$$(\delta_x u)_i = \frac{1}{\Delta x}\left[-\frac{1}{12}\left(u_{i+2} - u_{i-2}\right) + \frac{2}{3}\left(u_{i+1} - u_{i-1}\right)\right]$$

Derive the modified wavefunction and numerical phase speed relations.

$$u(x) = \sum_n a_n e^{i k_n x} \rightarrow \frac{du}{dx} = \sum_n a_n i k_n e^{i k_n x}$$

$$u_i = \cancel{\sum_n} \cancel{a_n} e^{i k x_i} \rightarrow \frac{du}{dx}\Big|_i = \cancel{\sum_n} \cancel{a_n}\, i k\, e^{i k x_i}$$

$$(\delta_x u)_i =$$

$$\frac{1}{\Delta x}\left[-\frac{1}{12}\left(e^{ik\Delta x(i+2)} - e^{ik\Delta x(i-2)}\right) + \frac{2}{3}\left(e^{ik\Delta x(i+1)} - e^{ik\Delta x(i-1)}\right)\right]$$

$$= \frac{1}{\Delta x}\left[\frac{-1}{12}\left(e^{ikx_i}\right)\overbrace{\left(e^{2ik\Delta x} - e^{-2ik\Delta x}\right)}^{2i\sin(2k\Delta x)} + \frac{2}{3}\left(e^{ikx_i}\right)\underbrace{\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)}_{2i\sin(k\Delta x)}\right]$$

$$= e^{ikx_i} \cdot i\,\frac{1}{\Delta x}\left[\frac{-1}{6}\sin(2k\Delta x) + \frac{4}{3}\sin(k\Delta x)\right]$$

1) $k^* = \frac{1}{\Delta x}\left[-\frac{1}{6}\sin(2k\Delta x) + \frac{4}{3}\sin(k\Delta x)\right]$

$$(\delta_x u)_i = \frac{1}{\Delta x}\left[-\frac{1}{4}(u_{i+2} - u_{i-2}) + (u_{i+1} - u_{i-1})\right]$$

Again: $u_i = e^{ikx_i} \rightarrow \frac{du_i}{dx} = ik e^{ikx_i}$

$$(\delta_x u)_i = \frac{1}{\Delta x}\left[-\frac{1}{4}\left(e^{ik\Delta x(i+2)} - e^{ik\Delta x(i-2)}\right) + \left(e^{ik\Delta x(i+1)} - e^{ik\Delta x(i-1)}\right)\right]$$

$$= \frac{1}{\Delta x}\left[-\frac{1}{4}\left(e^{ikx_i}\right)\left(e^{2ik\Delta x} - e^{-2ik\Delta x}\right) + \left(e^{ikx_i}\right)\left(e^{ik\Delta x} - e^{ik\Delta x}\right)\right]$$

$$= e^{ikx_i} \cdot \frac{1}{\Delta x}\left[-\frac{1}{4} \cdot 2i\sin(2k\Delta x) + i\sin(k\Delta x)\right]$$

2) $k^* = \frac{1}{\Delta x}\left[-\frac{1}{2}\sin(2k\Delta x) + \sin(k\Delta x)\right]$

Since the normalized phase speed and wavenumber are equal.

$$\frac{a^*}{a} = \frac{k^*}{k}$$

1) $a^* = \frac{a}{k\Delta x}\left[-\frac{1}{6}\sin(2k\Delta x) + \frac{4}{3}\sin(k\Delta x)\right]$

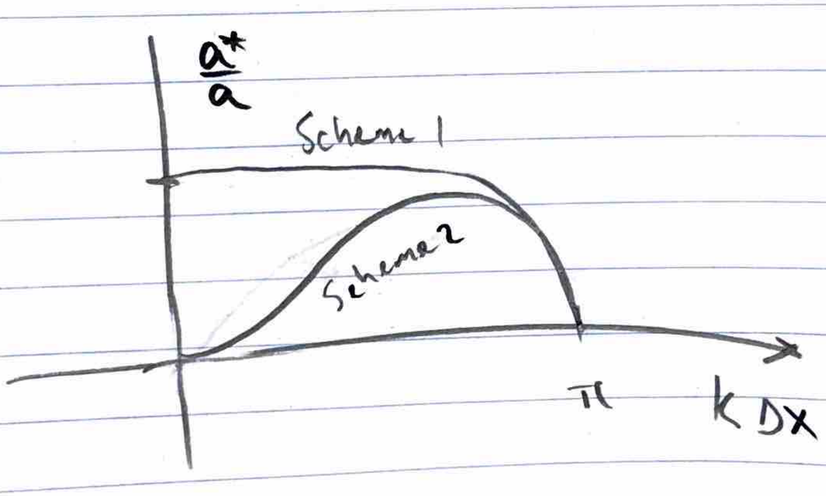
2) $a^* = \frac{a}{k\Delta x}\left[-\frac{1}{2}\sin(2k\Delta x) + \sin(k\Delta x)\right]$

For 1) $\frac{a^*}{a} = 0.95$ for

$k\Delta x = 1.15$ which is 5.46 grid points.

For 2) $\frac{a^*}{a}$ never exceeds

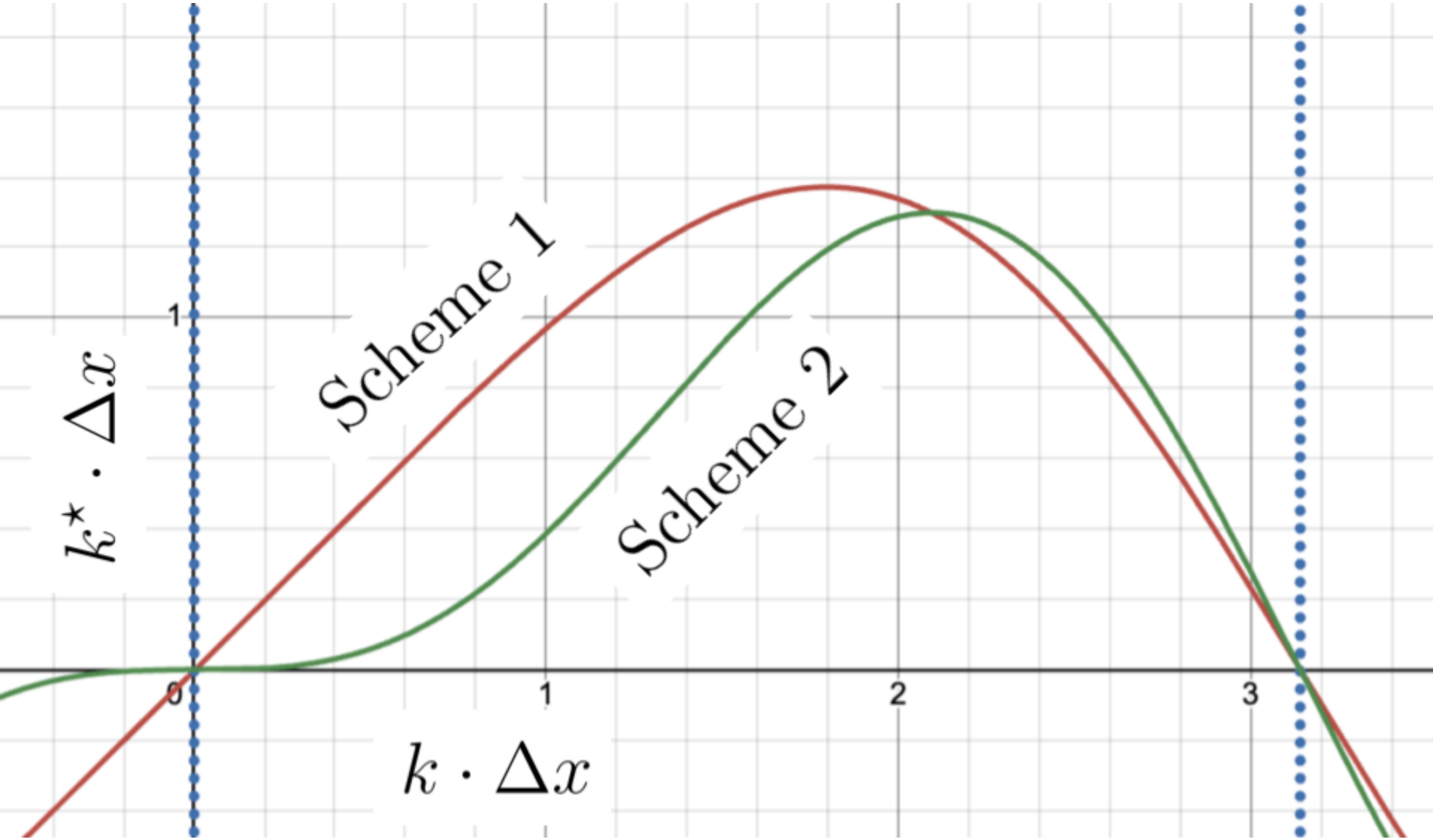
0.664 for $k\Delta x = 1.803$



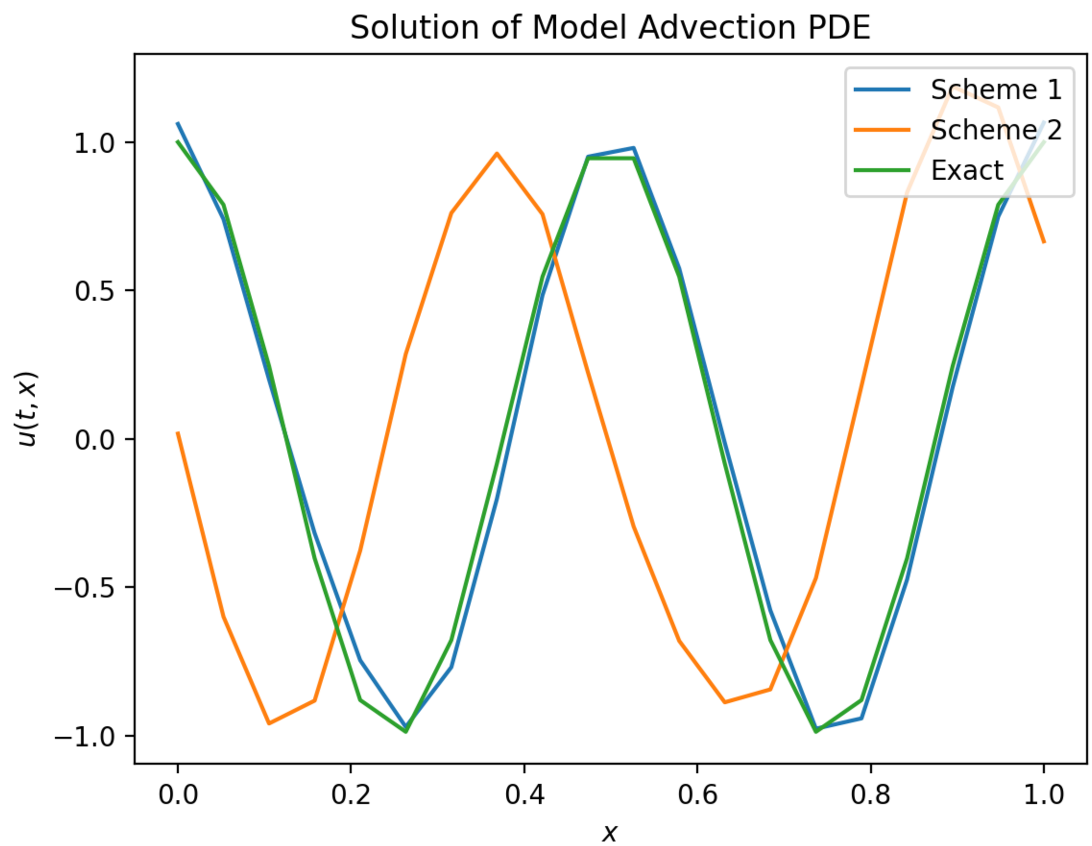
$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

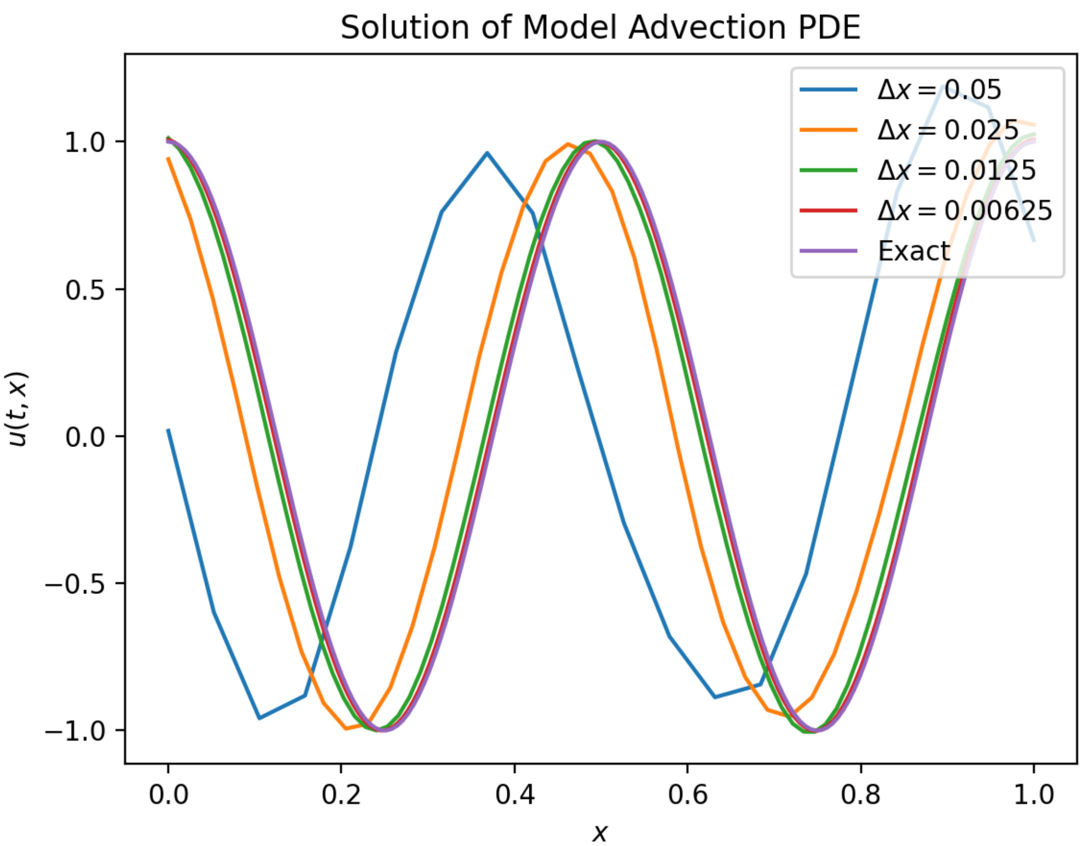
$$u(t=0, x) = \cos(4\pi(x))$$

$$u(t,x) = \cos(4\pi x - 4\pi t)$$



$k^\star \cdot \Delta x$
$k \cdot \Delta x$
Scheme 1
Scheme 2
0
1
2
3
1


Solution of Model Advection PDE
Scheme 1
Scheme 2
Exact
u(t, x)
x
1.0
0.5
0.0
−0.5
−1.0
0.0
0.2
0.4
0.6
0.8
1.0



Solution of Model Advection PDE
$\Delta x = 0.05$
$\Delta x = 0.025$
$\Delta x = 0.0125$
$\Delta x = 0.00625$
Exact
$u(t, x)$
x
1.0
0.5
0.0
−0.5
−1.0
0.0
0.2
0.4
0.6
0.8
1.0