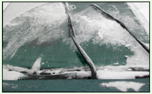
Experiment 4

Functionality of Windshield Wipers

MAT2002–Lab
Application of Differential and Difference Equations



Aim

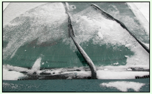
- Modeling and analyzing the working mechanism of the **windshield wipers** by solving governing equations

- To solve and visualize the solutions of **linear differential equations** using **dsolve** function

Functionality of Windshield Wipers

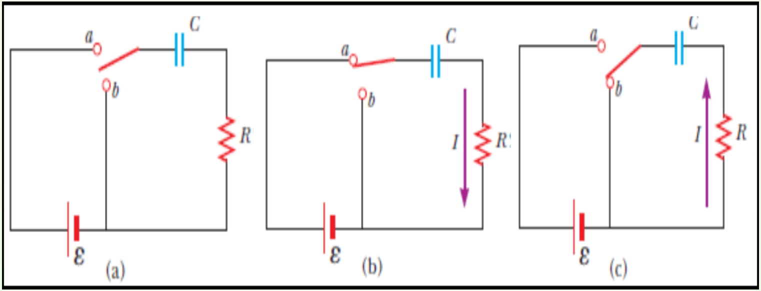
Many automobiles are equipped with windshield wipers that can operate intermittently during a light rainfall. **The operation of such wipers depends on the charging and discharging of a capacitor**. The wipers are part of an RC-circuit whose time constant can be varied by selecting different values of R (resistor) through a multi position switch. As the voltage across the capacitor increases, the capacitor reaches a point at which it discharges and triggers the wipers. The circuit then begins another charging cycle. The time interval between the individual sweeps of the wipers is determined by the value of the time constant. We divide this experiment in two parts

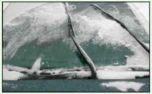
- Charging a Capacitor in an RC-Circuit
- Discharging a Capacitor in an RC-Circuit



Charging a Capacitor in an RC Circuit

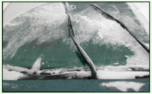
The following figures show a simple series RC circuit.





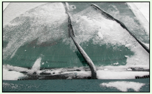
Charging a Capacitor in an RC Circuit

- Assume the capacitor in this circuit is initially uncharged. There is no current while the switch is open in figure (a).
- If the switch is thrown to position a at $t = 0$ as shown in figure (b), however, charge begins to flow, setting up a current in the circuit, and the capacitor begins to charge.
- During charging, charges do not jump across the capacitor plates because the gap between the plates represents an open circuit. Instead, charge is transferred between each plate and its connecting wires due to the electric field established in the wires by the battery until the capacitor is fully charged.



Charging a Capacitor in an RC Circuit

- As the plates are being charged, the potential difference across the capacitor increases. The value of the maximum charge on the plates depends on the voltage of the battery.
- Once the maximum charge is reached, the current in the circuit is zero because the potential difference across the capacitor matches that supplied by the battery.

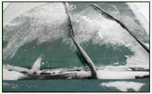


Charging a Capacitor in an RC Circuit

To analyse this circuit quantitatively, apply Kirchhoff's loop rule to the circuit after the switch is thrown to position . Traversing the loop in Figure (b) clockwise gives

$$\epsilon - \frac{q}{C} - IR = 0 \tag{1}$$

where $\frac{q}{C}$ is the potential difference across the capacitor and is the potential difference across the resistor. The capacitor is traversed in the direction from the positive plate to the negative plate, which represents a decrease in potential. Therefore, we used the negative sign for the potential difference.



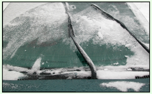
Charging a Capacitor in an RC Circuit

- Using equation (1) we can find the initial current in the circuit and the maximum charge on the capacitor.
- At the instant the switch is thrown to position $a (t = 0)$ the charge on the capacitor is zero.
- The initial current is a maximum and is given by $I_i = \frac{\epsilon}{R}$.
- At this time, the potential difference from the battery terminals appears entirely across the resistor.
- when the capacitor is charged to its maximum value Q, charges cease to flow, the current in the circuit is zero and the potential difference from the battery terminals appears entirely across the capacitor.
- Substituting $I = 0$ into the equation (1) gives the maximum charge on the capacitor $Q = C\epsilon$.

Charging a Capacitor in an RC Circuit

To determine analytical expressions for the time dependence of the charge and current, let us substitute $I = \frac{dq}{dt}$ and rearrange the equation (1), we will get

$$\frac{dq}{dt} = \frac{\epsilon}{R} - \frac{q}{RC} \qquad (2)$$



Charging a Capacitor in an RC Circuit

Solving the linear differential equation using variable separable method, we will get an expression for q as

$$q(t) = C\epsilon(1 - e^{-\frac{t}{RC}}) = Q(1 - e^{-\frac{t}{RC}}) \tag{3}$$

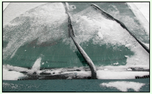
The above equation denotes the charge as a function of time for a capacitor being charged. We can find an expression for the charging current by differentiating equation (3) with respect to time. We get

$$I(t) = \frac{\epsilon}{R} e^{-\frac{t}{RC}} \tag{4}$$

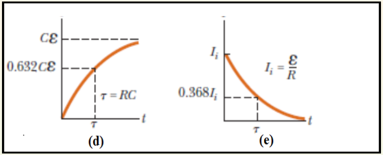
The above equation denotes the current as a function of time for a capacitor being charged. The quantity RC, which appears in the exponents of equations (3) and (4) is called **time constant** τ of the circuit i.e., $\tau = RC$.

Charging a Capacitor in an RC Circuit

The time constant represents the time interval during which the current decreases to $\frac{1}{e}$ of its initial value; that is after a time interva τ, the current decreases to $I = e^{-1}I_i = 0.3681I_i$. After a time interval 2τ, the current decreases to $I = e^{-2}I_i = 0.135I_i$ and so forth. Likewise in a time interval τ, the charge increases from zero to $C\epsilon[1-e^{-1}] = 0.632C\epsilon$.



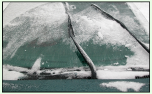
Charging a Capacitor in an RC Circuit



- Figure (d): Plot of capacitor charge versus time for the above circuit. After a time interval equal to one time constant has passed, the charge is 63.2% of the maximum value $C\epsilon$. The charge approaches its maximum value as t approaches infinity.
- Figure (e): Plot of current versus time for the above circuit. The current has its maximum value $I_i = \frac{\epsilon}{R}$ at $t = 0$ and decays to zero exponentially as t approaches infinity. After a time interval equal to one time constant τ has passed, the current is 36.8% initial value.

Dischararging a Capacitor in an RC Circuit

- If the capacitor in figure (b) is completely charged then a potential difference will exist across the capacitor and there will be zero potential difference across the resistor because $I = 0$.
- If the switch is now thrown to position b at $t = 0$ as shown in figure (c), the capacitor begins to discharge through the resistor.
- At some time during the discharge, the current in the circuit is I and the charge on the capacitor is q.
- The circuit in figure (c) is the same as the circuit in figure (b) except for the absence of the battery.



Dischararging a Capacitor in an RC Circuit

We eliminate the $e.m.f \epsilon$ from equation (1), we obtain the appropriate loop equation for the circuit.

$$-\frac{q}{C} - IR = 0 \tag{5}$$

Then $-R\frac{dq}{dt} = \frac{q}{C}$ and solving the differential equation, we will get the charge as a function of time for a discharging capacitor.

$$q(t) = Q(e^{-\frac{t}{RC}}) \tag{6}$$

Differentiating the equation (6), we will get the current as a function of time for a discharging capacitor

$$I(t) = -\frac{Q}{RC}e^{-\frac{t}{RC}} \tag{7}$$

where $I_i = \frac{Q}{RC}$ is the initial current. The negative sign indicates that as the capacitor discharges, the current direction is opposite its direction when the capacitor was being charged.

Linear Differential Equation

The general **linear differntial equation** of the nth order is of the form

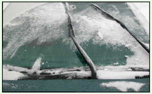
$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + p_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_n y = X$$

where $p_1, p_2, \cdots, p_n$ and X are function of x.

Linear differntial differential equation with constant co-efficient are of the form

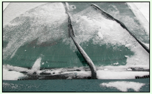
$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + k_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + k_n y = X$$

where $k_1, k_2, \cdots, k_n$ are constants.



MATLAB syntax used

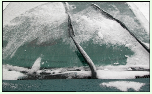
- **dsolve**:*dsolve(eqn,cond)* solves the ordinary differential equation eqn with the initial or boundary condition cond.
- **ezplot**:*ezplot(f)* plots the expression $f(x)$ over the default domain $-2\pi < x < 2\pi$, where $f(x)$ is not an implicit function of only one variable.



Example

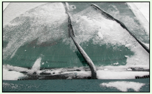
Solve the following differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}, \left.\frac{dy}{dt}\right]_{t=0} = 0 \text{ and } y(0) = 1$$



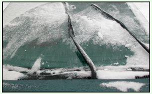
Solution

- To find $C.F.$
  Its $A.E.$ is $(D+2)^2 = 0$ where $D = -2, -2$
  $C.F. = (C_1 + C_2 t)e^{-2t}$
- To find $P.I.$
  $P.I. = \frac{1}{(D+2)^2)} e^{-t} = e^{-t}$
- To find $C.S.$
  $y(t) = C.F. + P.I. = (C_1 + C_2 t)e^{-2t} + e^{-t}$
- By using initial conditions
  $y(t) = e^{-2t}(t + e^t)$



Solution by Matlab Code

```
clc
clear all
close all
eqn=input(‘Enter the equation: ’);
inits=input(‘Enter the conditions: ’);
y=dsolve(eqn,inits, ‘t ’);
soln=[‘y(t)=’, char(simplify(y))];
disp(soln)
ezplot(y)
```



Input and Output of the Matlab Code

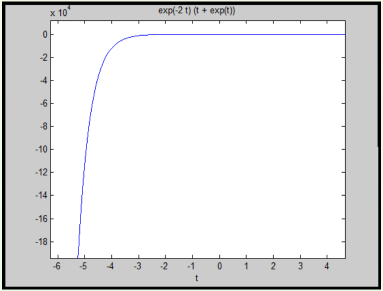
Input:

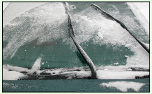
Enter the equation: 'D2y+4*Dy+4*y=exp(-t)'

Enter the conditions: 'y(0)=1,Dy(0)=0'

Output:

y(t)=exp(-2*t)*(t + exp(t))

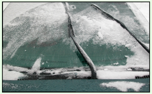




Exercise

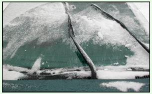
Solve the following differential equations by Matlab

- $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 10y = 0, \left.\frac{dy}{dt}\right]_{t=0} = 1$ and $y(0) = 4$
- $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 3\sin t + 4\cos t, \left.\frac{dy}{dt}\right]_{t=0} = 0$ and $y(0) = 1$
- $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 0, \left.\frac{dy}{dt}\right]_{t=0} = 2$ and $y(0) = 2$



Exercise

An uncharged capacitor and a resistor are connected in series to a battery as shown in the circuit (a), where $\epsilon = 12.0V$, $C = 5.00\mu F$ and $R = 8.00 \times 10^5 \Omega$. The switch is thrown to position a. Using MatLab find the time constant of the circuit, the maximum charge on the capacitor and the maximum current in the circuit. Plot the charge and the current in the circuit as functions of time.



Solution

Study of the above circuit and imagine throwing the switch to position a as shown in circuit (b).
The time constant of the circuit $\tau = RC = 4s$
The maximum charge on the capacitor $Q = C\epsilon = 60\mu C$
The maximum current in the circuit $I_i = \frac{\epsilon}{R} = 15\mu A$
The charge as a function of time is $q(t) = C\,\epsilon(1\text{-e}^{-\frac{t}{RC}}) = Q(1-e^{-\frac{t}{RC}}) = 60(1-e^{-\frac{t}{4}})$
The current as a function of time is $I(t) = \frac{\epsilon}{R}e^{-\frac{t}{RC}} = 15e^{-\frac{t}{4}}$