

# MATH 411: Week 5

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23 February 2026

## Problem 8.4

Let  $f$  be convex on  $(a, b)$  and let  $x, y, z$  be fixed such that  $a < x < y < z < b$ . Prove that

$$\frac{(t-x)f(y) - (t-y)f(x)}{y-x} \leq f(t) \leq \frac{(z-t)f(y) + (t-y)f(z)}{z-y}$$

for all  $t$  with  $y < t < z$ , and, similarly,

$$\frac{(z-s)f(y) - (y-s)f(z)}{z-y} \leq f(s) \leq \frac{(y-s)f(x) + (s-x)f(y)}{y-x}$$

for all  $s$  with  $x < s < y$ .

Let  $f$  be convex on  $(a, b)$  and let  $x, y, z$  be fixed such that  $a < x < y < z < b$ . Let  $y < t < z$ . Then by definition,

$$f(t) \leq \frac{f(y)(z-t) + f(z)(t-y)}{z-y}.$$

Also, since  $x < y < t$ ,

$$\begin{aligned} f(y) &\leq \frac{(t-y)f(x) + (y-x)f(t)}{t-x} \\ f(y)(t-x) &\leq (t-y)f(x) + (y-x)f(t) \\ f(y)(t-x) - (t-y)f(x) &\leq (y-x)f(t) \\ \frac{f(y)(t-x) - (t-y)f(x)}{y-x} &\leq f(t). \end{aligned}$$

Now let  $x < s < y$ . Similarly,

$$\frac{(z-s)f(y) - (y-s)f(z)}{z-y} \leq f(s) \leq \frac{(y-s)f(x) + (s-x)f(y)}{y-x}.$$

□

## Problem 8.6

*Let  $f$  be defined on  $[a, b]$ . Prove that if  $f$  is differentiable at a point  $x \in [a, b]$ , then  $f$  is continuous at  $x$ .*

Let  $f$  be defined on  $[a, b]$  and differentiable at some  $x \in [a, b]$ . We'll show that  $f$  is continuous at  $x$ .

Let  $c = f'(x)$ , which exists since  $f$  is differentiable at  $x$ . Let  $\epsilon > 0$ . By definition, there's some  $\gamma > 0$  such that if  $0 < |t - x| < \gamma$ , then

$$|\phi_x(t) - c| < 1.$$

Let

$$\delta = \min\left\{\gamma, \frac{\epsilon}{1 + |c|}\right\},$$

which exists and is positive since  $\gamma$ ,  $\epsilon$ , and  $1 + |c|$  are all positive.

Let  $t \in (a, b)$  be such that  $|t - x| < \delta$ . If  $t = x$ , then  $|f(t) - f(x)| = 0 < \epsilon$ . If  $t \neq x$ , then, by definition,

$$\begin{aligned} f(t) - f(x) &= \phi_x(t)(t - x) \\ &= (\phi_x(t) - c + c)(t - x) \\ |f(t) - f(x)| &= |(\phi_x(t) - c + c)(t - x)| \\ &= |\phi_x(t) - c + c||t - x| \\ &\leq |t - x|(|\phi_x(t) - c| + |c|) \\ &< |t - x|(1 + |c|) \\ &< \frac{\epsilon}{1 + |c|}(1 + |c|) \\ &= \epsilon, \end{aligned}$$

where the first inequality is justified by the triangle inequality, and the others are justified because  $|t - x| < \delta$ . We conclude that  $f$  is continuous at  $x$ .

□

## Problem 8.7

*Suppose that  $f$  and  $g$  are functions defined on  $[a, b]$  and are both differentiable at a point  $x \in [a, b]$ . Show that  $(f + g)'(x) = f'(x) + g'(x)$ .*

Let  $f$  and  $g$  be defined on  $[a, b]$  and differentiable at some  $x \in [a, b]$ . We'll show

that  $(f + g)'(x) = f'(x) + g'(x)$ . By definition,

$$\begin{aligned}
(f + g)'(x) &= \lim_{t \rightarrow x} \frac{(f + g)(t) - (f + g)(x)}{t - x} \\
&= \lim_{t \rightarrow x} \frac{f(t) + g(t) - f(x) - g(x)}{t - x} \\
&= \lim_{t \rightarrow x} \frac{f(t) - f(x) + g(t) - g(x)}{t - x} \\
&= \lim_{t \rightarrow x} \left( \frac{f(t) - f(x)}{t - x} + \frac{g(t) - g(x)}{t - x} \right) \\
&= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} + \lim_{t \rightarrow x} \frac{g(t) - g(x)}{t - x} \\
&= f'(x) + g'(x),
\end{aligned}$$

which exists since  $f$  and  $g$  are differentiable at  $x$ .

□

## Problem 8.8

*Suppose that  $f$  and  $g$  are functions defined on  $[a, b]$  and are both differentiable at a point  $x \in [a, b]$ . Show that  $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ .*

Let  $f$  and  $g$  be defined on  $[a, b]$  and differentiable at some  $x \in [a, b]$ . We'll show that  $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ .

For all  $t \in [a, b]$ , we denote

$$\begin{aligned}
F_t &= f(t) - f(x); \\
G_t &= g(t) - g(x).
\end{aligned}$$

We observe that

$$\begin{aligned}
f(t)g(t) &= (f(x) + f(t) - f(x))(g(x) + g(t) - g(x)) \\
&= (f(x) + F_t)(g(x) + G_t) \\
&= f(x)g(x) + f(x)G_t + g(x)F_t + F_tG_t.
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{t \rightarrow x} \frac{f(t)g(t) - f(x)g(x)}{t - x} &= \lim_{t \rightarrow x} \frac{f(x)G_t + g(x)F_t + F_tG_t}{t - x} \\
&= f(x) \lim_{t \rightarrow x} \frac{G_t}{t - x} + g(x) \lim_{t \rightarrow x} \frac{F_t}{t - x} + \lim_{t \rightarrow x} F_tG_t \\
&= f(x)g'(x) + g(x)f'(x) + \lim_{t \rightarrow x} F_tG_t.
\end{aligned}$$

Problem 8.6 ensures  $f$  and  $g$  are continuous at  $x$ . So  $t \mapsto F_t G_t$  is continuous by Problem 6.6, and we can say

$$\lim_{t \rightarrow x} F_t G_t = F_x G_x = (f(x) - f(x))(g(x) - g(x)) = 0.$$

So we have

$$\begin{aligned} (fg)'(x) &= \lim_{t \rightarrow x} \frac{(fg)(t) - (fg)(x)}{t - x} \\ &= \lim_{t \rightarrow x} \frac{f(t)g(t) - f(x)g(x)}{t - x} \\ &= f(x)g'(x) + g(x)f'(x). \end{aligned}$$

□