Abstract Mathematics Homework 3

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1 The How, When, and Why of Mathematics

1. **Polya's method**
   a. Understanding the problem
   b. Devising a plan
   c. Carrying out the plan
   d. Looking back

2 Logically Speaking

Theorem 2.7. Two statement forms P and Q are equivalent if and only if they have the same truth table.

Theorem 2.9. Let P and Q denote statement forms. The following are tautologies:

1. **DeMorgan's laws**
   $\neg(P \vee Q) \leftrightarrow (\neg P \wedge \neg Q)$
   $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$
2. **Implication and its negation**
   $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$
   $\neg(P \rightarrow Q) \leftrightarrow (P \vee \neg Q)$
3. **Double negation**
   $\neg(\neg P) \leftrightarrow P$

3 Introducing the Contrapositive and Converse

3.1 Definitions

Definition 3.1. An **integer** x **is odd** if there is an integer n such that $x = 2n + 1$.

Definition 3.2. An **integer** x **is even** if there is an integer n such that $x = 2n$

Definition 3.3. An integer p is **prime** if $p > 1$ and p cannot be written as a product of two positive integers, both different from p.

3.2 Theorems

Theorem 3.1. Let P, Q, and R denote statement forms. Then the following are tautologies:

1. **Distributive property**
   $(P \wedge (Q \vee R)) \leftrightarrow ((P \wedge Q) \vee (P \wedge R))$
   $(P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \vee R))$
2. **Associative property**
   $(P \wedge (Q \wedge R)) \leftrightarrow ((P \wedge Q) \wedge R)$
   $(P \vee (Q \vee R)) \leftrightarrow ((P \vee Q) \vee R)$
3. **Commutative property**
   $(P \wedge Q) \leftrightarrow (Q \wedge P)$
   $(P \vee Q) \leftrightarrow (Q \vee P)$

Theorem 3.3. Let x be an integer. If x^2 is odd, then x is odd.

Theorem (Contrapositive of the statement of Theorem 3.3). Let x be an integer. If x is even, then x^2 is even.

4 Set Notation and Quantifiers

4.1 Common Sets

The natural numbers : $\mathbb{N} = \{0, 1, 2, 3, ...\}$

The integers : $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$

The rational numbers : $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z} \ and \ q \neq 0\}$

The real numbers : $\mathbb{R}$

The complex numbers : $\mathbb{C} = \{a + bi : i^2 = -1 \ and \ a, b \in \mathbb{R}\}$

If A is one of the sets $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$, then the set of the positive elements is denoted by $A^+ = \{x \in A : x > 0\}$ and the set of the negative elements is denoted by $A^- = \{x \in A : x < 0\}$. Thus we have defined $\mathbb{Z}^+$, $\mathbb{Z}^-$, $\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{R}^+$, and $\mathbb{R}^-$

The plane $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$

For $n \in \mathbb{Z}^+$, Euclidean n-space $\mathbb{R}^n = \{(x_1, x_2, ..., x_n) : x_j \in \mathbb{R} \text{ for } j = 1, 2, ..., n\}$.

4.2 Symbols

For all $\forall$

There exists $\exists$

5 Proof Techniques

5.1 Definitions

Three methods discussed in this chapter:

- direct proof (just get started and keep going)
- proof by contradiction (show that the negation of the statement you wish to prove implies the impossible)
- proof in cases (which may be used when conditions dictate that different situations occur).

Definition 5.1. A nonzero integer a **divides** an integer b if there is an integer n such that $b = an$. We write this as $a|b$.

Definition 5.2. For a real number x, the **absolute value** of x is defined to be

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

5.2 Theorems

Theorem 5.1. If a, b and c are integers such that a divides b and a divides c, then a divides $b + c$.

Theorem 5.2. The number $\sqrt{2}$ is not rational.

Theorem 5.3. Let x and y be real numbers. Then $|xy| = |x||y|$.

6 Sets

6.1 Definitions

Definition 6.1. The **empty set**, denoted $\emptyset$ is the set with no elements.

Definition 6.2. The set A is a **subset** of the set B or, equivalently, A is **contained** in B, if every element of A is an element of B. We write $A \subseteq B$ to indicate that A is a subset of B.

Definition 6.3. The set A is a **proper subset** of B if $A \subseteq B$ and $A \neq B$, and we write $A \subset B$.

Definition 6.4. The set A is **equal** to B, written $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Definition 6.5. The **union** of the sets A and B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Definition 6.6. The **intersection** of the sets A and B is the set $A \cap B = \{x : x \in A \textit{ and } x \in B\}$.

Definition 6.7. Two sets A and B are **disjoint** if $A \cap B = \emptyset$.

Definition 6.8. The **set difference** of set B in set A is the set $A \setminus B = \{x \in A : x \notin B\}$.

Definition 6.9. If the set X is the universe and A is a subset of X, the **complement** of A is the set $A^c = X \setminus A$.

6.2 Theorem 6.11.

Let A be a set. Then $\emptyset \subseteq A$.