MATH 411: Week 5

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Problem 8.4

> *Let f be convex on (a,b) and let x,y,z be fixed such that $a < x < y < z < b$. Prove that*
>
> $$\frac{(t-x)f(y)-(t-y)f(x)}{y-x} \le f(t) \le \frac{(z-t)f(y)+(t-y)f(z)}{z-y}$$
>
> *for all t with $y < t < z$, and, similarly,*
>
> $$\frac{(z-s)f(y)-(y-s)f(z)}{z-y} \le f(s) \le \frac{(y-s)f(x)+(s-x)f(y)}{y-x}$$
>
> *for all s with $x < s < y$.*

Let f be convex on (a,b) and let x,y,z be fixed such that $a < x < y < z < b$. Let $y < t < z$. Then by definition,

$$f(t) \le \frac{f(y)(z-t)+f(z)(t-y)}{z-y}.$$

Also, since $x < y < t$,

$$\begin{aligned}
f(y) &\le \frac{(t-y)f(x)+(y-x)f(t)}{t-x} \\
f(y)(t-x) &\le (t-y)f(x)+(y-x)f(t) \\
f(y)(t-x)-(t-y)f(x) &\le (y-x)f(t) \\
\frac{f(y)(t-x)-(t-y)f(x)}{y-x} &\le f(t).
\end{aligned}$$

Now let $x < s < y$. Similarly,

$$\frac{(z-s)f(y)-(y-s)f(z)}{z-y} \le f(s) \le \frac{(y-s)f(x)+(s-x)f(y)}{y-x}.$$

□

Problem 8.6

> *Let f be defined on $[a, b]$. Prove that if f is differentiable at a point $x \in [a, b]$, then f is continuous at x.*

Let f be defined on $[a, b]$ and differentiable at some $x \in [a, b]$. We'll show that f is continuous at x.

Let $c = f'(x)$, which exists since f is differentiable at x. Let $\epsilon > 0$. By definition, there's some $\gamma > 0$ such that if $0 < |t - x| < \gamma$, then

$$|\phi_x(t) - c| < 1\,.$$

Let

$$\delta = \min\left\{\gamma, \frac{\epsilon}{1 + |c|}\right\},$$

which exists and is positive since γ, ϵ, and $1 + |c|$ are all positive.

Let $t \in (a, b)$ be such that $|t - x| < \delta$. If $t = x$, then $\big|f(t) - f(x)\big| = 0 < \epsilon$. If $t \neq x$, then, by definition,

$$\begin{aligned}
f(t) - f(x) &= \phi_x(t)(t - x) \\
&= \big(\phi_x(t) - c + c\big)(t - x) \\
\big|f(t) - f(x)\big| &= \big|\big(\phi_x(t) - c + c\big)(t - x)\big| \\
&= |\phi_x(t) - c + c||t - x| \\
&\leq |t - x|\big(|\phi_x(t) - c| + |c|\big) \\
&< |t - x|\big(1 + |c|\big) \\
&< \frac{\epsilon}{1 + |c|}\big(1 + |c|\big) \\
&= \epsilon\,,
\end{aligned}$$

where the first inequality is justified by the triangle inequality, and the others are justified because $|t - x| < \delta$. We conclude that f is continuous at x.

□

Problem 8.7

> *Suppose that f and g are functions defined on $[a, b]$ and are both differentiable at a point $x \in [a, b]$. Show that $(f + g)'(x) = f'(x) + g'(x)$.*

Let f and g be defined on $[a, b]$ and differentiable at some $x \in [a, b]$. We'll show that $(f+g)'(x) = f'(x) + g'(x)$. By definition,

$$\begin{aligned}
(f+g)'(x) &= \lim_{t \to x} \frac{(f+g)(t) - (f+g)(x)}{t-x} \\
&= \lim_{t \to x} \frac{f(t) + g(t) - f(x) - g(x)}{t-x} \\
&= \lim_{t \to x} \frac{f(t) - f(x) + g(t) - g(x)}{t-x} \\
&= \lim_{t \to x} \left(\frac{f(t) - f(x)}{t-x} + \frac{g(t) - g(x)}{t-x} \right) \\
&= \lim_{t \to x} \frac{f(t) - f(x)}{t-x} + \lim_{t \to x} \frac{g(t) - g(x)}{t-x} \\
&= f'(x) + g'(x)\,,
\end{aligned}$$

which exists since f and g are differentiable at x.

□