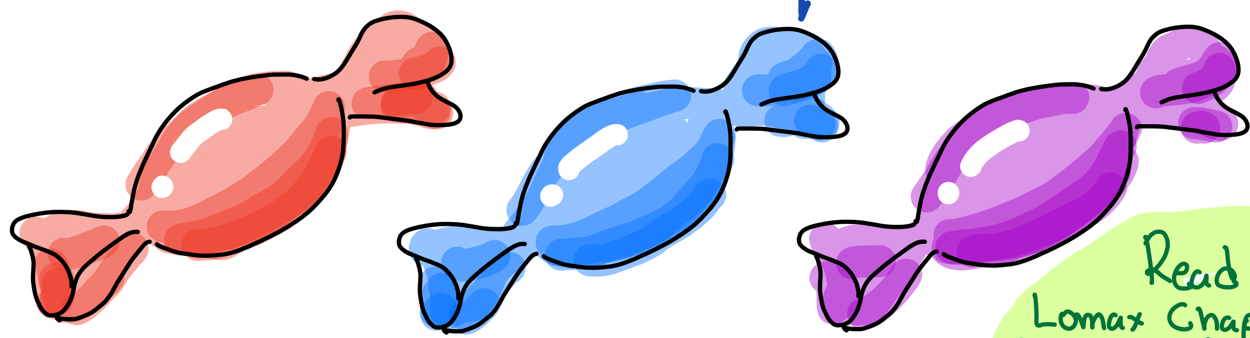
Flavors of the Navier-Stokes equations



Read also
Lomax Chapter 2
Ferziger & Perić Ch. 1

Compressible Navier-Stokes

		# variables	# equation
Mass:	$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} \rho u_i = 0$	4 (u, v, w, ρ)	1
Momentum:	$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}(u_i)}{\partial x_j}$	5 (p)	4
Equation of state (e.g. ideal gas)	$\rho = \frac{p}{RT}$ $\quad \rho(p, T)$	6 (T)	5
Energy	$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}(u_j(p+E)) = \frac{\partial}{\partial x_j}\left(-k\frac{\partial T}{\partial x_i} - \tau_{ij} u_j\right)$	6	6

$$E = \text{function}(T, p)$$

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix} + \frac{\partial}{\partial x}\left(F_{\text{convection}} + F_{\text{diffusion}}\right) + \frac{\partial}{\partial y}\left(G_c + G_d\right) + \frac{\partial}{\partial z}\left(H_c + H_d\right)$$

vector of state

Form: $\frac{\partial \vec{Q}}{\partial t} = RHS(\vec{Q}) \xrightarrow{\text{semi-discrete approach}} \frac{d}{dt}\vec{Q}_{ijk} = RHS(\vec{Q}_{ijk})$

Numerical Stability

Choice of Δt

$$CFL = a \frac{\Delta t}{\Delta x} \rightarrow \Delta t = \frac{\Delta x}{a} CFL$$

physical speed of information

What is "a" for the compressible NS

$$\frac{\partial \vec{Q}}{\partial t} + \frac{\partial \vec{F}}{\partial x} = 0$$

NS without viscous terms → Euler equation

$$\frac{\partial \vec{Q}}{\partial t} + \frac{\partial F}{\partial Q} \frac{\partial \vec{Q}}{\partial x} = 0$$

Flux Jacobian

$$\frac{\partial \vec{Q}}{\partial t} + A \frac{\partial \vec{Q}}{\partial x} = 0$$

diagonal matrix with eigenvalues of A

$$\frac{\partial \vec{W}}{\partial t} + \Lambda \frac{\partial \vec{W}}{\partial x} = 0$$

speed of sound

Eigenvalues: $c+u$, $c-u$, u

$$\Delta t = \frac{\Delta x}{|u|+c} CFL$$

340 m/s

Example: flow around a car:

car speed:

$u = 60$ mph → 28 m/s

$\Delta x = 0.01$

$CFL = 1.5$

→ $\Delta t = 0.00004$ sec

⇒ 24 533 time steps for 1 sec of flow evolution

Incompressible Navier-Stokes

Case of constant density : $\rho = \text{constant}$

		# Variables	# Equation
Mass :	$\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + \frac{\partial}{\partial x_i} \cancel{\rho} u_i = 0$	3	1
Momentum:	$\frac{\partial \cancel{\rho} u_i}{\partial t} + \frac{\partial}{\partial x_j}(\cancel{\rho} u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}(u_i)}{\partial x_j}$	4	4

Equation of state (e.g. ideal gas) $\cancel{\rho = \frac{p}{RT} \quad \rho(p,T)}$

Energy $$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}\left(u_j(p+E)\right) = \frac{\partial}{\partial x_j}\left(-k\frac{\partial T}{\partial x_i} - \tau_{ij} u_j\right)$$

$$E = \text{function}(T, p)$$

decouples

Our system is just: $$\frac{\partial}{\partial x_i} u_i = 0$$ ← **No $\frac{\partial}{\partial t}$**

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} u_i u_j = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\tau_{ij}$$

Cannot write the system as: $\frac{\partial \vec{Q}}{\partial t} = RHS$

$\frac{\partial u_i}{\partial x_i} = 0$ is satified by the correct choise of pressure

Fundamental Problem: $$\underline{\underline{u_{i,j}^{n+1}}} = \text{function}(u_{i,j}^{n}, \underline{\underline{p_{i,j}^{n+1}}})$$

velocity field at next time step

pressure at next time step

Fractional step method

Incompressible Navier-Stokes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Step 1: $u^* = u^n + \Delta t\left[-u^n \delta_x u^n - v^n \delta_y u^n + \nu(\delta_x\delta_x u^n + \delta_y\delta_y u^n)\right]$

$v^* = v^n + \Delta t\left[-u^n \delta_x v^n - v^n \delta_y v^n + \nu(\delta_x\delta_x v^n + \delta_y\delta_y v^n)\right]$

$t\ [u^*, v^*]$

Step 2: Poisson Equation

$$\delta_x\delta_x p + \delta_y\delta_y p = -\delta_x u^* - \delta_y v^*$$

$p(i,j)$

Step 3: Add pressure gradient to $[u^*, v^*]$

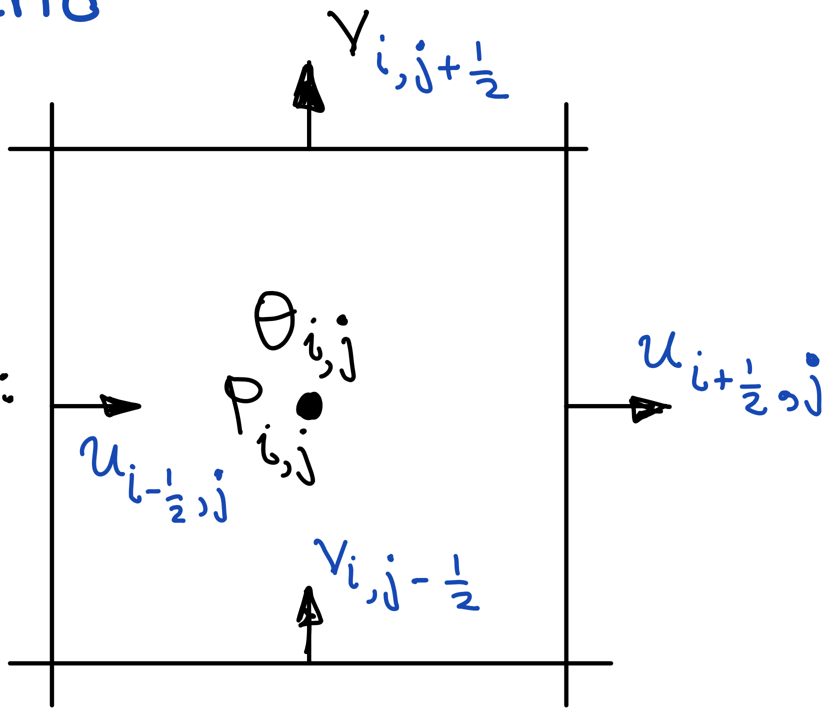
$$\left.\begin{aligned} u^{n+1} &= u^* + \delta_x p \\ v^{n+1} &= v^* + \delta_y p \end{aligned}\right\} \rightarrow u^{n+1}, v^{n+1}$$

sacifies mass conservation

Staggered Grid

Notation:

Grid cell (i,j):

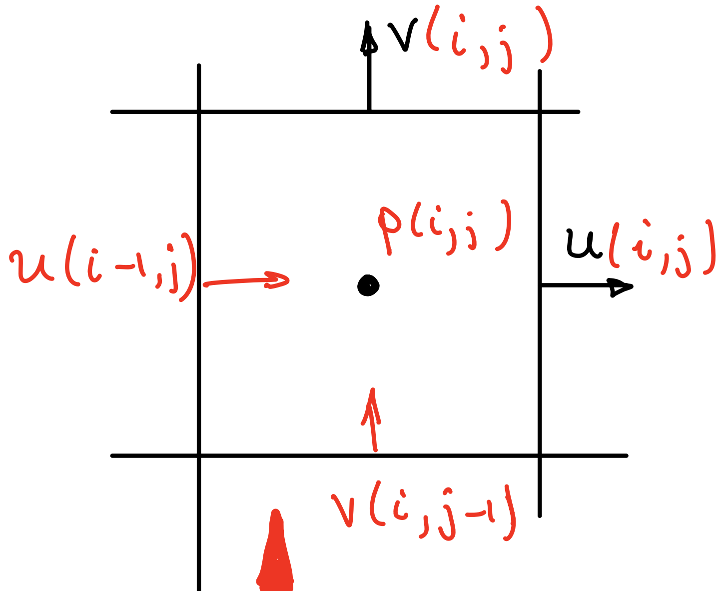


This notation is unambiguous

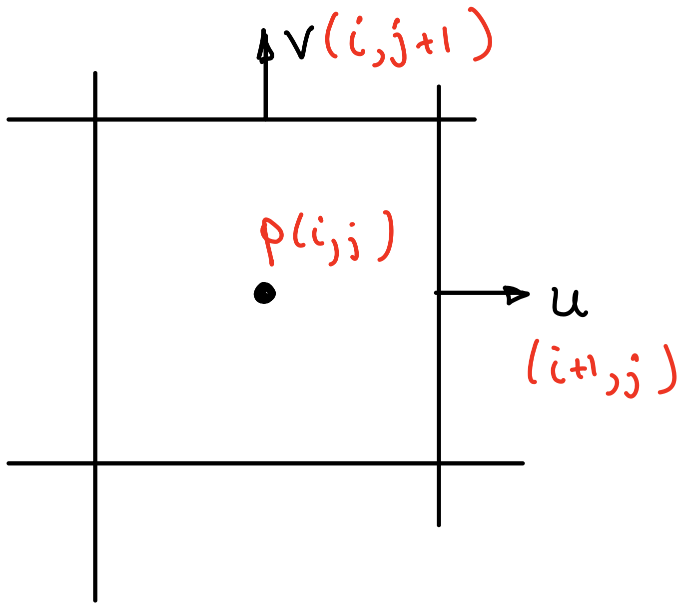
... however...

we cannot easily access $u(i+\frac{1}{2},j)$ on the computer

Option A: $i+\frac{1}{2} \to i$



Option B: $i+\frac{1}{2} \to i+1$



Class notes use this notation

Divergence and pressure gradient

1. Divergence

Location Location Location Location Location

$$\nabla^2 p = -\nabla\cdot\vec{u} \rightarrow \text{we need } \nabla\cdot\vec{u} \text{ at location of } p$$

Location Location Location Location

Need $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ at center of grid cell

Location

Use second-order differences:

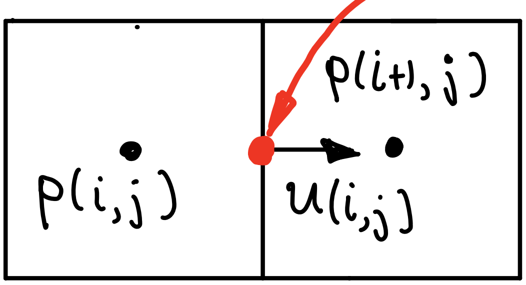
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \simeq \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + \frac{v_{i,j} - v_{i,j-1}}{\Delta y} + O(\Delta x^2, \Delta y^2)$$

$$\frac{P_{i+1,j} - 2P_{i,j} + P_{i-1,j}}{\Delta x^2} + \frac{P_{i,j+1} - 2P_{i,j} + P_{i,j-1}}{\Delta y^2} = \underbrace{-\frac{u_{i,j} - u_{i-1,j}}{\Delta x} - \frac{v_{i,j} - v_{i,j-1}}{\Delta y}}_{d_{i,j}}$$

Location

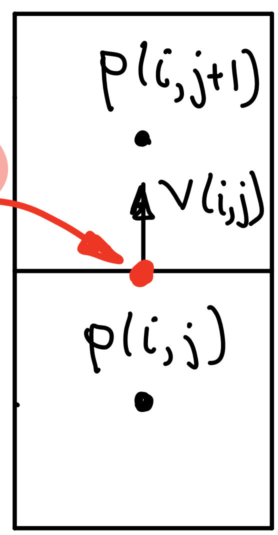
2a. Pressure gradient at location of u-velocity

Location Location



$$\frac{\partial p}{\partial x} \simeq \frac{p(i+1,j) - p(i,j)}{\Delta x}$$

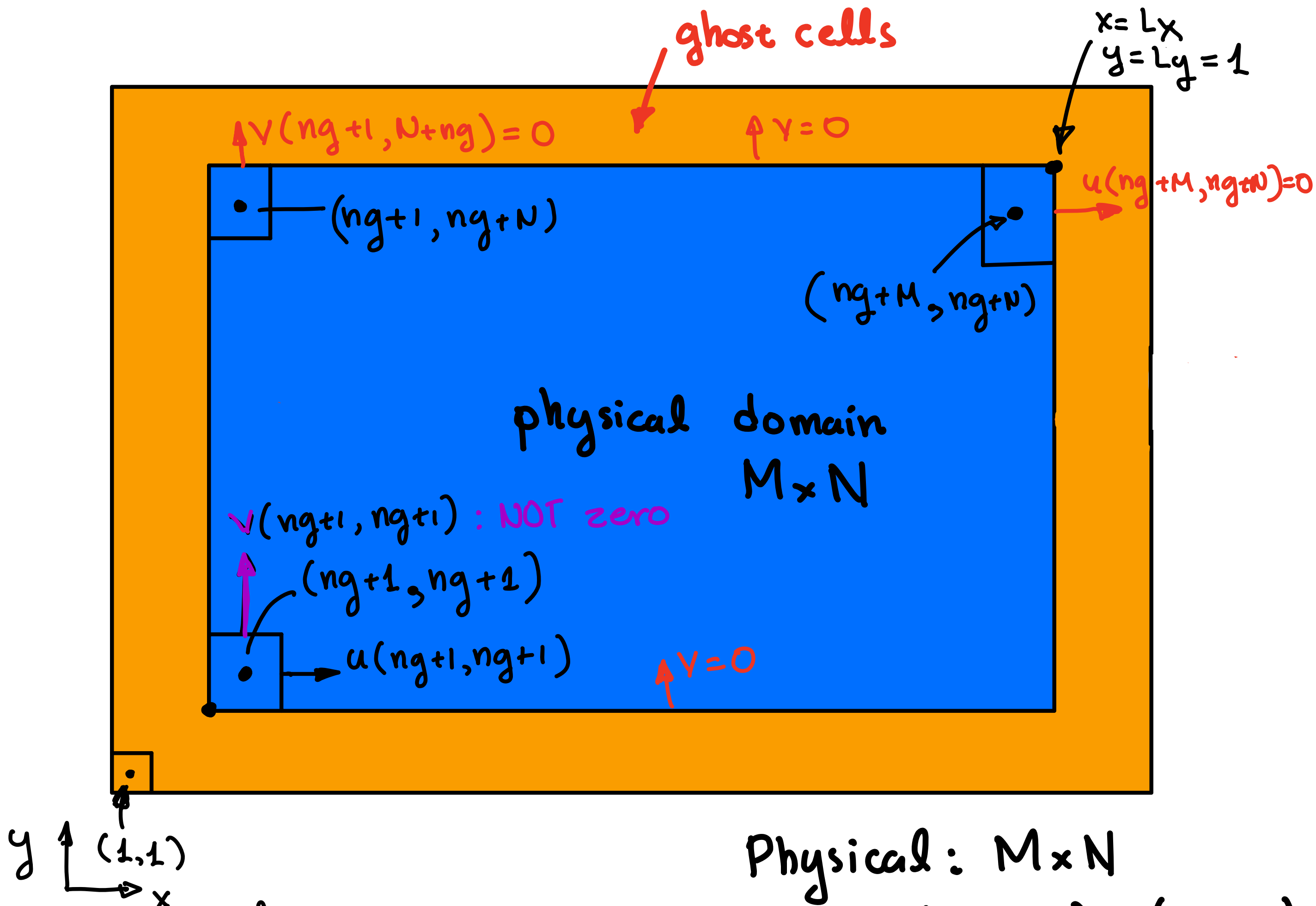
2b. Pressure gradient at location of v-velocity



$$\frac{\partial p}{\partial y} \simeq \frac{p(i,j+1) - p(i,j)}{\Delta y}$$

Computational Domain

- Periodic x-direction
- Walls top and bottom (at $y=0$ and $y=1$)



ng : number of ghost cells

Physical : $M \times N$

Total : $(M+2ng) \times (N+2ng)$

$M \times N$ grid cells in the physical domain

Total: $(M+2ng) \times (N+2ng)$

Pressure: $p(i,j)$: $M \times N$ unknowns

Making Divergence-free Velocity fields

Steps:

1. generate random $u^*(i,j)$ $v^*(i,j)$

   fill $u^*(i,j)$, $v^*(i,j)$ with random numbers

   - Simple Boundary condition for now
     - $v^*=0$ on top and bottom + periodic in x
     - u^* is periodic in x-direction
   - Important: the mean of v^* on and horizontal line should be zero

   Make $\sum_i v^*_{ij} = 0$ for all j in physical domain

2. Solve pressure Poisson equation to find $p(i,j)$

   $\nabla^2 p = -\nabla \cdot \vec{u}^* \rightarrow p(i,j)$

   Make pressure periodic in x

3. Add ∇p to $(u^*, v^*) \rightarrow \nabla \cdot \vec{u} = 0$

$$\left.\begin{aligned} u &= u^* + \frac{\partial p}{\partial x} \\ v &= v^* + \frac{\partial p}{\partial y} \end{aligned}\right\} \quad \nabla \cdot \vec{u} = 0$$

To verify correct implementation:

Compute $\nabla \cdot \vec{u}$ and show that $\max |\nabla \cdot \vec{u}| < 10^{-10}$

Mass Conservation and vertical velocity

Show that average vertical velocity is zero on horizontal lines:

Mass conservation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Average in the horizontal direction:

$$\left\langle \frac{\partial u}{\partial x} \right\rangle + \left\langle \frac{\partial v}{\partial y} \right\rangle = 0$$

$\left\langle \frac{\partial u}{\partial x} \right\rangle = 0$ zero because u is periodic in x

$$\frac{\partial \langle v \rangle}{\partial y} = 0$$

integrate from 0 to y: $\displaystyle\int_0^y \frac{d\langle v \rangle}{dy'} \, dy' = 0$

$$= \langle v \rangle (y) - \langle v \rangle (y=0) = 0$$

but $\langle v \rangle (y=0)$ is bottom wall wit $v(y=0) = 0$

thus $\langle v \rangle (y) = 0$