MATH 411: Week 4

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Problem 7.31

Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}\,.$$

(a) Explain (briefly) why the series converges. (Does it converge absolutely?)

(b) If we take the product of the series with itself, then we have

$$c_n = (-1)^n \sum_{k=0}^{\infty} \frac{1}{\sqrt{(n-k+1)(k+1)}}$$

(you should double check this). Show that each term in this sum is greater than or equal to $\frac{1}{\sqrt{n+1}}$*, and thus* $|c_n| \geq \sqrt{n+1}$.

(c) Explain (briefly) why the previous part implies $\sum c_n$ *diverges.*

Define the series $\sum_{n=0}^{\infty} a_n$ by

$$a_n = \frac{(-1)^n}{\sqrt{n+1}}\,.$$

(a) We'll show that $\sum a_n$ converges absolutely. We note that for any $n \in \mathbb{N}_0$,

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = 1\,.$$

Since the sequence $\left\{\frac{1}{n^{1/2}}\right\}$ is a convergent p-series, the limit comparison test ensures $\left\{\frac{1}{\sqrt{n+1}}\right\}$ also converges. Since

$$\left|\frac{(-1)^n}{\sqrt{n+1}}\right| = \frac{1}{\sqrt{n+1}}\,,$$

we conclude that $\sum a_n$ converges absolutely.

(b) We will show that we have $|c_n| > \sqrt{n+1}$ for all n, where $\sum_{n=0}^{\infty} c_n$ is the Cauchy product of $\sum a_n$ with itself. By definition,

$$c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}} \, .$$

If $n \in \mathbb{N}_0$ and $0 \le k \le n$, we have

$$\begin{aligned}
k &\le n \\
k \cdot k &\le nk \\
-k^2 &\ge -nk \\
nk - k^2 + n + 1 &\ge n + 1 \\
(n-k+1)(k+1) &\ge n + 1 \\
\sqrt{(n-k+1)(k+1)} &\ge \sqrt{n+1} \\
\frac{1}{\sqrt{(n-k+1)(k+1)}} &\ge \frac{1}{\sqrt{n+1}} \, ,
\end{aligned}$$

where the last line is justified by the fact that the square root function is nonnegative and that here its arguments are nonzero. Now we can say that for any $n \in \mathbb{N}_0$,

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}} &\ge \sum_{k=0}^{n} \frac{1}{\sqrt{n+1}} \\
&\ge \frac{n+1}{\sqrt{n+1}} \\
&= \sqrt{n+1} \, .
\end{aligned}$$

So for any $n \in \mathbb{N}_0$, we have $|c_n| \ge \sqrt{n+1}$.

(c) We'll show that $\sum c_n$ diverges. The series $\sum_{n=0}^{\infty} \sqrt{n+1}$ diverges, since its terms are unbounded. The comparison test then ensures $\sum c_n$ also diverges.

□

Problem 7.32

Let $c_n = \sum_{k=0}^{n} c_n$ (where c_n is defined as above). Prove that $C_n = a_0 B_{n-1} + \cdots + a_n B_0$ where $B_n = \sum_{k=0}^{n} b_n$. Define $\beta_n = B_n - B$ and $A_n = \sum_{k=0}^{n} a_n$. Explain why the following are true:

(a) $\lim_{n\to\infty} \beta_n = 0$

(b) For all $N \ge 0$, $C_n = A_n B + a_0 \beta_n + a_1 \beta_{n=1} + \cdots + a_n \beta_0$

(c) $\lim_{n\to\infty} A_n B = AB$

Let $\sum_{n=0}^{\infty} A_n$ be an absolutely convergent series with $\sum_{n=0}^{\infty} a_n = A$, and let $\sum_{n=0}^{\infty} b_n$ be a convergent series with $\sum_{n=0}^{\infty} b_n = B$. For all $n \geq 0$, define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, $C_n = \sum_{k=0}^{n} c_k$, and $\beta_n = B_n - B$.

Let $n \geq 0$. We'll show that $C_n = a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0$ where $B_n = \sum_{k=0}^{n} b_k$. We have:

$$\begin{aligned}
C_n &= \sum_{k=0}^{n} c_k \\
&= (a_0 b_0) + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\
&= a_0(b_0 + b_1 + \cdots + b_n) + a_1(b_0 + b_1 + \ldots b_{n-1}) + \cdots + a_n b_0 \\
&= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \,.
\end{aligned}$$

(a) We'll show that

$$\lim_{n\to\infty} \beta_n = 0 \,.$$

Since $\sum b_n = B$, we know $\{B_n\} \to B$, and thus $\{B_n - B\} \to B - B = 0$.

(b) Let $n \geq 0$. We'll show that

$$C_n = A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 \,.$$

We have:

$$\begin{aligned}
\sum_{k=0}^{n} a_k \beta_{n-k} &= \sum_{k=0}^{n} a_k (B_{n-k} - B) \\
&= \sum_{k=0}^{k} a_k B_{n-k} - \sum_{k=0}^{n} a_k B \\
&= \sum_{k=0}^{k} a_k B_{n-k} - A_n B \,.
\end{aligned}$$

We conclude:

$$C_n = \sum_{k_0}^{k} a_k B_{n-k} = \sum_{k=0}^{n} a_k \beta_{n-k} + A_n B \,.$$

(c) We will show that

$$\lim_{n\to\infty} A_n B = AB \,.$$

Since $\{A_n\} \to A$ and $\{B\}_{n=0}^{\infty} \to B$, Problem 5.13 ensures $\{A_n B\} \to AB$.

□

Problem 7.33

Let $\gamma = \sum_{n=0}^{\infty} |a_n|$. Use the first result in problem 7.32 to prove that for any $\delta_1 > 0$, there exists an N_1 such that for all $n > N_1$,

$$|\alpha_0\beta_n + \alpha_1\beta_{n-1} + \cdots + \alpha_{n-N_1}\beta_{N_1}| \leq \delta_1\gamma\,.$$

Let $\gamma = \sum_{n=0}^{\infty} |a_n|$ and $\delta_1 > 0$. We'll show that there exists an $N_1 \in \mathbb{N}$ such that for all $n > N_1$,

$$|\alpha_0\beta_n + \alpha_1\beta_{n-1} + \cdots + \alpha_{n-N_1}\beta_{N_1}| \leq \delta_1\gamma\,.$$

Problem 7.32 ensures that $\{\beta_n\} \to 0$, so there's an $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then

$$|\beta_n| < \delta_1\,.$$

Since $\gamma = \sum_{n=0}^{\infty} |a_n|$ is increasing and $\delta_1 > 0$, we know

$$\sum_{k=0}^{n-N_1} |a_k| \leq \gamma$$
$$\sum_{k=0}^{n-N_1} \delta_1 |a_k| \leq \delta_1\gamma\,,$$

for all $n \geq N_1$. If $0 \leq k \leq n - N_1$, then $n - k \geq N_1$, so

$$|\beta_{n-k}| < \delta_1\,.$$

Thus,

$$\sum_{k=0}^{n-N_1} |\beta_{n-k}||a_k| < \delta_1\gamma\,.$$

The triangle inequality now ensures

$$\left|\sum_{k=0}^{n-N_1} \beta_{n-k}a_k\right| \leq \sum_{k=0}^{n-N_1} |\beta_{n-k}a_k| < \delta_1\gamma\,.$$

□

Problem 7.34

Following along with the notation in the previous two problems:

(a) *Explain why we can choose an M with $M > |\beta_n|$ for all n.*

(b) Then use the fact that $\sum_{n=0}^{\infty} |a_n|$ converges to show that for any $\delta_2 > 0$, there exists an N_2 such that for any $n > N_2$ we have

$$|a_{N_2}\beta_{n-N_2} + a_{N_2+1}\beta_{n-N_2-1} + \cdots + a_n\beta_0| < M\delta_2 \,.$$

(a) Since $\{\beta_n\}$ converges, it must be bounded.

(b) Let $\delta_2 > 0$. Since $\sum_{n=0}^{\infty} |a_n|$ converges, the Cuachy criteron ensures

$$\sum_{k=N_2}^{n} |a_k| \leq \delta_2$$

$$\sum_{k=0}^{n-N_2} |a_{N_2+k}| \leq \delta_2$$

$$\sum_{k=0}^{n-N_2} |a_{N_2+k}| M \leq M\delta_2 \,.$$

By construction, $M > |\beta_{n-N_2-k}|$ for all $k \in \{0, 1, \ldots, n - N_2\}$. So we conclude:

$$\sum_{k=0}^{n-N_2} |a_{N_2+k}||\beta_{n-N_2-k}| < M\delta_2 \,.$$

The triangle inequality then ensures

$$\left| \sum_{k=0}^{n-N_2} a_{N_2+k}\beta_{n-N_2-k} \right| \leq \sum_{k=0}^{n-N_2} |a_{N_2+k}\beta_{n-N_2-k}| < M\delta_2 \,.$$

□

Problem 7.35

Now we use the work of the previous problems to finish the proof of Theorem 7.4. Choose appropriate values for δ_1 and δ_2 (as used in the previous two problems) to prove that for any $\epsilon > 0$ there exists an N (related to N_1 and N_2) such that for any $n \geq N$ we have

$$|a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0| \leq \epsilon \,.$$

Explain (using problem 7.32) why this means $\lim_{n\to\infty} C_n = AB$ and thus $\sum_{n=0}^{\infty} c_n = AB$.

We will show that $\sum_{n=0}^{\infty} c_n = AB$. Let $\epsilon > 0$. Let $\delta_1 = \frac{\epsilon}{2\gamma}$, and $\delta_2 = \frac{\epsilon}{2M}$. Let $N = \max\{N_1, N_2\}$ and $n \geq N$. We have:

$$\begin{aligned}
\left|\sum_{k=0}^{n} a_k \beta_{n-k}\right| &= \left|\sum_{k=0}^{n-N} a_k \beta_{n-k} + \sum_{k=n-M+1}^{n} a_k \beta_{n-k}\right| \\
&\leq \left|\sum_{k=0}^{n-N} a_k \beta_{n-k}\right| + \left|\sum_{k=n-M+1}^{n} a_k \beta_{n-k}\right| \\
&< \delta_1 \gamma + M\delta_2 \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon\,.
\end{aligned}$$

So $\lim_{n\to\infty} \sum_{k=0}^{n} a_k \beta_{n-k} = 0$. Then we have:

$$\begin{aligned}
\lim_{n\to\infty} C_n &= \lim_{n\to\infty} (A_n B) + \lim_{n\to\infty} \sum_{k=0}^{n} a_k \beta_{n-k} \\
&= AB + 0 = AB\,,
\end{aligned}$$

by our results in Problem 7.32.

□

Problem 7.36

> *Suppose that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both absolutely convergent series. Define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Use Theorem 7.4 to prove that $\sum_{n=0}^{\infty} c_n$ is absolutely convergent as well.*

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely. Define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, and $\overline{c_n} = \sum_{k=0}^{n} |a_k||b_{n-k}|$. We'll show that $\sum_{n=0}^{\infty} c_n$ converges absolutely. We notice that

$$|c_n| = \left|\sum_{k=0}^{n} a_k b_{n-k}\right| \leq \sum_{k=0}^{n} |a_k b_{n-k}| = \sum_{k=0}^{n} |a_k||b_{n-k}| = \overline{c_n}\,.$$

Since $\sum_{n=0}^{\infty} |a_n|$ and $\sum_{n=0}^{\infty} |b_n|$ both converge absolutely, Theorem 7.4 ensures that $\sum_{n=0}^{\infty} \overline{c_n}$ converges. The comparison test then ensures $\sum_{n=0}^{\infty} |c_n|$ converges. So $\sum_{n=0}^{\infty} c_n$ converges absolutely.

□

Problem 7.37

> *Let $\sum a_n$ be a divergent series where $a_n > 0$ for all n. Prove that $\sum \frac{a_n}{1+a_n}$ diverges.*

Let $\sum a_n$ be a divergent series with $a_n > 0$ for all n. We'll show that $\sum \frac{a_n}{1+a_n} = \sum \frac{1}{1/a_n + 1}$ diverges.

Assume $\limsup_{n\to\infty} a_n = \infty$. Then there's a subsequence $\{a'_n\}$ of $\{a_n\}$ with $\{a'_n\} \to \infty$. So $\{\frac{1}{a'_n}\} \to 0$, $\{\frac{1}{a'_n} + 1\} \to 1$, and $\frac{1}{1/a'_n + 1} \to 1$. Since there's a subsequence $\{\frac{1}{1/a'_n + 1}\}$ of $\{\frac{a_n}{1+a_n}\}$ that doesn't converge to 0, we know $\{\frac{a_n}{1+a_n}\} \not\to 0$.

Assume $\limsup_{n\to\infty} = L > 0$. Then there's some subsequence $\{a'_n\}$ of $\{a_n\}$ with $\{a'_n\} \to L$. Since $\{a'_n + 1\} \to L + 1$ and $1 + a'_n$ is never zero, we have

$$\left\{\frac{a'_n}{1 + a'_n}\right\} \to \frac{L}{L+1} \neq 0\,.$$

Since there's a subsequence $\left\{\frac{a'_n}{1+a'_n}\right\}$ of $\left\{\frac{a_n}{1+a_n}\right\}$ that doesn't converge to 0, we know $\left\{\frac{a_n}{1+a_n}\right\} \not\to 0$.

Assume $\limsup_{n\to\infty} a_n = 0$. Then there's some $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n < 1$, or $1 - a_n > 0$. So, if $n \geq N$, we have

$$\frac{a_n}{1 + a_n} > \frac{a_n}{1 + a_n + (1 - a_n)} = \frac{a_n}{2}\,.$$

Since $\sum a_n$ diverges, $\sum \frac{a_n}{2}$ also diverges, and the comparison test ensures $\sum \frac{a_n}{1+a_n}$ diverges.

$\square$