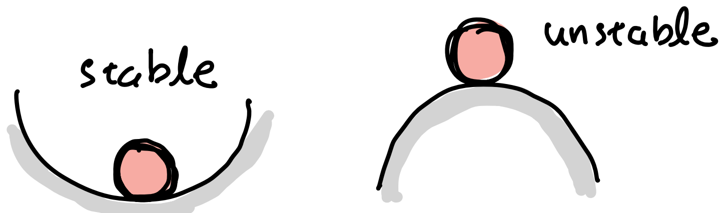
Properties

- **Convergence**: Numerical solution $\xrightarrow[\Delta x \to 0]{\Delta t \to 0}$ real solution
- **Consistency**: Discrete operator $\xrightarrow[\Delta x \to 0]{\Delta t \to 0}$ Continuous PDE
- **Stability**: Discrete operators do not amplify small errors / perturbations



Example: Consiseency:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad \leftarrow \text{PDE}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\,\frac{u_{i+1}^n - u_i^n}{\Delta x} = 0 \quad \leftarrow \text{discrete operator}$$

Taylor expansion about (i,n)

$$\times \frac{1}{\Delta t} \rightarrow \quad u_i^{n+1} = u^n + \Delta t \frac{\partial u}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2 u}{\partial t^2} + \cdots$$

$$\times \frac{1}{\Delta x} \quad u_{i+1}^n = u_i^n + \Delta x \frac{\partial u}{\partial x} + \frac{1}{2}\Delta x^2 \frac{\partial^2 u}{\partial x^2} + \cdots$$

$$\frac{u_i^{n+1} - u^n}{\Delta t} = \frac{\partial u}{\partial t} + \frac{1}{2}\Delta t \frac{\partial^2 u}{\partial t^2} \quad (1)$$

$$\frac{u_{i+1}^n - u_i^n}{\Delta x} = \frac{\partial u}{\partial x} + \frac{1}{2}\Delta x \frac{\partial^3 u}{\partial x^2} \quad (2)$$

① + a * ② $\Rightarrow$

$$\underbrace{\frac{u_i^{n+1}-u_i^n}{\Delta t} + a\,\frac{u_{i+1}^n - u_i^n}{\Delta x}}_{\text{discrete op.}} - \underbrace{\left(\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}\right)}_{\text{PDE}} = \frac{1}{2}\Delta t\frac{\partial^2 u}{\partial t^2} + \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2}$$

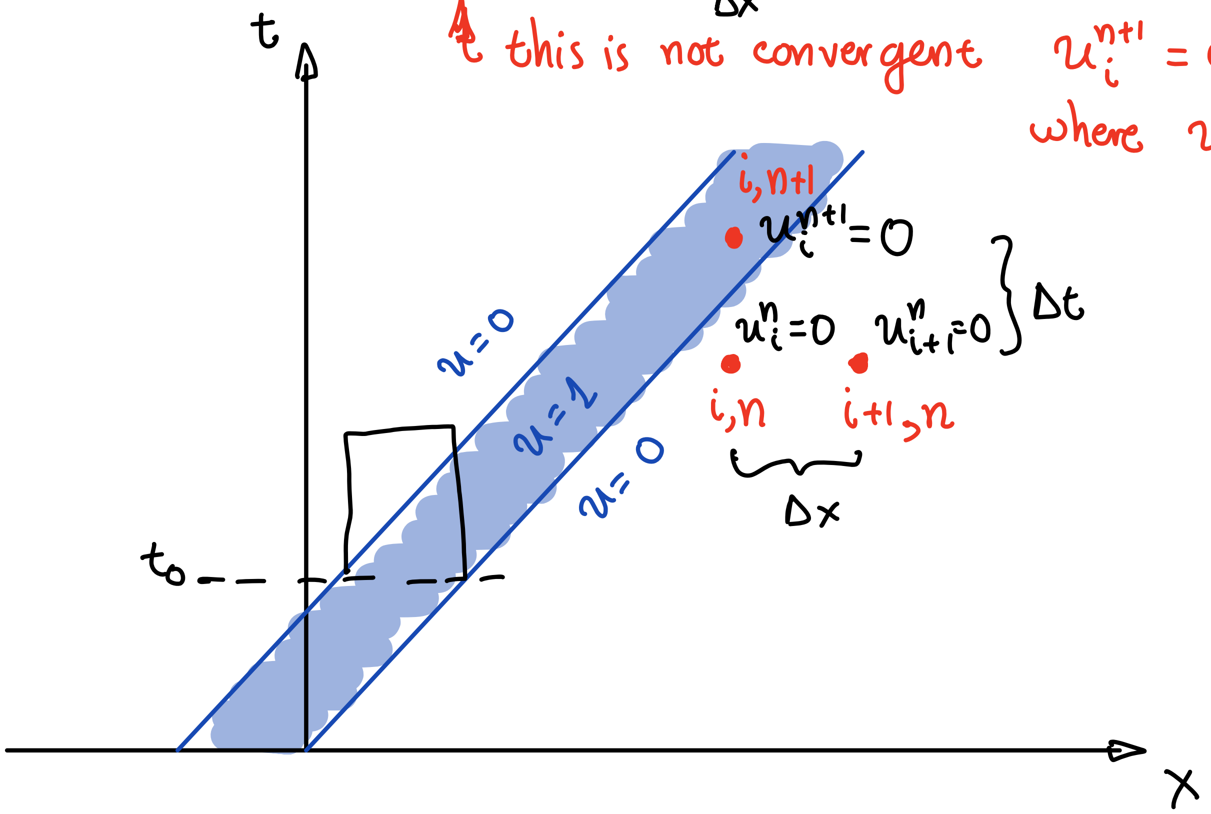
For consistency: $\displaystyle\lim_{\substack{\Delta t\to 0\\ \Delta x\to 0}}\left(\text{discrete op.} - \text{PDE}\right)$

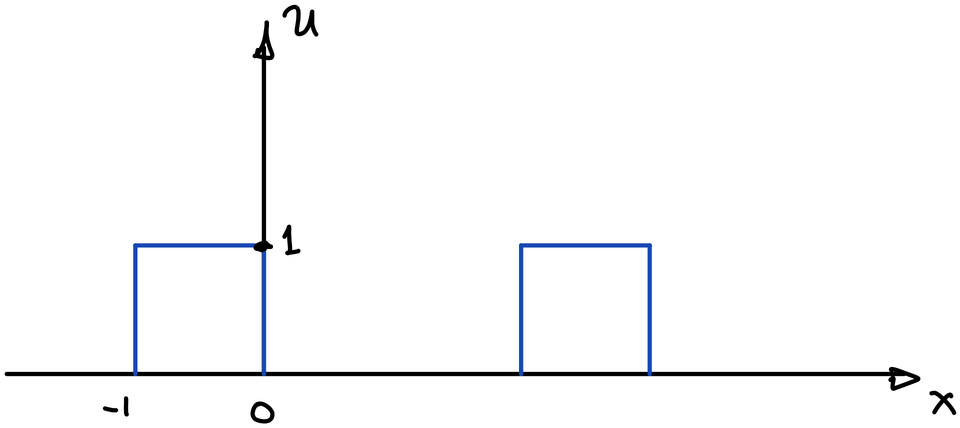
$$= \lim_{\substack{\Delta t\to 0\\ \Delta x\to 0}} \frac{1}{2}\Delta t\frac{\partial^2 u}{\partial t^2} + \frac{a}{2}\Delta x\frac{\partial^2 u}{\partial x^2} = 0$$

Does consistency imply convergence? Nope...

$$u_i^{n+1} = u_i^n - a\frac{\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right)$$

this is not convergent $u_i^{n+1} = 0$ where $u_i^{n+1} \neq 0$





Stability:

Definition: $|u^n| \leq C \sum_{j=0}^{J} |u^j|$ $\quad |u| = \left(\Delta x \sum_i |u_i|^2\right)^{1/2}$

the amount
of growth at time $t = n\Delta t$
is limited by a linear
combination of the norms
of previous steps

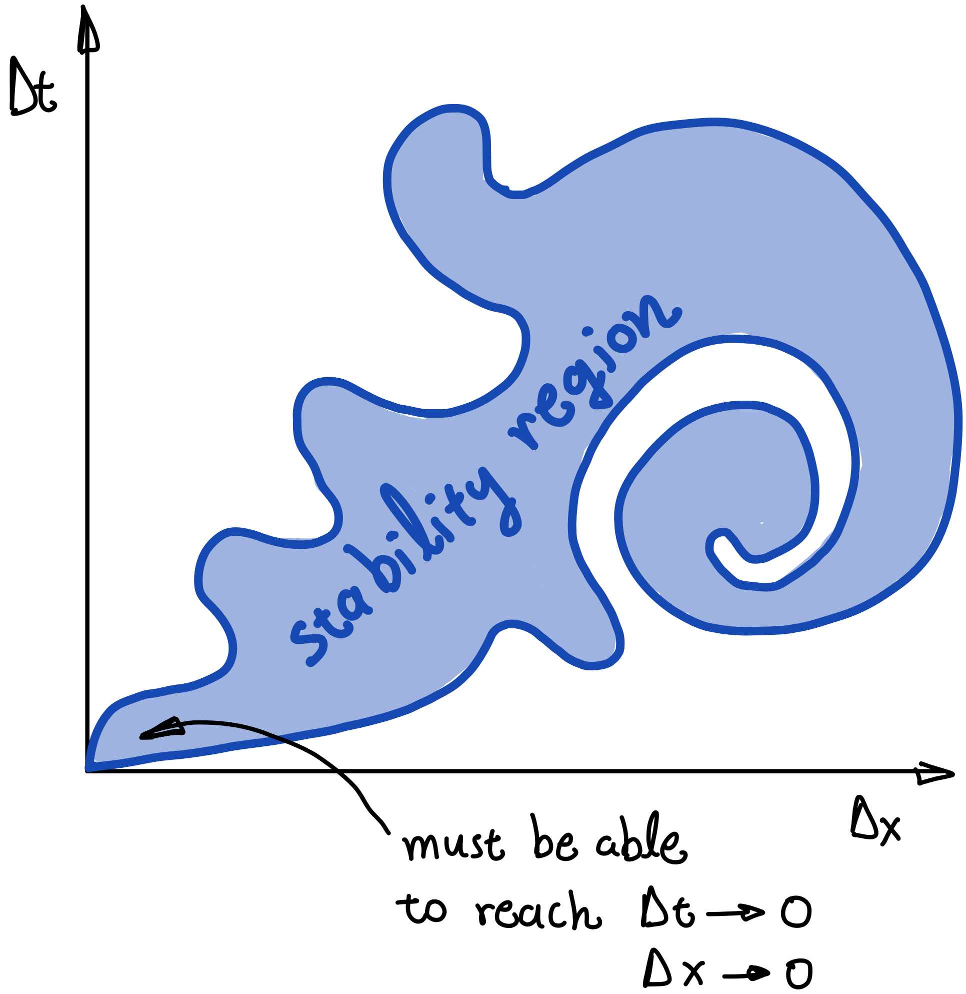
Important:

Definition of stability is for discrete operator $(u_i^n) = 0$

Key

Stability defined based on the homogenious problem

Stabily is a function of Δx , Δt



Stability Example

(similar to example 1.5.1 of Strikwerda)

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad -\infty < x < \infty$$

$$u_i^{n+1} = u_i^n - a\frac{\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right)$$

$$= \left(1 - a\frac{\Delta t}{\Delta x}\right)u_i^n + a\frac{\Delta t}{\Delta x}u_{i-1}^n$$

$$u_i^{n+1} = \alpha u_i^n + b u_{i-1}^n$$

$$\sum_{i=-\infty}^{\infty}|u_i^{n+1}|^2 = \sum_{i=-\infty}^{\infty}|\alpha u_i^n + b u_{i-1}^n|^2$$

$$\leq \sum_{i=-\infty}^{\infty}|a|^2|u_i^n|^2 + \underbrace{2|a||b||u_i||u_{i-1}|}_{2xy \leq x^2+y^2} + |b|^2|u_{i+1}|^2$$

$$\leq \sum_{i=-\infty}^{\infty}|a|^2|u_i^n|^2 + |a||b|\left(|u_i^n|^2 + |u_{i-1}^n|^2\right) + |b|^2|u_{i+1}^n|^2$$

- split i and $i-1$

$$= \sum_{i=-\infty}^{\infty}\left(|a|^2 + |a||b|\right)|u_i^n|^2 + \sum_{i=-\infty}^{\infty}\left(|a||b| + |b|^2\right)|u_{i-1}^n|^2$$

- sum over i or $i-1$ is the same

$$= \sum_{i=-\infty}^{\infty}\left(|a|^2 + 2|a||b| + |b|^2\right)|u_i^n|^2$$

$$= \sum_{i=-\infty}^{\infty}\left(|a| + |b|\right)^2|u_i^n|^2$$

Thus... $$\sum_{i=-\infty}^{\infty}|u_i^{n+1}|^2 \leq \left(|a| + |b|\right)^2\sum_{i=-\infty}^{\infty}|u_i^n|^2$$

It applies to all n, so...

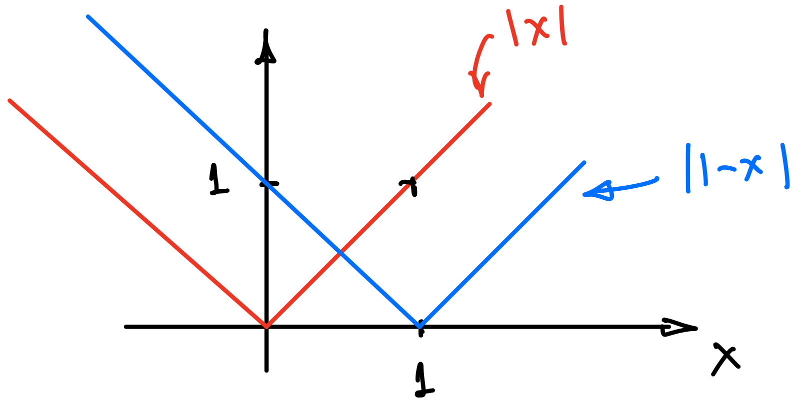
$$\sum_{i=-\infty}^{\infty}|u_i^{n+1}|^2 \leq \left(|a| + |b|\right)^{2n}\sum_{i=-\infty}^{\infty}|u_i^0|^2$$

Scheme is stable if $|a| + |b| \leq 1$

$$u_i^{n+1} = u_i^n - a\frac{\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right)$$

$$= \underbrace{\left(1 - a\frac{\Delta t}{\Delta x}\right)}_{a} u_i^n + \underbrace{a\frac{\Delta t}{\Delta x}}_{b} u_{i-1}^n$$

set $a\frac{\Delta t}{\Delta x} = x$ we need $|1-x| + |x| \leq 1$



$||1-x| + |x| \leq 1$

in interval $0 \leq x \leq 1$

$$0 \leq a \frac{\Delta t}{\Delta x} \leq 1$$

PDE

Numerical method

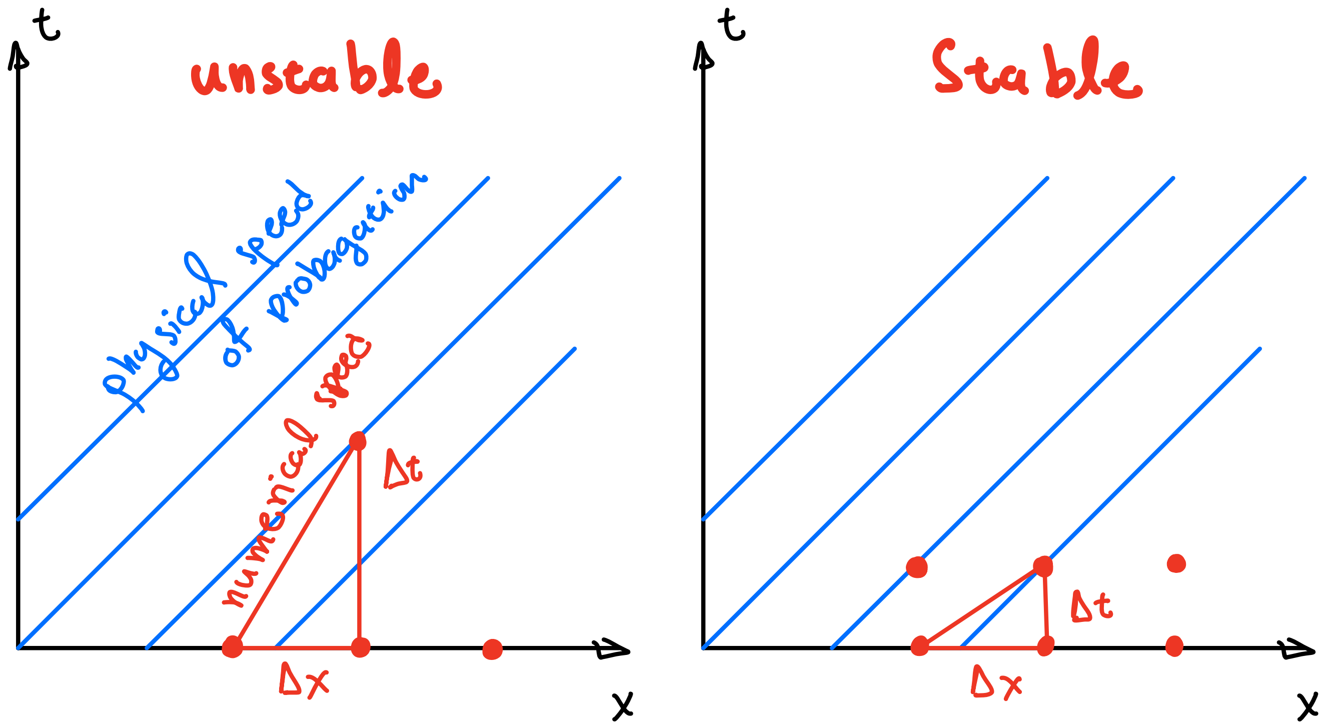
Very Important!

Courant - Friedrichs - Lewy Condition : $\left|a\frac{\Delta t}{\Delta x}\right| \leq 1$

(CFL)

physical speed

CFL number $= a\frac{\Delta t}{\Delta x}$



For stability: $a < \frac{\Delta x}{\Delta t}$ or $CFL < 1$

- What is the relation between consistency, stability and convergence?

The Lax-Richtmyer equivalence theorem:

consistency + stability $\longrightarrow$ convergence

for finite difference schemes

Stability of implicit scheme (example 1.6.1 Strikwerda)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a \frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x}$$

$$\left(1 + a\frac{\Delta t}{\Delta x}\right)^2 |u_i^{n+1}|^2 \leq |u_i^n|^2 + 2a\frac{\Delta t}{\Delta x}\, |u_{i-1}^{n+1}| + \left(a\frac{\Delta t}{\Delta x}\right)^2 |u_{i-1}^{n+1}|^2$$

$$\leq \left(1 + a\frac{\Delta t}{\Delta x}\right)|u_i^n|^2 + \left(a\frac{\Delta t}{\Delta x} + \left(a\frac{\Delta t}{\Delta x}\right)^2\right)|u_{i-1}^{n+1}|^2$$

Sum over all i :

$$\left(1 + a\frac{\Delta t}{\Delta x}\right)^2 \sum_{i=-\infty}^{\infty} |u_i^{n+1}|^2$$

$$\leq \left(1 + a\frac{\Delta t}{\Delta x}\right)\sum_{i=-\infty}^{\infty} |u_i^n|^2 + \left(a\frac{\Delta t}{\Delta x} + \left(a\frac{\Delta t}{\Delta x}\right)^2\right)\sum_{i=-\infty}^{i} |u_i^{n+1}|^2$$

$$\left[\left(1 + a\frac{\Delta t}{\Delta x}\right)^2 - a\frac{\Delta t}{\Delta x} - \left(a\frac{\Delta t}{\Delta x}\right)^2\right]\sum_{i=-\infty}^{\infty} |u_i^{n+1}|^2 \leq \left(1 + a\frac{\Delta t}{\Delta x}\right)\sum_{i=-\infty}^{\infty} |u_i^n|^2$$

$$\left(1 + a\frac{\Delta t}{\Delta x}\right)\sum_{i=-\infty}^{\infty} |u_i^{n+1}|^2 \leq \left(1 + a\frac{\Delta t}{\Delta x}\right)\sum |u_i^n|^2$$

$$\sum_{i=-\infty}^{\infty} |u_i^{n+1}|^2 \leq \sum_{i=-\infty}^{\infty} |u_i^n|^2 \quad \text{for any } \Delta t$$

Unconditionally stable!

Von Neuman Analysis (Chapter 2.2 Strikwerda)
(Chapter 7.7 Lomax)

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+a\,\frac{u_i^n-u_{i-1}^n}{\Delta x}=0$$

$$u_i^{n+1}=(1-a\lambda)u_i^n+a\lambda\,u_{i-1}^n \qquad \lambda=\frac{\Delta t}{\Delta x}$$

Fourier series $\quad u_i^n=\frac{1}{N}\sum_k \hat{u}_k^n\, e^{\underline{i}\,\underbrace{i\Delta x}_{x_i}\,k} \qquad \underline{i}=\sqrt{-1}$

$$u_{i-1}^n=\frac{1}{N}\sum_k \hat{u}_k^n\, e^{\underline{i}\,\underbrace{(i-1)\Delta x}_{x_{i-1}}\,k}$$

$$u_i^{n+1}=\frac{1}{N}\sum_k\left[(1-a\lambda)+a\lambda e^{-\underline{i}\Delta x k}\right]\hat{u}_k^n\, e^{\underline{i}\,i\Delta x k}$$

$$\frac{1}{N}\sum_k \hat{u}_k^{n+1}e^{\underline{i}\,i\Delta x k}=\frac{1}{N}\sum_k\left[(1-a\lambda)+a\lambda e^{-\underline{i}\Delta x k}\right]\hat{u}_k^n\, e^{\underline{i}\,i\Delta x k}$$

coefficients must be equal for each k

for each coefficient k:

$$\hat{u}_k^{n+1}=\left[(1-a\lambda)+a\lambda e^{-\underline{i}\Delta x k}\right]\hat{u}_k^n$$

function of $\Delta x k$

$$\hat{u}_k^{n+1}=g(\Delta x k)\,\hat{u}_k^n$$

Amplification Factor: $g(\Delta x k)=(1-a\lambda)+a\lambda e^{-\underline{i}\Delta x k}$

exponent

$$\hat{u}_k^n=g^n\,\hat{u}_k^0$$

Parceval's relation: $\sum_i |u_i^n|^2 = \sum_k |\hat{u}_k^n|^2$

$$\Rightarrow \sum_i |u_i^n|^2 = \sum_k |g|^{2n} |u_k^0|^2$$

Thus $|g|^2 < 1$

set $\Delta x \, k = \theta$:

$$g(\theta) = (1 - a\lambda) + a\lambda e^{-i\theta}$$

$$g(\theta) = (1 - a\lambda) + a\lambda \cos\theta - i\, a\lambda \sin\theta$$

$$|g(\theta)|^2 = (\text{real part})^2 + (\text{imaginary part})^2$$

$$|g(\theta)|^2 = 1 - 4a\lambda(1 - a\lambda)\sin^2 \tfrac{1}{2}\theta$$

$|g(\theta)| < 1$ if $0 \leq a\lambda \leq 1$

$$0 \leq a\frac{\Delta t}{\Delta x} \leq 1$$

<u>Another example</u>: $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$

Stability of : $\frac{u^{n+1}-u^n}{\Delta t} + a\,\frac{u^n_{i+1} - u^n_{i-1}}{2\Delta x} = 0$

Forward-time central-space

Take a shorcut in Von Neuman analysis:

replace u^n_i with $g^n e^{\underline{i} i\theta}$ x-index i

$$\frac{g^{n+1}e^{\underline{i} i\theta} - g^n e^{\underline{i} i\theta}}{\Delta t} + a\,\frac{g^n e^{\underline{i}(i+1)\theta} - g^n e^{\underline{i}(i-1)\theta}}{2\Delta x} = 0$$

$$g^n e^{\underline{i}k\theta}\left(\frac{g-1}{\Delta t} + a\,\frac{e^{\underline{i}\theta} - e^{-\underline{i}\theta}}{2\Delta x}\right) = 0$$

Remember: $e^{\underline{i}\theta} = \cos\theta + \underline{i}\sin\theta$

And $\frac{e^{\underline{i}\theta} - e^{-\underline{i}\theta}}{2} =$

$= \frac{1}{2}(\cancel{\cos\theta} + \underline{i}\sin\theta - \cancel{\cos\theta} - \underline{i}\sin(-\theta))$

$= \frac{1}{2}(+\underline{i}\sin\theta + \underline{i}\sin\theta) = \underline{i}\sin\theta$

$$g = 1 - \underline{i}\,a\,\frac{\Delta t}{\Delta x}\sin\theta$$

$$|g(\theta)|^2 = 1 + a^2\left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2\theta \geqslant 1 \text{ for all } \theta$$

scheme is unstable!

Reminder: if $z = x + \underline{i}y$ is a complex number

then: $|z|^2 = x^2 + y^2$

and $|z| = (x^2+y^2)^{1/2}$ is the absolute value of z

Note that $|z| \geqslant 0$ and $|z|$ is real!

$|z| = 0$ if and only if $x=0$ and $y=0$

What have we learned?

Not everything works...

PDE: $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$

Scheme:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i-1}^n - u_i^n}{\Delta x} = 0$$

- Convergent for $a > 0$ only
- $CFL < 1$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^n - u_i^n}{\Delta x} = 0$$

- Convergent for $a < 0$ only
- $CFL < 1$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^n - u_{i-1}^n}{\Delta x} = 0$$

- Unstable

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{\Delta x} = 0$$

- Unconditionally stable
- Implicit: must invert linear system