GATs, Cats, and CwFs

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- Background for my forthcoming PhD thesis [Neu25, Chapter 1]
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  - Restrict their signature language for QIITs to get a signature language for GATs
  - Matches the notion of GAT given by [Car86], except we don't allow for equations between sort. This provides an intrinsic presentation of GATs.

	GAT signature	Category of Algs.	Concrete CwF
Categories			
CwFs			
GAT signatures			

Categories of GAT Algebras

Generalized algebraic theories [Car86] are a syntax for specifying structures.

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def $\mathfrak{Cat} : \mathtt{GAT} := \{\!\![$

$$\text{def } \mathfrak{Cat} : \mathtt{GAT} := \{\!\!\{$$
$$\quad \mathtt{Obj} \quad : \mathbf{U},$$

$$
\begin{array}{l}
\mathsf{def}\ \mathfrak{Cat} : \mathtt{GAT} := \{\!\!\{ \\
\quad \mathtt{Obj} \quad : \mathbf{U}, \\
\quad \mathtt{Hom} \quad : \mathtt{Obj} \Rightarrow \mathtt{Obj} \Rightarrow \mathbf{U},
\end{array}
$$

$$
\begin{aligned}
&\text{def } \mathfrak{Cat} : \mathtt{GAT} := \{\!\!\{\\
&\quad \mathtt{Obj} \;\; : \mathbf{U},\\
&\quad \mathtt{Hom} \;\; : \mathtt{Obj} \Rightarrow \mathtt{Obj} \Rightarrow \mathbf{U},\\
&\quad \mathtt{id} \;\;\; : (\mathtt{I} : \mathtt{Obj}) \Rightarrow \mathtt{Hom}\ \mathtt{I}\ \mathtt{I},\\
&\quad \mathtt{comp} : \{\mathtt{I}\ \mathtt{J}\ \mathtt{K} : \mathtt{Obj}\} \Rightarrow\\
&\qquad\qquad \mathtt{Hom}\ \mathtt{J}\ \mathtt{K} \Rightarrow \mathtt{Hom}\ \mathtt{I}\ \mathtt{J} \Rightarrow \mathtt{Hom}\ \mathtt{I}\ \mathtt{K},
\end{aligned}
$$

$$\begin{aligned}
&\texttt{lunit} : \{\texttt{I}\ \texttt{J} : \texttt{Obj}\} \Rightarrow (\texttt{j} : \texttt{Hom}\ \texttt{I}\ \texttt{J}) \Rightarrow \\
&\qquad \texttt{comp}\ (\texttt{id}\ \texttt{J})\ \texttt{j} \equiv \texttt{j}, \\
&\texttt{runit} : \{\texttt{I}\ \texttt{J} : \texttt{Obj}\} \Rightarrow (\texttt{j} : \texttt{Hom}\ \texttt{I}\ \texttt{J}) \Rightarrow \\
&\qquad \texttt{comp}\ \texttt{j}\ (\texttt{id}\ \texttt{I}) \equiv \texttt{j}, \\
&\texttt{assoc} : \{\texttt{I}\ \texttt{J}\ \texttt{K}\ \texttt{L} : \texttt{Obj}\} \Rightarrow (\texttt{j} : \texttt{Hom}\ \texttt{I}\ \texttt{J}) \Rightarrow \\
&\qquad (\texttt{k} : \texttt{Hom}\ \texttt{J}\ \texttt{K}) \Rightarrow (\ell : \texttt{Hom}\ \texttt{K}\ \texttt{L}) \Rightarrow \\
&\qquad \texttt{comp}\ \ell\ (\texttt{comp}\ \texttt{k}\ \texttt{j}) \equiv \texttt{comp}\ (\texttt{comp}\ \ell\ \texttt{k})\ \texttt{j}
\end{aligned}$$

$\rrbracket$

whenever new abstract objects are constructed in a specified way out of given ones, it is advisable to regard the construction of the corresponding induced mappings on these new objects as an integral part of their definition

—[EM45, 236]

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Central Dogma of Category Theory

Every notion of "structure" comes equipped with a notion of "structure-preserving morphism", i.e. forms a category

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Central Dogma of Category Theory

Every notion of "structure" comes equipped with a notion of "structure-preserving morphism", i.e. forms a category

	GAT signature	Category of Algs.	Concrete CwF
Categories			
CwFs			
GAT signatures			

GAT signature	
Categories	$\mathfrak{Cat}$

	GAT signature	Category of Algs.
Categories	$\mathfrak{Cat}$	Cat

	GAT signature	Category of Algs.
Categories	$\mathfrak{Cat}$	Cat

EXTEND

$$
\begin{array}{l}
\text{def } \mathfrak{CwF} : \texttt{GAT} := \{\!\![\\
\quad \text{include } \mathfrak{Cat} \text{ as } (\texttt{Con},\texttt{Sub},\texttt{comp},\texttt{id},\_,\_,\_); \\
\quad\quad \texttt{empty} \quad : \texttt{Con}, \\
\quad\quad \epsilon \quad\quad\quad : (\Gamma : \texttt{Con}) \Rightarrow \texttt{Sub}\ \Gamma\ \texttt{empty}, \\
\quad\quad \epsilon\_\eta \quad\quad : (\Gamma : \texttt{Con}) \Rightarrow (\texttt{f} : \texttt{Sub}\ \Gamma\ \texttt{empty}) \Rightarrow \\
\quad\quad\quad\quad\quad \texttt{f} \equiv (\epsilon\ \Gamma),
\end{array}
$$

$$\mathtt{Ty} \qquad\qquad : \mathtt{Con} \Rightarrow \mathbf{U},$$

Categories with Families

$$\begin{aligned}
\mathtt{Ty} \quad &: \mathtt{Con} \Rrightarrow \mathbf{U}, \\
\mathtt{substTy} &: \{\Delta\ \Gamma : \mathtt{Con}\} \Rrightarrow \\
&\quad \mathtt{Sub}\ \Delta\ \Gamma \Rrightarrow \mathtt{Ty}\ \Gamma \Rrightarrow \mathtt{Ty}\ \Delta,
\end{aligned}$$

Categories with Families

$$
\begin{aligned}
&\mathtt{Ty} && : \mathtt{Con} \Rightarrow \mathbf{U}, \\
&\mathtt{substTy} && : \{\Delta\ \Gamma : \mathtt{Con}\} \Rightarrow \\
& && \quad \mathtt{Sub}\ \Delta\ \Gamma \Rightarrow \mathtt{Ty}\ \Gamma \Rightarrow \mathtt{Ty}\ \Delta, \\
&\mathtt{idTy} && : \{\Gamma : \mathtt{Con}\} \Rightarrow (\mathtt{A} : \mathtt{Ty}\ \Gamma) \Rightarrow \\
& && \quad \mathtt{substTy}\ (\mathtt{id}\ \Gamma)\ \mathtt{A} \equiv \mathtt{A}, \\
&\mathtt{compTy} && : \{\Theta\ \Delta\ \Gamma : \mathtt{Con}\} \Rightarrow (\mathtt{A} : \mathtt{Ty}\ \Gamma) \\
& && \quad (\delta : \mathtt{Sub}\ \Theta\ \Delta) \Rightarrow (\gamma : \mathtt{Sub}\ \Delta\ \Gamma) \Rightarrow \\
& && \quad \mathtt{substTy}\ \gamma\ (\mathtt{substTy}\ \delta\ \mathtt{A}) \\
& && \quad \equiv \mathtt{substTy}\ (\mathtt{comp}\ \gamma\ \delta)\ \mathtt{A},
\end{aligned}
$$

Categories with Families

$$\mathtt{Tm} \qquad : (\Gamma : \mathtt{Con}) \Rightarrow \mathtt{Ty}\ \Gamma \Rightarrow \mathbf{U},$$

Categories with Families

$$
\begin{aligned}
&\mathtt{Tm} \quad\quad : (\Gamma : \mathtt{Con}) \Rightarrow \mathtt{Ty}\ \Gamma \Rightarrow \mathbf{U},\\
&\mathtt{substTm} : \{\Delta\ \Gamma : \mathtt{Con}\} \Rightarrow \{\mathtt{A} : \mathtt{Ty}\ \Gamma\} \Rightarrow\\
&\quad\quad (\gamma : \mathsf{Sub}\ \Delta\ \Gamma) \Rightarrow\\
&\quad\quad \mathtt{Tm}\ \Gamma\ \mathtt{A} \Rightarrow \mathtt{Tm}\ \Delta\ (\mathtt{substTy}\ \gamma\ \mathtt{A}),
\end{aligned}
$$

$$
\begin{aligned}
&\mathtt{idTm} \quad : \{\Gamma : \mathsf{Con}\} \Rightarrow \{\mathtt{A} : \mathtt{Ty}\ \Gamma\} \Rightarrow (\mathtt{t} : \mathtt{Tm}\ \Gamma\ \mathtt{A}) \\
&\qquad \mathtt{substTm}\ (\mathtt{id}\ \Gamma)\ \mathtt{t} \qquad \#\langle \mathtt{idTy}\ \mathtt{A} \rangle \\
&\qquad \equiv \mathtt{t}, \\
&\mathtt{compTm} \ : \{\Theta\ \Delta\ \Gamma : \mathsf{Con}\} \Rightarrow \\
&\qquad \{\mathtt{A} : \mathtt{Ty}\ \Gamma\} \Rightarrow (\mathtt{t} : \mathtt{Tm}\ \Gamma\ \mathtt{A}) \Rightarrow \\
&\qquad (\delta : \mathsf{Sub}\ \Theta\ \Delta) \Rightarrow (\gamma : \mathsf{Sub}\ \Delta\ \Gamma) \Rightarrow \\
&\qquad \mathtt{substTm}\ \gamma\ (\mathtt{substTm}\ \delta\ \mathtt{t}) \\
&\qquad\quad \#\langle \mathtt{compTy}\ \mathtt{A}\ \gamma\ \delta \rangle \\
&\qquad \equiv \mathtt{substTm}\ (\mathtt{comp}\ \gamma\ \delta)\ \mathtt{t},
\end{aligned}
$$

$$
\begin{aligned}
&\mathtt{ext} && : (\Gamma : \mathtt{Con}) \Rightarrow \mathtt{Ty}\ \Gamma \Rightarrow \mathtt{Con},\\
&\mathtt{pair} && : \{\Delta\ \Gamma : \mathtt{Con}\} \Rightarrow \{\mathtt{A} : \mathtt{Ty}\ \Gamma\} \Rightarrow\\
& && \quad (\gamma : \mathtt{Sub}\ \Delta\ \Gamma) \Rightarrow\\
& && \quad \mathtt{Tm}\ \Delta\ (\mathtt{substTy}\ \gamma\ \mathtt{A}) \Rightarrow\\
& && \quad \mathtt{Sub}\ \Delta\ (\mathtt{ext}\ \Gamma\ \mathtt{A}),
\end{aligned}
$$

$$
\begin{aligned}
&\mathtt{pair\_nat} \colon \{\Theta\ \Delta\ \Gamma : \mathtt{Con}\} \Rightarrow \{\mathtt{A} : \mathtt{Ty}\ \Gamma\} \Rightarrow \\
&\qquad (\gamma : \mathtt{Sub}\ \Delta\ \Gamma) \Rightarrow \\
&\qquad (\mathtt{t} : \mathtt{Tm}\ \Delta\ (\mathtt{substTy}\ \gamma\ \mathtt{A})) \Rightarrow \\
&\qquad (\delta : \mathtt{Sub}\ \Theta\ \Delta) \Rightarrow \\
&\qquad \mathtt{comp}\ (\mathtt{pair}\ \gamma\ \mathtt{t})\ \delta \\
&\qquad \equiv \mathtt{pair}\ (\mathtt{comp}\ \gamma\ \delta) \\
&\qquad\quad (\mathtt{substTm}\ \delta\ \mathtt{t}\ \ \#\langle \mathtt{compTy}\ \mathtt{A}\ \gamma\ \delta \rangle),
\end{aligned}
$$

$$
\begin{aligned}
&\mathtt{p} \quad : \{\Gamma : \mathtt{Con}\} \Rightarrow (\mathtt{A} : \mathtt{Ty}\ \Gamma) \Rightarrow \\
&\qquad \mathtt{Sub}\ (\mathtt{ext}\ \Gamma\ \mathtt{A})\ \Gamma \\
&\mathtt{v} \quad : \{\Gamma : \mathtt{Con}\} \Rightarrow (\mathtt{A} : \mathtt{Ty}\ \Gamma) \Rightarrow \\
&\qquad \mathtt{Tm}\ (\mathtt{ext}\ \Gamma\ \mathtt{A})\ (\mathtt{substTy}\ (\mathtt{p}\ \mathtt{A})\ \mathtt{A}),
\end{aligned}
$$

Categories with Families

```
ext_β  : (Δ Γ : Con) ⇒ (A : Ty Γ) ⇒
    (γ : Sub Δ Γ) ⇒
    (t : Tm Δ (substTy γ A)) ⇒
    comp (p A) (pair γ t) ≡ γ,
ext_β  : (Δ Γ : Con) ⇒ (A : Ty Γ) ⇒
    (γ : Sub Δ Γ) ⇒
    (t : Tm Δ (substTy γ A)) ⇒
    substTm (pair γ t) (v A)
      #⟨compTy A (p A) (pair γ t)⟩
      #⟨ext_β  γ t⟩
    ≡ t,
ext_η  : (Γ : Con) ⇒ (A : Ty Γ) ⇒
    pair (p A) (v A)
    ≡ id (ext Γ A)
```

	GAT signature	Category of Algs.
Categories	$\mathfrak{Cat}$	Cat

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Categories	$\mathfrak{Cat}$	Cat
CwFs	$\mathfrak{CwF}$	

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Categories	$\mathfrak{Cat}$	Cat
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EXTEND

$\mathsf{def}\ \mathfrak{CwF}{+}{+} : \mathtt{GAT} := \{\!\!|$

$\quad \mathsf{include}\ \mathfrak{CwF};$

$$
\begin{array}{l}
\text{def } \mathfrak{CwF}{+}{+} : \mathtt{GAT} := \{\!\!\{ \\
\quad \text{include } \mathfrak{CwF}; \\
\qquad \mathtt{V} : \{\Gamma : \mathtt{Con}\} \Rightarrow \mathtt{Ty}\ \Gamma, \\
\qquad \mathtt{El} : \{\Gamma : \mathtt{Con}\} \Rightarrow \mathtt{Tm}\ \Gamma\ \mathtt{V} \Rightarrow \mathtt{Ty}\ \Gamma,
\end{array}
$$

$$
\begin{array}{l}
\textsf{def}\ \mathfrak{CwF}{+}{+} : \mathtt{GAT} := \{\!\!\{ \\
\quad \textsf{include}\ \mathfrak{CwF}; \\
\quad\quad \mathtt{V} : \{\Gamma : \mathtt{Con}\} \Rightarrow \mathtt{Ty}\ \Gamma, \\
\quad\quad \mathtt{El} : \{\Gamma : \mathtt{Con}\} \Rightarrow \mathtt{Tm}\ \Gamma\ \mathtt{V} \Rightarrow \mathtt{Ty}\ \Gamma, \\
\quad\quad \ldots \\
\quad\quad \mathtt{Eq} : \{\Gamma : \mathtt{Con}\} \Rightarrow \{\mathtt{A} : \mathtt{Ty}\ \Gamma\} \Rightarrow \\
\quad\quad\quad (\mathtt{t}\ \mathtt{t'} : \mathtt{Tm}\ \Gamma\ \mathtt{A}) \Rightarrow \mathtt{Ty}\ \Gamma, \\
\quad\quad \mathtt{reflect} : \{\Gamma : \mathtt{Con}\} \Rightarrow \{\mathtt{A} : \mathtt{Ty}\ \Gamma\} \Rightarrow \\
\quad\quad\quad \{\mathtt{t}\ \mathtt{t'} : \mathtt{Tm}\ \Gamma\ \mathtt{A}\} \Rightarrow \mathtt{Tm}\ \Gamma\ (\mathtt{Eq}\ \mathtt{t}\ \mathtt{t'}) \Rightarrow \\
\quad\quad\quad \mathtt{t} \equiv \mathtt{t'},
\end{array}
$$

$$
\begin{array}{l}
\text{def } \mathfrak{CwF}{+}{+} : \mathtt{GAT} := \{\!\!\{ \\
\quad \text{include } \mathfrak{CwF}; \\
\qquad \mathtt{V} : \{\Gamma : \mathtt{Con}\} \Rightarrow \mathtt{Ty}\ \Gamma, \\
\qquad \mathtt{El} : \{\Gamma : \mathtt{Con}\} \Rightarrow \mathtt{Tm}\ \Gamma\ \mathtt{V} \Rightarrow \mathtt{Ty}\ \Gamma, \\
\qquad \ldots \\
\qquad \mathtt{Eq} : \{\Gamma : \mathtt{Con}\} \Rightarrow \{\mathtt{A} : \mathtt{Ty}\ \Gamma\} \Rightarrow \\
\qquad\qquad (\mathtt{t}\ \mathtt{t'} : \mathtt{Tm}\ \Gamma\ \mathtt{A}) \Rightarrow \mathtt{Ty}\ \Gamma, \\
\qquad \ldots
\end{array}
$$

```
def CwF++ : GAT := {|
  include CwF;
    V : {Γ : Con} ⇒ Ty Γ,
    El : {Γ : Con} ⇒ Tm Γ V ⇒ Ty Γ,
    ...
    Eq : {Γ : Con} ⇒ {A : Ty Γ} ⇒
        (t t' : Tm Γ A) ⇒ Ty Γ,
    ...
    Pi : {Γ : Con} ⇒ (X : Tm Γ V) ⇒
        Ty (ext Γ (El X)) ⇒ Ty Γ,
    ...
```

Why this specific GAT?

1

The GAT Signature Language

Key Idea An *initial object* in the category of $\mathfrak{G}$-algebras represents the *pure syntax* of $\mathfrak{G}$, only the "stuff" which can be derived from the mere concept of $\mathfrak{G}$.

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- The initial monoid consists only of the identity element

Key Idea An *initial object* in the category of $\mathfrak{G}$-algebras represents the *pure syntax* of $\mathfrak{G}$, only the "stuff" which can be derived from the mere concept of $\mathfrak{G}$. E.g.

- The initial monoid consists only of the identity element
- The initial CwF+(constructors) is the syntax of a type theory with (constructors).

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Categories	$\mathfrak{Cat}$	Cat
CwFs	$\mathfrak{CwF}$	CwF

	GAT signature	Category of Algs.	Initial Algebra
Categories	$\mathfrak{Cat}$	Cat	
CwFs	$\mathfrak{CwF}$	CwF	

	GAT signature	Category of Algs.	Initial Algebra
Categories	$\mathfrak{Cat}$	Cat	$\emptyset$
CwFs	$\mathfrak{CwF}$	CwF	

	GAT signature	Category of Algs.	Initial Algebra
Categories	$\mathfrak{Cat}$	Cat	$\emptyset$
CwFs	$\mathfrak{CwF}$	CwF	$\{\diamond\}, K\emptyset$

	GAT signature	Category of Algs.	Initial Algebra
Categories	$\mathfrak{Cat}$	Cat	$\emptyset$
CwFs	$\mathfrak{CwF}$	CwF	$\{\diamond\}, K\emptyset$
CwF+V+Eq+Π	$\mathfrak{CwF}^{++}$		

	GAT signature	Category of Algs.	Initial Algebra
Categories	$\mathfrak{Cat}$	Cat	$\emptyset$
CwFs	$\mathfrak{CwF}$	CwF	$\{\diamond\}, K\emptyset$
GAT signatures	$\mathfrak{CwF}^{++}$		

	GAT signature	Category of Algs.	Initial Algebra
Categories	$\mathfrak{Cat}$	Cat	$\emptyset$
CwFs	$\mathfrak{CwF}$	CwF	$\{\diamond\}, K\emptyset$
GAT signatures	$\mathfrak{CwF}^{++}$	$\cdots$	$\textsc{one}\mathrm{GAT}$

The oneGAT language

We assume that the GAT $\mathfrak{CwF}{+}\mathsf{U}{+}\mathsf{Eq}{+}\Pi_{sd}$ has an initial algebra, given as a QIIT in the metatheory.

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We assume that the GAT $\mathfrak{CwF}$+U+Eq+Π_{sd} has an initial algebra, given as a QIIT in the metatheory.

$$
\begin{aligned}
\overline{\mathsf{Con}} &: \mathsf{Set} \\
\overline{\mathsf{Sub}} &: \overline{\mathsf{Con}} \to \overline{\mathsf{Con}} \to \mathsf{Set} \\
\overline{\mathsf{Ty}} &: \overline{\mathsf{Con}} \to \mathsf{Set} \\
\overline{\mathsf{Tm}} &: (\mathfrak{G} : \overline{\mathsf{Con}}) \to \overline{\mathsf{Ty}}\ \mathfrak{G}
\end{aligned}
$$

We assume that the GAT $\mathfrak{CwF}$+U+Eq+Π_{sd} has an initial algebra, given as a QIIT in the metatheory.

$$\begin{aligned}
\mathsf{GAT} := \overline{\mathsf{Con}} &: \mathsf{Set} \\
\overline{\mathsf{Sub}} &: \overline{\mathsf{Con}} \to \overline{\mathsf{Con}} \to \mathsf{Set} \\
\overline{\mathsf{Ty}} &: \overline{\mathsf{Con}} \to \mathsf{Set} \\
\overline{\mathsf{Tm}} &: (\mathfrak{G} : \overline{\mathsf{Con}}) \to \overline{\mathsf{Ty}}\ \mathfrak{G}
\end{aligned}$$

The oneGAT language

We assume that the GAT $\mathfrak{CwF}$+U+Eq+Π_{sd} has an initial algebra, given as a QIIT in the metatheory.

$$\begin{aligned}
\mathsf{GAT} := \overline{\mathsf{Con}} &: \mathsf{Set} \\
\overline{\mathsf{Sub}} &: \overline{\mathsf{Con}} \to \overline{\mathsf{Con}} \to \mathsf{Set} \\
\overline{\mathsf{Ty}} &: \overline{\mathsf{Con}} \to \mathsf{Set} \\
\overline{\mathsf{Tm}} &: (\mathfrak{G} : \overline{\mathsf{Con}}) \to \overline{\mathsf{Ty}}\ \mathfrak{G}
\end{aligned}$$

Use the DSL parsing utilities in LEAN4 to translate the intuitive syntax for GATs into ONEGAT.

$$
\begin{array}{l}
\mathsf{def}\ \mathfrak{N} : \mathtt{GAT} := \{\!\!\{ \\
\quad \mathtt{Nat} \quad : \mathbf{U}, \\
\quad \mathtt{zero} \ : \mathtt{Nat}, \\
\quad \mathtt{succ} \ : \mathtt{Nat} \Rightarrow \mathtt{Nat} \\
\}\!\!\}
\end{array}
$$

$$
\begin{array}{ll}
\mathsf{def}\ \mathfrak{N} : \mathtt{GAT} := \{\!\![& \diamond \\
\quad \mathtt{Nat} \quad : \mathbf{U}, & \triangleright\ \mathsf{V} \\
\quad \mathtt{zero} \ : \mathtt{Nat}, & \triangleright\ \mathsf{El}\ 0 \\
\quad \mathtt{succ} \ : \mathtt{Nat} \Rightarrow \mathtt{Nat} & \triangleright\ \Pi\ 1\ (\mathsf{El}\ 2) \\
]\!\!\} &
\end{array}
$$

Categories

```
def ℭ𝔞𝔱 : GAT := ⦃
  Obj   : U,
  Hom   : Obj ⇒ Obj ⇒ U,
  id    : (I : Obj) ⇒ Hom I I,
  comp  : {I J K : Obj} ⇒
          Hom J K ⇒ Hom I J ⇒ Hom I K,
  lunit : {I J : Obj} ⇒ (j : Hom I J) ⇒
          comp (id J) j ≡ j,
  runit : {I J : Obj} ⇒ (j : Hom I J) ⇒
          comp j (id I) ≡ j,
  assoc : {I J K L : Obj} ⇒
          (j : Hom I J) ⇒ (k : Hom J K) ⇒
          (ℓ : Hom K L) ⇒
          comp ℓ (comp k j)
          ≡ comp (comp ℓ k) j
⦄
```

Categories

$$
\begin{aligned}
&\mathsf{def}\ \mathfrak{Cat} : \mathtt{GAT} := \{\!\!\{\\
&\quad \mathtt{Obj} \ : \mathbf{U},\\
&\quad \mathtt{Hom} \ : \mathtt{Obj} \Rightarrow \mathtt{Obj} \Rightarrow \mathbf{U},\\
&\quad \mathtt{id} \ : (\mathtt{I} : \mathtt{Obj}) \Rightarrow \mathtt{Hom\ I\ I},\\
&\quad \mathtt{comp} \ : \{\mathtt{I\ J\ K} : \mathtt{Obj}\} \Rightarrow\\
&\qquad\quad \mathtt{Hom\ J\ K} \Rightarrow \mathtt{Hom\ I\ J} \Rightarrow \mathtt{Hom\ I\ K},\\
&\quad \mathtt{lunit} : \{\mathtt{I\ J} : \mathtt{Obj}\} \Rightarrow (\mathtt{j} : \mathtt{Hom\ I\ J}) \Rightarrow\\
&\qquad\quad \mathtt{comp\ (id\ J)\ j} \equiv \mathtt{j},\\
&\quad \mathtt{runit} : \{\mathtt{I\ J} : \mathtt{Obj}\} \Rightarrow (\mathtt{j} : \mathtt{Hom\ I\ J}) \Rightarrow\\
&\qquad\quad \mathtt{comp\ j\ (id\ I)} \equiv \mathtt{j},\\
&\quad \mathtt{assoc} : \{\mathtt{I\ J\ K\ L} : \mathtt{Obj}\} \Rightarrow\\
&\qquad\quad (\mathtt{j} : \mathtt{Hom\ I\ J}) \Rightarrow (\mathtt{k} : \mathtt{Hom\ J\ K}) \Rightarrow\\
&\qquad\quad (\ell : \mathtt{Hom\ K\ L}) \Rightarrow\\
&\qquad\quad \mathtt{comp}\ \ell\ \mathtt{(comp\ k\ j)}\\
&\qquad\quad \equiv \mathtt{comp\ (comp}\ \ell\ \mathtt{k)\ j}\\
&\}\!\!\}
\end{aligned}
$$

$\diamond$

$\triangleright$ V

$\triangleright$ Π 0 (Π 1 V)

$\triangleright$ Π 1 (El (1 @ 0 @ 0))

$\triangleright$ Π 2 (Π 3 (Π 4 (Π (4 @ 1 @ 0) (Π (5 @ 3 @ 2) (El (6 @ 4 @ 2))))))

$\triangleright$ Π 3 (Π 4 (Π (4 @ 1 @ 0) (Eq (3 @ 2 @ 1 @ 1 @ (4 @ 1) @ 0) 0)))

$\triangleright$ Π 4 (Π 5 (Π (5 @ 1 @ 0) (Eq (4 @ 2 @ 2 @ 1 @ 0 @ (5 @ 2)) 0)))

$\triangleright$ Π 5 (Π 6 (Π 7 (Π 8 (Π (8 @ 3 @ 2) (Π (9 @ 3 @ 2) (Π (10 @ 3 @ 2) (Eq (9 @ 6 @ 5 @ 3 @ 0 @ (9 @ 6 @ 5 @ 4 @ 1 @ 2)) (9 @ 6 @ 4 @ 3 @ (9 @ 5 @ 4 @ 3 @ 0 @ 1) @ 2))))))))

CwFs

$\mathfrak{Cat}$
- ▷ El 6
- ▷ Π 7 (El (7 @ 0 @ 1))
- ▷ Π 8 (Π (8 @ 0 @ 2) (Eq 0 (2 @ 1)))
- ▷ Π 9 V
- ▷ Π 10 (Π 11 (Π (11 @ 1 @ 0) (Π (3 @ 1) (El (4 @ 3)))))
- ▷ Π 11 (Π (2 @ 0) (Eq (2 @ 1 @ 1 @ (11 @ 1) @ 0) 0))
- ▷ Π 12 (Π 13 (Π 14 (Π (5 @ 0) (Π (15 @ 3 @ 2) (Π (16 @ 3 @ 2) (Eq (7 @ 4 @ 3 @ 0 @ (7 @ 5 @ 4 @ 1 @ 2)) (7 @ 5 @ 3 @ (15 @ 5 @ 4 @ 3 @ 0 @ 1) @ 2)))))))
- ▷ Π 13 (Π (4 @ 0) V)
- ▷ Π 14 (Π 15 (Π (6 @ 0) (Π (16 @ 2 @ 1) (Π (4 @ 2 @ 1) (El (5 @ 4 @ (8 @ 4 @ 3 @ 1 @ 2)))))))
- ▷ Π 15 (Π (6 @ 0) (Π (3 @ 1 @ 0) (Eq (3 @ 2 @ 2 @ 1 @ (16 @ 2) @ 0) 0)))
- ▷ Π 16 (Π 17 (Π 18 (Π (9 @ 0) (Π (6 @ 1 @ 0) (Π (20 @ 4 @ 3) (Π (21 @ 4 @ 3) (Eq (8 @ 5 @ 4 @ 3 @ 0 @ (8 @ 6 @ 5 @ (12 @ 5 @ 4 @ 0 @ 3) @ 1 @ 2)) (8 @ 6 @ 4 @ 3 @ (20 @ 6 @ 5 @ 4 @ 0 @ 1) @ 2))))))))

CwFs

- ▷ Π 17 (Π (8 @ 0) (El 19))
- ▷ Π 18 (Π 19 (Π (10 @ 0) (Π (20 @ 2 @ 1) (Π (8 @ 3 @ (11 @ 3 @ 2 @ 0 @ 1)) (El (22 @ 4 @ (5 @ 3 @ 2)))))))
- ▷ Π 19 (Π 20 (Π 21 (Π (12 @ 0) (Π (22 @ 2 @ 1) (Π (10 @ 3 @ (13 @ 3 @ 2 @ 0 @ 1)) (Π (24 @ 5 @ 4) (Eq (23 @ 6 @ 5 @ (8 @ 4 @ 3) @ (7 @ 5 @ 4 @ 3 @ 2 @ 1) @ 0) (7 @ 6 @ 4 @ 3 @ (23 @ 6 @ 5 @ 4 @ 2 @ 0) @ (11 @ 6 @ 5 @ (15 @ 5 @ 4 @ 2 @ 3) @ 0 @ 1)))))))))
- ▷ Π 20 (Π 21 (Π (12 @ 0) (Π (22 @ 2 @ (5 @ 1 @ 0)) (El (23 @ 3 @ 2)))))
- ▷ Π 21 (Π 22 (Π (13 @ 0) (Π (23 @ 2 @ (6 @ 1 @ 0)) (El (11 @ 3 @ (14 @ 3 @ 2 @ (4 @ 3 @ 2 @ 1 @ 0) @ 1))))))
- ▷ Π 22 (Π 23 (Π (14 @ 0) (Π (24 @ 2 @ 1) (Π (12 @ 3 @ (15 @ 3 @ 2 @ 0 @ 1)) (Eq (6 @ 4 @ 3 @ 2 @ (8 @ 4 @ 3 @ 2 @ 1 @ 0)) 1)))))
- ▷ Π 23 (Π 24 (Π (15 @ 0) (Π (25 @ 2 @ 1) (Π (13 @ 3 @ (16 @ 3 @ 2 @ 0 @ 1)) (Eq (6 @ 4 @ 3 @ 2 @ (9 @ 4 @ 3 @ 2 @ 1 @ 0)) 0)))))
- ▷ Π 24 (Π 25 (Π (16 @ 0) (Π (26 @ 2 @ (9 @ 1 @ 0)) (Eq (9 @ 3 @ 2 @ 1 @ (7 @ 3 @ 2 @ 1 @ 0) @ (6 @ 3 @ 2 @ 1 @ 0)) 0))))

To the logical extreme: oneGAT describes itself

Quotient Inductive-Inductive definitions over all GATs

The benefit of quotient inductive-inductive types is that we can reason about them by quotient induction induction.

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$$(\text{elim } m)_{\overline{\text{Con}}} : (\mathfrak{G} : \overline{\text{Con}}) \to M_{\overline{\text{Con}}}(\mathfrak{G})$$

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$$(\mathsf{elim}\ m)_{\overline{\mathsf{Con}}} \colon (\mathfrak{G} \colon \overline{\mathsf{Con}}) \to M_{\overline{\mathsf{Con}}}(\mathfrak{G})$$

$$(\mathsf{elim}\ m)_{\overline{\mathsf{Sub}}} \colon (\mathfrak{G}\ \mathfrak{H} \colon \overline{\mathsf{Con}}) \to (\mathfrak{s} \colon \overline{\mathsf{Sub}}\ \mathfrak{G}\ \mathfrak{H}) \to M_{\overline{\mathsf{Sub}}}((\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, (\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{H}, \mathfrak{s})$$

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$$
\begin{aligned}
(\mathrm{elim}\ m)_{\overline{\mathsf{Con}}} &: (\mathfrak{G} : \overline{\mathsf{Con}}) \to M_{\overline{\mathsf{Con}}}(\mathfrak{G}) \\
(\mathrm{elim}\ m)_{\overline{\mathsf{Sub}}} &: (\mathfrak{G}\ \mathfrak{H} : \overline{\mathsf{Con}}) \to (\mathfrak{s} : \overline{\mathsf{Sub}}\ \mathfrak{G}\ \mathfrak{H}) \to \\
&\qquad M_{\overline{\mathsf{Sub}}}((\mathrm{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, (\mathrm{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{H}, \mathfrak{s}) \\
(\mathrm{elim}\ m)_{\overline{\mathsf{Ty}}} &: (\mathfrak{G} : \overline{\mathsf{Con}}) \to (\mathcal{X} : \overline{\mathsf{Ty}}\ \mathfrak{G}) \to M_{\overline{\mathsf{Ty}}}((\mathrm{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, \mathcal{X})
\end{aligned}
$$

The benefit of quotient inductive-inductive types is that we can reason about them by quotient induction induction.
Given an appropriate *motive* $M_{\overline{\mathsf{Con}}}, M_{\overline{\mathsf{Sub}}}, M_{\overline{\mathsf{Ty}}}, M_{\overline{\mathsf{Tm}}}$ and appropriate *method* m, we get:

$$(\mathsf{elim}\ m)_{\overline{\mathsf{Con}}} \colon (\mathfrak{G} \colon \overline{\mathsf{Con}}) \to M_{\overline{\mathsf{Con}}}(\mathfrak{G})$$

$$\begin{aligned}(\mathsf{elim}\ m)_{\overline{\mathsf{Sub}}} \colon (\mathfrak{G}\ \mathfrak{H} \colon \overline{\mathsf{Con}}) \to (\mathfrak{s} \colon \overline{\mathsf{Sub}}\ \mathfrak{G}\ \mathfrak{H}) \to \\ M_{\overline{\mathsf{Sub}}}((\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, (\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{H}, \mathfrak{s})\end{aligned}$$

$$(\mathsf{elim}\ m)_{\overline{\mathsf{Ty}}} \colon (\mathfrak{G} \colon \overline{\mathsf{Con}}) \to (\mathcal{X} \colon \overline{\mathsf{Ty}}\ \mathfrak{G}) \to M_{\overline{\mathsf{Ty}}}((\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, \mathcal{X})$$

$$(\mathsf{elim}\ m)_{\overline{\mathsf{Tm}}} \colon (\mathfrak{G} \colon \overline{\mathsf{Con}}) \to (\mathcal{X} \colon \overline{\mathsf{Ty}}\ \mathfrak{G}) \to (X \colon \overline{\mathsf{Tm}}(\mathfrak{G}, \mathcal{X}))$$

Quotient Inductive-Inductive definitions over all GATs

The benefit of quotient inductive-inductive types is that we can reason about them by quotient induction induction.
Given an appropriate *motive* $M_{\overline{\mathsf{Con}}}, M_{\overline{\mathsf{Sub}}}, M_{\overline{\mathsf{Ty}}}, M_{\overline{\mathsf{Tm}}}$ and appropriate *method* m, we get:

$$
\begin{aligned}
(\mathsf{elim}\ m)_{\overline{\mathsf{Con}}} &: (\mathfrak{G} : \overline{\mathsf{Con}}) \to M_{\overline{\mathsf{Con}}}(\mathfrak{G}) \\
(\mathsf{elim}\ m)_{\overline{\mathsf{Sub}}} &: (\mathfrak{G}\ \mathfrak{H} : \overline{\mathsf{Con}}) \to (\mathfrak{s} : \overline{\mathsf{Sub}}\ \mathfrak{G}\ \mathfrak{H}) \to \\
&\qquad M_{\overline{\mathsf{Sub}}}((\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, (\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{H}, \mathfrak{s}) \\
(\mathsf{elim}\ m)_{\overline{\mathsf{Ty}}} &: (\mathfrak{G} : \overline{\mathsf{Con}}) \to (\mathcal{X} : \overline{\mathsf{Ty}}\ \mathfrak{G}) \to M_{\overline{\mathsf{Ty}}}((\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, \mathcal{X}) \\
(\mathsf{elim}\ m)_{\overline{\mathsf{Tm}}} &: (\mathfrak{G} : \overline{\mathsf{Con}}) \to (\mathcal{X} : \overline{\mathsf{Ty}}\ \mathfrak{G}) \to (X : \overline{\mathsf{Tm}}(\mathfrak{G}, \mathcal{X})) \\
&\qquad \to M_{\overline{\mathsf{Tm}}}((\mathsf{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, (\mathsf{elim}\ m)_{\overline{\mathsf{Ty}}}\ \mathcal{X}, X)
\end{aligned}
$$

Quotient Inductive-Inductive definitions over all GATs

The benefit of quotient inductive-inductive types is that we can reason about them by quotient induction induction.
Given an appropriate *motive* $M_{\overline{\mathsf{Con}}}, M_{\overline{\mathsf{Sub}}}, M_{\overline{\mathsf{Ty}}}, M_{\overline{\mathsf{Tm}}}$ and appropriate *method* m, we get:

$$
\begin{aligned}
&(\mathrm{elim}\ m)_{\overline{\mathsf{Con}}} : (\mathfrak{G} : \overline{\mathsf{Con}}) \to M_{\overline{\mathsf{Con}}}(\mathfrak{G}) \\
&(\mathrm{elim}\ m)_{\overline{\mathsf{Sub}}} : (\mathfrak{G}\ \mathfrak{H} : \overline{\mathsf{Con}}) \to (\mathfrak{s} : \overline{\mathsf{Sub}}\ \mathfrak{G}\ \mathfrak{H}) \to \\
&\qquad M_{\overline{\mathsf{Sub}}}((\mathrm{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, (\mathrm{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{H}, \mathfrak{s}) \\
&(\mathrm{elim}\ m)_{\overline{\mathsf{Ty}}} : (\mathfrak{G} : \overline{\mathsf{Con}}) \to (\mathcal{X} : \overline{\mathsf{Ty}}\ \mathfrak{G}) \to M_{\overline{\mathsf{Ty}}}((\mathrm{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, \mathcal{X}) \\
&(\mathrm{elim}\ m)_{\overline{\mathsf{Tm}}} : (\mathfrak{G} : \overline{\mathsf{Con}}) \to (\mathcal{X} : \overline{\mathsf{Ty}}\ \mathfrak{G}) \to (X : \overline{\mathsf{Tm}}(\mathfrak{G}, \mathcal{X})) \\
&\qquad \to M_{\overline{\mathsf{Tm}}}((\mathrm{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, (\mathrm{elim}\ m)_{\overline{\mathsf{Ty}}}\ \mathcal{X}, X)
\end{aligned}
$$

Quotient Inductive-Inductive definitions over all GATs

The benefit of quotient inductive-inductive types is that we can reason about them by quotient induction induction.

Given an appropriate *motive* $M_{\overline{\mathsf{Con}}}, M_{\overline{\mathsf{Sub}}}, M_{\overline{\mathsf{Ty}}}, M_{\overline{\mathsf{Tm}}}$ and appropriate *method* m, we get:

$$
\begin{aligned}
&(\mathrm{elim}\ m)_{\overline{\mathsf{Con}}} : (\mathfrak{G} : \mathtt{GAT}) \to M_{\overline{\mathsf{Con}}}(\mathfrak{G}) \\
&(\mathrm{elim}\ m)_{\overline{\mathsf{Sub}}} : (\mathfrak{G}\ \mathfrak{H} : \overline{\mathsf{Con}}) \to (\mathfrak{s} : \overline{\mathsf{Sub}}\ \mathfrak{G}\ \mathfrak{H}) \to \\
&\qquad\qquad M_{\overline{\mathsf{Sub}}}((\mathrm{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, (\mathrm{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{H}, \mathfrak{s}) \\
&(\mathrm{elim}\ m)_{\overline{\mathsf{Ty}}} : (\mathfrak{G} : \overline{\mathsf{Con}}) \to (\mathcal{X} : \overline{\mathsf{Ty}}\ \mathfrak{G}) \to M_{\overline{\mathsf{Ty}}}((\mathrm{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, \mathcal{X}) \\
&(\mathrm{elim}\ m)_{\overline{\mathsf{Tm}}} : (\mathfrak{G} : \overline{\mathsf{Con}}) \to (\mathcal{X} : \overline{\mathsf{Ty}}\ \mathfrak{G}) \to (X : \overline{\mathsf{Tm}}(\mathfrak{G}, \mathcal{X})) \\
&\qquad\qquad \to M_{\overline{\mathsf{Tm}}}((\mathrm{elim}\ m)_{\overline{\mathsf{Con}}}\ \mathfrak{G}, (\mathrm{elim}\ m)_{\overline{\mathsf{Ty}}}\ \mathcal{X}, X)
\end{aligned}
$$

$$\_\text{-}\mathsf{Alg} : (\mathfrak{G} : \mathtt{GAT}) \to \mathsf{Set}$$

From [KKA19, Appendix A]

Syntax	**Algebras**	
$\Gamma : \mathsf{Con}$	Γ^{A}	$: \mathsf{Set}$
$A : \mathsf{Ty}\,\Gamma$	A^{A}	$: \Gamma^{\mathsf{A}} \to \mathsf{Set}$
$\sigma : \mathsf{Sub}\,\Gamma\,\Delta$	σ^{A}	$: \Gamma^{\mathsf{A}} \to \Delta^{\mathsf{A}}$
$t : \mathsf{Tm}\,\Gamma\,A$	t^{A}	$: (\gamma : \Gamma^{\mathsf{A}}) \to A^{\mathsf{A}}\,\gamma$
$\cdot : \mathsf{Con}$	$\cdot^{\mathsf{A}}$	$:\equiv \top$
$\Gamma \rhd A : \mathsf{Con}$	$(\Gamma \rhd A)^{\mathsf{A}}$	$:\equiv (\gamma : \Gamma^{\mathsf{A}}) \times A^{\mathsf{A}}\,\gamma$
$(A : \mathsf{Ty}\,\Delta)[\sigma : \mathsf{Sub}\,\Gamma\,\Delta] : \mathsf{Ty}\,\Gamma$	$(A[\sigma])^{\mathsf{A}}\,\gamma$	$:\equiv A^{\mathsf{A}}\,(\sigma^{\mathsf{A}}\,\gamma)$
$\mathsf{id} : \mathsf{Sub}\,\Gamma\,\Gamma$	$\mathsf{id}^{\mathsf{A}}\,\gamma$	$:\equiv \gamma$
$(\sigma : \mathsf{Sub}\,\Theta\,\Delta) \circ (\delta : \mathsf{Sub}\,\Gamma\,\Theta) : \mathsf{Sub}\,\Gamma\,\Delta$	$(\sigma \circ \delta)^{\mathsf{A}}\,\gamma$	$:\equiv \sigma^{\mathsf{A}}\,(\delta^{\mathsf{A}}\,\gamma)$
$\epsilon : \mathsf{Sub}\,\Gamma\,\cdot$	$\epsilon^{\mathsf{A}}\,\gamma$	$:\equiv \mathsf{tt}$
$(\sigma : \mathsf{Sub}\,\Gamma\,\Delta), (t : \mathsf{Tm}\,\Gamma\,(A[\sigma])) : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)$	$(\sigma, t)^{\mathsf{A}}\,\gamma$	$:\equiv (\sigma^{\mathsf{A}}\,\gamma, t^{\mathsf{A}}\,\gamma)$
$\pi_1\,(\sigma : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)) : \mathsf{Sub}\,\Gamma\,\Delta$	$(\pi_1\,\sigma)^{\mathsf{A}}\,\gamma$	$:\equiv \mathsf{proj}_1\,(\sigma^{\mathsf{A}}\,\gamma)$
$\pi_2\,(\sigma : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)) : \mathsf{Tm}\,\Gamma\,(A[\pi_1\,\sigma])$	$(\pi_2\,\sigma)^{\mathsf{A}}\,\gamma$	$:\equiv \mathsf{proj}_2\,(\sigma^{\mathsf{A}}\,\gamma)$
$(t : \mathsf{Tm}\,\Delta\,A)[\sigma : \mathsf{Sub}\,\Gamma\,\Delta] : \mathsf{Tm}\,\Gamma\,(A[\sigma])$	$(t[\sigma])^{\mathsf{A}}\,\gamma$	$:\equiv t^{\mathsf{A}}\,(\sigma^{\mathsf{A}}\,\gamma)$
$[\mathsf{id}] : A[\mathsf{id}] = A$	$[\mathsf{id}]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$[\circ] : A[\sigma \circ \delta] = A[\sigma][\delta]$	$[\circ]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{ass} : (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$	$\mathsf{ass}^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{idl} : \mathsf{id} \circ \sigma = \sigma$	$\mathsf{idl}^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{idr} : \sigma \circ \mathsf{id} = \sigma$	$\mathsf{idr}^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\cdot\eta : \{\sigma : \mathsf{Sub}\,\Gamma\,\cdot\} \to \sigma = \epsilon$	$\cdot\eta^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\rhd\beta_1 : \pi_1\,(\sigma, t) = \sigma$	$\rhd\beta_1{}^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\rhd\beta_2 : \pi_2\,(\sigma, t) = t$	$\rhd\beta_2{}^{\mathsf{A}}$	$:\equiv \mathsf{refl}$

$\rhd\eta : (\pi_1\,\sigma, \pi_2\,\sigma) = \sigma$	$\rhd\eta^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$,\circ : (\sigma, t) \circ \delta = (\sigma \circ \delta, t[\delta])$	$,\circ^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{U} : \mathsf{Ty}\,\Gamma$	$\mathsf{U}^{\mathsf{A}}\,\gamma$	$:\equiv \mathsf{Set}$
$\mathsf{El}\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U}) : \mathsf{Ty}\,\Gamma$	$(\mathsf{El}\,a)^{\mathsf{A}}\,\gamma$	$:\equiv a^{\mathsf{A}}\,\gamma$
$\mathsf{U}[] : \mathsf{U}[\sigma] = \mathsf{U}$	$\mathsf{U}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{El}[] : (\mathsf{El}\,a)[\sigma] = \mathsf{El}\,(a[\sigma])$	$\mathsf{El}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\Pi\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U})(B : \mathsf{Ty}\,(\Gamma \rhd \mathsf{El}\,a)) : \mathsf{Ty}\,\Gamma$	$(\Pi\,a\,B)^{\mathsf{A}}\,\gamma$	$:\equiv (\alpha : a^{\mathsf{A}}\,\gamma) \to B^{\mathsf{A}}\,(\gamma, \alpha)$
$\mathsf{app}\,(t : \mathsf{Tm}\,\Gamma\,(\Pi\,a\,B)) : \mathsf{Tm}\,(\Gamma \rhd \mathsf{El}\,a)\,B$	$(\mathsf{app}\,t)^{\mathsf{A}}\,(\gamma, \alpha)$	$:\equiv t^{\mathsf{A}}\,\gamma\,\alpha$
$\Pi[] : (\Pi\,a\,B)[\sigma] = \Pi\,(a[\sigma])\,(B[\sigma^{\uparrow}])$	$\Pi[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{app}[] : (\mathsf{app}\,t)[\sigma \uparrow] = \mathsf{app}\,(t[\sigma])$	$\mathsf{app}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{Id}\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U})\,(t\,u : \mathsf{Tm}\,\Gamma\,(\mathsf{El}\,a)) : \mathsf{Ty}\,\Gamma$	$(\mathsf{Id}\,a\,t\,u)^{\mathsf{A}}\,\gamma$	$:\equiv (t^{\mathsf{A}}\,\gamma = u^{\mathsf{A}}\,\gamma)$
$\mathsf{reflect}\,(e : \mathsf{Tm}\,\Gamma\,(\mathsf{Id}\,a\,t\,u)) : t = u$	$(\mathsf{reflect}\,e)^{\mathsf{A}}$	$:\equiv \mathsf{funext}\,e^{\mathsf{A}}$
$\mathsf{Id}[] : (\mathsf{Id}\,a\,t\,u)[\sigma] = \mathsf{Id}\,(a[\sigma])\,(t[\sigma])\,(u[\sigma])$	$\mathsf{Id}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\hat{\Pi}\,(T : \mathsf{Set})\,(B : T \to \mathsf{Ty}\,\Gamma) : \mathsf{Ty}\,\Gamma$	$(\hat{\Pi}\,T\,B)^{\mathsf{A}}\,\gamma$	$:\equiv (\alpha : T) \to (B\,\alpha)^{\mathsf{A}}\,\gamma$
$(t : \mathsf{Tm}\,\Gamma\,(\hat{\Pi}\,T\,B))\,\hat{@}\,(\alpha : T) : \mathsf{Tm}\,\Gamma\,(B\,\alpha)$	$(t\,\hat{@}\,\alpha)^{\mathsf{A}}\,\gamma$	$:\equiv t^{\mathsf{A}}\,\gamma\,\alpha$
$\hat{\Pi}[] : (\hat{\Pi}\,T\,B)[\sigma] = \hat{\Pi}\,T\,(\lambda\alpha.(B\,\alpha)[\sigma])$	$\hat{\Pi}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\hat{@}[] : (t\,\hat{@}\,\alpha)[\sigma] = (t[\sigma])\,\hat{@}\,\alpha$	$\hat{@}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$

$$(\_)^{A} : (\mathfrak{G} : \mathtt{GAT}) \to \mathsf{Set}$$

$$
\begin{aligned}
\_\text{-}\mathsf{Alg} &: (\mathfrak{G} : \mathtt{GAT}) \to \mathsf{Set} \\
\mathsf{Hom} &: (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set}
\end{aligned}
$$

From [KKA19, Appendix A]

$\Gamma : \mathsf{Con}$	Γ^{M}	$: \Gamma^{\mathsf{A}} \to \Gamma^{\mathsf{A}} \to \mathsf{Set}$
$A : \mathsf{Ty}\,\Gamma$	A^{M}	$: \Gamma^{\mathsf{M}}\,\gamma^0\,\gamma^1 \to A^{\mathsf{A}}\,\gamma^0 \to A^{\mathsf{A}}\,\gamma^1 \mathsf{Set}$
$\sigma : \mathsf{Sub}\,\Gamma\,\Delta$	σ^{M}	$: \Gamma^{\mathsf{M}}\,\gamma^0\,\gamma^1 \to \Delta^{\mathsf{M}}\,(\sigma^{\mathsf{A}}\,\gamma^0)\,(\sigma^{\mathsf{A}}\,\gamma^1)$
$t : \mathsf{Tm}\,\Gamma\,A$	t^{M}	$: (\gamma^M : \Gamma^{\mathsf{M}}\,\gamma^0\,\gamma^1) \to A^{\mathsf{M}}\,\gamma^M\,(t^{\mathsf{A}}\,\gamma^0)\,(t^{\mathsf{A}}\,\gamma^1)$
$\cdot : \mathsf{Con}$	$\cdot^{\mathsf{M}}\ \mathsf{tt}\,\mathsf{tt}$	$:\equiv \top$
$\Gamma \rhd A : \mathsf{Con}$	$(\Gamma \rhd A)^{\mathsf{M}}\,(\gamma^0, \alpha^0)\,(\gamma^1, \alpha^1)$	$:\equiv (\gamma^M : \Gamma^{\mathsf{M}}\,\gamma^0\,\gamma^1) \times A^{\mathsf{M}}\,\gamma^M\,\alpha^0\,\alpha^1$
$(A : \mathsf{Ty}\,\Delta)[\sigma : \mathsf{Sub}\,\Gamma\,\Delta] : \mathsf{Ty}\,\Gamma$	$(A[\sigma])^{\mathsf{M}}\,\gamma^M\,\alpha^0\,\alpha^1$	$:\equiv A^{\mathsf{M}}\,(\sigma^{\mathsf{M}}\,\gamma^M)\,\alpha^0\,\alpha^1$
$\mathsf{id} : \mathsf{Sub}\,\Gamma\,\Gamma$	$\mathsf{id}^{\mathsf{M}}\,\gamma^M$	$:\equiv \gamma^m$
$(\sigma : \mathsf{Sub}\,\Theta\,\Delta) \circ (\delta : \mathsf{Sub}\,\Gamma\,\Theta) : \mathsf{Sub}\,\Gamma\,\Delta$	$(\sigma \circ \delta)^{\mathsf{M}}\,\gamma^M$	$:\equiv \sigma^{\mathsf{M}}\,(\delta^{\mathsf{M}}\,\gamma^M)$
$\epsilon : \mathsf{Sub}\,\Gamma\,\cdot$	$\epsilon^{\mathsf{M}}\,\gamma^M$	$:\equiv \mathsf{tt}$
$(\sigma : \mathsf{Sub}\,\Gamma\,\Delta), (t : \mathsf{Tm}\,\Gamma\,(A[\sigma])) : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)$	$(\sigma, t)^{\mathsf{M}}\,\gamma^M$	$:\equiv (\sigma^{\mathsf{M}}\,\gamma^M, t^{\mathsf{M}}\,\gamma^M)$
$\pi_1\,(\sigma : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)) : \mathsf{Sub}\,\Gamma\,\Delta$	$(\pi_1\,\sigma)^{\mathsf{M}}\,\gamma^M$	$:\equiv \mathsf{proj}_1\,(\sigma^{\mathsf{M}}\,\gamma^M)$
$\pi_2\,(\sigma : \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)) : \mathsf{Tm}\,\Gamma\,(A[\pi_1\,\sigma])$	$(\pi_2\,\sigma)^{\mathsf{M}}\,\gamma^M$	$:\equiv \mathsf{proj}_2\,(\sigma^{\mathsf{M}}\,\gamma^M)$
$(t : \mathsf{Tm}\,\Delta\,A)[\sigma : \mathsf{Sub}\,\Gamma\,\Delta] : \mathsf{Tm}\,\Gamma\,(A[\sigma])$	$(t[\sigma])^{\mathsf{M}}\,\gamma^M$	$:\equiv t^{\mathsf{M}}\,(\sigma^{\mathsf{M}}\,\gamma^M)$
$[\mathsf{id}] : A[\mathsf{id}] = A$	$[\mathsf{id}]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$[\circ] : A[\sigma \circ \delta] = A[\sigma][\delta]$	$[\circ]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{ass} : (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$	$\mathsf{ass}^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{idl} : \mathsf{id} \circ \sigma = \sigma$	$\mathsf{idl}^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{idr} : \sigma \circ \mathsf{id} = \sigma$	$\mathsf{idr}^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\cdot\eta : \{\sigma : \mathsf{Sub}\,\Gamma\,\cdot\} \to \sigma = \epsilon$		
$\rhd\beta_1 : \pi_1\,(\sigma, t) = \sigma$		
$\rhd\beta_2 : \pi_2\,(\sigma, t) = t$		
$\rhd\eta : (\pi_1\,\sigma, \pi_2\,\sigma) = \sigma$	$\rhd\eta^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$,\circ : (\sigma, t) \circ \delta = (\sigma \circ \delta, t[\delta])$	$,\circ^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{U} : \mathsf{Ty}\,\Gamma$	$\mathsf{U}^{\mathsf{M}}\,\gamma^M\,T^0\,T^1$	$:\equiv T^0 \to T^1$

$\mathsf{El}\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U}) : \mathsf{Ty}\,\Gamma$	$(\mathsf{El}\,a)^{\mathsf{M}}\,\gamma^M\,\alpha^0\,\alpha^1$	$:\equiv a^{\mathsf{M}}\,\gamma^M\,\alpha^0 = \alpha^1$
$\mathsf{U}[] : \mathsf{U}[\sigma] = \mathsf{U}$	$\mathsf{U}[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{El}[] : (\mathsf{El}\,a)[\sigma] = \mathsf{El}\,(a[\sigma])$	$\mathsf{El}[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\Pi\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U})(B : \mathsf{Ty}\,(\Gamma \rhd \mathsf{El}\,a)) : \mathsf{Ty}\,\Gamma$	$(\Pi\,a\,B)^{\mathsf{M}}\,\gamma^M\,f^0\,f^1$	$:\equiv (\alpha^0 : a^{\mathsf{A}}\,\gamma^0) \to B^{\mathsf{M}}\,(\gamma^M, \mathsf{refl})\,(f^0\,\alpha^0)\,(f^1\,(a^{\mathsf{M}}\,\gamma^M\,\alpha^0))$
$\mathsf{app}\,(t : \mathsf{Tm}\,\Gamma\,(\Pi\,a\,B)) : \mathsf{Tm}\,(\Gamma \rhd \mathsf{El}\,a)\,B$	$(\mathsf{app}\,t)^{\mathsf{M}}\,(\gamma^M, \alpha^M)$	$:\equiv \mathsf{J}\,(t^{\mathsf{M}}\,\gamma^M\,\alpha^0)\,\alpha^M$
$\Pi[] : (\Pi\,a\,B)[\sigma] = \Pi\,(a[\sigma])\,(B[\sigma^{\uparrow}])$	$\Pi[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{app}[] : (\mathsf{app}\,t)[\sigma\uparrow] = \mathsf{app}\,(t[\sigma])$	$\mathsf{app}[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\mathsf{Id}\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U})\,(t\,u : \mathsf{Tm}\,\Gamma\,(\mathsf{El}\,a)) : \mathsf{Ty}\,\Gamma$	$(\mathsf{Id}\,a\,t\,u)^{\mathsf{M}}\,\gamma^M\,e^0\,e^1$	$:\equiv \top$
$\mathsf{reflect}\,(e : \mathsf{Tm}\,\Gamma\,(\mathsf{Id}\,a\,t\,u)) : t = u$	$(\mathsf{reflect}\,e)^{\mathsf{M}}$	$:\equiv \mathsf{UIP}$
$\mathsf{Id}[] : (\mathsf{Id}\,a\,t\,u)[\sigma] = \mathsf{Id}\,(a[\sigma])\,(t[\sigma])\,(u[\sigma])$	$\mathsf{Id}[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\hat\Pi\,(T : \mathsf{Set})\,(B : T \to \mathsf{Ty}\,\Gamma) : \mathsf{Ty}\,\Gamma$	$(\hat\Pi\,T\,B)^{\mathsf{M}}\,\gamma^M\,f^0\,f^1$	$:\equiv (\alpha : T) \to (B\,\alpha)^{\mathsf{M}}\,\gamma^M\,(f^0\,\alpha)\,(f^1\,\alpha)$
$(t : \mathsf{Tm}\,\Gamma\,(\hat\Pi\,T\,B))\,\hat{@}(\alpha : T) : \mathsf{Tm}\,\Gamma\,(B\,\alpha)$	$(t\,\hat{@}\,\alpha)^{\mathsf{M}}\,\gamma^M$	$:\equiv t^{\mathsf{M}}\,\gamma^M\,\alpha$
$\hat\Pi[] : (\hat\Pi\,T\,B)[\sigma] = \hat\Pi\,T\,(\lambda\alpha.(B\,\alpha)[\sigma])$	$\hat\Pi[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$
$\hat{@}[] : (t\,\hat{@}\,\alpha)[\sigma] = (t[\sigma])\,\hat{@}\,\alpha$	$\hat{@}[]^{\mathsf{M}}$	$:\equiv \mathsf{refl}$

$$(\_)^M : (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-Alg} \to \mathfrak{G}\text{-Alg} \to \mathsf{Set}$$

	GAT signature	Category of Algs.	Initial Algebra
Categories	$\mathfrak{Cat}$		
CwFs	$\mathfrak{CwF}$		
GAT signatures	$\mathfrak{CwF}^{++}$	$\cdots$	ONEGAT

	GAT signature	Category of Algs.	Initial Algebra
Categories	$\mathfrak{Cat}$		
CwFs	$\mathfrak{CwF}$		
GAT signatures	$\mathfrak{CwF}^{++}$	$\cdots$	ONEGAT

GAT signature	Category of Algs.	Initial Algebra
Categories	$\mathfrak{Cat}$	
CwFs	$\mathfrak{CwF}$	
GAT signatures	$\mathfrak{CwF}^{++}$ $\cdots$	ONEGAT

Idea:

Elements of initial algebra are *only* those derivable from the *mere concept* of the structure

Construction of initial algebras

$$\text{def } \mathfrak{N} : \mathtt{GAT} := \{\!\!\{\ \mathtt{Nat} \quad : \mathbf{U},\ \mathtt{zero} \quad : \mathtt{Nat},\ \mathtt{succ} \quad : \mathtt{Nat} \Rightarrow \mathtt{Nat}\ \}\!\!\}$$

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$$\mathbb{N} := \overline{\mathsf{Tm}}(\mathfrak{N}, \mathtt{Nat})$$

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Forms a $\mathfrak{N}$-algebra with $\mathtt{zero}$ and $\lambda \mathtt{t} \to \mathtt{succ}\ \mathtt{t}$.

Construction of initial algebras

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Forms a $\mathfrak{N}$-algebra with $\mathtt{zero}$ and $\lambda \mathtt{t} \to \mathtt{succ}\ \mathtt{t}$.

Claim Can do this for arbitrary GAT $\mathfrak{G}$, by quotient induction-induction on ONEGAT.

Upshot: Any GAT extension of $\mathfrak{CwF}$ has an initial model

2

Concrete CwFs

Ahrens and Lumsdaine [AL19] introduce the notion of **displayed categories**

Displayed categories

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$$\mathsf{Obj}^{\mathsf{D}} : \mathsf{Obj} \to \mathsf{Set}$$
$$\mathsf{Hom}^{\mathsf{D}} : (I\ J : \mathsf{Obj}) \to \mathsf{Obj}^{\mathsf{D}}\ I \to \mathsf{Obj}^{\mathsf{D}}\ J \to \mathsf{Set}$$

Ahrens and Lumsdaine [AL19] introduce the notion of **displayed categories**

$$\begin{aligned}
\mathsf{Obj}^{\mathsf{D}} &\colon \mathsf{Obj} \to \mathsf{Set} \\
\mathsf{Hom}^{\mathsf{D}} &\colon (I\ J \colon \mathsf{Obj}) \to \mathsf{Obj}^{\mathsf{D}}\ I \to \mathsf{Obj}^{\mathsf{D}}\ J \to \mathsf{Set} \\
\mathsf{id}^{\mathsf{D}} &\colon (I \colon \mathsf{Obj}) \to (I^{\mathsf{D}} \colon \mathsf{Obj}^{\mathsf{D}}\ I) \to \mathsf{Hom}^{\mathsf{D}}\ I\ I^{\mathsf{D}}\ I\ I^{\mathsf{D}}
\end{aligned}$$

Displayed categories

Ahrens and Lumsdaine [AL19] introduce the notion of **displayed categories**

$$
\begin{aligned}
\mathrm{Obj}^{\mathrm{D}} &: \mathrm{Obj} \to \mathrm{Set} \\
\mathrm{Hom}^{\mathrm{D}} &: (I\ J : \mathrm{Obj}) \to \mathrm{Obj}^{\mathrm{D}}\ I \to \mathrm{Obj}^{\mathrm{D}}\ J \to \mathrm{Set} \\
\mathrm{id}^{\mathrm{D}} &: (I : \mathrm{Obj}) \to (I^{\mathrm{D}} : \mathrm{Obj}^{\mathrm{D}}\ I) \to \mathrm{Hom}^{\mathrm{D}}\ I\ I^{\mathrm{D}}\ I\ I^{\mathrm{D}} \\
&\dots
\end{aligned}
$$

Fact: We can define “displayed $\mathfrak{G}$-algebra” for every GAT $\mathfrak{G}$

$$\_\text{-}\mathsf{Alg} : (\mathfrak{G} : \mathtt{GAT}) \to \mathsf{Set}$$

$$\mathsf{Hom} : (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set}$$

$$\_\text{-}\mathsf{DAlg} : (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set}$$

From [KKA19, Appendix A]

Syntax	Displayed algebras	
$\Gamma : \mathsf{Con}$	Γ^{D}	$: \Gamma^{\mathsf{A}} \to \mathsf{Set}$
$A : \mathsf{Ty}\,\Gamma$	A^{D}	$: \Gamma^{\mathsf{D}}\,\gamma \to A^{\mathsf{A}}\,\gamma \to \mathsf{Set}$
$\sigma : \mathsf{Sub}\,\Gamma\,\Delta$	σ^{D}	$: \Gamma^{\mathsf{D}}\,\gamma \to \Delta^{\mathsf{D}}\,(\sigma^{\mathsf{A}}\,\gamma)$
$t : \mathsf{Tm}\,\Gamma\,A$	t^{D}	$: (\gamma^D : \Gamma^{\mathsf{D}}\,\gamma) \to A^{\mathsf{D}}\,\gamma^D\,(t^{\mathsf{A}}\,\gamma)$
$\cdot : \mathsf{Con}$	$\cdot^{\mathsf{D}}\,\mathsf{tt}$	$:\equiv \top$
$\Gamma \triangleright A : \mathsf{Con}$	$(\Gamma \triangleright A)^{\mathsf{D}}\,(\gamma, \alpha)$	$:\equiv (\gamma^D : \Gamma^{\mathsf{D}}\,\gamma) \times A^{\mathsf{D}}\,\gamma^D\,\alpha$
$(A : \mathsf{Ty}\,\Delta)[\sigma : \mathsf{Sub}\,\Gamma\,\Delta] : \mathsf{Ty}\,\Gamma$	$(A[\sigma])^{\mathsf{D}}\,\gamma^D\,\alpha$	$:\equiv A^{\mathsf{D}}\,(\sigma^{\mathsf{D}}\,\gamma^D)\,\alpha$
$\mathsf{id} : \mathsf{Sub}\,\Gamma\,\Gamma$	$\mathsf{id}^{\mathsf{D}}\,\gamma^D$	$:\equiv \gamma^D$
$(\sigma : \mathsf{Sub}\,\Theta\,\Delta) \circ (\delta : \mathsf{Sub}\,\Gamma\,\Theta) : \mathsf{Sub}\,\Gamma\,\Delta$	$(\sigma \circ \delta)^{\mathsf{D}}\,\gamma^D$	$:\equiv \sigma^{\mathsf{D}}\,(\delta^{\mathsf{D}}\,\gamma^D)$
$\epsilon : \mathsf{Sub}\,\Gamma\,\cdot$	$\epsilon^{\mathsf{D}}\,\gamma^D$	$:\equiv \mathsf{tt}$
$(\sigma : \mathsf{Sub}\,\Gamma\,\Delta), (t : \mathsf{Tm}\,\Gamma\,(A[\sigma])) : \mathsf{Sub}\,\Gamma\,(\Delta \triangleright A)$	$(\sigma, t)^{\mathsf{D}}\,\gamma^D$	$:\equiv (\sigma^{\mathsf{D}}\,\gamma^D, t^{\mathsf{D}}\,\gamma^D)$
$\pi_1\,(\sigma : \mathsf{Sub}\,\Gamma\,(\Delta \triangleright A)) : \mathsf{Sub}\,\Gamma\,\Delta$	$(\pi_1\,\sigma)^{\mathsf{D}}\,\gamma^D$	$:\equiv \mathsf{proj}_1\,(\sigma^{\mathsf{D}}\,\gamma^D)$
$\pi_2\,(\sigma : \mathsf{Sub}\,\Gamma\,(\Delta \triangleright A)) : \mathsf{Tm}\,\Gamma\,(A[\pi_1\,\sigma])$	$(\pi_2\,\sigma)^{\mathsf{D}}\,\gamma^D$	$:\equiv \mathsf{proj}_2\,(\sigma^{\mathsf{D}}\,\gamma^D)$
$(t : \mathsf{Tm}\,\Delta\,A)[\sigma : \mathsf{Sub}\,\Gamma\,\Delta] : \mathsf{Tm}\,\Gamma\,(A[\sigma])$	$(t[\sigma])^{\mathsf{D}}\,\gamma^D$	$:\equiv t^{\mathsf{D}}\,(\sigma^{\mathsf{D}}\,\gamma^D)$
$[\mathsf{id}] : A[\mathsf{id}] = A$	$[\mathsf{id}]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$[\circ] : A[\sigma \circ \delta] = A[\sigma][\delta]$	$[\circ]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\mathsf{ass} : (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$	$\mathsf{ass}^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\mathsf{idl} : \mathsf{id} \circ \sigma = \sigma$	$\mathsf{idl}^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\mathsf{idr} : \sigma \circ \mathsf{id} = \sigma$	$\mathsf{idr}^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\cdot\eta : \{\sigma : \mathsf{Sub}\,\Gamma\,\cdot\} \to \sigma = \epsilon$	$\cdot\eta^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\triangleright\beta_1 : \pi_1\,(\sigma, t) = \sigma$	$\triangleright\beta_1{}^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\triangleright\beta_2 : \pi_2\,(\sigma, t) = t$	$\triangleright\beta_2{}^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\triangleright\eta : (\pi_1\,\sigma, \pi_2\,\sigma) = \sigma$	$\triangleright\eta^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$,\circ : (\sigma, t) \circ \delta = (\sigma \circ \delta, t[\delta])$	$,\circ^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\mathsf{U} : \mathsf{Ty}\,\Gamma$	$\mathsf{U}^{\mathsf{D}}\,\gamma^D\,T$	$:\equiv T \to \mathsf{Set}$
$\mathsf{El}\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U}) : \mathsf{Ty}\,\Gamma$	$(\mathsf{El}\,a)^{\mathsf{D}}\,\gamma^D\,\alpha$	$:\equiv a^{\mathsf{D}}\,\gamma^D\,\alpha$
$\mathsf{U}[] : \mathsf{U}[\sigma] = \mathsf{U}$	$\mathsf{U}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\mathsf{El}[] : (\mathsf{El}\,a)[\sigma] = \mathsf{El}\,(a[\sigma])$	$\mathsf{El}[]^{\mathsf{A}}$	$:\equiv \mathsf{refl}$
$\Pi\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U})(B : \mathsf{Ty}\,(\Gamma \triangleright \mathsf{El}\,a)) : \mathsf{Ty}\,\Gamma$	$(\Pi\,a\,B)^{\mathsf{D}}\,\gamma^D\,f$	$:\equiv (\alpha^D : a^{\mathsf{D}}\,\gamma^D\,\alpha) \to B^{\mathsf{D}}\,(\gamma^D, \alpha^D)\,(f\,\alpha)$
$\mathsf{app}\,(t : \mathsf{Tm}\,\Gamma\,(\Pi\,a\,B)) : \mathsf{Tm}\,(\Gamma \triangleright \mathsf{El}\,a)\,B$	$(\mathsf{app}\,t)^{\mathsf{D}}\,(\gamma^D, \alpha^D)$	$:\equiv t^{\mathsf{D}}\,\gamma^D\,\alpha^D$
$\Pi[] : (\Pi\,a\,B)[\sigma] = \Pi\,(a[\sigma])\,(B[\sigma^{\uparrow}])$	$\Pi[]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\mathsf{app}[] : (\mathsf{app}\,t)[\sigma\uparrow] = \mathsf{app}\,(t[\sigma])$	$\mathsf{app}[]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\mathsf{Id}\,(a : \mathsf{Tm}\,\Gamma\,\mathsf{U})\,(t\,u : \mathsf{Tm}\,\Gamma\,(\mathsf{El}\,a)) : \mathsf{Ty}\,\Gamma$	$(\mathsf{Id}\,a\,t\,u)^{\mathsf{D}}\,\gamma^D\,e$	$:\equiv \mathsf{tr}_{(a^{\mathsf{D}}\,\gamma^D)}\,e\,(t^{\mathsf{D}}\,\gamma^D) = u^{\mathsf{D}}\,\gamma^D$
$\mathsf{reflect}\,(e : \mathsf{Tm}\,\Gamma\,(\mathsf{Id}\,a\,t\,u)) : t = u$	$(\mathsf{reflect}\,e)^{\mathsf{D}}$	$: t^{\mathsf{D}}\,\gamma^D \overset{e^{\mathsf{D}}\,\gamma^D}{=} u^{\mathsf{D}}\,\gamma^D$
$\mathsf{Id}[] : (\mathsf{Id}\,a\,t\,u)[\sigma] = \mathsf{Id}\,(a[\sigma])\,(t[\sigma])\,(u[\sigma])$	$\mathsf{Id}[]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\hat{\Pi}\,(T : \mathsf{Set})\,(B : T \to \mathsf{Ty}\,\Gamma) : \mathsf{Ty}\,\Gamma$	$(\hat{\Pi}\,T\,B)^{\mathsf{D}}\,\gamma^D\,f$	$:\equiv (\alpha : T) \to (B\,\alpha)^{\mathsf{D}}\,\gamma^D\,(f\,\alpha)$
$(t : \mathsf{Tm}\,\Gamma\,(\hat{\Pi}\,T\,B))\,\hat{@}(\alpha : T) : \mathsf{Tm}\,\Gamma\,(B\,\alpha)$	$(t\,\hat{@}\,\alpha)^{\mathsf{D}}\,\gamma^D$	$:\equiv t^{\mathsf{D}}\,\gamma^D\,\alpha$
$\hat{\Pi}[] : (\hat{\Pi}\,T\,B)[\sigma] = \hat{\Pi}\,T\,(\lambda\alpha.(B\,\alpha)[\sigma])$	$\hat{\Pi}[]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$
$\hat{@}[] : (t\,\hat{@}\,\alpha)[\sigma] = (t[\sigma])\,\hat{@}\,\alpha$	$\hat{@}[]^{\mathsf{D}}$	$:\equiv \mathsf{refl}$

$$(\_)^{\mathsf{D}} : (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set}$$

Displayed nat-algebras are “induction data”

$$\begin{aligned}\mathfrak{N}\text{-Alg} :=& (N \colon \mathsf{Set}) \\ &\times (z \colon N) \\ &\times (s \colon N \to N)\end{aligned}$$

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\end{aligned}
$$

$\mathfrak{N}\text{-DAlg}\ (N, z, s)$

Displayed nat-algebras are “induction data”

$$\begin{aligned}\mathfrak{N}\text{-}\mathsf{Alg} :=& (N : \mathsf{Set}) \\ & \times (z : N) \\ & \times (s : N \to N)\end{aligned}$$

$$\mathfrak{N}\text{-}\mathsf{DAlg}\ (N, z, s) := (N^{\mathsf{D}} : N \to \mathsf{Set})$$

Displayed nat-algebras are “induction data”

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Question: What kind of thing is the output of induction?

Displayed nat-algebras are "induction data"

$$
\begin{aligned}
\mathfrak{N}\text{-DAlg}\,(N, z, s) :=& (N^{\mathsf{D}} \colon N \to \mathsf{Set}) \\
&\times (z^{\mathsf{D}} \colon N^{\mathsf{D}}\ z) \\
&\times (s^{\mathsf{D}} \colon (n \colon N) \to N^{\mathsf{D}}\ n \to N^{\mathsf{D}}(s\ n))
\end{aligned}
$$

Displayed nat-algebras are "induction data"

$$\begin{aligned}\mathfrak{N}\text{-DAlg}\ (N, z, s) :=& (N^{\mathsf{D}} \colon N \to \mathsf{Set}) \\ &\times (z^{\mathsf{D}} \colon N^{\mathsf{D}}\ z) \\ &\times (s^{\mathsf{D}} \colon (n \colon N) \to N^{\mathsf{D}}\ n \to N^{\mathsf{D}}(s\ n))\end{aligned}$$

$$\mathfrak{N}\text{-Sect}\ (N, z, s)\ (N^{\mathsf{D}}, z^{\mathsf{D}}, s^{\mathsf{D}}) := (N \colon \mathsf{Set})$$

Displayed nat-algebras are “induction data”

$$\begin{aligned}\mathfrak{N}\text{-DAlg}\ (N, z, s) :=& (N^{\mathsf{D}} \colon N \to \mathsf{Set})\\ &\times (z^{\mathsf{D}} \colon N^{\mathsf{D}}\ z)\\ &\times (s^{\mathsf{D}} \colon (n \colon N) \to N^{\mathsf{D}}\ n \to N^{\mathsf{D}}(s\ n))\end{aligned}$$

$$\mathfrak{N}\text{-Sect}\ (N, z, s)\ (N^{\mathsf{D}}, z^{\mathsf{D}}, s^{\mathsf{D}}) := (N \colon \mathsf{Set})$$

Displayed nat-algebras are “induction data”

$$
\begin{aligned}
\mathfrak{N}\text{-DAlg}\,(N, z, s) :=\,& (N^{\mathsf{D}} \colon N \to \mathsf{Set}) \\
& \times (z^{\mathsf{D}} \colon N^{\mathsf{D}}\ z) \\
& \times (s^{\mathsf{D}} \colon (n \colon N) \to N^{\mathsf{D}}\ n \to N^{\mathsf{D}}(s\ n))
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{N}\text{-Sect}\,(N, z, s)\,(N^{\mathsf{D}}, z^{\mathsf{D}}, s^{\mathsf{D}}) :=\,& (N \colon \mathsf{Set}) \\
& \times (z \colon N)
\end{aligned}
$$

Displayed nat-algebras are “induction data”

$$
\begin{aligned}
\mathfrak{N}\text{-DAlg}\;(N, z, s) :=& (N^{\mathsf{D}} \colon N \to \mathsf{Set}) \\
&\times (z^{\mathsf{D}} \colon N^{\mathsf{D}}\; z) \\
&\times (s^{\mathsf{D}} \colon (n \colon N) \to N^{\mathsf{D}}\; n \to N^{\mathsf{D}}(s\; n))
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{N}\text{-Sect}\;(N, z, s)\;(N^{\mathsf{D}}, z^{\mathsf{D}}, s^{\mathsf{D}}) :=& (N \colon \mathsf{Set}) \\
&\times (z \colon N) \\
&\times (s \colon N \to N)
\end{aligned}
$$

Fact We can define the type of **sections** of a given displayed $\mathfrak{G}$-algebra, for all GATs $\mathfrak{G}$.

$$
\begin{aligned}
\_\text{-}\mathsf{Alg} &: (\mathfrak{G} : \mathtt{GAT}) \to \mathsf{Set} \\
\mathsf{Hom} &: (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set} \\
\_\text{-}\mathsf{DAlg} &: (\mathfrak{G} : \mathtt{GAT}) \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set} \\
\_\text{-}\mathsf{Sect} &: (\mathfrak{G} : \mathtt{GAT}) \to (\Gamma : \mathfrak{G}\text{-}\mathsf{Alg}) \to \mathfrak{G}\text{-}\mathsf{DAlg}\ \Gamma \\
&\qquad \to \mathsf{Set}
\end{aligned}
$$

Sections

Γ^{S}	$: (\gamma : \Gamma^{\mathsf{A}}) \to \Gamma^{\mathsf{D}}\,\gamma \to \mathsf{Set}$
A^{S}	$: \Gamma^{\mathsf{S}}\,\gamma\,\gamma^D \to (\alpha : A^{\mathsf{A}}\,\gamma) \to A^{\mathsf{D}}\,\gamma^D\,\alpha \to \mathsf{Set}$
σ^{S}	$: \Gamma^{\mathsf{S}}\,\gamma\,\gamma^D \to \Delta^{\mathsf{S}}\,(\sigma^{\mathsf{A}}\,\gamma)\,(\sigma^{\mathsf{D}}\,\gamma^D)$
t^{S}	$: (\gamma^S : \Gamma^{\mathsf{S}}\,\gamma\,\gamma^D) \to A^{\mathsf{S}}\,\gamma^S\,(t^{\mathsf{A}}\,\gamma)\,(t^{\mathsf{D}}\,\gamma^D)$
$\cdot^{\mathsf{S}}\ \mathsf{tt}\,\mathsf{tt}$	$:\equiv \top$
$(\Gamma \triangleright A)^{\mathsf{S}}\,(\gamma, \alpha)\,(\gamma^D, \alpha^D)$	$:\equiv (\gamma^S : \Gamma^{\mathsf{S}}\,\gamma\,\gamma^D) \times A^{\mathsf{S}}\,\gamma^S\,\alpha\,\alpha^D$
$(A[\sigma])^{\mathsf{S}}\,\gamma^S\,\alpha\,\alpha^D$	$:\equiv A^{\mathsf{S}}\,(\sigma^{\mathsf{S}}\,\gamma^S)\,\alpha\,\alpha^D$
$\mathsf{id}^{\mathsf{S}}\,\gamma^S$	$:\equiv \gamma^S$
$(\sigma \circ \delta)^{\mathsf{S}}\,\gamma^S$	$:\equiv \sigma^{\mathsf{S}}\,(\delta^{\mathsf{S}}\,\gamma^S)$
$\epsilon^{\mathsf{S}}\,\gamma^S$	$:\equiv \mathsf{tt}$
$(\sigma, t)^{\mathsf{S}}\,\gamma^S$	$:\equiv (\sigma^{\mathsf{S}}\,\gamma^S, t^{\mathsf{S}}\,\gamma^S)$
$(\pi_1\,\sigma)^{\mathsf{S}}\,\gamma^S$	$:\equiv \mathsf{proj}_1\,(\sigma^{\mathsf{S}}\,\gamma^S)$
$(\pi_2\,\sigma)^{\mathsf{S}}\,\gamma^S$	$:\equiv \mathsf{proj}_2\,(\sigma^{\mathsf{S}}\,\gamma^S)$
$(t[\sigma])^{\mathsf{S}}\,\gamma^S$	$:\equiv t^{\mathsf{S}}\,(\sigma^{\mathsf{S}}\,\gamma^S)$
$[\mathsf{id}]^{\mathsf{S}}$	$:\equiv \mathsf{refl}$
$[\circ]^{\mathsf{S}}$	$:\equiv \mathsf{refl}$
$\mathsf{ass}^{\mathsf{S}}$	$:\equiv \mathsf{refl}$
$\triangleright\beta_1$	$:\equiv \mathsf{refl}$
$\triangleright\beta_2{}^{\mathsf{S}}$	$:\equiv \mathsf{refl}$
$\triangleright\eta^{\mathsf{S}}$	$:\equiv \mathsf{refl}$
$,\circ^{\mathsf{S}}$	$:\equiv \mathsf{refl}$
$\mathsf{U}^{\mathsf{S}}\,\gamma^S\,T\,T^D$	$:\equiv (\alpha : T) \to T^D\,\alpha$

$(\mathsf{El}\,a)^{\mathsf{S}}\,\gamma^S\,\alpha\,\alpha^D$	$:\equiv a^{\mathsf{S}}\,\gamma^S\,\alpha = \alpha^D$
$\mathsf{U}[]^{\mathsf{S}}$	$:\equiv \mathsf{refl}$
$\mathsf{El}[]^{\mathsf{S}}$	$:\equiv \mathsf{refl}$
$(\Pi\,a\,B)^{\mathsf{S}}\,\gamma^S\,f\,f^D$	$:\equiv (\alpha : a^{\mathsf{A}}\,\gamma) \to B^{\mathsf{S}}\,(\gamma^S, \mathsf{refl}_{a^{\mathsf{S}}\,\gamma^S\,\alpha})\,(f\,\alpha)\,(f^D\,(a^{\mathsf{S}}\,\gamma^S\,\alpha))$
$(\mathsf{app}\,t)^{\mathsf{S}}\,(\gamma^S, \alpha^S)$	$:\equiv \mathsf{J}_{x.z.B^{\mathsf{S}}\,(\gamma^S, z)\,(t^{\mathsf{A}}\,\gamma\,\alpha)\,(t^{\mathsf{D}}\,\gamma^D\,x)}\,(t^{\mathsf{S}}\,\gamma^S\,\alpha)\,\alpha^S$
$\Pi[]^{\mathsf{S}}$	$:\equiv \mathsf{refl}$
$\mathsf{app}[]^{\mathsf{S}}$	$:\equiv \mathsf{refl}$
$(\mathsf{Id}\,a\,t\,u)^{\mathsf{S}}\,\gamma^S\,e\,e^D$	$:\equiv \top$
$(\mathsf{reflect}\,e)^{\mathsf{S}}$	$:\equiv \mathsf{UIP}$
$\mathsf{Id}[]^{\mathsf{S}}$	$:\equiv \mathsf{refl}$
$(\hat{\Pi}\,T\,B)^{\mathsf{S}}\,\gamma^S\,f\,f^D$	$:\equiv (\alpha : T) \to (B\,\alpha)^{\mathsf{S}}\,\gamma^S\,(f\,\alpha)\,(f^D\,\alpha)$
$(t\,\hat{@}\,\alpha)^{\mathsf{S}}\,\gamma^S$	$:\equiv t^{\mathsf{S}}\,\gamma^S\,\alpha$
$\hat{\Pi}[]^{\mathsf{S}}$	$:\equiv \mathsf{refl}$
$\hat{@}[]^{\mathsf{S}}$	$:\equiv \mathsf{refl}$

$$(\_)^S : (\mathfrak{G} : \mathtt{GAT}) \to (\Gamma : \mathfrak{G}\text{-Alg}) \to \mathfrak{G}\text{-DAlg} \to \mathsf{Set}$$

Sections

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- Principle of induction (e.g. $\mathbb{N}$)

Sections

Fact We can define the type of **sections** of a given displayed $\mathfrak{G}$-algebra, for all GATs $\mathfrak{G}$.

Theorem Every displayed algebra over the initial $\mathfrak{G}$-algebra admits a section

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- Syntax model is contextual

Sections

Fact We can define the type of **sections** of a given displayed $\mathfrak{G}$-algebra, for all GATs $\mathfrak{G}$.

Theorem Every displayed algebra over the initial $\mathfrak{G}$-algebra admits a section

- Principle of induction (e.g. $\mathbb{N}$)
- Syntax model is contextual
- Unary parametricity

$\mathfrak{G}$-Alg : Set

$$\mathfrak{G}\text{-}\mathsf{Alg} : \mathsf{Set}$$
$$\mathsf{Hom}_{\mathfrak{G}} : \mathfrak{G}\text{-}\mathsf{Alg} \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set}$$

$$
\begin{aligned}
\mathfrak{G}\text{-}\mathsf{Alg} &: \mathsf{Set} \\
\mathsf{Hom}_{\mathfrak{G}} &: \mathfrak{G}\text{-}\mathsf{Alg} \to \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set} \\
\mathfrak{G}\text{-}\mathsf{DAlg} &: \mathfrak{G}\text{-}\mathsf{Alg} \to \mathsf{Set}
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{G}\text{-Alg} &: \mathsf{Set} \\
\mathsf{Hom}_{\mathfrak{G}} &: \mathfrak{G}\text{-Alg} \to \mathfrak{G}\text{-Alg} \to \mathsf{Set} \\
\mathfrak{G}\text{-DAlg} &: \mathfrak{G}\text{-Alg} \to \mathsf{Set} \\
\mathfrak{G}\text{-Sect} &: (\Gamma : \mathfrak{G}\text{-Alg}) \to \mathfrak{G}\text{-DAlg}\ \Gamma \\
&\quad \to \mathsf{Set}
\end{aligned}
$$

Observation: Every GAT gives rise to a CwF of algebras and displayed algebras

Central Dogma of Category Theory

Every notion of "structure" comes equipped with a notion of "structure-preserving morphism", i.e. forms a category

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Central Dogma of Generalized Algebra

Every notion of “structure” comes equipped with a notion of “structure-preserving morphism”, “displayed structure”, and “section”, i.e. forms a category with families

	GAT signature	Category of Algs.	Initial Algebra
Categories	$\mathfrak{Cat}$		
CwFs	$\mathfrak{CwF}$		
GAT signatures	$\mathfrak{CwF}^{++}$	$\cdots$	ONEGAT

	GAT signature	Category of Algs.	Initial Algebra	Concrete CwF
Categories	$\mathfrak{Cat}$			
CwFs	$\mathfrak{CwF}$			
GAT signatures	$\mathfrak{CwF}^{++}$	$\cdots$	oneGAT	

Is the **setoid model**[1] the same thing as the concrete CwF of setoids?

Is the **groupoid model** the same thing as the concrete CwF of groupoids?

[1]We mean the setoid model à la Altenkirch [Alt99], which uses **equivalence relations**, not the setoid model of Hofmann [Hof95a, Hof95b], which uses **partial equivalence relations**.

No

From [ABK+21, 7]

$$\frac{A : \mathrm{Ty}\ \Gamma}{\begin{aligned}
|A| &: |\Gamma| \to \mathbf{Type} \\
A^{\sim} &: \{\gamma_0\ \gamma_1 : |\Gamma|\} \to \Gamma^{\sim}\ \gamma_0\ \gamma_1 \to |A|\gamma_0 \to |A|\gamma_1 \to \mathbf{Prop} \\
\textsf{refl*} &: \{\gamma : |\Gamma|\}(a : |A|\gamma) \to A^{\sim}\ (\textsf{refl}\ \Gamma\ \gamma)\ a\ a \\
\textsf{sym*} &: \forall\{\gamma_0\, \gamma_1\, a_0\, a_1\}\{p : \Gamma^{\sim}\ \gamma_0\ \gamma_1\} \to A^{\sim}\ p\ a_0\ a_1 \to A^{\sim}\ (\textsf{sym}\ \Gamma\ p)\ a_1\ a_0 \\
\textsf{trans*} &: A^{\sim}\ p_0\ a_0\ a_1 \to A^{\sim}\ p_1\ a_1\ a_2 \to A^{\sim}\ (\textsf{trans}\, \Gamma\, p_0\, p_1)\ a_0\ a_2 \\
\textsf{coe} &: \Gamma^{\sim}\ \gamma_0\ \gamma_1 \to |A|\gamma_0 \to |A|\gamma_1 \\
\textsf{coh} &: (p : \Gamma^{\sim}\ \gamma_0\ \gamma_1)(a : |A|\gamma_0) \to A^{\sim}\, p\, a\, (\textsf{coe}\, A\, p\, a)
\end{aligned}}$$

Fibrancy data for setoid model

From [ABK+21, 7]

Displayed Setoid

$$A : \mathrm{Ty}\ \Gamma$$

$$\begin{aligned}
|A| &: |\Gamma| \to \mathbf{Type} \\
A^{\sim} &: \{\gamma_0\ \gamma_1 : |\Gamma|\} \to \Gamma^{\sim}\ \gamma_0\ \gamma_1 \to |A|\gamma_0 \to |A|\gamma_1 \to \mathbf{Prop} \\
\mathsf{refl}^* &: \{\gamma : |\Gamma|\}(a : |A|\gamma) \to A^{\sim}\ (\mathsf{refl}\ \Gamma\ \gamma)\ a\ a \\
\mathsf{sym}^* &: \forall\{\gamma_0\,\gamma_1\,a_0\,a_1\}\{p : \Gamma^{\sim}\ \gamma_0\ \gamma_1\} \to A^{\sim}\ p\ a_0\ a_1 \to A^{\sim}\ (\mathsf{sym}\ \Gamma\ p)\ a_1\ a_0 \\
\mathsf{trans}^* &: A^{\sim}\ p_0\ a_0\ a_1 \to A^{\sim}\ p_1\ a_1\ a_2 \to A^{\sim}\ (\mathsf{trans}\,\Gamma\,p_0\,p_1)\ a_0\ a_2 \\
\mathsf{coe} &: \Gamma^{\sim}\ \gamma_0\ \gamma_1 \to |A|\gamma_0 \to |A|\gamma_1 \\
\mathsf{coh} &: (p : \Gamma^{\sim}\ \gamma_0\ \gamma_1)(a : |A|\gamma_0) \to A^{\sim}\,p\,a\,(\mathsf{coe}\,A\,p\,a)
\end{aligned}$$

From [ABK+21, 7]

Displayed Setoid

$$A : \mathrm{Ty}\ \Gamma$$

$$\boxed{\begin{aligned}
|A| &: |\Gamma| \to \mathbf{Type}\\
A^{\sim} &: \{\gamma_0\ \gamma_1 : |\Gamma|\} \to \Gamma^{\sim}\ \gamma_0\ \gamma_1 \to |A|\gamma_0 \to |A|\gamma_1 \to \mathbf{Prop}\\
\mathsf{refl}^* &: \{\gamma : |\Gamma|\}(a : |A|\gamma) \to A^{\sim}\ (\mathsf{refl}\ \Gamma\ \gamma)\ a\ a\\
\mathsf{sym}^* &: \forall\{\gamma_0\,\gamma_1\,a_0\,a_1\}\{p : \Gamma^{\sim}\ \gamma_0\ \gamma_1\} \to A^{\sim}\ p\ a_0\ a_1 \to A^{\sim}\ (\mathsf{sym}\ \Gamma\ p)\ a_1\ a_0\\
\mathsf{trans}^* &: A^{\sim}\ p_0\ a_0\ a_1 \to A^{\sim}\ p_1\ a_1\ a_2 \to A^{\sim}\ (\mathsf{trans}\,\Gamma\,p_0\,p_1)\ a_0\ a_2
\end{aligned}}$$

$$\left.\begin{aligned}
\mathsf{coe} &: \Gamma^{\sim}\ \gamma_0\ \gamma_1 \to |A|\gamma_0 \to |A|\gamma_1\\
\mathsf{coh} &: (p : \Gamma^{\sim}\ \gamma_0\ \gamma_1)(a : |A|\gamma_0) \to A^{\sim}\,p\,a\,(\mathsf{coe}\,A\,p\,a)
\end{aligned}\right\}$$

Fibrancy

Idea: Carve out sub-CwFs of concrete CwFs whose types are *fibrant*

	GAT signature	Category of Algs.	Initial Algebra	Concrete CwF
Categories	$\mathfrak{Cat}$			
CwFs	$\mathfrak{CwF}$			
GAT signatures	$\mathfrak{CwF}^{++}$	$\cdots$	ONEGAT	

	GAT signature	Category of Algs.	Initial Algebra	Concrete CwF	Fibrant Model
Categories	$\mathfrak{Cat}$				
CwFs	$\mathfrak{CwF}$				
GAT signatures	$\mathfrak{CwF}^{++}$ $\cdots$ ONEGAT				

	GAT signature	Category of Algs.	Initial Algebra	Concrete CwF	Fibrant Model
Categories	$\mathfrak{Cat}$				
CwFs	$\mathfrak{CwF}$				
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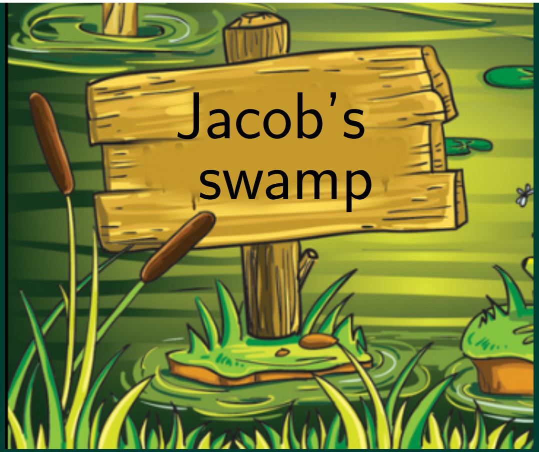
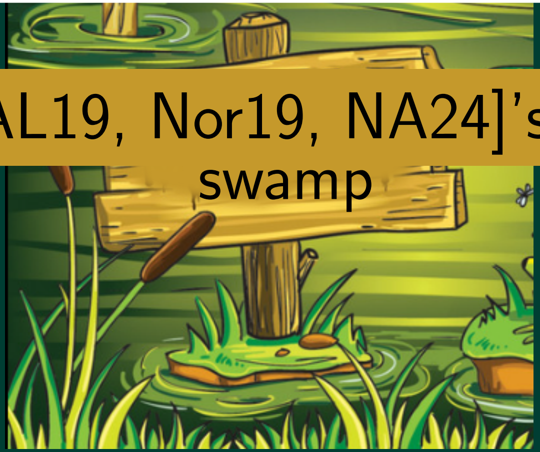


	GAT signature	Category of Algs.	Initial Algebra	Concrete CwF	Fibrant Model
Categories	$\mathfrak{Cat}$				
CwFs	$\mathfrak{CwF}$				
GAT signatures	$\mathfrak{CwF}^{++}$ $\cdots$ ONEGAT				



Autosynthesis

- The groupoid model provides a synthetic theory of groupoids

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- The groupoid model provides a synthetic theory of groupoids
- The setoid model provides a synthetic theory of setoids

Autosynthesis

- The groupoid model provides a synthetic theory of groupoids
- The setoid model provides a synthetic theory of setoids
- The preorder model provides a synthetic theory of preorders?
- The category model provides a synthetic theory of categories?

	GAT signature	Category of Algs.	Initial Algebra	Concrete CwF	Fibrant Model
Categories	$\mathfrak{Cat}$				
CwFs	$\mathfrak{CwF}$				
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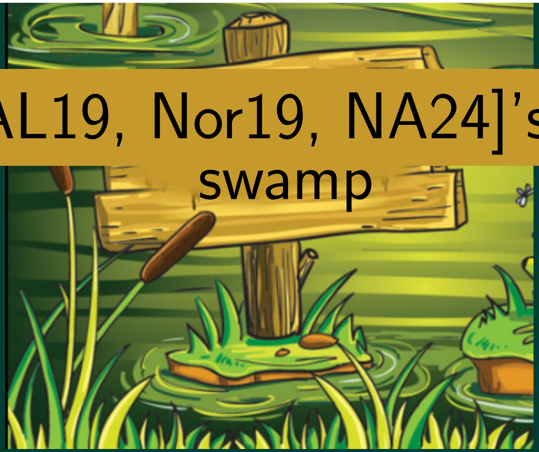
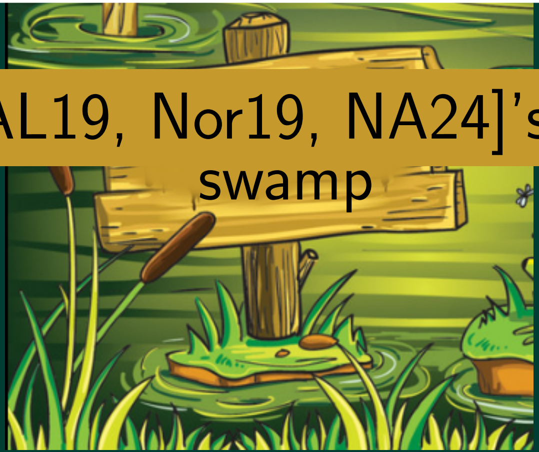


	GAT signature	Category of Algs.	Initial Algebra	Concrete CwF	Fibrant Model	Autosynthesis
Categories	$\mathfrak{Cat}$					
CwFs	$\mathfrak{CwF}$					
GAT signatures	$\mathfrak{CwF}^{++}$ $\cdots$ ONEGAT					



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Autosynthesis

- The groupoid model provides a synthetic theory of groupoids
- The setoid model provides a synthetic theory of setoids
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  - Yes: [Neu25]

	GAT signature	Category of Algs.	Initial Algebra	Concrete CwF	Fibrant Model	Autosynthesis
Categories	$\mathfrak{Cat}$					
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	GAT signature	Category of Algs.	Initial Algebra	Concrete CwF	Fibrant Model	Autosynthesis
Categories	$\mathfrak{Cat}$					
CwFs	$\mathfrak{CwF}$					
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here be dragons

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The End

(thank you!)