Presheaf Models of Polarized Higher-Order Abstract Syntax

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Contents

1 Preliminaries

Note 1.1

We'll work with three set universes in our metatheory, with implicit ("Russell-style") element operators.

$$\mathsf{Set}_0 \quad : \quad \mathsf{Set}_1 \quad : \quad \mathsf{Set}.$$

When convenient, we'll call elements of Set_0 *small sets* and elements of Set_1 *large sets*.[1] We freely make use of the following properties:

- These set universes are *cumulative*: any $X : \mathsf{Set}_0$ is also in Set_1
- Both Set_0 and Set_1 are *closed under dependent products and sums*: if $X : \mathsf{Set}_\ell$ and $P : X \to \mathsf{Set}_\ell$, then

$$\left(\prod_{x:X} P\ x\right) : \mathsf{Set}_\ell \qquad \text{and} \qquad \left(\sum_{x:X} P\ x\right) : \mathsf{Set}_\ell$$

- Both Set_0 and Set_1 form categories (which we'll also denote as Set_0 and Set_1, respectively), which are both full subcategories of Set

[1] We don't use these mutually exclusively: a large set could perhaps be a small set, and, by the cumulativity of the universes, all small sets are large.

- If $\mathbb{C}$ is a *small category*, that is, $|\mathbb{C}| : \mathsf{Set}_0$ and $\mathsf{Hom}_{\mathbb{C}}(I, J) : \mathsf{Set}_0$ for all $I, J : |\mathbb{C}|$, then the collection of *small presheaves on* $\mathbb{C}$ is a large set:

$$\left|\mathsf{Set}_0^{\mathbb{C}^{\mathrm{op}}}\right| : \mathsf{Set}_1$$

- The empty set $\emptyset$, the singleton set $\mathbb{1}$, and the set of natural numbers $\mathbb{N}$ are all small sets

For full generality, we might instead need to assume an infinite hierarchy $\mathsf{Set}_0 : \mathsf{Set}_1 : \mathsf{Set}_2 : \dots$ – see Note 3.1.

Definition 1.1

Given a category $\mathbb{C}$ and a presheaf $F : \mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}$, the **category of elements of** F – denoted $\int F$ – is the category whose

- objects are pairs (I, x), where I is an object of $\mathbb{C}$ and $x : F(I)$
- morphisms $(I, x) \to (J, y)$ are $\mathbb{C}$-morphisms $j : I \to J$ such that $F\ j\ y = x$

Morphisms in $\int F$ are uniquely identified by their underlying $\mathbb{C}$-morphism and their codomain: for any $j : I \to J$ and $y : F(J)$, we'll write $(j \downharpoonleft y)$ to indicate the $\int F$-morphism $(I, F\ j\ y) \to (J, y)$. We'll use the rule1 to type morphisms in categories of elements.

$$\frac{j \colon \mathbb{C}(I, J) \quad F \colon \mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set} \quad F\ j\ y = x}{(j \downharpoonleft y) \colon (\int F)\ (I, x)\ (J, y)} \tag{1}$$

Note 1.2

If $\blacklozenge$ is the constant-$\mathbb{1}$ presheaf $\mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}$, then

$$\int \blacklozenge \cong \mathbb{C} \tag{2}$$

Definition 1.2

Let $\mathbb{C}$ be any category, and consider the Yoneda embedding

$$\mathbf{y} : \mathbb{C} \Rightarrow \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}.$$

For any object $J : |\mathbb{C}|$, define the **slice category of** $\mathbb{C}$ **over** J to be the category of elements of $\mathbf{y}(J)$:

$$\mathbb{C}/J = \int(\mathbf{y} J).$$

Explicitly, the objects of $\mathbb{C}/J$ are pairs $(I \downharpoonleft j)$ where $I : |\mathbb{C}|$ and $j : \mathbb{C}(I, J)$. A morphism in $\mathbb{C}/J$ from (I_0, j_0) to (I_1, j_1) is a $\mathbb{C}$-morphism $i_2 : I_0 \to I_1$ such that the triangle

$$\begin{tikzcd} I_0 \arrow[rr, "i_2"] \arrow[dr, "j_0"'] & & I_1 \arrow[dl, "j_1"] \\ & J & \end{tikzcd}$$

commutes. So we specialize 1.

$$\frac{i_2 \colon \mathbb{C}(I_0, I_1) \quad j_1 \circ i_2 = j_0}{(i_2 \downharpoonleft j_1) \colon (\mathbb{C}/J)\ (I_0, j_0)\ (I_1, j_1)} \tag{3}$$

Proposition 1.1

If $\mathbb{C}$ is a small category, then, for any $J : |\mathbb{C}|$, the representable functor $\mathbf{y}J$ is a small presheaf on $\mathbb{C}$

$$\mathbf{y}J \colon \mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_0$$

and, moreover, the slice category $\mathbb{C}/J$ is also small.

Definition 1.3

A **category with families (cwf)** consists of the following data

- A category Con with terminal object $\bullet$. For $I, J : \mathsf{Con}$, write $\mathsf{Sub}\ I\ J$ for the hom-set $\mathsf{Hom}_{\mathsf{Con}}(I, J)$. We'll call objects $I : \mathsf{Con}$ *contexts*, and morphisms $j : \mathsf{Sub}\ I\ J$ *substitutions* or *context morphisms.*

- A presheaf
$$\mathsf{Ty} \colon \mathsf{Con}^{\mathrm{op}} \Rightarrow \mathsf{Set}$$
sending a context I to its *set of types* $\mathsf{Ty}\ I$, and a substitution $j : \mathsf{Sub}\ I\ J$ to the map
$$-[j] : \mathsf{Ty}\ J \to \mathsf{Ty}\ I$$

- A presheaf
$$\mathsf{Tm} : (\textstyle\int \mathsf{Ty})^{\mathrm{op}} \Rightarrow \mathsf{Set}$$
sending each $J : \mathsf{Con}$ and $Y : \mathsf{Ty}\ J$ to the set $\mathsf{Tm}(J, Y)$ of *terms of type A in context I*, and each $j : \mathsf{Sub}\ I\ J$ to the map
$$-[j] : \mathsf{Tm}(J, Y) \to \mathsf{Tm}(I, Y[j])$$

- For each $Y : \mathsf{Ty}\ J$, a context $J \triangleright Y : \mathsf{Con}$, called the *extension of J by Y*, such that there is an isomorphism
$$\mathsf{Sub}\ I\ (J \triangleright Y) \quad\cong\quad \sum_{j : \mathsf{Sub}\ I\ J} \mathsf{Tm}(I, Y[j])$$
natural in I.[2] We'll write $(\mathsf{p}_Y, \mathsf{v}_Y) : \sum_{\mathsf{Sub}\ (J \triangleright Y)\ J} \mathsf{Tm}(J \triangleright Y, Y[\mathsf{p}_Y])$ for the image of $\mathsf{id}_{J \triangleright Y}$ under this isomorphism (dropping the subscripts when possible), and write $\langle -, - \rangle$ for the reverse direction (e.g. $\langle \mathsf{p}, \mathsf{v} \rangle = \mathsf{id}_{J \triangleright Y}$).

Definition 1.4

A CwF $(\mathsf{Con}, \mathsf{Sub}, \mathsf{Ty}, \mathsf{Tm}, \ldots)$ is called *small* if all the sets involved are in Set_0, i.e.

- $|\mathsf{Con}| : \mathsf{Set}_0$
- $\mathsf{Sub}(I, J) \colon \mathsf{Set}_0$ for all I, J
- $\mathsf{Ty} \colon \mathsf{Con}^{\mathrm{op}} \Rightarrow \mathsf{Set}_0$
- $\mathsf{Tm} \colon (\textstyle\int \mathsf{Ty})^{\mathrm{op}} \Rightarrow \mathsf{Set}_0$

[2] That is,

$$\langle j_1, t \rangle \circ i_2 = \langle j_1 \circ i_2, t[i_2] \rangle \qquad \text{(extend-natural)}$$

where $t : \mathsf{Tm}(I_1, Y[j_1])$.

2 The Presheaf Model

Definition 2.1

Let $\mathbb{C}$ be any small category. The **presheaf model** (on $\mathbb{C}$) is the category $\widehat{\mathbb{C}} = \mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_1$, endowed with a CwF structure in the following way.

- $\widehat{\mathsf{Con}} = \mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_1$. A morphism $\sigma : \widehat{\mathsf{Sub}}\ \Delta\ \Gamma$ is a natural transformation of presheaves

$$\sigma : \int_{I:\mathbb{C}} \Delta I \to \Gamma I$$

- The constant-$\mathbb{1}$ presheaf is the terminal object, which we'll denote $\blacklozenge : \widehat{\mathsf{Con}}$

- For $\Gamma : \widehat{\mathsf{Con}}$, $\widehat{\mathsf{Ty}}(\Gamma)$ is defined as the set of small presheaves on the category of elements of Γ:

$$\widehat{\mathsf{Ty}}(\Gamma) = (\textstyle\int \Gamma)^{\mathrm{op}} \Rightarrow \mathsf{Set}_0 \qquad \text{(Defn. } \widehat{\mathsf{Ty}}\text{)}$$

  Explicitly, $A : \widehat{\mathsf{Ty}}(\Gamma)$ assigns to each $J : \mathbb{C}$, $\gamma : \Gamma J$ a small set $A(J, \gamma)$, and to each $j : I \to J$ a function

$$A(j \downharpoonleft \gamma) \colon A(J, \gamma) \to A(I, \Gamma\ j\ \gamma) \qquad (*)$$

- For a natural transformation $\sigma : \widehat{\mathsf{Sub}}\ \Delta\ \Gamma$ and some $A : \widehat{\mathsf{Ty}}(\Gamma)$, define $A[\sigma] \colon \widehat{\mathsf{Ty}}(\Delta)$ (i.e. $A[\sigma] : (\int \Delta)^{\mathrm{op}} \to \mathsf{Set}_0$) by

$$\begin{aligned} A[\sigma](I, \vartheta)\ &= A(I, \sigma_I \vartheta) \\ A[\sigma](j \downharpoonleft \delta) &= A(j \downharpoonleft \sigma_J \delta) \end{aligned} \qquad \text{(Defn. } A[\sigma]\text{)}$$

  The definition of the morphism part is well-typed, by Fig. A.1.

- Given $\Gamma : \widehat{\mathsf{Con}}$ and $A : \widehat{\mathsf{Ty}}(\Gamma)$, define

$$\widehat{\mathsf{Tm}}(\Gamma, A) = \int_{I:\mathbb{C}} (\phi : \Gamma I) \to A(I, \phi).$$

  Explicitly, a term $M : \widehat{\mathsf{Tm}}(\Gamma, A)$ consists of a dependent function

$$M : (I : \mathbb{C}) \to (\phi : \Gamma I) \to A(I, \phi)$$

  satisfying this condition: for any j and γ,

$$M_I(\Gamma\ j\ \gamma) = A\ (j \downharpoonleft \gamma)\ (M_J\ \gamma). \qquad \text{(Nat. } M\text{)}$$

- Given $\sigma : \widehat{\mathsf{Sub}}\ \Delta\ \Gamma$ and $M : \widehat{\mathsf{Tm}}(\Gamma, A)$, define

$$M[\sigma] : \int_{I:\mathbb{C}} (\vartheta : \Delta I) \to A[\sigma](I, \vartheta)$$

  by

$$(M[\sigma])_I\ \vartheta = M_I\ (\sigma_I \vartheta). \qquad \text{(Defn. } M[\sigma]\text{)}$$

  This is natural, by Fig. A.2.

- Given $A : \widehat{\mathsf{Ty}}(\Gamma)$, define $\Gamma \triangleright A : \widehat{\mathsf{Con}}$ by

$$\begin{aligned}
(\Gamma \triangleright A)\ I \quad &= \sum_{\phi : \Gamma I} A(I, \phi) \\
(\Gamma \triangleright A)\ j\ (\gamma, a) &= (\Gamma\ j\ \gamma,\ A\ (j \downharpoonleft \gamma)\ a)
\end{aligned}$$

Proposition 2.1

In the presheaf model, there is an isomorphism

$$\widehat{\mathsf{Sub}}\ \Delta\ (\Gamma.A) \cong \sum_{\sigma : \widehat{\mathsf{Sub}}\ \Delta\ \Gamma} \widehat{\mathsf{Tm}}(\Delta, A[\sigma])$$

natural in Δ.

Note 2.1

Given any closed type $E : \widehat{\mathsf{Ty}}\ \blacklozenge$, we can *weaken* E to be type in any context Γ by ignoring the elements of Γ:

$$\begin{aligned}
E \quad &: \widehat{\mathsf{Ty}}\ \Gamma \\
E(J, \gamma) \ &= E(J, \star) \\
E(j \downharpoonleft \gamma) &= E(j \downharpoonleft \mathsf{id}_\star)
\end{aligned}$$

where $\star$ is the unique element of $\mathbb{1}$. When we define *large types* (Defn. 3.1), the same will be true.

3 Type Formers

TODO: Dependent types in presheaf model

Note 3.1

The following construction will produce a universe $\mathbf{U}$ in the syntax of our type theory. However, $\mathbf{U}$ will not itself be a term of some larger universe. If we want $\mathbf{U}$ to live in a universe, we need to assume a further set-theoretic universe and repeat this construction to obtain $\mathbf{U}_1$. But then $\mathbf{U}_1$ will not live in a universe, and so on. To completely avoid this problem will require an infinite hierarchy

$$\mathsf{Set}_0 : \mathsf{Set}_1 : \mathsf{Set}_2 : \ \ldots : \mathsf{Set}.$$

We avoid doing this, for simplicity.

Definition 3.1

A **large type** in context Γ is a large presheaf on the category of elements of Γ:

$$\mathbf{X} : (\textstyle\int\ \Gamma)^{\mathrm{op}} \to \mathsf{Set}_1$$

Proposition 3.1

There is an isomorphism

$$\widehat{\mathsf{Con}} \cong \widehat{\mathsf{Ty}}\ \blacklozenge.$$

Proposition 3.2

We can extend the definition of $\widehat{\mathsf{Tm}}$ to large types: given any $\mathbf{X} : \widehat{\mathsf{Ty}}\ \Gamma$,

$$\widehat{\mathsf{Tm}}(\Gamma, \mathbf{X}) = \int_{I : \mathbb{C}} (\phi : \Gamma I) \to \mathbf{X}(I, \phi).$$

In the special case where the large type in question is some $\mathbf{E} : \widehat{\mathsf{Ty}}\ \blacklozenge$, weakened to be a large type in Γ (care of Note 2.1), then this definition is just natural transformations from Γ to $\mathbf{E}$ (taken as a context, care of Prop. 3.1):

$$\begin{aligned} \mathsf{Tm}(\Gamma, \mathbf{E}) &\cong \int_{I:\mathbb{C}} \Gamma I \to \mathbf{E} I && (4) \\ &= \widehat{\mathsf{Sub}}\ \Gamma\ \mathbf{E}. && (5) \end{aligned}$$

Theorem 3.3

There is a large closed type $\mathbf{U} : \widehat{\mathsf{Ty}}\ \blacklozenge$ such that

$$\widehat{\mathsf{Tm}}(\Gamma, \mathbf{U}) \cong \mathsf{Ty}\ \Gamma.$$

$$\frac{\widehat{\mathsf{Sub}}\ \Gamma\ \mathbf{U}}{(\int \Gamma)^{\mathrm{op}} \Rightarrow \mathsf{Set}_0} \qquad \text{(Fundamental Property of } \mathbf{U}\text{)}$$

Proposition 3.4

$$\frac{\widehat{\mathsf{Sub}}\ \blacklozenge\ (\Gamma \Rightarrow \Delta)}{\widehat{\mathsf{Sub}}\ \Gamma\ \Delta} \qquad (6)$$

4 Higher-Order Abstract Syntax

Definition 4.1

Observe that the base CwF type presheaf $\mathsf{Ty} : \mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_0$ can be regarded as a closed *type* in the presheaf model, $\mathsf{Ty} \colon \widehat{\mathsf{Ty}}\ \blacklozenge$, care of 2. A **closed base type** is a term $A \colon \widehat{\mathsf{Tm}}(\blacklozenge, \mathsf{Ty})$.

Given a closed base type A and some $I : \mathbb{C}$, write $A_I : \mathsf{Ty}\ I$ for the I-component of A (recalling that $\widehat{\mathsf{Tm}}(\blacklozenge, \mathsf{Ty})$ consists of natural transformations from $\blacklozenge$ to Ty).[3] Then define

$$\begin{aligned} &\mathsf{Tm}_A &&: \mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_0 \\ &\mathsf{Tm}_A\ I &&= \mathsf{Tm}(I, A_I) \\ &\mathsf{Tm}_A\ i_2 &&= \mathsf{Tm}(i_2 \downharpoonright A_{I_1}) \end{aligned}$$

See Fig. B.1.

[3]The naturality condition says: for all $i_2 \colon \mathbb{C}(I_0, I_1)$,

$$\mathsf{Ty}\ i_2\ A_{I_1} = A_{I_0} \qquad (7)$$

HOAS	Semantics in presheaf model	translated into category theory	Note
$\mathbf{U}$ type	$\mathbf{U}\colon \widehat{\mathsf{Ty}}\ \blacklozenge$	$\mathbf{U}\colon \mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_1$	
$\mathsf{Ty}\colon \mathbf{U}$	$\mathsf{Ty}\colon \widehat{\mathsf{Tm}}(\blacklozenge, \mathbf{U})$	$\mathsf{Ty}\colon \mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_0$	Fig. B.2
$\mathsf{Tm}\colon \mathsf{Ty} \to \mathbf{U}$	$\mathsf{Tm}\colon \widehat{\mathsf{Tm}}\,(\blacklozenge, \mathsf{Ty} \Rightarrow \mathbf{U})$	$\mathsf{Tm}\colon\ (\int \mathsf{Ty})^{\mathrm{op}} \Rightarrow \mathsf{Set}_0$	Fig. B.3
$\Sigma\colon (A : \mathsf{Ty}) \to (\mathsf{Tm}\ A \to \mathsf{Ty}) \to \mathsf{Ty}$	$\dfrac{A\colon \widehat{\mathsf{Tm}}(\blacklozenge, \mathsf{Ty}) \quad B\colon \widehat{\mathsf{Sub}}\ \mathsf{Tm}_A\ \mathsf{Ty}}{\Sigma AB\colon \widehat{\mathsf{Tm}}(\blacklozenge, \mathsf{Ty})}$		

5 Polarized Type Theory

Definition 5.1

A **polarized category with families (PCwF)** consists of a CwF with two 'copies' of each structure: a 'positive' and a 'negative' version. More precisely, it includes the following data.

- A category Con with terminal object $\bullet$ and hom-sets denoted Sub
- Two presheaves
$$\mathsf{Ty}\colon \mathsf{Con}^{\mathrm{op}} \Rightarrow \mathsf{Set} \qquad \mathsf{Ty}^-\colon \mathsf{Con}^{\mathrm{op}} \Rightarrow \mathsf{Set}$$
sending a context I to its set of positive types $\mathsf{Ty}\ I$ and set of negative types $\mathsf{Ty}^-\ I$, respectively, and sending a substitution $j : \mathsf{Sub}\ I\ J$ to the maps
$$-[j] : \mathsf{Ty}\ J \to \mathsf{Ty}\ I \qquad -[j^-] : \mathsf{Ty}^-\ J \to \mathsf{Ty}^-\ I$$
We require that Ty and Ty^- agree on the empty context:
$$\mathsf{Ty}\ \bullet = \mathsf{Ty}^-\ \bullet$$
- Two presheaves
$$\mathsf{Tm} : (\textstyle\int \mathsf{Ty})^{\mathrm{op}} \Rightarrow \mathsf{Set} \qquad \mathsf{Tm}^- : (\textstyle\int \mathsf{Ty}^-)^{\mathrm{op}} \Rightarrow \mathsf{Set}$$
sending each $J : \mathsf{Con}$ and $Y : \mathsf{Ty}^s\ J$ to the set $\mathsf{Tm}^s(J, Y)$, and each $j : \mathsf{Sub}\ I\ J$ to the map
$$-[j^s] : \mathsf{Tm}^s(J, Y) \to \mathsf{Tm}^s(I, Y[j^s])$$
where here, and henceforth, s is a metavariable for either negative or positive.
- For each $Y : \mathsf{Ty}^s\ J$, a context $J \triangleright^s Y : \mathsf{Con}$, called the *$s$-extension of J by Y*, such that there is an isomorphism
$$\mathsf{Sub}\ I\ (J \triangleright^s Y) \quad \cong \quad \sum_{j:\mathsf{Sub}\ I\ J} \mathsf{Tm}^s(I, Y[j^s])$$
natural in I.

Nomenclature

Base CwF

$\mathbb{C}$	: Small category	Category of contexts for the base CwF
Sub	: $\lvert\mathbb{C}\rvert \to \lvert\mathbb{C}\rvert \to \mathsf{Set}_0$	Substitutions in the base CwF, homsets of $\mathbb{C}$
$\mathbf{y}$	: $\mathbb{C} \Rightarrow \mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_0$	Yoneda embedding (object part is given by Sub)
I, J, I_0, I_1, K	: $\lvert\mathbb{C}\rvert$	Contexts in the base CwF
$\bullet$	: $\lvert\mathbb{C}\rvert$	Empty context in the base CwF
i_2	: $\mathbb{C}(I_0, I_1)$, a.k.a. $\mathsf{Sub}(I_0, I_1)$	Substitution, context morphism
j_0	: $\mathbb{C}(I_0, I_1)$, a.k.a. $\mathsf{Sub}(I_0, J)$	
j_1	: $\mathbb{C}(I_1, J)$, a.k.a. $\mathsf{Sub}(I_1, J)$	
k	: $\mathbb{C}(J, K)$, a.k.a. $\mathsf{Sub}(J, K)$	
Ty	: $\mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_0$	Family of Types of the base CwF
Tm	: $(\int \mathsf{Ty})^{\mathrm{op}} \Rightarrow \mathsf{Set}_0$	Family of Terms of the base CwF
Y	: $\mathsf{Ty}\ J$	Type in context J

Presheaf Model

$\widehat{\mathbb{C}}$	: Category	Category of contexts for the presheaf model, category of large presheaves on $\mathbb{C}$
$\blacklozenge$	: $\widehat{\mathsf{Con}}$, i.e. $\mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_1$	Empty context, the terminal presheaf
Δ, Γ	: $\widehat{\mathsf{Con}}$, i.e. $\mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_1$	Contexts in presheaf model, presheaves over $\mathbb{C}$
σ	: $\widehat{\mathsf{Sub}}\ \Delta\ \Gamma$	Substitutions in the presheaf model, natural transformations of presheaves
ϕ	: $\Gamma\ I$	Element of Γ at I
γ	: $\Gamma\ J$	Element of Γ at J
ϑ	: $\Delta\ I$	Element of Δ at I
δ	: $\Delta\ J$	Element of Δ at J
A	: $\widehat{\mathsf{Ty}}\ \Gamma$	Type in context Γ
E	: $\widehat{\mathsf{Ty}}\ \blacklozenge$	Closed type
M	: $\widehat{\mathsf{Tm}}(\Gamma, A)$	Term of type A in context Γ
a	: $A(I, \phi)$	Element of A at (I, ϕ)
$\mathbf{X}$	: $(\int \Gamma)^{\mathrm{op}} \Rightarrow \mathsf{Set}_1$	Large type in context Γ
$\mathbf{E}$	: $\mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_1$	Large closed type

A Presheaf Model Calculations & Proofs

$$\dfrac{A(j,\sigma_J\delta) : A(J,\sigma_J\delta) \to A(I,\Gamma\ j\ (\sigma_J\delta)) \qquad \dfrac{\dfrac{\sigma : \widehat{\mathsf{Sub}}\ \Delta\ \Gamma}{\Gamma\ j\ (\sigma_J\delta) = \sigma_I(\Delta\ j\ \delta)} \qquad \dfrac{(j \downharpoonleft \delta) : (\textstyle\int\ \Delta)\ (I,\vartheta)\ (J,\delta)}{\Delta\ j\ \delta = \vartheta}\ (1)}{\Gamma\ j\ (\sigma_J\delta) = \sigma_I\vartheta}}{A(j,\sigma_J\delta) : A(J,\sigma_J\delta) \to A(I,\sigma_I\vartheta)}$$

Figure A.1: Calculation: Well-typedness of type substitution in presheaf model

$$\begin{aligned}
(M[\sigma])_I(\Delta\ j\ \delta) &= M_I(\sigma_I(\Delta\ j\ \delta) && \text{(Defn. } M[\sigma]\text{)}\\
&= M_I(\Gamma\ j\ (\sigma_J\delta)) && \text{(Naturality of } \sigma\text{)}\\
&= A(j \downharpoonleft \sigma_J\delta)\ (M_J\ (\sigma_J\delta)) && \text{(Nat. } M\text{, with } \gamma = \sigma_J\delta\text{)}\\
&= A[\sigma]\ (j \downharpoonleft \delta)\ (M_J\ (\sigma_J\delta)) && \text{(Defn. } A[\sigma]\text{)}\\
&= A[\sigma]\ (j \downharpoonleft \delta)\ ((M[\sigma])_J\ \delta) && \text{(Defn. } M[\sigma]\text{)}
\end{aligned}$$

Thus $M[\sigma]$ satisfies the requisite "naturality" condition to be a term of type $A[\sigma]$.

Figure A.2: Calculation: Naturality of term substitution in presheaf model

Proposition A.1 (Theorem 3.3)
There is a large closed type $\mathbf{U} : \widehat{\mathsf{Ty}}\ \blacklozenge$ such that

$$\widehat{\mathsf{Tm}}(\Gamma, \mathbf{U}) \cong \mathsf{Ty}\ \Gamma.$$

Proof. —
We'll define $\mathbf{U}$ as a large presheaf on $\mathbb{C}$. To see what the definition of $\mathbf{U}(I)$ must be, we use the Yoneda Lemma:

$$\begin{aligned}
\mathbf{U}\ I &\cong \widehat{\mathsf{Sub}}\ (\mathbf{y}I)\ \mathbf{U} && \text{(Yoneda)}\\
&\cong \widehat{\mathsf{Tm}}(\mathbf{y}I, \mathbf{U}) && \text{(Prop. 3.2)}\\
&\cong \widehat{\mathsf{Ty}}(\mathbf{y}I) && \text{(Desired Property of } \mathbf{U}\text{)}\\
&= \mathsf{Set}_0^{(\int \mathbf{y}I)^{\mathrm{op}}} && \text{(Defn. } \widehat{\mathsf{Ty}}\text{)}\\
&= \mathsf{Set}_0^{(\mathbb{C}/I)^{\mathrm{op}}} && \text{(Defn. 1.2)}
\end{aligned}$$

This motivates this definition: on objects, $\mathbf{U}$ is defined as

$$\mathbf{U}\ I = \mathsf{Set}_0^{(\mathbb{C}/I)^{\mathrm{op}}}. \qquad \text{(Defn. } \mathbf{U}\text{)}$$

Then, for a $\mathbb{C}$-morphism $k : J \to K$, we must give $\mathbf{U}(k) : \mathbf{U}K \to \mathbf{U}J$. Given $F : (\mathbb{C}/K)^{\mathrm{op}} \Rightarrow \mathsf{Set}_0$, define $\mathbf{U}\ k\ F : (\mathbb{C}/J)^{\mathrm{op}} \Rightarrow \mathsf{Set}_0$ by

$$\mathbf{U}\ k\ F\ (I,j) = F(I, k \circ j) \tag{8}$$
$$\mathbf{U}\ k\ F\ (i_2 \downharpoonleft j_1) = F\ (i_2 \downharpoonleft k \circ j_1) \tag{9}$$

where the last line makes sense by Fig. A.3.

It remains to check that $\mathbf{U}$ is indeed a functor, and that we have the bijection

$$\widehat{\mathsf{Sub}}\ \Gamma\ \mathbf{U} \cong \widehat{\mathsf{Ty}}\ \Gamma$$

This bijection is given as a generalization of the proof of the Yoneda Lemma: given $\beta \colon \widehat{\mathsf{Sub}}\ \Gamma\ \mathbf{U}$, define $B : \widehat{\mathsf{Ty}}\ \Gamma$, that is, $B \colon (\int\ \Gamma)^{\mathrm{op}} \Rightarrow \mathsf{Set}_0$ by

$$\begin{aligned} B(I,\phi) &= \beta_I\ \phi\ (I, \mathsf{id}_I) && (10)\\ B(i_2 \downharpoonleft \phi_1) &= \beta_I\ \phi_1\ i_2 && (11) \end{aligned}$$

and, oppositely, given B, define

$$\begin{aligned} \beta_J\ \gamma\ &: (\mathbb{C}/J)^{\mathrm{op}} \Rightarrow \mathsf{Set}_0 && (12)\\ \beta_J\ \gamma\ (I,j) &= B(I, \Gamma\ j\ \gamma) && (13)\\ \beta_J\ \gamma\ (i_2 \downharpoonleft j_1) &= B(i_2 \downharpoonleft \Gamma\ j_1\ \gamma) && (14) \end{aligned}$$

That these definitions are well-typed and constitute a bijection is confirmed in Fig. A.4. . We can then conclude

$$\mathsf{Tm}(\Gamma, \mathbf{U}) \cong \widehat{\mathsf{Sub}}\ \Gamma\ \mathbf{U} \cong \mathsf{Ty}\ \Gamma.$$

□

$$\begin{tikzcd} I_0 \arrow[rr, "i_2"] \arrow[dr, "j_0"'] & & I_1 \arrow[dl, "j_1"] \\ & J \arrow[d, "k"] & \\ & K & \end{tikzcd}$$

$$\dfrac{F \colon (\mathbb{C}/K)^{\mathrm{op}} \Rightarrow \mathsf{Set}_0 \qquad \dfrac{i_2 \colon \mathbb{C}(I_0, I_1) \qquad \dfrac{\dfrac{(i_2 \downharpoonleft j_1) \colon (\mathbb{C}/J)\ (I_0, j_0)\ (I_1, j_1)}{j_1 \circ i_2 = j_0}\ (3)}{k \circ j_1 \circ i_2 = k \circ j_0}\ (3)}{(i_2 \downharpoonleft k \circ j_1) \colon (\mathbb{C}/K)\ (I_0, k \circ j_0)\ (I_1, k \circ j_1)}\ (3)}{F(i_2 \downharpoonleft k \circ j_1) \colon F(I_0, k \circ j_0) \to F(I_1, k \circ j_1)}$$

Figure A.3: Calculation: Well-typedness of the definition of the morphism part of $\mathbf{U}\ k\ F$

TODO

Figure A.4: Calculation: Correctness of the bijection proof of Theorem 3.3

B HOAS Calculations & Proofs

$$\dfrac{\mathsf{Tm}\colon (\textstyle\int \mathsf{Ty})^{\mathrm{op}} \Rightarrow \mathsf{Set}_0 \qquad \dfrac{i_2\colon \mathbb{C}(I_0, I_1) \qquad \overline{\mathsf{Ty}\ i_2\ A_{I_1} = A_{I_0}}\ (7)}{(i_2 \downharpoonleft A_{I_1})\colon (\textstyle\int \mathsf{Ty})\ (I_0, A_{I_0})\ (I_1, A_{I_1})}\ (1)}{\mathsf{Tm}(i_2 \downharpoonleft A_{I_1})\colon \mathsf{Tm}(I_1, A_{I_1}) \to \mathsf{Tm}(I_0, A_{I_0})}$$

Figure B.1: Calculation: Well-typedness of the Tm_A construction

$$\begin{aligned}
\widehat{\mathsf{Tm}}(\blacklozenge, \mathbf{U}) &\cong \widehat{\mathsf{Sub}}\ \blacklozenge\ \mathbf{U} && (5)\\
&\cong (\textstyle\int \blacklozenge)^{\mathrm{op}} \Rightarrow \mathsf{Set}_0 && (\text{Fundamental Property of } \mathbf{U})\\
&\cong \mathbb{C}^{\mathrm{op}} \Rightarrow \mathsf{Set}_0 && (2)
\end{aligned}$$

Figure B.2: Calculation: Meaning of $\mathsf{Ty}\colon \mathbf{U}$ in the presheaf model

$$\begin{aligned}
\widehat{\mathsf{Tm}}(\blacklozenge, \mathsf{Ty} \Rightarrow \mathbf{U}) &\cong \widehat{\mathsf{Sub}}\ \blacklozenge\ (\mathsf{Ty} \Rightarrow \mathbf{U}) && (5)\\
&\cong \widehat{\mathsf{Sub}}\ \mathsf{Ty}\ \mathbf{U} && (6)\\
&\cong (\textstyle\int \mathsf{Ty})^{\mathrm{op}} \Rightarrow \mathsf{Set}_0 && (\text{Fundamental Property of } \mathbf{U})
\end{aligned}$$

Figure B.3: Calculation: Meaning of $\mathsf{Tm}\colon \mathsf{Ty} \to \mathbf{U}$ in the presheaf model