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List of Runge–Kutta methods

Runge–Kutta methods are methods for the numerical solution of the ordinary differential equation

$$\frac{dy}{dt} = f(t, y).$$

Explicit Runge–Kutta methods take the form

$$\begin{aligned}
y_{n+1} &= y_n + h\sum_{i=1}^{s} b_i k_i \\
k_1 &= f(t_n, y_n), \\
k_2 &= f(t_n + c_2 h, y_n + h(a_{21}k_1)), \\
k_3 &= f(t_n + c_3 h, y_n + h(a_{31}k_1 + a_{32}k_2)), \\
&\ \ \vdots \\
k_i &= f\left(t_n + c_i h, y_n + h\sum_{j=1}^{i-1} a_{ij}k_j\right).
\end{aligned}$$

Stages for implicit methods of s stages take the more general form, with the solution to be found over all s

$$k_i = f\left(t_n + c_i h, y_n + h\sum_{j=1}^{s} a_{ij}k_j\right).$$

Each method listed on this page is defined by its Butcher tableau, which puts the coefficients of the method in a table as follows:

$$\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \dots & a_{1s} \\
c_2 & a_{21} & a_{22} & \dots & a_{2s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_s & a_{s1} & a_{s2} & \dots & a_{ss} \\
\hline
 & b_1 & b_2 & \dots & b_s
\end{array}$$

For adaptive and implicit methods, the Butcher tableau is extended to give values of b_i^*, and the estimated error is then

$$e_{n+1} = h\sum_{i=1}^{s}(b_i - b_i^*)k_i.$$

Explicit methods

The explicit methods are those where the matrix $[a_{ij}]$ is lower triangular.

Forward Euler

The Euler method is first order. The lack of stability and accuracy limits its popularity mainly to use as a simple introductory example of a numeric solution method.

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

Explicit midpoint method

The (explicit) midpoint method is a second-order method with two stages (see also the implicit midpoint method below):

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array}$$

Heun's method

Heun's method is a second-order method with two stages. It is also known as the explicit trapezoid rule, improved Euler's method, or modified Euler's method. (Note: The "eu" is pronounced the same way as in "Euler", so "Heun" rhymes with "coin"):

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

Ralston's method

Ralston's method is a second-order method[1] with two stages and a minimum local error bound:

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
2/3 & 2/3 & 0 \\
\hline
 & 1/4 & 3/4
\end{array}
$$

Generic second-order method

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
\alpha & \alpha & 0 \\
\hline
 & 1-\frac{1}{2\alpha} & \frac{1}{2\alpha}
\end{array}
$$

Kutta's third-order method

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 \\
1 & -1 & 2 & 0 \\
\hline
 & 1/6 & 2/3 & 1/6
\end{array}
$$

Generic third-order method

See Sanderse and Veldman (2019).[2]

for $\alpha \neq 0, \frac{2}{3}, 1$:

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
\alpha & \alpha & 0 & 0 \\
1 & 1+\frac{1-\alpha}{\alpha(3\alpha-2)} & -\frac{1-\alpha}{\alpha(3\alpha-2)} & 0 \\
\hline
 & \frac{1}{2}-\frac{1}{6\alpha} & \frac{1}{6\alpha(1-\alpha)} & \frac{2-3\alpha}{6(1-\alpha)}
\end{array}
$$

Heun's third-order method

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/3 & 1/3 & 0 & 0 \\
2/3 & 0 & 2/3 & 0 \\
\hline
 & 1/4 & 0 & 3/4
\end{array}
$$

Van der Houwen's/Wray third-order method

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
8/15 & 8/15 & 0 & 0 \\
2/3 & 1/4 & 5/12 & 0 \\
\hline
 & 1/4 & 0 & 3/4
\end{array}
$$

Ralston's third-order method

Ralston's third-order method[1] is used in the embedded Bogacki–Shampine method.

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 \\
3/4 & 0 & 3/4 & 0 \\
\hline
 & 2/9 & 1/3 & 4/9
\end{array}
$$

Third-order Strong Stability Preserving Runge-Kutta (SSPRK3)

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1/2 & 1/4 & 1/4 & 0 \\
\hline
 & 1/6 & 1/6 & 2/3
\end{array}
$$

Classic fourth-order method

The "original" Runge–Kutta method.[3]

0	0	0	0	0
1/2	1/2	0	0	0
1/2	0	1/2	0	0
1	0	0	1	0
	1/6	1/3	1/3	1/6

3/8-rule fourth-order method

This method doesn't have as much notoriety as the "classic" method, but is just as classic because it was proposed in the same paper (Kutta, 1901).[3]

0	0	0	0	0
1/3	1/3	0	0	0
2/3	−1/3	1	0	0
1	1	−1	1	0
	1/8	3/8	3/8	1/8

Ralston's fourth-order method

This fourth order method[1] has minimum truncation error.

0	0	0	0	0
.4	.4	0	0	0
.45573725	.29697761	.15875964	0	0
1	.21810040	−3.05096516	3.83286476	0
	.17476028	−.55148066	1.20553560	.17118478

Embedded methods

The embedded methods are designed to produce an estimate of the local truncation error of a single Runge–Kutta step, and as result, allow to control the error with adaptive stepsize. This is done by having two methods in the tableau, one with order p and one with order p-1.

The lower-order step is given by

$$y^*_{n+1} = y_n + h\sum_{i=1}^{s} b^*_i k_i,$$

where the k_i are the same as for the higher order method. Then the error is

$$e_{n+1} = y_{n+1} - y^*_{n+1} = h\sum_{i=1}^{s}(b_i - b^*_i)k_i,$$

which is $O(h^p)$. The Butcher Tableau for this kind of method is extended to give the values of b^*_i

$$
\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \dots & a_{1s} \\
c_2 & a_{21} & a_{22} & \dots & a_{2s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_s & a_{s1} & a_{s2} & \dots & a_{ss} \\
\hline
 & b_1 & b_2 & \dots & b_s \\
 & b_1^* & b_2^* & \dots & b_s^*
\end{array}
$$

Heun–Euler

The simplest adaptive Runge–Kutta method involves combining Heun's method, which is order 2, with the Euler method, which is order 1. Its extended Butcher Tableau is:

$$
\begin{array}{c|cc}
0 & & \\
1 & 1 & \\
\hline
 & 1/2 & 1/2 \\
 & 1 & 0
\end{array}
$$

The error estimate is used to control the stepsize.

Fehlberg RK1(2)

The Fehlberg method[4] has two methods of orders 1 and 2. Its extended Butcher Tableau is:

$$
\begin{array}{c|ccc}
0 & & & \\
1/2 & 1/2 & & \\
1 & 1/256 & 255/256 & \\
\hline
 & 1/512 & 255/256 & 1/512 \\
 & 1/256 & 255/256 & 0
\end{array}
$$

The first row of b coefficients gives the second-order accurate solution, and the second row has order one.

Bogacki–Shampine

The Bogacki–Shampine method has two methods of orders 2 and 3. Its extended Butcher Tableau is:

$$
\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
3/4 & 0 & 3/4 & & \\
1 & 2/9 & 1/3 & 4/9 & \\
\hline
 & 2/9 & 1/3 & 4/9 & 0 \\
 & 7/24 & 1/4 & 1/3 & 1/8
\end{array}
$$

The first row of b coefficients gives the third-order accurate solution, and the second row has order two.

Fehlberg

The Runge–Kutta–Fehlberg method has two methods of orders 5 and 4; it is sometimes dubbed RKF45 . Its extended Butcher Tableau is:

$$\begin{array}{r|cccccc}
0 & & & & & & \\
1/4 & 1/4 & & & & & \\
3/8 & 3/32 & 9/32 & & & & \\
12/13 & 1932/2197 & -7200/2197 & 7296/2197 & & & \\
1 & 439/216 & -8 & 3680/513 & -845/4104 & & \\
1/2 & -8/27 & 2 & -3544/2565 & 1859/4104 & -11/40 & \\
\hline
 & 16/135 & 0 & 6656/12825 & 28561/56430 & -9/50 & 2/55 \\
 & 25/216 & 0 & 1408/2565 & 2197/4104 & -1/5 & 0
\end{array}$$

The first row of *b* coefficients gives the fifth-order accurate solution, and the second row has order four. The coefficients here allow for an adaptive stepsize to be determined automatically.

Cash-Karp

Cash and Karp have modified Fehlberg's original idea. The extended tableau for the Cash–Karp method is

0						
1/5	1/5					
3/10	3/40	9/40				
3/5	3/10	−9/10	6/5			
1	−11/54	5/2	−70/27	35/27		
7/8	1631/55296	175/512	575/13824	44275/110592	253/4096	
	37/378	0	250/621	125/594	0	512/1771
	2825/27648	0	18575/48384	13525/55296	277/14336	1/4

The first row of *b* coefficients gives the fifth-order accurate solution, and the second row has order four.

Dormand–Prince

The extended tableau for the Dormand–Prince method is

0							
1/5	1/5						
3/10	3/40	9/40					
4/5	44/45	−56/15	32/9				
8/9	19372/6561	−25360/2187	64448/6561	−212/729			
1	9017/3168	−355/33	46732/5247	49/176	−5103/18656		
1	35/384	0	500/1113	125/192	−2187/6784	11/84	
	35/384	0	500/1113	125/192	−2187/6784	11/84	0
	5179/57600	0	7571/16695	393/640	−92097/339200	187/2100	1/40

The first row of *b* coefficients gives the fifth-order accurate solution, and the second row gives the fourth-order accurate solution.

Implicit methods

Backward Euler

The backward Euler method is first order. Unconditionally stable and non-oscillatory for linear diffusion problems.

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

Implicit midpoint

The implicit midpoint method is of second order. It is the simplest method in the class of collocation methods known as the Gauss-Legendre methods. It is a symplectic integrator.

$$\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}$$

Crank-Nicolson method

The Crank–Nicolson method corresponds to the implicit trapezoidal rule and is a second-order accurate and A-stable method.

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

Gauss–Legendre methods

These methods are based on the points of Gauss–Legendre quadrature. The Gauss–Legendre method of order four has Butcher tableau:

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{2} + \frac{\sqrt{3}}{2} & \frac{1}{2} - \frac{\sqrt{3}}{2} \end{array}$$

The Gauss–Legendre method of order six has Butcher tableau:

$$
\begin{array}{c|ccc}
\frac{1}{2}-\frac{\sqrt{15}}{10} & \frac{5}{36} & \frac{2}{9}-\frac{\sqrt{15}}{15} & \frac{5}{36}-\frac{\sqrt{15}}{30} \\
\frac{1}{2} & \frac{5}{36}+\frac{\sqrt{15}}{24} & \frac{2}{9} & \frac{5}{36}-\frac{\sqrt{15}}{24} \\
\frac{1}{2}+\frac{\sqrt{15}}{10} & \frac{5}{36}+\frac{\sqrt{15}}{30} & \frac{2}{9}+\frac{\sqrt{15}}{15} & \frac{5}{36} \\
\hline
 & \frac{5}{18} & \frac{4}{9} & \frac{5}{18} \\
 & -\frac{5}{6} & \frac{8}{3} & -\frac{5}{6}
\end{array}
$$

Diagonally Implicit Runge–Kutta methods

Diagonally Implicit Runge–Kutta (DIRK) formulae have been widely used for the numerical solution of stiff initial value problems; [5] the advantage of this approach is that here the solution may be found sequentially as opposed to simultaneously.

The simplest method from this class is the order 2 implicit midpoint method.

Kraaijevanger and Spijker's two-stage Diagonally Implicit Runge–Kutta method:

$$
\begin{array}{c|cc}
1/2 & 1/2 & 0 \\
3/2 & -1/2 & 2 \\
\hline
 & -1/2 & 3/2
\end{array}
$$

Qin and Zhang's two-stage, 2nd order, symplectic Diagonally Implicit Runge–Kutta method:

$$
\begin{array}{c|cc}
1/4 & 1/4 & 0 \\
3/4 & 1/2 & 1/4 \\
\hline
 & 1/2 & 1/2
\end{array}
$$

Pareschi and Russo's two-stage 2nd order Diagonally Implicit Runge–Kutta method:

$$
\begin{array}{c|cc}
x & x & 0 \\
1-x & 1-2x & x \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

This Diagonally Implicit Runge–Kutta method is A-stable if and only if $x \geq \frac{1}{4}$. Moreover, this method is L-stable if and only if x equals one of the roots of the polynomial $x^2 - 2x + \frac{1}{2}$, i.e. if $x = 1 \pm \frac{\sqrt{2}}{2}$. Qin and Zhang's Diagonally Implicit Runge–Kutta method corresponds to Pareschi and Russo's Diagonally Implicit Runge–Kutta method with $x = 1/4$.

Two-stage 2nd order Diagonally Implicit Runge–Kutta method:

$$
\begin{array}{c|cc}
x & x & 0 \\
1 & 1-x & x \\
\hline
 & 1-x & x
\end{array}
$$

Again, this Diagonally Implicit Runge–Kutta method is A-stable if and only if $x \geq \frac{1}{4}$. As the previous method, this method is again L-stable if and only if x equals one of the roots of the polynomial $x^2 - 2x + \frac{1}{2}$, i.e. if $x = 1 \pm \frac{\sqrt{2}}{2}$.

Crouzeix's two-stage, 3rd order Diagonally Implicit Runge–Kutta method:

$$\begin{array}{c|cc} \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{2}+\frac{\sqrt{3}}{6} & 0 \\ \frac{1}{2}-\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{3} & \frac{1}{2}+\frac{\sqrt{3}}{6} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Crouzeix's three-stage, 4th order Diagonally Implicit Runge–Kutta method:

$$\begin{array}{c|ccc} \frac{1+\alpha}{2} & \frac{1+\alpha}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\alpha}{2} & \frac{1+\alpha}{2} & 0 \\ \frac{1-\alpha}{2} & 1+\alpha & -(1+2\alpha) & \frac{1+\alpha}{2} \\ \hline & \frac{1}{6\alpha^2} & 1-\frac{1}{3\alpha^2} & \frac{1}{6\alpha^2} \end{array}$$

with $\alpha = \frac{2}{\sqrt{3}} \cos \frac{\pi}{18}$.

Three-stage, 3rd order, L-stable Diagonally Implicit Runge–Kutta method:

$$\begin{array}{c|ccc} x & x & 0 & 0 \\ \frac{1+x}{2} & \frac{1-x}{2} & x & 0 \\ 1 & -3x^2/2+4x-1/4 & 3x^2/2-5x+5/4 & x \\ \hline & -3x^2/2+4x-1/4 & 3x^2/2-5x+5/4 & x \end{array}$$

with $x = 0.4358665215$

Nørsett's three-stage, 4th order Diagonally Implicit Runge–Kutta method has the following Butcher tableau:

$$\begin{array}{c|ccc} x & x & 0 & 0 \\ 1/2 & 1/2-x & x & 0 \\ 1-x & 2x & 1-4x & x \\ \hline & \frac{1}{6(1-2x)^2} & \frac{3(1-2x)^2-1}{3(1-2x)^2} & \frac{1}{6(1-2x)^2} \end{array}$$

with x one of the three roots of the cubic equation $x^3 - 3x^2/2 + x/2 - 1/24 = 0$. The three roots of this cubic equation are approximately $x_1 = 1.06858$, $x_2 = 0.30254$, and $x_3 = 0.12889$. The root x_1 gives the best stability properties for initial value problems.

Four-stage, 3rd order, L-stable Diagonally Implicit Runge–Kutta method

$$
\begin{array}{c|cccc}
1/2 & 1/2 & 0 & 0 & 0 \\
2/3 & 1/6 & 1/2 & 0 & 0 \\
1/2 & -1/2 & 1/2 & 1/2 & 0 \\
1 & 3/2 & -3/2 & 1/2 & 1/2 \\
\hline
 & 3/2 & -3/2 & 1/2 & 1/2
\end{array}
$$

Lobatto methods

There are three main families of Lobatto methods,[6] called IIIA, IIIB and IIIC (in classical mathematical literature, the symbols I and II are reserved for two types of Radau methods). These are named after Rehuel Lobatto[6] as a reference to the Lobatto quadrature rule, but were introduced by Byron L. Ehle in his thesis.[7] All are implicit methods, have order $2s - 2$ and they all have $c_1 = 0$ and $c_s = 1$. Unlike any explicit method, it's possible for these methods to have the order greater than the number of stages. Lobatto lived before the classic fourth-order method was popularized by Runge and Kutta.

Lobatto IIIA methods

The Lobatto IIIA methods are collocation methods. The second-order method is known as the trapezoidal rule:

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
1 & 1/2 & 1/2 \\
\hline
 & 1/2 & 1/2 \\
 & 1 & 0
\end{array}
$$

The fourth-order method is given by

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/2 & 5/24 & 1/3 & -1/24 \\
1 & 1/6 & 2/3 & 1/6 \\
\hline
 & 1/6 & 2/3 & 1/6 \\
 & -\frac{1}{2} & 2 & -\frac{1}{2}
\end{array}
$$

These methods are A-stable, but not L-stable and B-stable.

Lobatto IIIB methods

The Lobatto IIIB methods are not collocation methods, but they can be viewed as discontinuous collocation methods (Hairer, Lubich & Wanner 2006, §II.1.4). The second-order method is given by

$$
\begin{array}{c|cc}
0 & 1/2 & 0 \\
1 & 1/2 & 0 \\
\hline
 & 1/2 & 1/2 \\
 & 1 & 0
\end{array}
$$

The fourth-order method is given by

$$\begin{array}{c|ccc} 0 & 1/6 & -1/6 & 0 \\ 1/2 & 1/6 & 1/3 & 0 \\ 1 & 1/6 & 5/6 & 0 \\ \hline & 1/6 & 2/3 & 1/6 \\ & -\frac{1}{2} & 2 & -\frac{1}{2} \end{array}$$

Lobatto IIIB methods are A-stable, but not L-stable and B-stable.

Lobatto IIIC methods

The Lobatto IIIC methods also are discontinuous collocation methods. The second-order method is given by

$$\begin{array}{c|cc} 0 & 1/2 & -1/2 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \\ & 1 & 0 \end{array}$$

The fourth-order method is given by

$$\begin{array}{c|ccc} 0 & 1/6 & -1/3 & 1/6 \\ 1/2 & 1/6 & 5/12 & -1/12 \\ 1 & 1/6 & 2/3 & 1/6 \\ \hline & 1/6 & 2/3 & 1/6 \\ & -\frac{1}{2} & 2 & -\frac{1}{2} \end{array}$$

They are L-stable. They are also algebraically stable and thus B-stable, that makes them suitable for stiff problems.

Lobatto IIIC* methods

The Lobatto IIIC* methods are also known as Lobatto III methods (Butcher, 2008), Butcher's Lobatto methods (Hairer et al., 1993), and Lobatto IIIC methods (Sun, 2000) in the literature.[6] The second-order method is given by

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

Butcher's three-stage, fourth-order method is given by

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ 1 & 0 & 1 & 0 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

These methods are not A-stable, B-stable or L-stable. The Lobatto IIIC* method for $s = 2$ is sometimes called the explicit trapezoidal rule.

Generalized Lobatto methods

One can consider a very general family of methods with three real parameters $(\alpha_A, \alpha_B, \alpha_C)$ by considering Lobatto coefficients of the form

$$a_{i,j}(\alpha_A, \alpha_B, \alpha_C) = \alpha_A a_{i,j}^A + \alpha_B a_{i,j}^B + \alpha_C a_{i,j}^C + \alpha_{C*} a_{i,j}^{C*},$$

where

$$\alpha_{C*} = 1 - \alpha_A - \alpha_B - \alpha_C.$$

For example, Lobatto IIID family introduced in (Nørsett and Wanner, 1981), also called Lobatto IIINW, are given by

$$\begin{array}{c|cc} 0 & 1/2 & 1/2 \\ 1 & -1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

and

$$\begin{array}{c|ccc} 0 & 1/6 & 0 & -1/6 \\ 1/2 & 1/12 & 5/12 & 0 \\ 1 & 1/2 & 1/3 & 1/6 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

These methods correspond to $\alpha_A = 2$, $\alpha_B = 2$, $\alpha_C = -1$, and $\alpha_{C*} = -2$. The methods are L-stable. They are algebraically stable and thus B-stable.

Radau methods

Radau methods are fully implicit methods (matrix A of such methods can have any structure). Radau methods attain order $2s - 1$ for s stages. Radau methods are A-stable, but expensive to implement. Also they can suffer from order reduction. The first order Radau method is similar to backward Euler method.

Radau IA methods

The third-order method is given by

$$\begin{array}{c|cc}
0 & 1/4 & -1/4 \\
2/3 & 1/4 & 5/12 \\
\hline
 & 1/4 & 3/4
\end{array}$$

The fifth-order method is given by

$$\begin{array}{c|ccc}
0 & \frac{1}{9} & \frac{-1-\sqrt{6}}{18} & \frac{-1+\sqrt{6}}{18} \\
\frac{3}{5}-\frac{\sqrt{6}}{10} & \frac{1}{9} & \frac{11}{45}+\frac{7\sqrt{6}}{360} & \frac{11}{45}-\frac{43\sqrt{6}}{360} \\
\frac{3}{5}+\frac{\sqrt{6}}{10} & \frac{1}{9} & \frac{11}{45}+\frac{43\sqrt{6}}{360} & \frac{11}{45}-\frac{7\sqrt{6}}{360} \\
\hline
 & \frac{1}{9} & \frac{4}{9}+\frac{\sqrt{6}}{36} & \frac{4}{9}-\frac{\sqrt{6}}{36}
\end{array}$$

Radau IIA methods

The c_i of this method are zeros of

$$\frac{d^{s-1}}{dx^{s-1}}(x^{s-1}(x-1)^s).$$

The third-order method is given by

$$\begin{array}{c|cc}
1/3 & 5/12 & -1/12 \\
1 & 3/4 & 1/4 \\
\hline
 & 3/4 & 1/4
\end{array}$$

The fifth-order method is given by

$$\begin{array}{c|ccc}
\frac{2}{5}-\frac{\sqrt{6}}{10} & \frac{11}{45}-\frac{7\sqrt{6}}{360} & \frac{37}{225}-\frac{169\sqrt{6}}{1800} & -\frac{2}{225}+\frac{\sqrt{6}}{75} \\
\frac{2}{5}+\frac{\sqrt{6}}{10} & \frac{37}{225}+\frac{169\sqrt{6}}{1800} & \frac{11}{45}+\frac{7\sqrt{6}}{360} & -\frac{2}{225}-\frac{\sqrt{6}}{75} \\
1 & \frac{4}{9}-\frac{\sqrt{6}}{36} & \frac{4}{9}+\frac{\sqrt{6}}{36} & \frac{1}{9} \\
\hline
 & \frac{4}{9}-\frac{\sqrt{6}}{36} & \frac{4}{9}+\frac{\sqrt{6}}{36} & \frac{1}{9}
\end{array}$$

Notes

1. Ralston, Anthony (1962). "Runge-Kutta Methods with Minimum Error Bounds" (https://doi.org/10.1090%2FS0025-5718-1962-0150954-0). *Math. Comput.* **16** (80): 431–437. doi:10.1090/S0025-5718-1962-0150954-0 (https://doi.org/10.1090%2FS0025-5718-1962-0150954-0).
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