# Category Theory (80-413/713) F20 HW9, Exercise 5 Solution

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## Problem:

> Let $I$ be a small category, and $\Delta : \mathsf{Set} \to \mathsf{Set}^I$ be the constant diagram functor. Prove that $\Delta$ is fully faithful iff $\operatorname{colim}_i \Delta 1 = 1$.

## Solution:

**1**

First, note the form of this problem: we have two statements – (a) $\Delta$ is fully faithful, and (b) $\operatorname{colim}_i \Delta 1 = 1$ – which we're proving equivalent. As we'll see going through the proof, (a) appears much stronger than (b), and the proof of (a) $\Longrightarrow$ (b) will be quick. Indeed, we'll see that (a) is a universal statement, and (b) just a single instance of (a). The difficulty of this proof is showing that (b) – which is a "local" property of just one particular set, 1 – is "universal" in the appropriate sense. Specifically, we'll have to show that $\operatorname{colim}_i \Delta 1 = 1$ implies $\epsilon_X : \operatorname{colim}_i \Delta X \cong X$ *for all* $X$.

**2**

Let's dispense with (a) $\Longrightarrow$ (b). Recall from lecture:

> **Fact 1** $U$ is fully faithful iff $\epsilon$ is a natural iso (For any adjunction $F \dashv U$)

Therefore, since $\Delta$ is a fully faithful right adjoint, the counit $\epsilon_X : \operatorname{colim}_i \Delta X \to X$ is an isomorphism (that is, a bijection) for all sets $X$. So, in particular,

$$\epsilon_1 : \operatorname*{colim}_i \Delta 1 \xrightarrow{\sim} 1.$$

Conclude that $\operatorname{colim}_i \Delta 1$ is a singleton set, i.e. a terminal object of $\mathsf{Set}$. As usual, we suppress the distinction between uniquely isomorphic objects, so

$$\operatorname*{colim}_i \Delta 1 = 1.$$

**3**

(b) $\Longrightarrow$ (a) is significantly more involved. Start with the naturality of $\epsilon$: for all functions $f : A \to B$, the square

$$\begin{tikzcd}
\operatorname{colim}_i \Delta A \arrow[r, "\epsilon_A"] \arrow[d, "\operatorname{colim}_i \Delta(f)"'] & A \arrow[d, "f"] \\
\operatorname{colim}_i \Delta B \arrow[r, "\epsilon_B"'] & B
\end{tikzcd}$$

commutes. **Recall 1** that we define $\operatorname{colim}_i \Delta(f)$ by taking the maps

$$\Delta(A)(j) \xrightarrow{(\Delta f)_j} \Delta(B)(j) \xrightarrow{\mathsf{inc}_j^{\Delta B}} \operatorname{colim}_i \Delta B$$

for each object $j$ of $I$, and then combining them together using the **"co-pairing"**

**operation of colimits:** [2]

$$\left[\mathsf{inc}_j^{\Delta B} \circ (\Delta f)_j \;\middle|\; j \in I\right] : \operatorname*{colim}_i \Delta A \to \operatorname*{colim}_i \Delta B.$$

But $\Delta(A)(j)$ is just $A$ by definition of $\Delta$, and likewise for $B$. And the $j$-component of the natural transform $\Delta f : \Delta A \to \Delta B$ is just $f$, for each $j$. So we can state the naturality of $\epsilon$ as

**Nat. $\epsilon$** $\quad f \circ \epsilon_A = \epsilon_B \circ [\mathsf{inc}_j^{\Delta B} \circ f]$ (For all $f : A \to B$)

4

Let's combine this with a **useful fact about coproducts in Set.** [3] Specifically this:

**Fact 2** $\quad \coprod_{x \in X} 1 = X$ (For all sets $X$)

If you spell out the details of this fact, you'll see that it basically involves defining another constant functor $\Delta_X : \mathsf{Set} \to \mathsf{Set}^X$ where $\Delta_X(Y)(x) = Y$ for all $x \in X$ and all sets $Y$, and then proving that $X$ satisfies the universal mapping property of the colimit over the diagram $\Delta_X(1)$. Note that for each $x \in X$ the inclusion map

$$\mathsf{inc}_x^{\Delta_X(1)} : \Delta_X(1)(x) \to X$$

is just the **element** $x : 1 \to X$**.** [4]

So for any set $X$ and any $x \in X$, apply **Nat. $\epsilon$** to $x : 1 \to X$:

$$\begin{tikzcd}
\operatorname{colim}_i \Delta 1 \arrow[r, "\epsilon_1"] \arrow[d, "\operatorname{colim}_i \Delta(x)"'] & 1 \arrow[d, "x"] \\
\operatorname{colim}_i \Delta X \arrow[r, "\epsilon_X"'] & X
\end{tikzcd}$$

Now we'll use the fact that $\operatorname{colim}_i \Delta 1 = 1$ (hence $\epsilon_1$ must be the identity on 1) to collapse this square into a triangle:

$$\begin{tikzcd}
1 \arrow[d, "\operatorname{colim}_i \Delta(x)"'] \arrow[dr, "x"] & \\
\operatorname{colim}_i \Delta X \arrow[r, "\epsilon_X"'] & X
\end{tikzcd}$$

**5**

Now recall we had the triangle above for *every* $x \in X$, hence we have

$$\begin{tikzcd}
1 \arrow[dr, "\operatorname{colim}_i \Delta(x)"'] \arrow[drr, "x"] & & \\
& \operatorname{colim}_i \Delta X \arrow[r, "\epsilon_X"'] & X \\
1 \arrow[ur, "\operatorname{colim}_i \Delta(x')"] \arrow[urr, "x'"'] & &
\end{tikzcd}$$

But recall Fact 2: $X$ is the coproduct of $X$-many copies of 1, where the inclusion maps are the elements of $X$:

$$\begin{tikzcd}
& 1 \arrow[dl, "x"'] \arrow[d] \arrow[dr, "x"] & \\
X \arrow[r, dashed, "{[\operatorname{colim}_i \Delta(x) \mid x \in X]}"] & \operatorname{colim}_i \Delta X \arrow[r, "\epsilon_X"'] & X \\
& 1 \arrow[ul, "x'"] \arrow[u] \arrow[ur, "x'"'] &
\end{tikzcd}$$

From this picture, it's quick to prove that $\epsilon_X^{-1} = [\operatorname{colim}_i \Delta(x)]$ is a right inverse for $\epsilon_X$: for any element $x : 1 \to X$, we know that

$$x = \epsilon_X \circ \epsilon_X^{-1} \circ x$$

and thus $\epsilon_X \circ \epsilon_X^{-1} = \mathbf{id}_X$. **5**

If we could prove that $\epsilon_X^{-1}$ is a left inverse for $\epsilon_X$, we would be done. But this proves difficult to do directly. **Instead we'll adopt a slightly different approach: 6** we'll show that $\epsilon_X^{-1}$ is an isomorphism by other means, and then, since we just showed that $\epsilon_X \circ \epsilon_X^{-1}$ is an isomorphism ($\mathrm{id}_X$), we'll have that $\epsilon_X$ is an iso by the **3-for-2 property of isomorphisms. 7**

$$\begin{tikzcd}
X \arrow[r, "\epsilon_X^{-1}"] \arrow[dr, equal] & \operatorname{colim}_i \Delta X \arrow[d, "\epsilon_X"] \\
& X
\end{tikzcd}$$

**6**

If we view **8** $\epsilon_X^{-1}$ as a morphism **9** $\coprod_x \operatorname{colim}_i 1 \to \operatorname{colim}_i \coprod_x 1$ and unfold the definition of $\epsilon_X^{-1}$, we get:

$$\begin{aligned}
\epsilon_X^{-1} &= \left[\operatorname*{colim}_i \Delta(x) \;\mid\; x \in X\right] \\
&= \left[\left[\mathsf{inc}_j^{\Delta X} \circ (\Delta x)_j \;\mid\; j \in I\right] \;\mid\; x \in X\right] \\
&= \left[\left[\mathsf{inc}_j^{\Delta X} \circ x \;\mid\; j \in I\right] \;\mid\; x \in X\right] && \text{(Defn of } \Delta\text{)} \\
&= \left[\left[\mathsf{inc}_j^{\Delta X} \circ \mathsf{inc}_x^{\Delta x(1)} \;\mid\; j \in I\right] \;\mid\; x \in X\right] && \text{(Defn of } \mathsf{inc}^{\Delta x(1)}\text{)}
\end{aligned}$$

But on Homework 6, Exercise 4 we proved that $\coprod_x \operatorname{colim}_i 1$ is isomorphic to $\operatorname{colim}_i \coprod_x 1$, and this is exactly **the isomorphism witnessing that fact** **10**. So $\epsilon_X^{-1}$ is an iso, and, as mentioned above, this gives us that $\epsilon_X$ is an iso. Since $X$ was arbitrary, we get that $\epsilon$ is a natural isomorphism and, by Fact 1, $\Delta$ is fully faithful.

## Notes:

**1** I'll be needing this fact later, so might as well mention it now

**2** Recall that if $z_i : X(i) \to Z$ is a cocone on a diagram $X$ with apex $Z$, then $[z_i \mid i \in I]$ is the unique map $\operatorname{colim}_i X(i) \to Z$ making all the triangles commute.

**3** If you know one of the categories you're working with is $\mathsf{Set}$, then don't hesitate to use the features of $\mathsf{Set}$ to solve the problem (it might be necessary to). As far as possible, I'd encourage you to try to stick to category-theoretic properties of $\mathsf{Set}$ (i.e. statements like Fact 2 which are phrased in terms of mappings and universal constructions – in this case, a coproduct – rather than anything too "nitty-gritty" about sets and the $\in$ relation.

**4** Though I try to be careful about when I'm using $x$ as an element of $X$ and when I'm viewing it as a morphism $1 \to X$, there isn't actually much of a distinction. Not only are the morphisms $1 \to X$ in canonical bijection with the elements of $X$, but they characterize the categorical behavior of $X$ as an object of $\mathsf{Set}$: if $f, g : X \to Y$ are functions and

$$f \circ x = g \circ x$$

for all $x : 1 \to X$, then $f = g$. This is just a categorical formulation of the principle of function extensionality.

**5** I'm using here the extensionality principle outlined in **4**. If I had shown that $f \circ h = g \circ h$ for some $f, g, h$, that would not imply $f = g$ in general. I'm allowed to conclude that here because the $h$ I proved it for was an arbitrary element $1 \to X$, i.e. for all $x \in X$.

**6** Sometimes workarounds like this will save you a lot of tedious morphism algebra. After all, I don't care *what* exactly the inverse of $\epsilon_X$ is (though it's nice to know that it's the $\epsilon_X^{-1}$ given), I just care *that* $\epsilon_X$ has an inverse. Of course, since we showed that $\epsilon_X^{-1}$ is a right inverse for $\epsilon_X$ and we'll show that $\epsilon_X$ is an iso, we could then confirm that $\epsilon_X^{-1}$ is the inverse of $\epsilon_X$ by uniqueness of inverses.

**7** You proved this on the first homework. The 3-for-2 property is not unique to isomorphisms, and indeed many of the notions of equivalence we care about exhibit a 3-for-2 property. It comes in handy often, like here.

**8** Since I'm viewing $\operatorname{colim}_i \Delta 1 = 1$, I can use the maps $\operatorname{colim}_i \Delta 1$ is known to have (e.g. its colimit inclusion maps) as maps into 1, and vice versa. Similarly with viewing $\coprod_x 1 = X$.

**9** Notice here that I suppress reference to $\Delta$. Writing $\operatorname{colim}_i 1$ means $\operatorname{colim}_i \Delta 1$, and $\coprod_x 1$ means $\coprod_x \Delta_X(1)$. Since $\Delta$ has constant value, it's justified to instead just write that value inside the colimit. I'm doing that here because the $\Delta$s would only serve to clutter the notation.

**10** This part wasn't too explicitly part of the HW6 solutions, so let me mention it. Suppose we have a category $\mathbb{C}$ with all $I$-shaped and $J$-shaped colimits, for some small $I$, $J$. Then if we have a diagram $X : I \times J \to \mathbb{C}$, write

- $\mathsf{inc}^1_{i_0,j_0}$ for the colimit inclusion $X(i_0, j_0) \to \operatorname{colim}_i X(i, j_0)$
- $\mathsf{inc}^2_{i_0,j_0}$ for the colimit inclusion $X(i_0, j_0) \to \operatorname{colim}_j X(i_0, j)$
- $\mathsf{inc}_{i_0}$ for the colimit inclusion $\operatorname{colim}_j X(i_0, j) \to \operatorname{colim}_i \operatorname{colim}_j X(i, j)$
- $\mathsf{inc}_{j_0}$ for the colimit inclusion $\operatorname{colim}_i X(i, j_0) \to \operatorname{colim}_j \operatorname{colim}_i X(i, j)$

So then we have:

$$\Big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\mid\; i \in I\big] \;\mid\; j \in J\Big] : \operatorname*{colim}_j \operatorname*{colim}_i X(i,j) \to \operatorname*{colim}_i \operatorname*{colim}_j X(i,j)$$
$$\Big[\big[\mathsf{inc}_j \circ \mathsf{inc}^1_{i,j} \;\mid\; j \in J\big] \;\mid\; i \in I\Big] : \operatorname*{colim}_i \operatorname*{colim}_j X(i,j) \to \operatorname*{colim}_j \operatorname*{colim}_i X(i,j)$$

To see why the first one is true, note that

$$\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} : X(i,j) \to \operatorname*{colim}_i \operatorname*{colim}_j X(i,j)$$

These maps form a cocone over $X(-, j)$, so we can pair them up along $I$:

$$\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\mid\; i \in I\big] : \operatorname*{colim}_i X(i,j) \to \operatorname*{colim}_i \operatorname*{colim}_j X(i,j)$$

But these maps form a cocone over $\operatorname{colim}_i X(i, -)$, so we can pair them up along $J$:

$$\Big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\mid\; i \in I\big] \;\mid\; j \in J\Big] : \operatorname*{colim}_j \operatorname*{colim}_i X(i,j) \to \operatorname*{colim}_i \operatorname*{colim}_j X(i,j).$$

Now, observe:

$$\begin{aligned}
&\big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\big]\big] \circ \big[\big[\mathsf{inc}_j \circ \mathsf{inc}^1_{i,j}\big]\big] \\
&= \big[\big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\big]\big] \circ \big[\mathsf{inc}_j \circ \mathsf{inc}^1_{i,j}\big]\big] \\
&= \big[\big[\big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\big]\big] \circ \mathsf{inc}_j \circ \mathsf{inc}^1_{i,j}\big]\big] \\
&= \big[\big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\big] \circ \mathsf{inc}^1_{i,j}\big]\big] \\
&= \big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\big]\big] \\
&= \big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\mid\; j \in J\big] \;\mid\; i \in I\big] \qquad (*)
\end{aligned}$$

This calculation is just a repeated application of the functoriality of the co-pairing operation (the dual of **12** from the HW6 solutions) and the fact that $[z_j \mid j \in J] \circ \mathsf{inc}_{j_0} = z_{j_0}$ by the definition of $[-]$, and likewise for $I$. Now, the morphism we got in line (*) is actually the identity on $\operatorname{colim}_i \operatorname{colim}_j X(i,j)$. To see this, pick any $i_0 \in I$ and $j_0 \in J$. Then observe:

$$\begin{aligned}
&\Big[\Big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\mid\; j \in J\Big] \;\mid\; i \in I\Big] \circ \mathsf{inc}_{i_0} \circ \mathsf{inc}^2_{i_0,j_0} \\
&= \Big[\mathsf{inc}_{i_0} \circ \mathsf{inc}^2_{i_0,j} \;\mid\; j \in J\Big] \circ \mathsf{inc}^2_{i_0,j_0} \\
&= \mathsf{inc}_{i_0} \circ \mathsf{inc}^2_{i_0,j_0} \\
&= \mathrm{id} \circ \mathsf{inc}_{i_0} \circ \mathsf{inc}^2_{i_0,j_0}
\end{aligned}$$

Since $j_0$ was arbitrary, we can conclude by **ColimInj** (see HW6 sols once again) that

$$\Big[\Big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\mid\; j \in J\Big] \;\mid\; i \in I\Big] \circ \mathsf{inc}_{i_0} \quad = \quad \mathrm{id} \circ \mathsf{inc}_{i_0}$$

but $i_0$ was arbitrary, so, again by **ColimInj**,

$$\Big[\Big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\mid\; j \in J\Big] \;\mid\; i \in I\Big] \quad = \quad \mathrm{id},$$

so

$$\Big[\Big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\Big]\Big] \circ \Big[\Big[\mathsf{inc}_j \circ \mathsf{inc}^1_{i,j}\Big]\Big] \quad = \quad \mathrm{id}.$$

A similar calculation in the other direction shows that

$$\Big[\Big[\mathsf{inc}_j \circ \mathsf{inc}^1_{i,j}\Big]\Big] \circ \Big[\Big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\Big]\Big] \quad = \quad \mathrm{id}.$$