EXPERIMENT 1

STRESS DISTRIBUTION IN A TOWER BRIDGE

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Aim: Calculating and visualizing the Eigen values of stress matrix for simply supported beam

Problem statement: Find principal stresses for a two-dimensional simply supported beam by finding the eigenvalues of the stress matrix with variable components.

Mathematical Background: The principal stresses are the eigenvalues of the stress matrix.

- The 2×2 stress matrix is given by $S = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$

- The principal stress at the point will be the eigenvalues of the stress matrix S

Definitions

A nonzero vector **x** is an ***eigenvector*** (or *characteristic vector*) of a square matrix **A** if there exists a scalar **λ** such that **Ax = λx**. Then **λ** is an ***eigenvalue*** (or *characteristic value*) of **A**.

Note: The zero vector can not be an eigenvector even though **A0 = λ0**. But **λ = 0** can be an eigenvalue.

Example:

$$\text{Show } X = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ is an eigenvector for } A = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$$

$$\text{Solution}: AX = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{But for } \lambda = 0, \;\; \lambda X = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Thus, } X \text{ is an eigen vector of } A, \text{and } \lambda = 0 \text{ is an eigenvalue.}$$

Eigenvalues

Let x be an eigenvector of the matrix **A.** Then there must exist an eigenvalue **λ** such that **$Ax = \lambda x$** or, equivalently,

$Ax - \lambda x = 0$ or

$(A - \lambda I)x = 0$

If we define a new matrix **$B = A - \lambda I$**, then **$Bx = 0$**

If **B** has an inverse then **$x = B^{-1}0 = 0$**. But an eigenvector cannot be zero.

Thus, it follows that x will be an eigenvector of **A** if and only if **B** does not have an inverse, or equivalently **$\det(B)=0$**, or

$$\det(A - \lambda I) = 0$$

This is called the **characteristic equation** of **A**. Its roots determine the eigenvalues of **A**.

Examples

Example 1: Find the eigenvalues of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = (\lambda - 2)(\lambda + 5) + 12$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

two eigenvalues: $-1, -2$

Note: The roots of the characteristic equation can be repeated. That is, $\lambda_1 = \lambda_2 = \ldots = \lambda_k$. If that happens, the eigenvalue is said to be of multiplicity k.

Example 2: Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

$\lambda = 2$ is an eigenvalue of multiplicity 3.

Eigenvectors

To each distinct eigenvalue of a matrix **A** there will correspond at least one eigenvector which can be found by solving the appropriate set of homogenous equations. If $\boldsymbol{\lambda_i}$ is an eigenvalue then the corresponding eigenvector $\mathbf{x_i}$ is the solution of **$(\mathbf{A} - \boldsymbol{\lambda_i}\mathbf{I})\mathbf{x_i} = \mathbf{0}$**

Example 1 (cont.):

$$\lambda = -1 : (-1)I - A = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_2 = 0 \Rightarrow x_1 = 4t,\ x_2 = t$$

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix},\ t \neq 0$$

$$\lambda = -2 : (-2)I - A = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \end{bmatrix},\ s \neq 0$$

Properties of Eigenvalues and Eigenvectors

1. The sum of the eigenvalues of a matrix equals the <u>trace</u> of the matrix.
2. A matrix is singular if and only if it has a zero eigenvalue.
3. The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal.
4. If λ is an eigenvalue of A and A is invertible, then $1/\lambda$ is an eigenvalue of matrix A^{-1}.
5. If λ is an eigenvalue of A then $k\lambda$ is an eigenvalue of kA where k is any arbitrary scalar.
6. If λ is an eigenvalue of A then λ^k is an eigenvalue of A^k for any positive integer k
7. If λ is an eigenvalue of A then λ is an eigenvalue of A^T

MATLAB syntax

p =poly(A)	where A is an nxn matrix returns an n+1 element row vector whose elements are the coefficients of the characteristic polynomial, det(λI - A), which are stored in p
r =roots(p)	Returns a column vector r whose elements are the roots of the polynomial p.
[V,D] = eig(A)	D=diagonal matrix with eigenvalues on its diagonal; V = modal matrix whose columns are the corresponding eigenvectors.
eye(n)	Returns an n x n identity matrix

Example

$$\text{If } A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \text{ Find}$$

a. Characteristic polynomial of A
b. Roots of Characteristic polynomial of A
c. Eigen values of A
d. Eigen vectors of A
e. Eigen values of A^{-1}
f. Eigen values of A^T
g. Eigen values of $B = A^2 + 3A + 2I$

MATLAB code

```
A=[3 0 -1; 0 1 0; 2 0 0]
Eigenvalues_of_A=eig(A)
[X,D]= eig(A)
p=poly(A)      % Coefficients of the characteristic polynomial
r = roots(p)   % To find the roots of the characteristic equation
sum_of_eigenvalues = sum(r)
Trace_of_A=trace(A)%Note that sum of eigenvalues is Trace of A
product_of_eigenvalues_A=prod(r)
Determinant_of_A =det(A)   % Note that product of the eigenvalues of A is  determinant of A
Inverse_A=inv(A)
Eigenvalues_of_invA= eig(Inverse_A)
Eigenvalues_Transpose_of_A= eig(A')
 % Note that eigenvalues of A and Transpose of A are same
Eigenvalues_of_B= eig(A^2+3*A+2*eye(3))
```

Output

```
>>
A =
    3     0    -1
    0     1     0
    2     0     0


Eigenvalues_of_A =

    2
    1
    1
```

```
X =
    0.7071    0.4472         0
         0         0    1.0000
    0.7071    0.8944         0

D =
    2     0     0
    0     1     0
    0     0     1

p =

    1    -4     5    -2
```

```
r =

   2.0000
   1.0000 + 0.0000i
   1.0000 - 0.0000i

sum_of_eigenvalues =     4.0000

Trace_of_A =      4

product_of_eigenvalues_A =     2.0000

Determinant_of_A =   2

Inverse_A =

         0         0    0.5000
         0    1.0000         0
   -1.0000         0    1.5000
```

```
Eigenvalues_of_invA =

    0.5000
    1.0000
    1.0000

Eigenvalues_Transpose_of_A =

     2
     1
     1
```

```
Eigenvalues_of_B =

    12
     6
     6
```

Practice Problem

If $A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$ Find

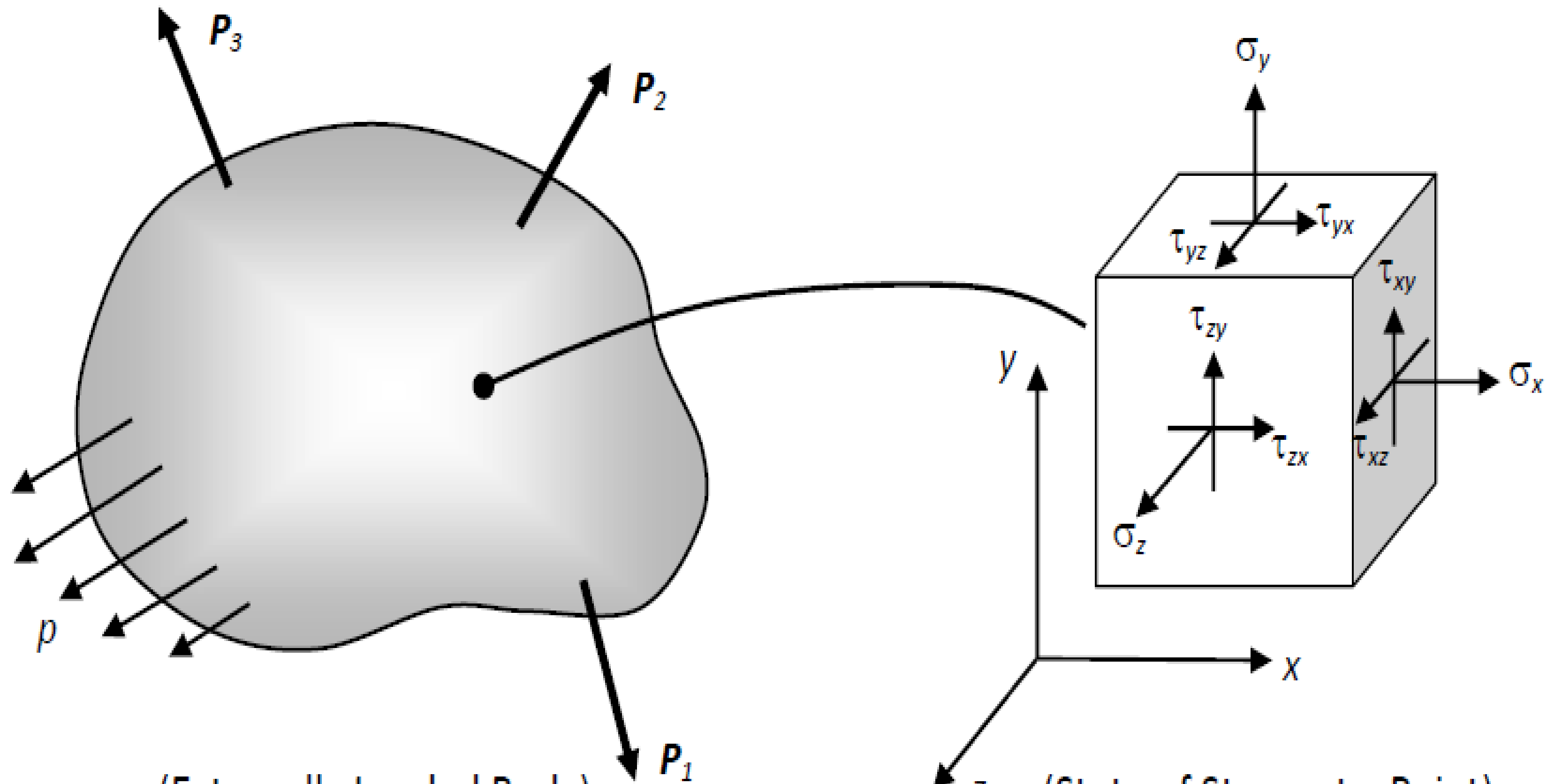
a. Characteristic polynomial of A
b. Roots of Characteristic polynomial of A
c. Eigen values of A
d. Eigen vectors of A
e. Eigen values of A^{-1}
f. Eigen values of A^T
g. Eigen values of $B = A^2 + 3A + 2I$

STRESS ANALYSIS

- An elastic body is subjected to applied loadings, stresses are created inside the body.
- In general these stresses often vary in complicated ways from point to point and from plane to plane within the structure.
- To help characterize this situation, stresses are normally defined with respect to a given coordinate system.
- The illustration below shows that at a typical point within a loaded body, the state of stress can be characterized on a small cube of material defined with respect to a Cartesian coordinate system.
- These nine components are called the *stress components, with*

  $\sigma_x, \sigma_y, \sigma_z$ *referred to as normal stresses and*

  $\tau_{zy}, \tau_{zx}, \tau_{xz}$ *called the shearing stresses.*



(Externally Loaded Body) (State of Stress at a Point)

STRESS ANALYSIS

- The number of components and some other transformation properties, the stress can be expressed as a 3 x 3 matrix

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

- Since the shearing stresses have the equalities $\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz}$, the stress matrix is symmetric.
- If we changed the orientation of a particular plane the normal stress component σ_x will vary
- There exists a special orientation where the normal stress will be a maximum, and these are called *principal planes and the normal stresses acting on them are called the principal stresses*

STRESS ANALYSIS

- The general three-dimensional case, the theory to determine principal stresses and the planes on which they act is formulated by the eigenvalue problem

$$[\sigma]\{n\} = \lambda\{n\}$$

- where σ is the stress matrix, {n} is the principal direction vector and λ (the eigenvalue) is the principal stress. Thus solving the eigenvalue problem will determine up to three distinct principal stresses and the corresponding three principal directions. It turns out for this application (3x3, symmetric real matrix) the principal directions are mutually orthogonal.
- The shear stress components will vanish on these three principal planes and so for a coordinate system that is aligned with the principal directions the stress matrix takes on the simplified diagonal form

STRESS ANALYSIS

$$\sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Where $\sigma_1, \sigma_2, \sigma_3$ are the problem's eigenvalues (roots of the characteristic equation) and are generally referred to as the principal stresses.

Example

Q Consider the bridge to be equivalent to a two-dimensional simply supported beam of thickness 1 meter. The bridge carries a concentration force P = 100 Newton at the mid-point. The length of the beam is 10 times of its thickness. If the individual stresses at the point (0.5, 0.5) are given by $\sigma_x = -4, \sigma_y = 0, \sigma_{xy} = -2.5$ then write the stress matrix and hence calculate the principal stresses.

Ans

The stress matrix at the point (0.5, 0.5) will be

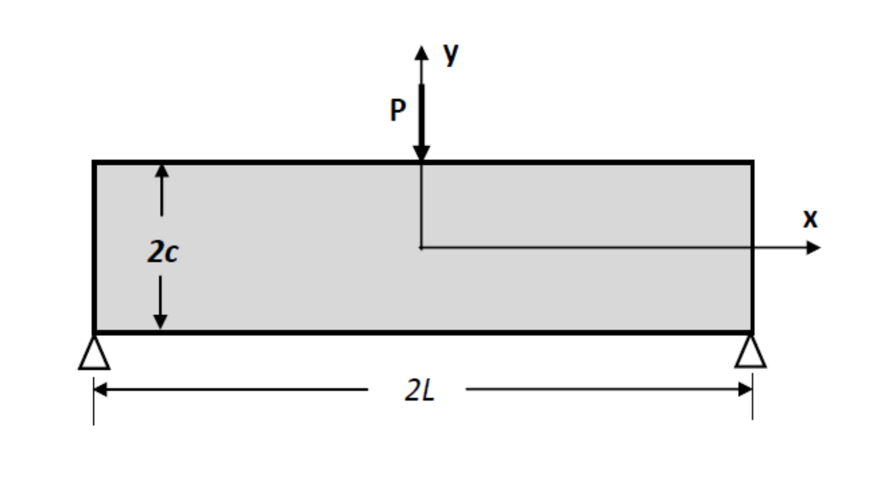
$$S = \begin{bmatrix} -4 & -2.5 \\ -2.5 & 0 \end{bmatrix}$$

Practice Problem

Q If the individual stresses in the beam at any point (x,y) are given by

$$\sigma_x = -\frac{3P}{4c^3}(L-x)y, \sigma_y = 0, \tau_{xy} = -\frac{3P}{4c^3}\left(c^2 - y^2\right)$$

Where P denotes the force 2c denotes the height and 2L denotes the length of the beam. Write the stress matrix and hence calculate the principal stresses.



Draw the stress distribution in the beam using contour plot

Solution

From mechanics of materials theory the in-plane stress components for the right half of the beam (x > 0) are given by

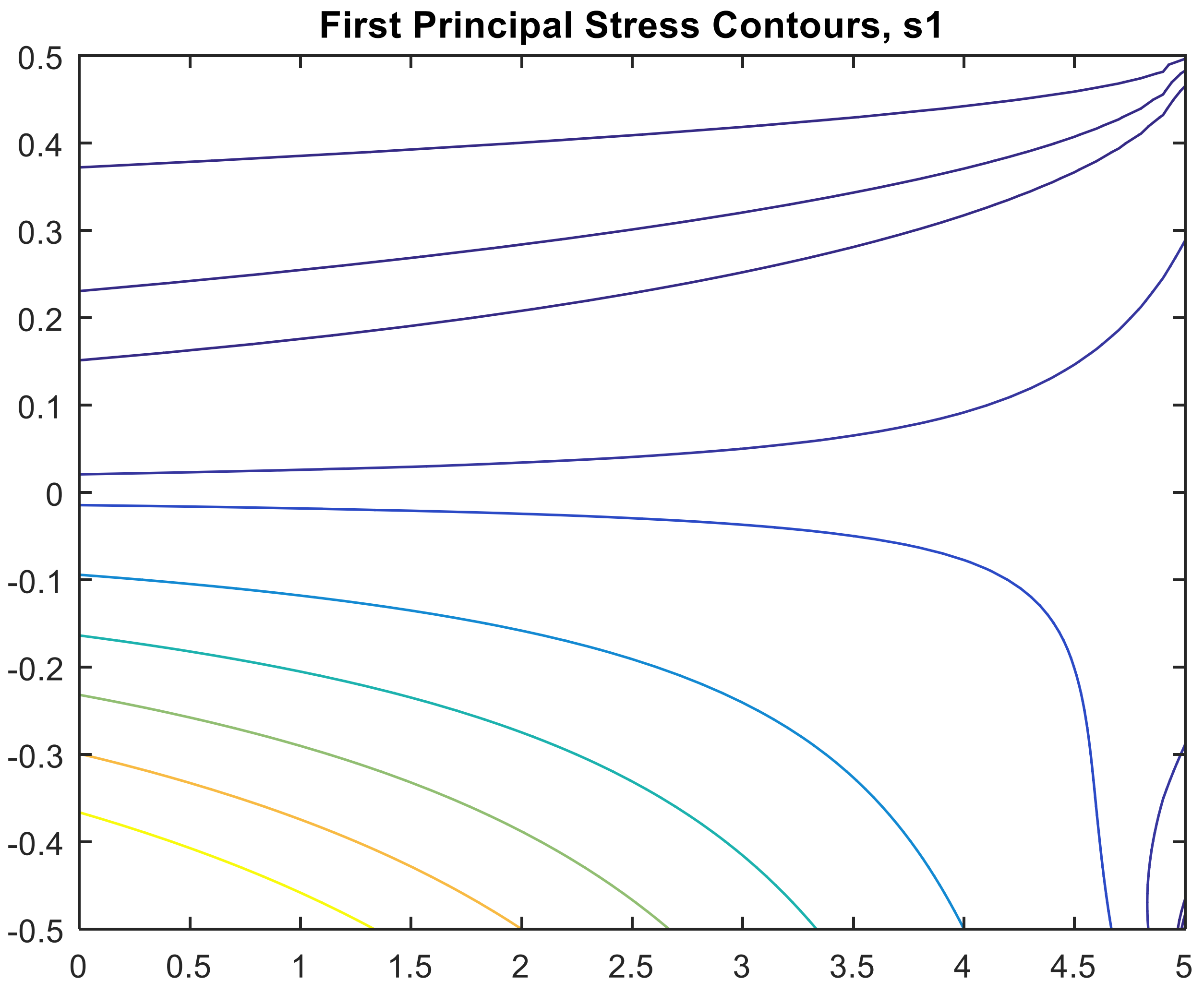
$$\sigma_x = -\frac{3P}{4c^3}(L-x)y, \sigma_y = 0, \tau_{xy} = -\frac{3P}{4c^3}\left(c^2 - y^2\right)$$

Thus the stress matrix for this problem reduces to the 2x2 form

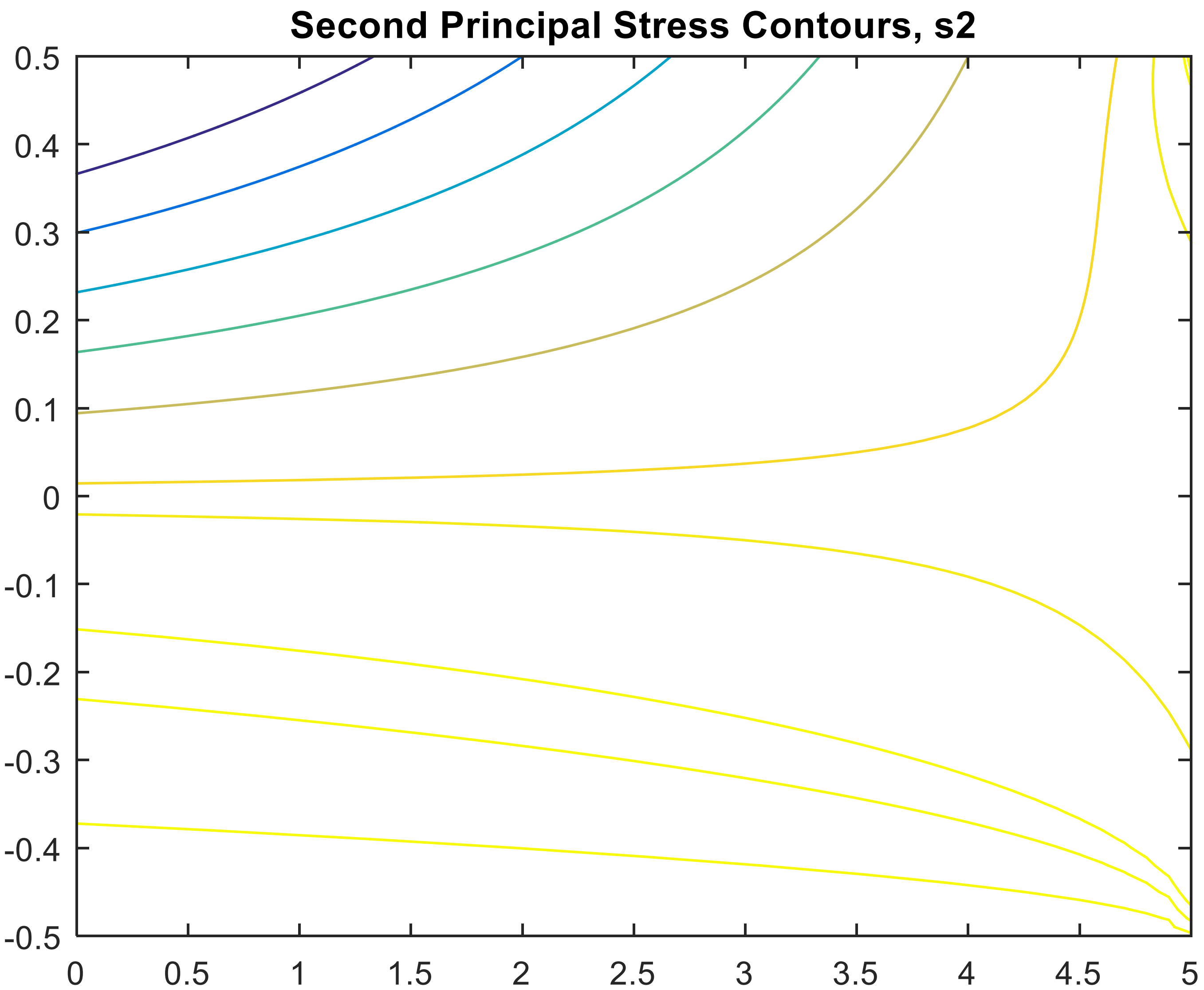
$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} -\frac{3P}{4c^3}(L-x)y & -\frac{3P}{4c^3}\left(c^2 - y^2\right) \\ -\frac{3P}{4c^3}\left(c^2 - y^2\right) & 0 \end{bmatrix}$$

```
% Determine and Plot Principal Stresses in Beam Exa
clc;clear all;clf;
% Input Beam Parameters
c=0.5;L=5;P=1;
% Set up grid of x and y values
x=[0:0.1:L];y=[-c:0.01:c];
[X,Y]=meshgrid(x,y);
% Calculate Individual Stresses
sx=-(3/(4*c^3))*(L-X).*Y;
sy=zeros(length(y),length(x));
txy=-(3/(8*c^3))*(c^2-Y.^2);
% Create Stress Matrix
for i=1:length(y)
    for j=1:length(x)
s=[sx(i,j),txy(i,j);txy(i,j),sy(i,j)];
% Determine Eigenvalues(Principal Stresses)
p=eig(s);
s1(i,j)=p(2);
s2(i,j)=p(1);
    end
end
% Plot Distributions of Principal Stresses s1 and s
figure(1)
contour(X,Y,s1,[0.01,0.05,0.1,0.5,1,3,5,7,9,11])
title('First Principal Stress Contours, s1')
axis tight
figure (2)
contour(X,Y,s2,[-0.01,-0.05,-0.1,-0.5,-1,-3,-5,-7,-
axis tight
title('Second Principal Stress Contours, s2')
```

Out put


First Principal Stress Contours, s1
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Out put


Second Principal Stress Contours, s2
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Lab Assignment for Exp-1

$$\text{If } A = \begin{bmatrix} 3 & -1 & 3 \\ 9 & -1 & 9 \\ 7 & -1 & 7 \end{bmatrix} \quad \text{Find}$$

a. Characteristic polynomial of A
b. Roots of Characteristic polynomial of A
c. Eigen values of A
d. Eigen vectors of A
e. Eigen values of A^T
f. Eigen values of A^{-1}
g. Eigen values of $B = A^2 + 3A + 2I$

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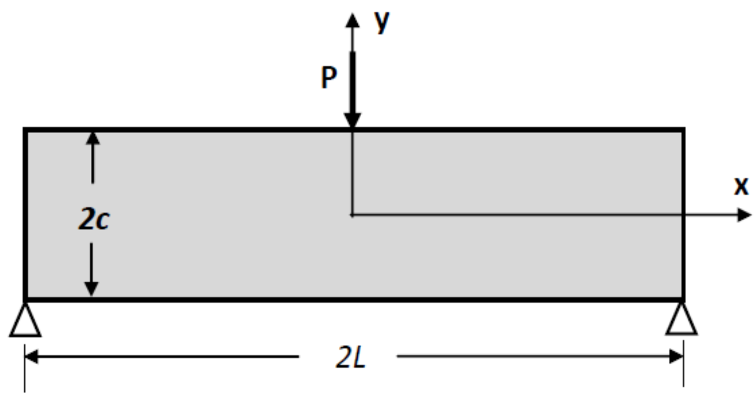

Where L = 6, C = 1.5, P = 2

Draw the stress distribution in the beam using contour plot