MATH 411: Week 5

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Problem 8.4

> *Let f be convex on (a, b) and let x, y, z be fixed such that $a < x < y < z < b$. Prove that*
>
> $$\frac{(t-x)f(y)-(t-y)f(x)}{y-x} \leq f(t) \leq \frac{(z-t)f(y)+(t-y)f(z)}{z-y}$$
>
> *for all t with $y < t < z$, and, similarly,*
>
> $$\frac{(z-s)f(y)-(y-s)f(z)}{z-y} \leq f(s) \leq \frac{(y-s)f(x)+(s-x)f(y)}{y-x}$$
>
> *for all s with $x < s < y$.*

Let f be convex on (a, b) and let x, y, z be fixed such that $a < x < y < z < b$. Let $y < t < z$. Then by definition,

$$f(t) \leq \frac{f(y)(z-t)+f(z)(t-y)}{z-y}\,.$$

Also, since $x < y < t$,

$$\begin{aligned} f(y) &\leq \frac{(t-y)f(x)+(y-x)f(t)}{t-x} \\ f(y)(t-x) &\leq (t-y)f(x)+(y-x)f(t) \\ f(y)(t-x)-(t-y)f(x) &\leq (y-x)f(t) \\ \frac{f(y)(t-x)-(t-y)f(x)}{y-x} &\leq f(t)\,. \end{aligned}$$

Now let $x < s < y$. Similarly,

$$\frac{(z-s)f(y)-(y-s)f(z)}{z-y} \leq f(s) \leq \frac{(y-s)f(x)+(s-x)f(y)}{y-x}\,.$$

□