The Category Interpretation of Polarized and Directed Type Theory

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Where to find more detail

- These slides: jacobneu.github.io/research/slides/HoTT-UF-2023.pdf
- In-progress preprint: jacobneu.github.io/research/preprints/polarTT.pdf
- Agda formalization coming soon (link will be added to preprint and slides)

Univalent Mathematics:

Groupoid Theory

versus

Category Theory

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A function $f : A \to B$ is automatically a functor w.r.t. this groupoid structure: using the J-rule, we can construct $\mathrm{ap}_f\ p \colon f(a) =_B f(a')$ for each $p : a =_A a'$ and prove this preserves identities (refl) and composition (path concatenation)

Key observation We don't need to inspect the definition of f to define ap_f or to prove it respects identities and composition – once we have f, we have its functoriality

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If we want to do ∞-category theory...

In summary: In univalent mathematics, groupoids are *synthetic* but categories are *analytic*

Why?

Goal: Design a variant of HoTT capable of synthetic (∞-)category theory

Some Existing Directed TT/Synthetic CT Projects

- Harper and Licata – *2-Dimensional Directed Type Theory* (2011) $\dagger\star$
- Nuyts – *Towards a Directed Homotopy Type Theory based on 4 Kinds of Variance* (2015) $\dagger\star$
- Riehl and Shulman – *A type theory for synthetic ∞-categories* (2017)
- Ahrens, North, and van der Wiede – *Semantics for two-dimensional type theory* (2022) $\star$
- Cisinski, Nguyen, and Walde – *Univalent Directed Type Theory* (2023)

$\dagger$ No model theory

$\star$ Includes a directed version of *judgmental equality*

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  - Proved the independence of the *Uniqueness of Identity Proofs*

- Hofmann and Streicher's *Lifting Grothendieck Universes* (1999, unpublished)
  - Established a technique for modelling universes in *presheaf models* of type theory
- Hofmann's *Semantical analysis of higher-order abstract syntax* (1999)
  - Gave presheaf semantics for a **higher-order abstract syntax**, which abstracts away cumbersome details about substitution and binding

Categories with Families

Defn. A **category with families (CwF)** is a (generalized) algebraic structure, consisting of:

- A category Con of *contexts* and *substitutions*, with a terminal object $\bullet$, the *empty context*
- A presheaf $\mathsf{Ty} \colon \mathsf{Con}^{\mathsf{op}} \to \mathsf{Set}$ of *types*
- A presheaf $\mathsf{Tm} \colon (\int \mathsf{Ty})^{\mathsf{op}} \to \mathsf{Set}$ of *terms*
- An operation of *context extension*:

$$\frac{\Gamma \colon \mathsf{Con} \qquad A \colon \mathsf{Ty}\,\Gamma}{\Gamma \rhd A \colon \mathsf{Con}}$$

satisfying a 'local representability' condition.

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Further structure Can interpret dependent types and identity types in the groupoid model, and find types whose identity types violate UIP

Main Idea: Replace groupoids with categories!

The Category Interpretation of Type Theory

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- For each $A : \mathsf{Ty}\ \Gamma$, there is a type $A^- : \mathsf{Ty}\ \Gamma$

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- A functor $(\_)^{-} : \mathrm{Con} \to \mathrm{Con}$ such that $(\Gamma^{-})^{-} = \Gamma$ and $\bullet^{-} = \bullet$
- For each $\Gamma : \mathrm{Con}$, a function $(\_)^{-} : \mathrm{Ty}\,\Gamma \to \mathrm{Ty}\,\Gamma$ such that $(A^{-})^{-} = A$
- Two operations of *context extension*: for s either $+$ or $-$,

$$\frac{\Gamma : \mathrm{Con} \qquad A : \mathrm{Ty}\,\Gamma^{s}}{\Gamma \triangleright^{s} A : \mathrm{Con}}$$

The Local Representability Condition

For any Δ, Γ and any A: Ty Γ,

$$\mathsf{Con}(\Delta, \Gamma \triangleright^s A) \quad \cong \quad \sum_{\gamma : \, \mathsf{Con}(\Delta,\Gamma)} \mathsf{Tm}(\Delta^s, A[\gamma^s]^s)$$

natural in Δ.

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Notice This is the essential ingredient in making our types into **synthetic categories**.

Further structure

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$$\frac{A : \mathrm{Ty}\,\Gamma}{A^0 : \mathrm{Ty}\,\Gamma} \qquad \frac{a : \mathrm{Tm}(\Gamma, A^0)}{+a : \mathrm{Tm}(\Gamma, A) \qquad -a : \mathrm{Tm}(\Gamma, A^-)}$$

Refl and J

Core types allow us to state the **introduction rule** for hom types:

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as well as the appropriate J-rules:

TODO

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- Core types and the $+$ and $-$ operations on terms
- The $\_ \Rightarrow \_$ type former with refl constructor and J eliminators

Thank you!

Appendix: Proof of concept: Composition

TODO

Appendix: Arrow Types

TODO

Appendix: Map and Functoriality

TODO