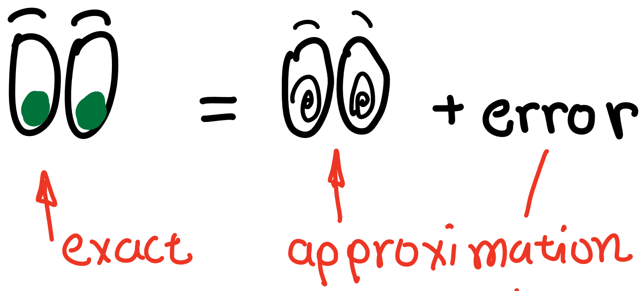
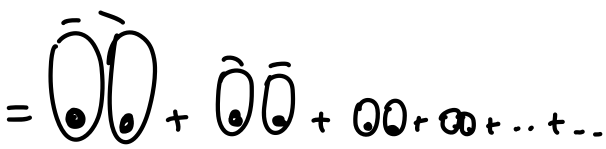
Approximation

- We will use our computers to approximate "stuff"
  e.g. integrals, derivatives, solutions to differential equation
- We say "approximation" because there is (almost) always some deviation from exactness

In principle:



error = sum of many terms



largest = leading error term

$= \text{coefficient} * \Delta x^n$

power n

Δx: measure of granularity of approximation

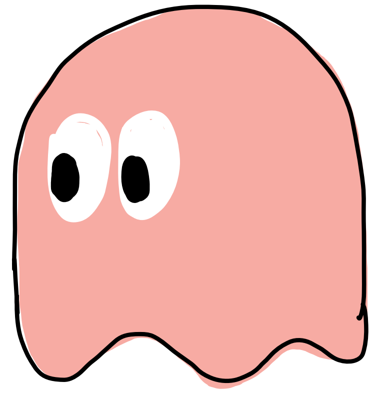
$$\Delta x = \frac{L}{N}$$

L: size (length)

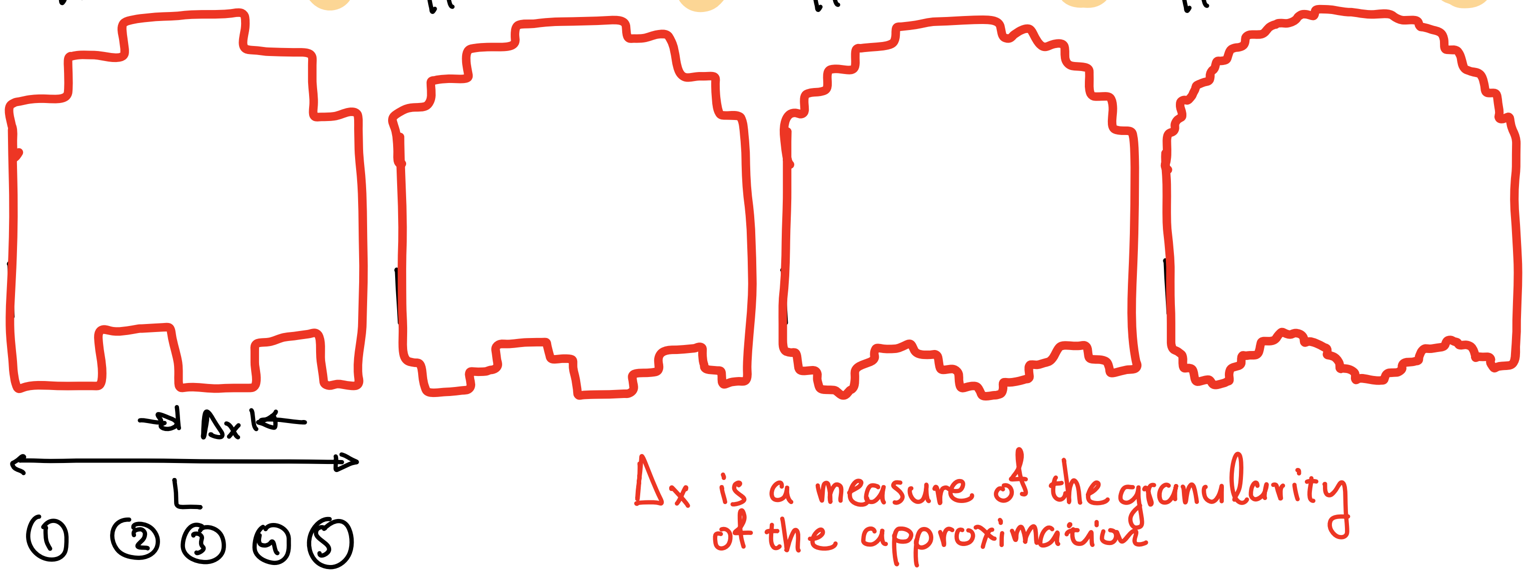
N: number of granules

Example:

Pac-Man Red Ghost



Approximation 1 Approximation 2 Approximation 3 Approximation 4



$N_1 = 5$

$\Delta x_1 = \frac{L}{N_1}$

Δx is a measure of the granularity of the approximation

Δx has dimentions (typically distance)

$N = \frac{L}{\Delta x}$ is also a measure of granularity

N has no dimensions

Approximation	Error	Δx
1	$error_1$	$\Delta x_1 = \frac{L}{N_1}$
2	$error_2$	$\Delta x_2 = \frac{L}{N_2}$
3	$error_3$	$\Delta x_3 = \frac{L}{N_3}$
4	$error_4$	$\Delta x_5 = \frac{L}{N_3}$

$$\text{error} = C\,\Delta_x^{\,n} = C\left(\frac{L}{N}\right)^n = c'\,N^{-n}$$

we are interested in the coefficient n

n is called the rate of convergence or convergence rate

- How to determine n from computer simulations

- Vary Δx (or N) and compute the error each time

e.g.

Δx_1	error_1
Δx_2	error_2
$\vdots$	$\vdots$

the relation is $\text{error} = c\,\Delta x^n$ or $c_2 N^{-n}$

coefficient does not depent on Δx or n

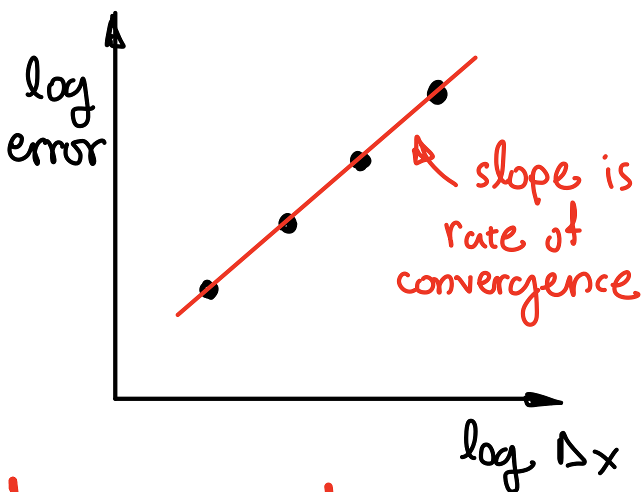
- How to find exponent?

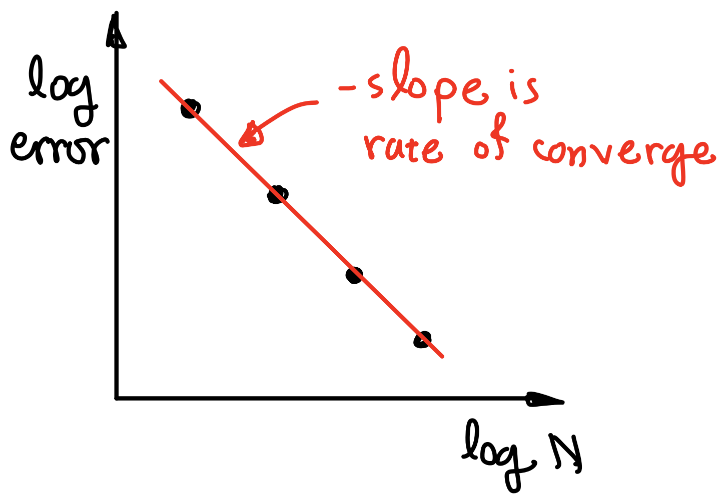
trick: $\text{error} = c\,\Delta x^n$

$$\log(\text{error}) = \underbrace{n}_{\text{slope}} \log(c\Delta x)$$ ← easier!

$$\log(\text{error}) = -n \log(c_2 N)$$

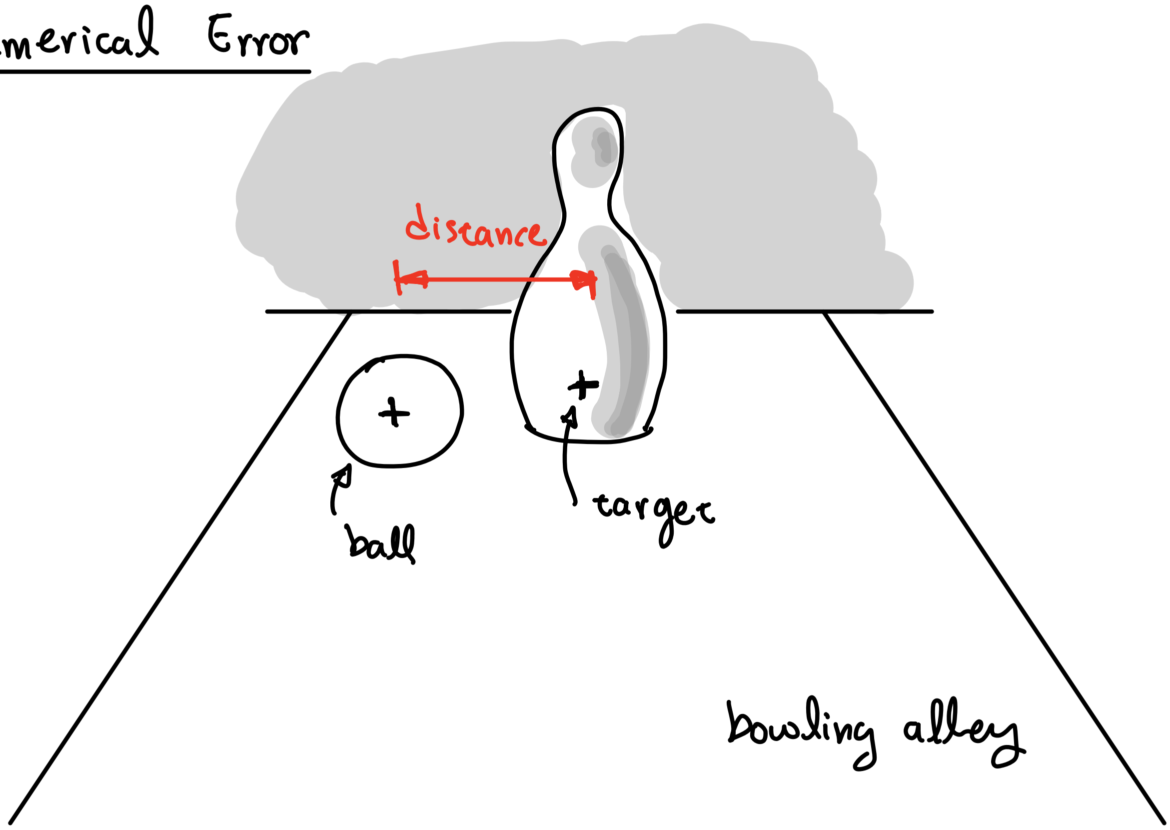
In practice:





Important: limit is asymptotic: need to take $\Delta x \to 0$ or $N \to \infty$

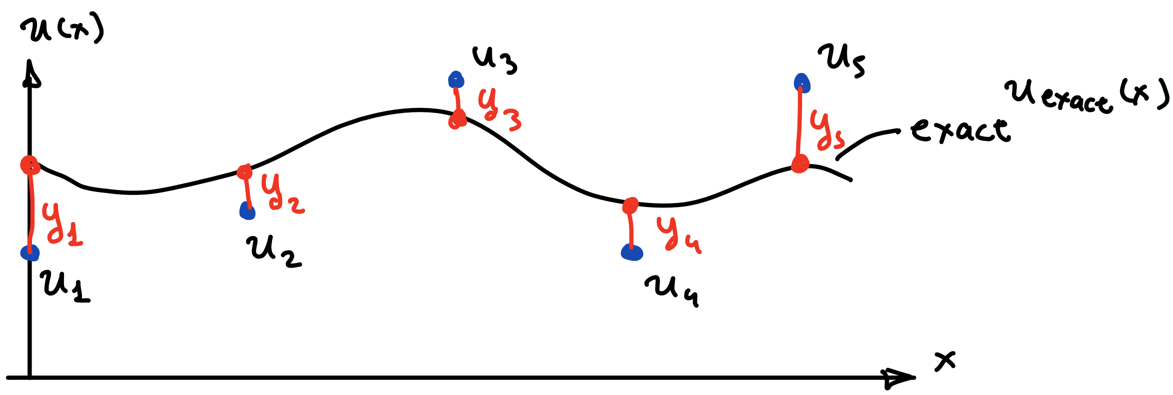
Numerical Error



Definition: error = measure of the "distance" between an exact solution "target" and an approximation

error for scalars : error = $|u_{approximation} - u_{exact}|$

absolute value of difference



error vector : $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_5 \end{bmatrix} = \begin{bmatrix} u_1 - u_{1,exact} \\ u_2 - u_{2,exact} \\ \vdots \\ \vdots \end{bmatrix}$

Vector Norms

For a vector y_i we define its p-norm:

p-norm $\quad |\vec{y}|_p = \left(\sum_i^n |y_i|^p \right)^{1/p} \quad p = 1, 2, 3, \ldots$ ← DO NOT use this equation for convergence studies

2-norm $\quad$ for 3D vectors $\quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$|y|_2 = \sqrt{y_1^2 + y_2^2 + y_3^2}$$

$p = \infty$, infinity norm $\quad |\vec{y}|_\infty = \max_i |y_i|$

Vector norm properties:

1. $|\vec{y}| > 0$ when $\vec{y} \neq 0$ and $|\vec{y}| = 0$ if and only if $\vec{y} = 0$

2. $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$

3. $|c\vec{y}| = |c|\,|\vec{y}|$ for any scalar c

   ↑ scalar

Vector Norms for CFD: error norms

Consider two vectors of N elements on a grid x_i, $i = 1, \dots, N$ with equal spacing Δx

u_i is the numerical approximation

v_i is the exact solution

Important: vectors u_i and v_i are at the same location

remember that $u_i = u(x_i)$ and $v_i = v(x_i)$

Note Δx

exact solution

p-norm
$$L_p = \left[\Delta x \sum_{i=1}^{N} |u_i - v_i|^p \right]^{\frac{1}{p}}$$

$p = 1, 2, 3, \dots$ integer > 0

special case : infinity norm

$$L_\infty = \max_i |u_i - v_i|$$

We need to include Δx because we will be comparing norms of vectors of different lengths.

Similarly for two-dimensional "vectors"

$$L_p = \left[\Delta x \Delta y \sum_{j=1}^{N} \sum_{i=1}^{M} \left| u_{ij} - v_{ij} \right|^p \right]^{\frac{1}{p}}$$

$$L_\infty = \max_{ij} \left| u_{ij} - v_{ij} \right|$$