Intro to
Homotopy Type Theory

Martin-Löf Type Theory

The Language of Homotopy Type Theory

What is MLTT?

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Martin-Löf Type Theory is

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Martin-Löf Type Theory is a **formal language** and **deductive system** which has the form of an abstract **typed programming language** and can be used to reason about both the **topology of higher-dimensional spaces** and **higher-order intuitionistic logic**.

0 Speaking the Language

Proof. The proof is by chasing the element $\mathrm{Id}_c \in C(c,c)$ around both legs of a naturality square for a natural transformation $\eta\colon C(-,c) \to X$ (hence a homomorphism of presheaves):

$$\begin{array}{ccccccc} C(c,c) & \overset{\eta_c}{\to} & X(c) & & \mathrm{Id}_c & \mapsto & \eta_c(\mathrm{Id}_c) \overset{\mathrm{def}}{=} \xi \\ {}^{C(f,c)}\downarrow & & \downarrow^{X(f)} & & \downarrow & & \downarrow^{X(f)} \\ C(b,c) & \underset{\eta_b}{\to} & X(b) & & f & \mapsto & \eta_b(f) \end{array}$$

What this diagram shows is that the entire transformation $\eta\colon C(-,c) \to X$ is completely determined from the single value $\xi \coloneqq \eta_c(\mathrm{Id}_c) \in X(c)$, because for each object b of C, the component $\eta_b\colon C(b,c) \to X(b)$ must take an element $f \in C(b,c)$ (i.e., a morphism $f\colon b \to c$) to $X(f)(\xi)$, according to the commutativity of this diagram.

The crucial point is that the naturality condition on any natural transformation $\eta\colon C(-,c) \Rightarrow X$ is sufficient to ensure that η is already entirely fixed by the value $\eta_c(\mathrm{Id}_c) \in X(c)$ of its component $\eta_c\colon C(c,c) \to X(c)$ on the identity morphism Id_c. And every such value extends to a natural transformation η.

More in detail, the bijection is established by the map

$$[C^{\mathrm{op}}, \mathrm{Set}](C(-,c), X) \overset{|_c}{\to} \mathrm{Set}(C(c,c), X(c)) \xrightarrow{\mathrm{ev}_{\mathrm{Id}_c}} X(c)$$

where the first step is taking the component of a natural transformation at $c \in C$ and the second step is evaluation at $\mathrm{Id}_c \in C(c,c)$.

The inverse of this map takes $f \in X(c)$ to the natural transformation η^f with components

$$\eta^f_d \colon = X(-)(f)\colon C(d,c) \to X(d)\,.$$

▮

https://ncatlab.org/nlab/show/Yoneda+lemma

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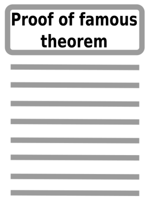
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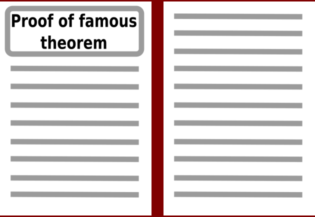
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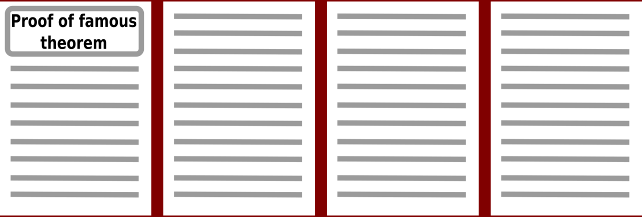
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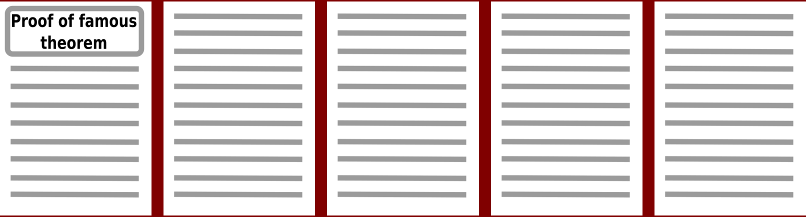
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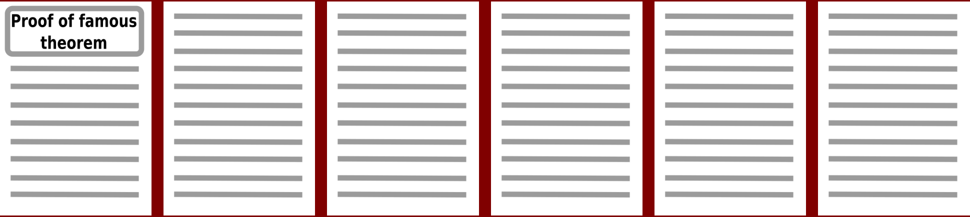
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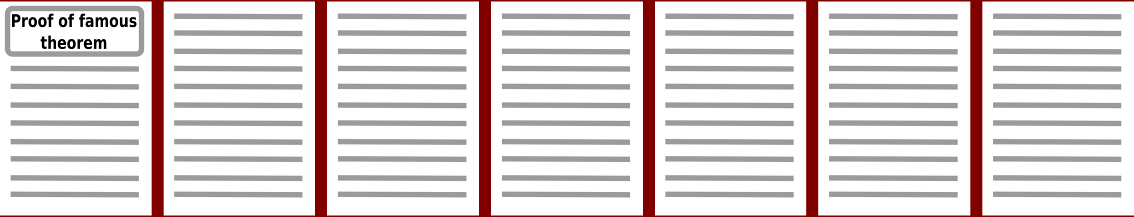
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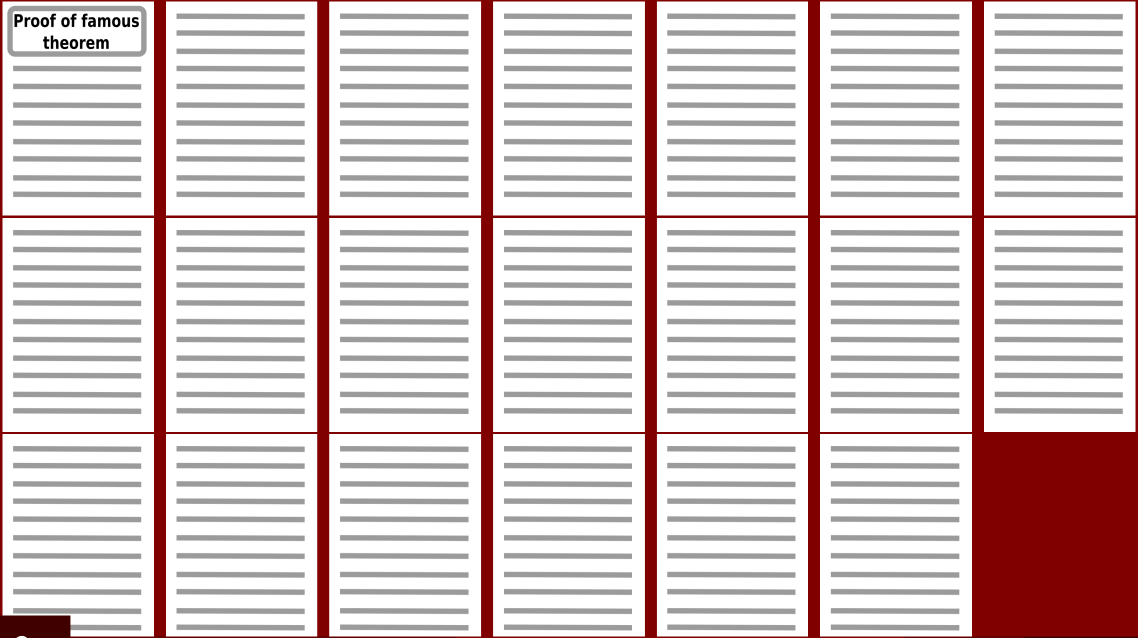
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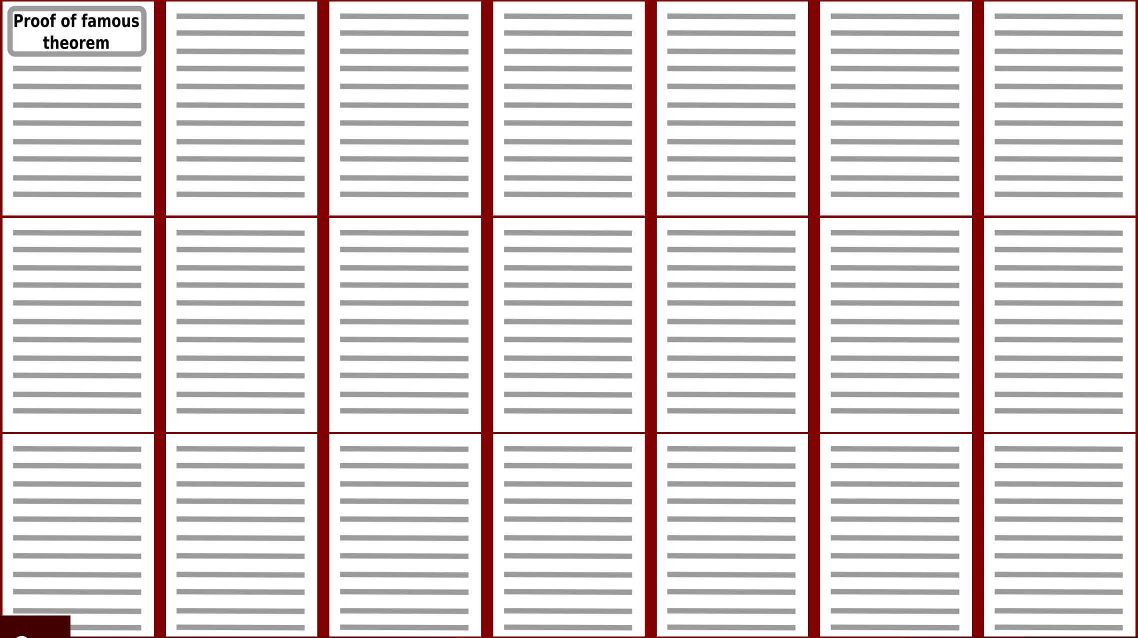
Proof of famous theorem

Titans of Mathematics Clash Over Epic Proof of ABC Conjecture

39

Two mathematicians have found what they say is a hole at the heart of a proof that has convulsed the mathematics community for nearly six years.

Despite multiple conferences dedicated to explicating Mochizuki's proof, number theorists have struggled to come to grips with its underlying ideas. His series of papers, which total more than 500 pages, are written in an impenetrable style, and refer back to a further 500 pages or so of previous work by Mochizuki, creating what one mathematician, Brian Conrad of Stanford University, has called "a sense of infinite regress."

But the meeting led to an oddly unsatisfying conclusion: Mochizuki couldn't convince Scholze and Stix that his argument was sound, but they couldn't convince him that it was unsound. Mochizuki has now posted Scholze's and Stix's report on his website, along with several reports of his own in rebuttal. (Mochizuki and Hoshi did not respond to requests for comments for this article.)

Language & Deduction

Language & Deduction

Language & Deduction

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Language & Deduction

The fundamental group of a type T is the zero-truncation of T, viewed as a group

S^1 is defined by
    base : S^1
    loop : base = base

Therefore...

The fundamental group of S^1 is the integers

There are certain general conditions under which the structure of a language is regarded as *exactly specified*. Thus, to specify the structure of a language, we must characterize unambiguously the class of those words and expressions which are to be considered *meaningful*. In particular, we must indicate all words which we decide to use without defining them, and which are called "*undefined* (or *primitive*) terms"; and we must give the so-called *rules of definition* for introducing new or *defined terms*. Furthermore, we must set up criteria for distinguishing within the class of expressions those which we call "*sentences*." Finally, we must formulate the conditions under which a sentence of the language can be *asserted*. In particular, we must indicate all *axioms* (or *primitive sentences*), i.e., those sentences which we decide to assert without proof; and we must give the so-called *rules of inference* (or *rules of proof*) by means of which we can deduce new asserted sentences from other sentences which have been previously asserted. Axioms, as well as sentences deduced from them by means of rules of inference, are referred to as "*theorems*" or "*provable sentences*."

- Alfred Tarski, The Semantic Conception of Truth (1944)

Terms

```
>     b=0
>     if (b=4 or b=5):
>       do_thing1()
>     else:
>       do_thing2()
```

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```
> b=0
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```

Two Judgments of MLTT

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$$\underset{\text{Term}}{x} : T$$

Two Judgments of MLTT

$$\underset{\text{Term}}{x} : \underset{\text{Type}}{T}$$

Two Judgments of MLTT

$$\underset{\text{Term}}{x} : \underset{\text{Type}}{T}$$

$$x \doteq x' : T$$

Two Judgments of MLTT

$$\underset{\text{Term}}{x} : \underset{\text{Type}}{T}$$

$$x \underset{\substack{\text{Judgmental}\\ \text{Equality}}}{\doteq} x' : T$$

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Therefore...

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Therefore...

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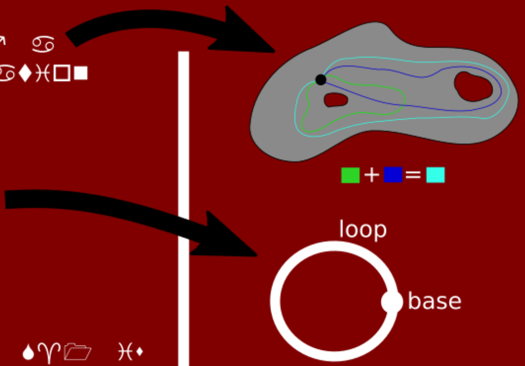

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Therefore...

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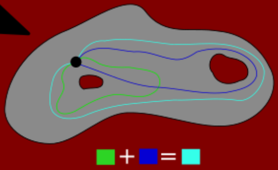

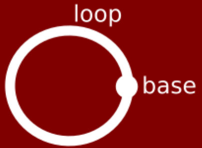

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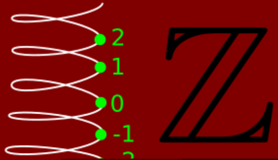

Interpretation 1: Spaces

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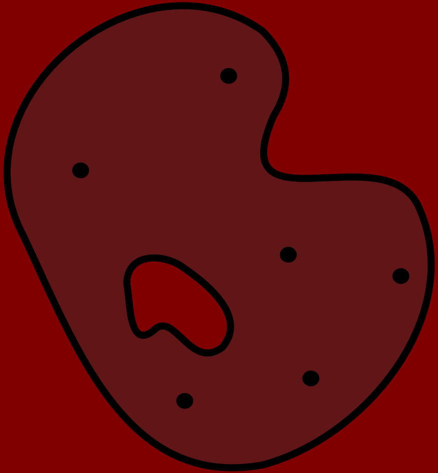

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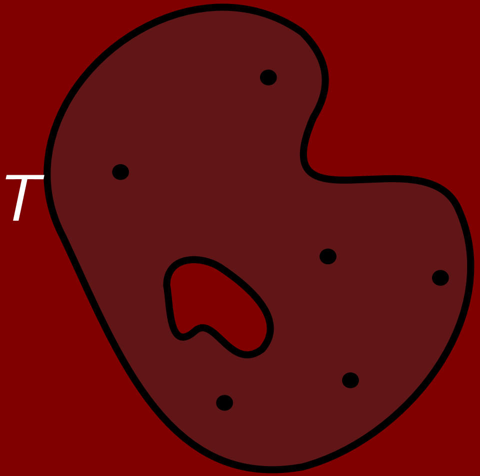
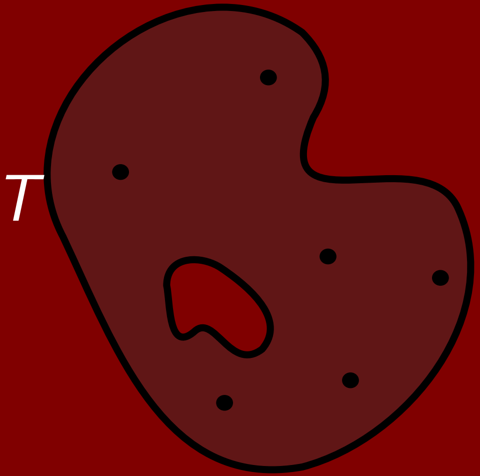


Interpretation 1: Spaces

$$x : T$$



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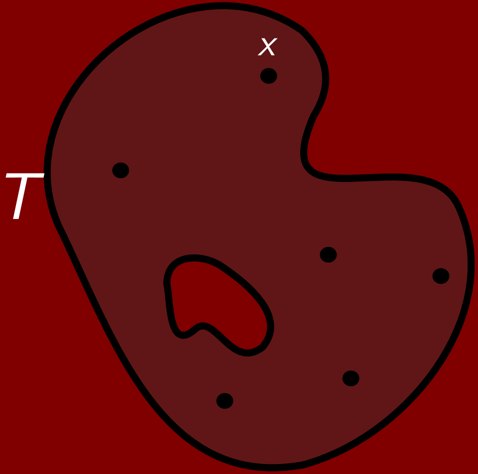

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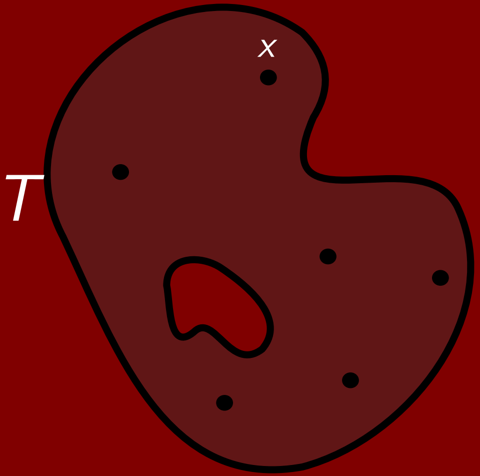

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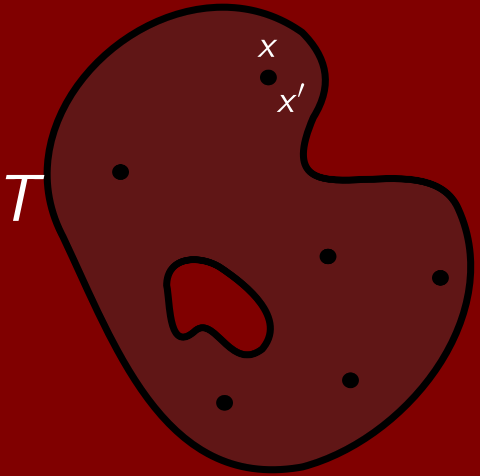

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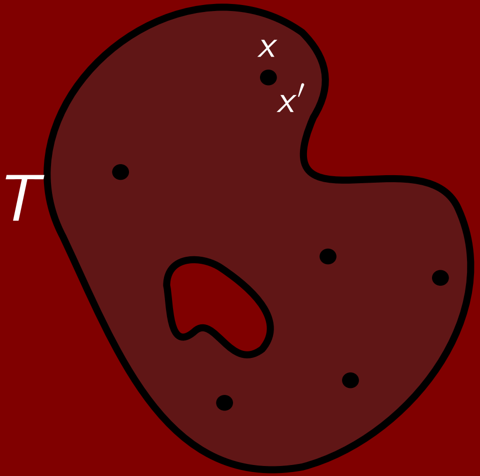
Types – Spaces



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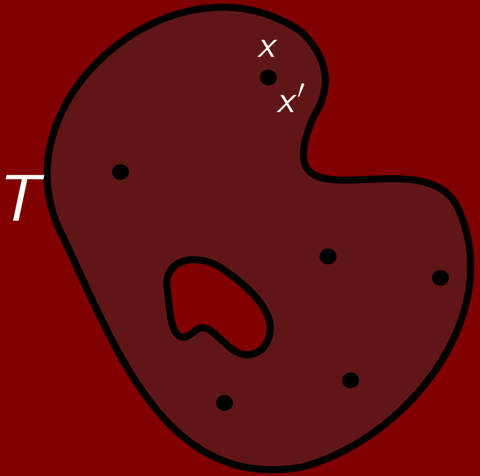
Types – Spaces
Terms – Points



Interpretation 2: Logic (Curry-Howard)

$$w : P$$

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$$\underset{\text{witness}}{w} : P$$

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$$\underset{\text{witness}}{w} : \underset{\text{Proposition}}{P}$$

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Terms – Witnesses

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By discussing these in a common language, we can

- identify similarities
- "transpose" concepts

1 Judgments, Contexts, and Type Families

Four Judgments of MLTT

$$T \text{ type} \qquad\qquad x : T$$

$$T \doteq T' \text{ type} \qquad\qquad x \doteq x' : T$$

What MLTT is made of

What MLTT is made of

- Types

What MLTT is made of

- Types (built up recursively)

What MLTT is made of

- Types (built up recursively)
- Contexts

What MLTT is made of

- Types (built up recursively)
- Contexts
- Terms-in-Context

What MLTT is made of

- Types (built up recursively)
- Contexts
- Terms-in-Context (built up recursively, along with the types)

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- Types (built up recursively)
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- Inference Rules

What MLTT is made of

- Types (built up recursively)
- Contexts
- Terms-in-Context (built up recursively, along with the types)
- Inference Rules
- Derivations

Contexts give MLTT "memory"

A **context** consists of a finite (possibly empty), ordered list of typing judgments

$$x_1 : T_1, x_2 : T_2, \ldots, x_n : T_n$$

Contexts give MLTT "memory"

A **context** consists of a finite (possibly empty), ordered list of typing judgments

$$x_1 : T_1, x_2 : T_2, \ldots, x_n : T_n$$

- Type Theory: Declaring some typed variables
- Logic: Assuming the truth of some propositions (with witnesses)
- Homotopy: Declaring names for points of given spaces

Judgments-in-Context

Let Γ be a context.

$$\Gamma \vdash T \ \text{type}$$

$$\Gamma \vdash x : T$$

$$\Gamma \vdash T \doteq T' \ \text{type}$$

$$\Gamma \vdash x \doteq x' : T$$

Inference Rules

An **inference rule** is of the form

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$$\frac{\mathcal{J}_1 \quad \mathcal{J}_2 \quad \cdots \quad \mathcal{J}_k}{\mathcal{J}_{k+1}}$$

which says: "if $\mathcal{J}_1$ and $\mathcal{J}_2$ and ... and $\mathcal{J}_k$, then deduce $\mathcal{J}_{k+1}$",

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For instance,

$$\frac{\Gamma \vdash T \text{ type}}{\Gamma \vdash T \doteq T \text{ type}} \qquad \frac{\Gamma \vdash T \text{ type} \quad \Gamma \vdash \mathcal{J}}{\Gamma, x : T \vdash \mathcal{J}}$$

Example: Booleans

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```
$ true
```

Example: Booleans

```
$  true
>      true :  bool
```

Example: Booleans

```
$  true
>      true :  bool
$  false
```

Example: Booleans

```
$  true
>      true :  bool
$  false
>      false :  bool
```

Example: Booleans

```
$  true
>      true :  bool
$  false
>      false :  bool
$  (if true
```

Example: Booleans

```
$  true
>      true :  bool
$  false
>      false :  bool
$  (if true
$   then 5
```

Example: Booleans

```
$  true
>      true :  bool
$  false
>      false :  bool
$  (if true
$   then 5
$   else 4)
```

Example: Booleans

```
$  true
>      true :  bool
$  false
>      false :  bool
$  (if true
$   then 5
$   else 4)
>      5
```

Example: Booleans

```
$  (if true
$   then 5
$   else 4)
>      5
$  (if false then 5 else 4)
```

Example: Booleans

```
$  (if true
$   then 5
$   else 4)
>      5
$  (if false then 5 else 4)
>      4
```

Example: Booleans

The type of booleans will be denoted $\mathbb{2}$ and contain exactly two values, $0_{\mathbb{2}}$ and $1_{\mathbb{2}}$. We'll formally express this using inference rules.

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- **Formation**:

$$\frac{}{\Gamma \vdash \mathbb{2}\ \mathsf{type}}$$

- **Introduction**:

$$\frac{}{\Gamma \vdash 0_{\mathbb{2}} : \mathbb{2}} \qquad \frac{}{\Gamma \vdash 1_{\mathbb{2}} : \mathbb{2}}$$

Boolean Elimination & Computation (non-dependent)

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- **Elimination**

$$\frac{\Gamma \vdash T \text{ type} \quad \Gamma \vdash p_0 : T \quad \Gamma \vdash p_1 : T}{\Gamma, x : \mathbb{2} \vdash \mathrm{ind}_{\mathbb{2}}(p_0, p_1, x) : T}$$

Boolean Elimination & Computation (non-dependent)

- **Elimination**

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- **Computation**:

$$\frac{\Gamma \vdash T \text{ type} \quad \Gamma \vdash p_0 : T \quad \Gamma \vdash p_1 : T}{\Gamma \vdash \mathrm{ind}_{\mathbb{2}}(p_0, p_1, 0_{\mathbb{2}}) \doteq p_0 : T}$$

$$\frac{\Gamma \vdash T \text{ type} \quad \Gamma \vdash p_0 : T \quad \Gamma \vdash p_1 : T}{\Gamma \vdash \mathrm{ind}_{\mathbb{2}}(p_0, p_1, 1_{\mathbb{2}}) \doteq p_1 : T}$$

Example: Binary Products

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- Formation:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \times B \text{ type}}$$

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- Introduction:

$$\frac{\Gamma \vdash x : A \quad \Gamma \vdash y : B}{\Gamma \vdash (x, y) : A \times B}$$

(also need "Congruence Rule" to state that if $x \doteq x'$ and $y \doteq y'$, then $(x, y) \doteq (x', y')$)

Example: Binary Products

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(also need "Congruence Rule" to state that if $x \doteq x'$ and $y \doteq y'$, then $(x, y) \doteq (x', y')$)

- Elimination and Computation: Next time!

Check Your Understanding

- List the terms of type $\mathbb{2} \times \mathbb{2}$
- Given terms $b_1 : \mathbb{2}$ and $b_2 : \mathbb{2}$, use $\mathrm{ind}_{\mathbb{2}}$ to come up with
  - a term $b_3 : \mathbb{2}$ which is $1_{\mathbb{2}}$ if b_1 is $0_{\mathbb{2}}$ and is $0_{\mathbb{2}}$ if b_1 is $1_{\mathbb{2}}$
  - a term $b_4 : \mathbb{2}$ which is $1_{\mathbb{2}}$ if both b_1 and b_2 are $1_{\mathbb{2}}$, and $0_{\mathbb{2}}$ otherwise
  - a term $b_5 : \mathbb{2}$ which is $1_{\mathbb{2}}$ if either b_1 or b_2 is $1_{\mathbb{2}}$, and $0_{\mathbb{2}}$ otherwise

Example: Arrow Types

Check Your Understanding

Write terms of the following types

- $P \to P$
- $P \to (Q \to P)$
- $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$
- $(Q \to R) \to ((P \to Q) \to (P \to R))$

Example: Arrow Types

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- Formation:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \to B \text{ type}}$$

Example: Arrow Types

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$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \to B \text{ type}}$$

- Introduction:

$$\frac{\Gamma, x : A \vdash e(x) : B}{\Gamma \vdash (\lambda x . e(x)) : A \to B}\ \lambda$$

(also need "Congruence Rule" to state that if $e(x) \doteq e'(x)$ for arbitrary x, then $(\lambda x . e(x)) \doteq (\lambda x . e'(x))$)

- Elimination:

$$\frac{\Gamma \vdash f : A \to B}{\Gamma, x : A \vdash f(x) : B}\ \mathsf{ev}$$

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$$\frac{\Gamma \vdash f : A \to B}{\Gamma, x : A \vdash f(x) : B}\ \mathsf{ev}$$

- Computation:

$$\frac{\Gamma, x : A \vdash e(x) : B}{\Gamma, x : A \vdash (\lambda y.e(y))(x) \doteq e(x) : B}\ \beta$$

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash (\lambda x.f(x)) \doteq f : A \to B}\ \eta$$

Dependence and Type Families

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- Terms that depend on variables in the context:

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$$\Gamma, x : A \vdash e(x) : B$$

- *Types* that depend on variables in the context:

$$\Gamma, x : A \vdash B(x) \text{ type}$$

B is called a **type family** over A.

2 Deduction in MLTT

Idea

$$
\dfrac{
\dfrac{\begin{matrix}\mathcal{H}_1 \quad \mathcal{H}_2 \\ \vdots \\ \mathcal{J}_{2,1}\end{matrix} \qquad \mathcal{H}_3}{\mathcal{J}_{1,1}}
\quad
\dfrac{\begin{matrix}\mathcal{H}_4 \\ \vdots \\ \mathcal{J}_{2,2}\end{matrix}}{\mathcal{J}_{1,2}}
\quad
\dfrac{\mathcal{H}_5 \quad \mathcal{H}_6 \quad \mathcal{H}_7}{\mathcal{J}_{1,3}}
\quad
\dfrac{}{\mathcal{J}_{1,4}}
\quad
\dfrac{
\dfrac{\dfrac{\mathcal{H}_8}{\mathcal{J}_{3,1}} \quad \mathcal{H}_9}{\mathcal{J}_{2,3}}
\qquad
\dfrac{\mathcal{H}_{10} \quad \mathcal{H}_{11}}{\mathcal{J}_{2,4}}
}{\mathcal{J}_{1,5}}
}{\mathcal{C}}
$$

Idea

$$\dfrac{\dfrac{\begin{matrix}\mathcal{H}_1 \quad \mathcal{H}_2\\ \vdots \\ \mathcal{J}_{2,1}\end{matrix} \qquad \mathcal{H}_3}{\mathcal{J}_{1,1}} \quad \dfrac{\begin{matrix}\mathcal{H}_4\\ \vdots\\ \mathcal{J}_{2,2}\end{matrix}}{\mathcal{J}_{1,2}} \quad \dfrac{\mathcal{H}_5 \quad \mathcal{H}_6 \quad \mathcal{H}_7}{\mathcal{J}_{1,3}} \quad \dfrac{}{\mathcal{J}_{1,4}} \quad \dfrac{\dfrac{\dfrac{\mathcal{H}_8}{\mathcal{J}_{3,1}} \quad \mathcal{H}_9}{\mathcal{J}_{2,3}} \quad \dfrac{\mathcal{H}_{10} \quad \mathcal{H}_{11}}{\mathcal{J}_{2,4}}}{\mathcal{J}_{1,5}}}{\mathcal{C}}$$

is a deduction of

$$\dfrac{\mathcal{H}_1 \quad \mathcal{H}_2 \quad \ldots \mathcal{H}_{11}}{\mathcal{C}}$$

Idea

A derived rule

$$\frac{\mathcal{H}_1 \quad \mathcal{H}_2 \quad \dots \quad \mathcal{H}_k}{\mathcal{C}}$$

can

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$$\frac{\mathcal{H}_1 \quad \mathcal{H}_2 \quad \ldots \quad \mathcal{H}_k}{\mathcal{C}}$$

can

- Be used to derive more rules

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A derived rule

$$\frac{\mathcal{H}_1 \quad \mathcal{H}_2 \quad \ldots \quad \mathcal{H}_k}{\mathcal{C}}$$

can

- Be used to derive more rules
- Serve as a formally-proven theorem about how our type theory works

Idea

A derived rule

$$\frac{\mathcal{H}_1 \quad \mathcal{H}_2 \quad \ldots \quad \mathcal{H}_k}{\mathcal{C}}$$

can

- Be used to derive more rules
- Serve as a formally-proven theorem about how our type theory works

We'll need some simple rules to make our deduction system work.

Judgmental Equality is an equivalence relation

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$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A \doteq A \text{ type}}$$

Judgmental Equality is an equivalence relation

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A \doteq A \text{ type}} \qquad \frac{\Gamma \vdash A \doteq B \text{ type}}{\Gamma \vdash B \doteq A \text{ type}}$$

Judgmental Equality is an equivalence relation

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A \doteq A \text{ type}} \qquad \frac{\Gamma \vdash A \doteq B \text{ type}}{\Gamma \vdash B \doteq A \text{ type}} \qquad \frac{\Gamma \vdash A \doteq B \text{ type} \quad \Gamma \vdash B \doteq C \text{ type}}{\Gamma \vdash A \doteq C \text{ type}}$$

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$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \doteq a : A}$$

Judgmental Equality is an equivalence relation

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$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \doteq a : A} \qquad \frac{\Gamma \vdash a \doteq b : A}{\Gamma \vdash b \doteq a : A}$$

Judgmental Equality is an equivalence relation

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$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \doteq a : A} \qquad \frac{\Gamma \vdash a \doteq b : A}{\Gamma \vdash b \doteq a : A} \qquad \frac{\Gamma \vdash a \doteq b : A \quad \Gamma \vdash b \doteq c : A}{\Gamma \vdash a \doteq c : A}$$

Variable Rule and Weakening

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x : A \vdash x : A}\ \delta$$

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$$\frac{\Gamma \vdash A \ \mathsf{type}}{\Gamma, x : A \vdash x : A} \ \delta$$

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma, \Delta \vdash \mathcal{J}}{\Gamma, x : A, \Delta \vdash \mathcal{J}} \ W$$

Variable Rule and Weakening

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x : A \vdash x : A}\ \delta$$

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, \Delta \vdash \mathcal{J}}{\Gamma, x : A, \Delta \vdash \mathcal{J}}\ W$$

Allows us to define the **constant type family** B over A:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma, x : A \vdash B \text{ type}}\ W$$

Moving variables around

Moving variables around

- Variable Conversion Rule

$$\frac{\Gamma \vdash A \doteq A' \quad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, x : A', \Delta \vdash \mathcal{J}}$$

Substitution

Moving variables around

Moving variables around

- Substitution Rule

$$\frac{\Gamma \vdash a : A \quad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]}\ S$$

Moving variables around

- Substitution Rule

$$\frac{\Gamma \vdash a : A \quad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]} \; S$$

- Substitution Congruence Rules

$$\frac{\Gamma \vdash a \doteq a' : A \quad \Gamma, x : A, \Delta \vdash B \text{ type}}{\Gamma, \Delta[a/x] \vdash B[a/x] \doteq B[a'/x] \text{ type}}$$

$$\frac{\Gamma \vdash a \doteq a' : A \quad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[a/x] \vdash b[a/x] \doteq b[a'/x] : B[a/x]}$$

Derived Structural Rules

Derived Structural Rules

- Substituting with a fresh variable

$$\frac{\Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, x' : A, \Delta[x'/x] \vdash \mathcal{J}[x'/x]} \; x'/x$$

Derived Structural Rules

- Substituting with a fresh variable

$$\frac{\Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, x' : A, \Delta[x'/x] \vdash \mathcal{J}[x'/x]}\ x'/x$$

- Interchange rule

$$\frac{\Gamma \vdash B \text{ type} \quad \Gamma, x : A, y : B, \Delta \vdash \mathcal{J}}{\Gamma, y : B, x : A, \Delta \vdash \mathcal{J}}$$

Derivation

$\mathbb{2} \to \mathbb{2}$

$\mathbb{2} \to \mathbb{2}$

$\mathbb{2} \to \mathbb{2}$

$$\dfrac{\dfrac{\overline{\Gamma \vdash 0_{\mathbb{2}} : \mathbb{2}}}{\Gamma, x : \mathbb{2} \vdash 0_{\mathbb{2}} : \mathbb{2}}\ W}{\Gamma \vdash (\lambda x.0_{\mathbb{2}}) : \mathbb{2} \to \mathbb{2}}\ \lambda$$

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$\mathbb{2} \to \mathbb{2}$

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$$\dfrac{\dfrac{\overline{\Gamma \vdash \mathbb{2} \text{ type}}}{\Gamma, x : \mathbb{2} \vdash x : \mathbb{2}}\, \delta}{\Gamma \vdash (\lambda x.x) : \mathbb{2} \to \mathbb{2}}\, \lambda$$

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$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \mathbb{2}\ \mathsf{type}} \quad \dfrac{}{\Gamma \vdash 1_{\mathbb{2}} : \mathbb{2}} \quad \dfrac{}{\Gamma \vdash 0_{\mathbb{2}} : \mathbb{2}}}{\Gamma, x : \mathbb{2} \vdash \mathsf{ind}_{\mathbb{2}}(1_{\mathbb{2}}, 0_{\mathbb{2}}, x) : \mathbb{2}}}{\Gamma \vdash (\lambda x.\mathsf{ind}_{\mathbb{2}}(1_{\mathbb{2}}, 0_{\mathbb{2}}, x)) : \mathbb{2} \to \mathbb{2}}\,\lambda$$

Appending Definitions To Derivations

$$\dfrac{\dfrac{\begin{array}{cccc}\mathcal{H}_1 & \mathcal{H}_2 & & \mathcal{H}_k \\ \vdots & \vdots & \dots & \vdots\end{array}}{\Gamma \vdash a : A}}{}$$

Appending Definitions To Derivations

$$\dfrac{\dfrac{\begin{matrix}\mathcal{H}_1 & \mathcal{H}_2 & & \mathcal{H}_k \\ \vdots & \vdots & \dots & \vdots\end{matrix}}{\Gamma \vdash a : A}}{\Gamma \vdash c :\equiv a : A}$$

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Appending Definitions To Derivations

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$$\dfrac{\mathcal{H}_1 \quad \mathcal{H}_2 \quad \cdots \quad \mathcal{H}_k}{\Gamma \vdash c : A} \qquad \dfrac{\mathcal{H}_1 \quad \mathcal{H}_2 \quad \cdots \quad \mathcal{H}_k}{\Gamma \vdash c \doteq a : A}$$

Example: The Identity Function

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A \text{ type}}{\Gamma, x : A \vdash x : A}\,\delta}{\Gamma \vdash (\lambda x.x) : A \to A}\,\lambda}{\Gamma \vdash \mathrm{id}_A := (\lambda x.x) : A \to A}$$

Example: Composition

$$\text{comp} := (\lambda g.\lambda f.\lambda x.g(f(x))) \ : \ (B \to C) \to (A \to B) \to (A \to C)$$

(See book for formal derivation)

$$g \circ f := ((\text{comp}\ g)\ f) \ : \ A \to C$$

Example: The Left Unit Law

Check Your Understanding Derive:

$$\frac{\Gamma \vdash f : A \to B}{\Gamma, x : A \vdash \mathrm{id}_B(f(x)) \doteq f(x) : B}\ (a)$$

Example: The Left Unit Law

Check Your Understanding Derive:

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Then...

$$\frac{}{\Gamma \vdash \mathrm{id}_B \circ f \doteq f}$$

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$$\frac{\dfrac{\Gamma \vdash f : A \to B}{\Gamma \vdash \lambda x.f(x) \doteq f}\,\eta}{\Gamma \vdash \mathrm{id}_B \circ f \doteq f}$$

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Check Your Understanding Derive:

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Then...

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash f : A \to B}{\Gamma, x : A \vdash \mathrm{id}_B(f(x)) \doteq f(x)}\,(a)}{\Gamma \vdash \lambda x.\mathrm{id}_B(f(x)) \doteq \lambda x.f(x)} \qquad \dfrac{\Gamma \vdash f : A \to B}{\Gamma \vdash \lambda x.f(x) \doteq f}\,\eta}{\Gamma \vdash \mathrm{id}_B \circ f \doteq f}$$

3 How we'll use MLTT

Blending with interpretations

Moving forward, we'll be more casual about interpretations, switching between them as suits our purposes

Informal Type Theory

The formal framework of contexts, type judgments, etc. can often be too clunky and get in the way. So we'll work in an **informal** style, e.g.

- The context is usually implicit

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- "Assume T" means "Assume T is inhabited"
- "Let X be a gadget" means "Let X be a term (of the appropriate type) such that $\text{is\_gadget}(X)$ is inhabited"
- We'll have informal ways of reading (and using) the formal inference rules we use to define our types

Formalization

A key benefit of HoTT is its amenability to **formalization**: even though we usually work informally, our informal methods closely mirror our formal rules so it's easily to "translate" into formal derivations.

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Interactice proof assistants (like Agda or Coq) allow us to write our formal proofs in a computer-readable format, so the computer can check our proofs and verify their correctness!

Next Time. . .

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- More discussion of type families

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- Dependent Types & their interpretations

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Next Time…

- More discussion of type families
- Dependent Types & their interpretations
- "Official" rules for $\mathbb{2}$, $\times$, etc.
- More types

Thanks for watching!

Designed, written, and performed by
Jacob Neumann

Based on the textbook
Introduction to Homotopy Type Theory
by
Egbert Rijke

Music:
"*Wholesome*" and "*Fluidscape*"
Kevin MacLeod (incompetech.com)

Next video

Full lecture

Next video

Full lecture