MATH 411: Week 4

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13 February 2026

Problem 7.31

Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}\,.$$

(a) Explain (briefly) why the series converges. (Does it converge absolutely?)

(b) If we take the product of the series with itself, then we have

$$c_n = (-1)^n \sum_{k=0}^{\infty} \frac{1}{\sqrt{(n-k+1)(k+1)}}$$

(you should double check this). Show that each term in this sum is greater than or equal to $\frac{1}{\sqrt{n+1}}$, and thus $|c_n| \geq \sqrt{n+1}$.

(c) Explain (briefly) why the previous part implies $\sum c_n$ diverges.

Define the series $\sum_{n=0}^{\infty} a_n$ by

$$a_n = \frac{(-1)^n}{\sqrt{n+1}}\,.$$

(a) We'll show that $\sum a_n$ converges absolutely. We note that for any $n \in \mathbb{N}_0$,

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = 1\,.$$

Since the sequence $\left\{\frac{1}{n^{1/2}}\right\}$ is a convergent p-series, the limit comparison test ensures $\left\{\frac{1}{\sqrt{n+1}}\right\}$ also converges. Since

$$\left|\frac{(-1)^n}{\sqrt{n+1}}\right| = \frac{1}{\sqrt{n+1}}\,,$$

we conclude that $\sum a_n$ converges absolutely.

(b) We will show that we have $|c_n| > \sqrt{n+1}$ for all n, where $\sum_{n=0}^{\infty} c_n$ is the Cauchy product of $\sum a_n$ with itself. By definition,

$$c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}} .$$

If $n \in \mathbb{N}_0$ and $0 \leq k \leq n$, we have

$$\begin{aligned} k &\leq n \\ k \cdot k &\leq nk \\ -k^2 &\geq -nk \\ nk - k^2 + n + 1 &\geq n+1 \\ (n-k+1)(k+1) &\geq n+1 \\ \sqrt{(n-k+1)(k+1)} &\geq \sqrt{n+1} \\ \frac{1}{\sqrt{(n-k+1)(k+1)}} &\geq \frac{1}{\sqrt{n+1}} , \end{aligned}$$

where the last line is justified by the fact that the square root function is nonnegative and that here its arguments are nonzero. Now we can say that for any $n \in \mathbb{N}_0$,

$$\begin{aligned} \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}} &\geq \sum_{k=0}^{n} \frac{1}{\sqrt{n+1}} \\ &\geq \frac{n+1}{\sqrt{n+1}} \\ &= \sqrt{n+1} . \end{aligned}$$

So for any $n \in \mathbb{N}_0$, we have $|c_n| \geq \sqrt{n+1}$.

(c) We'll show that $\sum c_n$ diverges. The series $\sum_{n=0}^{\infty} \sqrt{n+1}$ diverges, since its terms are unbounded. The comparison test then ensures $\sum c_n$ also diverges.

□

Problem 7.32

Let $c_n = \sum_{k=0}^{n} c_n$ (where c_n is defined as above). Prove that $C_n = a_0 B_{n-1} + \cdots + a_n B_0$ where $B_n = \sum_{k=0}^{n} b_n$. Define $\beta_n = B_n - B$ and $A_n = \sum_{k=0}^{n} a_n$. Explain why the following are true:

(a) $\lim_{n\to\infty} \beta_n = 0$

(b) For all $N \geq 0$, $C_n = A_n B + a_0\beta_n + a_1\beta_{n=1} + \cdots + a_n\beta_0$

(c) $\lim_{n\to\infty} A_n B = AB$

Let $\sum_{n=0}^{\infty} A_n$ be an absolutely convergent series with $\sum_{n=0}^{\infty} a_n = A$, and let $\sum_{n=0}^{\infty} b_n$ be a convergent series with $\sum_{n=0}^{\infty} b_n = B$. For all $n \geq 0$, define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, $C_n = \sum_{k=0}^{n} c_k$, and $\beta_n = B_n - B$.

Let $n \geq 0$. We'll show that $C_n = a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0$ where $B_n = \sum_{k=0}^{n} b_k$. We have:

$$\begin{aligned}
C_n &= \sum_{k=0}^{n} c_k \\
&= (a_0 b_0) + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\
&= a_0(b_0 + b_1 + \cdots + b_n) + a_1(b_0 + b_1 + \ldots b_{n-1}) + \cdots + a_n b_0 \\
&= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \,.
\end{aligned}$$

(a) We'll show that

$$\lim_{n\to\infty} \beta_n = 0 \,.$$

Since $\sum b_n = B$, we know $\{B_n\} \to B$, and thus $\{B_n - B\} \to B - B = 0$.

(b) Let $n \geq 0$. We'll show that

$$C_n = A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 \,.$$

We have:

$$\begin{aligned}
\sum_{k=0}^{n} a_k \beta_{n-k} &= \sum_{k=0}^{n} a_k (B_{n-k} - B) \\
&= \sum_{k=0}^{k} a_k B_{n-k} - \sum_{k=0}^{n} a_k B \\
&= \sum_{k=0}^{k} a_k B_{n-k} - A_n B \,.
\end{aligned}$$

We conclude:

$$C_n = \sum_{k_0}^{k} a_k B_{n-k} = \sum_{k=0}^{n} a_k \beta_{n-k} + A_n B \,.$$

(c) We will show that

$$\lim_{n\to\infty} A_n B = AB \,.$$

Since $\{A_n\} \to A$ and $\{B\}_{n=0}^{\infty} \to B$, Problem 5.13 ensures $\{A_n B\} \to AB$.