoint", we must first prove that $A^c \cup B^c = X \to A \cap B = \emptyset$, and then prove $A \cap B = \emptyset \to A^c \cup B^c = X$.

Let $A \cap B = \emptyset$. When we take the complement of both sides of the statement we get $(A \cap B)^c = \emptyset^c = A^c \cup B^c = X$. So $A^c \cup B^c = X \to A \cap B = \emptyset$.

Let $A^c \cup B^c = X$. When we take the complement of both sides of the statement we get $(A^c \cup B^c)^c = X^c = A \cap B = \emptyset$. So $A \cap B = \emptyset \to A^c \cup B^c = X$.

We conclude that $A^c \cup B^c = X$ if and only if A and B are disjoint.

$\square$