MATH 411: Week 1

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Definition

Let $A \subseteq [-\infty, \infty]$. If $\infty \in A$, we define $\operatorname{lub} A = \infty$. If $A = \{-\infty\}$, we define $\operatorname{lub} A = -\infty$. Otherwise, we define $\operatorname{lub} A = \operatorname{lub}\big(A \setminus \{-\infty\}\big)$.

Problem 7.2

> *Let $\{a_n\}$ be a sequence of real numbers and let E be its set of subsequential limits. Prove that $\limsup_{n\to\infty} a_n \in E$.*

Let $\{a_n\}$ be a sequence of real numbers and let $E \subseteq [-\infty, \infty]$ be its set of subsequential limits. We will show that $\limsup_{n\to\infty} a_n = \operatorname{lub} E \in E$.

Our definition ensures that if $\infty \notin E$, then $\operatorname{lub} E = -\infty$ or $\operatorname{lub} E \in \mathbb{R}$. So if $\operatorname{lub} E = \infty$, then $\infty \in E$. If $E = \{-\infty\}$, our definition ensures $\operatorname{lub} E = -\infty \in E$.

If $\infty \notin E$ and $E \neq \{-\infty\}$, our definition ensures that $\operatorname{lub} E = \operatorname{lub} A$, where A is the set of accumulation points of $\{a_n\}$. Since $\infty \notin E$, there is no subsequence of $\{a_n\}$ that converges to ∞, so $\{a_n\}$ and thus A is bounded above. Problem 5.18 ensures A is closed. Problem 4.9 then ensures $\operatorname{lub} E = \operatorname{lub} A \in A \subseteq E$. $\square$

Problem 7.3

> *Let $\{a_n\}$ be a sequence of real numbers. If $\limsup_{n\to\infty} \in \mathbb{R}$, and $x > \limsup_{n\to\infty} a_n \in \mathbb{R}$, prove that there exists a natural number N such that for all $n \geq N$ we have $a_n < x$.*

Let $\{a_n\}$ be a sequence of real numbers with $\alpha = \limsup_{n\to\infty} \in \mathbb{R}$, and let $x > \alpha \in \mathbb{R}$. We will show that there exists a natural number N such that for all $n \geq N$ we have $a_n < x$. Note that since $\limsup_{n\to\infty} \in \mathbb{R}$, there is no subsequence of $\{a_n\}$ that converges to ∞, so $\{a_n\}$ is bounded.

Since $\alpha \in \mathbb{R}$, it is the least upper bound of the set of accumulation points of $\{a_n\}$. Since $\{a_n\}$ is bounded above, there exists some $M \in \mathbb{R}$ such that $a_n \leq M$ for all n. If $M < x$, then $a_n < x$ for all $n \geq 1$ and we're done. So we assume $M \geq x$.

Problem 5.17 ensures that if there exists a subsequence of $\{a_n\}$ with terms in $[x, M]$, then that subsequence—and thus $\{a_n\}$ itself—has an accumulation point in $[x, M]$. But since α is an upper bound on the set of accumulation points of $\{a_n\}$ and $x > \alpha$, we know that $[x, M]$ has no accumulation points of $\{a_n\}$. We conclude that there is no subsequence of $\{a_n\}$ with terms from $[x, M]$. Further, since by definition there are no terms of $\{a_n\}$ greater than M, we know that there is no subsequence of $\{a_n\}$ with terms from $[x, \infty)$. So there are only finitely many terms of $\{a_n\}$ greater than or equal to x. $\square$

Problem 7.5

If $\sum a_k$ converges then $\lim_{k\to\infty} a_k = 0$.

Let $\sum a_k$ be a convergent series. We will show that $\lim_{k\to\infty} a_k = 0$.

Let $\epsilon > 0$. Problem 7.4 ensures that there is a natural N such that for all $m \geq n > N$, we have

$$\left|\sum_{k=n}^{m} a_k\right| < \epsilon\,.$$

If $m = n$, then

$$\begin{aligned}\left|\sum_{k=n}^{n} a_k\right| &< \epsilon \\ |a_n| &< \epsilon \\ -\epsilon < a_n &< \epsilon \\ 0 - \epsilon < a_n &< 0 + \epsilon \\ a_n &\in (0 - \epsilon, 0 + \epsilon)\,.\end{aligned}$$

So we've shown that for all $\epsilon > 0$, there's a natural number $N + 1$ such that if $n \geq N + 1 > N$, we have $a_n \in (0 - \epsilon, 0 + \epsilon)$. We conclude that $\lim_{k\to\infty} a_k = 0$. $\square$

Problem 7.6

If the series $\sum |a_k|$ converges then the series $\sum a_k$ converges as well.

Let $\sum a_k$ be a series such that $\sum |a_k|$ converges. We will show that $\sum a_k$ also converges.

Since $\sum |a_k|$ converges, Problem 7.4 ensures that there is a natural N such that for all $m \geq n > N$, we have

$$\left| \sum_{k=n}^{m} |a_k| \right| < \epsilon$$

Applying the triangle inequality, we get:

$$\epsilon > \left| \sum_{k=n}^{m} |a_k| \right| \geq \left| \left| \sum_{k=n}^{m} a_k \right| \right| = \left| \sum_{k=n}^{m} a_k \right| .$$

Problem 7.4 then ensures that $\sum a_k$ converges. $\square$

Problem 7.7

> *If $|a_k| \leq c_k$ for all $k \geq N_0$ where N_0 is a fixed integer, and $\sum c_k$ converges, then $\sum a_k$ converges as well.*

Let $N_0 \in \mathbb{N}$. Let $\sum a_k$ be a series and $\sum c_k$ be a convergent series, with $|a_k| \leq c_k$ for all $k \geq N_0$. We will show that $\sum a_k$ is also convergent.

Let $\epsilon > 0$. Since $\sum c_k$ converges, Problem 7.4 ensures that there exists an $N \in \mathbb{N}$ such that for all $m \geq n > N$, we have

$$\left| \sum_{k=n}^{m} c_k \right| < \epsilon .$$

Let $M = \max\{N_0, N\}$. Then if $m \geq n > M$, since $0 \leq |a_k| \leq c_k$ for all $k \geq N_0$, we have:

$$\sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k$$
$$\left| \sum_{k=n}^{m} |a_k| \right| \leq \left| \sum_{k=n}^{m} c_k \right| < \epsilon .$$

Since for every $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that for every $m \geq n > M$ we have $\left| \sum_{k=n}^{m} |a_k| \right| < \epsilon$, Problem 7.4 ensures $\sum |a_k|$ converges. Problem 7.6 then ensures $\sum a_k$ converges. $\square$

Problem 7.8

> *If $a_k \geq d_k \geq 0$ for all $k \geq N_0$ where N_0 is a fixed integer, and $\sum d_k$ diverges, then $\sum a_k$ diverges as well.*

Let $N_0 \in \mathbb{N}$. Let $\sum a_k$ be a series and $\sum d_k$ be a divergent series, with $a_k \geq d_k \geq 0$ for all $k \geq N_0$. We will show that $\sum a_k$ is also divergent.

Since $d_k \geq 0$, we know that $d_k = |d_k|$ and so $|d_k| \leq a_k$ for all $k \geq N_0$. If $\sum a_k$ converges, then Problem 7.7 ensures $\sum d_k$ converges. Since $\sum d_k$ does not converge, we conclude that $\sum a_k$ does not converge. $\square$

Problem 7.9

> *Suppose that $a_k \geq 0$ for all k. Then the series $\sum a_k$ converges if and only if the seqeunce of partial sums is bounded.*

Let $\sum a_k$ be a series with $a_k \geq 0$ for all k. We will show that $\sum a_k$ converges if and only if its seqeunce of partial sums $\{s_k\}$ is bounded.

Let $n \in \mathbb{N}$. Then:

$$s_{n+1} = \sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1} = s_n + a_{n+1} \geq s_n \,,$$

since $a_{n+1} \geq 0$. So $\{s_k\}$ is monotonically increasing. Problem 5.23 then ensures $\{s_k\}$ converges if and only if it is bounded. $\square$

Problem 7.10

> *Let $r \in \mathbb{R}$ be fixed. Use mathematical induction to prove that for all $n \in \mathbb{N}$ we have*
>
> $$(1-r)(1+r+r^2+\cdots+r^n) = 1 - r^{n+1} \,.$$

Let $r \in \mathbb{R}$. For all $n \in \mathbb{N}$, let P_n be the statement that

$$(1-r)(1+r+r^2+\cdots+r^n) = 1 - r^{n+1} \,.$$

We will show by induction that P_n holds for all natural n. P_1 holds:

$$(1-r)(1+r^1) = (1-r)(1+r) = 1 - r^2 = 1 - r^{1+1} \,.$$

Now assume that P_n holds for some $n \in \mathbb{N}$, such that

$$(1-r)(1+r+r^2+\cdots+r^n) = 1 - r^{n+1} \,.$$

Then:

$$
\begin{aligned}
&(1-r)(1+r+r^2+\cdots+r^n+r^{n+1}) \\
&\qquad = (1-r)(1+r+r^2+\cdots+r^n)+(1-r)r^{n+1} \\
&\qquad = 1-r^{n+1}+(1-r)r^{n+1} \\
&\qquad = 1-r^{n+1}+r^{n+1}-r\cdot r^{n+1} \\
&\qquad = 1-r\cdot r^{n+1} \\
&\qquad = 1-r^{(n+1)+1}\,.
\end{aligned}
$$

We've shown that P_n implies P_{n+1}. By induction we conclude that P_n holds for all natural n. $\square$