

MAT2002 - Applications of Differential and Difference Equations MATLAB Lab

z -transform

- ▶ The z -transform of a sequence x_n is given by

$$Zx_n = X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$

- ▶ The inverse Z -transform is

$$x_n = Z^{-1} \{X(z)\}$$

- ▶ The z -transform maps a discrete sequence x_n from a sample domain $[n]$ into a complex plane z .

Transfer Function

- Consider the generalised input-output difference equation

$$(a_0 E^N + a_1 E^{N-1} + \dots + a_{N-1} E_1 + a_N) x_n = (b_0 E^M + b_1 E^{M-1} + \dots + n_{M-1} E_1 + a_M) u_n$$

- Taking z -transform on both sides of the above difference equation, the given difference equations transforms into a transfer function, also known as the Pulse Transfer Function $H(z)$, which is the ratio of the z -transform of the state-response $\{x_n\}$ to the z -transform of the input $\{u_n\}$ when all initial conditions are zero. The transfer function can be expressed mathematically as

$$\begin{aligned} H(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}} \\ &= z^{N-M} \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \dots + b_M}{a_0 z^N + a_1 z^{N-1} + a_2 z^{N-2} + \dots + a_N} \end{aligned}$$

Poles, Zeros, and Gain

- ▶ If the numerator and denominator of the proper form of a transfer function are factored,
 - ▶ the roots of the numerator polynomial are called *zeros*
 - ▶ the roots of the denominator polynomial are called *poles*
 - ▶ the leading coefficient in the numerator is called *gain*
- ▶ A system is said to be *stable* if all of its poles lie inside of a unit circle on the z -plane.

Finding Poles, Zeros and Gain using MATLAB

- ▶ The MATLAB provides a number of functions to assist in the zero-pole-gain analysis of the system.
- ▶ The command `[z p k] = tf2zpk(b,a)` finds the zeros z , poles p , and gain k from the coefficients b and a of the transfer function $H(z)$.
- ▶ The command `zplane(b,a)` plots the zeros and poles of the system along with a reference unit circle. A \circ is used for zeros and \times is used for poles. If all the \times lies inside the unit circle, then the given system is stable.

Example

Consider the difference equation given by

$$x_n + 0.2x_{n-2} = u_n + 2u_{n-1}$$

with all zero initial conditions. Applying the z -transform on both sides, we obtain

$$X(z) + 0.2z^{-2}X(z) = U(z) + 2z^{-1}U(z).$$

Thus we have,

$$H(z) = \frac{X(z)}{U(z)} = \frac{1 + 2z^{-1}}{1 + 0.2z^{-2}} = \frac{z^2 + 2z}{z^2 + 0.2}$$

MATLAB Implementation

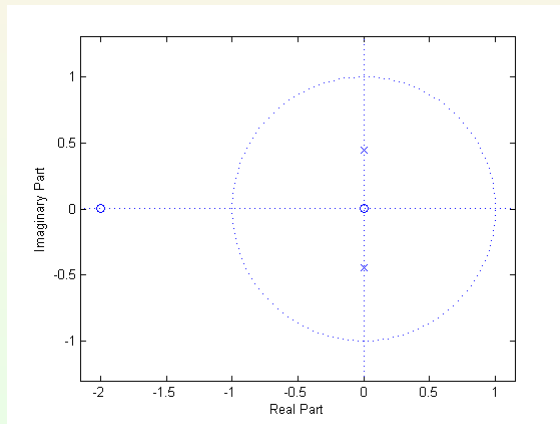
The matlab representation of the above transfer function is

```
num = [1 2 0];  
den = [1 0 0.2];
```

The zero-pole-gain analysis and plotting of zeros-poles of the system can be done by using the commands

```
[z p k] = tf2zpk(num,den)  
zplane(z,p)
```

MATLAB Output



Inverse z -transform

- ▶ Inverse z -transform can be found using three techniques:
 - ▶ Partial Fraction Method
 - ▶ Power Series/Long Division Method
- ▶ We can use the MATLAB built-in functions to find the inverse z -transform by using the both of the above methods.

Partial Fraction Method

Consider the transfer function

$$H(z) = \frac{18z^3}{18z^3 + 3z^2 - 4z - 1}.$$

To find the partial fraction expansion of $H(z)$ we use MATLAB's built-in function `residuez(num, den)`

```
num = [18 0 0 0];  
den = [18 3 -4 -1];  
[r,p,k] = residuez(num,den);  
disp('Residues'); disp(r');  
disp('Poles'); disp(p');  
disp('Constants'); disp(k)
```

MATLAB Output

Residues

0.3600 0.2400 0.4000

Poles

0.5000 -0.3333 -0.3333

Constants

0

The result of the program can be read as:

$$H(z) = \frac{0.36}{1 - 0.5z^{-1}} + \frac{0.24}{1 + 0.3333z^{-1}} + \frac{0.4}{(1 + 0.3333z^{-1})^2}$$

Power Series/Long Division Method

To find the z -transform by power series/long division method, one may use the built-in MATLAB function `filter`. To illustrate this, consider the transfer function

$$H(z) = \frac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}.$$

We wish to find the first 11 system responses i.e., $\{x_n\}_{n=1}^{11}$.

MATLAB Code

```
L = 11;  
num = [1 2];  
den = [1 0.4 -0.12];  
u = [1 zeros(1,L-1)];  
x = filter(num,den,u);  
disp('Coefficients of the power series expansion: ');  
disp(x);
```

MATLAB Output

Coefficients of the power series expansion:

Columns 1 through 10

1.0000	1.6000	-0.5200	0.4000	-0.2224	
0.1370	-0.0815	0.0490	-0.0294	0.0176	

Column 11

-0.0106

From the output we obtain that

$x_0 = 0, x_1 = 1, x_2 = 1.6, \dots, x_{10} = 0.0176, x_{11} = -0.0106.$

Impulse and Step Responses

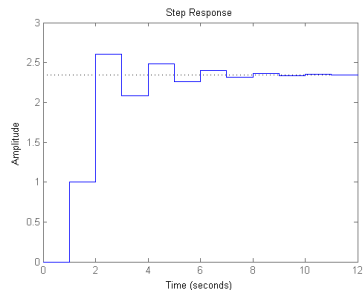
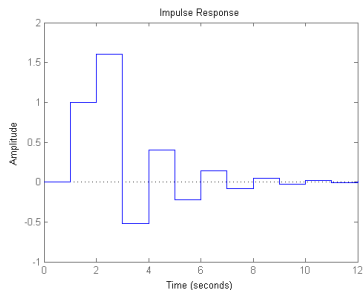
- ▶ In electronic engineering and control theory, step response is the time behaviour of the outputs of a general system when its inputs change from zero to one in a very short time.
- ▶ In signal processing, the impulse response, of a dynamic system is its output when presented with a brief input signal, called an impulse.
- ▶ If the transfer function of a system is given by $H(z)$, then the impulse response of a system is given by $h(n)$ where $h(n)$ is the inverse z -transform of $H(z)$.

Impulse and Step Responses using MATLAB

The impulse response response and unit step response of difference equation represented by the transfer function $H(z)$, can be plotted using MATLAB with the help of the command `dimpulse(num,den,nooftimesteps)` (for impulse response) and `dstep(num,den,nooftimesteps)` (for unit step response). For the previous transfer function the impulse response and step response for 13 time steps can be plotted using

```
figure; dimpulse(num,den,13);  
figure; dstep(num,den,13)
```


MATLAB Output



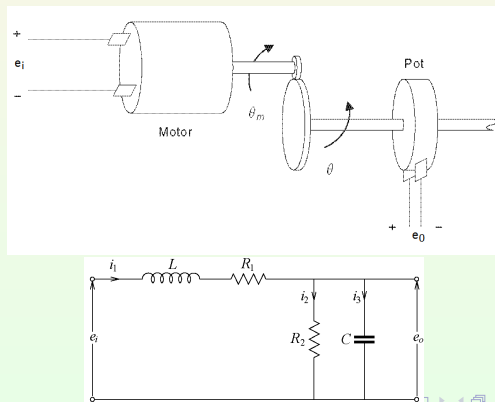
Stability Analysis of Motor Driven Gear Train

Introduction

- ▶ A gear train is a mechanical system formed by mounting gears on a frame so that the teeth of the gears engage.
- ▶ Gear train system occurs in many of the automobile drivetrains, which generally have two or more major areas where gearing is used.
- ▶ Gearing is employed in the transmission, which contains a number of different sets of gears that can be changed to allow a wide range of vehicle speeds.
- ▶ Gearing is also used in the differential, which facilitates the splitting torque equally between two wheels while permitting them to have different speeds when travelling in a curved path.

Mathematical Formulation

Consider a motor driving a rotary load through a gear train and the simple RLC circuit representing the motor connection, as shown in below Figure.



Mathematical Formulation (Cont...)

Applying the basic circuit laws for voltage and currents, we obtain

$$e_i(t) = L \frac{di_1(t)}{dt} + R_1 i_1(t) + e_0(t) \quad (1)$$

and

$$e_0(t) = R_2 i_2(t) = \frac{1}{C} \int_0^t i_3(\tau) d\tau + v_c(0) \implies i_3(t) = C \frac{de_0(t)}{dt} \quad (2)$$

But, we have

$$i_1(t) = i_2(t) + i_3(t) = \frac{1}{R_2} e_0(t) + C \frac{de_0(t)}{dt} \quad (3)$$

From which, we obtain the following second order differential equation, relating input and output of the system, representing mathematical model of the circuit given in the Figure 1.

$$\frac{d^2 e_0(t)}{dt^2} + \left(\frac{L + R_1 R_2 C}{R_2 LC} \right) \frac{de_0(t)}{dt} + \left(\frac{R_1 + R_2}{R_2 LC} \right) e_0(t) = \frac{1}{LC} e_i(t) \quad (4)$$

Mathematical Formulation (Cont...)

Denoting $e_0(t)$ by $x(t)$ and $e_i(t)$ by $u(t)$, and using the approximations $\frac{dx}{dt} \approx x_t - x_{t-1}$ and $\frac{d^2x}{dt^2} \approx x_t - 2x_{t-1} + x_{t-2}$, the above differential equation (4) reduces to the following second order difference equation,

$$x_t = (a + 1)x_{t-1} - ax_{t-2} + bu_{t-1} \quad (5)$$

where

$$a = \frac{R_2LC}{(R_1 + L)(R_2C + 1) + R_2} \text{ and } b = \frac{R_2}{(R_1 + L)(R_2C + 1) + R_2}$$

The transfer function for the above difference equation (5) is given by

$$H(z) = \frac{bz}{z^2 - (1 + a)z + a} \quad (6)$$

Exercise

Determine the poles, zeros and stability of the above transfer function for $R_1 = 1$ ohms, $R_2 = 1$ ohms, $L = 2$ Henry and $C = 2$ farad. Also plot the impulse and unit step responses.

MATLAB Output

z =

0

p =

1.0000

0.4000

k =

0.1000

coefficients of the power series expansion:

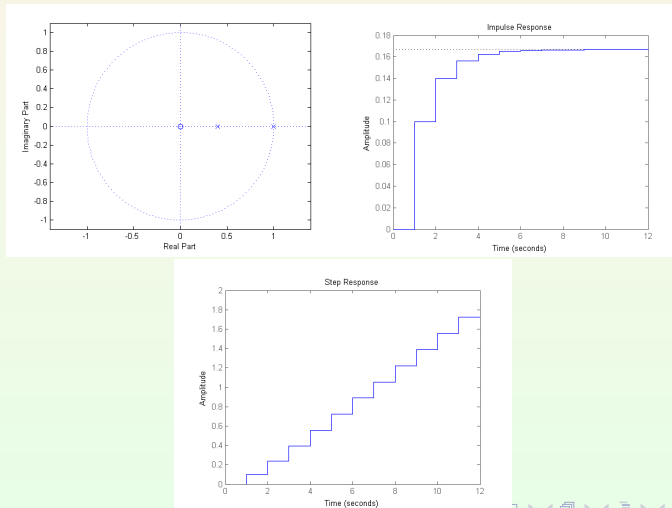
Columns 1 through 10

0	0.1000	0.1400	0.1560	0.1624	0.1650	0.1660
	0.1664	0.1666	0.1666			

Column 11

0.1666

MATLAB Output



References

- ▶ The Z Transform, System Transfer Function, Poles and Stability - Lecture Notes by Dr. David Corrigan available at www.mee.tcd.ie/~corrigan
- ▶ Advanced Control Using MATLAB by David I. Wilson available at www.i2c2.aut.ac.nz/PHP/getDownload.php?File=cnotes.pdf&Type=Pub
- ▶ Applied Control Theory for Embedded Systems by Tim Wescott, Elsevier (2006).