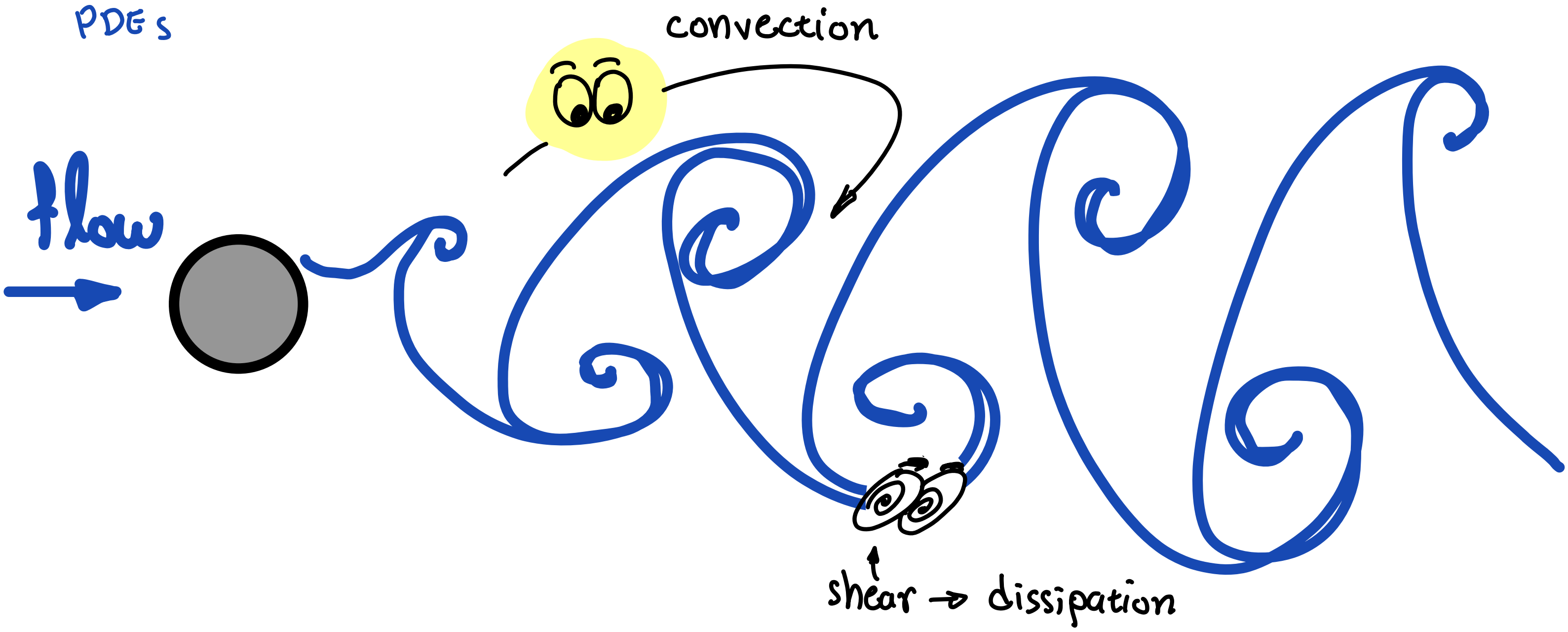
Model PDEs:

From Navier–Stokes to one-dimensional PDEs



Momentum :

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\nabla\tilde{p} + \nu\,\nabla\cdot\nabla\vec{u}$$

Kinematic viscosity

$\nu = \frac{\mu}{\rho}$: diffusivity coefficient of momentum

3 dimensions → 1 dimension

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial \tilde{p}}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

3 components (u,v,w) → 1 component

$$\frac{\partial u}{\partial t} + \underbrace{u\frac{\partial u}{\partial x}}_{\text{convection}} = \underbrace{\nu\frac{\partial^2 u}{\partial x^2}}_{\text{diffusion}} \quad\longrightarrow\quad \frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2}$$

convection velocity

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad\longrightarrow\quad \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

nonlinear equation

linear equation

Key idea:

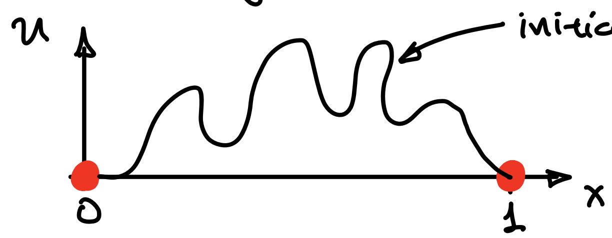
- if a methods does not work for $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$ then it does not work for $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$
- if a methods does work for $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$ then it may work for $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$

One-dimensional convection

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$ "Our favorite PDE"

a is a number

- 1 initial condition $u(t=0,x)$
- 1 boundary condition
  - → $a>0$ on left boundary
  - → $a<0$ on right boundary



For example: stuff u travel with constant velocity = 2

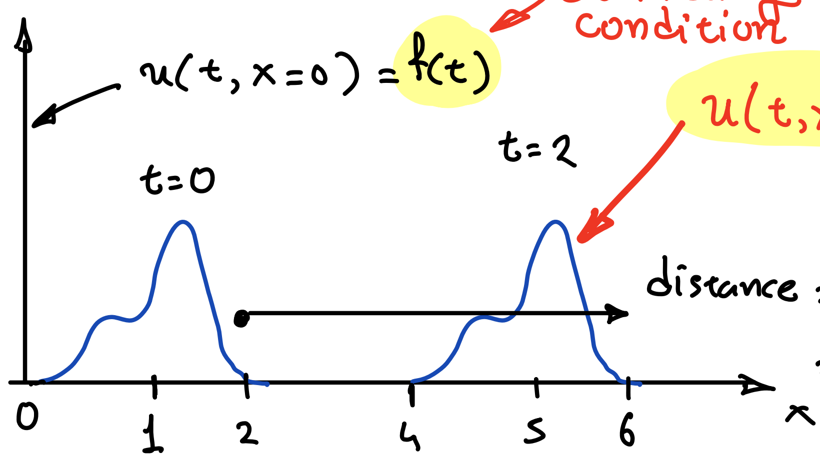
$$\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 0$$

Boundary condition

$u(t, x=0) = f(t)$

exact solution

$u(t,x) = f(x-at)$

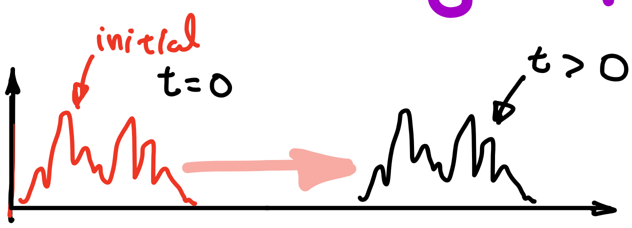


distance = velocity * time

$= 2 \times 2 = 4$

→ Please read: 2.3 from Lomax

- Shape is preserved! **Very Important!**



→ not distored

→ All wavelengths travel with the same speed = a

→ No <u>dispersion</u>

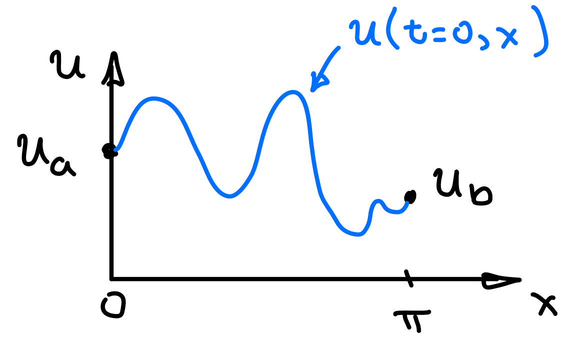
One-dimensional diffusion

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

1 initial condition

2 boundary conditions

Following Lomax 2.4.2:



$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \quad \text{on} \quad x \text{ in } [0,\pi]$$

initial condition $u(t=0,x)$

boundary conditions: $u(x=0) = u_a$
$u(x=\pi) = u_b$

Steady state solution: $\frac{\partial u}{\partial t} = 0 = \nu \frac{\partial^2 u}{\partial x^2}$

can integrate w.r.t. x $\Rightarrow h(x) = u_a + \frac{u_b - u_a}{\pi} x$

$$\text{Solution: } u(t,x) = \underbrace{\sum_m \overset{\text{time}}{f_m(t)} \overset{\text{space}}{\sin k_m x}}_{\text{No cosines for Dirichlet BC}} + h(x)$$

Substitute in PDE: $\frac{df_m}{dt} = -k_m^2 \nu f_m$

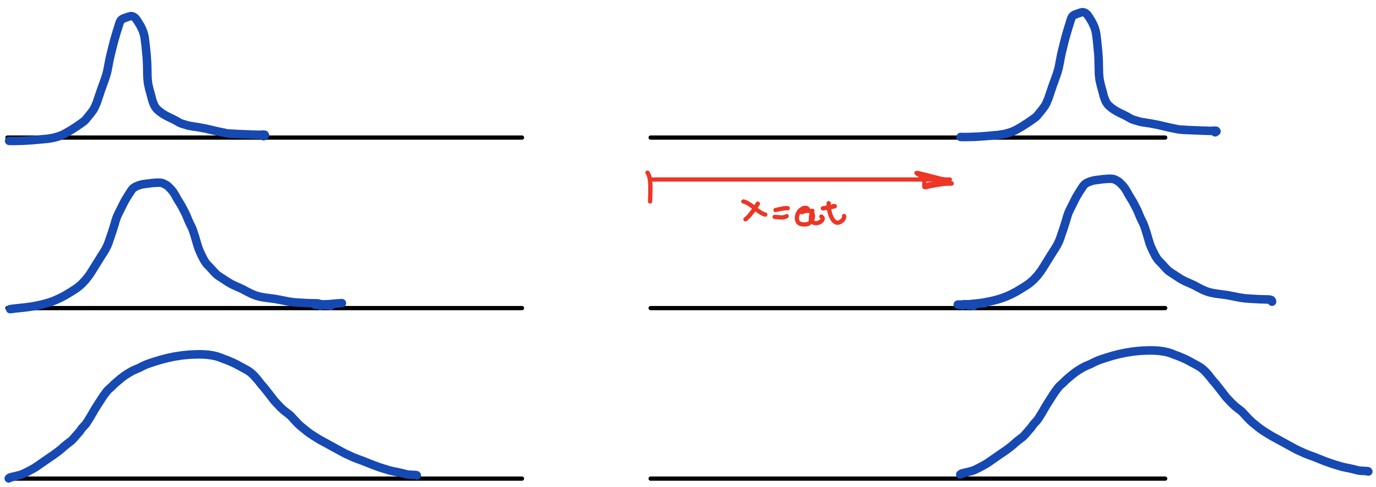
$$f_m(t) = f_m(t=0) e^{-k_m^2 \nu t}$$

$f_m(t=0)$ determined from I.C.: $u(t=0) = \sum_m f_m(0) \sin k_m t + h(x)$

$$\text{Solution: } u(t,x) = \sum_m \underbrace{\underbrace{f_m(t=0)}_{\text{constant}} \underbrace{e^{-k_m^2 \nu t}}_{\text{depends on } k_m \text{ and } \nu}}_{\text{amplitude as function of time!}} \sin k_m x + h(x)$$

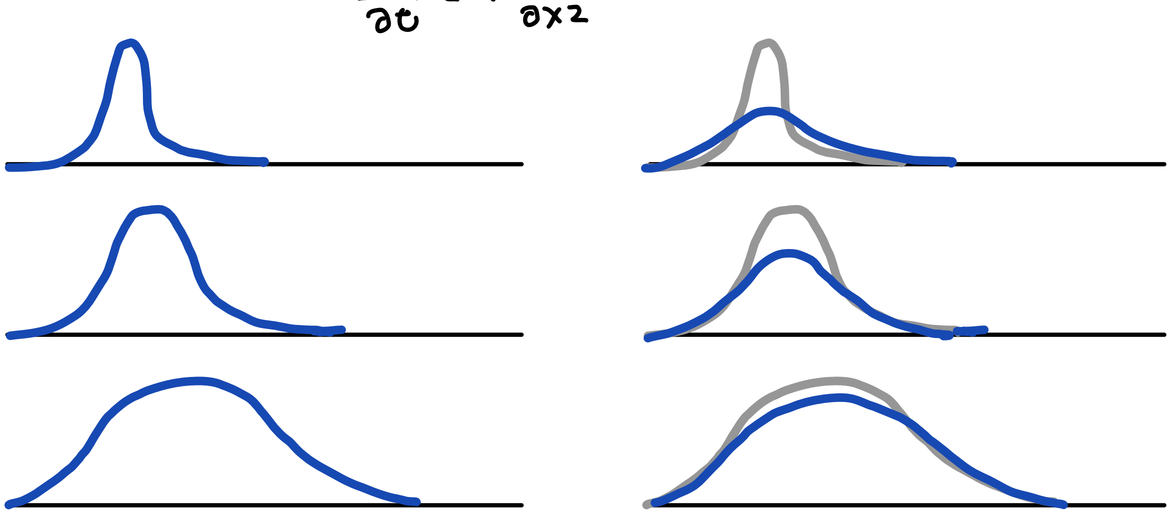
1D convection $\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$

Initial condition After time Δt



- shape is the same
- no change in amplitute

1D diffusion $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$



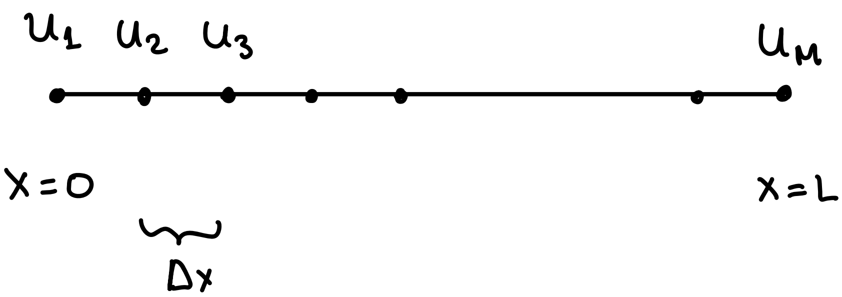
- finer scales diffuse faster
- change in amplitude depending on wavelength

Solving $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$ on the computer

- Remember notation: u_i^n means $u(n\Delta t, i\Delta x)$

  when $i = 0, \dots M-1$

- Discretize in space: $\Delta x = \frac{L}{M}$



Very important: each u_i only depents on time

We will find a solution when we figure out the time history of each u_i

- Discretize space derivative: $\left.\frac{\partial u}{\partial x}\right|_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2)$

PDE becomes: $\frac{du_i}{dt} + a\frac{u_{i+1} - u_{i-1}}{2\Delta x} = 0 \qquad i = 1, \dots, M$

- Initial condition will set: $u_i^{n=0} = f(x) = f((i-1)\Delta x)$

- Boundary condition will set u_1^n $(a>0)$

  $u_1^n = b(t) = b((n-1)\Delta t)$

- I have $u_2, u_3, \dots u_M$ unknowns for $n > 1$
- I can write this in vector form :

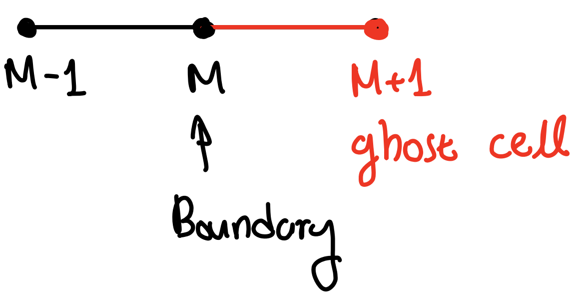
known from boundary condition

$$\frac{d}{dt}\begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_M \end{bmatrix} = -a \begin{bmatrix} \frac{u_3 - u_1}{2\Delta x} \\ \frac{u_4 - u_2}{2\Delta x} \\ \vdots \\ \frac{u_{M+1} - u_{M-2}}{2\Delta x} \end{bmatrix}$$

Same form as Homework 3 Problem 3 : $\frac{d\vec{u}}{dt} = f(\vec{u})$

Use Runge-Kutta to perform the time-integration and find the values of u_i^n for $n > 1$

At right boundary:



Two options: • take one-sided difference

$$\left.\frac{\partial u}{\partial x}\right|_{i=M} = \frac{u_M - u_{M-1}}{\Delta x}$$

• extrapolate to the right and use same stencil as the interior

Lomax p.43 eq. 3.70

$$(1-E^{-1})^p u_{M+1} = 0$$

$E u_i = u_{i+1}$

order of appox. = p-1

$p=2 \rightarrow (1-E^{-1})^2 u_{M+1} = 0$

$$(1+E^{-2}-2E^{-1})\, u_{M+1} = 0$$

$$u_{M+1} + u_{M-1} - 2u_M = 0$$

$$u_{M+1} = 2u_M - u_{M-1}$$