Project Parts:

1. Poisson equation solution
2. Time integration of incompressible Euler
3. Time integration of incompressible Navier-Stokes

Equations:

Part 1

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

viscous terms

buoyancy term

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} uu - \frac{\partial}{\partial y} vu + \frac{\partial p}{\partial x} + Pr\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial x} uv - \frac{\partial}{\partial y} vv + \frac{\partial p}{\partial y} + Pr\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + Ra\, Pr\, \theta$$

$$\frac{\partial \theta}{\partial t} = -\frac{\partial}{\partial x} u\theta - \frac{\partial}{\partial y} v\theta + \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}$$

Part 2 Part 3

CFD Program Structure

1. Setup
   a. Input: Δx, Δy, L_x, Δt or CFL, Ra, Pr
   b. Allocate variables
   c. Initialize and make variables physical
      - Must have physical initial condition
      - Velocity field must be divergence free

2. Time Integration $\frac{d}{dt}\begin{bmatrix} u \\ v \\ \theta \end{bmatrix} = \vec{f}(u,v,\theta)$

   - Each Runge Kutta substep:

     1. Form RHS for u, v, and θ
     2. Update u, v, θ to get u^* and v^*, θ

     Time advance momentum and θ equation except the ∇p term

     3. Apply Boundary Condition for u^* and v^* and θ
     4. Remove horizontal mean from v^*
     5. Find pressure $(\nabla^2 p = \nabla \cdot \vec{u}^*)$
     6. Apply Boundary Condition to p
     7. Add ∇p to $\vec{u}^*$
     8. Apply Boundary Condition for u and v

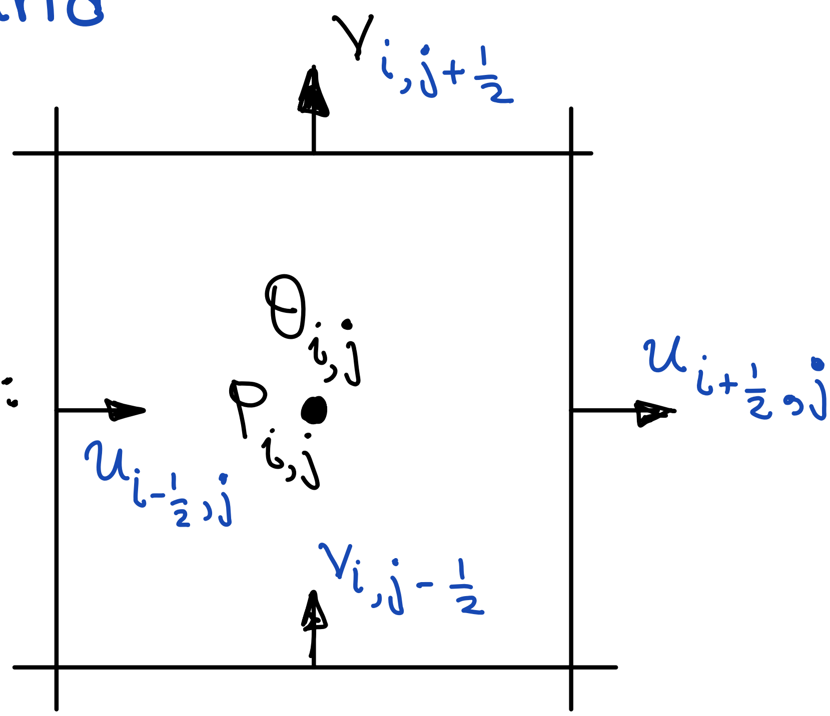
     Fully Time Advance Mass and Momentum

3. Output

Staggered Grid

Notation:

Grid cell (i,j):

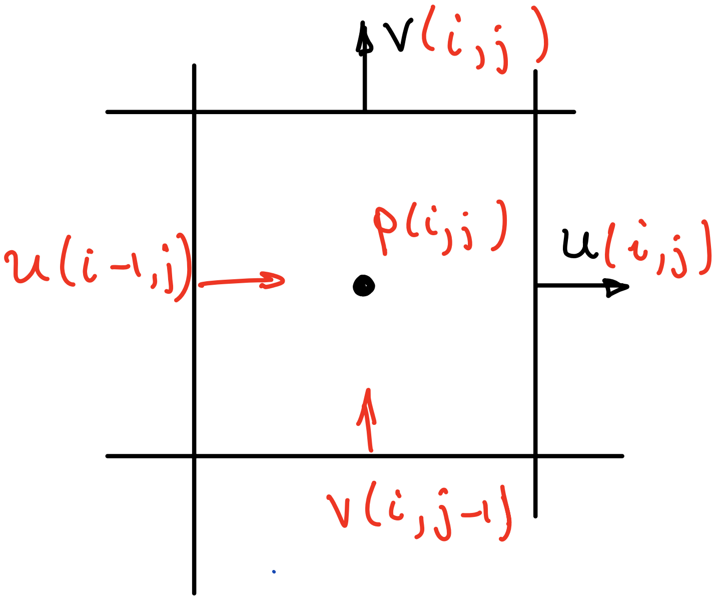


This notation is unambiguous

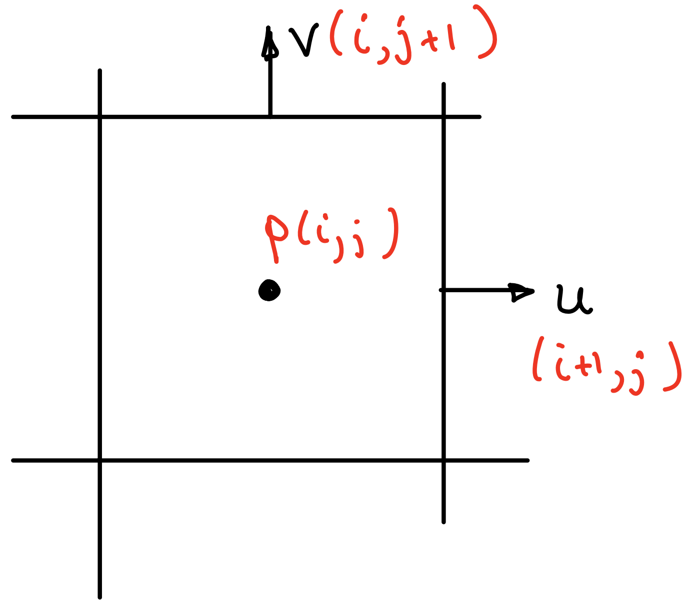
... however...

we cannot easily access $u(i+\frac{1}{2},j)$ on the computer

Option A: $i+\frac{1}{2} \to i$



Option B: $i+\frac{1}{2} \to i+1$



Divergence and pressure gradient

1. Divergence

$$\nabla^2 p = \nabla \cdot \vec{u} \rightarrow \text{we need } \nabla \cdot \vec{u} \text{ at location of } p$$

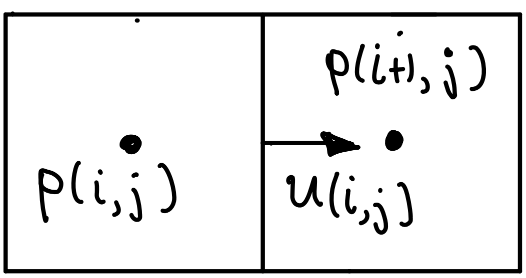
Need $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ at center of grid cell

Use second-order differences:

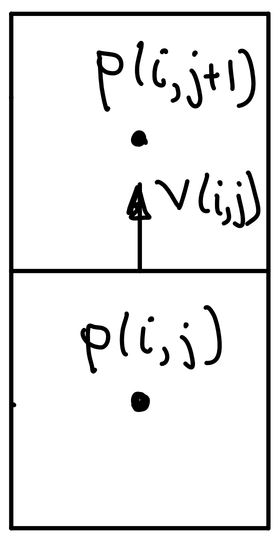
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \simeq \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + \frac{v_{i,j+1} - v_{i,j}}{\Delta y} + O(\Delta x^2, \Delta y^2)$$

$$\frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{\Delta x^2} + \frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{\Delta y^2} = \underbrace{\frac{u_{i+1,j} - u_{i,j}}{\Delta x} + \frac{v_{i,j+1} - v_{i,j}}{\Delta y}}_{d_{i,j}}$$

2a. Pressure gradient at u-velocity



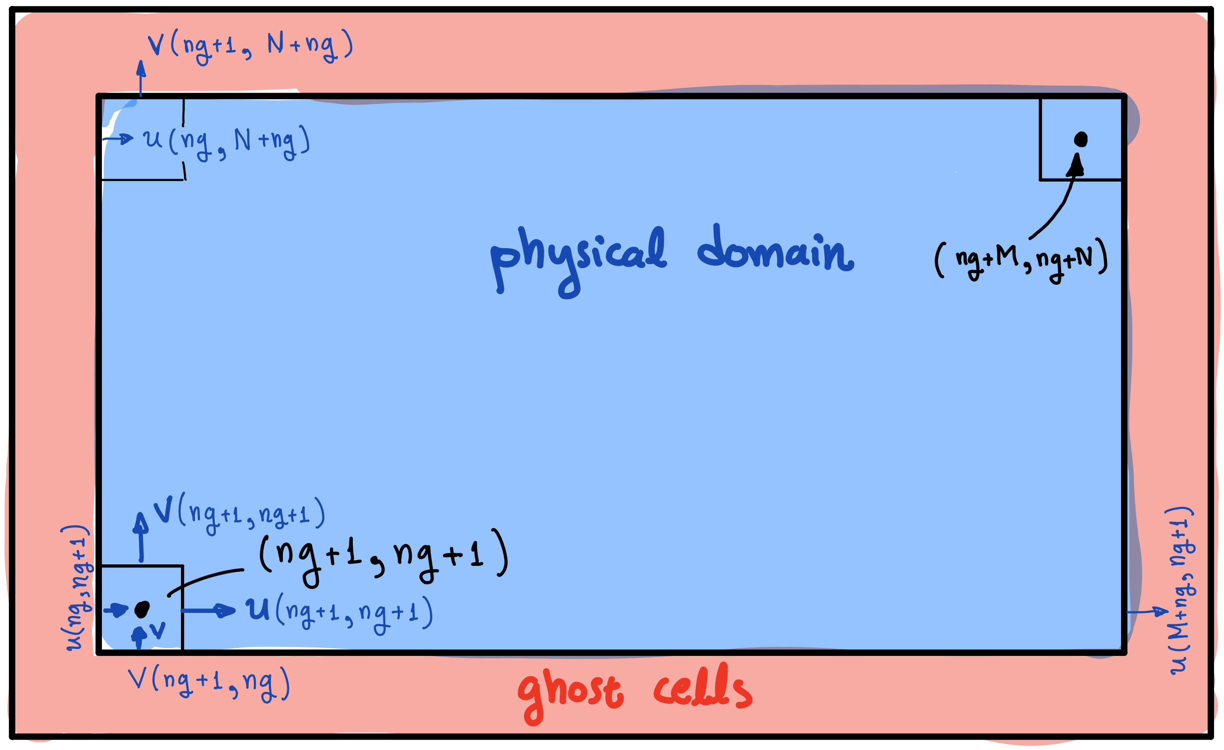
$$\frac{\partial p}{\partial x} \simeq \frac{p(i+1,j) - p(i,j)}{\Delta x}$$



2b. Pressure gradient at v-velocity

$$\frac{\partial p}{\partial y} \simeq \frac{p(i,j+1) - p(i,j)}{\Delta y}$$

Computational Domain



$M \times N$ u and θ unknowns

$M \times N-1$ v unknowns (because $v=0$ on walls)

$u = \text{zeros}(M+2*ng,\ N+2*ng)$

$V =$

$\theta =$

$\rightarrow$ suggestion : $d(i,j) = (M+2ng) \times (N+2ng)$

such that $d(i,j) = \dfrac{u(i,j)-u(i-1,j)}{\Delta x} + \dfrac{v(i,j)-v(i,j-1)}{\Delta y}$

no index shift

call $p = \text{poisson}(\Delta x, \Delta y, d(ng+1:ng-M,\ ng+1:ng+N))$

then apply BC to p: $p(1,j) = p(1+M,j)$

$p(M+2\quad ,j) = p(\quad 1,j)$

Time integration – Convection Term discretization

get equations for u, v, θ in semi-discrete form

$$\frac{du_{i,j}}{dt} = rhs_u(u_i, v_i)$$

$$\frac{dv_{i,j}}{dt} = rhs_v(u_i, v_i)$$

$$\frac{d\theta_{i,j}}{dt} = rhs_\theta(u_i, v_i, \theta_i)$$

What are the rhs?

y-direction flux of u

x-direction flux of u

$$rhs_u(u,v) = -\frac{\partial}{\partial x}(uu) - \frac{\partial}{\partial y}(vu) = -\frac{\partial}{\partial x} f^u - \frac{\partial}{\partial y} g^u$$

$$rhs_v(u,v) = -\frac{\partial}{\partial x}(uv) - \frac{\partial}{\partial y}(vv) = -\frac{\partial}{\partial x} f^v - \frac{\partial}{\partial y} g^v$$

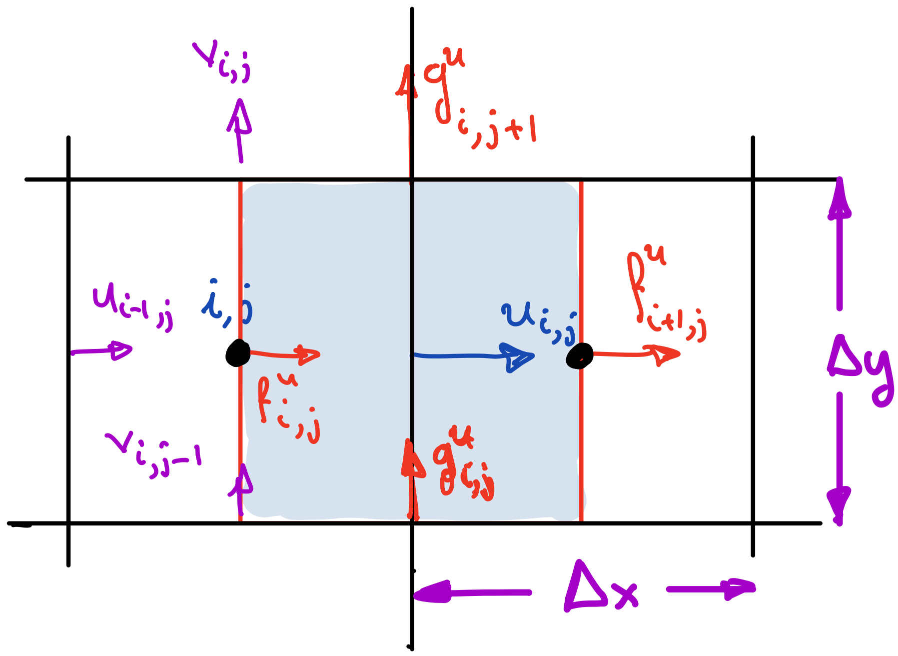
$$rhs_\theta(u,v,\theta) = -\frac{\partial}{\partial x}(u\theta) - \frac{\partial}{\partial y}(v\theta) = -\frac{\partial}{\partial x} f^\theta - \frac{\partial}{\partial y} g^\theta$$

Approximate derivatives:

$$rhs_u(u_{i,j}, v_{i,j}) = -\frac{f^u_{i+1,j} - f^u_{i,j}}{\Delta x} - \frac{g^u_{i,j+1} - g^u_{i,j}}{\Delta y}$$

$$rhs_v(u_{i,j}, v_{i,j}) = -\frac{f^v_{i+1,j} \quad f^v_{i,j}}{\Delta x} - \frac{g^v_{i,j+1} - g^v_{i,j}}{\Delta y}$$

$$rhs_\theta(u_{i,j}, v_{i,j}, \theta_{i,j}) = -\frac{f^\theta_{i+1,j} - f^\theta_{i,j}}{\Delta x} - \frac{g^\theta_{i,j+1} - g^\theta_{i,j}}{\Delta y}$$



$$f^u_{i,j} = \left(\frac{u_{i-1,j} + u_{i,j}}{2}\right)^2$$

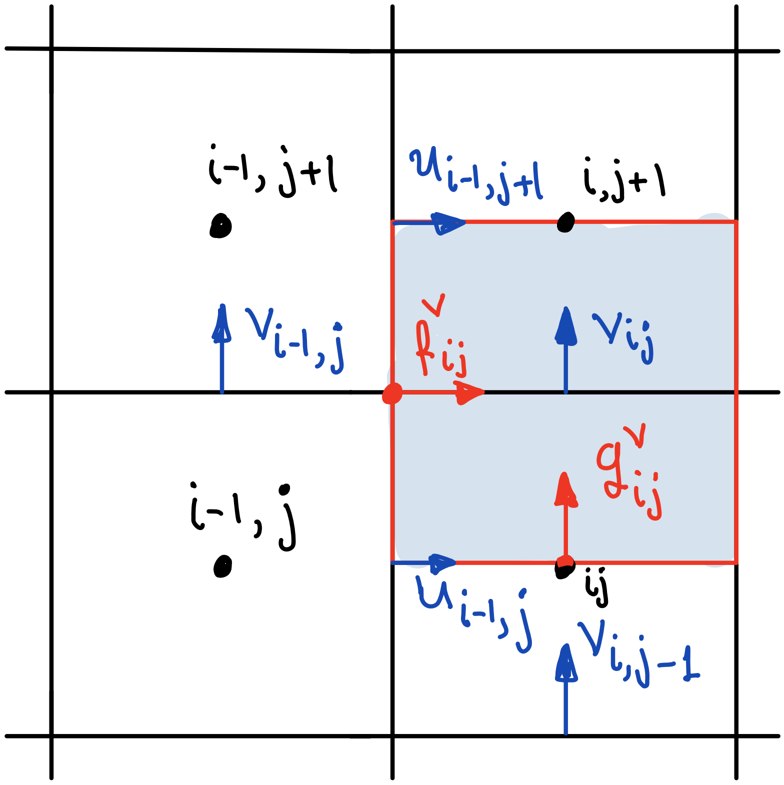
Approximates uu at center of grid cell

$$g^u_{i,j} = \frac{u_{i,j} + u_{i,j-1}}{2} \cdot \frac{v_{i,j-1} + v_{i+1,j-1}}{2}$$

Approximates vu at NE edge

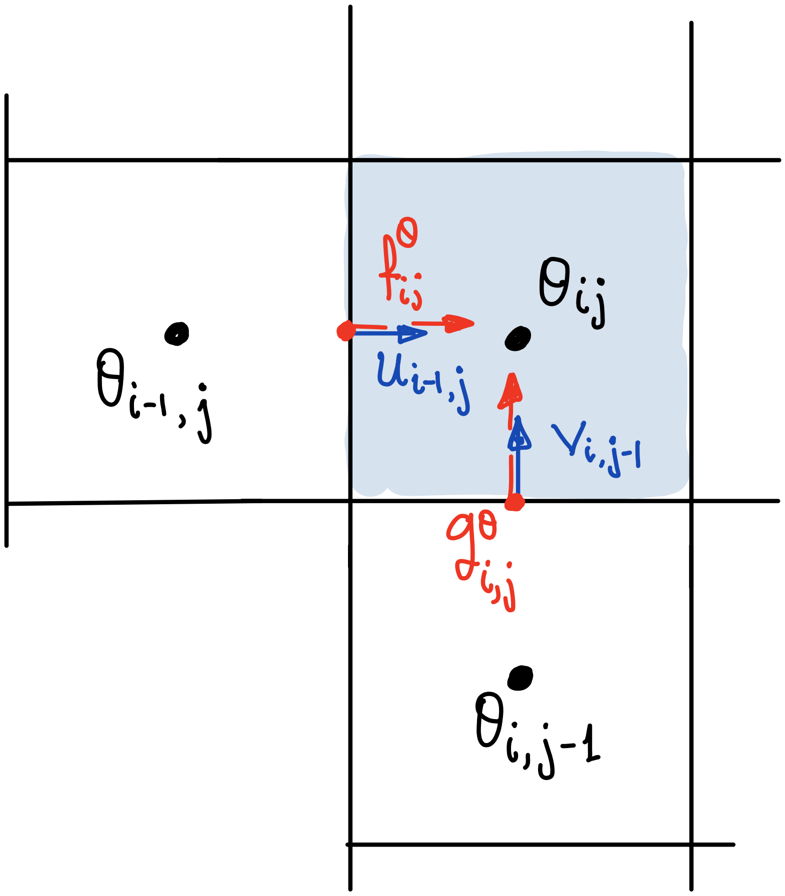
For example, index $(i,j+1)$:

$$g_{i,j+1} = \frac{u_{i,j+1} + u_{i,j}}{2} \cdot \frac{v_{i,j} + v_{i+1,j}}{2}$$



$$f^v_{ij} = \frac{u_{i-1,j} + u_{i-1,j+1}}{2} \cdot \frac{v_{i-1,j} + v_{i,j}}{2}$$

$$g^v_{ij} = \left(\frac{v_{i,j-1} + v_{i,j}}{2}\right)^2$$



$$f^{\theta}_{i,j} = \frac{\theta_{i-1,j} + \theta_{i,j}}{2} \cdot u_{i-1,j}$$

Approximates $u\theta$ at left face

$$g^{\theta}_{i,j} = \frac{\theta_{i,j-1} + \theta_{i,j}}{2} \cdot v_{i,j-1}$$

Approximates $v\theta$ at bottom face

Viscous terms

Viscous terms are straightforward because only the variable of the corresponding equation appears.

We will use second order centered differences

$$\frac{\partial u}{\partial t} = \text{stuff} - \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)$$

$$\frac{du_{i,j}}{dt} = \underbrace{\underbrace{rhs_{convection}(u,v)}_{\text{Part 2}} + \underbrace{rhs_{viscous}(u)}_{\text{Part 3}}}_{\text{Full RHS}}$$

$$rhs^{u}_{viscous} = Pr\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

rhs(i,j) =

$$= Pr\left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2}\right)$$

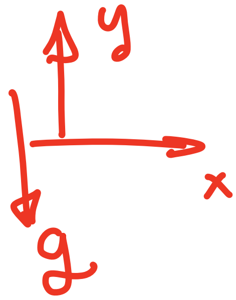
$$rhs^{v}_{viscous} = Pr\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

No changes in v - BC

$$= Pr\left(\frac{v_{i-1,j} - 2v_{i,j} + v_{i+1,j}}{\Delta x^2} + \frac{v_{i,j-1} - 2v_{i,j} + v_{i,j+1}}{\Delta y^2}\right)$$

$$rhs^{\theta}_{viscous} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \frac{\theta_{i-1,j} - 2\theta_{i,j} + \theta_{i+1,j}}{\Delta x^2} + \frac{\theta_{i,j-1} - 2\theta_{i,j} + \theta_{i,j+1}}{\Delta y^2}$$

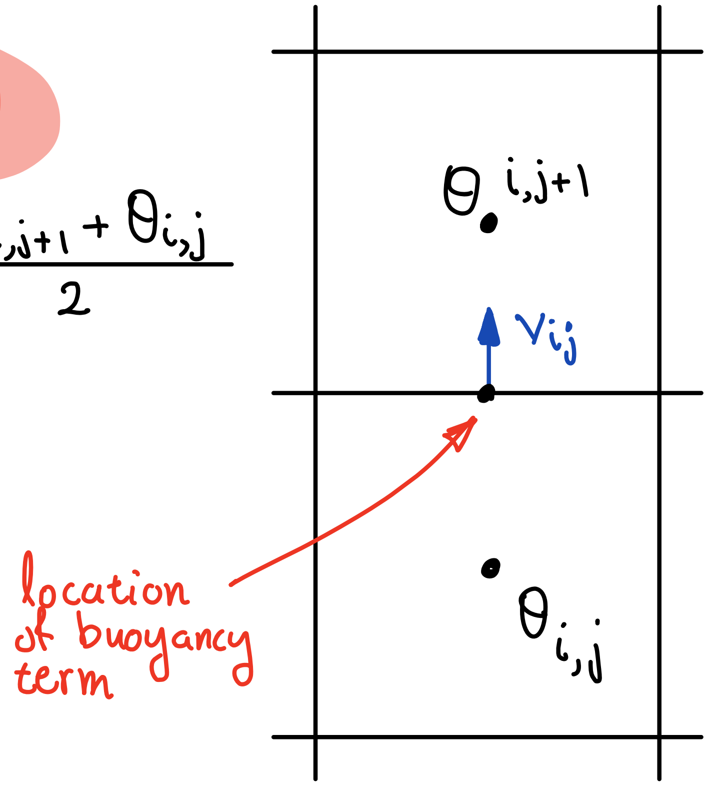
Buoyancy Term (only for v-velocity)



$$\frac{dv_{i,j}}{dt} = rhs^v_{convection} + rhs^v_{viscous} + rhs^v_{buoyancy}$$

$$rhs^v_{buoyancy} = Ra\,Pr\,\theta$$

$$\simeq Ra\,Pr\,\frac{\theta_{i,j+1} + \theta_{i,j}}{2}$$

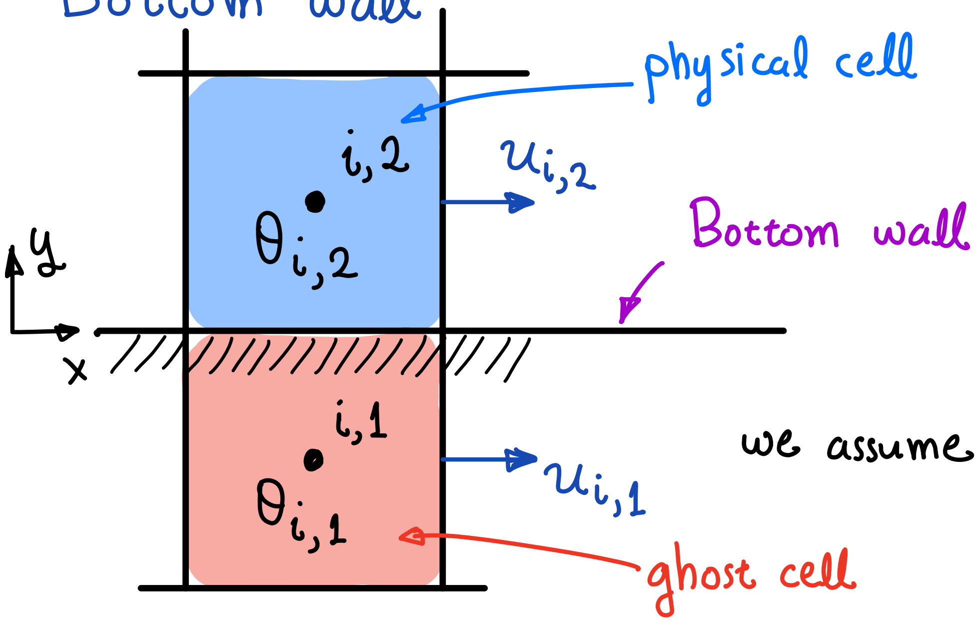


Boundary Conditions - General stuff

- We need BCs for u and θ on bottom and top walls
- All fields are periodic in x-direction
  - we did that in Part 2, don't need anything else
- $v = 0$ on bottom and top walls
  - we did that in Part 2, don't need anything else

Discrete Boundary Conditions on walls

Bottom wall



we assume one ghost cell

- Need to set $u_{i,1}$ and $\theta_{i,1}$ to satisfy BC at $y=0$
- $u(x, y=0) = 0$
- $\theta(x, y=0) = 1$ ←

BC for u: $\quad u_{i,1+\frac{1}{2}} = \dfrac{u_{i,1} + u_{i,2}}{2} = 0 \Rightarrow u_{i,1} = -u_{i,2}$

BC for θ: $\quad \theta_{i,1+\frac{1}{2}} = \dfrac{\theta_{i,1} + \theta_{i,2}}{2} = 1 \Rightarrow \theta_{i,1} = 2 - \theta_{i,2}$

if not using ghost cells to set the BC

the second-derivative approximations $\frac{\partial^2 \theta}{\partial y^2}$ and $\frac{\partial^2 u}{\partial y^2}$

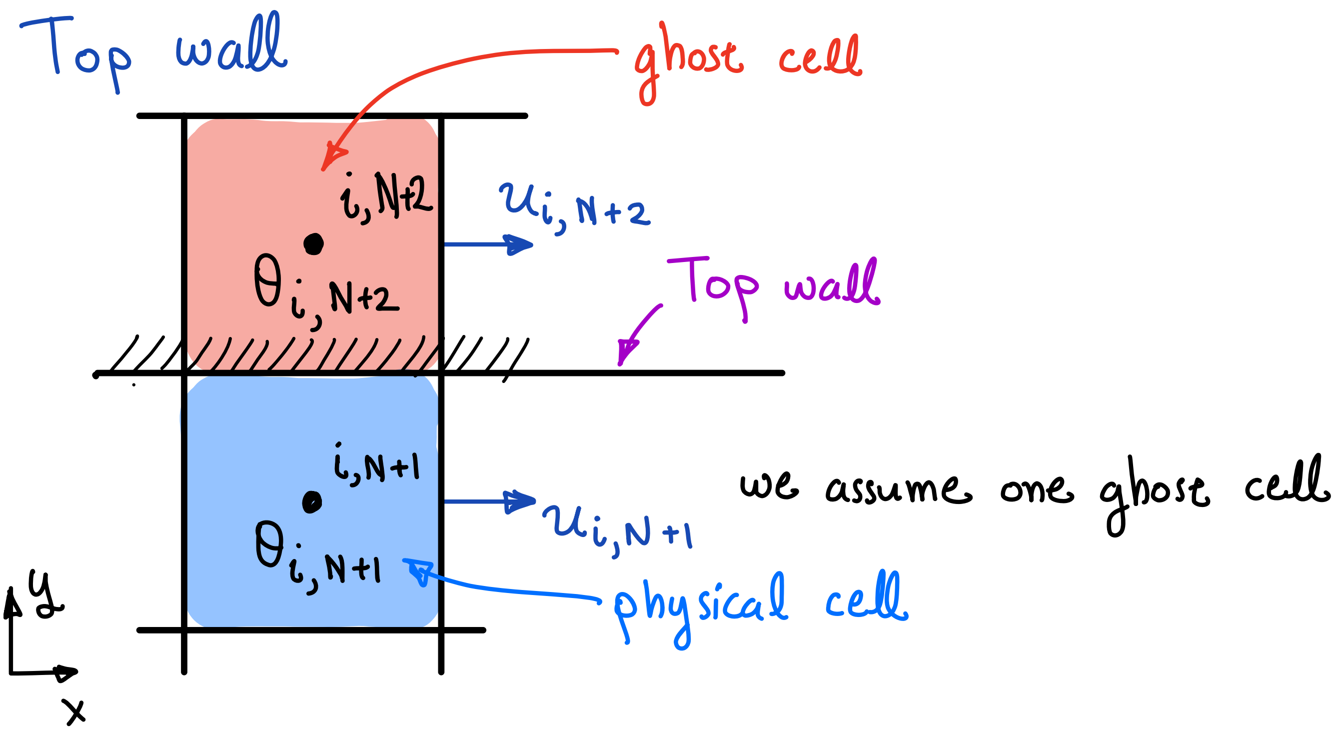
at $j=2$ must be modified

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{j=2} = \frac{1}{\Delta y^2}\left(u_{i,3} - 2u_{i,2} + u_{i,1}\right)$$

using $u_{i,1} = -u_{i,2}$

$$= -\frac{1}{\Delta y^2}\left(u_{i,3} - 3u_{i,2}\right)$$

similarly for $\frac{\partial^2 \theta}{\partial y^2} = \frac{1}{\Delta y^2}\left(\theta_{i,3} - 3\theta_{i,2} + 2\right)$



- Need to set $u_{i,N+2}$ and $\theta_{i,N+2}$ to satisfy BC at $y=1$
- $u(x, y=1) = 0$
- $\theta(x, y=1) = 0$

BC for u: $$u_{i,N+\frac{3}{2}} = \frac{u_{i,N+1} + u_{i,N+2}}{2} = 0$$

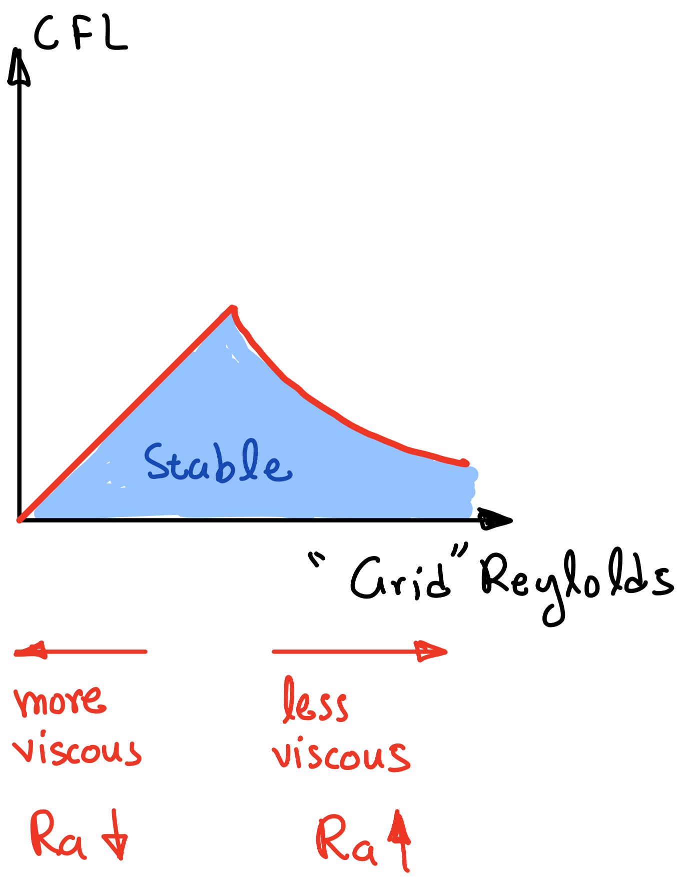
$$\Rightarrow u_{i,N+2} = -u_{i,N+1}$$

BC for θ: $$\theta_{i,N+\frac{3}{2}} = \frac{\theta_{i,N+1} + \theta_{i,N+2}}{2} = 0 \Rightarrow \theta_{i,N+2} = -\theta_{i,N+1}$$

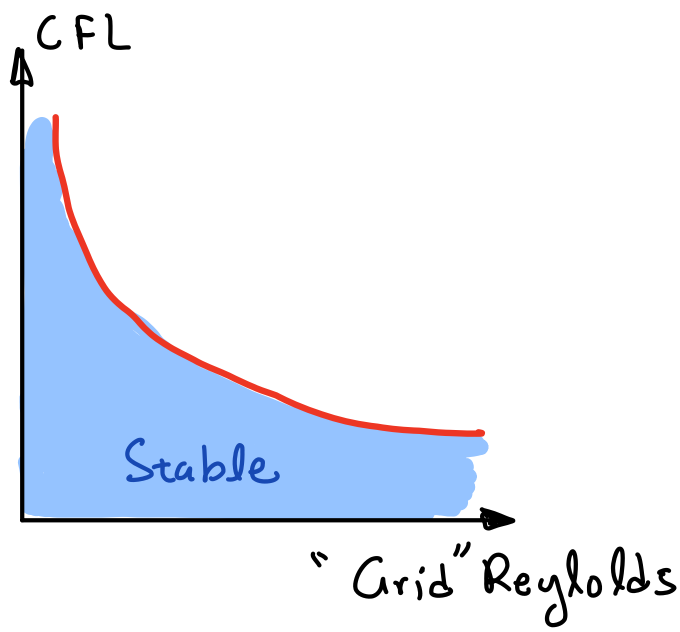
Choosing Δt - Preliminaries

Stability for Convection-Diffusion Problem

Explicit Convection
Explicit Diffusion



Explicit Convection
Implicit Diffusion



Choosing Δt

- Convection stability limit $CFL_c = \dfrac{u\,\Delta t}{\Delta x}$

Diffusion coefficient

- Diffusion stability limit $CFL_d = \nu \dfrac{\Delta t}{\Delta x^2}$

Convection:

$$CFL_c(i,j) = \frac{|u(i,j)|}{\Delta x}\Delta t + \frac{|v(i,j)|}{\Delta y}\Delta t$$

Don't bother with grid staggering

Diffusion $CFL_d = Pr \dfrac{\Delta t}{\Delta x^2}$ $\leftarrow$ $[\min(\Delta x, \Delta y)]^2$

To find Δt of next time step:

- Find $\max\limits_{i,j} (CFL_c , CFL_d)$

- Choose $\Delta t_{new} = (CFL_{target} / CFL_{max})\, \Delta t_{old}$

Even better option:

- $$\Delta t_{new} = \left(\frac{1}{2}(CFL_{target} / CFL_{max}) + \frac{1}{2}\right) \Delta t_{old}$$