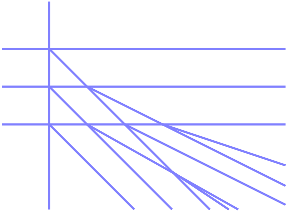
Finite-order approximations of scattering diagrams



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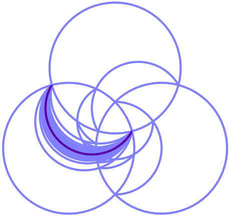


Motivation

Scattering diagrams are piece-wise linear geometric objects which can be used to visualize the exchange graph of a cluster algebra and construct a canonical basis (in many cases).

Yet they may be defined without ever referring to cluster algebras!

At heart, they are a geometric visualization of commutation relations inside a group $\widehat{\mathbb{E}}(B)$; equivalently, a commutative diagram involving ring automorphisms called elementary transformations.

The initial ingredient is a skew-symmetric $r \times r$ integral matrix B.

$$\widehat{\mathcal{F}}(\mathrm{B}) := \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_r^{\pm 1}][[y_1, y_2, \ldots, y_r]]$$

Some notation! Let $m \in \mathbb{Z}^r$ and $n \in \mathbb{N}^r$.

$$x^m := x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}, \quad \gcd(n) := \gcd(n_1, n_2, \ldots, n_r)$$

Def: Formal elementary transformations

For non-zero $n \in \mathbb{N}^r$, the formal elementary transformation $E_{n,\mathrm{B}}$ is the automorphism of $\widehat{\mathcal{F}}(\mathrm{B})$ given by

$$E_{n,\mathrm{B}}(x^m) = (1 + x^{\mathrm{B}n} y^n)^{\frac{n \cdot m}{\gcd(n)}} x^m, \quad E_{n,\mathrm{B}}(y^{n'}) = y^{n'}$$

While $\frac{n \cdot m}{\gcd(n)}$ must be an integer, it may be negative (that's ok!).

Throughout, $\mathrm{J} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the simplest non-trivial B.

Examples!

Let $\mathrm{B} = \mathrm{J}$.

$$E_{(1,0)}(x_1) = (1 + x_2 y_1)x_1, \quad E_{(1,0)}(x_2) = x_2$$

$$E_{(1,0)}(x_1^{-1}) = x_1(1 - x_2 y_1 + x_2^2 y_1^2 - x_2^3 y_1^3 + \cdots)$$

$$E_{(0,1)}E_{(1,0)}(x_2) = (1 + (1 + x_1^{-1} y_2)x_2 y_1)x_2$$

$$E_{(1,0)}E_{(0,1)}(x_2) = (1 + x_2 y_1)x_2$$

Exercise

For any B, let $n, n' \in \mathbb{N}^r$ be such that $n \cdot \mathrm{B}n' = 1$. Prove that

$$E_n E_{n'} = E_{n'} E_{n+n'} E_n$$

as automorphisms of $\widehat{\mathcal{F}}$.

This fundamental relation implies others. Let B, n, n' as above.

$$\begin{aligned} E_n^2 E_{n'} &= E_n(E_{n'} E_{n+n'} E_n) \\ &= (E_{n'} E_{n+n'} E_n) E_{n+n'} E_n \\ &= E_{n'} E_{n+n'}^2 E_{2n+n'} E_n^2 \end{aligned}$$

Exercise

Let B, n, n' as above. Prove that

$$E_n^3 E_{n'} = E_{n'} E_{n+n'}^3 E_{3n+2n'} E_{2n+n'}^3 E_{3n+n'} E_n^3$$

by repeatedly using the previous exercise.

We also want to have infinite limits of automorphisms. Since $\widehat{\mathcal{F}}$ is a topological ring, $Aut(\widehat{\mathcal{F}})$ has a topology of pointwise convergence.

$$\widehat{\mathbb{E}}(\mathrm{B}) := \overline{\text{group generated by } \{E_{n,\mathrm{B}} \mid n \in \mathbb{N}^r\}} \subset Aut(\widehat{\mathcal{F}}(\mathrm{B}))$$

Elements of $\widehat{\mathbb{E}}(\mathrm{B})$ are infinite products of FETs and their inverses, which have finitely many copies of any given element.

Exercise

Let B and n be arbitrary. Prove that

$$E_n E_{2n} E_{4n} E_{8n} \cdots E_{2^k n} \cdots$$

converges to the automorphism of $\widehat{\mathcal{F}}$ which sends

$$x^m \mapsto (1 - x^{\mathrm{B}n} y^n)^{-\frac{n \cdot m}{\gcd(n)}} x^m \text{ and } y^{n'} \mapsto y^{n'}$$

It will often be useful to work with $\widehat{\mathbb{E}}(\mathrm{B})$ to finite order. Let

$$\mathfrak{m} := \langle y_1, y_2, \ldots, y_r \rangle \subset \widehat{\mathcal{F}}$$

Each $E_{n,\mathrm{B}}$ descends to an automorphism of $\widehat{\mathcal{F}}/\mathfrak{m}^d$ for all d. Then

$$\widehat{\mathbb{E}}(\mathrm{B}) = \varprojlim \left(\text{group gen. by } \{E_{n,\mathrm{B}} \mid n \in \mathbb{N}^r\} \subset Aut(\widehat{\mathcal{F}}(\mathrm{B})/\mathfrak{m}^d) \right)$$

That is, we only need finite products when working to finite order.

Exercise

Let B, n, n' be arbitrary. Prove that

$$E_n E_{n'} = E_{n'} E_n \text{ in } Aut(\widehat{\mathcal{F}}/\mathfrak{m}^d)$$

if $y^{n+n'} \in \mathfrak{m}^d$, and that

$$E_n E_{n'} = E_{n'} E_{n+n'}^{\lambda} E_n \text{ in } Aut(\widehat{\mathcal{F}}/\mathfrak{m}^d), \; \lambda = \frac{n \cdot \mathrm{B} n' \gcd(n+n')}{\gcd(n)\gcd(n')}$$

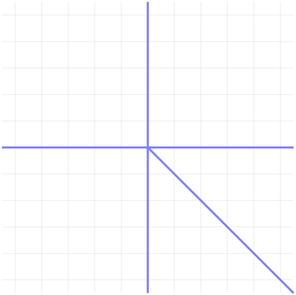
if $y^{2n+n'}, y^{n+2n'} \in \mathfrak{m}^d$.

Goal

Use affine geometric objects to visualize relations in $\widehat{\mathbb{E}}(\mathrm{B})$.

Commutative diagrams will become scattering diagrams!

$$\begin{tikzcd}
\widehat{\mathcal{F}} \arrow[dd, "E_{(0,1)}"'] & & \widehat{\mathcal{F}} \arrow[ll, "E_{(1,0)}"'] \arrow[d, "E_{(0,1)}"] \\
& & \widehat{\mathcal{F}} \arrow[dl, "E_{(1,1)}"] \\
\widehat{\mathcal{F}} & \widehat{\mathcal{F}} \arrow[l, "E_{(1,0)}"] &
\end{tikzcd}$$



Elementary walls

Given B, an **(affine elementary) wall** is a pair (n, W) of

- a non-zero $n \in \mathbb{N}^r$, and
- an affine polyhedral cone $W \subset \mathbb{R}^r$ which spans an affine hyperplane normal to n.

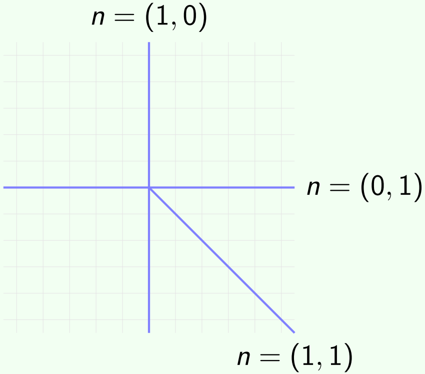
If $r = 2$, W must be a line or a ray in $\mathbb{R}^2$.

Scattering diagrams

Given B, an **(affine) scattering diagram** is a multiset of walls which, for each n, has only finitely many walls with that n.

Examples

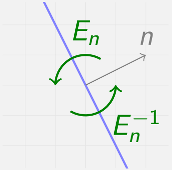
Let $\mathrm{B} = \mathrm{J}$. Then an example scattering diagram is below.



Note that n is determined by W and $\gcd(n)$.
Lazyness: For unlabeled walls, assume n has $\gcd(n) = 1$.

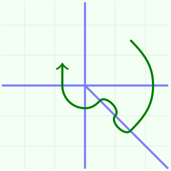
Geometric/algebraic correspondence: idea

A wall is a 'prism' which acts by E_n as we pass through it from side n points in, and by E_n^{-1} the other way.



Following this rule, we associate a path-ordered product to any path p in $\mathfrak{D}$ which avoids collisions of non-parallel walls.

Example



Path-ordered product:

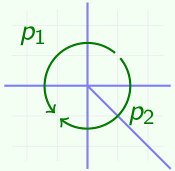
$$E_{(0,1)}^{-1} E_{(1,0)} E_{(1,1)} E_{(1,1)}^{-1} E_{(1,1)} E_{(0,1)}$$

Consistency

A scattering diagram is **consistent** (resp. **consistent mod $\mathfrak{m}^d$**) if every pair of paths with the same end points have the same path-ordered product in $Aut(\widehat{\mathcal{F}})$ (resp. $Aut(\widehat{\mathcal{F}}/\mathfrak{m}^d)$).

Sufficient: the POP around every small loop is the identity.

Example



Path-ordered prod. of $p_1 = E_{(0,1)}E_{(1,0)}$

Path-ordered prod. of $p_2 = E_{(1,0)}E_{(1,1)}E_{(0,1)}$

Consistent by fund. relation ✓

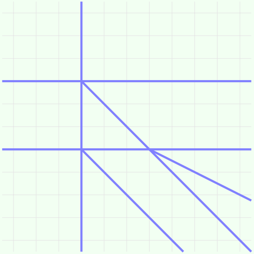
Exercise

Prove that a scattering diagram consisting of walls supported on hyperplanes is consistent mod $\mathfrak{m}^2$.

Consistent scattering diagrams encode multiple identities in $\widehat{\mathbb{E}}(\mathsf{B})$.

Example

The following scattering diagram with $\mathsf{B} = \mathsf{J}$ is consistent.



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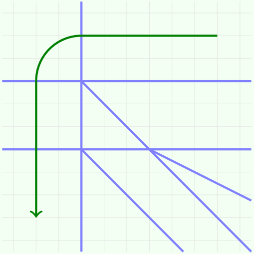

$$E_{(0,1)}^2 E_{(1,0)}$$

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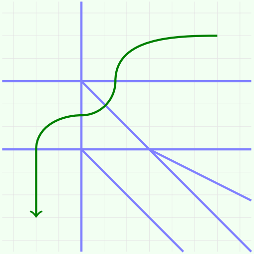

$$
\begin{aligned}
&E_{(0,1)}^2 E_{(1,0)} \\
&\qquad = E_{(0,1)} E_{(1,0)} E_{(1,1)} E_{(0,1)}
\end{aligned}
$$

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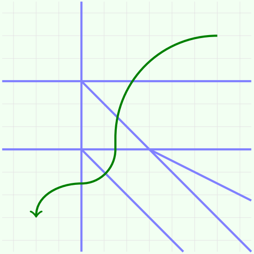

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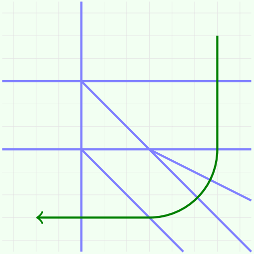

$$
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&E_{(0,1)}^2 E_{(1,0)} \\
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&\qquad = E_{(1,0)} E_{(1,1)}^2 E_{(1,2)} E_{(0,1)}^2
\end{aligned}
$$

A wall (n, W) is outgoing if $\{p + \mathbb{R}_{\geq 0} Bn\} \not\subset W$ for all p.

Consistent completion theorem [GSP, KS, GHKK]

Given a scattering diagram consistent mod $\mathfrak{m}^d$, there is an essentially unique way to add outgoing walls to make it consistent.

The proof is constructive, and adds new walls order-by-order.

- Given a scattering diagram consistent mod $\mathfrak{m}^d$, compute the path-ordered product around tiny loops mod $\mathfrak{m}^{d+1}$.
- Add outgoing walls to make these products trivial. (It should not be obvious how to do this yet!)
- Repeat, and take the limit as $d \to \infty$.

For consistency mod $\mathfrak{m}^{d'}$, stop after $(d' - d)$-many steps.

Sage Goal 1

Implement the consistent completion algorithm to finite-order.

Sage Goal 1

Implement the consistent completion algorithm to finite-order.

Goal 1 Status: Crudely implemented for $B = J$.

Class: `ScatteringDiagram(walls)`

A finite-order scattering diagram for J, where `walls` is a list of

- `SDWall(n,point=p)`: a wall normal to `n` through `p`.

Some associated methods:

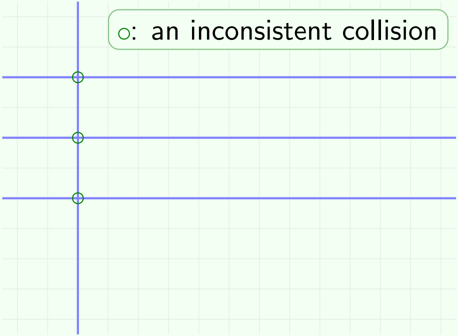
- `.improve()` adds outgoing walls to increase order of consistency by one.
- `.draw()` plots the walls and collisions.

How do we actually find the outgoing walls for `.improve()`?

If two walls collide with n_1, n_2 such that $n_1 \cdot \mathrm{B} n_2 = \pm 1$, the fundamental relation says: add a wall with normal $n_1 + n_2$.

Example

Let $\mathrm{B} = \mathrm{J}$, as usual. Start with 4 hyperplane walls.



Consistent mod $\mathfrak{m}^2$

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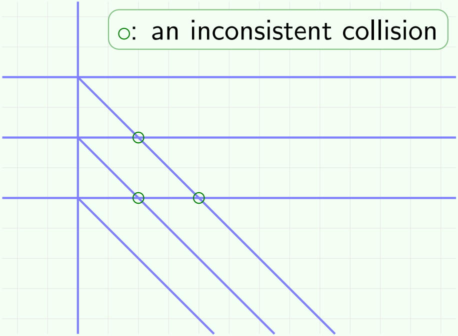

Consistent mod $\mathfrak{m}^3$

If two walls collide with n_1, n_2 such that $n_1 \cdot \mathrm{B} n_2 = \pm 1$, the fundamental relation says: add a wall with normal $n_1 + n_2$.

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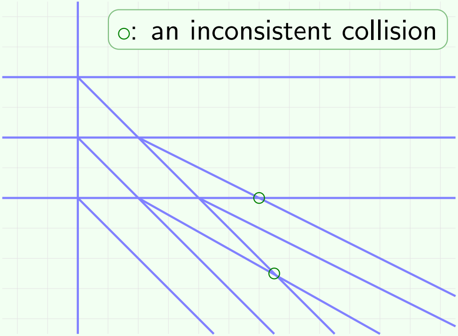

Consistent mod $\mathfrak{m}^4$

If two walls collide with n_1, n_2 such that $n_1 \cdot \mathrm{B} n_2 = \pm 1$, the fundamental relation says: add a wall with normal $n_1 + n_2$.

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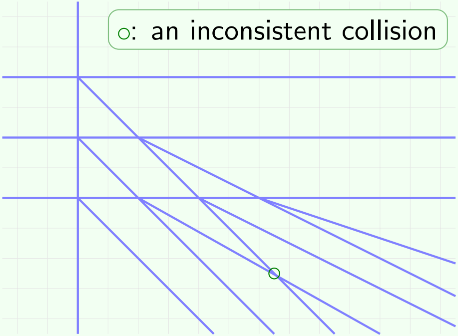

Consistent mod $\mathfrak{m}^5$

If two walls collide with n_1, n_2 such that $n_1 \cdot \mathrm{B} n_2 = \pm 1$, the fundamental relation says: add a wall with normal $n_1 + n_2$.

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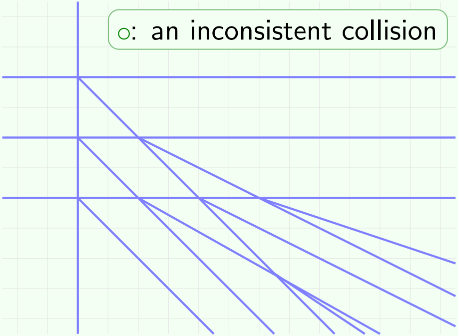

Consistent mod $\mathfrak{m}^6$

If two walls collide with n_1, n_2 such that $n_1 \cdot \mathrm{B} n_2 = \pm 1$, the fundamental relation says: add a wall with normal $n_1 + n_2$.

Example

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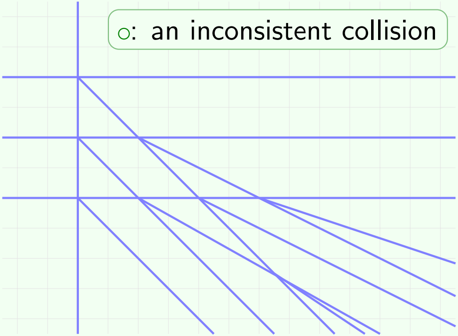
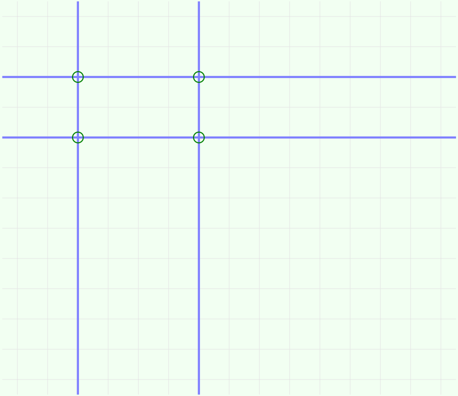


In fact, consistent!

Example

Let $B = J$, as usual. Start with 4 hyperplane walls.



Consistent mod $\mathfrak{m}^2$

Example

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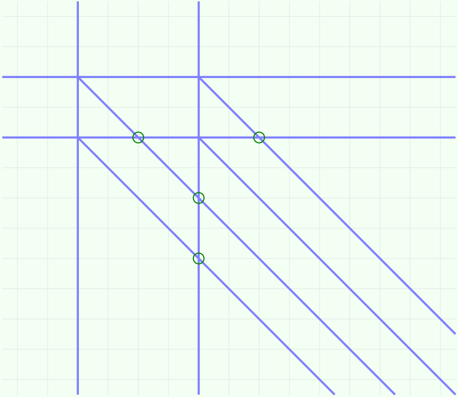

Consistent mod $\mathfrak{m}^3$

Example

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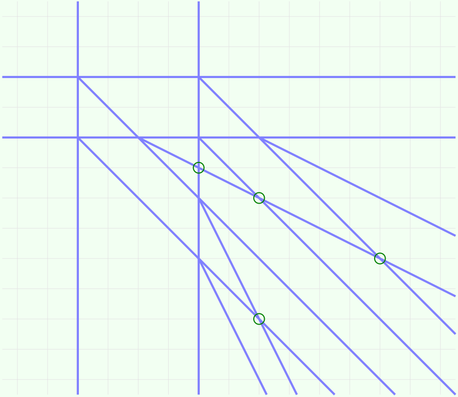

Consistent mod $\mathfrak{m}^4$

Example

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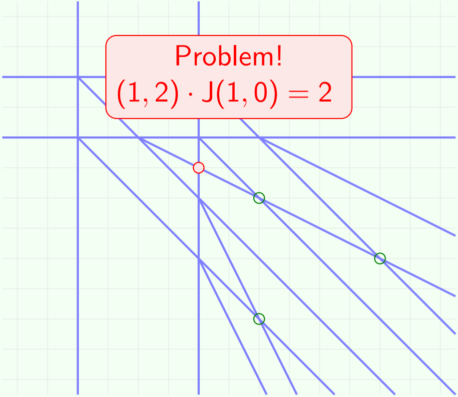


Consistent mod $\mathfrak{m}^4$

We have gone as far as the fund. relation will take us...or have we?

What walls do we need to add to an arbitrary collision?

Key trick: all consistent collisions between pairs of walls reduces to understanding certain consistent scattering diagrams for $\mathrm{B} = \mathrm{J}$.

$$\mathfrak{D}(b, c) := \text{cons. comp. of } \{b \cdot (e_1, e_1^{\perp}), c \cdot (e_2, e_2^{\perp})\} \text{ for } \mathrm{B} = \mathrm{J}$$

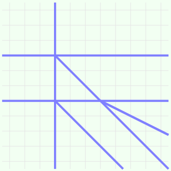
These diagrams help us understand generic collisions as follows.

Local models for generic collisions (rough idea)

A collision between two walls (n_1, W_1) and (n_2, W_2) in a consistent scattering diagram is locally equivalent to an affine transformation of $\mathfrak{D}\left(\frac{n_1 \cdot \mathrm{B} n_2}{\gcd(n_1)}, \frac{n_1 \cdot \mathrm{B} n_2}{\gcd(n_2)}\right)$, though the wall multiplicities can change.

Simple example

Consider the consistent scattering diagram below.



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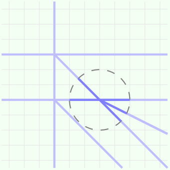

$n_1 = (0, 1)$

$n_2 = (1, 1)$

Simple example

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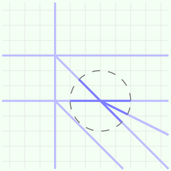
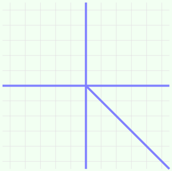


$n_1 = (0, 1)$

$n_2 = (1, 1)$

$$\frac{n_1 \cdot \mathrm{B} n_2}{\gcd(n_1)} = \frac{n_1 \cdot \mathrm{B} n_2}{\gcd(n_2)} = 1$$

So, the collision should look locally like $\mathfrak{D}(1, 1)$.



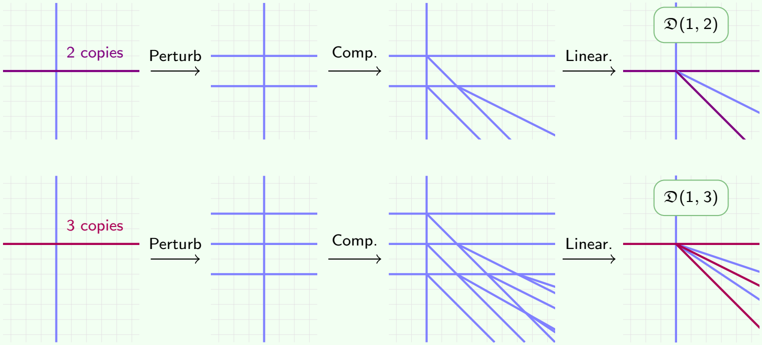
$\mathfrak{D}(1, 1)$

In general, collisions which look locally like $\mathfrak{D}(1, 1)$ are an instance of the fund. relation.

Great! So, how can we compute the $\mathfrak{D}(b, c)$?

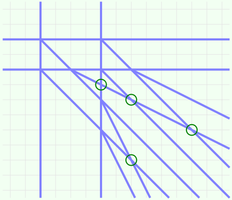
We can find $\mathfrak{D}(b, c)$ by taking the input walls, perturbing them, computing the cons. comp., and then linearizing the walls.

Examples



Let's return to the problem from before!

Example (resumed)



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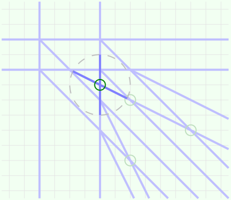

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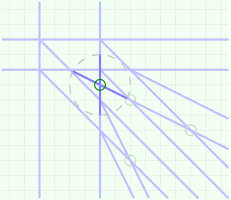
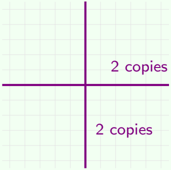


Local model $\longrightarrow$



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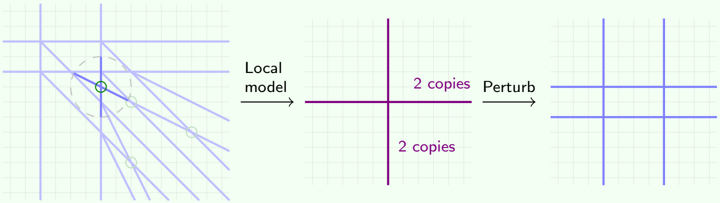
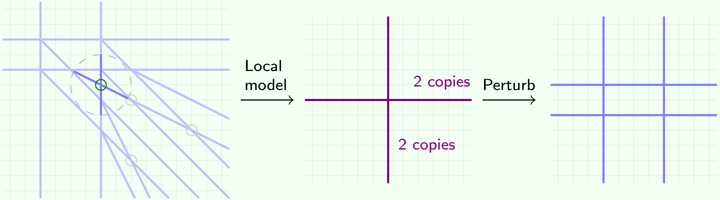


Dang it! Back where we started! Problem:

$$\text{We need } \mathfrak{D}(2,2) \text{ to compute } \mathfrak{D}(2,2)$$

Let's return to the problem from before!

Example (resumed)



Dang it! Back where we started! Solution:

$$\text{We need } \mathfrak{D}(2,2) \bmod \mathfrak{m}^{d} \text{ to compute } \mathfrak{D}(2,2) \bmod \mathfrak{m}^{2d}$$

A giant recursive computation

$\mathfrak{D}(b, c)$ may be computed to any finite order, using only finitely many scattering diagrams of the form $\mathfrak{D}(b', c')$ to lower order.

Hence, approximating any $\mathfrak{D}(b, c)$ to any finite order is suitable to computer implementation!

Since these are the building blocks of all consistent scattering diagrams, this is a great place to start.

Sage Goal 1.A

Implement a table of finite-order approximations of scattering diagrams of the form $\mathfrak{D}(b, c)$, which dynamically increases each diagram's order as needed by internal and external computations.

Goal 1.A status: Crudely implemented.

Class: `SDTable()`

Initializes a dictionary of model scattering diagrams.

- `.diagrams`: A dictionary with key:value pairs

  $$\texttt{(b,c)}\ :\ \text{the current finite-order approx. of } \mathfrak{D}(b,c)$$

- `.multiplicity((b,c),n)`: Returns the multiplicity of the wall with normal `n` in $\mathfrak{D}(b,c)$.
- `.mtable((b,c),d)`: Prints a table of multiplicities in $\mathfrak{D}(b,c)$ with order $\leq d$.

Both methods create and improve diagrams as needed to achieve the required order of consistency.

Sage Goal 1.B

Implement linear scattering diagrams with $r = 3$ with corresponding `.improve()`.

Reasons linear scattering diagrams with $r = 3$ shouldn't be so bad:

- Collisions between walls are a line or ray.
- Maybe visualized using stereographic projection.
- Are completely determined by a certain 2-dimensional 'slice'.

Intuitively, linear $r = 3$ is still 'essentially 2 dimensional'.

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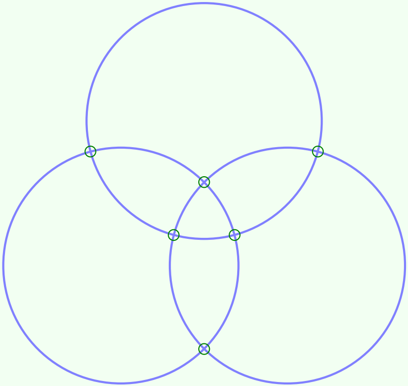
Goal 1.B Status: Not implemented (some stereo. proj. code).

Example

Consider a scattering diagram in $\mathbb{R}^3$ with a wall for each coordinate plane, visualized with a stereographic projection.

$$B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$



Consistent mod $\mathfrak{m}^2$

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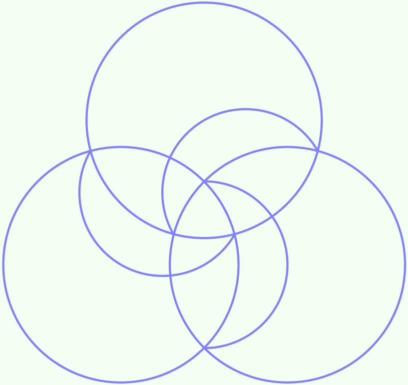

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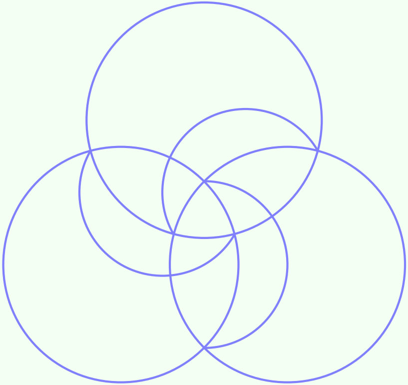


Consistent ✓

What about cluster algebras? Given B, let

$$\mathfrak{D}(\mathrm{B}) := \text{cons. comp. } \{(e_i, e_i^{\perp}) \mid 1 \leq i \leq r\} \text{ for B}$$

$$\mathcal{A}(\mathrm{B}) := \text{cluster algebra of B}$$

Chamber: connected component in the complement of the walls.
Reachable: connected to positive orthant by a path which crosses finitely-many walls.

Cluster combinatorics from $\mathfrak{D}(\mathrm{B})$ [GHKK]

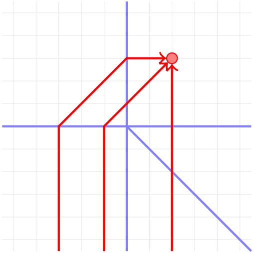
There is a bijection

$$\text{clusters of } \mathcal{A}(\mathrm{B}) \xrightarrow{\sim} \text{reachable chambers of } \mathfrak{D}(\mathrm{B})$$

which sends a cluster to its cone of g-vectors.

Equivalently, the g-fan is the union of the reachable chambers.

For each $m \in \mathbb{Z}^r$, there is a formal series Θ_m called a theta function whose coefficients count certain broken lines in $\mathfrak{D}(\mathrm{B})$.



$$\Theta_{(0,-1)} = x^{(-1,0)} + x^{(-1,-1)} + x^{(0,-1)}$$
$$= \frac{x_2 + 1 + x_1}{x_1 x_2}$$

Cluster algebra from $\mathfrak{D}(\mathrm{B})$ [GHKK]

Every cluster monomial is the theta function of its g-vector, and (in many cases) the theta functions are a basis for $\mathcal{A}(\mathrm{B})$.

Convergence of theta functions is still an open question.

Sage Goal 2

Use finite-order approximations of $\mathfrak{D}(B)$ to study cluster algebras.

I have two specific research questions in mind.

Sage Goal 2.A

Is there a B such that $\mathfrak{D}(B)$ has more than two reachable components of open chambers?

Sage Goal 2.B

When B corresponds to the once-punctured torus, do the non-reachable theta functions coincide with the notched arc elements of Fomin, Shapiro, and Thurston?

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Goal 2 status: 'tis a consummation devoutly to be wished.