Presheaf Models of Polarized Higher-Order Abstract Syntax

Thorsten Altenkirch and Jacob Neumann

University of Nottingham

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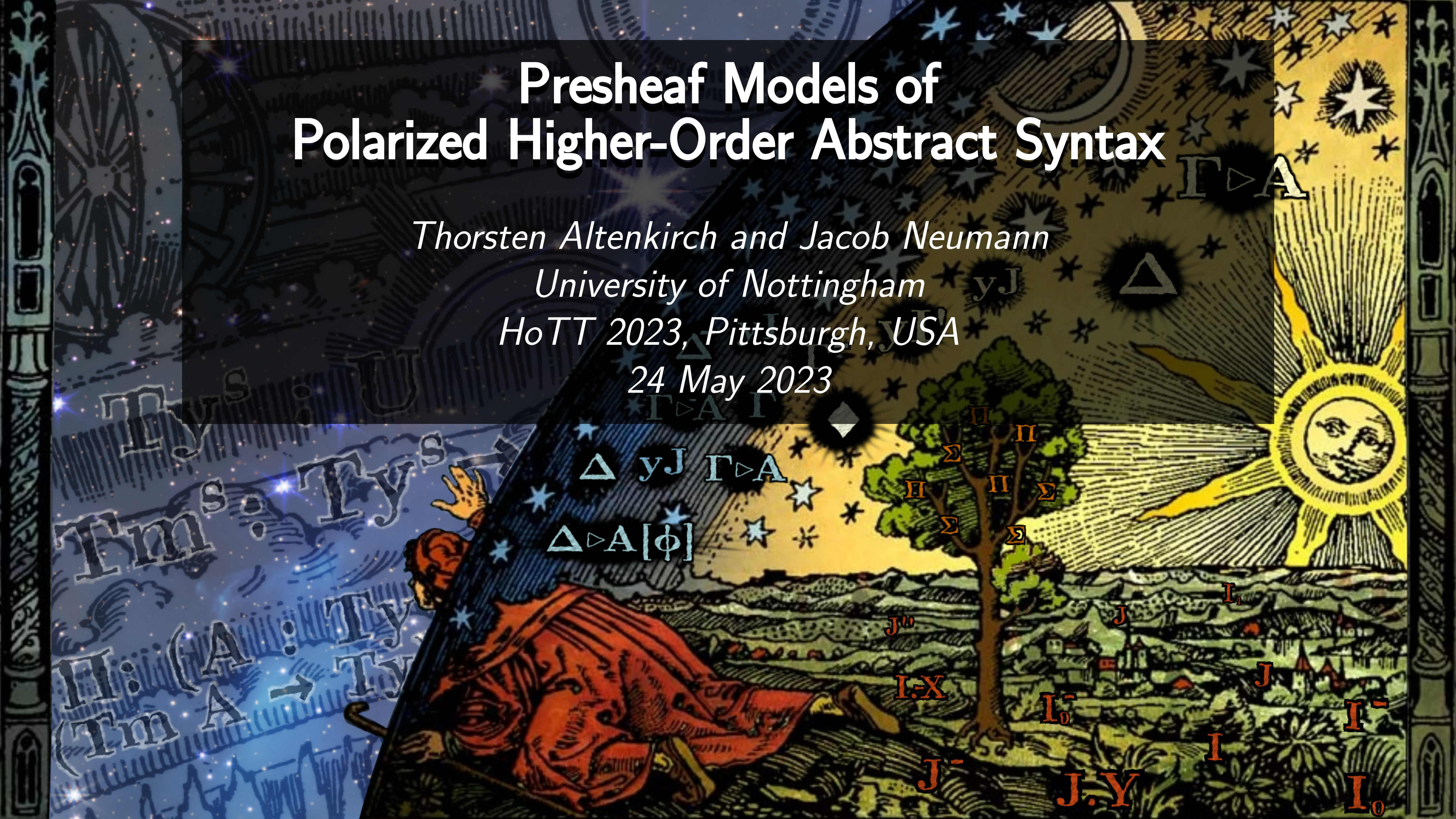

Where to find more detail

jacobneu.github.io/research/directedTT/landing.html

What I’m interested in:

What I’m interested in:

Directed TT

What I'm interested in:

Directed TT

$\times$

Higher Observational TT

What I'm interested in:

Directed TT

$\times$

Higher Observational TT?

Key Component

Key Component :

HOAS with polaritys

0 Polarized Type Theory

Three Important Models of Type Theory

Set	Setoid	Grpd
The Set Model	*The Setoid Model*	*The Groupoid Model*

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The Set Model

- Contexts are **sets**

Setoid
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Grpd
The Groupoid Model

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[Dyb95, Hof97]

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[Hof94, Alt99]

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$$\begin{tikzcd}
\mathsf{Set} \arrow[d, dash] & \\
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\end{tikzcd}$$

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Univalence/Antisymm $\rightarrow$

Dimension $\rightarrow$

$$
\begin{tikzcd}
\mathsf{Set} \arrow[r, dash] \arrow[d, dash] & \text{U-Grpd} \arrow[d, dash] \\
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\end{tikzcd}
$$

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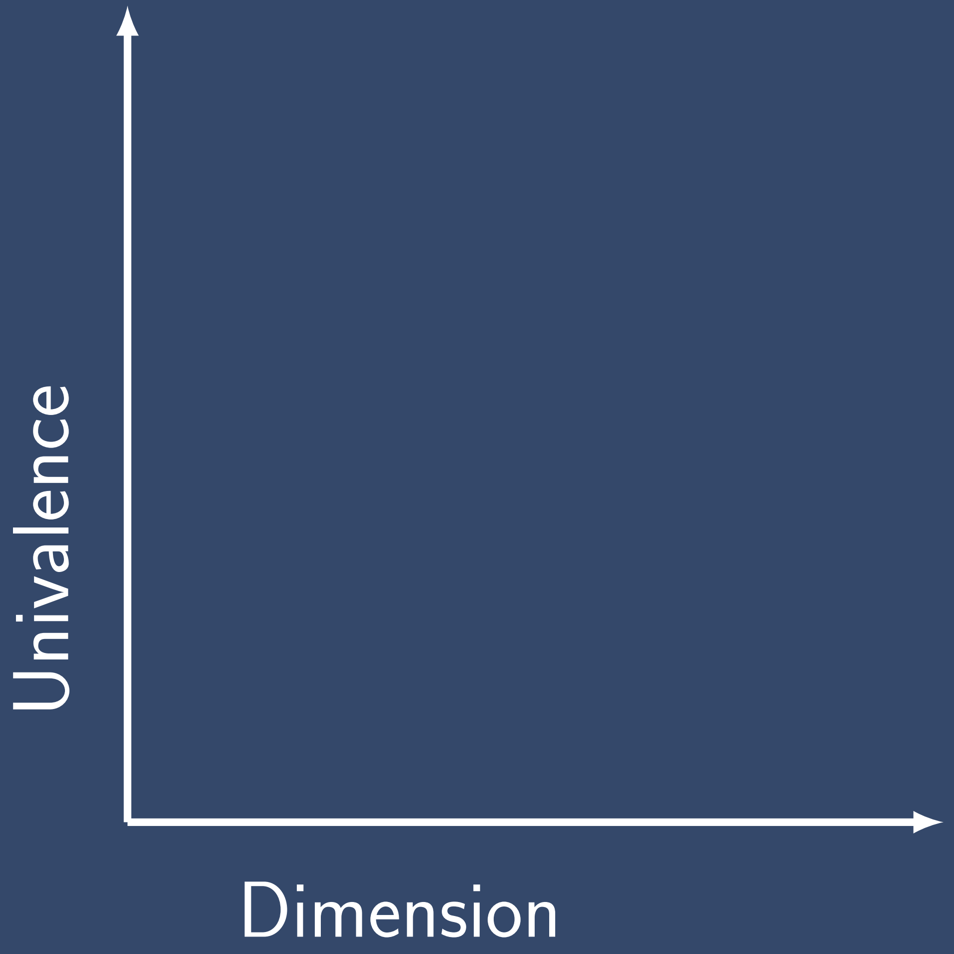
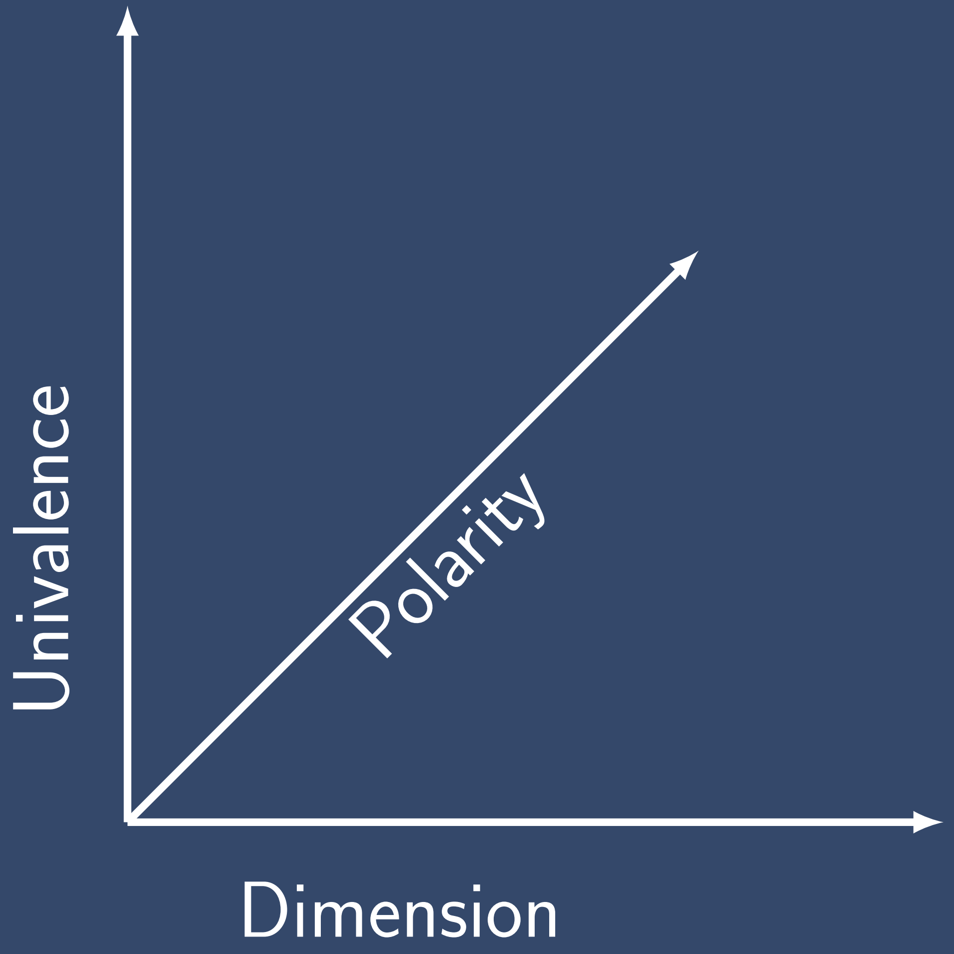
Dimension $\rightarrow$

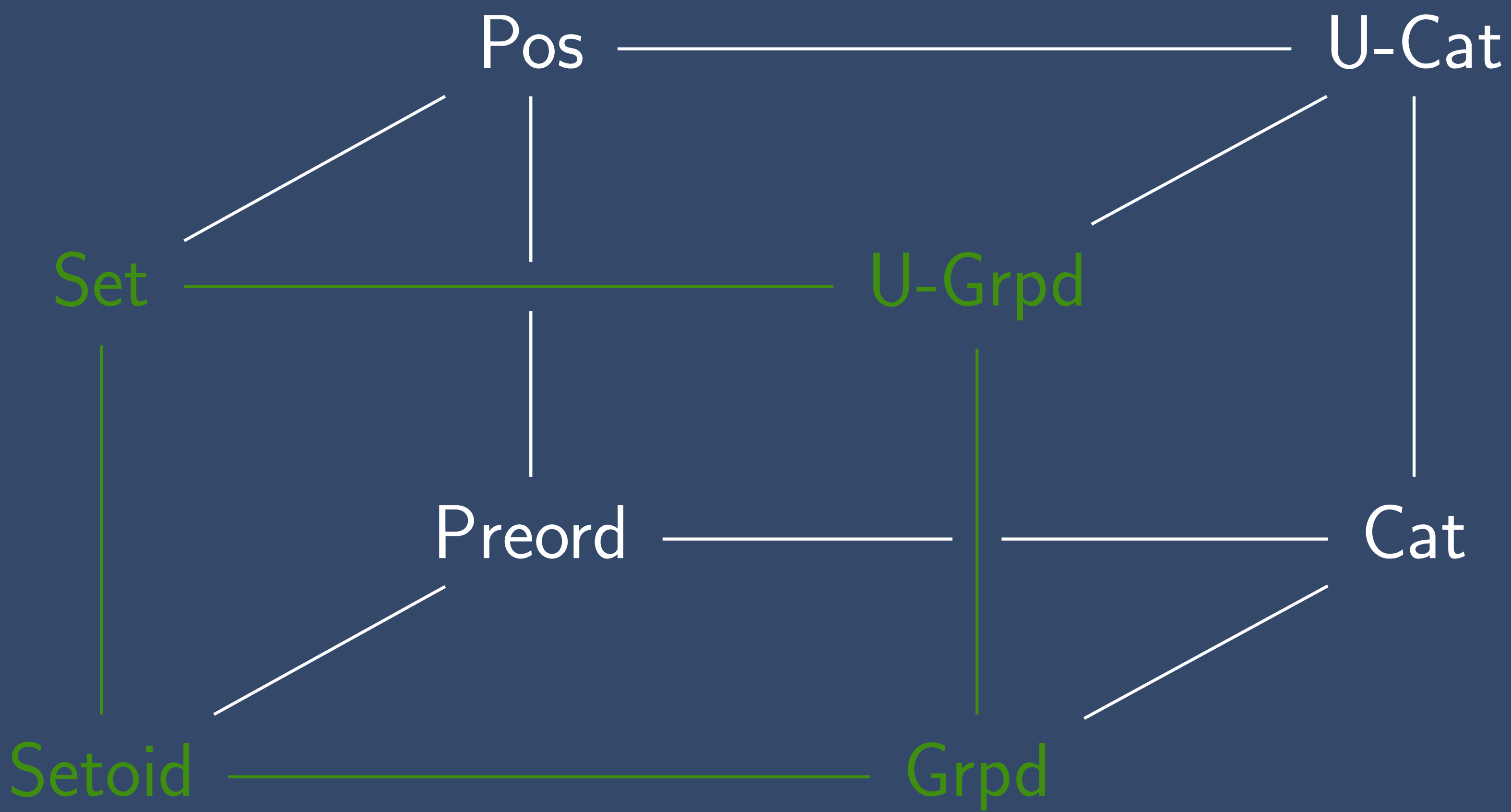

Univalence
Dimension


Univalence
Polarity
Dimension

$$
\begin{tikzcd}
 & \mathsf{Pos} \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & \mathsf{U\text{-}Cat} \arrow[dl, dash] \arrow[dd, dash] \\
\mathsf{Set} \arrow[rr, dash] \arrow[dd, dash] & & \mathsf{U\text{-}Grpd} \arrow[dd, dash] & \\
 & \mathsf{Preord} \arrow[rr, dash] \arrow[dl, dash] & & \mathsf{Cat} \arrow[dl, dash] \\
\mathsf{Setoid} \arrow[rr, dash] & & \mathsf{Grpd} &
\end{tikzcd}
$$

Model theory of the front face:

Categories with Families



CwFs

Defn. A **category with families (CwF)** is a (generalized) algebraic structure, consisting of:

- A category Con of *contexts* and *substitutions*, with a terminal object $\bullet$, the *empty context*
- A presheaf $\mathrm{Ty} : \mathrm{Con}^{\mathrm{op}} \to \mathrm{Set}$ of *types*
- A presheaf $\mathrm{Tm} : (\int \mathrm{Ty})^{\mathrm{op}} \to \mathrm{Set}$ of *terms*
- An operation of *context extension*:

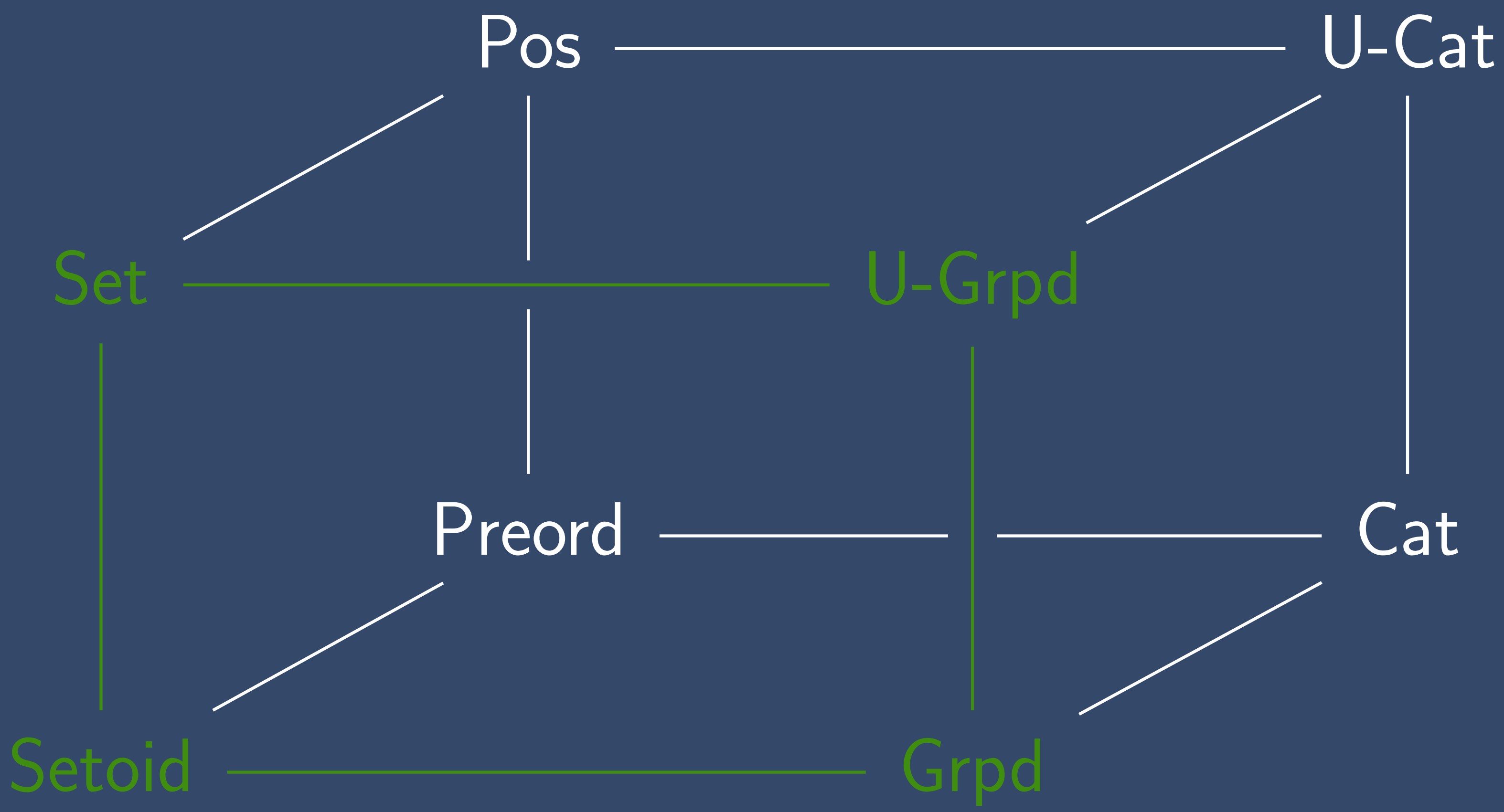
$$\frac{J : \mathrm{Con} \qquad Y : \mathrm{Ty}\ J}{J \triangleright Y : \mathrm{Con}}$$

  so that $J \triangleright Y$ is a 'locally representing object' (in the sense spelled out on the next slide)

For any I, J: Con and any J: Ty Γ,

$$\mathrm{Con}(I, J \triangleright Y) \quad \cong \quad \sum_{j\,:\,\mathrm{Con}(I,J)} \mathrm{Tm}(I, Y[j])$$

natural in Δ.



CwFs

$$\begin{tikzcd}
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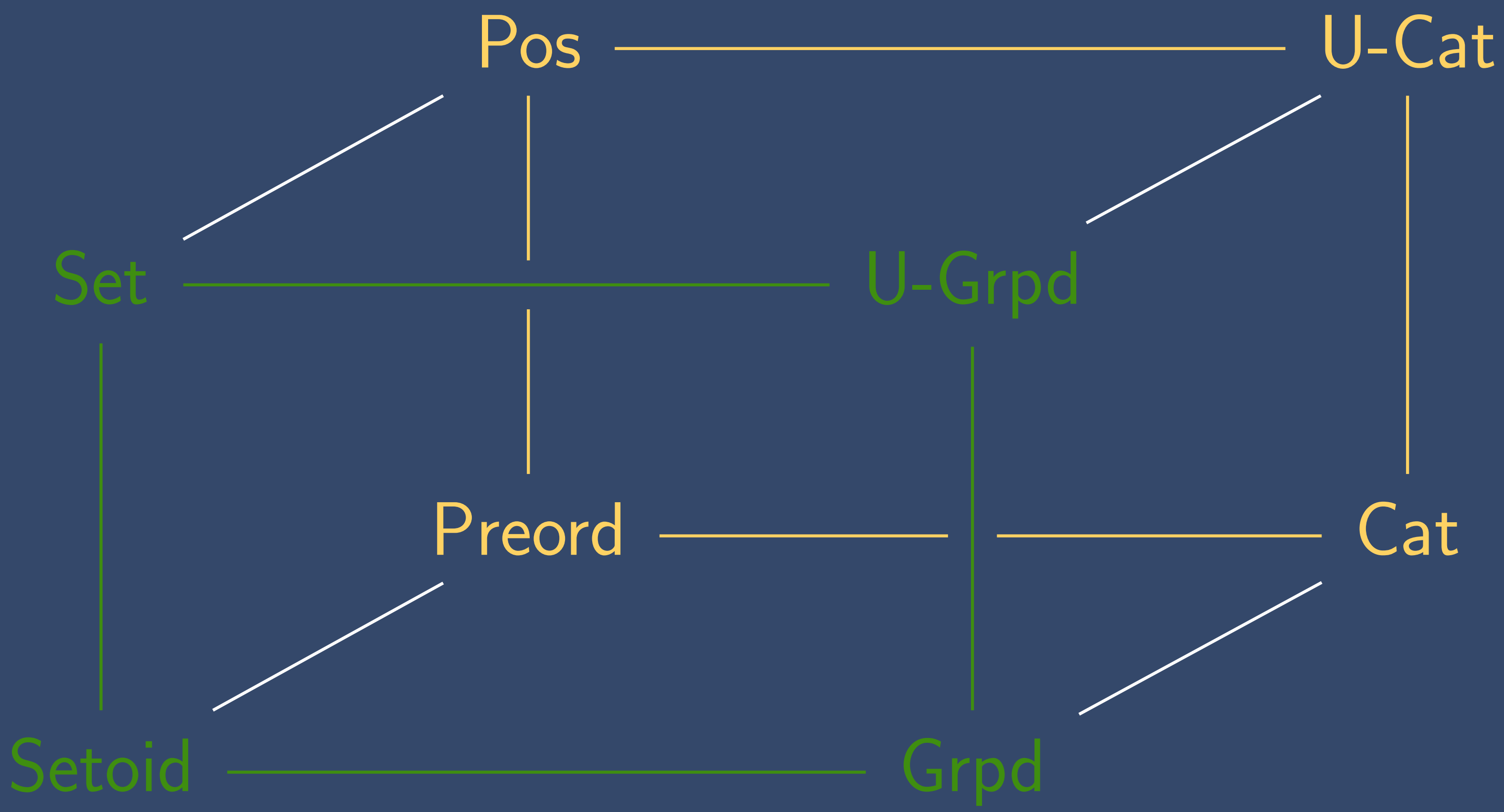
CwFs



CwFs PCwFs

What is a polarized CwF?

The Category Interpretation of Type Theory

The category model of type theory is a CwF where

- Con is the category of categories and functors
- Ty J is the set of J-indexed families of categories (i.e. pseudofunctors $J \to \mathrm{Cat}$)
- ...

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Further structure The category of categories comes equipped with the **opposite category** operation, which we can view as a functor $\mathsf{Cat} \to \mathsf{Cat}$.

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- For each context Γ, there is a context Γ^-
- For each $A : \mathsf{Ty}\ \Gamma$, there is a type $A^- : \mathsf{Ty}\ \Gamma$

1 Presheaf Semantics of HOAS

2 Polarized HOAS

Blah

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Thank you!