Experiment #01 - STRESS DISTRIBUTION IN A TOWER BRIDGE

Aim: Calculating and visualizing the eigenvalues of stress matrix for a simply supported beam.

Problem statement: Find the principal stresses for a two-dimensional simply supported beam by finding the eigenvalues of the stress matrix with variable components.

Mathematical background: The principal stresses are eigenvalues of the stress matrix.

The 2×2 stress matrix is given by:

$$S = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

The principal stress at the point will be the eigenvalues of the stress matrix S.

Properties of eigenvalues and eigenvectors:

1. The sum of the eigenvalues of a matrix equals the trace of the matrix.
2. A matrix is singular if and only if it has a zero eigenvalue.
3. The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal.

MATLAB Syntax

`p = poly(A)`	Where A is an $n \times n$ matrix returns an n+1 element row vector whose elements are the coefficients of the characteristic polynomial, $\det(A - \lambda I)$, which are stored in p.
`r = roots(p)`	Returns a column vector r whose elements are the roots of the polynomial p.
`[V,D] = eig(A)`	D = diagonal matrix with eigenvalues on its diagonal; V = modal matrix whose columns are the corresponding eigenvectors.
`eye(n)`	Returns an $n \times n$ identity matrix.

Example #1:

```
A =

     3     0    -1
     0     1     0
     2     0     0

>> poly(A)

ans =

     1    -4     5    -2

>> roots(poly(A))

ans =

   2.0000 + 0.0000i
   1.0000 + 0.0000i
   1.0000 - 0.0000i

>> eig(A)

ans =

     2
     1
     1

>> [V,D] = eig(A)

V =

    0.7071    0.4472         0
         0         0    1.0000
    0.7071    0.8944         0


D =

     2     0     0
     0     1     0
     0     0     1

>> eig(trace(A))

ans =

     4

>> det(A)

ans =

     2

>> [V,D] = eig(inv(A))
```

```
V =

   -0.7071   -0.4472         0
         0         0    1.0000
   -0.7071   -0.8944         0


D =

    0.5000         0         0
         0    1.0000         0
         0         0    1.0000


>> [V,D] = eig(A')

V =

    0.8944   -0.7071         0
         0         0    1.0000
   -0.4472    0.7071         0


D =

     2     0     0
     0     1     0
     0     0     1

>> I = eye(3)

I =

     1     0     0
     0     1     0
     0     0     1

>> B = A^2 + 3*A + 2*I

B =

    18     0    -6
     0     6     0
    12     0     0

>> [V,D] = eig(B)

V =

    0.7071    0.4472         0
         0         0    1.0000
    0.7071    0.8944         0


D =

    12     0     0
     0     6     0
     0     0     6
```

```
>> eig(B)

ans =

    12
     6
     6

>>
```

Example #2

```
>> A = [2 0 -1; 0 2 -2; 1 -1 2]

A =

     2     0    -1
     0     2    -2
     1    -1     2

>> eig(A)

ans =

    3.0000
    2.0000
    1.0000

>> roots(poly(A))

ans =

    3.0000
    2.0000
    1.0000
```

Questions

1. Perform basics matrix tasks using MATLAB.

```
>> A = zeros(3)

A =

     0     0     0
     0     0     0
     0     0     0

>> A = [1,2,3]

A =

     1     2     3

>> B = [1;2;3]

B =

     1
     2
     3

>> C = [1 2 3; 10 20 30; 3 6 9]

C =

     1     2     3
    10    20    30
     3     6     9

>> length(C)

ans =

     3

>> D = [1 2 3; 4 5 6]

D =

     1     2     3
     4     5     6

>> length(D)
```

```
ans =

     3

>> size(D)

ans =

     2     3
```

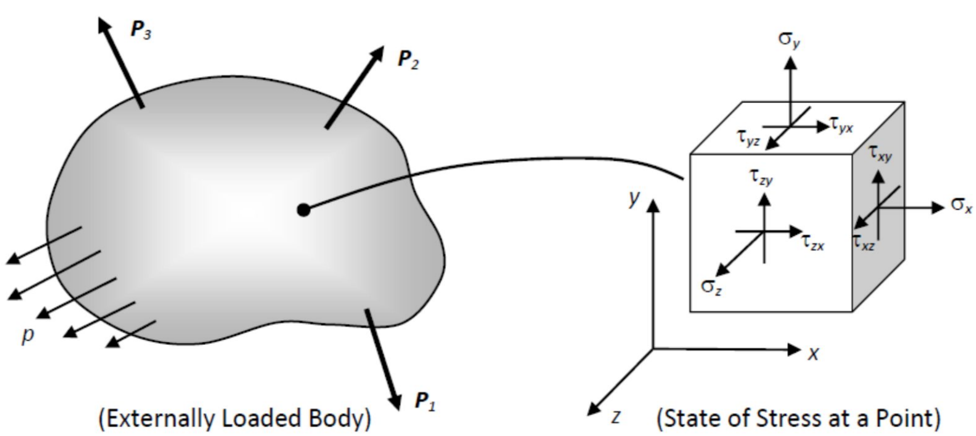
2. Find the following for A:
    a. Characteristic polynomial of A.
    b. Roots of characteristic polynomial of A.
    c. Eigenvalues for A.
    d. Eigenvectors for A.
    e. Eigenvalues of A^{-1} .
    f. Eigenvalues of A^{T} .
    g. Eigenvalues of $B = A^{2} + 3A + 2I$.

```
>> A = [1 2 1; 6 -1 0; -1 -2 -1]

A =

     1     2     1
     6    -1     0
    -1    -2    -1

>> poly(A)

ans =

    1.0000    1.0000  -12.0000    0.0000

>> roots(poly(A))

ans =

   -4.0000
    3.0000
    0.0000

>> eig(A)

ans =

   -4.0000
    3.0000
    0.0000

>> [V,D]=eig(A)
```

```
V =

    0.4082   -0.4851   -0.0697
   -0.8165   -0.7276   -0.4180
   -0.4082    0.4851    0.9058


D =

   -4.0000         0         0
         0    3.0000         0
         0         0    0.0000

>> eig(trace(A))

ans =

    -1

>> det(A)

ans =

     0

>> [V,D] = eig(inv(A))
Warning: Matrix is singular to working precision.
Error using eig
Input to EIG must not contain NaN or Inf.

>> [V,D] = eig(A')

V =

   -0.7252    0.9117    0.7071
    0.6447    0.3419    0.0000
    0.2417    0.2279    0.7071


D =

   -4.0000         0         0
         0    3.0000         0
         0         0   -0.0000

>> eig(A')

ans =

   -4.0000
    3.0000
   -0.0000

>> B = A^2 + 3*A + 2*eye(3)

B =

    17     4     3
    18    12     6
```

```
   -15    -4    -1

>> [V,D] = eig(B)

V =

   -0.4851   -0.4082   -0.0697
   -0.7276    0.8165   -0.4180
    0.4851    0.4082    0.9058


D =

   20.0000         0         0
         0    6.0000         0
         0         0    2.0000

>> eig(B)

ans =

   20.0000
    6.0000
    2.0000
```

Stress Analysis



The number of components and some other transformation properties, the stress can be expressed as a 3×3 matrix:

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

Since the shearing stresses have the equalities, $\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz}$,the stress matrix is symmetric.

If we changed the orientation of a particular plane the normal stress component σ_x will vary.

Total surface force $\overline{p}$ per unit volume is:
$\overline{p} = \frac{\delta p}{\delta x} + \frac{\delta p}{\delta y} + \frac{\delta p}{\delta z}$
$\overline{p}_x$,$\overline{p}_y$ and $\overline{p}_z$ are vectors which can be decomposed into the components perpendicular to each surface elements, i.e., normal surface σ and by giving the direction of normal stresses as index for normal stresses. The components in the plane of surface elements are called tangential stresses τ. They acquire the double index. The first position indicates to which access the surface element is perpendicular and second position states in which direction the stress τ is pointing.

In notation, τ is pointing:
$\overline{p}_x = \sigma_x\overline{e}_x + \tau_{xy}\overline{e}_y + \tau_{xz}\overline{e}_z$
$\overline{p}_y = \tau_{yx}\overline{e}_x + \sigma_y\overline{e}_y + \tau_{yz}\overline{e}_z$
$\overline{p}_z = \tau_{zx}\overline{e}_x + \tau_{zy}\overline{e}_y + \sigma_z\overline{e}_z$

The normal stresses are known as principal stresses:
$\sigma_l = \sigma_x l + \tau_{xy} m + \tau_{xz} n$ …(1)
$\sigma_m = \tau_{yx} l + \sigma_y m + \tau_{yz} n$ …(2)
$\sigma_n = \tau_{zx} l + \tau_{zy} m + \sigma_z n$ …(3)

Where, σ_l , σ_m and σ_n are components of principle stress σ .

We know that $l^2 + m^2 + n^2 = 1$ …(4)

Thus above system of homogenous equations does not admit a trivial solution. {a trivial solution would be $(l = m = n = 0)$ } as (4) is true.

Since, $\Delta = 0$, the characteristic equation is,

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{vmatrix} = 0$$

Example 1:

Q: Consider the bridge to be equivalent to a two-dimensional simple supported beam of thickness 1 meter. The bridge carries a concentrated force P = 100N at the mid-point. The length of the beam is 10 times its thickness. If the individual stresses at the point (0.5, 0.5) are given by $\sigma_x = -4$, $\sigma_y = 0$ and $\sigma_{xy} = -2.5$, then write the stress matrix and hence calculate the principal stresses.

A: The stress matrix is at the point (0.5,0.5) will be:

```
>> S = [-4 -2.5, -2.5 0]

S =

   -4.0000   -2.5000   -2.5000         0

>> S = [-4 -2.5; -2.5 0]

S =

   -4.0000   -2.5000
   -2.5000         0

>> eig(S)

ans =

   -5.2016
    1.2016

>> [V,D] = eig(S)

V =

   -0.9013    0.4332
   -0.4332   -0.9013


D =

   -5.2016         0
         0    1.2016
```

Example 2:

Q: If the individual stresses in the beam at any point (x, y) are given by:

$\sigma_x = \frac{-3P}{4c^3}$, $\sigma_y = 0$ and $\tau_{xy} = \frac{-3}{4c^3}(c^2 - y^2)$

When P denotes the force 2c denotes the height and 2L denotes the length of the beam. Write the stress matrix and hence calculate the principal stresses. Draw the stress distribution in the beam using contour plot.

A:

```
clc;
clear all;
c = 0.5; L = 5; P = 1;
x = [0:0.1:L]; y = [-c:0.01:c];
[X,Y] = meshgrid(x,y);
sx =-(3*P/(4*c^3))*(L-X).*Y;
sy =zeros(length(y),length(x));
txy =-(3/(4*c^3))*(c^2-Y.^2);
for i = 1:length(y)
    for j = 1:length(x)
        s = [sx(i,j),txy(i,j);txy(i,j),sy(i,j)]
        z = eig(s);
        s1(i,j) = z(2);
        s2(i,j) = z(1);
    end
end

figure(1)
contour(X,Y,s1,[0.01,0.05,0.1,0.5,1,3,5,7,9,11]);

figure(2)
contour(X,Y,s2,[-0.01,-0.05,-0.1,-0.5,-1,-3,-5,-7,-9,-11]);
```

Figure 1:

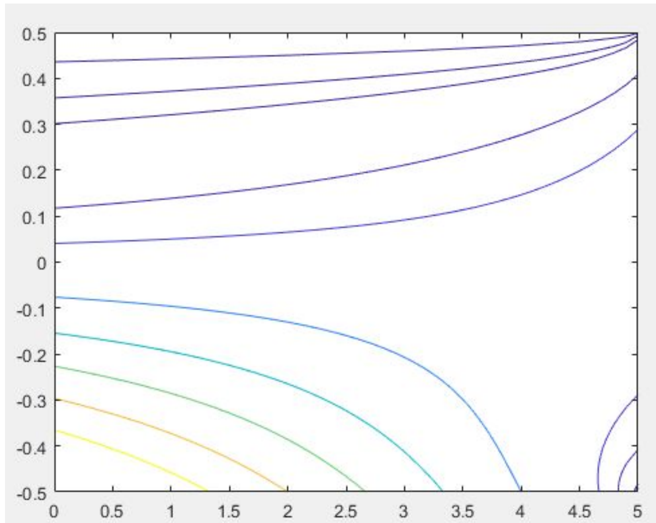
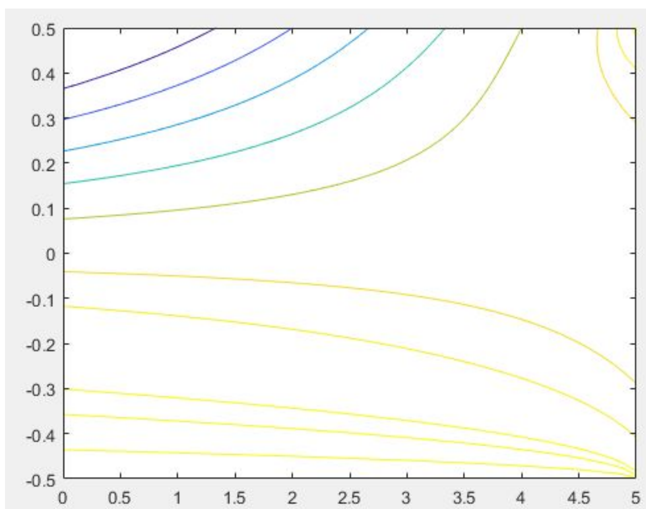


Figure 2:



Exercise - 1

Q: Create a 3 x 3 matrix:

1. Find the eigenvalues and eigenvectors.
2. Verify the properties of eigenvalues.

A:

```
>> A  = [2 0 4;0 6 0;4 0 2]

A =

     2     0     4
     0     6     0
     4     0     2

>> [V,D] = eig(A)

V =

    0.7071    0.7071         0
         0         0   -1.0000
   -0.7071    0.7071         0


D =

    -2     0     0
     0     6     0
     0     0     6
```

```
>> A = [2 0 4;0 6 0;4 0 2];
>> %Eigen values of A and A transpose are the same
>> eig(A)

ans =

    -2
     6
     6

>> eig(transpose(A))

ans =

    -2
     6
     6

>> %trace of A = sum of eigenvectors
>> trace(A)
```

```
ans =

    10

>> sum(eig(A))

ans =

    10

>> %det = product of eigenvalues of A
>> det(A)

ans =

   -72

>> prod(eig(A))

ans =

   -72

>> %lamda^2 are the eigenvectors of A^2
>> A^2

ans =

    20     0    16
     0    36     0
    16     0    20

>> eig(A^2)

ans =

     4
    36
    36

>> eig(A)

ans =

    -2
     6
     6

>> %The diagonal elements are the eigen vectors of a diagonal matrix
>> A = [1 0 0;0 2 0;0 0 3];
>> A

A =

     1     0     0
     0     2     0
     0     0     3

>> eig(A)
```

```
ans =

     1
     2
     3
```

Attestation:

EXPERIMENT – 1 : Stress distribution in a tower bridge

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix} \quad \text{eig } A = \begin{bmatrix} -2 \\ 6 \\ 6 \end{bmatrix}$$

EXPERIMENT – 2 : Google's page ranking algorithm

$$A = \begin{bmatrix} 0 & 1/3 & 1 & 1/3 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 1/3 & 1/3 \\ 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 \end{bmatrix}$$

$$x = \begin{matrix} 0.3286 \\ 0.1801 \\ 0.1797 \\ 0.1316 \end{matrix}$$

11/9/17