MATH 430: HW 3

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Exercise 3.38

> *There is an isomorphism of U_7 with $\mathbb{Z}_7$ in which $\zeta = e^{i(2\pi/7)} \leftrightarrow 4$. Find the element in $\mathbb{Z}_7$ to which ζ^m must correspond for $m = 0,2,3,4,5,$ and 6.*

We have:

$$\begin{aligned}
\zeta^0 = 1 &\leftrightarrow 0 \\
\zeta^1 = \zeta &\leftrightarrow 4 \\
\zeta^2 = \zeta \cdot \zeta &\leftrightarrow 4 +_7 4 = 1 \\
\zeta^3 = \zeta^2 \cdot \zeta &\leftrightarrow 1 +_7 4 = 5 \\
\zeta^4 = \zeta^3 \cdot \zeta &\leftrightarrow 5 +_7 4 = 2 \\
\zeta^5 = \zeta^4 \cdot \zeta &\leftrightarrow 2 +_7 4 = 6 \\
\zeta^6 = \zeta^5 \cdot \zeta &\leftrightarrow 6 +_7 4 = 3\,.
\end{aligned}$$

Exercise 3.42

> *a. Derive a formula for $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$ using Euler's formula.*
>
> *b. Derive the formula $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ from part (a) and the identity $\sin^2\theta + \cos^2\theta = 1$.*

(a) By Euler's formula,

$$\begin{aligned}
e^{i\cdot 3\theta} &= \left(e^{i\theta}\right)^3 \\
\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\
&= (\cos\theta + i\sin\theta)(\cos^2\theta + i\cdot 2\sin\theta\cos\theta - \sin^2\theta)\,.
\end{aligned}$$

We have:

$$
\begin{aligned}
\cos 3\theta &= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta \\
\cos 3\theta &= \cos^3\theta - 3\sin^2\theta\cos\theta\,.
\end{aligned}
$$

(b) We have:

$$
\begin{aligned}
\cos 3\theta &= \cos^3\theta - 3\sin^2\theta\cos\theta \\
&= \cos\theta(\cos^2\theta - 3\sin^2\theta)
\end{aligned}
$$

???

Exercise 4.5

We have:

$$
\sigma^{-1}\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 5 & 4 & 3 \end{pmatrix}\,.
$$

Exercise 4.8

$$
\begin{aligned}
\sigma^{100} = \sigma^{96+4} = \left(\sigma^6\right)^{16}\sigma^4 = e^{16}\sigma^4 = \sigma^4 \\
= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}\,.
\end{aligned}
$$

Exercise 4.10

We have:

$$
\begin{aligned}
\sigma &= (1,3,4,5,6,2) \\
\tau &= (1,2,4,3)(5,6) \\
\mu &= (1,5)(3,4)\,.
\end{aligned}
$$

Exercise 4.13a

We have:

$$
\mu\rho^2\mu\rho^8 = \mu^2\rho^{12-2}\rho^8 = \rho^{10+8} = \rho^6\,.
$$

Exercise 4.32

Let $n \geq 3$, and denote $\Sigma = \{1, 2, \ldots, n\}$. We will show that the only element σ of S_n satisfying $\sigma\gamma = \gamma\sigma$ for all $\gamma \in S_n$ is $\sigma = \iota$. Let $\sigma \in S_n$ with $\sigma \neq \iota$. Then there's some $x \in \Sigma$ such that $\sigma(x) = y \neq x$. Let $\gamma : \Sigma \to \Sigma$ be such that $\gamma(y) = z$, $\gamma(z) = y$, and $\gamma(v) = v$ for all other $v \in \Sigma$. γ is clearly bijective, so $\gamma \in S_n$. Then,

$$\sigma\gamma(x) = \sigma(x) = y \neq z = \gamma(y) = \gamma\sigma(x) \,.$$

We've shown that for any $\sigma \neq \iota$ there's a $\gamma \in S_n$ such that $\sigma\gamma \neq \gamma\sigma$. We conclude that ι is the only element of S_n satisfying $\sigma\gamma = \gamma\sigma$ for all $\gamma \in S_n$.

□

Exercise 5.59

Let S by any subset of a group G. We will show that

$$H_S = \{s \in G \mid xs = sx \text{ for all } s \in S\}$$

is a subgroup of G. Let $a, b \in H_S$, and let $s \in S$. Then we have:

$$(ab)s = a(bs) = a(sb) = (as)b = (sa)b = s(ab) \,.$$

So $ab \in H_S$. Let e be the identity in G, and let $s \in S$. Then we have:

$$es = s = se \,,$$

by the definition of the identity. So $e \in H_S$. Since $H_S \subseteq G$, we conclude that H_S is a subgroup of G.

□