1. Spectral Differentiation Matrix

$$\hat{\delta}_n = \text{DFT}\left[\delta_i\right] = \sum_{i=0}^{N-1}\left[\delta_i e^{-\underline{i}k_n x_i}\right] \tag{1}$$

k_n can be simplifed as $k_n = 2\pi n/(2\pi) = n$

$$\begin{aligned}\hat{\delta}_n &= \left(\delta_0 e^{-\underline{i}nx_0}\right) + \left(\delta_1 e^{-\underline{i}nx_1}\right) + \ldots + \left(\delta_{N-1} e^{-\underline{i}nx_{N-1}}\right) \\ &= \left(\delta_0 e^{-\underline{i}nx_0}\right) = 1e^0 = 1\end{aligned} \tag{2}$$

We can then construct the polynomial interpolations as the following:

$$p(x) = \frac{1}{2N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1}\left[\hat{\delta}_n e^{\underline{i}nx}\right] + \frac{1}{2N}\sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}}\left[\hat{\delta}_n e^{\underline{i}nx}\right] \tag{3}$$

By using general formula for an exponential sum, we can simplify this to:

$$p(x) = \frac{1}{N}\left[\frac{e^{\underline{i}x\left(-\frac{N}{2}\right)} - e^{\underline{i}x\left(+\frac{N}{2}\right)}}{2\left(1 - e^{\underline{i}x}\right)} + \frac{e^{\underline{i}x\left(-\frac{N}{2}+1\right)} - e^{\underline{i}x\left(+\frac{N}{2}+1\right)}}{2\left(1 - e^{\underline{i}x}\right)}\right] \tag{4}$$

By using Euler's Identity, this can be changed to:

$$p(x) = \frac{1}{N}\frac{\sin\left(\frac{Nx}{2}\right)\left(\sin(x) - \underline{i}\cos(x)\right) - \underline{i}\sin\left(\frac{Nx}{2}\right)}{1 - \underline{i}\sin(x) - \cos(x)} \tag{5}$$

$$p(x) = \frac{1}{N}\sin\left(\frac{Nx}{2}\right)\left[\frac{\sin(x) - \underline{i}\cos(x) - \underline{i}}{1 - \underline{i}\sin(x) - \cos(x)}\right] \tag{6}$$

$$p(x) = \frac{1}{N}\sin\left(\frac{Nx}{2}\right)\cot\left(\frac{x}{2}\right) \tag{7}$$

From which, the continuous derivative can be found to be:

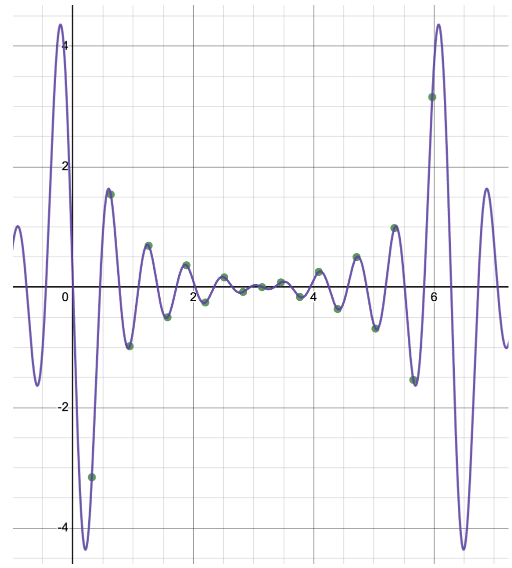
$$p'(x) = \frac{N\cot\left(\frac{x}{2}\right)\cos\left(\frac{Nx}{2}\right) - \csc^2\left(\frac{x}{2}\right)\sin\left(\frac{Nx}{2}\right)}{2N} \tag{8}$$

And the discrete derivative can be found as follows:

$$p'_{\text{TS}}(x_i) = -\frac{1}{12}(N^2+2)x_i + \mathcal{O}(x_i^2) \rightarrow p'_{\text{TS}}(0) = 0 \tag{9}$$

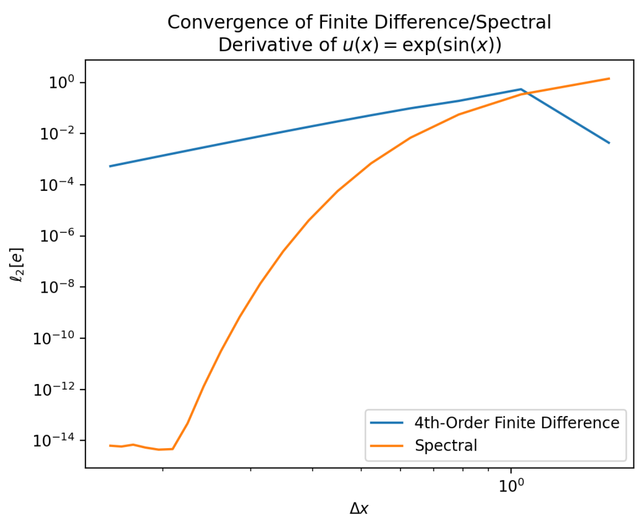
Though the continous function away from $x_i = 0$ can be visually shown to agree with the discrete version, I was not completely sure how to manipulate them to show that algebraicly.

$$\frac{N\cot\left(\frac{x}{2}\right)\cos\left(\frac{Nx}{2}\right) - \csc^2\left(\frac{x}{2}\right)\sin\left(\frac{Nx}{2}\right)}{2N} = \frac{1}{2}(-1)^i\cot\left(\frac{\pi i}{N}\right) \tag{10}$$



$$p'(x_i) = \begin{cases} 0, & i = 0 \\ \frac{1}{2}(-1)^i\cot\left(\frac{i\Delta x}{2}\right), & i \neq 0 \end{cases} \tag{11}$$

2. Comparison of Derivative Approximations



As can be seen from the figure above, the spectral derivative error rapidly converges to machine epsilon with a non-constant order, whereas the 4th Order Finite Difference has a constant order. It should be noted that the error in the FDM for larger timesteps is due to the fact that convergence is asymptotic. When Δx is large, it can vary from the actual convergence rate.