MATH 411: Week 2

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Lemma

> *Let $\sum_{k=k_0}^{\infty} a_k$ be a series. Let c be a nonzero real number. Then $\sum_{k=k_0}^{\infty} a_k$ converges if and only if $\sum_{k=k_0}^{\infty} ca_k$ converges.*

Let $\{s_n\}$ and $\{t_n\}$ be the sequences of partial sums of $\sum a_k$ and $\sum ca_k$, respectively. By Problem 5.11, if $\{s_n\}$ converges, then $\{cs_n\}_{n=k_0}^{\infty}$ converges. On the other hand, if $\{cs_n\}_{n=k_0}^{\infty}$ converges, then since $c \neq 0$, Problem 5.11 ensures that $\{\frac{1}{c}cs_n\} = \{s_n\}$ converges. So we've shown that $\{s_n\}$ converges if and only if $\{cs_n\}$ converges.

We observe that, for all $n \geq k_0$,

$$cs_n = c\sum_{k=k_0}^{n} a_k = \sum_{k=k_0}^{n} ca_k = t_n \,.$$

We conclude that $\{s_n\}$ converges if and only if $\{t_n\}$ converges, and hence that $\sum a_k$ converges if and only if $\sum ca_k$ converges. $\square$

Problem 7.15

> *Use Theorem 7.1 to prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.*

Let $p \in \mathbb{R}$. We will show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. Problem 7.1 ensures that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if

$$\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=0}^{\infty} \frac{1}{(2^n)^{p-1}} = \sum_{n=0}^{\infty} \frac{1}{(2^{p-1})^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$$

converges. By Problem 7.11, this series converges if and only if

$$\begin{aligned}
\left|\frac{1}{2^{p-1}}\right| &< 1 \\
\left|2^{p-1}\right| &> 1 \\
p-1 &> 0 \\
p &> 1\,.
\end{aligned}$$

□

Problem 7.16

> *Use Theorem 7.1 to prove that the series $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$.*

Let $p \in \mathbb{R}$. We will show that the series $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$. Problem 7.1 ensures that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^n \frac{1}{2^n(\ln 2^n)^p} &= \sum_{n=0}^{\infty} \frac{1}{(\ln 2^n)^p} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n \ln 2)^p} \\
&= \sum_{n=0}^{\infty} \frac{1}{n^p(\ln 2)^p} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{(\ln 2)^p} \cdot \frac{1}{n^p}\right)
\end{aligned}$$

converges. By our lemma above and Problem 7.15, this series converges if and only if $p > 1$, since $\frac{1}{(\ln 2)^p} \neq 0$.

□

Problem 7.17

> *For a series $\sum a_n$ suppose that there exists some $M > 0$ such that for each partial sum s_n we have $|s_n| < M$. Prove that for any $0 < p < q$ we have*
>
> $$\left|\sum_{k=p}^{q} a_k\right| < 2M\,.$$

Let $M > 0$, and let $\sum a_n$ be a series such that for each partial sum s_n we have $|s_n| < M$. Let $0 < p < q$. We'll show that

$$\left|\sum_{k=p}^{q} a_k\right| < 2M\,.$$

We have:

$$\begin{aligned}
\left|\sum_{k=p}^{q} a_k\right| &= \left|\sum_{k=1}^{q} a_k - \sum_{k=1}^{p} a_k\right| \\
&= |s_q + (-s_p)| \\
&< |s_q| + |-s_p| \\
&= |s_q| + |s_p| \\
&< M + M \\
&< 2M\,.
\end{aligned}$$

□

Problem 7.19

> *Suppose that $\{c_n\}$ is a monotonically decreasing sequence such that $\lim_{n\to\infty} c_n = 0$. Use Theorem 7.2 to prove that the series $\sum(-1)^n c_n$ converges.*

Suppose that $\{c_n\}$ is a monotonically decreasing sequence such that $\lim_{n\to\infty} c_n = 0$. We'll show that the series $\sum(-1)^n c_n$ converges.

If n is even, then $\sum_{k=1}^{n}(-1)^k = 0$, and when n is odd, $\sum_{k=1}^{n}(-1)^k = -1$. In either case, $\left|\sum_{k=1}^{n}(-1)^k\right| < 2$, so the sequence of partial sums of $\{(-1)^n\}$ is bounded. Then since $\{c_n\}$ is monotically decreasing with $\{c_n\} \to 0$, Theorem 7.2 ensures $\sum(-1)^n c_n$ converges.

□