

# MATH 411: Week 4

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## Problem 7.31

Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}.$$

(a) Explain (briefly) why the series converges. (Does it converge absolutely?)

(b) If we take the product of the series with itself, then we have

$$c_n = (-1)^n \sum_{k=0}^{\infty} \frac{1}{\sqrt{(n-k+1)(k+1)}}$$

(you should double check this). Show that each term in this sum is greater than or equal to  $\frac{1}{\sqrt{n+1}}$ , and thus  $|c_n| \geq \sqrt{n+1}$ .

(c) Explain (briefly) why the previous part implies  $\sum c_n$  diverges.

Define the series  $\sum_{n=0}^{\infty} a_n$  by

$$a_n = \frac{(-1)^n}{\sqrt{n+1}}.$$

(a) We'll show that  $\sum a_n$  converges absolutely. We note that for any  $n \in \mathbb{N}_0$ ,

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n+1}{n}} = \lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n}} = 1.$$

Since the sequence  $\left\{ \frac{1}{n^{1/2}} \right\}$  is a convergent  $p$ -series, the limit comparison test ensures  $\left\{ \frac{1}{\sqrt{n+1}} \right\}$  also converges. Since

$$\left| \frac{(-1)^n}{\sqrt{n+1}} \right| = \frac{1}{\sqrt{n+1}},$$

we conclude that  $\sum a_n$  converges absolutely.

(b) We will show that we have  $|c_n| > \sqrt{n+1}$  for all  $n$ , where  $\sum_{n=0}^{\infty} c_n$  is the Cauchy product of  $\sum a_n$  with itself. By definition,

$$c_n = (-1)^n \sum_{k=0}^n \frac{1}{\sqrt{(n-k+1)(k+1)}}.$$

If  $n \in \mathbb{N}_0$  and  $0 \leq k \leq n$ , we have

$$\begin{aligned} k &\leq n \\ k \cdot k &\leq nk \\ -k^2 &\geq -nk \\ nk - k^2 + n + 1 &\geq n + 1 \\ (n - k + 1)(k + 1) &\geq n + 1 \\ \sqrt{(n - k + 1)(k + 1)} &\geq \sqrt{n + 1} \\ \frac{1}{\sqrt{(n - k + 1)(k + 1)}} &\geq \frac{1}{\sqrt{n + 1}}, \end{aligned}$$

where the last line is justified by the fact that the square root function is non-negative and that here its arguments are nonzero. Now we can say that for any  $n \in \mathbb{N}_0$ ,

$$\begin{aligned} \sum_{k=0}^n \frac{1}{\sqrt{(n-k+1)(k+1)}} &\geq \sum_{k=0}^n \frac{1}{\sqrt{n+1}} \\ &\geq \frac{n+1}{\sqrt{n+1}} \\ &= \sqrt{n+1}. \end{aligned}$$

So for any  $n \in \mathbb{N}_0$ , we have  $|c_n| \geq \sqrt{n+1}$ .

(c) We'll show that  $\sum c_n$  diverges. The series  $\sum_{n=0}^{\infty} \sqrt{n+1}$  diverges, since its terms are unbounded. The comparison test then ensures  $\sum c_n$  also diverges.

□

## Problem 7.32

Let  $c_n = \sum_{k=0}^n c_n$  (where  $c_n$  is defined as above). Prove that  $C_n = a_0 B_{n-1} + \dots + a_n B_0$  where  $B_n = \sum_{k=0}^n b_k$ . Define  $\beta_n = B_n - B$  and  $A_n = \sum_{k=0}^n a_k$ . Explain why the following are true:

(a)  $\lim_{n \rightarrow \infty} \beta_n = 0$

(b) For all  $N \geq 0$ ,  $C_n = A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \dots + a_n \beta_0$

(c)  $\lim_{n \rightarrow \infty} A_n B = AB$

Let  $\sum_{n=0}^{\infty} A_n$  be an absolutely convergent series with  $\sum_{n=0}^{\infty} a_n = A$ , and let  $\sum_{n=0}^{\infty} b_n$  be a convergent series with  $\sum_{n=0}^{\infty} b_n = B$ . For all  $n \geq 0$ , define  $c_n = \sum_{k=0}^n a_k b_{n-k}$ ,  $C_n = \sum_{k=0}^n c_k$ , and  $\beta_n = B_n - B$ .

Let  $n \geq 0$ . We'll show that  $C_n = a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0$  where  $B_n = \sum_{k=0}^n b_k$ . We have:

$$\begin{aligned} C_n &= \sum_{k=0}^n c_k \\ &= (a_0 b_0) + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0(b_0 + b_1 + \cdots + b_n) + a_1(b_0 + b_1 + \cdots + b_{n-1}) + \cdots + a_n b_0 \\ &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0. \end{aligned}$$

(a) We'll show that

$$\lim_{n \rightarrow \infty} \beta_n = 0.$$

Since  $\sum b_n = B$ , we know  $\{B_n\} \rightarrow B$ , and thus  $\{B_n - B\} \rightarrow B - B = 0$ .

(b) Let  $n \geq 0$ . We'll show that

$$C_n = A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0.$$

We have:

$$\begin{aligned} \sum_{k=0}^n a_k \beta_{n-k} &= \sum_{k=0}^n a_k (B_{n-k} - B) \\ &= \sum_{k=0}^n a_k B_{n-k} - \sum_{k=0}^n a_k B \\ &= \sum_{k=0}^n a_k B_{n-k} - A_n B. \end{aligned}$$

We conclude:

$$C_n = \sum_{k=0}^n a_k B_{n-k} = \sum_{k=0}^n a_k \beta_{n-k} + A_n B.$$

(c) We will show that

$$\lim_{n \rightarrow \infty} A_n B = AB.$$

Since  $\{A_n\} \rightarrow A$  and  $\{B\}_{n=0}^{\infty} \rightarrow B$ , Problem 5.13 ensures  $\{A_n B\} \rightarrow AB$ .

□

### Problem 7.33

Let  $\gamma = \sum_{n=0}^{\infty} |a_n|$ . Use the first result in problem 7.32 to prove that for any  $\delta_1 > 0$ , there exists an  $N_1$  such that for all  $n > N_1$ ,

$$|\alpha_0\beta_n + \alpha_1\beta_{n-1} + \cdots + \alpha_{n-N_1}\beta_{N_1}| \leq \delta_1\gamma.$$

Let  $\gamma = \sum_{n=0}^{\infty} |a_n|$  and  $\delta_1 > 0$ . We'll show that there exists an  $N_1 \in \mathbb{N}$  such that for all  $n > N_1$ ,

$$|\alpha_0\beta_n + \alpha_1\beta_{n-1} + \cdots + \alpha_{n-N_1}\beta_{N_1}| \leq \delta_1\gamma.$$

Problem 7.32 ensures that  $\{\beta_n\} \rightarrow 0$ , so there's an  $N_1 \in \mathbb{N}$  such that if  $n \geq N_1$ , then

$$|\beta_n| < \delta_1.$$

Since  $\gamma = \sum_{n=0}^{\infty} |a_n|$  is increasing and  $\delta_1 > 0$ , we know

$$\begin{aligned} \sum_{k=0}^{n-N_1} |a_k| &\leq \gamma \\ \sum_{k=0}^{n-N_1} \delta_1 |a_k| &\leq \delta_1 \gamma, \end{aligned}$$

for all  $n \geq N_1$ . If  $0 \leq k \leq n - N_1$ , then  $n - k \geq N_1$ , so

$$|\beta_{n-k}| < \delta_1.$$

Thus,

$$\sum_{k=0}^{n-N_1} |\beta_{n-k}| |a_k| < \delta_1 \gamma.$$

The triangle inequality now ensures

$$\left| \sum_{k=0}^{n-N_1} \beta_{n-k} a_k \right| \leq \sum_{k=0}^{n-N_1} |\beta_{n-k} a_k| < \delta_1 \gamma.$$

□

### Problem 7.34

Following along with the notation in the previous two problems:

- (a) Explain why we can choose an  $M$  with  $M > |\beta_n|$  for all  $n$ .

(b) Then use the fact that  $\sum_{n=0}^{\infty} |a_n|$  converges to show that for any  $\delta_2 > 0$ , there exists an  $N_2$  such that for any  $n > N_2$  we have

$$|a_{N_2}\beta_{n-N_2} + a_{N_2+1}\beta_{n-N_2-1} + \cdots + a_n\beta_0| < M\delta_2.$$

(a) Since  $\{\beta_n\}$  converges, it must be bounded.

(b) Let  $\delta_2 > 0$ . Since  $\sum_{n=0}^{\infty} |a_n|$  converges, the Cauchy criterion ensures

$$\begin{aligned} \sum_{k=N_2}^n |a_k| &\leq \delta_2 \\ \sum_{k=0}^{n-N_2} |a_{N_2+k}| &\leq \delta_2 \\ \sum_{k=0}^{n-N_2} |a_{N_2+k}|M &\leq M\delta_2. \end{aligned}$$

By construction,  $M > |\beta_{n-N_2-k}|$  for all  $k \in \{0, 1, \dots, n - N_2\}$ . So we conclude:

$$\sum_{k=0}^{n-N_2} |a_{N_2+k}| |\beta_{n-N_2-k}| < M\delta_2.$$

The triangle inequality then ensures

$$\left| \sum_{k=0}^{n-N_2} a_{N_2+k} \beta_{n-N_2-k} \right| \leq \sum_{k=0}^{n-N_2} |a_{N_2+k} \beta_{n-N_2-k}| < M\delta_2.$$

□

### Problem 7.36

Suppose that  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  are both absolutely convergent series. Define  $c_n = \sum_{k=0}^n a_k b_{n-k}$ . Use Theorem 7.4 to prove that  $\sum_{n=0}^{\infty} c_n$  is absolutely convergent as well.

Let  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  converge absolutely. Define  $c_n = \sum_{k=0}^n a_k b_{n-k}$ , and  $\bar{c}_n = \sum_{k=0}^n |a_k||b_{n-k}|$ . We'll show that  $\sum_{n=0}^{\infty} c_n$  converges absolutely. We notice that

$$|c_n| = \left| \sum_{k=0}^n a_k b_{n-k} \right| \leq \sum_{k=0}^n |a_k b_{n-k}| = \sum_{k=0}^n |a_k||b_{n-k}| = \bar{c}_n.$$

Since  $\sum_{n=0}^{\infty} |a_n|$  and  $\sum_{n=0}^{\infty} |b_n|$  both converge absolutely, Theorem 7.4 ensures that  $\sum_{n=0}^{\infty} \bar{c}_n$  converges. The comparison test then ensures  $\sum_{n=0}^{\infty} |c_n|$  converges. So  $\sum_{n=0}^{\infty} c_n$  converges absolutely.

□

## Problem 7.37

Let  $\sum a_n$  be a divergent series where  $a_n > 0$  for all  $n$ . Prove that  $\sum \frac{a_n}{1+a_n}$  diverges.

Let  $\sum a_n$  be a divergent series with  $a_n > 0$  for all  $n$ . We'll show that  $\sum \frac{a_n}{1+a_n} = \sum \frac{1}{1/a_n + 1}$  diverges.

Assume  $\limsup_{n \rightarrow \infty} a_n = \infty$ . Then there's a subsequence  $\{a'_n\}$  of  $\{a_n\}$  with  $\{a'_n\} \rightarrow \infty$ . So  $\{\frac{1}{a'_n}\} \rightarrow 0$ ,  $\{\frac{1}{a'_n} + 1\} \rightarrow 1$ , and  $\frac{1}{1/a'_n + 1} \rightarrow 1$ . Since there's a subsequence  $\{\frac{1}{1/a'_n + 1}\}$  of  $\{\frac{a_n}{1+a_n}\}$  that doesn't converge to 0, we know  $\{\frac{a_n}{1+a_n}\} \not\rightarrow 0$ .

Assume  $\limsup_{n \rightarrow \infty} a_n = L > 0$ . Then there's some subsequence  $\{a'_n\}$  of  $\{a_n\}$  with  $\{a'_n\} \rightarrow L$ . Since  $\{a'_n + 1\} \rightarrow L + 1$  and  $1 + a'_n$  is never zero, we have

$$\left\{ \frac{a'_n}{1 + a'_n} \right\} \rightarrow \frac{L}{L + 1} \neq 0.$$

Since there's a subsequence  $\left\{ \frac{a'_n}{1 + a'_n} \right\}$  of  $\left\{ \frac{a_n}{1 + a_n} \right\}$  that doesn't converge to 0, we know  $\left\{ \frac{a_n}{1 + a_n} \right\} \not\rightarrow 0$ .

Assume  $\limsup_{n \rightarrow \infty} a_n = 0$ . Then there's some  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $a_n < 1$ , or  $1 - a_n > 0$ . So, if  $n \geq N$ , we have

$$\frac{a_n}{1 + a_n} > \frac{a_n}{1 + a_n + (1 - a_n)} = \frac{a_n}{2}.$$

Since  $\sum a_n$  diverges,  $\sum \frac{a_n}{2}$  also diverges, and the comparison test ensures  $\sum \frac{a_n}{1+a_n}$  diverges.