addition, the rates of exponential convergence decrease with Reynolds number (although the curves are not all well separated for $y_0 = 1$). These observations are also consistent with the expected convergence properties of the vorticity and the irrotational component of the velocity.

APPENDIX: Time Integration

Method B uses a low-storage third-order Runge–Kutta scheme designed by A. Wray (submitted to *J. Comput. Phys.*). Method A uses a slight generalization of that scheme, in order to provide implicit treatment of the viscous term, which is presented here.

The system of equations for the unknown vector function $\mathbf{u}$ can be written as

$$\frac{\partial \mathbf{u}}{\partial t} = L(\mathbf{u}) + N(\mathbf{u}) \equiv R(\mathbf{u}), \tag{A1}$$

where L is a linear operator and N is a nonlinear operator. They do not depend explicitly on t. Typically L comprises the viscous and pressure terms and N the convection term; but any linear term can be included in either operator. In particular, for method B, $L = 0$ and we include all terms in N. With a no-slip condition, as in method A, L is stiff and requires an implicit scheme if a reasonable time step is to be used. N is always hard to linearize in a spectral method and thus requires an explicit scheme if one wants to avoid iterating. We wish to extend Wray's analysis to obtain a low-storage, hybrid implicit/explicit, three-substep scheme of highest possible order.

Let $\mathscr{D}$ and $\mathscr{E}$ be the first and second derivatives of N with respect to $\mathbf{u}$. Then $\mathscr{D}$ is a linear operator, $\mathscr{E}$ is a bilinear operator which is symmetric with respect to its arguments, and

$$N(\mathbf{u} + d\mathbf{u}) = N(\mathbf{u}) + \mathscr{D}(d\mathbf{u}) + \tfrac{1}{2}\mathscr{E}(d\mathbf{u}, d\mathbf{u}) + O(d\mathbf{u}^3). \tag{A2}$$

Thus

$$\mathbf{u}(t + \Delta t) = \mathbf{u} + \Delta t\, R(\mathbf{u}) + \frac{\Delta t^2}{2}(L + \mathscr{D})(R(\mathbf{u}))$$

$$\times \frac{\Delta t^3}{6}[\mathscr{E}(R(\mathbf{u}), R(\mathbf{u})) + (L + \mathscr{D})^2(R(\mathbf{u}))] + O(\Delta t^4). \tag{A3}$$

To achieve the desired accuracy the numerical scheme should match this Taylor expansion to third order. The scheme to advance from $\mathbf{u}_n$, at time t, to $\mathbf{u}_{n+1}$, at time $t + \Delta t$, has three substeps:

$$\mathbf{u}' = \mathbf{u}_n + \Delta t[L(\alpha_1 \mathbf{u}_n + \beta_1 \mathbf{u}') + \gamma_1 N_n] \tag{A4a}$$

$$\mathbf{u}'' = \mathbf{u}' + \Delta t[L(\alpha_2 \mathbf{u}' + \beta_2 \mathbf{u}'') + \gamma_2 N' + \zeta_1 N_n] \tag{A4b}$$

$$\mathbf{u}_{n+1} = \mathbf{u}'' + \Delta t[L(\alpha_3 \mathbf{u}'' + \beta_3 \mathbf{u}_{n+1}) + \gamma_3 N'' + \zeta_2 N'], \tag{A4c}$$

where $N_n \equiv N(\mathbf{u}_n)$, $N' \equiv N(\mathbf{u}')$, and $N'' \equiv N(\mathbf{u}'')$. For N this is Wray's scheme (for each substep it is like Euler explicit or second-order Adams–Bashforth, but with different coefficients γ and ζ). For L this is like the Crank–Nicolson scheme on each substep but again with different coefficients. This extra freedom should allow us to obtain third-order accuracy. The advantage of the scheme described here is that it needs only the minimum of storage, actually no more than the Euler explicit scheme for the present methods. A disadvantage compared with Adams–Bashforth is that there are three different implicit operators ($[1 - \Delta t\, \beta L]$ in (A4)) so it would be more costly to precompute them, preinvert them, and store them. Currently we do not precompute the operators because the time step is adjusted continually to keep the peak CFL number constant.

The general scheme (A4) has 11 unknown coefficients and must satisfy 17 equations for third-order accuracy (expand (A3) and recall that L and $\mathscr{D}$ do not commute). However, Leonard (personal communication) in a similar scheme requires

$$\alpha_1 + \beta_1 = \gamma_1, \qquad \alpha_2 + \beta_2 = \gamma_2 + \zeta_1, \qquad \alpha_3 + \beta_3 = \gamma_3 + \zeta_2, \tag{A5}$$

which mean that the length of the substeps is the same in the scheme for L and the scheme for N. This seems reasonable and it reduces the system to eight equations in eight unknowns (because the terms $L(\mathbf{u})$ and $N(\mathbf{u})$ do not get separated). From here on we replace α_1 by $\gamma_1 - \beta_1$, and so on. The equations are, for first order,

$$\gamma_1 + \gamma_2 + \gamma_3 + \zeta_1 + \zeta_2 = 1; \tag{A6}$$

for second order,

$$\gamma_3(\gamma_1 + \gamma_2 + \zeta_1) + \gamma_1(\zeta_2 + \gamma_2) = \tfrac{1}{2}, \tag{A7a}$$

$$\gamma_1\beta_1 + \gamma_1(\gamma_2 + \zeta_1) + (\gamma_2 + \zeta_1)\,\beta_2 + (\gamma_3 + \zeta_2)(\gamma_1 + \gamma_2 + \zeta_1) + \beta_3(\gamma_3 + \zeta_2) = \tfrac{1}{2}; \tag{A7b}$$

for third order,

$$\gamma_1\gamma_2\gamma_3 = \tfrac{1}{6}, \tag{A8a}$$

$$\gamma_1^2(\gamma_2 + \zeta_2) + \gamma_3(\gamma_1 + \zeta_1 + \gamma_2)^2 = \tfrac{1}{3}, \tag{A8b}$$

$$\beta_3[\gamma_3(\gamma_1 + \gamma_2 + \zeta_1) + \gamma_1\zeta_2] + \gamma_1\gamma_2(\gamma_3 + \zeta_2) + \gamma_1\gamma_2\beta_2 = \tfrac{1}{6}, \tag{A8c}$$

$$\gamma_3[\gamma_1\beta_1 + (\gamma_2 + \zeta_1)(\gamma_1 + \beta_2)] + \gamma_1\beta_1(\gamma_2 + \zeta_2) = \tfrac{1}{6}, \tag{A8d}$$

$$\begin{aligned}&[(\gamma_1 + \beta_3 + \beta_2)(\zeta_1 + \gamma_2) + (\beta_3 + \beta_1)\,\gamma_1 + \beta_3^2]\,\zeta_2 \\ &\quad + [(\gamma_1 + \beta_3 + \beta_2)\,\gamma_3 + (\beta_2 + \beta_1)\,\gamma_1 + \beta_2^2]\,\zeta_1 \\ &\quad + [(\gamma_1 + \beta_3 + \beta_2)\,\gamma_2 + (\beta_3 + \beta_1)\,\gamma_1 + \beta_3^2]\,\gamma_3 \\ &\quad + [(\beta_2 + \beta_1)\,\gamma_1 + \beta_2^2]\,\gamma_2 + \beta_1^2\gamma_1 = \tfrac{1}{6}.\end{aligned} \tag{A8e}$$

Wray had obtained (A6), (A7a), and (A8a)–(A8b) (i.e., four equations for five unknowns) and found a one-parameter family of explicit third-order schemes.

Unfortunately this nonlinear system of equations (A6)–(A8) apparently does not have a solution. We sacrificed the last equation (A8e) because it involves only the viscous term. The scheme is still third-order on the pure convection terms and on the cross-terms. There is then a one-parameter family of such schemes, as found by Wray, and the mismatch in the last equation gets as low as 0.018. A good compromise between this mismatch and the desire to have fairly even substeps is

$$\gamma_1 = \tfrac{8}{15}, \quad \gamma_2 = \tfrac{5}{12}, \quad \gamma_3 = \tfrac{3}{4}, \quad \zeta_1 = -\tfrac{17}{60}, \quad \zeta_2 = -\tfrac{5}{12},$$
$$\alpha_1 = \tfrac{29}{96}, \quad \alpha_2 = -\tfrac{3}{40}, \quad \alpha_3 = \tfrac{1}{6},$$
$$\beta_1 = \tfrac{37}{160}, \quad \beta_2 = \tfrac{5}{24}, \quad \beta_3 = \tfrac{1}{6}.$$

Presumably full third-order accuracy could be obtained with a four-substep scheme, but our main interest is in the stability of the third-order Runge–Kutta scheme (the theoretical limit is a CFL number of $\sqrt{3}$). In practice, with the help of the viscous term and the intermittency of the high-velocity conditions, peak CFL numbers up to two are routinely used. The definition of CFL used here is

$$\mathrm{CFL} \equiv \frac{2}{3}\pi \left[\frac{|u|}{\Delta x} + \frac{|v|}{\Delta y} + \frac{|w|}{\Delta z} \right] \Delta t, \tag{A9}$$

where Δx, Δy, and Δz are quadrature grid spacings, the factor of $\frac{2}{3}$ is used to account for the fact that the quadrature mesh has $\frac{3}{2}$ as many points as modes in each coordinate direction for dealiasing and the definition for the y component is purely by analogy with the Fourier directions. This translates into a CFL number of $\frac{2}{3}$ per evaluation, appreciably larger than the CFL numbers used in similar simulations with the Adams–Bashforth scheme.

Another stability advantage of the new scheme is that when the eigenvalues of L are large in magnitude, their eigenvectors are damped by a ratio approaching $\frac{87}{185}$ at each full step. In contrast, with Crank–Nicolson there is little damping since the ratio approaches -1. However, this extreme situation is not encountered in our simulations because the viscosity is small.

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