1.

Jacob Miller   Discrete Test 2   question 1

Use the quotient-remainder theorem with divisor equal to 3 to prove that the product of two consecutive integers has the form $3K$ or $3K+2$ for some integer K.

let the product of two consecutive integers be represented by $n(n+1)$

By the quotient-remainder theorem we have 3 possible cases:

$3a \qquad 3a+1 \qquad 3a+2$

case 1:

Assume $n=3a$. substitute $n(n+1)=3a(3a+1)$

Distribute a. $3(3a^2+a)$ notice the format. let $K=3a^2+a$

so that $3(3a^2+a)=3K$

so we see that $n(n+1)=3K$

case 2:

Assume $n=3a+1$ substitute $n(n+1)=(3a+1)(3a+2)=9a^2+9a+2$

take out a 3, $3(3a^2+3a)+2$ notice the format. let $K=3a^2+3a$

so that $3(3a^2+3a)+2=3K+2$

we see that $n(n+1)=3K+2$

case 3:

Assume $n=3a+2$ substitute $n(n+1)=(3a+2)(3a+3)=9a^2+15a+6$

take out a 3, $3(3a^2+5a+2)$ notice the format. let $K=3a^2+5a+2$

so that $3(3a^2+5a+2)=3K$

we see that $n(n+1)=3K$

Therefore, by these three cases, we can see that the product of two consecutive integers $n(n+1)\in\mathbb{Z}$ have the form

$3K$ or $3K+2$ by the quotient-remainder theorem

2.

Jacob Miller Discrete Test 2 Question 2.

Use Mathematical Induction to prove that for $n \geq 2$

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$$

i) Base case let $n=2$ ✓

$$\frac{n(n-1)(n+1)}{3} = \frac{2(2-1)(2+1)}{3} = 2 = \sum_{i=1}^{1} i(i+1)$$

$k \geq 2 \in \mathbb{Z}^+$

ii) Assume $\displaystyle\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$ for all

for $k+1$

$$\sum_{i=1}^{k+1-1} i(i+1) = \frac{k(k+1)(k+2)}{3}$$

$$= \underbrace{[1(1+1)] + \dots + [(k-1)k]}_{\text{notice} \;\to\; = \frac{k(k-1)(k+1)}{3}} + [k(k+1)] = \frac{k(k+1)(k+2)}{3}$$

so $$\frac{k(k-1)(k+1)}{3} + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

$$\Rightarrow \frac{k(k-1)(k+1) + 3k(k+1)}{\cancel{3}} = \frac{k(k+1)(k+2)}{\cancel{3}}$$

$$\Rightarrow \cancel{k(k+1)}(k-1+3) = \cancel{k(k+1)}(k+2)$$

$$\Rightarrow k+2 = k+2$$

By Mathematical Induction, for $n \geq 2$

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$$

3.

Jacob Miller Discrete Test 2 question 3

a) Show that n^3-7n+3 is divisible by 3 for every integer $n \geq 0$

i) Base case let $n=0$. $3 \mid n^3-7n+3 \Rightarrow 3 \mid (27-21+3) \Rightarrow 3 \mid 9$ ✓

ii) Assume $n=K$ $\quad = (K+1)^3 - 7(K+1)+3$

K^3-7K+3 $\quad = K^3+3K^2+3K+1-7K-7+3$

For $n=K+1$ $\quad = \underbrace{(K^3-7K+3)}_{} + 3K^2+3K+-6$

Divisible by 3 $\quad 3 \mid 3(K^2+K-2)$?

Rewrite remaining terms: let $a = K^2+K-2$

$3 \mid 3a$? by divisibility rules we know $3 \mid a$ and that

$c \mid a \wedge c \mid b \rightarrow c \mid (a+b)$

let $c=3$

Proof by Mathematical Induction.

$a = K^3-7K+3 \quad b = 3(K^2+K-2)$

b) A sequence $a_1, a_2, a_3, \ldots$ is defined by the recursive formula $a_1 = 2$ and $a_K = \frac{a_{K-1}}{K}$ for each integer $K \geq 2$. For $n \geq 1$ show that $a_n = \frac{2}{n!}$

i) Base case $K=2$ $n=1$. $\frac{a_1}{2} = \frac{2}{1!} \Rightarrow \frac{2}{2} = \frac{2}{1} \Rightarrow 2=2$ ✓

ii) Assume $n=j$ $\quad$ Now look at $j+1$ term

$\frac{a_{j-1}}{j} = \frac{2}{j!}$ $\quad \frac{a_{j+1-1}}{(j+1)} = \frac{2}{(j+1)!} \Rightarrow \frac{a_j}{(j+1)} = \frac{2}{(j+1)\cdot j!}$

$\Rightarrow a_j = \frac{2}{j!}$ Plug in RHS

$a_j = \frac{a_{j-1}}{j}$ Notice this is the definition of our recursive sequence. $a_K = \frac{a_{K-1}}{K}$

Proof by Mathematical Induction

Jacob Miller Discrete Test 2 question 4

Suppose $P(n)$ is a property such that $P(0), P(1), P(2)$ are true and for each integer $K \geq 0$, if $P(k)$ is true, $P(3k)$ is true. Must it follow that $P(n)$ is true for every integer ≥ 0? If yes, explain why; if no give a counter example

If the given relationships are true, $P(3K)$ may be true, but it may not always be true ~~and~~ depending on the predicate.

Consider the Predicate $P(n)$: $n \leq 10$ and let $K = 10$

So

$P(0): 0 \leq 10$ true
$P(1): 1 \leq 10$ true
$P(2): 2 \leq 10$ true
$\vdots$
$P(k): K \leq 10 \Leftrightarrow 10 \leq 10$ true

but notice how $P(3k)$

$P(30): 30 \leq 10$ False.

With a different predicate, this assumption may hold true, but one cannot be certain with the given information.

5.

Jacob Miller Discrete Test question 5

a) Prove the statement if it is true, and give a counter example if it is not true.

For every real number x, $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$

This statement is False

take $\frac{7}{2}$ as a counter example

$$\left\lfloor \frac{7}{2}^2 \right\rfloor = \left\lfloor \frac{49}{4} \right\rfloor = \left\lfloor 12\tfrac{1}{4} \right\rfloor = 12$$

but

$$\left\lfloor \frac{7}{2} \right\rfloor^2 = 3^2 = 9 \quad \text{so} \quad \lfloor x^2 \rfloor \neq \lfloor x \rfloor^2$$

b) For any odd integer n, $\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2+3}{4}$

This statement is true, rewrite

where n is odd such that $n = 2k+1$

$$\left\lceil \frac{(2k+1)^2}{4} \right\rceil = \left\lceil \frac{4k^2+4k+1}{4} \right\rceil = \left\lceil k^2+k+\frac{1}{4} \right\rceil \quad k^2+k \in \mathbb{Z}$$

so

$$\left\lceil k^2+k+\frac{1}{4} \right\rceil = k^2+k+1 \quad \text{then look at } \frac{n^2+3}{4}$$

$$= \frac{(2k+1)^2+3}{4} = \frac{4k^2+4k+4}{4} = k^2+k+1$$

so

$$\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2+3}{4}$$

$$\therefore \quad k^2+k+1 = k^2+k+1$$

6.

Jacob Miller   Discrete Test 2   question 6

a) Use Well-Ordering Principle to prove that every positive integer $n \geq 2$ can be factored as a product of primes.

let S be the set of all positive integers ≥ 2 that can be factored by a product of Prime numbers. This set has at least one element: $n=2$ $2=2\cdot 1$ which are both prime.

By well-ordering principle, every positive integer $n \geq 2$ is also in S, the set of positive integers ≥ 2 that can be written as a product of prime numbers

b). Now Prove a) by Strong Mathematical Induction

$n = p$ where p is a product of primes

$n=2 = 2\cdot 1$ 2 and 1 are prime

$n=3 = 3\cdot 1$ 3 and 1 are prime

$n=4 = 2\cdot 2$ 2 is prime $n=K=p'$

Assume $P(i)$ is true for all $2 \leq i \leq K$

Show $P(K+1)$ is true. $n=K+1$

$K+1 = p$

Case 1: $K+1$ is prime

then it can be written as $(K+1)\cdot 1$ a product of primes

Case 2: $K+1$ isn't prime

then $K+1 = p''$ where p'' is a product of prime numbers

7.

Jacob Miller Discrete Test 2 Question 7

a) Prove that for all positive real numbers r and s,

$$\sqrt{r+s} \neq \sqrt{r} + \sqrt{s}$$

Suppose r and s are arbitrary positive real numbers p, and q such that

$$\sqrt{p+q} = \sqrt{p} + \sqrt{q}$$

$$(\sqrt{p+q})^2 = (\sqrt{p} + \sqrt{q})^2$$

$$p+q = p+q+\sqrt{p}\cdot\sqrt{q}$$ thus for all positive real numbers p, and q $\sqrt{p+q} \neq \sqrt{p} + \sqrt{q}$

b) Prove that

For every real number r, if r^2 is irrational then r is irrational.

Suppose r is some arbitrary rational number such that $r = x$. Since x is rational it can be written as the ratio of $k, l \in \mathbb{Z}$ such that $x = \frac{k}{l}$, then $x^2 = \frac{k^2}{l^2}$. The square of an integer is also an integer so $\frac{k^2}{l^2}$ is rational.

Proof By Contraposition.

~~Since~~ So if x^2 is irrational, then x must be irrational.

So if r^2 is irrational then r must also be irrational

8.

Jacob Miller Discrete Test 2 Question 8

a. $\boxed{\text{If } a|c \text{ and } b|c \text{ then } ab|c}$

Disprove by giving counterexample

let $a = 2$, $b = 4$, $c = 4$

$2|4$ yes $4|4$ yes, but $8 \nmid 4$

4 does not divide 8 so this is false $\divideontimes$

b. $\boxed{\text{If } c|n \text{ and } d|n \text{ then } (c+d)|n}$

Disproof by giving counterexample.

let $c = 2$, $d = 4$, $n = 4$

$2|4$ yes, $4|4$ yes, but $(2+4) = 6 \nmid 4$

6 does not divide 8 so this statement is false $\divideontimes$

c. $\boxed{\text{If } a|c \text{ and } b|d \text{ then } ab|cd}$

$a|c \rightarrow c = a \cdot l$ for some integer l

$b|d \rightarrow d = b \cdot m$ for some integer m

then, product of c and d, $c \cdot d = (a \cdot b)(l \cdot m)$

so

$ab|cd$ can be written as $ab|[(ab)(l \cdot m)]$

the product of two integers is also an integer such that

$l \cdot m = p \in \mathbb{Z}$

by rules of divisibility

$ab | p \cdot ab$

therefore if $a|c$ and $b|d$ then $ab|cd$