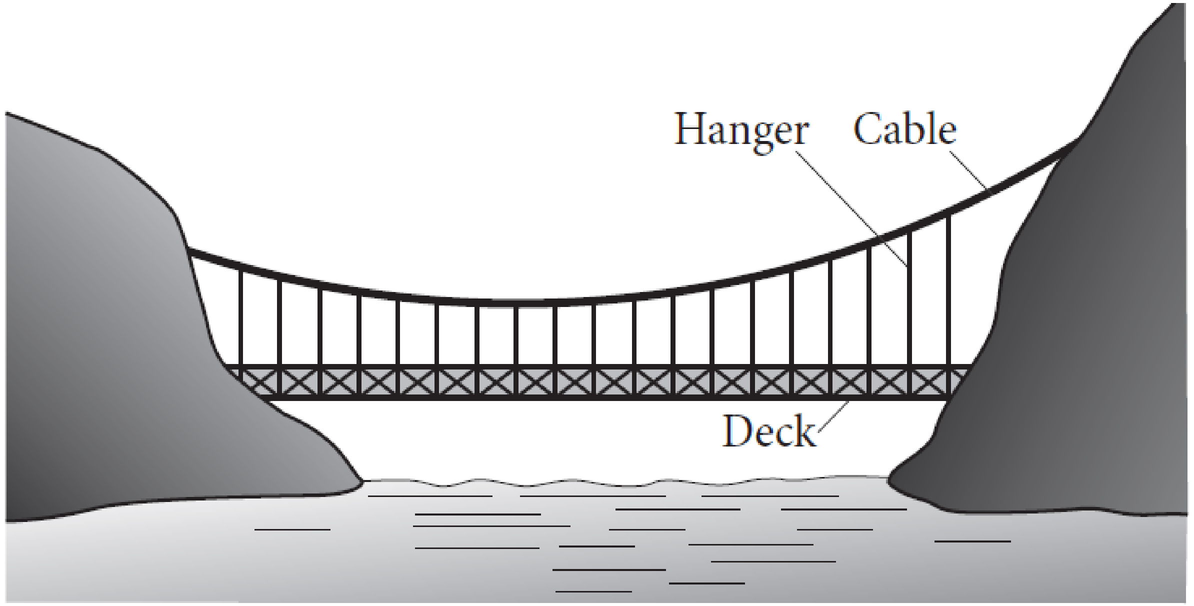
Determining the Shape of a Suspension Bridge Cable

Aim

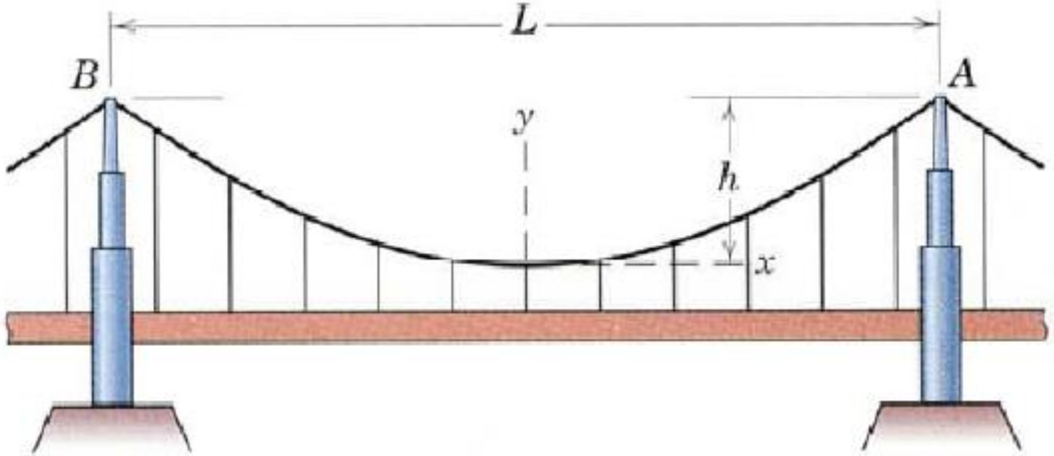
- A Suspension Bridge consists of the main cable, the hangers, and the deck as shown in the figure.



- The self-weight of the deck and the loads applied on the deck are transferred to the cable through the hangers.
- The purpose of this experiment is to determine the shape of the cable subject to different load functions and tension in the cable.

Mathematical Modelling

- We consider a main cable to be inextensible, hanging between the two fixed ends A and B.



- The vertical distance between the highest and the lowest points of the cable is called Sag, while the horizontal distance between two supports A and B is called Span.
- We assume that the cable is of length S and of constant self-weight per unit length $W = mg$, where m is the mass per unit length of the cable, and g is the acceleration due to gravity.

Determination of the static equilibrium shape of a cable can, in some cases, be simplified if the downwardly directed external loads, which are approximated as point forces in Fig. a, can be further approximated by a continuously distributed load, as shown in Fig. b.

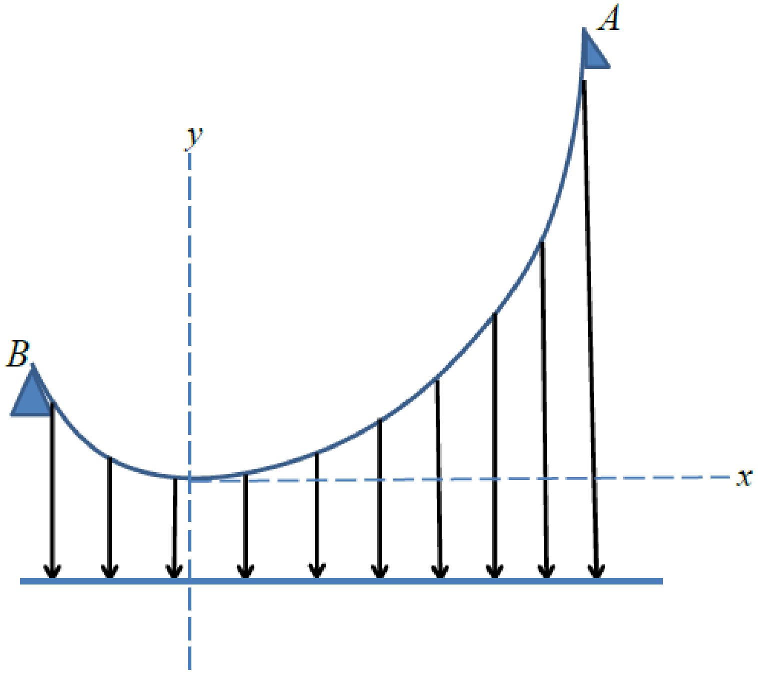


Fig a. Downwardly directed external loads.

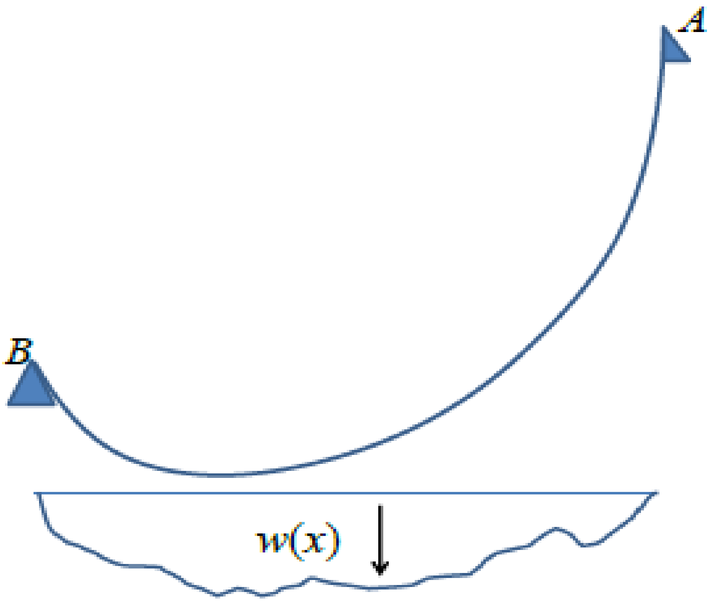
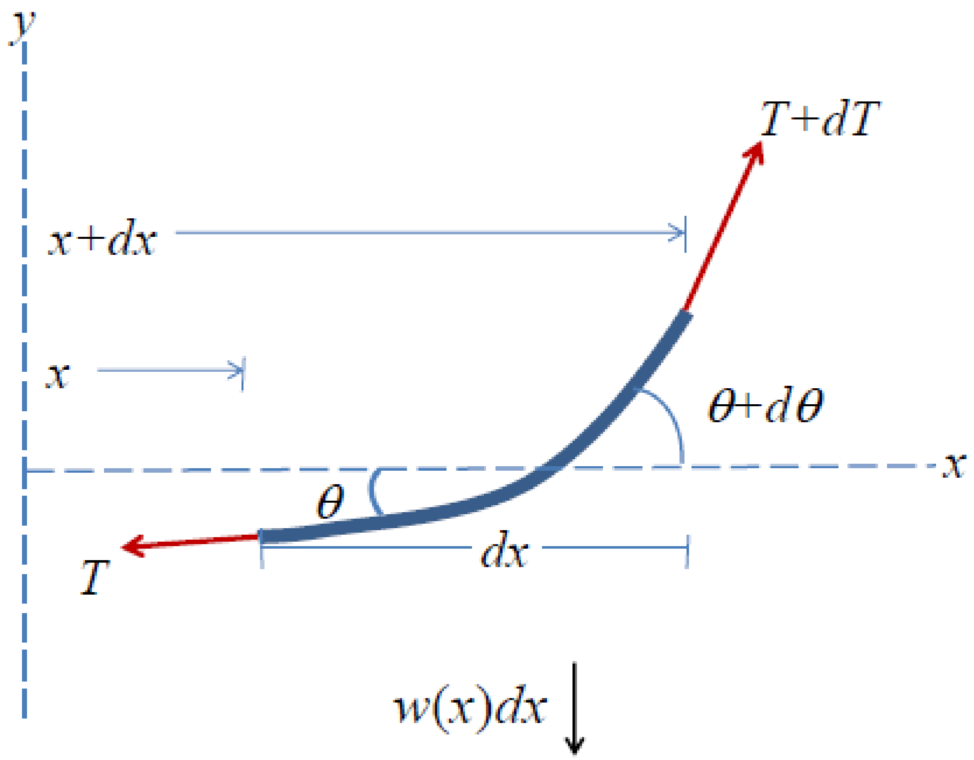


Fig b. Continuously distributed load

- We assume that the cable is loaded by the distributed vertical external load $w(x)$. Now to derive the equation of the cable, we consider a cable element as shown in the Fig. 3 below:



- From the figure, it is clear that the elongations in the vertical direction are

$$y_1 = T\sin\theta \text{ and } y_2 = (T + dT)\sin(\theta + d\theta).$$

- For static equilibrium, the sum of the forces must be equal to zero. That is, the sum of the vertical forces $\sum F_y = 0$.

$$\Rightarrow -y_1 + y_2 - w(x)dx = 0$$

$$\Rightarrow (T + dT)\sin(\theta + d\theta) = T\sin\theta + w(x)dx \qquad \textbf{(1)}$$

The sum of the horizontal forces $\sum F_x = 0 \Rightarrow (T + dT)\cos(\theta + d\theta) = T\cos\theta$. **(2)**

Assuming that $d\theta \longrightarrow 0$, we have $sin\ d\theta\ \backsimeq d\theta$, and $cos\ d\theta = 1$. Thus Eq. (1) and (2) read as

$$(T + dT)\,[\sin\theta + \cos\theta\, d\theta] = T\sin\theta + w(x)dx \qquad \textbf{(3)}$$

$$(T + dT)\,[\cos\theta\ - \sin\theta\, d\theta] = T\cos\ \theta \qquad \textbf{(4)}$$

- On simplification, we get

$$dT\sin\theta + T\cos\theta\, d\theta + dT\cos\theta\, d\theta = w(x)dx$$

$$dT\ \cos\theta\ - T\sin\theta\, d\theta - dT\sin\theta\, d\theta = 0$$

- Ignoring the second order terms $dTd\theta$ in these equations, we get

$$dT\sin\theta + T\cos\theta\, d\theta = w(x)dx \qquad \textbf{(5)}$$

$$dT\ \cos\theta\ - T\sin\theta\, d\theta = 0 \qquad \textbf{(6)}$$

- Equations (5) and (6) can be written as

$$d(T\sin\theta) = w(x)dx \qquad \textbf{(7)}$$

$$d(T\cos\theta) = 0 \ \Rightarrow T\cos\theta = T_H \quad (a\ constant) \ \Longrightarrow T = \frac{T_H}{\cos\theta}. \qquad \textbf{(8)}$$

- Substituting the value of in (7) we get

$$d(T_H\tan\theta) = w(x)dx$$

- But we know that $\tan\theta = \frac{dy}{dx}$. Hence we have

$$\frac{d^2y}{dx^2} = \frac{w(x)}{T_H}.$$

This is the differential equation for a flexible cable.

Solution by method of variation of parameters:

Method of variation of parameters enables to find solution of any linear non homogeneous differential equation of second order, provided its complimentary function (C.F.) is given / known. The particular integral of the non-homogeneous equation is obtained by varying the parameters, i.e. by replacing the arbitrary constants in the C.F. by variable functions.

Consider a linear non-homogeneous second order differential equation with constant coefficients

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x), \text{ where } p,\ q \text{ are constants.} \qquad (13)$$

Let the complimentary function be of the form $y_c = C_1 y_1(x) + C_2 y_2(x)$, where $C_1,\ C_2$ are arbitrary constants. This is the solution of the homogeneous equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0.$$

In the method of variation of parameters, the arbitrary constants C_1, C_2 are replaced with two unknown functions $u(x)$ and $v(x)$.

Let us assume that the particular integral is of the form $y_p = u(x)y_1(x) + v(x)y_2(x)$ (14)

where $u(x) = -\int \frac{y_2(x)f(x)}{y_1 y_2' - y_1' y_2} dx$ and $v(x) = \int \frac{y_1(x)f(x)}{y_1 y_2' - y_1' y_2} dx$.

On putting the values of $u(x)$ and $v(x)$ in (14), we get the particular integral y_p.

Hence the required solution $y(x) = y_c + y_p$.

MATLAB CODE

```
% Program for solving differential equation of the form
% ay"+by'+cy=f(x), for a, b and c as constants.
clear all
close all
clc
syms A B x m
p=input('Enter the coefficients a,b,c');
f=input('Enter the RHS function f(x)');
a=p(1);b=p(2);c=p(3);
disc=b^2-4*a*c;
m=subs(solve('a*m^2+b*m+c') );
if(disc>0)
    CF= A*exp(m(1)*x)+B*exp(m(2)*x);
    u=exp(m(1)*x);v=exp(m(2)*x);
elseif (disc==0)
    CF=(A+B*x)*exp(m(1)*x);
        u=exp(m(1)*x);v=x*exp(m(1)*x);
else
    alfa=real(m(1));
    beta=imag(m(1));
    CF=exp(alfa*x)*(A*cos(beta*x)+B*sin(beta*x));
        u=exp(alfa*x)*cos(beta*x);v=exp(alfa*x)*sin(beta*x);
end
```

```
% Method of variation of parameters.
f1=f/a;
jac=u*diff(v,x)-diff(u,x)*v;  %Jacobian of u and v
P=int(-v*f1/jac,x);
Q=int(u*f1/jac,x);
PI=P*u+Q*v;
y_gen=CF+PI;
dy_gen=diff(y_gen);
cond=input('Enter the initial conditions x0, y(x0) and Dy(x0)');
eq1=(subs(y_gen,x,cond(1))-cond(2));
eq2=(subs(dy_gen,x,cond(1))-cond(3));
[A B]=solve(eq1,eq2);
y=subs(CF+PI)
```

Example 1: Solve the equation $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = \sin 3x$, $y(0) = 0$, $y'(0) = 1$.

MATLAB input

Enter the coefficients a,b,c [1 -5 6]
Enter the RHS function f(x) sin(3*x)
Enter the initial conditions x0, y(x0) and Dy(x0) [0 0 1]

MATLAB output

y =
(5*cos(3*x))/78 - (16*exp(2*x))/13 + (7*exp(3*x))/6 - sin(3*x)/78

Example 2: Consider the problem of suspension cable $\frac{d^2y}{dx^2}=\frac{w(x)}{T_H}$

with the conditions $y(0)=0$, $\left(\frac{dy}{dx}\right)_{x=0}=0$.

MATLAB code

```
%Program for differential equation of a suspension cable
clear all
close all
clc
syms A B x m
W=input('Enter the external load:  ');
T=input('Enter the horizontal tension:  ');
f=W/T;
a=1;b=0;c=0;
```

```
disc=b^2-4*a*c;
m=subs(solve('a*m^2+b*m+c') );
if(disc>0)
    CF= A*exp(m(1)*x)+B*exp(m(2)*x);
    u=exp(m(1)*x);v=exp(m(2)*x);
elseif (disc==0)
    CF=(A+B*x)*exp(m(1)*x);
        u=exp(m(1)*x);v=x*exp(m(1)*x);
else
    alfa=real(m(1));
    beta=imag(m(1));
    CF=exp(alfa*x)*(A*cos(beta*x)+B*sin(beta*x));
        u=exp(alfa*x)*cos(beta*x);v=exp(alfa*x)*sin(beta*x);
end
```

```
% Method of variation of parameters.
f1=f/a;
jac=u*diff(v,x)-diff(u,x)*v;  %Jacobian of u and v
P=int(-v*f1/jac,x);
Q=int(u*f1/jac,x);
PI=P*u+Q*v;
y_gen=CF+PI;
dy_gen=diff(y_gen);
cond=[0 0 0];
eq1=(subs(y_gen,x,cond(1))-cond(2));
eq2=(subs(dy_gen,x,cond(1))-cond(3));
A=solve(eq1);
B=solve(eq2);
y=subs(CF+PI)
```

MATLAB input
Enter the external load: 1
Enter the horizontal tension: 1

MATLAB output
y =
x^2/2

MATLAB input
Enter the external load: x
Enter the horizontal tension: 1

MATLAB output
y =
x^3/6

Exercise problems:

1. Solve the equation $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = \sin 2x, \quad y(0) = \dfrac{1}{8}, \; y'(0) = 4.$

2. Solve the equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = x, \quad y(0) = 0, \; y'(0) = 1.$