MATH 411: Week 5

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Problem 8.4

Let f be convex on (a,b) and let x, y, z be fixed such that $a < x < y < z < b$. Prove that

$$\frac{(t-x)f(y)-(t-y)f(x)}{y-x} \le f(t) \le \frac{(z-t)f(y)+(t-y)f(z)}{z-y}$$

for all t with $y < t < z$, and, similarly,

$$\frac{(z-s)f(y)-(y-s)f(z)}{z-y} \le f(s) \le \frac{(y-s)f(x)+(s-x)f(y)}{y-x}$$

for all s with $x < s < y$.

Let f be convex on (a,b) and let x, y, z be fixed such that $a < x < y < z < b$. Let $y < t < z$. Then by definition,

$$f(t) \le \frac{f(y)(z-t)+f(z)(t-y)}{z-y}\,.$$

Also, since $x < y < t$,

$$\begin{aligned}
f(y) &\le \frac{(t-y)f(x)+(y-x)f(t)}{t-x} \\
f(y)(t-x) &\le (t-y)f(x)+(y-x)f(t) \\
f(y)(t-x)-(t-y)f(x) &\le (y-x)f(t) \\
\frac{f(y)(t-x)-(t-y)f(x)}{y-x} &\le f(t)\,.
\end{aligned}$$

Now let $x < s < y$. Similarly,

$$\frac{(z-s)f(y)-(y-s)f(z)}{z-y} \le f(s) \le \frac{(y-s)f(x)+(s-x)f(y)}{y-x}\,.$$

□

Problem 8.6

> *Let f be defined on $[a, b]$. Prove that if f is differentiable at a point $x \in [a, b]$, then f is continuous at x.*

Let f be defined on $[a, b]$ and differentiable at some $x \in [a, b]$. We'll show that f is continuous at x.

Let $c = f'(x)$, which exists since f is differentiable at x. Let $\epsilon > 0$. Since f is differentiable at x, there's some $\gamma > 0$ such that if $0 < |t - x| < \gamma$, then

$$|\phi_x(t) - c| < \sqrt{\epsilon}\,.$$

If $c \neq 0$, let

$$\delta = \min\left\{\gamma, \frac{\epsilon}{2|c|}, \frac{\sqrt{\epsilon}}{2}\right\}.$$

In this case, $\delta > 0$, since γ, ϵ, and $|c|$ are all positive. Otherwise, let $\delta = \gamma$, in which case obviously $\delta > 0$.

Let $t \in (a, b)$ be such that $|t - x| < \delta$. If $t = x$, then $\big|(f) - f(x)\big| = 0 < \epsilon$. If $t \neq x$, then, by definition,

$$\begin{aligned}
f(t) - f(x) &= \phi_x(t)(t - x) \\
&= \big(\phi_x(t) - c\big)(t - x) + c(t - x) \\
|f(t) - f(x)| &= \big|\big(\phi_x(t) - c\big)(t - x) + c(t - x)\big| \\
&\leq \big|\big(\phi_x(t) - c\big)(t - x)\big| + \big|c(t - x)\big| \\
&= \big|\phi_x(t) - c\big||t - x| + |c||t - x| \\
&< \sqrt{\epsilon} \cdot \frac{\sqrt{\epsilon}}{2} + |c|\frac{\epsilon}{2|c|} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.
\end{aligned}$$

We conclude that f is continuous at x.

□