EXPERIMENT - VI

Lateral Vibration of Hanging Rope

Aim: To find the power series solution of the Bessel's equation of order zero by the method of Frobenius and visualize it using MATLAB.

Series Solution of Differential Equations

•Many differential equations arising from physical problems are linear with variable co-efficient.

•A general solution in terms of known function does not exist for these types of equations.

•Such equations can be solved by finding the solution in the form of an infinite convergent series.

Basic Definition- Singular Point

Consider the Differential Equation of the form,

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \qquad (1)$$

If $P_0(a) \neq 0$, then $x = a$ is called an *ordinary point* of (1), otherwise *singular point*.

Basic Definition - Regular Singular Point

A singular point $x = a$ of (1) is called *regular* if (1) is expressed in the form

$$\frac{d^2y}{dx^2} + \frac{Q_1(x)}{x-a}\frac{dy}{dx} + \frac{Q_2(x)}{(x-a)^2}y = 0$$

Where $Q_1(x)$ *and* $Q_2(x)$ possess derivatives of all orders in the neighbourhood of ‘*a*’.

- A singular point which is not regular is called an *irregular singular point.*
- *When* $x = a$ *is a regular singular point of (1), then it can be solved using the method of Frobenius.*

Frobenius Method

- Let $b(x)$ and $c(x)$ be any functions that are analytic at $x=0$ ('x' is regular singular point). Then the ODE

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 ------(2)$$

has at least one solution that can be represented in the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m ------(3)$$

where $a_0 \neq 0$, the exponent $'r'$ may be real or complex.

- Bessel's equation is given by,

$$y'' + \frac{1}{x} y' + \left(\frac{x^2 - n^2}{x^2} \right) y = 0 - - - - -(4)$$

And eq.(4) is identical to eq. (2) with $b(x) = 1$ and $c(x) = x^2 - n^2$ analytic at $x = 0.$

Steps involved in the Frobenius method for solving (2).

Step : 1

Multiplying equation (2) by x^2,

$$x^2 y'' + xb(x)y' + c(x)y = 0 ------(4)$$

Equation (3) can be written as

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} ------(5)$$

Step : 2

Taking derivative of equation (5)

$$y'(x) = \sum_{m=0}^{\infty} a_m (m+r) x^{m+r-1}$$

$$y''(x) = \sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r-2}$$

Substituting these values in equation (4) and equating the sum of coefficients of each power $x^r, x^{r+1}, x^{r+2},\ldots$ to zero, we obtain a system of equations involving the unknown coefficients a_m.

The corresponding equation is ,

$$\left[r(r-1)+b_0 r+c_0\right]a_0=0 \quad ------(6)$$

since $a_0 \neq 0$,

$$r(r-1)+b_0 r+c_0=0 \quad ------(7)$$

called as ***Indicial equation.***

Basis of Solutions

Case : 1 Distinct roots (r_1 and r_2) not differing by an integer. Basis is,

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \ldots\right) \quad \text{and}$$
$$y_2(x) = x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \ldots\right)$$

Case : 2 Double root (r_1=r_2=r, equal roots). A basis is

$$y_1(x) = x^{r}\left(a_0 + a_1 x + a_2 x^2 + \ldots\right) \quad \text{and}$$
$$y_2(x) = y_1(x)\ \ln x + x^{r}\left(A_1 x + A_2 x^2 + \ldots\right) \quad (x > 0)$$

Case : 3 Roots differing by an integer. A basis is

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \ldots\right) \quad \text{and}$$
$$y_2(x) = k y_1(x)\ \ln x + x^{r_2}\left(A_1 x + A_2 x^2 + \ldots\right)$$

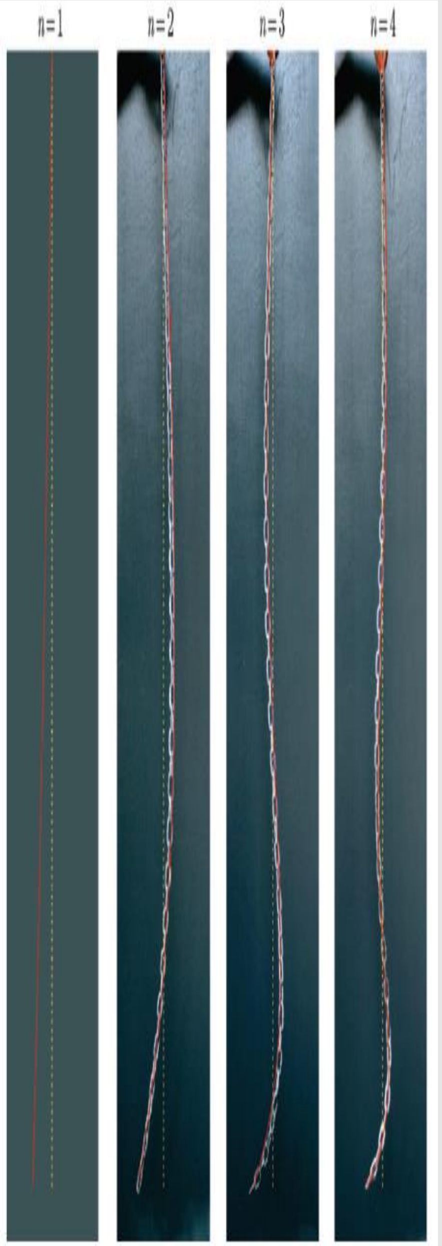
, where the roots are so denoted that r_1-r_2>0 and k may turn out to be zero.

Problem statement

•A flexible uniform chain/rope/cable of length L and constant linear density (mass/unit length) of ρ gm/cm is fixed at the upper end ($x = L$). The x-axis is vertical, measured up from the equilibrium position of the free end of the chain.

•$u(x,t)$ represents the displacement function for a point x on the chain.

•The displacements are small compared with the length of the chain, so that the displacement can be neglected.



- The tension in the chain is due to the weight below point x, ***w(x) = ρgx,*** and the difference in the horizontal components of the tension at the ends of a small interval Δx of chain is the accelerating force.

- For any displacement of angle α, the restoring force is

$$F(x) = W \sin\alpha \sim Wu_x$$

- The difference in force between points on the change at x and $x + \Delta x$ is thus $\Delta F = \Delta x\,(Wu_x)_x$

- From Newton's 2nd law, $f = ma = mu_{tt}$,

$$\Delta x\,\rho g\,[x\,u_x]_x = \rho \Delta x\,u_{tt} \quad \textbf{Or} \quad u_{tt} = g(u_x + x u_{xx})$$

The Vertical Solution

- Separate the variables with the solution function of the form

$$u(x,t) = F(x)G(t) \text{------------} (A)$$

$$u_{tt} = F(x)\, G''(t),\ u_x = F'(x)\, G(t),\ u_{xx} = F''(x)\, G(t)$$

using (A), we get,

$$\frac{G''(t)}{G(t)} = \frac{gxF(x) + gF'(x)}{F(x)}$$

•The time function is thus just a cosine function of the form $G(t) = \cos(\omega t + \varphi)$.

• The angular frequency *'ω'* has units of $time^{-1}$.

• The chain will probably begin its oscillation at maximum displacement (initial velocity = 0), so that the phase angle *'φ'* will be 0.

•The governing equation is therefore given by,

$$xF''(x) + F'(x) + \frac{\omega^2}{g} F(x) = 0 \quad ----(8)$$

•This ODE represents the Bessel's equation, which has a typical form:

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$$

Where '*p*' is the order of the Bessel function (in our case $p = 0$).

<u>Note</u>:

Eq. (8) has equal roots and hence one have to use the **case:2** type of solution.

MATLAB Commands

coeffs(P, var)	returns coefficients of the polynomial 'P' with respect to the variable 'var'
collect(P, var)	rewrites 'P' in terms of the powers of the variable 'var'
n=numel(A)	returns the number of elements 'n', in array 'A' , equivalent to prod(size(A)).
simplify(S)	performs an algebraic simplification of S.
J = besselj(nu,Z)	computes the Bessel function of the first kind, where 'nu' is order and 'Z' is an argument
Y = bessely(nu,Z)	computes Bessel function of the second kind, where 'nu' represents order and 'Z' is an argument

MATLAB Code

```
clc
clear all
syms x a0 a1 a2 a3 a4 m c1 c2
y=a0*x^m+a1*x^(m+1)+a2*x^(m+2)
                    +a3*x^(m+3)+a4*x^(m+4)
eq=x^2*diff(y,x,2)+x*diff(y,x,1)+x^2*y
eq1=collect(eq)
eq2=coeffs(simplify(eq1*x^(1-m)),x)
```

```
eq3=solve(eq2(1),m) % roots of indicial equation
a1=solve(eq2(2),a1)
a2=solve(eq2(3),a2)
a3=subs(solve(eq2(4),a3))
a4=subs(solve(eq2(5),a4))
ss=a0*x^m+a1*x^(m+1)+a2*x^(m+2)
                    +a3*x^(m+3)+a4*x^(m+4)
y1=subs(ss,m,eq3(1))
```

```
y2=subs(diff(ss,m),m,eq3(1))
gs=c1*y1+c2*y2
%% visualisation of Bessel's (order zero) first and second kind solutions
X = 0:0.1:20;
Y = zeros(5,numel(X));
J = zeros(5,numel(X));
Y0 = bessely(0,X);
J0=besselj(0,X);
```

```
subplot(1,2,1),plot(X,J0)
title('First kind')
xlabel('X')
ylabel('J_0(X)')
subplot(1,2,2),plot(X,Y0)
title('second kind')
xlabel('X')
ylabel('Y_0(X)')
```