ME 5311 Project (ver. 1.1)

Two-dimensional turbulence

March 22, 2023

1 Problem description

The problem is to simulate a two-dimensional incompressible flow in a doubly-periodic domain with dimensions $[2\pi \times 2\pi]$. The governing equations are the conservation of mass and x- and y-momentum:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

where $p = p'/\rho$ is the scaled pressure and ν the kinematic viscosity. With appropriate initial conditions for the velocity vector and pressure. We will base our initial condition on the Taylor–Green vortex.

2 Numerical solution

We will use a vorticity–streamfunction method to numerically integrate (1)–(3). We will replace u, v and p with two new variables, the vorticity

$$\omega = \nabla \times \mathbf{u} \tag{4}$$

and the streamfunction ψ, where $\mathbf{u} = [u, v]$ is the velocity vector. The velocity components are related to the streamfunction

$$u = \frac{\partial \psi}{\partial y}, \tag{5}$$

$$v = -\frac{\partial \psi}{\partial x}. \tag{6}$$

In two-dimensions, there is only one vorticity component, and (2)–(3) become a single evolution equation for the scalar ω

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \tag{7}$$

The streamfunction is related to vorticity

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\omega \tag{8}$$

The boundary conditions are periodic for all variables, which significantly simplifies the problem. In general, boundary conditions are needed for (7)–(8), which can be challenging to derive depending on the type of boundary condition for the velocity or pressure.

Equation (4)–(8) replace the original system (1)–(3). The new system (4)–(8) has the advantage of a single scalar PDE for the evolution of ω in time, instead of the evolution equation for the velocity vector (2) and (3).

We will use the RK3 of Homework 3 to perform the time integration. For each substep of the RK3 the following steps are performed, starting from u, v, and ω:

1. Time-advance ω to the next substep
2. Solve (8) to obtain ψ
3. Use (5) and (6) to obtain u and v

We will use a pseudo-spectral method to solve (4)–(8). Thus, we will use the two-dimensional Discrete Fourier Transform (DFT) $\hat{\omega}_{pq}$ of ω_{ij}. The DFT-ed equation (7) is

$$\frac{\partial I_{pq}\hat{\omega}_{pq}}{\partial t} + \mathbf{i}k_p\widehat{u_{ij}\omega_{ij}}I_{pq} + \mathbf{i}k_q\widehat{v_{ij}\omega_{ij}}I_{pq} = 0, \tag{9}$$

where

$$I_{pq} = e^{\nu(k_p^2+k_q^2)t}, \tag{10}$$

is an integrating factor. Equation (8) becomes

$$k_p^2\hat{\psi}_{pq} + k_q^2\hat{\psi}_{pq} = \hat{\omega}_{pq} \tag{11}$$

We will carry out the project in two stages:

1. Poisson equation solution: due March 29.
2. Full solution: due April 27.

2.1 Poisson equation solution [16 points]

The first solution stage, due March 29, is to implement the numerical solution of the Poisson equation for the stremfunction

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\omega(x, y), \tag{12}$$

in $[L_x \times L_y]$ with periodic boundary conditions.

To verify the accuracy of the method, choose L_x and L_y and use

$$\omega(x, y) = \frac{4\pi^2(L_x^2 + L_y^2)}{L_x^2 L_y^2}\sin\left(\frac{2\pi x}{L_x}\right)\cos\left(\frac{2\pi y}{L_y}\right). \tag{13}$$

The exact solution is

$$\psi(x,y) = \sin\left(\frac{2\pi x}{L_x}\right)\cos\left(\frac{2\pi y}{L_y}\right). \tag{14}$$

Show that the numerical solution agrees with the exact solution for different number of grid points M and N and L_x and L_y.

Perform a second, and more rigorous verification test, using

$$\begin{aligned}\omega(x,y) = \frac{4\pi^2}{L_x^2 L_y^2}\mathrm{Exp}\left[\sin\left(\frac{2\pi x}{L_x}\right)+\cos\left(\frac{2\pi y}{L_y}\right)\right]\Bigg(L_y^2\left(-\cos\left(\frac{2\pi x}{L_x}\right)^2+\sin\left(\frac{2\pi x}{L_x}\right)\right)\\ + L_x^2\left(\cos\left(\frac{2\pi y}{L_y}\right)^2-\sin\left(\frac{2\pi y}{L_y}\right)\right)\Bigg)\end{aligned} \tag{15}$$

The exact solution is

$$\psi(x,y) = \mathrm{Exp}\left[\sin\left(\frac{2\pi x}{L_x}\right)+\cos\left(\frac{2\pi y}{L_y}\right)\right]. \tag{16}$$

Show that the convergence rate is spectral.

2.2 Full solution [32 points]

The last step is the full solution of the two-dimensional Navier–Stokes using the vorticity–streamfunction method. We will use the Taylor–Green vortex solution to verify our implementation.

The initial condition is

$$u(x,y) = \cos x \sin y \tag{17}$$

$$v(x,y) = -\sin x \cos y \tag{18}$$

in domain $[2\pi\times 2\pi]$. The Taylor–Green vortex is one of the few exact solutions of the Navier–Stokes:

$$u(t,x,y) = e^{-2\nu t}\cos x \sin y \tag{19}$$

$$v(t,x,y) = -e^{-2\nu t}\sin x \cos y \tag{20}$$

We will perform the following tasks:

a. Verify that your numerical solution agrees with the exact solution. When using the integrating factor the numerical solution agrees with the exact solution within roundoff error accuracy.

b. Show that the velocity field is always divergence-free.

c. The Taylor–Green flow is unstable. Add random perturbations to the velocity or vorticity field and plot the vorticity field after instability develops.

Because this is a "fundamental" flow you can write your code assuming that the domain is always $[2\pi \times 2\pi]$ and that the number of grid points is the same in both directions, $M = N$. If in the future you want to use your code with $M \neq N$, do not forget to carry out the verification steps!

Also, **important**: You must submit your computer code, all required plots, and all documentation and information regarding your simulations. If your implementation allows, please submit a single file with all the code. You need to assume that you are presenting this information to colleagues, e.g., through a paper or presentation. You DO NOT need to write a paper and DO NOT need to make a presentation, but you need to provide all the information for someone to reproduce your results. The information should be concise and clear.

Therefore, based on the information you submit, the instructor will attempt to reproduce your plots with his code. If your simulation setup writeup is incomplete, incorrect, or the instructor takes more than 60 seconds to comprehend your documentation, then you'll loose 8 points from this item.

Table 1: Values of grid points M and viscosity ν for well-resolved flow.

M	ν
64	5×10^{-3}
128	2×10^{-3}
256	9×10^{-4}
512	2×10^{-4}
1024	8×10^{-5}

2.3 Extra credit A [2 points]

Add a passive scalar in the problem and integrate an additional equation for the evolution of the passive scalar fluctuation ϕ:

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + bv = \frac{\nu}{Sc}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right), \tag{21}$$

where Sc is the Schmidt number and b the constant scalar gradient, such that the scalar field is $\Phi(t, x, y) = by + \phi(t, x, y)$. Note that ϕ has periodic boundary conditions, but Φ does not.

Deliverables: submit your code and two images of the scalar field ϕ at different times.

2.4 Extra credit B [4 points]

Compute forced two-dimensional turbulence. To sustain turbulence we need to force the flow. Because the flow is periodic we need an "artificial" method to provide stirring. The simplest way to force the flow is to change the sign of the viscosity coefficient for a band of the wavenumber vector magnitude $k_{\mathrm{low}} < (k_p^2+k_q^2)^{\frac{1}{2}} < k_{\mathrm{hi}}$. Also, we will need to remove the energy that accumulates at the large-scale. An easy way to do that is to increase the viscosity coefficient for the highest wavenumbers, e.g., $(k_p^2 + k_q^2)^{\frac{1}{2}} > 4$.

Choose a viscosity coefficient, such that $3 < k_{\mathrm{max}}\eta < 1.5$, where k_{max} is the largest dealiased wavenumber and η is the Kolmogorov scale.

Deliverables: submit your code, the list of parameters you have used, and images of the vorticity, u, v and dissipation fields at a time when the flow is in a stationary state.

You can submit extra credit A or B or both.

2.5 Testing, assessment and debugging

It is important to verify different parts of the implementation independently. When all the elements are put together, it is very difficult to diagnose potential mistakes. Imagine you are baking a cake.

After you baked the cake, if the cake tastes bad, it is difficult to find out if you used a spoiled ingredient and which is the bad ingredient. If you verify the freshness of each ingredient as you mix the cake batter, then it is easy to make a great-tasting cake. Let's pretend to be good cooks, engineers, scientist — or whatever you aspire to be — for this assignment!

Some tests you can try and words of wisdom:

a. Verify that you can compute first and second derivatives of functions of two variables spectrally. For instance, discretize $\phi(x, y) = \sin x \cos y$ and compute different derivatives.

b. Solve the two-dimensional convection equation $\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = 0$. You can simplify this test even more by solving one-dimensional convection equations for the 2D function ϕ by alternatively setting $u = 0$ or $v = 0$.

c. Verify that the velocity field is always divergence-free.

d. Time-advance the vorticity equation without the viscous term and check that the velocity field remains divergence free. You can only do this for a few time steps.

e. Do not use the integrating factor and use a separate viscous term.

f. Use a debugger.

g. Plot fields instead of looking at numbers on screen. Debug like it's 2023!

h. Write simple and straightforward code. Spend some time organizing and designing. The method is simple and elegant and it is possible to write very compact code (about 100 lines of code for the entire project). However, please avoid fancy coding until you have everything working.

i. **VERY IMPORTANT**: Please make sure you completely understand the indexing and plotting of your programing environment or the specific plotting command you are using. It is typical to index two-dimensional variables as ω_{ij} with indexes increasing in the direction of the Cartesian coordinates x and y. But on the computer we use "array" `omega(i,j)` and how the object `omega` is printed or plotted depends on the programing language or function.

2.6 Figures

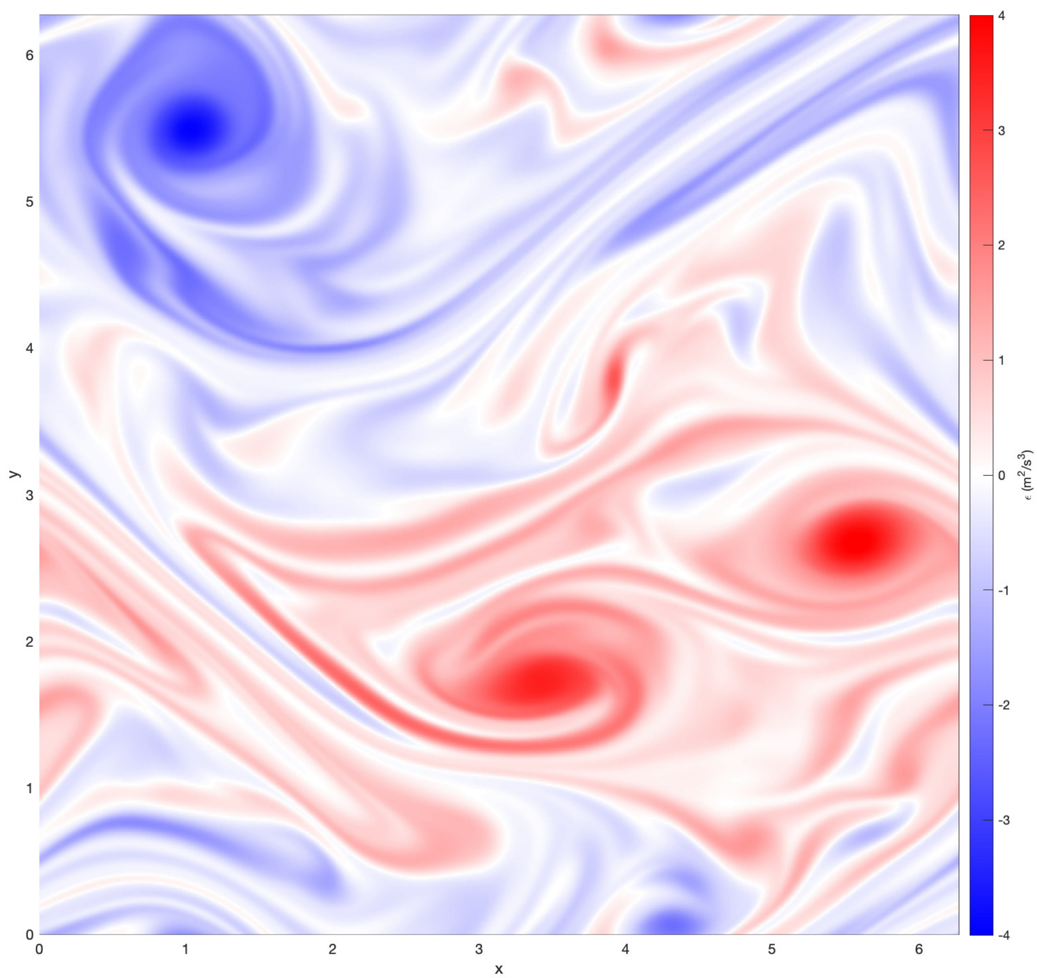

Figure 1: Vorticity field of forced two-dimensional turbulence with 512^2 grid.

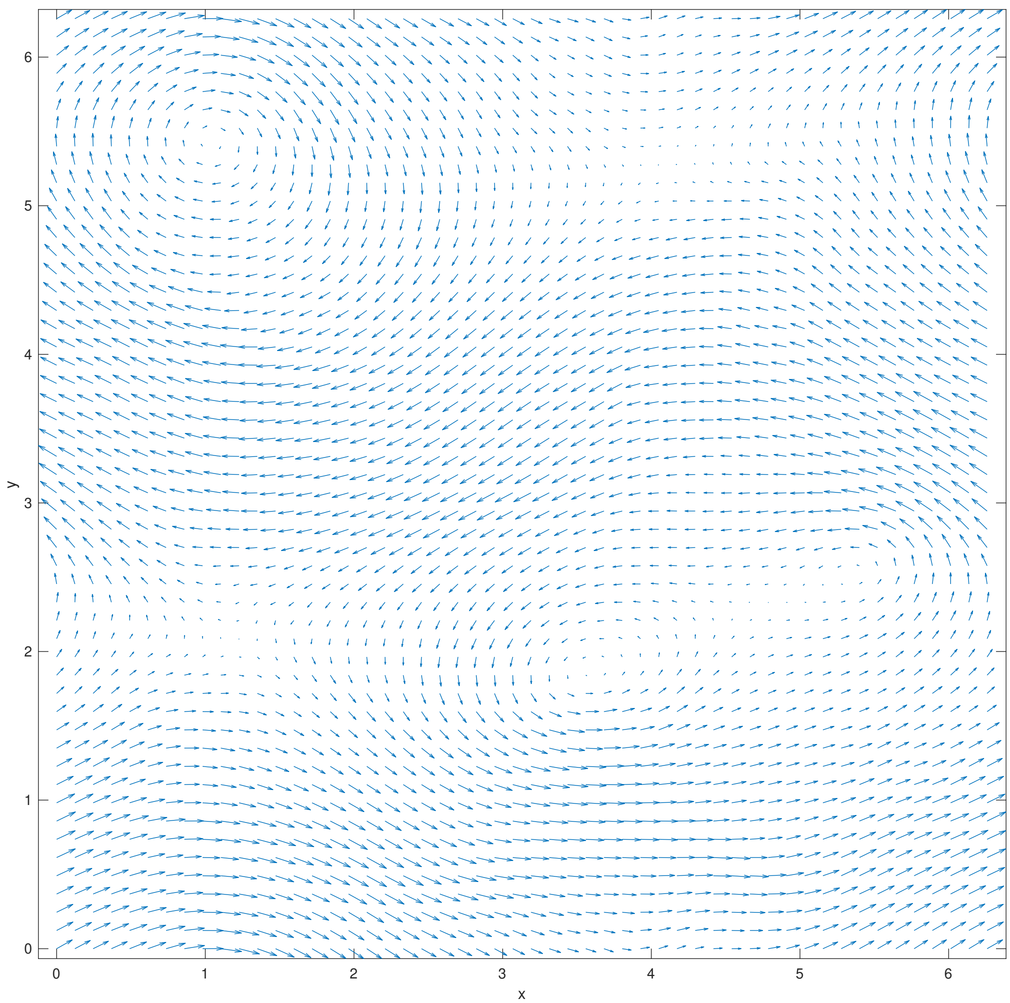

Figure 2: Velocity field of forced two-dimensional turbulence with 512^2 grid.

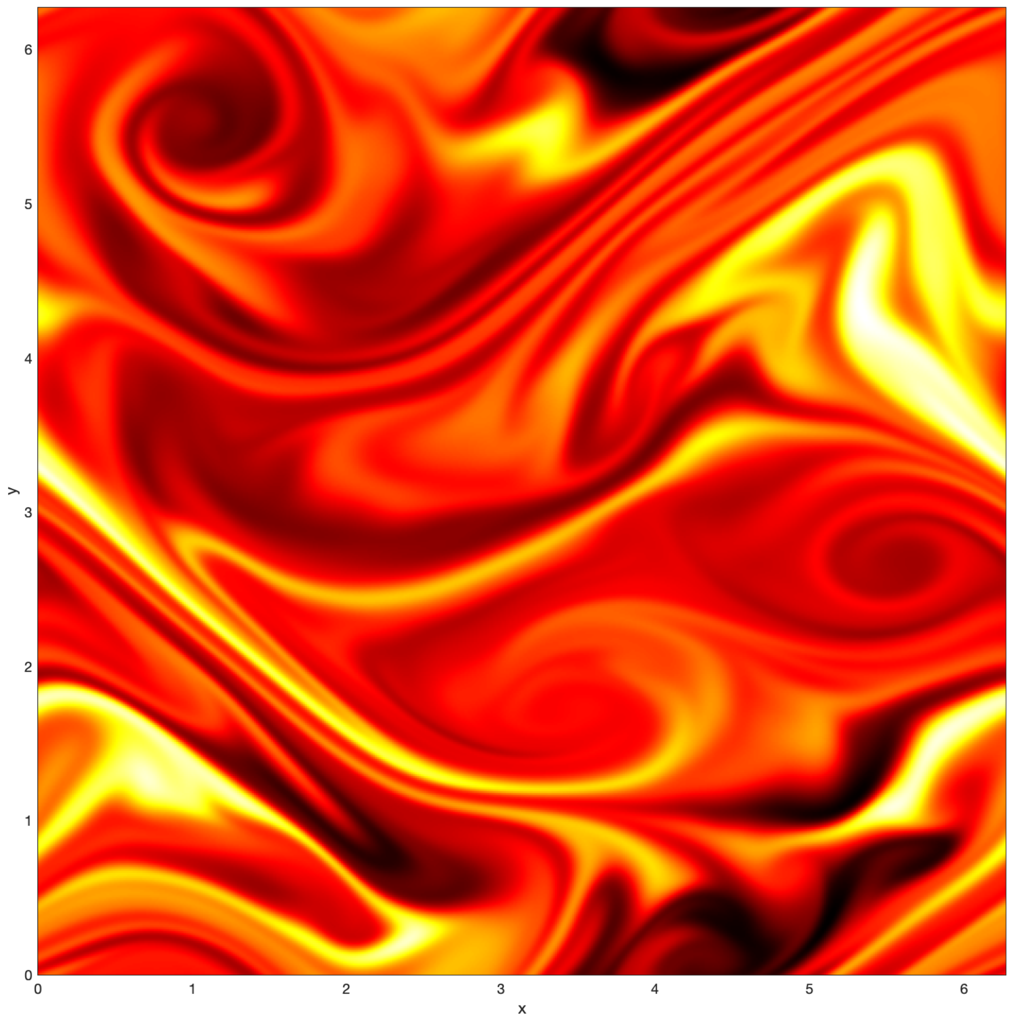

Figure 3: Scalar field of forced two-dimensional turbulence with 512^2 grid.

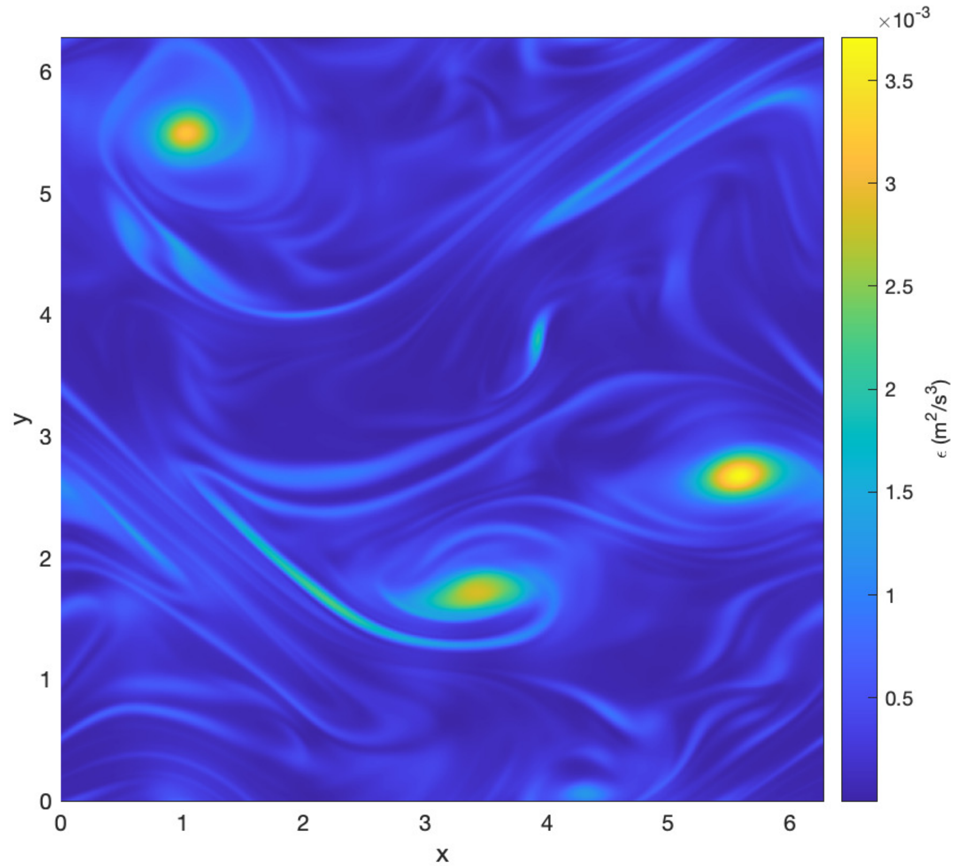

Figure 4: Dissipation field of forced two-dimensional turbulence with 512^2 grid.