

MATH 411: Week 3

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10 February 2026

Lemma 1

Let $a_1, a_2, \dots, a_n \in \mathbb{R}$ be such that for all $i \in \{1, 2, \dots, n\}$ we have $0 \leq a_i < M$ for some $M \in \mathbb{R}^+$. Then,

$$\prod_{i=1}^n a_i < M^n.$$

We proceed by induction. For all $n \in \mathbb{N}$, let P_n be the statement that if $a_1, a_2, \dots, a_n \in \mathbb{R}$ and $0 \leq a_i < M$ for all $i \in \{1, 2, \dots, n\}$ for some $M \in \mathbb{R}^+$, then $\prod_{i=1}^n a_i < M^n$.

If $a_1 \in \mathbb{R}$ and $0 \leq a_1 < M$ for some $M \in \mathbb{R}^+$, then we have:

$$\prod_{i=1}^1 a_i = a_1 < M = M^1.$$

So P_1 holds.

Now assume that P_n holds for some $n \in \mathbb{N}$. Let $a_1, a_2, \dots, a_n, a_{n+1} \in \mathbb{R}$ with $0 \leq a_i < M$ for all $i \in \{1, 2, \dots, n, n+1\}$ for some $M \in \mathbb{R}^+$. If $a_{n+1} = 0$, then since M is positive,

$$\prod_{i=1}^{n+1} a_i = (a_{n+1}) \prod_{i=1}^n a_i = 0 < M^{n+1}.$$

On the other hand, if $a_{n+1} > 0$, we have:

$$\begin{aligned} & \prod_{i=1}^n a_i < M^n \\ & (a_{n+1}) \prod_{i=1}^n a_i < (a_{n+1}) M^n \\ & \prod_{i=1}^{n+1} a_i < (a_{n+1}) M^n. \end{aligned}$$

Since $a_{n+1} < M$ and $M^n > 0$, it follows that $(a_{n+1})M^n < M \cdot M^n = M^{n+1}$. We conclude that

$$\prod_{i=1}^{n+1} a_i < M^{n+1}.$$

We've shown that P_n implies P_{n+1} . So P_n holds for all natural n by induction.

□

Lemma 2

Let $\{a_n\}$ be a monotonic sequence. If $\{a_n\}$ is increasing, then $\{a_n\} \rightarrow \infty$ if and only if $\{a_n\}$ is not bounded above. Likewise, if $\{a_n\}$ is decreasing, then $\{a_n\} \rightarrow -\infty$ if and only if $\{a_n\}$ is not bounded below.

Clearly, if $\{a_n\} \rightarrow \infty$, then $\{a_n\}$ is not bounded above, and if $\{a_n\} \rightarrow -\infty$, then $\{a_n\}$ is not bounded below.

Suppose $\{a_n\}$ is increasing and not bounded above. Then, for any $M > 0$, there's a natural N such that $a_N > M$. Since $\{a_n\}$ is increasing, if $n > N$, then we still have $a_n > M$. So $\{a_n\} \rightarrow \infty$.

Now suppose $\{a_n\}$ is decreasing and not bounded below. Then, for any $M < 0$, there's a natural N such that $a_N < M$. Since $\{a_n\}$ is decreasing, if $n > N$, then we still have $a_n < M$. So $\{a_n\} \rightarrow -\infty$.

□

Lemma 3

Let $\sum a_n$ be a series with sequence of partial sums $\{s_n\}$. If $a_n \geq 0$ for all natural n , then $\{s_n\} \rightarrow \infty$ if and only if $\sum a_n$ diverges. Likewise, if $a_n \leq 0$ for all natural n , then $\{s_n\} \rightarrow -\infty$ if and only if $\sum a_n$ diverges.

If $a_n \geq 0$ for all natural n , then $\{s_n\}$ is clearly monotonically increasing. The proof of Problem 5.23 ensures that $\{s_n\}$ is not bounded above if and only if $\sum a_n$ diverges.

If $a_n \leq 0$ for all natural n , then $\{s_n\}$ is clearly monotonically decreasing. The proof of Problem 5.23 ensures that $\{s_n\}$ is not bounded below if and only if $\sum a_n$ diverges.

Lemma 2 finishes the proof.

□

Problem 7.21

For the series $\sum a_n$, suppose that there exists a real number β with $0 < \beta < 1$ and an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\sqrt[n]{|a_n|} \leq \beta$.
Prove that $\sum a_n$ converges absolutely.

Let $\sum_{n=1}^{\infty} a_n$ be a series, let $\beta \in (0, 1)$, and let $N \in \mathbb{N}$, with $\sqrt[n]{|a_n|} \leq \beta$ for all $n \geq N$. We'll show that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

For all $n \in \mathbb{N}$, since $|a_n|$ is nonnegative, we notice that

$$\begin{aligned}\sqrt[n]{|a_n|} &\leq \beta \\ |a_n| &\leq \beta^n.\end{aligned}$$

Since $\beta \in (0, 1)$, Problem 7.11 ensures the series $\sum_{n=1}^{\infty} \beta^n$ converges. So Problem 7.7 ensures $\sum_{n=1}^{\infty} |a_n|$ converges, since for all $n \in \mathbb{N}$ we certainly have $||a_n|| = |a_n| \leq \beta^n$. So $\sum_{n=1}^{\infty} a_n$ converges absolutely.

□

Problem 7.22

For the series $\sum a_n$, suppose that there exists a real number β with $0 < \beta < 1$ and an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\left| \frac{a_{n+1}}{a_n} \right| < \beta$.

(a) Prove that for all $n \geq N$ we have

$$\left| \frac{a_n}{a_N} \right| \beta^N < \beta^n.$$

(b) Use the result above to prove that $\sum a_n$ converges absolutely.

Let $\sum_{n=1}^{\infty} a_n$ be a series, let $\beta \in (0, 1)$, and let $N \in \mathbb{N}$, with $\left| \frac{a_{n+1}}{a_n} \right| < \beta$ for all $n \geq N$.

(a) Let $n \in \mathbb{N}$ with $n \geq N$. We'll show that

$$\left| \frac{a_n}{a_N} \right| \beta^N < \beta^n.$$

We have:

$$\left| \frac{a_n}{a_N} \right| = \left| \prod_{i=N}^{n-1} \frac{a_{i+1}}{a_i} \right| = \prod_{i=N}^{n-1} \left| \frac{a_{i+1}}{a_i} \right| < \prod_{i=N}^{n-1} \beta = \beta^{n-N}.$$

where the first statement can be easily proven by induction, the second uses a well known fact about the absolute value of a product, and the third is justified by Lemma 1 above, since for each i we have $0 \leq \left| \frac{a_{i+1}}{a_i} \right| < \beta$, and $\beta > 0$.

Since $\beta^N > 0$, we conclude that

$$\left| \frac{a_n}{a_N} \right| \beta^N < \beta^{n-N} \beta^N = \beta^n.$$

(b) We will show that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

We observe that by part (a), for all $n \geq N$, since $\beta^N > 0$,

$$\sqrt[n]{\left| \frac{a_n}{a_N} \beta^N \right|} < \beta.$$

So Problem 7.21 ensures the series $\sum_{n=1}^{\infty} \left| \frac{a_n}{a_N} \beta^N \right|$ converges. A simple consequence of Problem 7.20 is that if a series converges, the series formed by multiplying each of its term by a constant also converges. Hence, since $\beta > 0$, we can say that

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{\beta^N} \right| \left| \frac{a_n}{a_N} \beta^N \right| = \sum_{n=1}^{\infty} |a_n|$$

converges. So $\sum_{n=1}^{\infty} a_n$ converges absolutely.

□

Problem 7.23

Use the comparison test to prove that $\sum a_n$ converges absolutely if and only if both $\sum a_n^+$ and $\sum a_n^-$ converge absolutely.

Let $\sum a_n$ be a series. We'll show that $\sum a_n$ converges absolutely if and only if both $\sum a_n^+$ and $\sum a_n^-$ converge absolutely.

Assume $\sum a_n$ converges absolutely. We observe that for any natural n ,

$$|a_n^+| \leq |a_n|,$$

because if $a_n \geq 0$, then $|a_n^+| = |a_n|$, and if $a_n < 0$, then $|a_n| > 0 = |0| = |a_n^+|$. Likewise, we have

$$|a_n^-| \leq |a_n|,$$

because if $a_n \geq 0$, then $|a_n| \geq 0 = |0| = |a_n^-|$, and if $a_n < 0$, then $|a_n^-| = |a_n|$. The comparison test ensures that both $\sum a_n^+$ and $\sum a_n^-$ converge absolutely.

Now assume $\sum a_n^+$ and $\sum a_n^-$ converge absolutely. We note that, by construction, for any natural n ,

$$|a_n| = |a_n^+ + a_n^-| \leq |a_n^+| + |a_n^-|.$$

Since $\sum a_n^+$ and $\sum a_n^-$ converge absolutely, $\sum(|a_n^+| + |a_n^-|)$ converges. The comparison test ensures $\sum a_n$ converges absolutely.

□

Problem 7.24

Prove that if $\sum a_n$ converges conditionally then $\sum a_n^+$ must go off to ∞ and $\sum a_n^-$ must go off to $-\infty$.

Let $\sum a_n$ be a series that converges conditionally to L . We'll show that the sequence of partial sums $\{s_n\}$ of $\sum a_n^+$ converges to ∞ and the sequence of partial sums $\{t_n\}$ of $\sum a_n^-$ converges to $-\infty$.

We first note that, for any natural n ,

$$\sum_{k=1}^n a_k = \sum_{k=1}^n (a_k^+ + a_k^-) = \sum_{k=1}^n a_k^+ + \sum_{k=1}^n a_k^- = s_n + t_n.$$

Let $\epsilon > 0$. Then there's an N_1 such that if $n \geq N_1$, then $|s_n + t_n - L| < \epsilon$. Let $M < 0$, and suppose $\{s_n\}$ is not bounded above. Then there's an N_2 such that if $n \geq N_2$, then $s_n > -M + \epsilon + L$. Let $n = \max\{N_1, N_2\}$. Then we have

$$\begin{aligned} s_n &> -M + \epsilon + L \\ s_n - L + M &> \epsilon. \\ s_n - L + M &> s_n + t_n - L \\ M &> t_n. \end{aligned}$$

So $\{t_n\}$ is not bounded below. On the other hand, let $M > 0$, and suppose $\{t_n\}$ is not bounded below. Then there's an N_2 such that if $n \geq N_2$, then $t_n < -M + \epsilon + L$. Let $n = \max N_1, N_2$. Then we have

$$\begin{aligned} t_n &< -M + \epsilon + L \\ t_n - LM &< \epsilon. \\ t_n - L + M &< s_n + t_n - L \\ M &< s_n. \end{aligned}$$

So $\{s_n\}$ is not bounded above.

Now we observe that $\sum a_n^+$ and $\sum a_n^-$ each converge if and only if they converge absolutely. The case of $\sum a_n^+$ is trivial, since $|a_n^+| = a_n^+$ for all n . In the case of $\sum a_n^-$, we note that $|a_n^-| = -a_n^-$ for all n . Problem 7.20 ensures that if $\sum a_n^-$ converges, then so does $\sum -a_n^- = \sum |a_n^-|$. Problem 7.23 ensures that at least one of $\sum a_n^+$ and $\sum a_n^-$ does not converge absolutely. So one of them must diverge.

If $\sum a_n^+$ diverges, then Lemma 3 ensures $\{s_n\}$ converges to ∞ , which as we just noted implies $\{t_n\}$ is not bounded below, which by Lemma 2 means $\{t_n\}$ converges to $-\infty$. Likewise, if $\sum a_n^-$ diverges, then Lemma 3 ensures $\{s_n\}$ converges to $-\infty$, which as we just noted implies $\{s_n\}$ is not bounded above, which by Lemma 2 means $\{s_n\}$ converges to ∞ . In either case, $\{s_n\} \rightarrow \infty$ and $\{t_n\} \rightarrow -\infty$.

□

Problem 7.25

Let $\sum a_n$ be a conditionally convergent series.

- (a) Prove that for any $n \in \mathbb{N}$ and any $M > 0$, there exists a $q > n$ such that

$$\sum_{k=n}^q a_n^+ > M.$$

- (b) Prove that for any $M > 0$ and $\epsilon > 0$, that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$ there exists a $q > n$ with

$$M < \sum_{k=n}^q a_n^+ < M + \epsilon.$$

Let $\sum a_n$ be a conditionally convergent series. Let s_n denote the n -th partial sum of $\sum a_n^+$.

- (a) Let $n \in \mathbb{N}$ and $M > 0$. We will show there exists a $q > n$ such that

$$s_q - s_{n-1} = \sum_{k=n}^q a_k^+ > M.$$

Problem 7.24 ensures that the sequence of partial sums of $\sum a_n^+$ converges to ∞ . So there is a $q \in \mathbb{N}$ such that $s_q > M + s_{n-1}$. We have:

$$s_q - s_{n-1} > M + s_{n-1} - s_{n-1} = M.$$

- (b) Let $M > 0$ and $\epsilon > 0$. We will show that there exists an $N \in \mathbb{N}$ such that for any $n \geq N$ there exists a $q > n$ with

$$M < \sum_{k=n}^q a_n^+ < M + \epsilon.$$

Let $\{p_{n_k}\}$ be the subsequence of $\{a_n\}$ whose terms are positive. Since $\sum a_n$ converges, Problem 7.5 and Problem 5.15 ensure that $\{p_{n_k}\} \rightarrow 0$. We note

that every term in $\{a_n^+\}$ is either a term of $\{p_{n_k}\}$ or is equal to 0. Since every neighborhood around 0 contains all but finitely many terms of $\{p_{n_k}\}$ and certainly contains 0, we conclude that $\{a_n^+\} \rightarrow 0$.

Let $n \in \mathbb{N}$ be such that if $p \geq n$, then we have

$$a_p^+ = |a_p^+ - 0| < \min\{\epsilon, M\},$$

which exists because ϵ and M are positive. Let $Q > n$ be such that

$$\sum_{k=n}^Q a_k^+ > M,$$

as guaranteed by part (a). Let m be the largest element of $\{n, n+1, \dots, Q\}$ that satisfies

$$\sum_{k=n}^m a_k^+ \leq M.$$

m exists, since $a_n^+ < M$. We know that

$$\sum_{k=n}^{m+1} a_k^+ > M,$$

since $m \neq Q$ and thus $m+1 \in \{n, n+1, \dots, Q\}$, but $m+1 > m$. We also have:

$$\begin{aligned} \sum_{k=n}^m a_k^+ &\leq M \\ \sum_{k=n}^m a_k^+ + a_{m+1}^+ &\leq M + a_{m+1}^+ \\ \sum_{k=n}^{m+1} a_k^+ &\leq M + a_{m+1}^+. \end{aligned}$$

Further, $a_{m+1}^+ < \epsilon$, since $m+1 > m \geq n$. So we have:

$$\sum_{k=n}^{m+1} a_k^+ \leq M + a_{m+1}^+ < M + \epsilon.$$

So $q = m+1$ has the desired properties.

□

Problem 7.26

Prove that for any $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, if $\sum a_n$ converges conditionally, there exists a rearrangement of the series $\sum a'_n$ with partial sums s'_n such that

$$\liminf_{n \rightarrow \infty} s'_n = \alpha, \text{ and } \limsup_{n \rightarrow \infty} s'_n = \beta.$$

We will not prove this formally, but will instead informally construct such a rearrangement and justify why it has the desired properties.

We construct our sequence $\{a'_n\}$ by alternating between two steps:

1. taking enough terms of $\{a_n^+\}$ so that their sum is sufficiently large in magnitude to make the whole sum greater than β but as small as possible; and
2. taking enough terms of $\{a_n^-\}$ so that their sum is sufficiently large in magnitude to make the whole sum less than α but as large as possible.

In this process, we skip terms from $\{a_n^+\}$ and $\{a_n^-\}$ where $a_n^+ \neq a_n$ or $a_n^- \neq a_n$. Since this can only occur when $a_n = 0$, it does not change the incremental sums. Since $\sum a_n^+$ goes off to ∞ and $\sum a_n^-$ goes off to $-\infty$, we are assured that we can always take enough terms of each to produce the stated behavior.

Every term of $\{a_n\}$ is either in $\{a_n^+\}$ or in $\{a_n^-\}$. Since any term of $\{a_n^+\}$ and $\{a_n^-\}$ that is in $\{a_n\}$ is eventually represented, we know that every term of $\{a_n\}$ is represented. Further, no term is repeated, since neither $\{a_n^+\}$ nor $\{a_n^-\}$ repeat terms from $\{a_n\}$, and neither one contains terms of $\{a_n\}$ that the other contains. So we are justified in claiming that $\{a'_n\}$ is a rearrangement of $\{a_n\}$.

There are an infinite number of terms of $\{s'_n\}$ greater than β —namely, every partial sum obtained by summing up to the end of the terms taken in any instance of step (1) above. Likewise, there are an infinite number of terms of $\{s'_n\}$ less than α . Moreover, these are the only terms of $\{s'_n\}$ above β and below α , since within a step the sum is monotonic, we made each post-step sum as small in magnitude as possible, and after each step, the terms of $\{a'_n\}$ change sign.

We've shown in previous problems that since $\sum a_n$ converges, $\{a_n^+\} \rightarrow 0$ and $\{a_n^-\} \rightarrow 0$. Let $\epsilon > 0$. So there's an N such that if $n \geq N$, we have $a_n^+ < \epsilon$ and $a_n^- > -\epsilon$. Consider a term s'_n of $\{s'_n\}$ with $n \geq N$ such that $s'_n > \beta$. We know that $s'_{n-1} \leq \beta$, since s'_n is as small as possible. So we have:

$$\begin{aligned} s'_n - \beta &\leq s'_n - s'_{n-1} = a_n^+ < \epsilon \\ s'_n &< \beta + \epsilon. \end{aligned}$$

Likewise, for a term $s'_n < \alpha$, we have:

$$\begin{aligned}s'_n - \alpha &\geq s'_n - s'_{n-1} = a_n^- > -\epsilon \\ s'_n &> \alpha - \epsilon.\end{aligned}$$

We've shown that for any given $\epsilon > 0$, there are an infinite number of terms of $\{s'_n\}$ in the ϵ -neighborhood around α and β —in particular, all the partial sums after a certain point obtained by summing up to the end of the terms taken in any step above. We conclude that α and β are accumulation points for $\{s'_n\}$.

Further, since no other terms—besides the ones that we've just shown converge to α and β —are above β or below α , we know that β is the biggest accumulation point and α is the smallest accumulation point.

□

Problem 7.28

Suppose that $\sum a_n$ converges and $a_n \geq 0$ for all n . Prove that $\sum \frac{\sqrt{a_n}}{n}$ converges as well.

Suppose that $\sum a_n$ converges and $a_n \geq 0$ for all n . We'll show that $\sum \frac{\sqrt{a_n}}{n}$ converges as well. The Schurz inequality ensures that for any N ,

$$\left(\sum_{n=1}^N \sqrt{a_n} \cdot \frac{1}{n} \right)^2 \leq \left(\sum_{n=1}^N a_n \right) \left(\sum_{n=1}^N \frac{1}{n^2} \right).$$

$\sum a_n$ converges by assumption and $\sum \frac{1}{n^2}$ converges since it's a p -series with $p > 1$. Problem 5.13 then ensures that

$$\left\{ \left(\sum_{n=1}^N a_n \right) \left(\sum_{n=1}^N \frac{1}{n^2} \right) \right\}_{n=1}^\infty$$

converges. We note that, for all N ,

$$\sum_{n=1}^N \sqrt{a_n} \cdot \frac{1}{n} \geq 0,$$

since $a_n \geq 0$. Thus, the squeeze theorem ensures

$$\left\{ \left(\sum_{n=1}^N \sqrt{a_n} \cdot \frac{1}{n} \right)^2 \right\}_{n=1}^\infty$$

converges. ???

□

Problem 7.29

Let $\sum a_n x^n$ be a power series for some x . Let

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

and

$$R = \begin{cases} 0 & \alpha = \infty \\ \frac{1}{\alpha} & 0 < \alpha < \infty \\ \infty & \alpha = 0 \end{cases} .$$

We will show that $\sum a_n x^n$ converges absolutely for all $|x| < R$ and diverges for all $|x| > R$.

Assume $\alpha = \infty$. Then $\{\sqrt[n]{|a_n|}\}$ is unbounded and for any $M > 0$ there's an N such that if $n \geq N$ then

$$\begin{aligned} \sqrt[n]{|a_n|} &> M \\ |a_n| &> M^n. \end{aligned}$$

So $\{|a_n|\}$ is unbounded and hence $\sum a_n$