Uniform Integrability, Pathwise Lebesgue-Stieltjes Integration and Limit Equalities of Stochastic Integrals

Jacob Miller

This paper will prove a result in the theory of stochastic integration (Theorem 0.1, below). In the service of the proof, it will also provide short expositions on uniform integrability, L^p convergence of martingales and properties of Lebesgue-Stieltjes intergration w.r.t. quadratic variation paths of $\mathscr{M}_2^c$ martingales. The goal is to provide ample cross-reference among sources on measure theory, stochastic analysis and probability theory. All proofs and examples are original unless otherwise stated and as such all mistakes are mine alone.

Let $(\Omega, \mathscr{F}, P)$ be a complete, filtered probability space with a right-continuous filtration $\{\mathscr{F}_t : 0 \leq t < \infty\}$ such that $\mathscr{F}_0$ contains every P-null set in $\mathscr{F}$ (that is, suppose the filtration satisfies the usual hypotheses). Following the notation of [K & S] (References) we will let $\mathscr{M}_2$ denote the class of real-valued, right-continuous martingales M w.r.t $\{\mathscr{F}_t\}$ having $E\left[M_t^2\right] < \infty$ for all $0 \leq t < \infty$ and $M_0 = 0$ a.s. P. We will further define $\mathscr{M}_2^c \triangleq \{M \in \mathscr{M}_2 : M \text{ continuous}\}$ and for an $M \in \mathscr{M}_2$ will let $\langle M \rangle$ denote the unique, natural increasing process in the Doob-Meyer decomposition of M^2 ([K & S] Definition 1.5.3), the uniqueness being up to indistinguishability. That is, we let $\langle M \rangle$ be the unique (up to indistinguishability) adapted, natural increasing process such that $\langle M \rangle_0 = 0$ a.s. and $M^2 - \langle M \rangle$ is a martingale. We will call $\langle M \rangle$ the quadratic variation process of such an M. Definitions of natural and increasing processes are taken as those in given in [K & S] and are provided in Appendix A. For an $M \in \mathscr{M}_2^c$, we will abuse the notation of [K & S] and let $\mathscr{L}^*(M)$ denote the set containing every progressively measureable process $X = \{X_t : 0 \leq t < \infty\}$ satisfying

$$E \int_{[0,t]} X_s^2 \, d\langle M \rangle_s < \infty$$

for all $t > 0$, where the integral is to be understood in the Lebesgue-Stieltjes sense. For an $M \in \mathscr{M}_2^c$ and an $X \in \mathscr{L}^*(M)$, we will denote the Itô stochastic integral of X w.r.t. M by $I_t(X) = \int_0^t X_s dM_s$.

A positive measure will refer to a measure taking values in $[0, \infty]$. Letting $(X, \mathscr{M}, \mu)$ be a positive measure space and letting $\mathscr{B}(Y)$ denote the Borel σ-algebra on a topological space Y, a function $f : X \to \mathbb{R}$ which is measurable $\mathscr{M}/\mathscr{B}(\mathbb{R})$ will simply be referred to as a real measurable function on X. Let $\mathbb{R}_+ \triangleq [0, \infty)$. We would like to prove the following:

Theorem 0.1. *([K & S] Problem 3.2.18) Let $M = \{M_t : 0 \leq t < \infty\}$ and $N = \{N_t : 0 \leq t < \infty\}$ be in $\mathscr{M}_2^c$ and suppose that X and Y are progressively measurable w.r.t $\{\mathscr{F}_t\}$ satisfying*

$$E \int_{\mathbb{R}_+} X_s^2 d\langle M \rangle_s < \infty, \;\; E \int_{\mathbb{R}_+} Y_s^2 d\langle N \rangle_s < \infty.$$

Then:

a) The processes $I_t^M(X) \triangleq \int_0^t X_s \, dM_s$ and $I_t^N(Y) \triangleq \int_0^t Y_s \, dN_s$ are UI (Definition 1.1);

b) $I_t^M(X)$ and $I_t^N(Y)$ converge P-a.s. (as $t \to \infty$) to random variables $I_\infty^M(X)$ and $I_\infty^N(Y)$ respectively s.t. $\{I_t^M(X) : 0 \leq t \leq \infty\}$ and $\{I_t^N(Y) : 0 \leq t \leq \infty\}$ are martingales w.r.t. $\{\mathscr{F}_t : 0 \leq t \leq \infty\}$, where

$\mathscr{F}_\infty \triangleq \sigma\left(\cup_{0\le t<\infty} \mathscr{F}_t\right)$;

c) The cross-variation process $\langle I^M(X), I^N(Y)\rangle_t$ (Definition 2.1) converges to a real number as $t \to \infty$ a.s., and

$$E\left[I^M_\infty(X) I^N_\infty(Y)\right] = E\left[\langle I^M(X), I^N(Y)\rangle_\infty\right] = E\int_{\mathbb{R}_+} X_s Y_s d\langle M, N\rangle_s \ .$$

1 Uniform Integrability

Uniform integrability is ubiquitous in martingale theory; the powerful concept is widespread across the proofs of fundamental theorems. However, it is also a concept that receives differing treatment among measure theory and probability theory sources, where different definitions and perspectives are sometimes taken. This note attempts to bridge the gap in a small way. This note also provides a proof of L^p convergence for certain martingale processes, which we will use in the proof of Theorem 0.1. In this section, we will take A to be an arbitrary index set.

Definition 1.1. *(Uniformly Integrable (UI)) Let $(X, \mathscr{M}, \mu)$ be a positive measure space and let $\{f_\alpha\}_{\alpha\in A}$ be a family of real $L^1(\mu)$ functions on X. We will call this family uniformly integrable if*

$$\lim_{c\to\infty} \sup_\alpha \int_{\{|f_\alpha|>c\}} |f_\alpha| d\mu = 0. \tag{1}$$

We note that UI implies a decay of measure uniformly in α for sets of form $\{|f_\alpha| > c\}$ at a rate such that $\sup_\alpha \left[c\mu\left(\{|f_\alpha| > c\}\right)\right] \to 0$ as $c \to \infty$. We also note the lack of care we are using in calling a function f_α a member of $L^1(\mu)$, following custom.

Definition 1.2. *(Uniformly Integrable Second Type (UIST)) Let $(X, \mathscr{M}, \mu)$ be a positive measure space. Let $\{f_\alpha\}_{\alpha\in A}$ be a family of real $L^1(\mu)$ functions on X. Call this family UIST if for all $\epsilon > 0$ there exists a $\delta > 0$ s.t.*

$$\int_E |f_\alpha| d\mu < \epsilon$$

whenever $\alpha \in A$, $E \in \mathscr{M}$ and $\mu(E) < \delta$.

Remark 1.3. *We can note as in [Billingsly] pg. 216 that if we let $\{f_\alpha\}_{\alpha\in A}$ be a family of real, measurable functions on a positive finite measure space and if we assume it satisfies* (1)*, we obtain for C large enough:*

$$\int_X |f_\alpha| d\mu = \int_{|f_\alpha|>C} |f_\alpha| d\mu + \int_{|f_\alpha|\le C} |f_\alpha| d\mu < 1 + C\mu(X) \tag{2}$$

for all $\alpha \in A$, a uniform L^1 bound in α. Hence, taking $(X, \mathscr{M}, \mu)$ to be a probability space in Definition 1.1, we obtain a definition about random variables that is equivalent to those found in [Chung], [Doob] and [Billingsly]; although these sources do not explicitly assume $L^1(\mu)$ for all $\alpha \in A$, we see from (2) *that this property is implied for the r.v.s in these definitions.*

UIST is found in [Bogachev] and therein is referred to as having "Uniformly Absolutely Continuous Integrals." In some sources such as [Rudin] and [Folland], UIST is weakened by replacing $\int_E |f_\alpha| d\mu < \epsilon$ with $\left|\int_E f_\alpha \, d\mu\right| < \epsilon$ in Definition 1.2; in these sources, this weakened version of UIST is taken as the definition of "Uniformly Integrable." Though we will make infrequent use of this weakened version, we will refer to it as UIWST.

Remark 1.4. *Let $\{f_\alpha\}_{\alpha\in A}$ be a family of real $L^1(\mu)$ functions on a positive measure space $(X, \mathscr{M}, \mu)$. If $\{f_\alpha\}$ is UI then it is also UIST. We can see this by noting as in [Bogachev] Proposition 4.5.3 that for fixed $\epsilon > 0$ and for all $\alpha \in A$ we have,*

$$\int_E |f_\alpha| d\mu = \int_{E\cap\{|f_\alpha|>C\}} |f_\alpha| d\mu + \int_{E\cap\{|f_\alpha|\le C\}} |f_\alpha| d\mu < \epsilon/2 + \epsilon/2$$

for C large enough and $\mu(E)$ small enough. However, the converse is not true, even if we restrict to the case of $\mu(X) < \infty$. To see this, simply take $X = \{1, 2\}$, $\mathscr{M} = \mathscr{P}(X)$,

$$\mu(\{x\}) = 1/2 \;\; (x \in X) \; ; \; \mu(\{1, 2\}) = 1 \; ; \; \mu(\emptyset) = 0,$$

and the family $\{f_n\}_{n\in\mathbb{N}}$ such that

$$f_n(x) = \begin{cases} n & x = 1 \\ 0 & x = 2 \end{cases} .$$

$\{f_n\}$ is UIST (for any $\epsilon > 0$ simply take $\delta = 1/3$) but for all $c \ge 0$ we have

$$\sup_n \int_{\{|f_n|>c\}} |f_n| d\mu \ge \sup_n \int_{\{|f_n|>\lceil c\rceil\}} |f_n| d\mu \ge \int_{\{|f_{\lceil c\rceil+1}|>\lceil c\rceil\}} |f_{\lceil c\rceil+1}| d\mu = (\lceil c\rceil + 1)(1/2) \ge (1/2),$$

so $\{f_n\}$ is not UI.

To obtain a version of the converse for $\{f_\alpha\}_{\alpha\in A}$, then, we must make additional assumptions. There exist two options when $\mu(X) < \infty$. The first is to assume that μ has no atoms. This, together with UIST, obtains a uniform L^1 bound on f_α (see the proof of [Bogachev] Proposition 4.5.3) and using the following proposition then obtains UI:

Proposition 1.5. *Let $(X', \mathscr{M}', \mu')$ be a finite positive measure space and let $\{f_\phi\}_{\phi\in\Phi}$ be a family of real $L^1(\mu')$ functions on X' over arbitrary index set Φ. The family is UI iff the following hold:*
a) The family is uniformly bounded in L^1;
b) The family is UIST.

Proposition 1.5 is a generalized version of [Chung] Theorem 4.5.3 where Chung is of course dealing with random variables in the case of $\mu(X') = 1$. We have already shown the forward direction, and one can follow the proof in [Chung] to obtain the backward direction.

Proposition 1.5 also tells us of an alternative assumption that we can make when $\mu(X) < \infty$ and $\{f_\alpha\}_{\alpha\in A}$ is UIST; we may assume a uniform L^1 bound on f_α and this additional assumption will imply $\{f_\alpha\}$ UI.

Lastly, we state and discuss a very important result which combines [Rudin] Exercise 6.11 and [Chung] Exercise 4.5.8. This is a powerful result and will be used twice in our proof of Theorem 0.1.

Lemma 1.6. *Let $(X, \mathscr{M}, \mu)$ be a positive measure space. Suppose that $\mu(X) < \infty$, and let $\{f_\alpha\}_{\alpha\in A}$ be a family of real measurable functions on X. If there exists a real $p > 1$ and $0 \le C < \infty$ such that $\int_X |f_\alpha|^p d\mu \le C$ for all α, then $\{f_\alpha\}_{\alpha\in A}$ is UI and UIST.*

Proof. Fix $\epsilon > 0$ and let $q = p/(p-1)$ (conjugate to p). $\forall\alpha \in A$ and $\forall E \in \mathscr{M}$, we use Hölder's Inequality to obtain,

$$\int_E |f_\alpha| d\mu = \int_X |f_\alpha| I_E I_E d\mu \le \left(\int_X (|f_\alpha| I_E)^p \, d\mu\right)^{1/p} \left(\int_X (I_E)^q d\mu\right)^{1/q} =$$

$$\left(\int_E |f_\alpha|^p d\mu\right)^{1/p} \mu(E)^{1/q} \le C^{1/p}\mu(E)^{1/q} ,$$

and the last term can be brought less than ϵ for $\mu(E)$ small enough. Hence, $\{f_\alpha\}$ is UIST. Replacing E with X in the proceeding chain obtains a uniform bound in α on $\int_X |f_\alpha| d\mu$. Applying Proposition 1.5 we see $\{f_\alpha\}$ is UI.

□

L^p boundedness such as that discussed in Lemma 1.6 is very important in martingale theory and if Lemma 1.6 holds for a right-continuous martingale then quite a bit is said about that martingale. As such, let M be a real-valued, right-continuous martingale w.r.t. our filtration $\{\mathscr{F}_t\}$. Suppose that there exists a real $p > 1$ and $0 \le C < \infty$ s.t. $E[|M_t|^p] \le C$ for all $t \in \mathbb{R}_+$. We will show that $\lim_{t\to\infty}\left(E[|M_t - M_\infty|^p]\right)^{1/p} = 0$ for some r.v. $M_\infty \in L^p$ and we will use this fact in the proof of Theorem 0.1.

First, we note that since $|M|$ is a right-continuous and non-negative submartingale, [R & Y] Theorem 2.1.7 (Doob's L^p-inequality) tells us that $||\ \sup_t |M_t|\ ||_p \le \frac{p}{p-1}\sup_t ||M_t||_p \le \frac{p}{p-1}C^{1/p}$ (the measurability of $\sup_t |M_t|$ is also therein justified owing to the right-continuity). It follows that $(\sup_t |M_t|)^p \in L^1$. Fix $\epsilon > 0$. We now have for all $t \in \mathbb{R}_+$ and $c \ge 0$,

$$\int_{\{|M_t|^p > c\}} |M_t|^p dP \le \int_{\{(\sup_t |M_t|)^p > c\}} \left(\sup_t |M_t|\right)^p dP. \tag{3}$$

We state the useful result of [Rudin] Exercise 1.12:

Lemma 1.7. *([Rudin] Exercise 1.12) Let $(X, \mathscr{M}, \mu)$ be a positive measure space and let $\{f\}$ be a family consisting of a single $L^1(\mu)$ function. Then $\{f\}$ is UIST. (End Lemma 1.7)*

Again using Doob's L^p-inequality ([R & Y] Theorem 2.1.7), we have for $C > 0$

$$P\left(\left\{\left(\sup_t |M_t|\right)^p \ge C\right\}\right) = P\left(\left\{\sup_t |M_t| \ge C^{1/p}\right\}\right) \le \frac{\sup_t E[|M_t|^p]}{C}$$

and hence by Lemma 1.7 we may choose C large enough that $c \ge C$ implies that the right-hand side of (3) is less than ϵ, implying the same for the left-hand side for arbitrary $t \in \mathbb{R}_+$. Therefore, we see that $\{|M_t|^p\}$ is UI. It follows also from Lemma 1.6 that $\{M_t\}$ is UI. We can therefore invoke [K & S] Theorem 3.15 (Submartingale Convergence) to see that $M_t \to M_\infty$ a.s. for some L^1 r.v. M_∞. We now state Vitali's Convergence Theorem as presented in [Folland]

(Vitali Convergence Theorem) Let $(X, \mathscr{M}, \mu)$ be a positive measure space. Suppose $1 \le p < \infty$ and $\{f_n\}_1^\infty \subset L^p$. In order for $\{f_n\}$ to be Cauchy in the L^p norm it is necessary and sufficient for the following to hold:

(i) For every $\epsilon > 0$, $\mu(\{x : |f_n(x) - f_m(x)| \ge \epsilon\}) \to 0$ as $m, n \to \infty$;
(ii) $\{|f_n|^p\}$ is UIWST;
(iii) For every $\epsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \infty$ and $\int_{E^c} |f_n|^p d\mu < \epsilon$ for all n.

(End Vitali Convergence Theorem)

Fix $\{t_n\}_{n=1}^\infty \subset \mathbb{R}_+$ s.t. $t_n \uparrow \infty$. We have the following:

a) The almost sure convergence of M_{t_n} to M_∞ as $n \to \infty$ implies (i) for $\{M_{t_n}\}$ ([Folland] Page 62);
b) $\{|M_{t_n}|^p\}$ is UI $\Rightarrow$ UIST $\Rightarrow$ UIWST by our earlier remarks;

c) By finiteness in measure of our probability space, (iii) holds for $\{M_{t_n}\}$.

It follows that $\{M_{t_n}\}$ is Cauchy in the L^p norm and therefore

$$\lim_{n\to\infty} ||M_{t_n} - M_\infty||_p = 0,$$

with $M_\infty \in L^p$.
Since $\{t_n\}_{n=1}^\infty$ was arbitrary, we have

$$\lim_{t\to\infty} ||M_t - M_\infty||_p = 0$$

as desired.

We lastly remark that a martingale can be uniformly bounded in L^1 and simultaneously not be UI; for an example, see the remark following Theorem 2.1.7 in [R & Y]. We therefore see the profound interest in the sharpness of the condition $p > 1$ in Lemma 1.6.

2 Pathwise Lebesgue-Stieltjes Integration w.r.t. Quadratic Variation

We provide a short note on properties of Lebesgue-Stieltjes integrals w.r.t. quadratic variation paths of $\mathscr{M}_2^c$ martingales.

Following [K & S] we make:

Definition 2.1. *(Cross-Variation) For M, N in $\mathscr{M}_2$, define their cross-variation process* $\{\langle M, N\rangle_t : 0 \le t < \infty\}$ *by*

$$\langle M, N\rangle_t \triangleq \frac{1}{4}\left[\langle M+N\rangle_t - \langle M-N\rangle_t\right].$$

For a real-valued stochastic process $\{X_t : 0 \le t < \infty\}$, define $TV^X : \Omega \times \mathbb{R}_+ \to [0, \infty]$ as

$$TV^X(\omega, t) \triangleq \sup\left\{\sum_1^n \left|X_{t_j}(\omega) - X_{t_{j-1}}(\omega)\right| : n \in \mathbb{N},\ 0 = t_0 \le \cdots \le t_n = t\right\}$$

([Folland] Section 3.5).

Let $M, N \in \mathscr{M}_2^c$. We can first note that by definition $(M+N)^2 - \langle M+N\rangle$ and $(M-N)^2 - \langle M-N\rangle$ are martingales. Subtracting the right difference from the left difference and expanding the square terms, we see the important fact that $MN - \langle M, N\rangle$ is a martingale; this will be used in our proof of Theorem 0.1 (c). We can also see from this that $\langle M\rangle$ and $\langle M, M\rangle$ are indistinguishable.

We state some important observations, the analogous results of course also being true for $\langle N\rangle$.

(i) $\langle M\rangle$ and $\langle M, N\rangle$ have continuous sample paths. This is the content of [K & S] Theorem 1.5.13 and the proof therein.

(ii) $t \to \langle M\rangle_t(\omega)$ is a.s. a non-decreasing and real-valued function on $\mathbb{R}_+$ and hence we have a.s.

$$TV^{\langle M\rangle}(\omega, t) < \infty \ \ \forall \ \ 0 \le t < \infty. \qquad (*)$$

We note the difference of two a.s. non-decreasing, real and continuous functions in $\langle M,N\rangle$ to see that $(*)$ is also a.s. satisfied if one replaces $\langle M\rangle$ with $\langle M,N\rangle$ ([Folland] Theorem 3.2.7 (b)).

(iii) Almost surely, $\langle M,N\rangle_0 = 0$ and $\langle M\rangle_0 = 0$.

Taking the definition of signed measure found in [Folland] so as to allow the signed measure to take one of $+/-\infty$, it follows ([R & Y] Sections 4.1 and 0.4) that a.s. the sample path of $\langle M,N\rangle$ corresponds uniquely to a signed measure $\mu_\omega^{\langle M,N\rangle}$ on $\mathscr{B}(\mathbb{R}_+)$ satisfying $\forall\, 0 \le t < \infty$,

$$\mu_\omega^{\langle M,N\rangle}([0,t]) \;\; = \langle M,N\rangle_t(\omega) \;\; ;$$

$$\left|\mu_\omega^{\langle M,N\rangle}\right|([0,t]) = TV^{\langle M,N\rangle}(\omega,t) \;\; ;$$

$$\mu_\omega^{\langle M,N\rangle}(\{t\}) = 0 \; ,$$

where $\left|\mu_\omega^{\langle M,N\rangle}\right|$ is the total variation of the measure $\mu_\omega^{\langle M,N\rangle}$ and where analogous results hold for $\langle M\rangle$ and $\langle N\rangle$.

Hence, for a measurable process X, the Lebesgue-Stieltjes integrals $\int_{[0,t]} X_s^{\pm} d\langle M\rangle_s$ are well defined for all $t \ge 0$ a.s., and also for $|d\langle M\rangle_s|$, $d\langle M,N\rangle_s$ and $|d\langle M,N\rangle_s|$. Further, if X is progressively measurable s.t.

$$\int_{[0,t]} X_s d\langle M\rangle_s = \int_{[0,t]} X_s^+ d\langle M\rangle_s - \int_{[0,t]} X_s^- d\langle M\rangle_s$$

is well-defined and finite for all $t \ge 0$ a.s., then the above Lebesgue-Stieltjes integral is a progressively measurable process ([K & S] Remark 4.6) and analagously for the other integrators listed above. We note the $\mu_\omega^{\langle M,N\rangle}$ - neglibability of singleton sets in $\mathbb{R}_+$ as a consequence of continuity.

3 Last Lemmas

We will now use the notation $\int_{[\alpha,\beta]} = \int_\alpha^\beta$ for $0 \le \alpha \le \beta < \infty$ and $\int_{\mathbb{R}_+} = \int_0^\infty$.

We will also use the following classical result:

Lemma 3.1. *([K & S] Problem 1.3.20) Let $(\Omega, \mathscr{F}, P)$ be a filtered probability space with filtration $\{\mathscr{F}_t : 0 \le t < \infty\}$. Let $X = \{X_t : 0 \le t < \infty\}$ be a right-continuous martingale w.r.t $\{\mathscr{F}_t\}$. If X is UI, then X_t converges P a.s. (as $t \to \infty$) to an r.v. $X_\infty \in L^1(P)$ s.t. $\{X_t : 0 \le t \le \infty\}$ is a martingale w.r.t. $\{\mathscr{F}_t : 0 \le t \le \infty\}$.*

Lastly, we state two fundamental theorems and one useful result:

Theorem 3.2. *(The Kunita-Watanabe Inequality; [K & S] Theorem 3.2.14) If $M, N \in \mathscr{M}_2^c, X \in \mathscr{L}^*(M)$ and $Y \in \mathscr{L}^*(N)$, then a.s.*

$$\int_0^t |X_s Y_s|\,|d\langle M,N\rangle_s| \le \sqrt{\int_0^t X_s^2 d\langle M\rangle_s}\;\sqrt{\int_0^t Y_s^2 d\langle N\rangle_s} \;\; ; \;\; 0 \le t < \infty \; .$$

Theorem 3.3. *([K & S] Theorem 3.2.10) Let $M \in \mathscr{M}_2^c$ and $X \in \mathscr{L}^*(M)$. Then,*

$$E\left[\left(I_t^M(X)\right)^2\right] = E\int_0^t X_s^2 d\langle M\rangle_s \;\; ; \;\; 0 \le t < \infty.$$

Lemma 3.4. *([K & S] Proposition 3.2.17) Let* $M, N \in \mathscr{M}_2^c$, $X \in \mathscr{L}^*(M)$ *and* $Y \in \mathscr{L}^*(N)$*. Then a.s.,*

$$\left\langle I^M(X), I^N(Y) \right\rangle_t = \int_0^t X_s Y_s d\left\langle M, N \right\rangle_s \ ; \ 0 \le t < \infty.$$

4 Proof of Theorem 0.1

Here is the result restated for ease of reference:

Let $M = \{M_t : 0 \le t < \infty\}$ and $N = \{N_t : 0 \le t < \infty\}$ be in $\mathscr{M}_2^c$ and suppose that X and Y are progressively measurable w.r.t $\{\mathscr{F}_t\}$ satisfying

$$E\int_0^\infty X_s^2 d\langle M\rangle_s < \infty, \ \ E\int_0^\infty Y_s^2 d\langle N\rangle_s < \infty.$$

Then:

a) The processes $I_t^M(X) \triangleq \int_0^t X_s \, dM_s$ and $I_t^N(Y) \triangleq \int_0^t Y_s \, dN_s$ are UI;

b) $I_t^M(X)$ and $I_t^N(Y)$ converge P-a.s. (as $t \to \infty$) to random variables $I_\infty^M(X)$ and $I_\infty^N(Y)$ respectively s.t. $\{I_t^M(X) : 0 \le t \le \infty\}$ and $\{I_t^N(Y) : 0 \le t \le \infty\}$ are martingales w.r.t. $\{\mathscr{F}_t : 0 \le t \le \infty\}$, where $\mathscr{F}_\infty \triangleq \sigma\left(\cup_{0 \le t < \infty} \mathscr{F}_t\right)$;

c) The cross-variation process $\langle I^M(X), I^N(Y)\rangle_t$ converges to a real number as $t \to \infty$ a.s., and

$$E\left[I_\infty^M(X) I_\infty^N(Y)\right] = E\left[\langle I^M(X), I^N(Y)\rangle_\infty\right] = E\int_0^\infty X_s Y_s d\langle M, N\rangle_s \ .$$

Proof. We will suppress the notational dependence of $I^M(X)$ and $I^N(Y)$ on X and Y wherever it is not useful.

Set $\rho_X = E\int_0^\infty X_s^2 d\langle M\rangle_s$, $\rho_Y = E\int_0^\infty Y_s^2 d\langle N\rangle_s$. $\rho_X,\ \rho_Y < \infty$ by assumption.

Using Theorem 3.3, for all $t \ge 0$ we have

$$E[|I_t^M|^2] = E\int_0^t X_s^2 d\langle M\rangle_s \le E\int_0^\infty X_s^2 d\langle M\rangle_s = \rho_X. \tag{4}$$

By Lemma 1.6, I^M is UI as is I^N analogously. Lemma 3.1 then obtains (b).

Almost surely, $\forall\ 0 \le \alpha \le \beta < \infty$ we have

(∗)

$$\int_\alpha^\beta |X_s Y_s|\, |d\langle M, N\rangle_s| = \int_0^\beta \left|X_s I_{\{s \ge \alpha\}}\right| \left|Y_s I_{\{s \ge \alpha\}}\right| |d\langle M, N\rangle_s| \le$$

$$\sqrt{\int_0^\beta (X_s I_{\{s\ge\alpha\}})^2 d\langle M\rangle_s}\ \sqrt{\int_0^\beta (Y_s I_{\{s\ge\alpha\}})^2 d\langle N\rangle_s} = \sqrt{\int_\alpha^\beta X_s^2 d\langle M\rangle_s}\ \sqrt{\int_\alpha^\beta Y_s^2 d\langle N\rangle_s} \ \ \le$$

$$\sqrt{\int_\alpha^\infty X_s^2 d\langle M\rangle_s}\ \sqrt{\int_\alpha^\infty Y_s^2 d\langle N\rangle_s} \ ,$$

where we have used Theorem 3.2.

Since $\rho_X,\ \rho_Y < \infty$, we have the following a.s.:

i) The right-hand integrals of $(*)$ and hence the left-hand integral of $(*)$ all converge to 0 as $\beta \geq \alpha \to \infty$.

ii) For all $0 \leq \alpha \leq \beta < \infty$,

$$\text{the function } [\alpha,\beta] \ni t \to X_t(\omega)Y_t(\omega) \text{ is } L^1\left([\alpha,\beta],\ \mathscr{B}([\alpha,\beta]),\ \mu_\omega^{\langle M,N\rangle}\right)$$

where we have used [Folland] Exercise 3.1.3.

Hence, a.s. we have $\forall\ 0 \leq \alpha \leq \beta < \infty$

$(**)$

$$\left|\left\langle I^M(X), I^N(Y)\right\rangle_\beta - \left\langle I^M(X), I^N(Y)\right\rangle_\alpha\right| = \left|\int_0^\beta X_sY_s d\langle M,N\rangle_s - \int_0^\alpha X_sY_s d\langle M,N\rangle_s\right| =$$

$$\left|\int_\alpha^\beta X_sY_s d\langle M,N\rangle_s\right| \leq \int_\alpha^\beta |X_sY_s|\,|d\langle M,N\rangle_s|\,,$$

with convergence of the right-hand term to 0 as $\beta \geq \alpha \to \infty$, where we have used Lemma 3.4 and again [Folland] Exercise 3.1.3. The first claim in (c) follows.

For the final part of (c) we expound on the argument given on page 227 of [K & S]. We recall that since $I^M, I^N \in \mathscr{M}_2^c$, we have $\Gamma = I^M I^N - \langle I^M, I^N\rangle$ a martingale with both terms on the right-hand side equal to 0 almost surely for $t = 0$. Therefore for $0 \leq t < \infty$,
$E\left[I_t^M I_t^N\right] = E\left[\Gamma_t\right] + E\left[\langle I^M, I^N\rangle_t\right] = E\left[\langle I^M, I^N\rangle_t\right]$.

Almost surely $\langle I^M, I^N\rangle_\infty = \int_0^\infty X_sY_s\ d\langle M,N\rangle_s$ and we have shown in $(*)$ and $(**)$ that a.s.

$$\left|\langle I^M, I^N\rangle_t\right| = \left|\int_0^t X_sY_s\ d\langle M,N\rangle_s\right| \leq \left(\int_0^\infty X_s^2 d\langle M\rangle_s\right)^{1/2}\left(\int_0^\infty Y_s^2 d\langle N\rangle_s\right)^{1/2}$$

with the right-hand r.v. in $L^1(P)$ by Cauchy-Schwarz. Hence by dominated convergence we obtain,

$$\lim_{t\to\infty} E\left[I_t^M I_t^N\right] = \lim_{t\to\infty} E\left[\langle I^M, I^N\rangle_t\right] = E\left[\langle I^M, I^N\rangle_\infty\right] = E\left[\int_0^\infty X_sY_s\ d\langle M,N\rangle_s\right].$$

Now we can write for arbitrary t,

$$E\left[I_\infty^M I_\infty^N\right] = E\left[\left(\left(I_\infty^M - I_t^M\right) + I_t^M\right)\left(\left(I_\infty^N - I_t^N\right) + I_t^N\right)\right] =$$

$$E\left[\left(I_\infty^M - I_t^M\right)\left(I_\infty^N - I_t^N\right)\right] + E\left[\left(I_\infty^M - I_t^M\right)I_t^N\right] + E\left[\left(I_\infty^N - I_t^N\right)I_t^M\right] + E\left[I_t^M I_t^N\right]. \tag{5}$$

For the leftmost term in (5), we can apply Cauchy-Schwarz and recall that (4) implies L^2 convergence of I^M and I^N to see that $\lim_{t\to\infty} E\left[\left(I_\infty^M - I_t^M\right)\left(I_\infty^N - I_t^N\right)\right] = 0$. For the middle two terms, we can again apply Cauchy-Schwarz and use the uniform L^2 bound and L^2 convergence of the martingales to see $\lim_{t\to\infty} E\left[\left(I_\infty^M - I_t^M\right)I_t^N\right] = 0,\ \lim_{t\to\infty} E\left[\left(I_\infty^N - I_t^N\right)I_t^M\right] = 0$. Therefore, $E\left[I_\infty^M I_\infty^N\right] = \lim_{t\to\infty} E\left[I_t^M I_t^N\right]$.
□

5 References

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6 Appendix A

Definition 6.1. *(Increasing Process) We will call an adapted process A increasing if for P-a.e. $\omega \in \Omega$ we have*

(a) $A_0(\omega) = 0$
(b) $t \to A_t(\omega)$ *is a nondecreasing, right-continuous function,*

and $E[A_t] < \infty$ *holds for every* $t \in [0, \infty)$.

Definition 6.2. *(Natural Process) An increasing process A is called natural if for every bounded, right-continuous martingale M we have*

$$E \int_{(0,t]} M_s \, dA_s = E \int_{(0,t]} M_{s^-} \, dA_s \;\; ; \;\; 0 < t < \infty.$$