MATH 430: HW 3

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Exercise 3.38

> *There is an isomorphism of U_7 with $\mathbb{Z}_7$ in which $\zeta = e^{i(2\pi/7)} \leftrightarrow 4$. Find the element in $\mathbb{Z}_7$ to which ζ^m must correspond for $m = 0,2,3,4,5,$ and 6.*

We have:

$$\begin{aligned}
\zeta^0 = 1 &\leftrightarrow 0 \\
\zeta^1 = \zeta &\leftrightarrow 4 \\
\zeta^2 = \zeta \cdot \zeta &\leftrightarrow 4 +_7 4 = 1 \\
\zeta^3 = \zeta^2 \cdot \zeta &\leftrightarrow 1 +_7 4 = 5 \\
\zeta^4 = \zeta^3 \cdot \zeta &\leftrightarrow 5 +_7 4 = 2 \\
\zeta^5 = \zeta^4 \cdot \zeta &\leftrightarrow 2 +_7 4 = 6 \\
\zeta^6 = \zeta^5 \cdot \zeta &\leftrightarrow 6 +_7 4 = 3\,.
\end{aligned}$$

Exercise 3.42

> a. *Derive a formula for $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$ using Euler's formula.*
>
> b. *Derive the formula $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ from part (a) and the identity $\sin^2\theta + \cos^2\theta = 1$.*

(a) By Euler's formula,

$$\begin{aligned}
e^{i\cdot 3\theta} &= \left(e^{i\theta}\right)^3 \\
\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\
&= (\cos\theta + i\sin\theta)(\cos^2\theta + i\cdot 2\sin\theta\cos\theta - \sin^2\theta)\,.
\end{aligned}$$

We have:

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta\\ \cos 3\theta &= \cos^3\theta - 3\sin^2\theta\cos\theta\,.\end{aligned}$$

(b) We have:

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\sin^2\theta\cos\theta\\ &= \cos\theta(\cos^2\theta - 3\sin^2\theta)\end{aligned}$$

???

Exercise 4.5

We have:

$$\sigma^{-1}\tau\sigma = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 2 & 6 & 1 & 5 & 4 & 3\end{pmatrix}\,.$$

Exercise 4.8

$$\begin{aligned}\sigma^{100} = \sigma^{96+4} = \left(\sigma^6\right)^{16}\sigma^4 = e^{16}\sigma^4 = \sigma^4\\ = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 6 & 5 & 2 & 1 & 3 & 4\end{pmatrix}\,.\end{aligned}$$

Exercise 4.10

We have:

$$\begin{aligned}\sigma &= (1,3,4,5,6,2)\\ \tau &= (1,2,4,3)(5,6)\\ \mu &= (1,5)(3,4)\,.\end{aligned}$$

Exercise 4.13a

We have:

$$\mu\rho^2\mu\rho^8 = \mu^2\rho^{12-2}\rho^8 = \rho^{10+8} = \rho^6\,.$$

Exercise 4.32

Let $n \geq 3$. We will show that the only element σ of S_n satisfying $\sigma\gamma = \gamma\sigma$ for all $\gamma \in S_n$ is $\sigma = \iota$.

Let $\sigma \in S_n$ with $\sigma \neq \iota$. Then there's some $x \in \{1, 2, \ldots, n\}$ such that $\sigma(x) = y \neq x$. Let $\gamma \in S_n$ be some bijection that sends $x \mapsto x$, $y \mapsto z$, and $z \mapsto y$, where $z \neq x$ and $z \neq y$. γ exists, since $n \geq 3$. Then,

$$\sigma\gamma(x) = \sigma(x) = y \neq z = \gamma(y) = \gamma\sigma(x)\,.$$

We've shown that for any $\sigma \neq \iota$ there's a $\gamma \in S_n$ such that $\sigma\gamma \neq \gamma\sigma$. We conclude that ι is the only element of S_n satisfying $\sigma\gamma = \gamma\sigma$ for all $\gamma \in S_n$.

□

Exercise 5.59

Let S by any subset of a group G. We will show that

$$H_S = \{s \in G \mid xs = sx \text{ for all } s \in S\}$$

is a subgroup of G. Let $a, b \in H_S$, and let $s \in S$. Then we have:

$$(ab)s = a(bs) = a(sb) = (as)b = (sa)b = s(ab)\,.$$

So $ab \in H_S$. Let e be the identity in G, and let $s \in S$. Then we have:

$$es = s = se\,,$$

by the definition of the identity. So $e \in H_S$. Since $H_S \subseteq G$, we conclude that H_S is a subgroup of G.

□