Fundamental Concept: Taylor exansion

Equation (3.8): Taylor expansion for function values

$$u_{j+k} = u_j + (k\Delta x)\left(\frac{\partial u}{\partial x}\right)_j + \frac{1}{2}(k\Delta x)^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \frac{1}{6}(k\Delta x)^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j \qquad (1)$$

expand around j

distance $k\Delta x$ from j

Taylor expansion for derivatives:

Equation (3.46) for first derivative, that is $m=1$

$$\left(\frac{\partial u}{\partial x}\right)_{j+k} = \left(\frac{\partial u}{\partial x}\right)_j + (k\Delta x)\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \frac{1}{2}(k\Delta x)^2\left(\frac{\partial^3 u}{\partial x^3}\right)_j + \frac{1}{6}(k\Delta x)^3\left(\frac{\partial^4 u}{\partial x^4}\right)_j \qquad (2)$$

Example of Table 3.3:

We want to construct a finite difference scheme of the form :

note sign

$$d\left(\frac{\partial u}{\partial x}\right)_{j-1} + \left(\frac{\partial u}{\partial x}\right)_j + e\left(\frac{\partial u}{\partial x}\right)_{j+1} = \frac{1}{\Delta x}\left(a\,u_{j-1} + b\,u_j + c\,u_{j+1}\right)$$

we need to find a, b, c, d, e

Taylor Table:

Step 1:

	u_j	$\Delta x \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5 \left(\frac{\partial^5 u}{\partial x^5}\right)_j$

1 This row is alway the same. It's the terms of the Taylor series expansion except the coefficients

Step 2:

take the finite differece terms...

$$d\left(\frac{\partial u}{\partial x}\right)_{j-1} + \left(\frac{\partial u}{\partial x}\right)_j + e\left(\frac{\partial u}{\partial x}\right)_{j+1} - \frac{1}{\Delta x}\left(a u_{j-1} + b u_j + c u_{j+1}\right)$$

... and multiply each by Δx...

$$\Delta x\, d\left(\frac{\partial u}{\partial x}\right)_{j-1} \quad \Delta x\left(\frac{\partial u}{\partial x}\right)_j \quad \Delta x\, e\left(\frac{\partial u}{\partial x}\right)_{j+1} \quad a u_{j-1} \quad b u_j \quad c u_{j+1}$$

then write them vertically in the first column...

	u_j	$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4\left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5\left(\frac{\partial^5 u}{\partial x^5}\right)_j$
$\Delta x\, d\left(\frac{\partial u}{\partial x}\right)_{j-1}$						
$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$						
$\Delta x\, e\left(\frac{\partial u}{\partial x}\right)_{j+1}$						
$-a u_{j-1}$						
$-b u_j$						
$-c u_{j+1}$						

note sign

Step 3: Expand each term in the first column using a Taylor series

	u_j	$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4\left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5\left(\frac{\partial^5 u}{\partial x^5}\right)_j$
$\Delta x\, d\left(\frac{\partial u}{\partial x}\right)_{j-1}$						
$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$						
$\Delta x\, e\left(\frac{\partial u}{\partial x}\right)_{j+1}$						
$-a\, u_{j-1}$						
$-b\, u_j$						
$-c\, u_{j+1}$						

expand this using equation ② with $k=-1$

$$\Delta x\, d\left(\frac{\partial u}{\partial x}\right)_{j-1} =$$

$$\Delta x\, d\Bigg[\left(\frac{\partial u}{\partial x}\right)_j + (-1\Delta x)\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \frac{1}{2}(-1\Delta x)^2\left(\frac{\partial^3 u}{\partial x^3}\right)_j + \frac{1}{6}(-1\Delta x)^3\left(\frac{\partial^4 u}{\partial x^4}\right)_j + \frac{1}{24}(-1\Delta x)^4\left(\frac{\partial^5 u}{\partial x^5}\right)_j\Bigg]$$

$$\Delta x\, d\left[\left(\frac{\partial u}{\partial x}\right)_j + (-1\Delta x)\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \frac{1}{2}(-1\Delta x)^2\left(\frac{\partial^3 u}{\partial x^3}\right)_j + \frac{1}{6}(-1\Delta x)^3\left(\frac{\partial^4 u}{\partial x^4}\right)_j + \frac{1}{24}(-1\Delta x)^4\left(\frac{\partial^5 u}{\partial x^5}\right)_j\right]$$

red and green stuff go to second row

this times this equal this

	u_j	$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4\left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5\left(\frac{\partial^5 u}{\partial x^5}\right)_j$
$\Delta x\, d\left(\frac{\partial u}{\partial x}\right)_{j-1}$	0	d	$(-1)d$	$\frac{1}{2}(-1)^2 d$	$\frac{1}{6}(-1)^3 d$	$\frac{1}{24}(-1)^4 d$
$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$						
$\Delta x\, e\left(\frac{\partial u}{\partial x}\right)_{j+1}$						
$-\Delta x\, a\, u_{j-1}$						
$-\Delta x\, b\, u_j$						
$-\Delta x\, c\, u_{j+1}$						

each row corresponds to the Taylor expansion of the term on first column

Similarly, Taylor expansion of $\Delta x\, c\, u_{j+1}$ using equation ① using $k=1$

$$-c\, u_{j+1} =$$
$$-c\left[u_j + (1\Delta x)\left(\frac{\partial u}{\partial x}\right)_j + \frac{1}{2}(1\Delta x)^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \frac{1}{6}(1\Delta x)^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j + \frac{1}{24}(1\Delta x)^4\left(\frac{\partial^4 u}{\partial x^4}\right)_j\right]$$

Red and green stuff go to row seven:

Note sign !!!

	u_j	$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4\left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5\left(\frac{\partial^5 u}{\partial x^5}\right)_j$
$\Delta x\, d\left(\frac{\partial u}{\partial x}\right)_{j-1}$	0	d	$(-1)d$	$\frac{1}{2}(-1)^2 d$	$\frac{1}{6}(-1)^3 d$	$\frac{1}{24}(-1)^4 d$
$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$						
$\Delta x\, e\left(\frac{\partial u}{\partial x}\right)_{j+1}$						
$-a\, u_{j-1}$						
$-b\, u_j$						
$-c\, u_{j+1}$	$-c$	$-c$	$-c(1)^2\frac{1}{2}$	$-c(1)^3\frac{1}{6}$	$-c(1)^4\frac{1}{24}$	$-c(1)^5\frac{1}{120}$

We do the same for the other terms...

	u_j	$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4\left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5\left(\frac{\partial^5 u}{\partial x^5}\right)_j$
$\Delta x\, d\left(\frac{\partial u}{\partial x}\right)_{j-1}$	0	d	$(-1)d$	$\frac{1}{2}(-1)^2 d$	$\frac{1}{6}(-1)^3 d$	$\frac{1}{24}(-1)^4 d$
$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$		1				
$\Delta x\, e\left(\frac{\partial u}{\partial x}\right)_{j+1}$	0	e	$(1)e$	$\frac{1}{2}(1)^2 e$	$\frac{1}{6}(1)^3 e$	$\frac{1}{24}(1)^4 e$
$-a u_{j-1}$	$-a$	$-a(-1)$	$-a(-1)^2\frac{1}{2}$	$-a(-1)^3\frac{1}{6}$	$-a(-1)^4\frac{1}{24}$	$-a(-1)^5\frac{1}{120}$
$-b u_j$	$-b$					
$-c u_{j+1}$	$-c$	$-c(1)^2$	$-c(1)^2\frac{1}{2}$	$-c(1)^3\frac{1}{6}$	$-c(1)^4\frac{1}{24}$	$-c(1)^5\frac{1}{120}$

Step 4: We set the sum of each interior column to zero

$$\left.\begin{aligned}
&-a-b-c=0\\
&d+1+e+a-c=0\\
&-d+e-\tfrac{1}{2}a-\tfrac{1}{2}c=0\\
&\tfrac{1}{2}d+\tfrac{1}{2}e+\tfrac{1}{6}a-\tfrac{1}{6}c=0\\
&-\tfrac{1}{6}d+\tfrac{1}{6}e-\tfrac{1}{24}a-\tfrac{1}{24}c=0
\end{aligned}\right\}\ \text{Solve for } a, b, c, d, e$$

Step 5: leading error term

	u_j	$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4\left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5\left(\frac{\partial^5 u}{\partial x^5}\right)_j$
$\Delta x\, d\left(\frac{\partial u}{\partial x}\right)_{j-1}$	0	d	$(-1)d$	$\frac{1}{2}(-1)^2 d$	$\frac{1}{6}(-1)^3 d$	$\frac{1}{24}(-1)^4 d$
$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$		1				
$\Delta x\, e\left(\frac{\partial u}{\partial x}\right)_{j+1}$	0	e	$(1)e$	$\frac{1}{2}(1)^2 e$	$\frac{1}{6}(1)^3 e$	$\frac{1}{24}(1)^4 e$
$-a u_{j-1}$	$-a$	$-a(-1)$	$-a(-1)^2\frac{1}{2}$	$-a(-1)^3\frac{1}{6}$	$-a(-1)^4\frac{1}{24}$	$-a(-1)^5\frac{1}{120}$
$-b u_j$	$-b$					
$-c u_{j+1}$	$-c$	$-c(1)^2$	$-c(1)^2\frac{1}{2}$	$-c(1)^3\frac{1}{6}$	$-c(1)^4\frac{1}{24}$	$-c(1)^5\frac{1}{120}$

sum last column and multiply by this and divide by Δx to find leading error term:

$$er_t = \left(\frac{1}{24}d + \frac{1}{24}e + \frac{1}{120}a - \frac{1}{120}c\right)\Delta x^5\left(\frac{\partial^5 u}{\partial x^5}\right)_j \frac{1}{\Delta x}$$

$$= \frac{\Delta x^4}{120}\left(\frac{\partial^5 u}{\partial x^5}\right)_j$$