MATH 411: Week 3

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Lemma 1

Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ *be such that for all* $i \in \{1, 2, \ldots, n\}$ *we have* $0 \leq a_i < M$ *for some* $M \in \mathbb{R}^+$*. Then,*

$$\prod_{i=1}^{n} a_i < M^n .$$

We proceed by induction. For all $n \in \mathbb{N}$, let P_n be the statement that if $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $0 \leq a_i < M$ for all $i \in \{1, 2, \ldots, n\}$ for some $M \in \mathbb{R}^+$, then $\prod_{i=1}^{n} a_i < M^n$.

If $a_1 \in \mathbb{R}$ and $0 \leq a_1 < M$ for some $M \in \mathbb{R}^+$, then we have:

$$\prod_{i=1}^{1} a_1 = a_1 < M = M^1 .$$

So P_1 holds.

Now assume that P_n holds for some $n \in \mathbb{N}$. Let $a_1, a_2, \ldots, a_n, a_{n+1} \in \mathbb{R}$ with $0 \leq a_i < M$ for all $i \in \{1, 2, \ldots, n, n+1\}$ for some $M \in \mathbb{R}^+$. If $a_{n+1} = 0$, then since M is positive,

$$\prod_{i=1}^{n+1} a_i = (a_{n+1}) \prod_{i=1}^{n} a_i = 0 < M^{n+1} .$$

On the other hand, if $a_{n+1} > 0$, we have:

$$\begin{aligned} \prod_{i=1}^{n} a_i &< M^n \\ (a_{n+1}) \prod_{i=1}^{n} a_i &< (a_{n+1}) M^n \\ \prod_{i=1}^{n+1} a_i &< (a_{n+1}) M^n . \end{aligned}$$

Since $a_{n+1} < M$ and $M^n > 0$, it follows that $(a_{n+1})M^n < M \cdot M^n = M^{n+1}$. We conclude that

$$\prod_{i=1}^{n+1} a_i < M^{n+1}.$$

We've shown that P_n implies P_{n+1}. So P_n holds for all natural n by induction.

□

Lemma 2

> *Let $\{a_n\}$ be a monotonic sequence. If $\{a_n\}$ is increasing, then $\{a_n\} \to \infty$ if and only if $\{a_n\}$ is not bounded above. Likewise, if $\{a_n\}$ is decreasing, then $\{a_n\} \to -\infty$ if and only if $\{a_n\}$ is not bounded below.*

Clearly, if $\{a_n\} \to \infty$, then $\{a_n\}$ is not bounded above, and if $\{a_n\} \to -\infty$, then $\{a_n\}$ is not bounded below.

Suppose $\{a_n\}$ is increasing and not bounded above. Then, for any $M > 0$, there's a natural N such that $a_N > M$. Since $\{a_n\}$ is increasing, if $n > N$, then we still have $a_n > M$. So $\{a_n\} \to \infty$.

Now suppose $\{a_n\}$ is decreasing and not bounded below. Then, for any $M < 0$, there's a natural N such that $a_N < M$. Since $\{a_n\}$ is decreasing, if $n > N$, then we still have $a_n < M$. So $\{a_n\} \to -\infty$.

□

Lemma 3

> *Let $\sum a_n$ be a series with sequence of partial sums $\{s_n\}$. If $a_n \geq 0$ for all natural n, then $\{s_n\} \to \infty$ if and only if $\sum a_n$ diverges. Likewise, if $a_n \leq 0$ for all natural n, then $\{s_n\} \to -\infty$ if and only if $\sum a_n$ diverges.*

If $a_n \geq 0$ for all natural n, then $\{s_n\}$ is clearly monotonically increasing. The proof of Problem 5.23 ensures that $\{s_n\}$ is not bounded above if and only if $\sum a_n$ diverges.

If $a_n \leq 0$ for all natural n, then $\{s_n\}$ is clearly monotonically decreasing. The proof of Problem 5.23 ensures that $\{s_n\}$ is not bounded below if and only if $\sum a_n$ diverges.

Lemma 2 finishes the proof.

□

Lemma 4

If $c > 0$ and $\{a_n\}$ is a sequence bounded above, then

$$\limsup_{n\to\infty} ca_n = c \limsup_{n\to\infty} a_n \,.$$

Let $c \geq 0$ and $\{a_n\}$ be a sequence. Let $\epsilon > 0$. Then there's an N such that if $n \geq N$, we have

$$|a_n - \limsup_{n\to\infty} a_n| < \frac{\epsilon}{c}$$
$$|ca_n - c\limsup_{n\to\infty} a_n| < \epsilon \,.$$

So $c \limsup_{n\to\infty} a_n$ is an accumulation point of $\{ca_n\}$.

Let $y > c\limsup_{n\to\infty} a_n$. Then $\frac{y}{c} > \limsup_{n\to\infty} a_n$, so $\frac{y}{c}$ is not an accumulation point of $\{a_n\}$. There must exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists an $n \geq N$ with

$$|a_n - \frac{y}{c}| > \epsilon$$
$$|ca_n - y| > c\epsilon \,.$$

So y is not an accumulation point of $\{ca_n\}$.

□

Problem 7.21

For the series $\sum a_n$, suppose that there exists a real number β with $0 < \beta < 1$ and an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\sqrt[n]{|a_n|} \leq \beta$. Prove that $\sum a_n$ converges absolutely.

Let $\sum_{n=1}^{\infty} a_n$ be a series, let $\beta \in (0,1)$, and let $N \in \mathbb{N}$, with $\sqrt[n]{|a_n|} \leq \beta$ for all $n \geq N$. We'll show that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

For all $n \in \mathbb{N}$, since $|a_n|$ is nonnegative, we notice that

$$\sqrt[n]{|a_n|} \leq \beta$$
$$|a_n| \leq \beta^n \,.$$

Since $\beta \in (0,1)$, Problem 7.11 ensures the series $\sum_{n=1}^{\infty} \beta^n$ converges. So Problem 7.7 ensures $\sum_{n=1}^{\infty} |a_n|$ converges, since for all $n \in \mathbb{N}$ we certainly have $\big||a_n|\big| = |a_n| \leq \beta^n$. So $\sum_{n=1}^{\infty} a_n$ converges absolutely.

□

Problem 7.22

For the series $\sum a_n$*, suppose that there exists a real number* β *with* $0 < \beta < 1$ *and an* $N \in \mathbb{N}$ *such that for all* $n \geq N$ *we have* $\left|\frac{a_{n+1}}{a_n}\right| < \beta$*.*

(a) Prove that for all $n \geq N$ *we have*

$$\left|\frac{a_n}{a_N}\right| \beta^N < \beta^n .$$

(b) Use the result above to prove that $\sum a_n$ *converges absolutely.*

Let $\sum_{n=1}^{\infty} a_n$ be a series, let $\beta \in (0,1)$, and let $N \in \mathbb{N}$, with $\left|\frac{a_{n+1}}{a_n}\right| < \beta$ for all $n \geq N$.

(a) Let $n \in \mathbb{N}$ with $n \geq N$. We'll show that

$$\left|\frac{a_n}{a_N}\right| \beta^N < \beta^n .$$

We have:

$$\left|\frac{a_n}{a_N}\right| = \left|\prod_{i=N}^{n-1} \frac{a_{i+1}}{a_i}\right| = \prod_{i=N}^{n-1} \left|\frac{a_{i+1}}{a_i}\right| < \prod_{i=N}^{n-1} \beta = \beta^{n-N} .$$

where the first statement can be easily proven by induction, the second uses a well known fact about the absolute value of a product, and the third is justified by Lemma 1 above, since for each i we have $0 \leq \left|\frac{a_{i+1}}{a_i}\right| < \beta$, and $\beta > 0$.

Since $\beta^N > 0$, we conclude that

$$\left|\frac{a_n}{a_N}\right| \beta^N < \beta^{n-N} \beta^N = \beta^n .$$

(b) We will show that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

We observe that by part (a), for all $n \geq N$, since $\beta^N > 0$,

$$\sqrt[n]{\left|\frac{a_n}{a_N} \beta^N\right|} < \beta .$$

So Problem 7.21 ensures the series $\sum_{n=1}^{\infty} \left|\frac{a_n}{a_N} \beta^N\right|$ converges. A simple consequence of Problem 7.20 is that if a series converges, the series formed by multiplying each of its term by a constant also converges. Hence, since $\beta > 0$, we can say that

$$\sum_{n=1}^{\infty} \left|\frac{a^N}{\beta^N}\right| \left|\frac{a_n}{a_N} \beta^N\right| = \sum_{n=1}^{\infty} |a_n|$$

converges. So $\sum_{n=1}^{\infty} a_n$ converges absolutely.

□

Problem 7.23

> *Use the comparison test to prove that $\sum a_n$ converges absolutely if and only if both $\sum a_n^+$ and $\sum a_n^-$ converge absolutely.*

Let $\sum a_n$ be a series. We'll show that $\sum a_n$ converges absolutely if and only if both $\sum a_n^+$ and $\sum a_n^-$ converge absolutely.

Assume $\sum a_n$ converges absolutely. We observe that for any natural n,

$$|a_n^+| \leq |a_n|\,,$$

because if $a_n \geq 0$, then $|a_n^+| = |a_n|$, and if $a_n < 0$, then $|a_n| > 0 = |0| = |a_n^+|$. Likewise, we have

$$|a_n^-| \leq |a_n|\,,$$

because if $a_n \geq 0$, then $|a_n| \geq 0 = |0| = |a_n^-|$, and if $a_n < 0$, then $|a_n^-| = |a_n|$. The comparison test ensures that both $\sum a_n^+$ and $\sum a_n^-$ converge absolutely.

Now assume $\sum a_n^+$ and $\sum a_n^-$ converge absolutely. We note that, by construction, for any natural n,

$$|a_n| = |a_n^+ + a_n^-| \leq |a_n^+| + |a_n^-|\,.$$

Since $\sum a_n^+$ and $\sum a_n^-$ converge absolutely, $\sum\big(|a_n^+| + |a_n^-|\big)$ converges. The comparison test ensures $\sum a_n$ converges absolutely.

□

Problem 7.24

> *Prove that if $\sum a_n$ converges conditionally then $\sum a_n^+$ must go off to ∞ and $\sum a_n^-$ must go off to $-\infty$.*

Let $\sum a_n$ be a series that converges conditionally to L. We'll show that the sequence of partial sums $\{s_n\}$ of $\sum a_n^+$ converges to ∞ and the sequence of partial sums $\{t_n\}$ of $\sum a_n^-$ converges to $-\infty$.

We first note that, for any natural n,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(a_k^+ + a_k^-) = \sum_{k=1}^{n} a_k^+ + \sum_{k=1}^{n} a_k^- = s_n + t_n\,.$$

Let $\epsilon > 0$. Then there's an N_1 such that if $n \geq N_1$, then $|s_n + t_n - L| < \epsilon$. Let $M < 0$, and suppose $\{s_n\}$ is not bounded above. Then there's an N_2 such that if $n \geq N_2$, then $s_n > -M + \epsilon + L$. Let $n = \max\{N_1, N_2\}$. Then we have

$$
\begin{aligned}
s_n &> -M + \epsilon + L \\
s_n - L + M &> \epsilon \,. \\
s_n - L + M &> s_n + t_n - L \\
M &> t_n \,.
\end{aligned}
$$

So $\{t_n\}$ is not bounded below. On the other hand, let $M > 0$, and suppose $\{t_n\}$ is not bounded below. Then there's an N_2 such that if $n \geq N_2$, then $t_s < -M + \epsilon + L$. Let $n = \max N_1, N_2$. Then we have

$$
\begin{aligned}
t_n &< -M + \epsilon + L \\
t_n - LM &< \epsilon \,. \\
t_n - L + M &< s_n + t_n - L \\
M &< s_n \,.
\end{aligned}
$$

So $\{s_n\}$ is not bounded above.

Now we observe that $\sum a_n^+$ and $\sum a_n^-$ each converge if and only if they converge absolutely. The case of $\sum a_n^+$ is trivial, since $|a_n^+| = a_n^+$ for all n. In the case of $\sum a_n^-$, we note that $|a_n^-| = -a_n^-$ for all n. Problem 7.20 ensures that if $\sum a_n^-$ converges, then so does $\sum -a_n^- = \sum |a_n^-|$. Problem 7.23 ensures that at least one of $\sum a_n^+$ and $\sum a_n^-$ does not converge absolutely. So one of them must diverge.

If $\sum a_n^+$ diverges, then Lemma 3 ensures $\{s_n\}$ converges to ∞, which as we just noted implies $\{t_n\}$ is not bounded below, which by Lemma 2 means $\{t_n\}$ converges to $-\infty$. Likewise, if $\sum a_n^-$ diverges, then Lemma 3 ensures $\{t_n\}$ converges to $-\infty$, which as we just noted implies $\{s_n\}$ is not bounded above, which by Lemma 2 means $\{s_n\}$ converges to ∞. In either case, $\{s_n\} \to \infty$ and $\{t_n\} \to -\infty$.

$\square$

Problem 7.25

Let $\sum a_n$ be a conditionally convergent series.

(a) *Prove that for any $n \in \mathbb{N}$ and any $M > 0$, there exists a $q > n$ such that*

$$\sum_{k=n}^{q} a_n^+ > M \,.$$

(b) Prove that for any $M > 0$ and $\epsilon > 0$, that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$ there exists a $q > n$ with

$$M < \sum_{k=n}^{q} a_n^+ < M + \epsilon\,.$$

Let $\sum a_n$ be a conditionally convergent series. Let s_n denote the n-th partial sum of $\sum a_n^+$.

(a) Let $n \in \mathbb{N}$ and $M > 0$. We will show there exists a $q > n$ such that

$$s_q - s_{n-1} = \sum_{k=n}^{q} a_k^+ > M\,.$$

Problem 7.24 ensures that the sequence of partial sums of $\sum a_n^+$ converges to ∞. So there is a $q \in \mathbb{N}$ such that $s_q > M + s_{n-1}$. We have:

$$s_q - s_{n-1} > M + s_{n-1} - s_{n-1} = M\,.$$

(b) Let $M > 0$ and $\epsilon > 0$. We will show that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$ there exists a $q > n$ with

$$M < \sum_{k=n}^{q} a_n^+ < M + \epsilon\,.$$

Let $\{p_{n_k}\}$ be the subsequence of $\{a_n\}$ whose terms are positive. Since $\sum a_n$ converges, Problem 7.5 and Problem 5.15 ensure that $\{p_{n_k}\} \to 0$. We note that every term in $\{a_n^+\}$ is either a term of $\{p_{n_k}\}$ or is equal to 0. Since every neighborhood around 0 contains all but finitely many terms of $\{p_{n_k}\}$ and certainly contains 0, we conclude that $\{a_n^+\} \to 0$.

Let $n \in \mathbb{N}$ be such that if $p \geq n$, then we have

$$a_p^+ = |a_p^+ - 0| < \min\{\epsilon, M\}\,,$$

which exists because ϵ and M are positive. Let $Q > n$ be such that

$$\sum_{k=n}^{Q} a_k^+ > M\,,$$

as guaranteed by part (a). Let m be the largest element of $\{n, n+1, \ldots, Q\}$ that satisfies

$$\sum_{k=n}^{m} a_k^+ \leq M\,.$$

m exists, since $a_n^+ < M$. We know that

$$\sum_{k=n}^{m+1} a_k^+ > M\,,$$

since $m \neq Q$ and thus $m+1 \in \{n, n+1, \ldots, Q\}$, but $m+1 > m$. We also have:

$$\sum_{k=n}^{m} a_k^+ \leq M$$

$$\sum_{k=n}^{m} a_k^+ + a_{m+1}^+ \leq M + a_{m+1}^+$$

$$\sum_{k=n}^{m+1} a_k^+ \leq M + a_{m+1}^+\,.$$

Further, $a_{m+1}^+ < \epsilon$, since $m+1 > m \geq n$. So we have:

$$\sum_{k=n}^{m+1} a_k^+ \leq M + a_{m+1}^+ < M + \epsilon\,.$$

So $q = m+1$ has the desired properties.

□

Problem 7.26

> *Prove that for any $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, if $\sum a_n$ converges conditionally, there exists a rearrangement of the series $\sum a'_n$ with partial sums s'_n such that*
>
> $$\liminf_{n\to\infty} s'_n = \alpha, \text{ and } \limsup_{n\to\infty} s'_n = \beta\,.$$

We will not prove this formally, but will instead informally construct such a rearrangment and justify why it has the desired properties.

We construct our sequence $\{a'_n\}$ by alternating between two steps:

1. taking enough terms of $\{a_n^+\}$ so that their sum is sufficiently large in magnitude to make the whole sum greater than β but as small as possible; and
2. taking enough terms of $\{a_n^-\}$ so that their sum is sufficiently large in magnitude to make the whole sum less than α but as large as possible.

In this process, we skip terms from $\{a_n^+\}$ and $\{a_n^-\}$ where $a_n^+ \neq a_n$ or $a_n^- \neq a_n$. Since this can only occur when $a_n = 0$, it does not change the incremental sums. Since $\sum a_n^+$ goes off to ∞ and $\sum a_n^-$ goes off to $-\infty$, we are assured that we can always take enough terms of each to produce the stated behavior.

Every term of $\{a_n\}$ is either in $\{a_n^+\}$ or in $\{a_n^-\}$. Since any term of $\{a_n^+\}$ and $\{a_n^-\}$ that is in $\{a_n\}$ is eventually represented, we know that every term of $\{a_n\}$ is represented. Further, no term is repeated, since neither $\{a_n^+\}$ nor $\{a_n^-\}$ repeat terms from $\{a_n\}$, and neither one contains terms of $\{a_n\}$ that the other contains. So we are justified in claiming that $\{a'_n\}$ is a rearrangment of $\{a_n\}$.

There are an infinite number of terms of $\{s'_n\}$ greater than β—namely, every partial sum obtained by summing up to the end of the terms taken in any instance of step (1) above. Likewise, there are an infinite number of terms of $\{s'_n\}$ less than α. Moreover, these are the only terms of $\{s'_n\}$ above β and below α, since within a step the sum is monotonic, we made each post-step sum as small in magnitude as possible, and after each step, the terms of $\{a'_n\}$ change sign.

We've shown in previous problems that since $\sum a_n$ converges, $\{a_n^+\} \to 0$ and $\{a_n^-\} \to 0$. Let $\epsilon > 0$. So there's an N such that if $n \geq N$, we have $a_n^+ < \epsilon$ and $a_n^- > -\epsilon$. Consider a term s'_n of $\{s'_n\}$ with $n \geq N$ such that $s'_n > \beta$. We know that $s'_{n-1} \leq \beta$, since s'_n is as small as possible. So we have:

$$\begin{aligned} s'_n - \beta \leq s'_n - s'_{n-1} = a_n^+ < \epsilon \\ s'_n < \beta + \epsilon \,. \end{aligned}$$

Likewike, for a term $s'_n < \alpha$, we have:

$$\begin{aligned} s'_n - \alpha \geq s'_n - s'_{n-1} = a_n^- > -\epsilon \\ s'_n > \alpha - \epsilon \,. \end{aligned}$$

We've shown that for any given $\epsilon > 0$, there are an infinite number of terms of $\{s'_n\}$ in the ϵ-neighborhood around α and β—in particular, all the partial sums after a certain point obtained by summing up to the end of the terms taken in any step above. We conclude that α and β are accumulation points for $\{s'_n\}$.

Further, since no other terms—besides the ones that we've just shown converge to α and β—are above β or below α, we know that β is the biggest accumulation point and α is the smallest accumulation point.

□

Problem 7.28

Suppose that $\sum a_n$ converges and $a_n \geq 0$ for all n. Prove that $\sum \frac{\sqrt{a_n}}{n}$ converges as well.

Suppose that $\sum a_n$ converges and $a_n \geq 0$ for all n. We'll show that $\sum \frac{\sqrt{a_n}}{n}$ converges as well. The Scharz inequality ensures that for any N,

$$\left(\sum_{n=1}^{N} \sqrt{a_n} \cdot \frac{1}{n}\right)^2 \leq \left(\sum_{n=1}^{N} a_n\right)\left(\sum_{n=1}^{N} \frac{1}{n^2}\right).$$

$\sum a_n$ converges by assumption and $\sum \frac{1}{n^2}$ converges since it's a p-series with $p > 1$. Problem 5.13 then ensures that

$$\left\{\left(\sum_{n=1}^{N} a_n\right)\left(\sum_{n=1}^{N} \frac{1}{n^2}\right)\right\}_{N=1}^{\infty}$$

converges. We note that, for all N,

$$\sum_{n=1}^{N} \sqrt{a_n} \cdot \frac{1}{n} \geq 0\,,$$

since $a_n \geq 0$. Thus, the squeeze theorem ensures

$$\left\{\left(\sum_{n=1}^{N} \sqrt{a_n} \cdot \frac{1}{n}\right)^2\right\}_{N=1}^{\infty}$$

converges. So

$$\left\{\sum_{n=1}^{N} \sqrt{a_n} \cdot \frac{1}{n}\right\}_{N=1}^{\infty}$$

also converges. We haven't proven this, but since we know the square root function is continuous, it seems like a reasonable conclusion.

□

Problem 7.29

Let $\sum a_n x^n$ be a power series for some x. Let

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

and

$$R = \begin{cases} 0 & \alpha = \infty \\ \frac{1}{\alpha} & 0 < \alpha < \infty \\ \infty & \alpha = 0 \end{cases}.$$

We will show that $\sum a_n x^n$ converges absolutely for all $|x| < R$ and diverges for all $|x| > R$.

Assume $\alpha = \infty$ and $|x| > 0$. Then $\{\sqrt[n]{|a_n|}\}$ is unbounded and for any $M > 0$ there's an N such that if $n \geq N$ then

$$\begin{aligned} \sqrt[n]{|a_n|} &> \sqrt[n]{M}\left|x^{-1}\right| \\ |a_n| &> M\left|x^{-n}\right| \\ |a_n x^n| &> M\,. \end{aligned}$$

So $\{a_n x^n\}$ is unbounded and hence $\sum a_n x^n$ diverges.

Now assume $0 < \alpha < \infty$. Lemma 4 ensures that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = |x| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = |x|\alpha\,.$$

If $|x| < \frac{1}{\alpha}$, then $|x|\alpha < 1$ and the root test ensures $\sum a_n x^n$ converges. If $|x| > \frac{1}{\alpha}$, then $|x|\alpha > 1$ and the root test ensures $\sum a_n x^n$ diverges.

Finally, assume $\alpha = 0$. Then Lemma 4 ensures that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = |x| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0 < 1\,.$$

The root test ensures $\sum a_n x^n$ converges.

□