ME 5311 ASSIGNMENT 4 JACOB IVANOV

1. Model Diffusion Half-Life

The original PDE Diffusion model is defined as follows:

$$\begin{cases} \partial_t[u_k] = \alpha \cdot \partial_x^2[u_k], \quad x \in [0, 2\pi], \quad t \geq 0 \\ u_k(t=0, x) = \sin(kx), \quad x \in [0, 2\pi] \\ u_k(t, x=0) = u_k(t, x=2\pi) = 0, \quad t \geq 0 \end{cases} \tag{1}$$

By assuming Seperation of Variables:

$$u_k(t, x) = T(t) \cdot X(x) \tag{2}$$

$$T'X = \alpha T X'' \tag{3}$$

$$\frac{T'}{T} = \alpha \frac{X''}{X} = c_1 \tag{4}$$

where c_1 is an arbitrary constant since T'/T can only equal X''/X if they are both the same constant.

From which we can solve for T, X independently:

$$T' - c_1 T = 0 \tag{5}$$

$$X'' - \frac{\alpha}{c_1} X = 0 \tag{6}$$

if we assume an exponential form of $T = e^{\lambda t}$, it can be shown that $c_1 = \lambda$. As for X, if we assume a trigonometric form $X(x) = b_1 \cos(kx) + b_2 \sin(kx)$, find X' and X'', plug into Eq. (6) and split by trigonometric term, it can be shown that $c_1 = -\alpha k^2$. As a result, our general solution (restricted to where $X(x)$ is trigonometric, rather than exponential) is as follows:

$$\begin{aligned} u_k(t, x) &= T(t) X(x) \\ &= e^{-\alpha k^2 t} \left[b_1 \cos(kx) + b_2 \sin(kx) \right] \end{aligned} \tag{7}$$

If we apply our initial conditions (and indirectly our boundary conditions), it can be seen that $b_1 = 0$ and $b_2 = 1$. As a result, we are left with:

$$u_k(t, x) = e^{-\alpha k^2 t} \sin(kx) \tag{8}$$

By defining a half-life time $t_{1/2}$ as follows, it can be directly found with some logarithm algebra.

$$u_k(t = t_{1/2}, x) = \frac{1}{2} u_k(t=0, x) \tag{9}$$

$$t_{1/2} = \frac{\ln(2)}{\alpha k^2} \rightarrow t \propto \frac{1}{k^2} \tag{10}$$

Therefore, the half-life decay time will be inversely proportional to the square of the wavenumber in the initial condition. For a linear combination of initial conditions, finer perturbations will decay quadratically faster than those on the large scale.

2. Model Advection

The advection phenomenon will be modelled as follows:

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) = 0 \\ u(t, x=0) = \sin(8t), \quad t \in [0, 2] \\ t \in [0, 2], \quad x \in [0, 1] \end{cases} \tag{11}$$

Though the analytical solution can be found with the Method of Characteristics, we will instead solve it numerically with a Runge-Kutta 3rd Integration Method. To handle the right boundary, we will use a one sided finite difference. The solution at various times is shown below: