ME 5311 Semester Project: Part 2

Buoyant Convection

March 27, 2024

1 Problem description

The problem is to study buoyant convection in a two-dimensional rectangular domain with dimensions $[l \times d]$ in the horizontal and vertical directions. The flow is also called Rayleigh–Bénard convection from the two fluid dynamicists who studied the flow. The computational domain is periodic in the streamwise, x, direction. The governing equations are the conservation of mass, streamwise and vertical momentum, and temperature,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + g\alpha(T - T_{\mathrm{ref}}) + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{3}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right). \tag{4}$$

where $p = p'/\rho$ is the scaled pressure, g the acceleration of gravity, α the coefficient of thermal expansion at the reference temperature T_{ref}, and ν and κ the diffusivity coefficients for momentum and temperature, respectively. The top and bottom walls are at temperatures T_c and T_h, respectively. The governing equations follow the Boussinesq approximation, which mean that we do not take into account density variations for mass and momentum transport, but we do take into account the density variation in the fluid's buoyancy. Therefore the vertical momentum equation includes a buoyancy term $g\alpha(T - T_{\mathrm{ref}})$, which is proportional to flow temperature variations with respect to a reference temperature.

We will numerically integrate the non-dimensional form of the equations. First, we define a scaled temperature

$$\Theta = \frac{T - T_c}{T_h - T_c}. \tag{5}$$

We use the diffusivity κ and the vertical gap d for form length, velocity and time scales: d, κ/d,

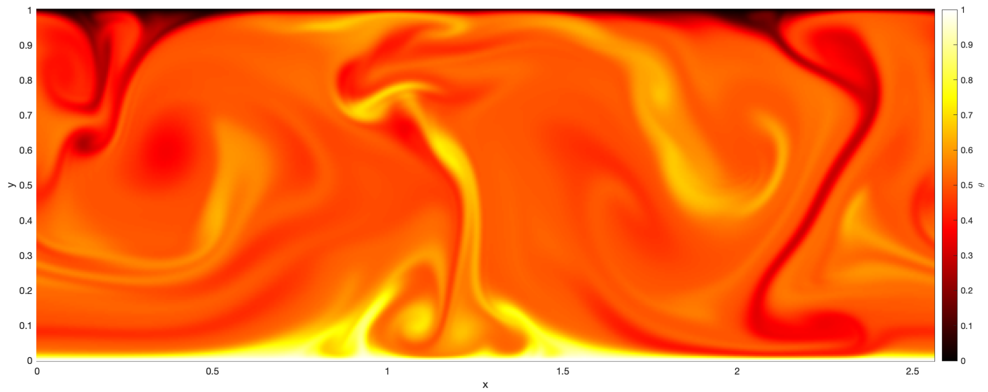

Figure 1: Scaled temperature contours.

and d^2/κ. The pressure is normalized by κ^2/d^2. The non-dimensional equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mathrm{Pr}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{7}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mathrm{Ra}\,\mathrm{Pr}\,\Theta + \mathrm{Pr}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{8}$$

$$\frac{\partial \Theta}{\partial t} + u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y} = \frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2}, \tag{9}$$

where

$$\mathrm{Pr} = \frac{\nu}{\kappa} \tag{10}$$

is the Prandtl number, and

$$\mathrm{Ra} = \frac{g\alpha(T_h - T_c)d^3}{\nu\kappa} \tag{11}$$

is the Rayleigh number. The boundary conditions at the bottom and top boundary are

$$u(t, x, 0) = 0, \tag{12}$$
$$u(t, x, 1) = 0, \tag{13}$$

for the streamwise velocity;

$$v(t, x, 0) = 0, \tag{14}$$
$$v(t, x, 1) = 0, \tag{15}$$

for the vertical velocity; and

$$\Theta(t, x, 0) = 1, \tag{16}$$
$$\Theta(t, x, 1) = 0, \tag{17}$$

for the scaled temperature.

2 Numerical solution

We will discuss in class a numerical solution method through the remainder of the semester. We will implement a fractional step method to numerically solve the two dimensional Navier–Stokes equations using second-order finite differences.

We will carry out the project in three stages:

1. Poisson equation solution: due April 3.
2. Convection term discretization: due April 17.
3. Full solution: due May 1.

2.1 Poisson equation for pressure solution

The first solution stage, due April 3, is to implement the numerical solution of the Poisson equation for the pressure

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = d(x, y), \tag{18}$$

in $[L \times 1]$ with Neumann boundary conditions at $y = 0$ and $y = 1$ and periodic horizontal boundary. You must verify the implementation by:

a. Verify the order of accuracy of the method [12 points]

b. Create a divergence-free velocity field from a randomly-generated velocity field [4 points]

To verify the accuracy of the method, choose L, and use

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\frac{16\pi^2(4 + L^2)}{L^2} \sin\left(8\pi\frac{x}{L}\right) \cos\left(4\pi y\right). \tag{19}$$

The exact solution is

$$p(x, y) = \sin\left(8\pi\frac{x}{L}\right) \cos\left(4\pi y\right). \tag{20}$$

Please submit:

- Your computer code
- The convergence plot for the Poisson equation solutions
- Your code output of the initial divergence of the velocity field and the divergence after ∇p was added to $\vec{u}$.

2.2 Convection term discretization

The second project milestone is to implement the convection term discretization for x and y momentum and the temperature and perform a time integration of the incompressible Euler equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{21}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}, \tag{22}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}, \tag{23}$$

$$\frac{\partial \Theta}{\partial t} + u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y} = 0. \tag{24}$$

Note that the buoyancy term is missing from (24), thus Θ is a passive scalar in the system (21–24).

We will perform the following tasks:

- Implement the convection term discretization for momentum and temperature.
- Use the Runge–Kutta method of Assignment 3 to perform the time integration of momentum and temperature. Make sure to choose a time step such that the CFL number $< \sqrt{3}$. Use random numbers for u, v, and Θ initial conditions. Remember to make the random v field zero-mean in the horizontal.

Deliverables:

a. Show that you can run for at least 100 time steps maintaining a divergence-free flow. Please submit your code and the output showing that the simulation does not "blow up" and the flow is divergence free. A straightforward way to accomplish the "deliverable" is to have the code print the time step number, time, maximum CFL and maximum divergence at each time step. You can also make plots (e.g., using Matlab's `contourf` or `pcolor`) of u, v, and Θ at the initial condition and after 100 time steps. Make sure you include a colorbar to show the values in the contour plot. [12 points]

b. Verify conservation of kinetic energy and scalar variance. That is, show that kinetic energy and scalar variance error is proportional to Δt^3 (third-order convergence with respect to Δt). [4 points]

Please review class notes and recording before attempting to write code. It is important to understand the method and what is required for b. Also, please start early, there is a fair amount of work required to discretize six non-linear terms.

There will be no posted solutions for Part 2 or the project.