Experiment #06 - LATERAL VIBRATIONS OF HANGING ROPE

Aim: To find the power series solution of the Bessel's equation of order zero by the method of Frobenius and visualize it using MATLAB.

Mathematical Background:

Series solution of differential equations

- Many differential equations arising from physical problems are linear with variable coefficients.
- A general solution in terms of known function does not exist for types of equations.
- Such equations can be solved by finding the solution in the form of an infinite convergent series.

Singular Point

Consider the differential equation of the form,

$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$ …(1)

If $P_0(a) \neq 0$, then $x = a$ is called an *ordinary point* of (1), otherwise *singular point.* When $x = a$ is a regular singular point of (1), then it can be solved using method of Frobenius.

Frobenius Method

Let *b(x)* and *c(x)* be any functions that are analytic at x = 0. ('x' is regular singular point). Then the ODE:

$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0$ …(2)

Has at least one solution that can be represented in the form:

$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$ …(3)

Where $a_0 \neq 0$, the exponent r may be real or complex.

Matlab syntax

`coeffs(P,var)`	Returns coefficients of the polynomial 'P' with respect to the variable 'var'.
`collect(P,var)`	Rewrites 'P' in terms of the powers of the variable 'var'.
`n=numel(A)`	Returns the number of elements 'n' in array 'A', equivalent to prod(size(A)).
`simplify(S)`	Performs an algebraic simplification of S.
`J=besselj(nu,Z)`	Computes the Bessel function of the first kind, where 'nu' is order and 'Z' is an argument.
`Y=bessely(nu,Z)`	Computes Bessel function of the second kind, where 'nu' represents order and 'Z' is an argument.

MATLAB Code:

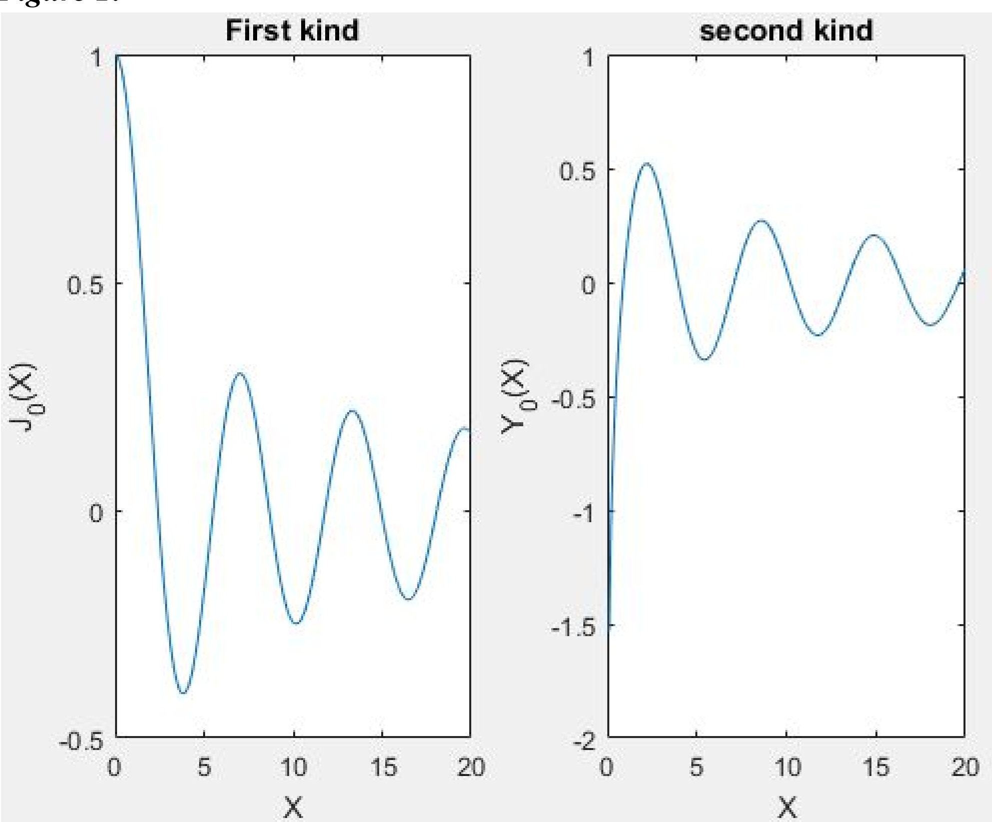
```
clc;
clear all;
syms x a0 a1 a2 a3 a4 m c1 c2
y=a0*x^m+a1*x^(m+1)+a2*x^(m+2)+a3*x^(m+3)+a4*x^(m+4)
eq=x^2*diff(y,x,2)+x*diff(y,x,1)+x^2*y
eq1=collect(eq)
eq2=coeffs(simplify(eq1*x^(1-m)),x)
eq3=solve(eq2(1),m) %roots of indicial equation
a1=solve(eq2(2),a1)
a2=solve(eq2(3),a2)
a3=subs(solve(eq2(4),a3))
a4=subs(solve(eq2(5),a4))
ss=a0*x^m+a1*x^(m+1)+a2*x^(m+2)+a3*x^(m+3)+a4*x^(m+4)
y1=subs(ss,m,eq3(1))
y2=subs(diff(ss,m),m,eq3(1))
gs=c1*y1+c2*y2
%%visualization of bessel's (order zero)
X =0:0.1:20;
%Y= zeros(5,numel(X));
%J= zeros(5,numel(X));
Y0 = bessely(0,X);
J0 = besselj(0,X);
subplot(1,2,1),plot(X,J0)
title('First kind')
xlabel('X')
ylabel('J_0(X)')
subplot(1,2,2),plot(X,Y0)
title('second kind')
xlabel('X')
ylabel('Y_0(X)')
```

Output:

```
y =

a0*x^m + a1*x^(m + 1) + a2*x^(m + 2) + a3*x^(m + 3) + a4*x^(m + 4)


eq =

x^2*(a0*x^m + a1*x^(m + 1) + a2*x^(m + 2) + a3*x^(m + 3) + a4*x^(m + 4)) + x^2*(a0*m*x^(m -
2)*(m - 1) + a1*m*x^(m - 1)*(m + 1) + a2*x^m*(m + 1)*(m + 2) + a3*x^(m + 1)*(m + 2)*(m + 3)
+ a4*x^(m + 2)*(m + 3)*(m + 4)) + x*(a2*x^(m + 1)*(m + 2) + a3*x^(m + 2)*(m + 3) + a4*x^(m +
3)*(m + 4) + a0*m*x^(m - 1) + a1*x^m*(m + 1))


eq1 =

(a0*x^m + a1*x^(m + 1) + a2*x^(m + 2) + a3*x^(m + 3) + a4*x^(m + 4) + a0*m*x^(m - 2)*(m - 1)
+ a1*m*x^(m - 1)*(m + 1) + a2*x^m*(m + 1)*(m + 2) + a3*x^(m + 1)*(m + 2)*(m + 3) + a4*x^(m +
2)*(m + 3)*(m + 4))*x^2 + (a2*x^(m + 1)*(m + 2) + a3*x^(m + 2)*(m + 3) + a4*x^(m + 3)*(m +
4) + a0*m*x^(m - 1) + a1*x^m*(m + 1))*x


eq2 =

[ a0*m^2, a1*m^2 + 2*a1*m + a1, a2*m^2 + 4*a2*m + a0 + 4*a2, a3*m^2 + 6*a3*m + a1 + 9*a3,
a4*m^2 + 8*a4*m + a2 + 16*a4, a3, a4]


eq3 =

 0
 0


a1 =

0


a2 =

-a0/(m^2 + 4*m + 4)


a3 =

0


a4 =

a0/((m^2 + 4*m + 4)*(m^2 + 8*m + 16))


ss =
```

```
a0*x^m - (a0*x^(m + 2))/(m^2 + 4*m + 4) + (a0*x^(m + 4))/((m^2 + 4*m + 4)*(m^2 + 8*m + 16))


y1 =

(a0*x^4)/64 - (a0*x^2)/4 + a0


y2 =

(a0*x^2)/4 - (3*a0*x^4)/128 + a0*log(x) - (a0*x^2*log(x))/4 + (a0*x^4*log(x))/64


gs =

c1*((a0*x^4)/64 - (a0*x^2)/4 + a0) + c2*((a0*x^2)/4 - (3*a0*x^4)/128 + a0*log(x) -
(a0*x^2*log(x))/4 + (a0*x^4*log(x))/64)

>>
```

Figure 1:



Hanging Chain

- Chain is perfectly flexible.
- The flexible uniform chain of length l and constant linear density ρ is fixed at the upper end ($x = l$).
- x-axis is the vertical axis.
- $u(x, t)$ is displacement function for the point x on the chain.
- The displacements are small compared to the length of the chain.
- The chain's equilibrium position is due to the gravitational force.
- No air resistance.
- The tension in the chain is due to the weight below point x. $w(x) = u \cdot g \cdot x$

$$sin(\alpha) = \frac{\Delta u(x)}{\Delta x} = u_x(x)$$

- For any displacement of angle α the restoring force:
  $= f(x) \propto u(x)$
  $= wu(x)$
  $= \rho gx \cdot u(x)$

$F(x) = w \cdot sin(x) = wu_x(x)$

Therefore, the net force:

$F(x + \Delta x) - F(x) = wu_x(x + \Delta x) - wu_x(x) = w\left[\frac{u_x(x+\Delta x)-u_x(x)}{\Delta x}\right] = \rho gx \cdot [u_x]_x \cdot \Delta x$

$F = ma$

$\rho gx \cdot [u_x]_x \cdot \Delta x = \rho \Delta x \cdot u_{xx}$

(i.e.) $u_{tt} = g(u_x + xu_{xx}) \ldots (1)$

$y_{tt} = ay_{xx}$

$y(x, t) = X(x) \cdot T(t)$

$y_{tt} = XT''$

$y_{xx} = X''T$

$XT'' = aX''T$

$\frac{T''}{T} = \frac{aX''}{X} = k$

$T'' - kT = 0 \Rightarrow T(t)$

$aX'' - kx = 0 \Rightarrow X(t)$

$u(x, t) = F(x)G(t)$

$u_x = F'(x)G(t)$

$u_{xx} = F''(x)G(t)$

$u_{tt} = F(x)G''(t)$
Substituting into (1)
$F(x)G''(t) = g\,[F'(x)G(t) + xF''(x)G(t)]$
(i.e.) $\frac{G''}{F} = \frac{gG(F'+xF'')}{F}$
(i.e.) $\frac{G''}{G} = \frac{gF'+gxF''}{F} = -w^2$
(i.e.) $G'' + w^2G = 0 \ldots(2)$
$xF'' + F' + \frac{w^2}{g}F = 0 \ldots(3)$

By substituting $z = 2\sqrt{\frac{x}{g}}$ in (3), we have:
$\frac{d^2F}{dz^2} + \frac{1}{z}\frac{dF}{dz} + w^2F = 0$
(i.e.) $z^2F'' + zF' + w^2z^2F = 0$ ...which is a kind of Bessel equation.

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EXPERIMENT – 6

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2'(x)\,y = 0$$

$$y'' + \frac{b(x)}{x}\,y' + \frac{c(x)}{x^2}\,y = 0 \quad \text{-- (2)}$$

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m \quad \text{– (3)}$$

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EXPERIMENT – 7

$$\begin{bmatrix} M_s & 0 \\ 0 & M_{us} \end{bmatrix} \begin{bmatrix} \frac{d^2y_s}{dt^2} \\ \frac{d^2y_{us}}{dt^2} \end{bmatrix} + \begin{bmatrix} c_s & -c_s \\ -c_s & c_s \end{bmatrix} \begin{bmatrix} \frac{dy_s}{dt} \\ \frac{dy_{us}}{dt} \end{bmatrix}$$

$$+ \begin{bmatrix} k_s & -k_s \\ -k_s & k_s + k_t \end{bmatrix} \begin{bmatrix} y_s \\ y_{us} \end{bmatrix} = \begin{bmatrix} 0 \\ k_t u \end{bmatrix}$$

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