MATH 411: Week 5

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Problem 8.4

Let f be convex on (a, b) and let x, y, z be fixed such that $a < x < y < z < b$. Prove that

$$\frac{(t-x)f(y) - (t-y)f(x)}{y-x} \le f(t) \le \frac{(z-t)f(y) + (t-y)f(z)}{z-y}$$

for all t with $y < t < z$, and, similarly,

$$\frac{(z-s)f(y) - (y-s)f(z)}{z-y} \le f(s) \le \frac{(y-s)f(x) + (s-x)f(y)}{y-x}$$

for all s with $x < s < y$.

Let f be convex on (a, b) and let x, y, z be fixed such that $a < x < y < z < b$. Let $y < t < z$. Then by definition,

$$f(t) \le \frac{f(y)(z-t) + f(z)(t-y)}{z-y} \, .$$

Also, since $x < y < t$,

$$f(y) \le \frac{(t-y)f(x) + (y-x)f(t)}{t-x}$$
$$f(y)(t-x) \le (t-y)f(x) + (y-x)f(t)$$
$$f(y)(t-x) - (t-y)f(x) \le (y-x)f(t)$$
$$\frac{f(y)(t-x) - (t-y)f(x)}{y-x} \le f(t) \, .$$

Now let $x < s < y$. Similarly,

$$\frac{(z-s)f(y) - (y-s)f(z)}{z-y} \le f(s) \le \frac{(y-s)f(x) + (s-x)f(y)}{y-x} \, .$$

$\square$

Problem 8.6

> *Let f be defined on $[a, b]$. Prove that if f is differentiable at a point $x \in [a, b]$, then f is continuous at x.*

Let f be defined on $[a, b]$ and differentiable at some $x \in [a, b]$. We'll show that f is continuous at x.

Let $c = f'(x)$, which exists since f is differentiable at x. Let $\epsilon > 0$. Since f is differentiable at x, there's some $\gamma > 0$ such that if $0 < |t - x| < \gamma$, then

$$|\phi_x(t) - c| < 1 \,.$$

Let

$$\delta = \min\left\{\gamma, \frac{\epsilon}{1 + |c|}\right\},$$

which exists and is positive, since γ, ϵ, and $1 + |c|$ are all positive.

Let $t \in (a, b)$ be such that $|t - x| < \delta$. If $t = x$, then $\big|(f) - f(x)\big| = 0 < \epsilon$. If $t \neq x$, then, by definition,

$$
\begin{aligned}
f(t) - f(x) &= \phi_x(t)(t - x) \\
&= \big(\phi_x(t) - c + c\big)(t - x) \\
\big|f(t) - f(x)\big| &= \big|\big(\phi_x(t) - c + c\big)(t - x)\big| \\
&= |\phi_x(t) - c + c||t - x| \\
&\leq |t - x|\big(|\phi_x(t) - c| + |c|\big) \\
&< |t - x|\big(1 + |c|\big) \\
&< \frac{\epsilon}{1 + |c|}\big(1 + |c|\big) \\
&= \epsilon \,,
\end{aligned}
$$

where the first inequality is justified by the triangle inequality, and the others are justified because $|t - x| < \delta$. We conclude that f is continuous at x.

$\square$