Experiment No. 9: POPULATION DYNAMICS

Aim:

Analyzing and solving Difference equation for the population growth.

Motivation:

Differential equations arise while modeling physical phenomena in which the independent variable or space variable or both are continuous. Difference equation are the discrete analogues of Differential equation. For example

i) In an experiment we may take measurements on some physical variable, say, temperature at equally spaced time intervals.
ii) We may be interested in population growth of a certain species at discrete time intervals or
iii) We may like to approximate differential equations by writing them in difference form.

Infact difference equations are essential for systems with discrete or digital data.

Problem Statement:

We divide this experiment in two parts

i) Formulation of population growth (Human population) by first order difference equation.
ii) Formulation of population growth of rabbits in terms of second order difference equation (through Fibonacci numbers).

Modeling of Human Population growth:

In adopting a growth model to human populations, we must take into account births as well as deaths. A plausible assumptions is that births and deaths both occur at a rate which is proportional to the size of N of the population at any time t.

The governing equation of Human population can be written in Difference form as

$$\frac{dN}{dt} = \frac{N_{n+1} - N_n}{\Delta t} = \left(\alpha N_n - \beta N_n\right) = kN_n \qquad \text{where } k = \left(\alpha - \beta\right), \Delta t = 1$$

$$N_{n+1} = N_n(1+k) \quad \textbf{(1)}$$

Solution: This is a first order difference equation. It can be easily solved:

$$N_1 = (1+k)\,N_0, \qquad N_2 = (1+k)\,N_1 = (1+k)^2\,N_0, \ldots \qquad N_n = (1+k)^n\,N_0.$$

Where N_0 represents the initial population

A general method for solving such difference equations may now be explained. Consider the same equation $N_{n+1} = (1+k)N_n$.

We assume $N_n = a^n$. Substituting into the given equation, we get

$$a^{n+1} = (1+k)a^n \Rightarrow a = (1+k)$$

Thus, the solution satisfying the condition $N(0) = N_0$ is $N_n = N_0(1+k)^n$. **(2)**

Differential form: Note: (Above problem is now discussed in Differential form)

$$\frac{dN}{dt} = \alpha N - \beta N = (\alpha - \beta)N = kN\,,\ k>0 \quad \textbf{(3)}$$

Where α and β are positive constants denoting the average rate of births and deaths.

This simple equation was proposed in 1798 by the English economist Thomas Mathews as a basic model for population growth. If $N(t_0) = N_0$, we have

$$N(t) = N_0\,e^{k(t-t_0)} \quad \textbf{(4)}$$

Note: If the population doubles

$$2N_0 = N_0 e^{k(t-t_0)} \Rightarrow \frac{\ln 2}{k} = t - t_0 = d \text{ (say)}$$

The population doubles every d years; it does not depend on the initial value. Therefore,

t_0	t_0+d	t_0+2d	t_0+3d	. . .
N_0	$2N_0$	$4N_0$	$8N_0$	. . .

Population growth for every d years given by $N_0(2^0+2^1+2^2+2^3+\ldots)$

Thus, the population grows in geometric progression as the time changes in arithmetic progression.

Example: Census data gathered over last 200 years in a certain country gave the following data:

Year	Population
1801	16345646
1851	27533755
1901	41609091

Find the proportionality constant k. Predict the population for 1901.

Solution by solving Difference equation: The solution of **(1)** satisfying the condition $N(0) = N_0$, can be written as

$$N_n = N_0 (1+k)^n = N_0 (a)^n$$

From the given data, $N(1801) = N_0 = 16345646$; $N(1851) = N_1 = 27533755$ and n=100

Therefore from (2), we have, a=1.01. The population after 100 years is <u>44 million</u>.

Solution by solving Differential equation: The solution of **(3)** satisfying the condition $N(0) = N_0$, can be written as: $N(1851) = N(1801)\,e^{50k}$

$$k = \frac{\ln(27533755) - \ln(16345646)}{50} \approx 0.010$$

Our model predicts, by equation ( ), N (1901) =N (1801) exp (100k) $\approx$ <u>46 million</u>

<u>Second Order Difference Equation</u>

<u>Fibonacci Numbers</u>

The original problem that Fibonacci investigated was about how fast rabbits could breed in ideal circumstances. It is not only rabbits but also cows, bees and birds and perhaps several other animals and insects breed the same way. It follows a particular pattern:

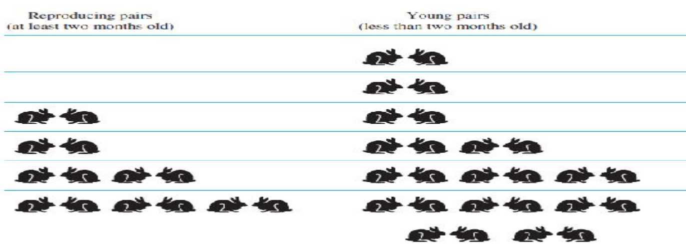
In case of Rabbits

t=0	1 month	2	3	. . .
1 pair	1 pair	2	3	. . .

If we change months into years, we get how cow pair population changes.

<u>Example: Rabbit Population</u>

Consider this problem, which was originally, posed by Leonardo Pisano, also known as Fibonacci, in the thirteenth century in his book Liber abaci. A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month, as shown in Figure 1. Find a difference equation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

FIGURE 1 Rabbits on an Island.

Solution: Denote by f_n the number of pairs of rabbits after n months. We will show that f_n, n = 1, 2, 3,..., are the terms of the Fibonacci sequence.

The rabbit population can be modeled using a recurrence relation. At the end of the first month, the number of pairs of rabbits on the island is $f_1 = 1$. Because this pair does not breed during the second month, $f_2 = 1$ also. To find the number of pairs after n months, add the number on the island the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least 2 months old. Consequently, the sequence $\{f_n\}$ satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ together with the initial conditions $f_1 = 1$ and $f_2 = 1$. Because this recurrence relation and the initial conditions uniquely determine this sequence, the number of pairs of rabbits on the island after n months is given by the nth Fibonacci number.

The Rabbit population can be predicted by solving the following second order Fibonacci difference equation

$$F_n = F_{n-1} + F_{n-2} \quad \text{(Or)} \quad F_{n+2} = F_n + F_{n+1} \quad \text{with } F_0 = 0, F_1 = 1 \tag{5}$$

The complete solution of the Fibonacci equation is: $$F_n = \frac{1}{2^n\sqrt{5}}\left[\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right] \tag{6}$$

Note: (Petals on a flower)

On many plants, the number of petals is a Fibonacci number. Leaf arrangement like branches on a tree-the sun flower, vegetable like Cauli flowers follows Fibonacci sequence. In fact, the Fibonacci numbers are natural numbers.

Note: (Golden Number)

Fibonacci numbers and the golden number: 1,1 2,3,5,8,13,…

If we take the ratio of two successive numbers in Fibonacci series, we obtain the following series of numbers.

1/1=1; 2/1=2; 3/2=1.5; 5/3=1.666; 8/5=1.6,…

The ratio seems to be settling down to a particular value, which we call the golden ratio or the golden number. It has a value approximately equal to 1.618034.

We shall solve the Fibonacci difference equation using ***MATLAB****in two ways.*

First method: Here, we solve the Fibonacci equation ***using z-transform***

Second method: Here, we have written the general code for finding the solution of any second order difference equation.

MATLAB CODE: {for finding solution of Fibonacci Difference equation using z-transform }

```
clc
clear all
syms z Y n positive
LHS=ztrans(sym('y(n+2)')-sym('y(n+1)')-sym('y(n)'),n,z);
RHS=ztrans(0,n,z)
newLHS=subs(LHS,{'ztrans(y(n),n,z)','y(0)','y(1)'},{Y,0,1});
Y=solve(newLHS-RHS,Y);
y=iztrans(Y,z,n)
```

OUTPUT:

```
(5^(1/2)/5)*(5^(1/2)/2 + 1/2)^n + (-5^(1/2)/5)*(1/2 - 5^(1/2)/2)^n
```

MATLAB CODE: {for finding solution of linear second order Non-homogenous Difference equation}

```
clc
clear all
symsnk1k2m
assume(n,'integer')
a = input('Enter the coefficient of y(n+2): ');
b = input('Enter the coefficient of y(n+1): ');
c = input('Enter the coefficient of y(n): ');
g = input('Enter the non-homogeneous part: ');
r = subs(solve(a*m^2+b*m+c,m));
ifimag(r)~=0
rho = sqrt(real(r(1))^2 + imag(r(1))^2);
theta = atan(abs(imag(r(1)))/real(r(1)));
    y1 = (rho^n)*cos(n*theta);
    y2 = (rho^n)*sin(n*theta);
elseif r(1)==r(2)
    y1 = r(1)^n;
    y2 = n*r(1)^n;
else
    y1 = r(1)^n;
    y2 = r(2)^n;
end
Co = det([y1, y2;subs(y1,n,n+1), subs(y2,n,n+1)]); %Casoratian of the solutions
y_c = k1*y1 + k2*y2;

disp('Complementary Solution is: ');
disp(y_c);
if(g ~= 0)
    y11 = subs(y1,n,n+1);
    y21 = subs(y2,n,n+1);
    Co1 = subs(Co,n,n+1);
    u1 = simplify(symsum(-g*y21/Co1,n))
    u2 = simplify(symsum(g*y11/Co1,n))
```

```
y_p = simplify(u1*y1+u2*y2);
    y = y_c + y_p;
else
    y = y_c;
end
check = input('If the given problem has initial conditions then enter 1 else enter 0: ');
if (check == 1)
    yval1 = input('Enter the initial condition at n = 0: ');
    yval2 = input('Enter the initial condition at n = 1: ');
    cond1 = strcat(char(subs(y,n,0)),'=',num2str(yval1));
    cond2 = strcat(char(subs(y,n,1)),'=',num2str(yval2));
    [k1,k2] = solve(cond1,cond2);
    y = subs(y);
end
disp('Complete Solution is: ')
disp(collect(collect(y,y1),y2))

if(check ~= 0)
nrange = 0:10;
    Y = subs(y,n,nrange);
    stem(nrange,Y);
    set(gca,'XTick',linspace(0,10,11))
xlabel('n');
ylabel('y(n)');
end
```

Fibonacci Difference equation OUTPUT:

```
Enter the coefficient of y(n+2): 1
Enter the coefficient of y(n+1): -1
Enter the coefficient of y(n): -1
Enter the non-homogeneous part: 0
```

Complementary Solution is:

k1*(1/2 - 5^(1/2)/2)^n + k2*(5^(1/2)/2 + 1/2)^n

If the given problem has initial conditions then enter 1 else enter 0: 1

Enter the initial condition at n = 0: 0
Enter the initial condition at n = 1: 1

Complete Solution is:

(5^(1/2)/5)*(5^(1/2)/2 + 1/2)^n + (-5^(1/2)/5)*(1/2 - 5^(1/2)/2)^n

Fibonacci sequence