ME 5311 Semester Project: Part 1

Buoyant Convection

March 20, 2024

1 Problem description

The problem is to study buoyant convection in a two-dimensional rectangular domain with dimensions $[l \times d]$ in the horizontal and vertical directions. The flow is also called Rayleigh–Bénard convection from the two fluid dynamicists who studied the flow. The computational domain is periodic in the streamwise, x, direction. The governing equations are the conservation of mass, streamwise and vertical momentum, and temperature,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + g\alpha(T - T_{\text{ref}}) + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{3}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right). \tag{4}$$

where $p = p'/\rho$ is the scaled pressure, g the acceleration of gravity, α the coefficient of thermal expansion at the reference temperature T_{ref}, and ν and κ the diffusivity coefficients for momentum and temperature, respectively. The top and bottom walls are at temperatures T_c and T_h, respectively. The governing equations follow the Boussinesq approximation, which mean that we do not take into account density variations for mass and momentum transport, but we do take into account the density variation in the fluid's buoyancy. Therefore the vertical momentum equation includes a buoyancy term $g\alpha(T - T_{\text{ref}})$, which is proportional to flow temperature variations with respect to a reference temperature.

We will numerically integrate the non-dimensional form of the equations. First, we define a scaled temperature

$$\Theta = \frac{T - T_c}{T_h - T_c}. \tag{5}$$

We use the diffusivity κ and the vertical gap d for form length, velocity and time scales: d, κ/d,

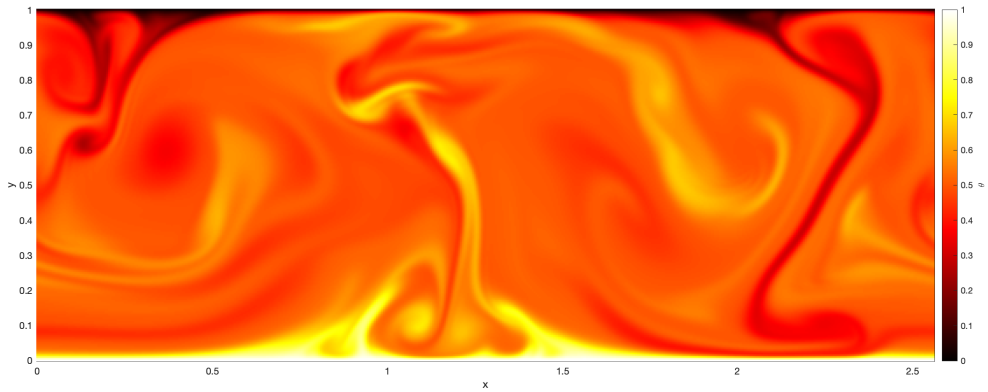

Figure 1: Scaled temperature contours.

and d^2/κ. The pressure is normalized by κ^2/d^2. The non-dimensional equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mathrm{Pr}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{7}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mathrm{Ra}\,\mathrm{Pr}\,\Theta + \mathrm{Pr}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{8}$$

$$\frac{\partial \Theta}{\partial t} + u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y} = \frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2}, \tag{9}$$

where

$$\mathrm{Pr} = \frac{\nu}{\kappa} \tag{10}$$

is the Prandtl number, and

$$\mathrm{Ra} = \frac{g\alpha(T_h - T_c)d^3}{\nu\kappa} \tag{11}$$

is the Rayleigh number. The boundary conditions at the bottom and top boundary are

$$u(t, x, 0) = 0, \tag{12}$$
$$u(t, x, 1) = 0, \tag{13}$$

for the streamwise velocity;

$$v(t, x, 0) = 0, \tag{14}$$
$$v(t, x, 1) = 0, \tag{15}$$

for the vertical velocity; and

$$\Theta(t, x, 0) = 1, \tag{16}$$
$$\Theta(t, x, 1) = 0, \tag{17}$$

for the scaled temperature.

2 Numerical solution

We will discuss in class a numerical solution method through the remainder of the semester. We will implement a fractional step method to numerically solve the two dimensional Navier–Stokes equations using second-order finite differences.

We will carry out the project in three stages:

1. Poisson equation solution: due April 3.
2. Convection term discretization: due April 17.
3. Full solution: due May 1.

2.1 Poisson equation for pressure solution

The first solution stage, due April 3, is to implement the numerical solution of the Poisson equation for the pressure

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = d(x, y), \tag{18}$$

in $[L \times 1]$ with Neumann boundary conditions at $y = 0$ and $y = 1$ and periodic horizontal boundary. You must verify the implementation by:

a. Verify the order of accuracy of the method [12 points]

b. Create a divergence-free velocity field from a randomly-generated velocity field [4 points]

To verify the accuracy of the method, choose L, and use

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\frac{16\pi^2(4 + L^2)}{L^2} \sin\left(8\pi \frac{x}{L}\right) \cos\left(4\pi y\right). \tag{19}$$

The exact solution is

$$p(x, y) = \sin\left(8\pi \frac{x}{L}\right) \cos\left(4\pi y\right). \tag{20}$$

Please submit:

- Your computer code
- The convergence plot for the Poisson equation solutions
- Your code output of the initial divergence of the velocity field and the divergence after ∇p was added to $\vec{u}$.