Two dimensional Incompressible Navier-Stokes

2 dimensions: $[x, y]$ $\quad \vec{u} = [u, v]$

incompressible means $\underset{\text{density}}{\rho} = \text{constant}$

Mass : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ or $\nabla \cdot \vec{u} = 0$

Conservation form

Conservation of mass in volume $\mathcal{V}$: $\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho \, d\mathcal{V} + \int_S \rho \, \vec{u} \cdot \hat{n} \, ds = 0$

if $\rho = \text{const} \Rightarrow \int_S \vec{u} \cdot \hat{n} \, ds = 0$

$$\int_{\mathcal{V}} \nabla \cdot \vec{u} \, d\mathcal{V} = 0 \Rightarrow \nabla \cdot \vec{u} = 0$$

Kinematic condition for velocity vector

x-momentum: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

y-momentum: $\underbrace{\underbrace{\frac{\partial v}{\partial t}}_{\text{unsteady term}} + \underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}}_{\text{convection}}}_{\text{Fluid acceleration } \frac{D\vec{u}}{Dt}} = \underbrace{\underbrace{-\frac{\partial p}{\partial y}}_{\text{isotropic stress}} + \underbrace{\nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}_{\text{deviatoric stress}}}_{\text{force from stress}} \quad \underbrace{\rho\vec{g}}_{\text{body forces}}$

Momentum is a vector equation: $\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \nabla \cdot \underset{\text{dev. stress tensor}}{\tau_{ij}}$

diffusivity coeff.

Scalar transport : $\frac{\partial \varphi}{\partial t} + u\frac{\partial \varphi}{\partial x} + v\frac{\partial \varphi}{\partial y} = \nu\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right)$

The role of pressure

Chapters 7.1.5, 8.3.3 Ferziger

variables

coefficients or constants

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = K\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right)$$

- How many equations: 4
- How many variables: 4

Where is the equation for pressure?

$$\nabla\cdot\underbrace{\left(\frac{\partial}{\partial t}\vec{u} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \nu\nabla^2\vec{u}\right)}_{\text{vector}}$$

$$\frac{\partial}{\partial t}\underbrace{\nabla\cdot\vec{u}}_{=0} + \nabla\cdot(\vec{u}\cdot\nabla\vec{u}) = -\nabla\cdot\nabla p = \nu\nabla^2\underbrace{(\nabla\cdot\vec{u})}_{=0}$$

$$\nabla\cdot\nabla p = -\nabla\cdot(\vec{u}\cdot\nabla\vec{u})$$

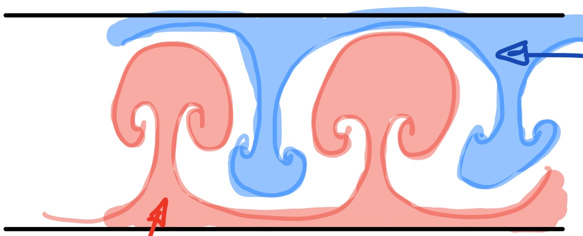
$$\nabla^2 p = -\nabla\cdot(\vec{u}\cdot\nabla\vec{u}) \leftarrow \text{Poisson Equation for pressure}$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \cdots$$

Boundary condition on walls and inflow/outflow: $\frac{\partial p}{\partial n} = 0$

Buoyant Convection: Introduction

cold wall



fluid cools near the top wall and its buoyancy decreases

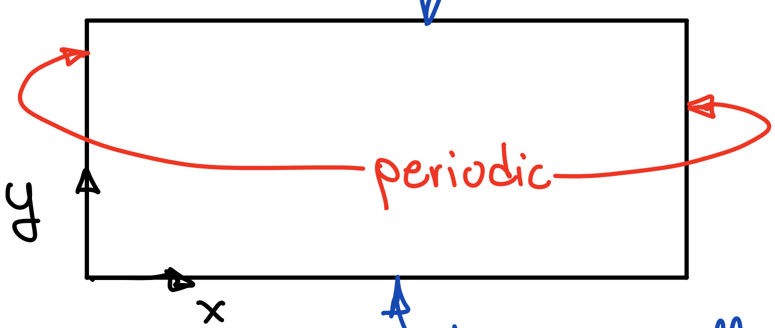
hot wall

fluid warms up near the bottom hot wall and rises (becomes positively-buoyant)

Because we will consider flow between two vertical plates with no horizontal boundaries the horizontal direction will be periodic

Computational Domain:

Top wall • no slip
• cold temperature



bottom wall : • no slip
• warm temperature

The role of pressure: The sequel

given a velocity field $\vec{u}^* = [u^*, v^*]$

find a vector $\vec{\varphi}$ that makes $\underbrace{\nabla \cdot (\vec{u}^* + \vec{\varphi}\,)}_{\vec{u}^* + \vec{\varphi} \text{ is divergence free}} = 0$

we want $\nabla \cdot (\vec{u}^* + \vec{\varphi}\,) = 0$

$$\nabla \cdot (\vec{u}^* + \nabla p) = 0$$

scalar

$$\nabla \cdot \vec{u}^* + \nabla \cdot \nabla p = 0$$

$$\nabla^2 p = -\nabla \cdot \vec{u}^*$$

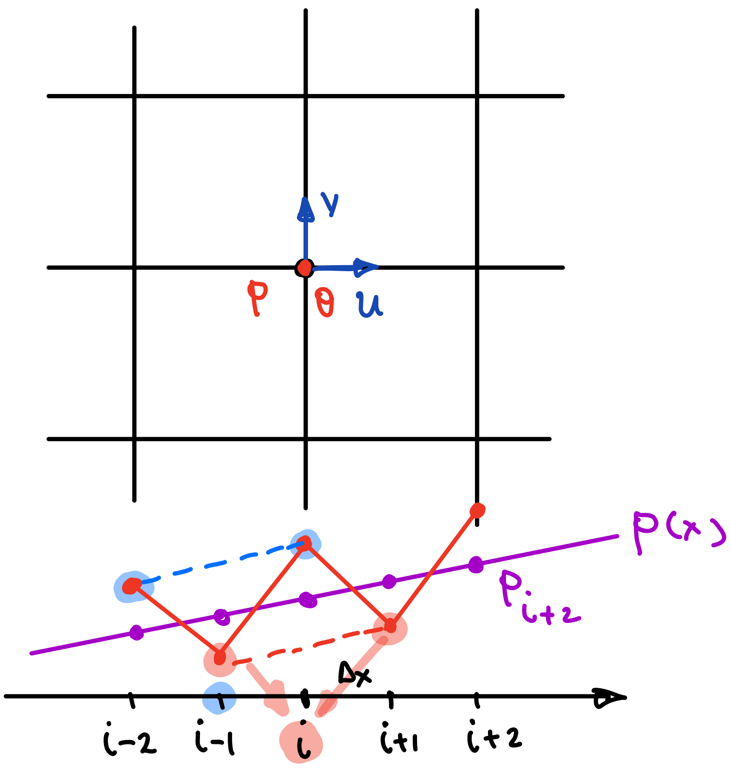
$$\frac{\partial \vec{u}}{\partial t} : \vec{u}^n \xrightarrow{\text{Runge Kutta}} \vec{u}^{n+a} : \vec{u}^* \rightarrow \nabla^2 p = -\nabla \cdot \vec{u}^* \rightarrow \vec{u}^{n+a} = \vec{u}^* + \nabla p$$

Variable Arrangement: u, v, p, θ

Chapter 7.1.4 Ferziger

Location, Location, Location!

Collocated



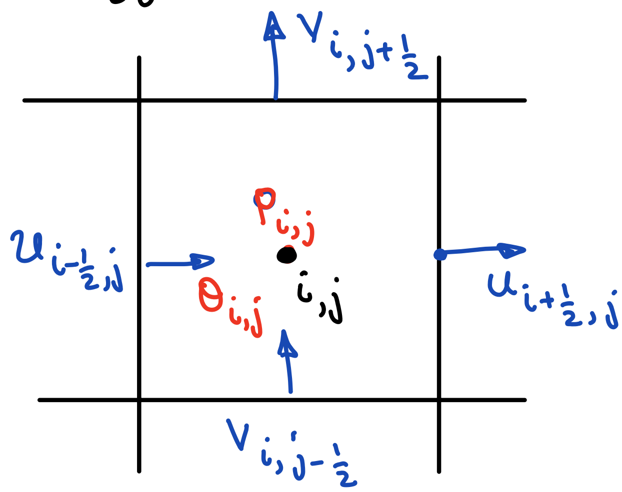
$$\frac{dp}{dx}\bigg|_i = \frac{p_{i+1} - p_{i-1}}{2\Delta x}$$

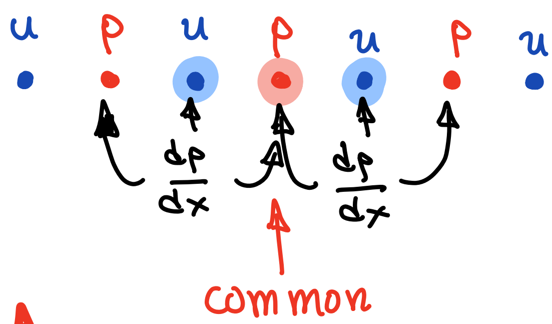
$$\frac{dp}{dx}\bigg|_{i-1} = \frac{p_i - p_{i-2}}{2\Delta x}$$

Even-Odd decoupling

results in pressure oscillations

Staggered

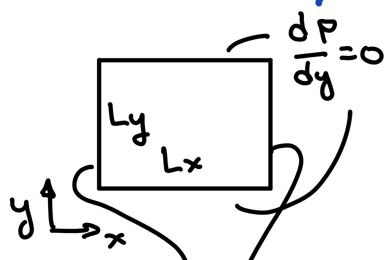




use this grid in our project

Discretization of Pressure Poisson Equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = d(x,y)$$



periodic x-direction

second order finite differences

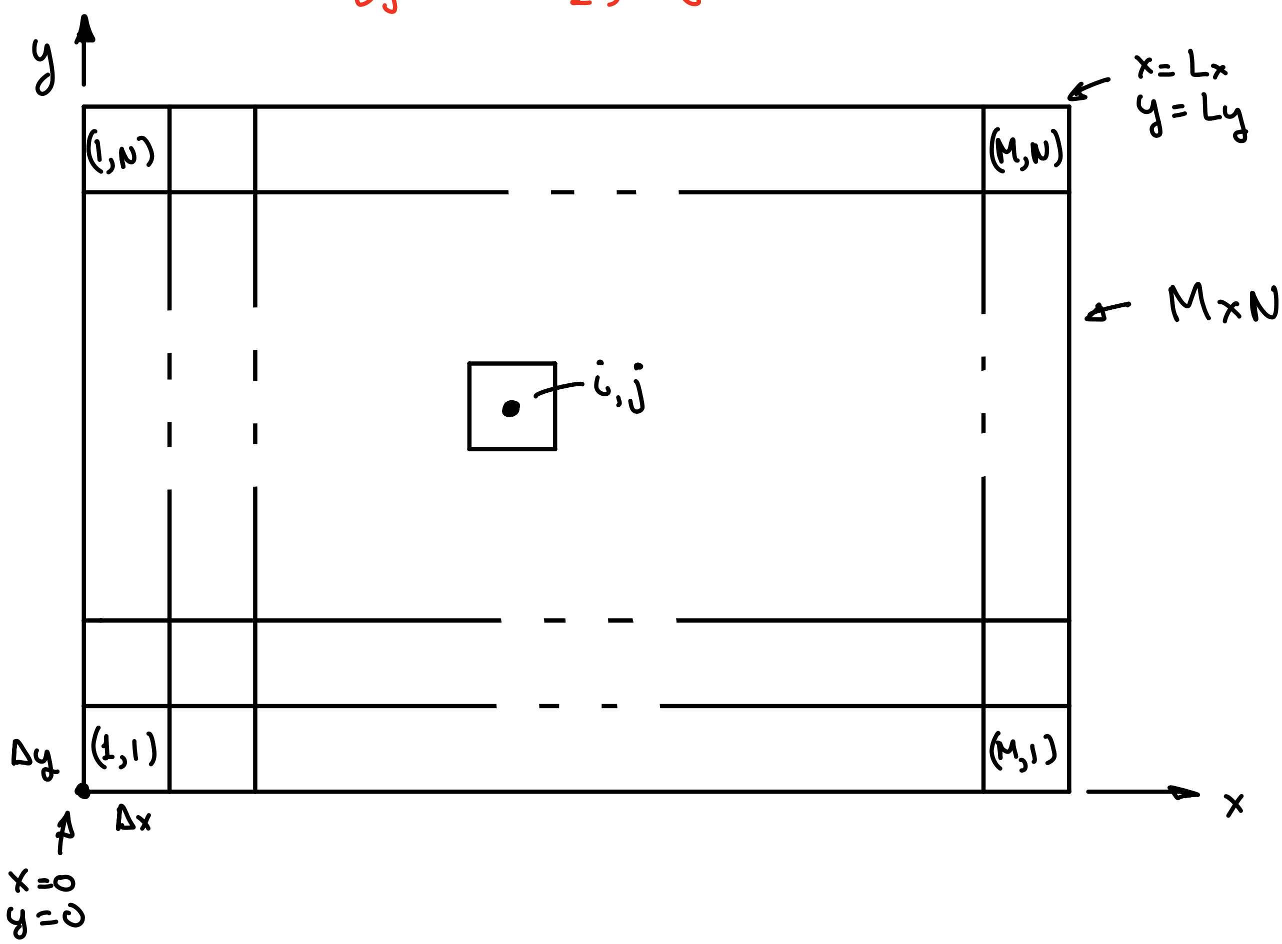
$$\frac{p_{i-1,j} - 2p_{i,j} + p_{i+1,j}}{\Delta x^2} + \frac{p_{i,j-1} - 2p_{i,j} + p_{i,j+1}}{\Delta y^2} = d(i,j)$$

known

Important: pressure variable location is at the center of grid cell

p_{ij} means $x_i = (i - \frac{1}{2})\Delta x$ for $i = 1, 2, \dots M$

$y_j = (j - \frac{1}{2})\Delta y$ for $j = 1, 2, \dots N$



Poisson equation for pressure:

$$\frac{1}{\Delta x^2}\left(p_{i-1,j} - 2p_{i,j} + p_{i+1,j} \right) + \frac{1}{\Delta y^2}\left(p_{i,j-1} - 2p_{i,j} + p_{i,j+1} \right) = d_{i,j} \quad (1)$$

where : $\left.\begin{array}{l} i = 1, \dots, M \\ j = 1, \dots, N \end{array}\right\}$ $M \times N$ grid with constant $\Delta x, \Delta y$

that is : $\Delta x = L_x / M$

$\Delta y = 1/N$

Boundary conditions:

- periodic in x-direction
- $\frac{dp}{dy} = 0$ at $y=0$, $y=1$ on top and bottom walls

Solution method:

We will take advantage of the periodicity in x and utilize a Fourier expansion

Remember : $p_{i,j} = \sum_{k=0}^{M-1} \hat{p}_{k,j} \, e^{i k 2\pi \frac{\Delta x i}{L}}$ (2)

Fourier coefficients from x-direction

Discrete Fourier Transform

Use 2 in 1:

$$\sum_{k=0}^{M-1}\left[\frac{1}{\Delta x^2}\hat{p}_{k,j}\left(e^{ik\frac{2\pi}{L}\Delta x(i-1)}-2e^{ik\frac{2\pi}{L}\Delta x i}+e^{ik\frac{2\pi}{L}\Delta x(i+1)}\right)\right.$$

$$\left.+\frac{1}{\Delta y^2}\left(\hat{p}_{k,j-1}-2\hat{p}_{k,j}+\hat{p}_{k,j+1}\right)e^{ik\frac{2\pi}{L}\Delta x i}\right]=\sum_{k=0}^{M-1}\hat{d}_{k,j}\,e^{ik\frac{2\pi}{L}\Delta x i}$$

Because of the orthogonality property of the $e^{ik\frac{2\pi}{L}\Delta x i}$ we can write separate equations for each k

Thus...

$$\frac{1}{\Delta x^2}\hat{p}_{k,j}\underbrace{\left(e^{-ik\frac{2\pi\Delta x}{L}}-2+e^{ik\frac{2\pi\Delta x}{L}}\right)}_{\left(-2+2\cos\left(2\pi\frac{k}{M}\right)\right)}+\frac{1}{\Delta y^2}\left(\hat{p}_{k,j-1}-2\hat{p}_{k,j}+\hat{p}_{k,j+1}\right)=\hat{d}_{k,j}$$

for $k=0,\dots,M-1$

we replaced $\frac{\Delta x}{L}=\frac{1}{M}$

... and we have M equations:

3

$$\frac{1}{\Delta y^2}\hat{p}_{k,j-1}+\left[-\frac{2}{\Delta y^2}+\frac{1}{\Delta x^2}\left(-2+2\cos\left(2\pi\frac{k}{M}\right)\right)\right]\hat{p}_{k,j}+\frac{1}{\Delta y^2}\hat{p}_{k,j+1}=\hat{d}_{k,j}$$

where $j=1,\dots,N$

$k=0,\dots,M-1$

$$\frac{1}{\Delta y^2}\hat{p}_{k,j-1} + \left[-\frac{2}{\Delta y^2} + \frac{1}{\Delta x^2}\left(-2+2\cos\left(2\pi\frac{k}{M}\right)\right)\right]\hat{p}_{k,j} + \frac{1}{\Delta y^2}\hat{p}_{k,j+1} = \hat{d}_{k,j}$$

where $j = 1, \dots, N$
$k = 0, \dots, M-1$

Two important observations:

- There are no $k-1$ or $k+1$ indexes
  This is because the periodic boundary condition is "encapsulated" in the Fourier transform.
  In other words, we can only use a Fourier transform if the function $p(x,y)$ is periodic in x

- The j-index includes $j-1$ and $j+1$ terms:

when $j = 1$:

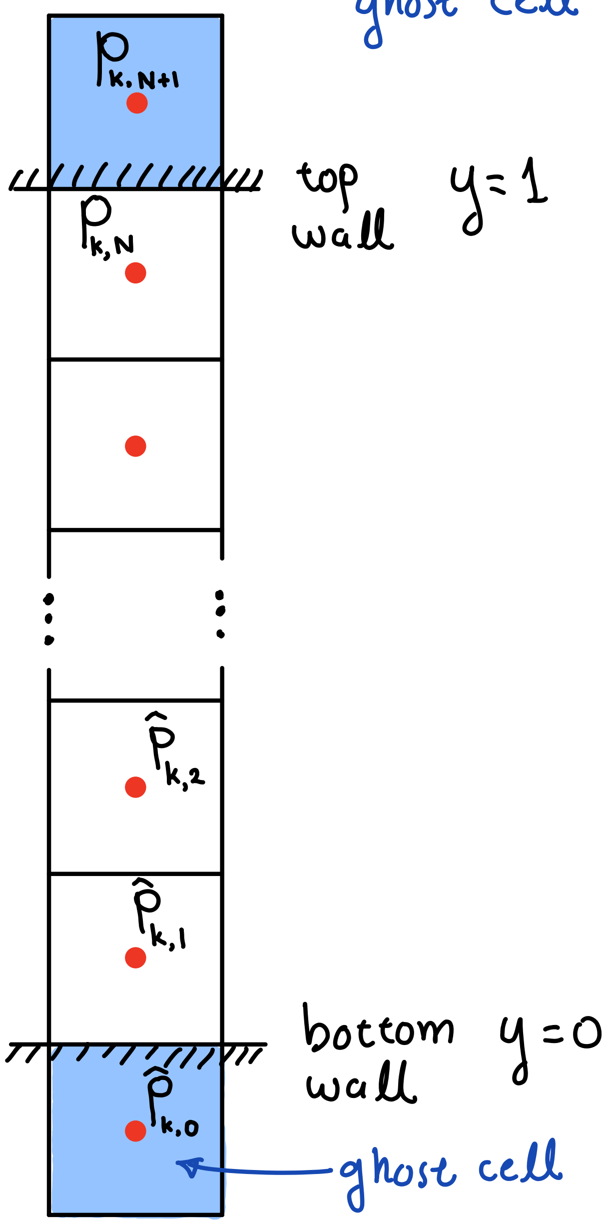
$$\frac{1}{\Delta y^2}\hat{p}_{k,0} + \left[-\frac{2}{\Delta y^2} + \frac{1}{\Delta x^2}\left(-2+2\cos\left(2\pi\frac{k}{M}\right)\right)\right]\hat{p}_{k,1} + \frac{1}{\Delta y^2}\hat{p}_{k,1} = \hat{d}_{k,1}$$

when $j = N$:

$$\frac{1}{\Delta y^2}\hat{p}_{k,N-1} + \left[-\frac{2}{\Delta y^2} + \frac{1}{\Delta x^2}\left(-2+2\cos\left(2\pi\frac{k}{M}\right)\right)\right]\hat{p}_{k,N} + \frac{1}{\Delta y^2}\hat{p}_{k,N+1} = \hat{d}_{k,N}$$

we will use the boundary condition to find these

Boundary conditions for 3



at walls: $\frac{\partial p}{\partial y} = 0$

$y=0$: $\frac{\hat{p}_{k,1} - \hat{p}_{k,0}}{\Delta y} = 0$

$\Rightarrow \hat{p}_{k,1} = \hat{p}_{k,0}$

$y=1$: $\hat{p}_{k,N+1} = \hat{p}_{k,N}$

Solution of 3 :

we have M tridiagonal systems to solve

... actually we will set $p_{0,j} = 0$ for all j

so we will solve only M-1 systems

The systems are of the form:

$$\begin{bmatrix} \ddots & \ddots & & 0 \\ \ddots & b & c & \\ & a & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \cdot \hat{p}_{k,j} = \hat{d}_{k,j} \qquad 4$$

Vectors: $a(:) = \frac{1}{\Delta y^2}$

$$c(:) = \frac{1}{\Delta y^2}$$

depends on k!

$$b(2:N-1) = \left[-\frac{2}{\Delta y^2} + \frac{1}{\Delta x^2}\left(-2+2\cos\left(2\pi \frac{k}{M}\right)\right)\right]$$

$$b(1) = b(N) = \left[-\frac{2}{\Delta y^2} + \frac{1}{\Delta x^2}\left(-2+2\cos\left(2\pi \frac{k}{M}\right)\right)\right] + \frac{1}{\Delta y^2}$$

Algorithm:

1. Input
   a. RHS $d_{i,j}$ is $M \times N$ array
   b. Δx
   c. Δy

2. Compute $\hat{d}_{k,j}$ by taking N FFTs along direction i

3. Compute $\hat{\varphi}_{k,j}$

   Form and solve $M-1$ tridiagonal systems 4

   Set $\hat{\varphi}_{0,j} = 0$

4. Compute $\varphi_{i,j}$

   Perform N inverse FFTs along direction i

Verification

Use the method of manufactured solutions

"manufactured solution" must satisfy boundary conditions:

- periodic in x with period L_x
- $\frac{dp}{dy} = 0$ at $y=0$ and $y=L_y$

take $p(x,y) = \sin\left(n \frac{2\pi}{L_x} x\right) \cos\left(m \frac{2\pi}{L_y} y\right)$ (5)

integers

PDE: $\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = d(x,y)$ (6)

Use (5) in LHS of (6):

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\left[\left(n \frac{2\pi}{L_x}\right)^2 + \left(m \frac{2\pi}{L_y}\right)^2\right] \sin\left(n \frac{2\pi}{L_x} x\right) \cos\left(m \frac{2\pi}{L_y} y\right) \quad (7)$$

thus...

$$d(x,y) = -\left[\left(n \frac{2\pi}{L_x}\right)^2 + \left(m \frac{2\pi}{L_y}\right)^2\right] \sin\left(n \frac{2\pi}{L_x} x\right) \cos\left(m \frac{2\pi}{L_y} y\right)$$

when $p(x,y)$ is that of (5)

So, numerically integrate (7), and verify second-order convergence to the exact solution (5) for m, n of your choise