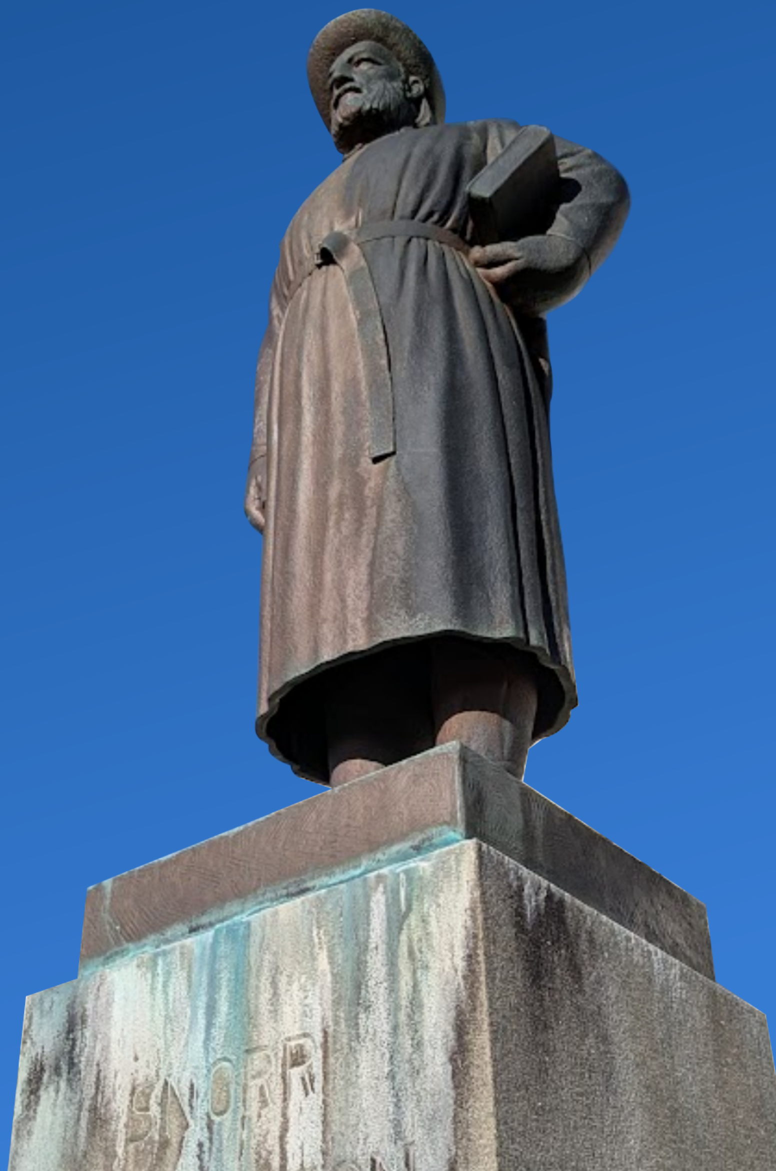
Polarized and Directed Type Theory

Jacob Neumann
25 April 2025
Reykholt, Iceland



0 Background

Standard Martin-Löf Type Theory includes **identity types**:

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Basic results

Symmetry

$$\frac{\Gamma \vdash p \colon \mathsf{Id}(t, t')}{\Gamma \vdash p^{-1} := J\ \mathsf{refl}_t\ t'\ p \colon \mathsf{Id}(t', t)}$$

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- Assuming the **uniqueness of identity proofs** (if $p\ p' \colon \mathsf{Id}(t, t')$, there's some $\mathsf{UIP}(p, p') \colon \mathsf{Id}(p, p')$), this makes A into an **equivalence relation**.

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- Assuming UIP for the *identity types of* A, this makes A into a **synthetic groupoid** (terms of A are objects, Id-terms are morphisms, refl_t is the identity morphism at t)

Inverses

$$\frac{\Gamma \vdash p \colon \mathsf{Id}(t, t')}{\Gamma \vdash p^{-1} := J\ \mathsf{refl}_t\ t'\ p \colon \mathsf{Id}(t', t)}$$

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Question: Can we have a synthetic *category* theory?

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Key Idea: Need *semantics* to prove that symmetry is unprovable

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- Assuming the **uniqueness of identity proofs** (if $p\ p' : \mathsf{Id}(t, t')$, there's some $\mathsf{UIP}(p, p') : \mathsf{Id}(p, p')$)—**which can't be proved in general from the rules of MLTT**—, this makes A into an **equivalence relation**.

The groupoid interpretation of type theory

Hofmann and Streicher prove [HS95] that UIP is *independent from the rules of Martin-Löf Type Theory*.

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- A term t (in context Γ) of type A is a section sending objects $\gamma : |\Gamma|$ to objects $t\ \gamma : |A(\gamma)|$ and morphisms $\gamma_{01} : \Gamma\,[\gamma_0, \gamma_1]$ to morphisms in $(A\ \gamma_1)\,[A\ \gamma_{01}\ (t\ \gamma_0), t\ \gamma_1]$.

$$
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\mathsf{Id}(t, t') \quad &: \Gamma \to \mathbf{Grpd} \\
\mathsf{Id}(t, t')\ \gamma \quad &= (A\ \gamma)\ [t\ \gamma, t'\ \gamma] \\
\mathsf{Id}(t, t')\ \gamma_{01}\ x_0 &= (t'\ \gamma_{01}) \circ A\ \gamma_{01}\ x_0 \circ (t\ \gamma_{01})^{-1}
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This type has multiple terms in general, hence UIP cannot be proved from the rules.

Idea: Replace groupoids with categories

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The **category model** of type theory is an interpretation of **directedTT** as a mathematical structure:

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- A type in context Γ is a Γ-indexed family of **categories**, i.e. a functor $\Gamma \to$ **Cat**
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1 Polarity Calculus

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  - What we do [NA24, Neu25], following [LH11, Nuy15, Nor19]
- OR: a little of both
  - E.g. virtual double category approaches [NL23, Nas25, Nas24]

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$$\Gamma \xrightarrow{A} \mathbf{Cat} \xrightarrow{(\_)^{\mathrm{op}}} \mathbf{Cat}$$

Negative structure

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$\Gamma^{-} := \Gamma^{\mathrm{op}}$

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- **Negative context extension**

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Polarizing Pi types

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash B \text{ type}}{\Gamma \vdash \Pi(A, B) \text{ type}}$$

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Problem: We can’t do anything with these

- No identity function

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  - $A \to A$ isn't a well-formed type (domain has to be in Γ^-, codomain in $\Gamma ,_- \; x : A$)

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  - $\lambda x^A.x$ isn't a well-formed term (x is a term in $(\Gamma ,_- x : A)^-$, not $\Gamma ,_- x \colon A$)

Polarizing Pi types

$$\frac{\Gamma^{-} \vdash A \text{ type} \quad \Gamma \,,_{-}\, x : A \vdash B \text{ type}}{\Gamma \vdash \Pi(A, B) \text{ type}}$$

$$\frac{\Gamma \,,_{-}\, x : A \vdash e : B}{\Gamma \vdash \lambda x^{A}.e : \Pi(A, B)} \qquad \frac{\Gamma \vdash f : \Pi(A, B) \quad \Gamma^{-} \vdash t : A^{-}}{\Gamma \vdash f(t) : B(t)}$$

We annotate **variances/polarities** by adding operations to MLTT

- **Type negation**

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A^- \text{ type}} \qquad \frac{}{(A^-)^- = A}$$

- **Context negation**

$$\frac{\Gamma \text{ ctx}}{\Gamma^- \text{ ctx}} \qquad \frac{}{(\Gamma^-)^- = \Gamma}$$

- **Negative context extension**

$$\frac{\Gamma^- \vdash A \text{ type}}{\Gamma \,,_- \; x : A \vdash} \qquad \frac{\Gamma^- \vdash A \text{ type}}{(\Gamma \,,_- \; x : A)^- \vdash x : A^-}$$

- No identity function
  - $A \to A$ isn't a well-formed type (domain has to be in Γ^-, codomain in $\Gamma ,_- x\colon A$)
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- Forget about composition...

We instead work in a **neutral context**, i.e. a groupoid (but types are still valued in categories)

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2 Directed Equality

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\mathsf{Id}(t, t') &: \Gamma \to \mathbf{Grpd} \\
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Directed Type Theory includes **hom-types**:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash t' : A}{\Gamma \vdash \mathrm{Hom}_{\mathbf{A}}(t, t') \text{ type}}$$

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We cannot negate x, since $\Gamma, x \colon A$ is not neutral!

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Key Idea: Use model to prove symmetry is independent

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This type has multiple terms in general, hence UIP cannot be proved from the rules.

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Applications

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  - Analogous to interpretation of types in HoTT as homotopy spaces
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- Synthetic rewriting systems
  - Can have intensional and extensional identity types in the system simultaneously, with hom-types sitting between them

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Conclusion

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- Synthetic category theory, directed homotopy theory, concurrency and rewriting

Thank you!