Non dimensional form of equations of motion

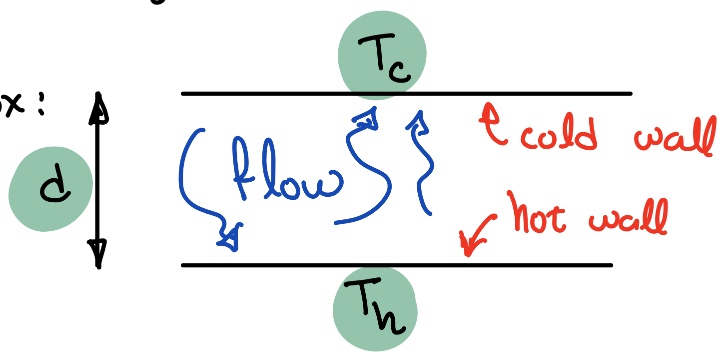
we will use primes to denote dimensional variables
for example u' is x-component of velocity in m/s

mass: $$\frac{\partial u_i'}{\partial x_i'} = 0$$

momentum $$\frac{\partial u_i'}{\partial t'} + u_j' \frac{\partial u_i'}{\partial x_j'} = -\frac{\partial p'}{\partial x_i'} + \nu \frac{\partial^2 u_i'}{\partial x_j'^2}$$

temperature $$\frac{\partial \theta'}{\partial t'} + u_j' \frac{\partial \theta'}{\partial x_j'} = \kappa \frac{\partial^2 \theta'}{\partial x_j'^2}$$

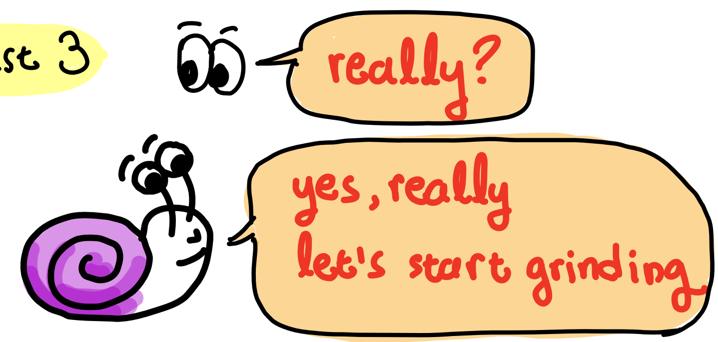
We are solving these in a box:



How many problem parameters?
Can we reduce the number of problem parameters?

<u>Total 5</u> parameters: ν, κ, d, T_c, T_h ← All these are dimensional

We will show that we can reduce the number of parameters to just 3



We will divide each variable with an appropriate quantity to form non-dimentional variables

Scale for:			
length	d	$x = \frac{x'}{d}$	$x' = xd \quad y' = yd$
velocity	$\frac{d}{\kappa}$	$u = u'\frac{d}{\kappa}$	$u' = \frac{\kappa}{d}u$
		$v = v' d/\kappa$	$v' = \frac{\kappa}{d}v$
time	$\frac{d}{u}$	$t = t'\frac{\kappa}{d^2}$	$t' = \frac{d^2}{\kappa}t$
temperature	$\frac{T_h - T_c}{T_c}$	$\theta = \frac{T' - T_c}{T_h - T_c}$	$T' = \Delta T\,\theta + T_c$
pressure	$\left(\frac{d}{\kappa}\right)^2$	$p = p'\frac{d^2}{\kappa^2}$	$p' = \frac{\kappa^2}{d^2}p \qquad \Delta T = T_h - T_c$

these (without prime) are non-dimensional

we will use these relations to replace dimensional quantities with non-dimensional

Note that θ: $0 < \theta < 1$

cold wall

warm wall

1. mass:

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \rightarrow \frac{\kappa}{d}\frac{1}{d}\frac{\partial u}{\partial x} + \frac{\kappa}{d}\frac{1}{d}\frac{\partial v}{\partial y} = 0$$

from u' from x' from v' from y'

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

same form as dimensional equation

2a. x-momentum:

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \nu\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right)$$

$$\frac{\kappa}{d}\frac{\kappa}{d^2}\frac{\partial u}{\partial t} + \frac{\kappa^2}{d^2}\frac{1}{d}u\frac{\partial u}{\partial x} + \frac{\kappa^2}{d^2}\frac{1}{d}v\frac{\partial u}{\partial y} = -\frac{\kappa^2}{d^2}\frac{1}{d}\frac{\partial p}{\partial x} + \nu\frac{\kappa}{d}\frac{1}{d^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\nu}{\kappa}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + Pr\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Prandtl number $Pr \equiv \frac{\nu}{\kappa}$

2b. y-momentum

$$\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} = -\frac{\partial p'}{\partial y'} + g\alpha\left(T' - \underset{\uparrow\, T_0 = T_c}{T_0'}\right) + \nu\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right)$$

$$\frac{K^2}{d^3}\frac{\partial v}{\partial t} + \frac{K^2}{d^3}u\frac{\partial v}{\partial x} + \frac{K^2}{d^3}v\frac{\partial v}{\partial y} = -\frac{K^2}{d^3}\frac{\partial p}{\partial y} + g\alpha\Delta T\theta + \frac{\nu K}{d^3}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \underbrace{g\alpha\Delta T\frac{d^3}{K^2}}\theta + Pr\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\downarrow$$

$$\underbrace{\frac{g\alpha\Delta T d^3}{K\nu}}_{\text{Rayleigh number: Ra}}\ \underbrace{\frac{\nu}{K}}_{Pr}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + Ra\,Pr\,\theta + Pr\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

3. temperature $\quad \frac{\partial \theta'}{\partial t} + u\frac{\partial \theta'}{\partial x} + v\frac{\partial \theta'}{\partial y} = K\left(\frac{\partial^2 \theta'}{\partial x^2} + \frac{\partial^2 \theta'}{\partial y^2}\right)$

$$\Delta T\frac{K}{d^2}\frac{\partial \theta}{\partial t} + \frac{K}{d}\Delta T\frac{1}{d}u\frac{\partial \theta}{\partial x} + \frac{K}{d}\Delta T\frac{1}{d}u\frac{\partial \theta}{\partial y} = K\Delta T\frac{1}{d^2}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right)$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}$$

Non-dimensional system of equations:

mass $\quad \frac{\partial u_i}{\partial x_i} = 0$

$u_i = 0$ on walls

2 parameters

momentum $\quad \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + Ra\,Pr\,\theta\,\delta_{i2} + Pr\frac{\partial^2 u_i}{\partial x_j^2}$

temperature $\quad \frac{\partial \theta}{\partial t} + u_j\frac{\partial \theta}{\partial x_j} = \frac{\partial^2 \theta}{\partial x_j^2}$

$\theta(y=0) = 1$

$\theta(y=1) = 0$