



# GATs, Cats, and CwFs

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# What is this talk?

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# What is this talk?

- Background for my forthcoming PhD thesis [Neu25, Chapter 1]
- Interpretation of [KKA19]
  - ▶ Restrict their signature language for QIITs to get a signature language for GATs
  - ▶ Matches the notion of GAT given by [Car86], except we don't allow for equations between sort. This provides an intrinsic presentation of GATs.

# *GAT signature*

# *Category of Algs.*

# *Concrete CwF*

Categories

CwFs

GAT signatures



# Categories of GAT Algebras

**Generalized algebraic theories** [Car86] are a syntax for specifying structures.

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```
def Cat : GAT := {
```

```
def Cat : GAT := {  
  Obj : U,
```

```
def Cat : GAT := {  
    Obj : U,  
    Hom : Obj ⇒ Obj ⇒ U,
```

```
def Cat : GAT := {
  Obj  : U,
  Hom  : Obj ⇒ Obj ⇒ U,
  id   : (I : Obj) ⇒ Hom I I,
  comp : {I J K : Obj} ⇒
            Hom J K ⇒ Hom I J ⇒ Hom I K,
```

## Categories

$\text{lunit} : \{\text{I J : Obj}\} \Rightarrow (\text{j : Hom I J}) \Rightarrow$   
 $\text{comp} (\text{id J}) \text{j} \equiv \text{j},$

$\text{runit} : \{\text{I J : Obj}\} \Rightarrow (\text{j : Hom I J}) \Rightarrow$   
 $\text{comp j} (\text{id I}) \equiv \text{j},$

$\text{assoc} : \{\text{I J K L : Obj}\} \Rightarrow (\text{j : Hom I J}) \Rightarrow$   
 $(\text{k : Hom J K}) \Rightarrow (\ell : \text{Hom K L}) \Rightarrow$   
 $\text{comp} \ell (\text{comp k j}) \equiv \text{comp} (\text{comp} \ell \text{k}) \text{j}$

}

whenever new abstract objects are constructed in a specified way out of given ones, it is advisable to regard the construction of the corresponding induced mappings on these new objects as an integral part of their definition

—[EM45, 236]

**Generalized algebraic theories** [Car86] are a syntax for specifying structures. Every GAT  $\mathfrak{G}$  gives rise to a notion of structure,  $\mathfrak{G}$ -**algebras**.

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### Central Dogma of Category Theory

Every notion of “structure” comes equipped with a notion of “structure-preserving morphism”, i.e. forms a category

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### Central Dogma of Category Theory

Every notion of “structure” comes equipped with a notion of “structure-preserving morphism”, i.e. forms a category

# *GAT signature*

# *Category of Algs.*

# *Concrete CwF*

Categories

CwFs

GAT signatures

# GAT signature

---

---

Categories

Cat

# GAT signature

## Category of Algs.

Categories

Cat

Cat

# GAT signature

## Category of Algs.

Categories



Cat

Cat

# Categories with Families

```
def Cwf : GAT := {
  include Cat as (Con,Sub,comp,id,_,_,_);
  empty   : Con,
   $\epsilon$        : ( $\Gamma$  : Con)  $\Rightarrow$  Sub  $\Gamma$  empty,
   $\epsilon_\eta$    : ( $\Gamma$  : Con)  $\Rightarrow$  (f : Sub  $\Gamma$  empty)  $\Rightarrow$ 
  f  $\equiv$  ( $\epsilon$   $\Gamma$ ),
```

# Categories with Families

$\text{Ty} : \text{Con} \Rightarrow \mathbf{U}$ ,

# Categories with Families

$\text{Ty} : \text{Con} \Rightarrow \mathbf{U}$ ,  
 $\text{substTy} : \{\Delta \vdash \Gamma : \text{Con}\} \Rightarrow$   
 $\text{Sub } \Delta \vdash \Gamma \Rightarrow \text{Ty } \Gamma \Rightarrow \text{Ty } \Delta$ ,

# Categories with Families

$\text{Ty} : \text{Con} \Rightarrow \mathbf{U}$ ,

$\text{substTy} : \{\Delta \Gamma : \text{Con}\} \Rightarrow$

$\text{Sub } \Delta \Gamma \Rightarrow \text{Ty } \Gamma \Rightarrow \text{Ty } \Delta$ ,

$\text{idTy} : \{\Gamma : \text{Con}\} \Rightarrow (A : \text{Ty } \Gamma) \Rightarrow$

$\text{substTy } (\text{id } \Gamma) A \equiv A$ ,

$\text{compTy} : \{\Theta \Delta \Gamma : \text{Con}\} \Rightarrow (A : \text{Ty } \Gamma)$

$(\delta : \text{Sub } \Theta \Delta) \Rightarrow (\gamma : \text{Sub } \Delta \Gamma) \Rightarrow$

$\text{substTy } \gamma (\text{substTy } \delta A)$

$\equiv \text{substTy } (\text{comp } \gamma \delta) A$ ,

# Categories with Families

$\text{Tm} : (\Gamma : \text{Con}) \Rightarrow \text{Ty } \Gamma \Rightarrow \textcolor{brown}{U},$

# Categories with Families

$\text{Tm} : (\Gamma : \text{Con}) \Rightarrow \text{Ty } \Gamma \Rightarrow \mathbf{U}$ ,  
 $\text{substTm} : \{\Delta \mid \Gamma : \text{Con}\} \Rightarrow \{A : \text{Ty } \Gamma\} \Rightarrow$   
 $(\gamma : \text{Sub } \Delta \mid \Gamma) \Rightarrow$   
 $\text{Tm } \Gamma \mid A \Rightarrow \text{Tm } \Delta \mid (\text{substTy } \gamma \mid A)$ ,

# Categories with Families

$$\begin{aligned} \text{idTm} &: \{\Gamma : \text{Con}\} \Rightarrow \{A : \text{Ty } \Gamma\} \Rightarrow (t : \text{Tm } \Gamma A) \\ &\text{substTm } (\text{id } \Gamma) t \quad \# \langle \text{idTy } A \rangle \\ &\equiv t, \\ \text{compTm} &: \{\Theta \Delta \Gamma : \text{Con}\} \Rightarrow \\ &\{A : \text{Ty } \Gamma\} \Rightarrow (t : \text{Tm } \Gamma A) \Rightarrow \\ &(\delta : \text{Sub } \Theta \Delta) \Rightarrow (\gamma : \text{Sub } \Delta \Gamma) \Rightarrow \\ &\text{substTm } \gamma (\text{substTm } \delta t) \\ &\# \langle \text{compTy } A \gamma \delta \rangle \\ &\equiv \text{substTm } (\text{comp } \gamma \delta) t, \end{aligned}$$

# Categories with Families

`ext` :  $(\Gamma : \text{Con}) \Rightarrow \text{Ty } \Gamma \Rightarrow \text{Con}$ ,  
`pair` :  $\{\Delta \mid \Gamma : \text{Con}\} \Rightarrow \{A : \text{Ty } \Gamma\} \Rightarrow$   
 $(\gamma : \text{Sub } \Delta \mid \Gamma) \Rightarrow$   
 $\text{Tm } \Delta \mid (\text{substTy } \gamma \ A) \Rightarrow$   
 $\text{Sub } \Delta \mid (\text{ext } \Gamma \ A)$ ,

## Categories with Families

$$\begin{aligned} \text{pair\_nat}: & \{\Theta \Delta \Gamma : \text{Con}\} \Rightarrow \{A : \text{Ty } \Gamma\} \Rightarrow \\ & (\gamma : \text{Sub } \Delta \Gamma) \Rightarrow \\ & (t : \text{Tm } \Delta (\text{substTy } \gamma A)) \Rightarrow \\ & (\delta : \text{Sub } \Theta \Delta) \Rightarrow \\ & \text{comp}(\text{pair } \gamma t) \delta \\ & \equiv \text{pair}(\text{comp } \gamma \delta) \\ & (\text{substTm } \delta t \# \langle \text{compTy } A \gamma \delta \rangle), \end{aligned}$$

# Categories with Families

$$\begin{aligned} p &: \{\Gamma : \text{Con}\} \Rightarrow (A : \text{Ty } \Gamma) \Rightarrow \\ &\quad \text{Sub}(\text{ext } \Gamma A) \Gamma \\ v &: \{\Gamma : \text{Con}\} \Rightarrow (A : \text{Ty } \Gamma) \Rightarrow \\ &\quad \text{Tm}(\text{ext } \Gamma A) (\text{substTy}(p A) A), \end{aligned}$$

# Categories with Families

$\text{ext\_}\beta : (\Delta \Gamma : \text{Con}) \Rightarrow (A : \text{Ty } \Gamma) \Rightarrow$   
 $(\gamma : \text{Sub } \Delta \Gamma) \Rightarrow$   
 $(t : \text{Tm } \Delta (\text{substTy } \gamma A)) \Rightarrow$   
 $\text{comp } (\text{p } A) (\text{pair } \gamma t) \equiv \gamma,$   
 $\text{ext\_}\beta : (\Delta \Gamma : \text{Con}) \Rightarrow (A : \text{Ty } \Gamma) \Rightarrow$   
 $(\gamma : \text{Sub } \Delta \Gamma) \Rightarrow$   
 $(t : \text{Tm } \Delta (\text{substTy } \gamma A)) \Rightarrow$   
 $\text{substTm } (\text{pair } \gamma t) (v A)$   
 $\#\langle \text{compTy } A (\text{p } A) (\text{pair } \gamma t) \rangle$   
 $\#\langle \text{ext\_}\beta \gamma t \rangle$   
 $\equiv t,$   
 $\text{ext\_}\eta : (\Gamma : \text{Con}) \Rightarrow (A : \text{Ty } \Gamma) \Rightarrow$   
 $\text{pair } (\text{p } A) (v A)$   
 $\equiv \text{id } (\text{ext } \Gamma A)$

# GAT signature

## Category of Algs.

Categories

Cat

Cat

# GAT signature

## Category of Algs.

Categories

CwFs

$\mathfrak{Cat}$

$\mathfrak{CwF}$

$\mathbf{Cat}$

*GAT signature*

*Category of Algs.*

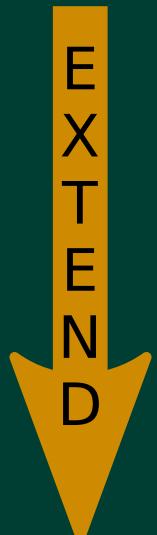
Categories	$\mathfrak{Cat}$	Cat
CwFs	$\mathfrak{CwF}$	CwF

# GAT signature

## Category of Algs.

Categories

CwFs



Cat

CwF

Cat

CwF

```
def ℰwF++ : GAT := {  
  include ℰwF;
```

```
def ℂWF++ : GAT := {
  include ℂWF;
  V : {Γ : Con} ⇒ Ty Γ,
  El : {Γ : Con} ⇒ Tm Γ V ⇒ Ty Γ,
```

```

def  $\mathfrak{EF}++$  : GAT := {
  include  $\mathfrak{EF}$ ;
  V :  $\{\Gamma : \text{Con}\} \Rightarrow \text{Ty } \Gamma,$ 
  El :  $\{\Gamma : \text{Con}\} \Rightarrow \text{Tm } \Gamma \ V \Rightarrow \text{Ty } \Gamma,$ 
  ...
  Eq :  $\{\Gamma : \text{Con}\} \Rightarrow \{A : \text{Ty } \Gamma\} \Rightarrow$ 
     $(t \ t' : \text{Tm } \Gamma \ A) \Rightarrow \text{Ty } \Gamma,$ 
  reflect :  $\{\Gamma : \text{Con}\} \Rightarrow \{A : \text{Ty } \Gamma\} \Rightarrow$ 
     $\{t \ t' : \text{Tm } \Gamma \ A\} \Rightarrow \text{Tm } \Gamma \ (\text{Eq } t \ t') \Rightarrow$ 
     $t \equiv t',$ 
}

```

```

def  $\mathfrak{CwF}^{++}$  : GAT := {
  include  $\mathfrak{CwF}$ ;
  V :  $\{\Gamma : \text{Con}\} \Rightarrow \text{Ty } \Gamma$ ,
  El :  $\{\Gamma : \text{Con}\} \Rightarrow \text{Tm } \Gamma \ V \Rightarrow \text{Ty } \Gamma$ ,
  ...
  Eq :  $\{\Gamma : \text{Con}\} \Rightarrow \{A : \text{Ty } \Gamma\} \Rightarrow$ 
     $(t \ t' : \text{Tm } \Gamma \ A) \Rightarrow \text{Ty } \Gamma$ ,
  ...
}

```

```

def CwF++ : GAT := [
  include CwF;
  V : {Γ : Con} ⇒ Ty Γ,
  El : {Γ : Con} ⇒ Tm Γ V ⇒ Ty Γ,
  ...
  Eq : {Γ : Con} ⇒ {A : Ty Γ} ⇒
    (t t' : Tm Γ A) ⇒ Ty Γ,
  ...
  Pi : {Γ : Con} ⇒ (X : Tm Γ V) ⇒
    Ty (ext Γ (El X)) ⇒ Ty Γ,
]

```

**Why this specific  
GAT?**



# The GAT Signature Language

**Key Idea** An *initial object* in the category of  $\mathfrak{G}$ -algebras represents the *pure syntax* of  $\mathfrak{G}$ , only the “stuff” which can be derived from the mere concept of  $\mathfrak{G}$ .

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- The initial monoid consists only of the identity element

**Key Idea** An *initial object* in the category of  $\mathfrak{G}$ -algebras represents the *pure syntax* of  $\mathfrak{G}$ , only the “stuff” which can be derived from the mere concept of  $\mathfrak{G}$ . E.g.

- The initial monoid consists only of the identity element
- The initial CwF+(constructors) is the syntax of a type theory with (constructors).

# *GAT signature*

## Category of Algs.

Categories

CwFs

$\mathfrak{Cat}$

$\mathfrak{CwF}$

$\text{Cat}$

$\text{CwF}$

*GAT signature*      *Category of Algs.*      *Initial Algebra*

---

---

Categories	$\mathfrak{Cat}$	$\text{Cat}$
$\text{CwFs}$	$\mathfrak{CwF}$	$\text{CwF}$

*GAT signature*      *Category of Algs.*      *Initial Algebra*

---

---

Categories	$\mathfrak{Cat}$	Cat	$\emptyset$
CwFs	$\mathfrak{CwF}$	CwF	

*GAT signature*      *Category of Algs.*      *Initial Algebra*

---

---

Categories

$\mathfrak{Cat}$       Cat       $\emptyset$

CwFs

$\mathfrak{CwF}$       CwF       $\{\diamond\}, K\emptyset$

*GAT signature*      *Category of Algs.*      *Initial Algebra*

---

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Categories	$\mathfrak{Cat}$	Cat	$\emptyset$
CwFs	$\mathfrak{CwF}$	CwF	$\{\diamond\}, K\emptyset$
CwF+V+Eq+Π	$\mathfrak{CwF}^{++}$		

*GAT signature*      *Category of Algs.*      *Initial Algebra*

---

---

Categories	$\mathfrak{Cat}$	Cat	$\emptyset$
CwFs	$\mathfrak{CwF}$	CwF	$\{\diamond\}, K\emptyset$
GAT signatures	$\mathfrak{CwF}^{++}$		

*GAT signature*      *Category of Algs.*      *Initial Algebra*

---

---

Categories	$\mathfrak{Cat}$	Cat	$\emptyset$
CwFs	$\mathfrak{CwF}$	CwF	$\{\diamond\}, K\emptyset$
GAT signatures	$\mathfrak{CwF}^{++}$	...	ONEGAT

# The oneGAT language

We assume that the GAT  $\mathfrak{CwF} + \mathbf{U} + \mathbf{Eq} + \Pi_{sd}$  has an initial algebra, given as a QIIT in the metatheory.

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$$\overline{\mathbf{Con}} : \mathbf{Set}$$

$$\overline{\mathbf{Sub}} : \overline{\mathbf{Con}} \rightarrow \overline{\mathbf{Con}} \rightarrow \mathbf{Set}$$

$$\overline{\mathbf{Ty}} : \overline{\mathbf{Con}} \rightarrow \mathbf{Set}$$

$$\overline{\mathbf{Tm}} : (\mathfrak{G} : \overline{\mathbf{Con}}) \rightarrow \overline{\mathbf{Ty}} \mathfrak{G}$$

# The oneGAT language

We assume that the GAT  $\mathfrak{CwF} + \mathbf{U} + \mathbf{Eq} + \Pi_{sd}$  has an initial algebra, given as a QIIT in the metatheory.

$$\begin{aligned}\text{GAT} := & \overline{\text{Con}} : \text{Set} \\ & \overline{\text{Sub}} : \overline{\text{Con}} \rightarrow \overline{\text{Con}} \rightarrow \text{Set} \\ & \overline{\text{Ty}} : \overline{\text{Con}} \rightarrow \text{Set} \\ & \overline{\text{Tm}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow \overline{\text{Ty}} \mathfrak{G}\end{aligned}$$

# The oneGAT language

We assume that the GAT  $\mathfrak{CwF} + \mathbf{U} + \mathbf{Eq} + \Pi_{sd}$  has an initial algebra, given as a QIIT in the metatheory.

$$\begin{aligned}\text{GAT} &:= \overline{\text{Con}} : \text{Set} \\ \overline{\text{Sub}} &: \overline{\text{Con}} \rightarrow \overline{\text{Con}} \rightarrow \text{Set} \\ \overline{\text{Ty}} &: \overline{\text{Con}} \rightarrow \text{Set} \\ \overline{\text{Tm}} &: (\mathfrak{G} : \overline{\text{Con}}) \rightarrow \overline{\text{Ty}} \mathfrak{G}\end{aligned}$$

Use the DSL parsing utilities in LEAN4 to translate the intuitive syntax for GATs into ONEGAT.

```
def  $\mathfrak{N}$  : GAT := {  
    Nat : U,  
    zero : Nat,  
    succ : Nat  $\Rightarrow$  Nat  
}
```

# Nat algebras

```
def N : GAT := {  
    Nat : U,  
    zero : Nat,  
    succ : Nat  $\Rightarrow$  Nat  
}  
    ◇  
    ▷ V  
    ▷ El 0  
    ▷  $\Pi 1 (El 2)$ 
```

# Categories

```
def Cat : GAT := {
  Obj  : U,
  Hom  : Obj  $\Rightarrow$  Obj  $\Rightarrow$  U,
  id   : (I : Obj)  $\Rightarrow$  Hom I I,
  comp : {I J K : Obj}  $\Rightarrow$ 
    Hom J K  $\Rightarrow$  Hom I J  $\Rightarrow$  Hom I K,
  lunit : {I J : Obj}  $\Rightarrow$  (j : Hom I J)  $\Rightarrow$ 
    comp (id J) j  $\equiv$  j,
  runit : {I J : Obj}  $\Rightarrow$  (j : Hom I J)  $\Rightarrow$ 
    comp j (id I)  $\equiv$  j,
  assoc : {I J K L : Obj}  $\Rightarrow$ 
    (j : Hom I J)  $\Rightarrow$  (k : Hom J K)  $\Rightarrow$ 
    ( $\ell$  : Hom K L)  $\Rightarrow$ 
    comp  $\ell$  (comp k j)
     $\equiv$  comp (comp  $\ell$  k) j
}
```

# Categories

```
def Cat : GAT := {
  Obj  : U,
  Hom  : Obj  $\Rightarrow$  Obj  $\Rightarrow$  U,
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  comp : {I J K : Obj}  $\Rightarrow$ 
    Hom J K  $\Rightarrow$  Hom I J  $\Rightarrow$  Hom I K,
  lunit : {I J : Obj}  $\Rightarrow$  (j : Hom I J)  $\Rightarrow$ 
    comp (id J) j  $\equiv$  j,
  runit : {I J : Obj}  $\Rightarrow$  (j : Hom I J)  $\Rightarrow$ 
    comp j (id I)  $\equiv$  j,
  assoc : {I J K L : Obj}  $\Rightarrow$ 
    (j : Hom I J)  $\Rightarrow$  (k : Hom J K)  $\Rightarrow$ 
    ( $\ell$  : Hom K L)  $\Rightarrow$ 
    comp  $\ell$  (comp k j)
     $\equiv$  comp (comp  $\ell$  k) j
}
```

◇  
▷ V  
▷  $\Pi 0 (\Pi 1 V)$   
▷  $\Pi 1 (\mathrm{El}(1 @ 0 @ 0))$   
▷  $\Pi 2 (\Pi 3 (\Pi 4 (\Pi (4 @ 1 @ 0) (\Pi (5 @ 3 @ 2) (\mathrm{El}(6 @ 4 @ 2))))))$   
▷  $\Pi 3 (\Pi 4 (\Pi (4 @ 1 @ 0) (\mathrm{Eq}(3 @ 2 @ 1 @ 1 @ (4 @ 1) @ 0))))$   
▷  $\Pi 4 (\Pi 5 (\Pi (5 @ 1 @ 0) (\mathrm{Eq}(4 @ 2 @ 2 @ 1 @ 0 @ (5 @ 2)) 0)))$   
▷  $\Pi 5 (\Pi 6 (\Pi 7 (\Pi 8 (\Pi (8 @ 3 @ 2) (\Pi (9 @ 3 @ 2) (\Pi (10 @ 3 @ 2) (\mathrm{Eq}(9 @ 6 @ 5 @ 3 @ 0 @ (9 @ 6 @ 5 @ 4 @ 1 @ 2)) (9 @ 6 @ 4 @ 3 @ (9 @ 5 @ 4 @ 3 @ 0 @ 1) @ 2))))))))$

## Cat

- ▷ El 6
- ▷ Π 7 (El (7 @ 0 @ 1))
- ▷ Π 8 (Π (8 @ 0 @ 2) (Eq 0 (2 @ 1)))
- ▷ Π 9 V
- ▷ Π 10 (Π 11 (Π (11 @ 1 @ 0) (Π (3 @ 1) (El (4 @ 3)))))
- ▷ Π 11 (Π (2 @ 0) (Eq (2 @ 1 @ 1 @ (11 @ 1) @ 0) 0))
- ▷ Π 12 (Π 13 (Π 14 (Π (5 @ 0) (Π (15 @ 3 @ 2) (Π (16 @ 3 @ 2) (Eq (7 @ 4 @ 3 @ 0 @ (7 @ 5 @ 4 @ 1 @ 2)) (7 @ 5 @ 3 @ (15 @ 5 @ 4 @ 3 @ 0 @ 1) @ 2)))))))
- ▷ Π 13 (Π (4 @ 0) V)
- ▷ Π 14 (Π 15 (Π (6 @ 0) (Π (16 @ 2 @ 1) (Π (4 @ 2 @ 1) (El (5 @ 4 @ (8 @ 4 @ 3 @ 1 @ 2)))))))
- ▷ Π 15 (Π (6 @ 0) (Π (3 @ 1 @ 0) (Eq (3 @ 2 @ 2 @ 1 @ (16 @ 2) @ 0) 0)))
- ▷ Π 16 (Π 17 (Π 18 (Π (9 @ 0) (Π (6 @ 1 @ 0) (Π (20 @ 4 @ 3) (Π (21 @ 4 @ 3) (Eq (8 @ 5 @ 4 @ 3 @ 0 @ (8 @ 6 @ 5 @ (12 @ 5 @ 4 @ 0 @ 3) @ 1 @ 2)) (8 @ 6 @ 4 @ 3 @ (20 @ 6 @ 5 @ 4 @ 0 @ 1) @ 2)))))))

- ▷  $\Pi 17 (\Pi (8 @ 0) (\text{El } 19))$
- ▷  $\Pi 18 (\Pi 19 (\Pi (10 @ 0) (\Pi (20 @ 2 @ 1) (\Pi (8 @ 3 @ (11 @ 3 @ 2 @ 0 @ 1)) (\text{El } (22 @ 4 @ (5 @ 3 @ 2)))))))$
- ▷  $\Pi 19 (\Pi 20 (\Pi 21 (\Pi (12 @ 0) (\Pi (22 @ 2 @ 1) (\Pi (10 @ 3 @ (13 @ 3 @ 2 @ 0 @ 1)) (\Pi (24 @ 5 @ 4) (\text{Eq } (23 @ 6 @ 5 @ (8 @ 4 @ 3) @ (7 @ 5 @ 4 @ 3 @ 2 @ 1) @ 0) (7 @ 6 @ 4 @ 3 @ (23 @ 6 @ 5 @ 4 @ 2 @ 0) @ (11 @ 6 @ 5 @ (15 @ 5 @ 4 @ 2 @ 3) @ 0 @ 1))))))))$
- ▷  $\Pi 20 (\Pi 21 (\Pi (12 @ 0) (\Pi (22 @ 2 @ (5 @ 1 @ 0)) (\text{El } (23 @ 3 @ 2))))))$
- ▷  $\Pi 21 (\Pi 22 (\Pi (13 @ 0) (\Pi (23 @ 2 @ (6 @ 1 @ 0)) (\text{El } (11 @ 3 @ (14 @ 3 @ 2 @ (4 @ 3 @ 2 @ 1 @ 0) @ 1))))))$
- ▷  $\Pi 22 (\Pi 23 (\Pi (14 @ 0) (\Pi (24 @ 2 @ 1) (\Pi (12 @ 3 @ (15 @ 3 @ 2 @ 0 @ 1)) (\text{Eq } (6 @ 4 @ 3 @ 2 @ (8 @ 4 @ 3 @ 2 @ 1 @ 0)) 1))))))$
- ▷  $\Pi 23 (\Pi 24 (\Pi (15 @ 0) (\Pi (25 @ 2 @ 1) (\Pi (13 @ 3 @ (16 @ 3 @ 2 @ 0 @ 1)) (\text{Eq } (6 @ 4 @ 3 @ 2 @ (9 @ 4 @ 3 @ 2 @ 1 @ 0)) 0))))))$
- ▷  $\Pi 24 (\Pi 25 (\Pi (16 @ 0) (\Pi (26 @ 2 @ (9 @ 1 @ 0)) (\text{Eq } (9 @ 3 @ 2 @ 1 @ (7 @ 3 @ 2 @ 1 @ 0) @ (6 @ 3 @ 2 @ 1 @ 0)) 0))))))$

To the logical  
**extreme: oneGAT**  
describes itself

# Quotient Inductive-Inductive definitions over all GATs

The benefit of quotient inductive-inductive types is that we can reason about them by quotient induction induction.

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Given an appropriate *motive*  $M_{\overline{\text{Con}}}$ ,  $M_{\overline{\text{Sub}}}$ ,  $M_{\overline{\text{Ty}}}$ ,  $M_{\overline{\text{Tm}}}$  and appropriate *method*  $m$ , we get:

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$$(\text{elim } m)_{\overline{\text{Con}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow M_{\overline{\text{Con}}}(\mathfrak{G})$$

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$$\begin{aligned} (\text{elim } m)_{\overline{\text{Con}}} : (\mathfrak{G} : \overline{\text{Con}}) &\rightarrow M_{\overline{\text{Con}}}(\mathfrak{G}) \\ (\text{elim } m)_{\overline{\text{Sub}}} : (\mathfrak{G} \mathfrak{H} : \overline{\text{Con}}) &\rightarrow (\mathfrak{s} : \overline{\text{Sub}} \mathfrak{G} \mathfrak{H}) \rightarrow \\ &M_{\overline{\text{Sub}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, (\text{elim } m)_{\overline{\text{Con}}} \mathfrak{H}, \mathfrak{s}) \end{aligned}$$

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Given an appropriate *motive*  $M_{\overline{\text{Con}}}$ ,  $M_{\overline{\text{Sub}}}$ ,  $M_{\overline{\text{Ty}}}$ ,  $M_{\overline{\text{Tm}}}$  and appropriate *method*  $m$ , we get:

$$(\text{elim } m)_{\overline{\text{Con}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow M_{\overline{\text{Con}}}(\mathfrak{G})$$

$$\begin{aligned} (\text{elim } m)_{\overline{\text{Sub}}} : (\mathfrak{G} \mathfrak{H} : \overline{\text{Con}}) \rightarrow (\mathfrak{s} : \overline{\text{Sub}} \mathfrak{G} \mathfrak{H}) \rightarrow \\ M_{\overline{\text{Sub}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, (\text{elim } m)_{\overline{\text{Con}}} \mathfrak{H}, \mathfrak{s}) \end{aligned}$$

$$(\text{elim } m)_{\overline{\text{Ty}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow (\mathcal{X} : \overline{\text{Ty}} \mathfrak{G}) \rightarrow M_{\overline{\text{Ty}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, \mathcal{X})$$

# Quotient Inductive-Inductive definitions over all GATs

The benefit of quotient inductive-inductive types is that we can reason about them by quotient induction induction.

Given an appropriate *motive*  $M_{\overline{\text{Con}}}$ ,  $M_{\overline{\text{Sub}}}$ ,  $M_{\overline{\text{Ty}}}$ ,  $M_{\overline{\text{Tm}}}$  and appropriate *method*  $m$ , we get:

$$(\text{elim } m)_{\overline{\text{Con}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow M_{\overline{\text{Con}}}(\mathfrak{G})$$

$$\begin{aligned} (\text{elim } m)_{\overline{\text{Sub}}} : (\mathfrak{G} \mathfrak{H} : \overline{\text{Con}}) \rightarrow (\mathfrak{s} : \overline{\text{Sub}} \mathfrak{G} \mathfrak{H}) \rightarrow \\ M_{\overline{\text{Sub}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, (\text{elim } m)_{\overline{\text{Con}}} \mathfrak{H}, \mathfrak{s}) \end{aligned}$$

$$(\text{elim } m)_{\overline{\text{Ty}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow (\mathcal{X} : \overline{\text{Ty}} \mathfrak{G}) \rightarrow M_{\overline{\text{Ty}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, \mathcal{X})$$

$$(\text{elim } m)_{\overline{\text{Tm}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow (\mathcal{X} : \overline{\text{Ty}} \mathfrak{G}) \rightarrow (\mathcal{X} : \overline{\text{Tm}}(\mathfrak{G}, \mathcal{X}))$$

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$$(\text{elim } m)_{\overline{\text{Con}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow M_{\overline{\text{Con}}}(\mathfrak{G})$$

$$\begin{aligned} (\text{elim } m)_{\overline{\text{Sub}}} : (\mathfrak{G} \mathfrak{H} : \overline{\text{Con}}) \rightarrow (\mathfrak{s} : \overline{\text{Sub}} \mathfrak{G} \mathfrak{H}) \rightarrow \\ M_{\overline{\text{Sub}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, (\text{elim } m)_{\overline{\text{Con}}} \mathfrak{H}, \mathfrak{s}) \end{aligned}$$

$$(\text{elim } m)_{\overline{\text{Ty}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow (\mathcal{X} : \overline{\text{Ty}} \mathfrak{G}) \rightarrow M_{\overline{\text{Ty}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, \mathcal{X})$$

$$\begin{aligned} (\text{elim } m)_{\overline{\text{Tm}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow (\mathcal{X} : \overline{\text{Ty}} \mathfrak{G}) \rightarrow (X : \overline{\text{Tm}}(\mathfrak{G}, \mathcal{X})) \\ \rightarrow M_{\overline{\text{Tm}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, (\text{elim } m)_{\overline{\text{Ty}}} \mathcal{X}, X) \end{aligned}$$

# Quotient Inductive-Inductive definitions over all GATs

The benefit of quotient inductive-inductive types is that we can reason about them by quotient induction induction.

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$$(\text{elim } m)_{\overline{\text{Con}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow M_{\overline{\text{Con}}}(\mathfrak{G})$$

$$\begin{aligned} (\text{elim } m)_{\overline{\text{Sub}}} : (\mathfrak{G} \mathfrak{H} : \overline{\text{Con}}) \rightarrow (\mathfrak{s} : \overline{\text{Sub}} \mathfrak{G} \mathfrak{H}) \rightarrow \\ M_{\overline{\text{Sub}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, (\text{elim } m)_{\overline{\text{Con}}} \mathfrak{H}, \mathfrak{s}) \end{aligned}$$

$$(\text{elim } m)_{\overline{\text{Ty}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow (\mathcal{X} : \overline{\text{Ty}} \mathfrak{G}) \rightarrow M_{\overline{\text{Ty}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, \mathcal{X})$$

$$\begin{aligned} (\text{elim } m)_{\overline{\text{Tm}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow (\mathcal{X} : \overline{\text{Ty}} \mathfrak{G}) \rightarrow (X : \overline{\text{Tm}}(\mathfrak{G}, \mathcal{X})) \\ \rightarrow M_{\overline{\text{Tm}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, (\text{elim } m)_{\overline{\text{Ty}}} \mathcal{X}, X) \end{aligned}$$

# Quotient Inductive-Inductive definitions over all GATs

The benefit of quotient inductive-inductive types is that we can reason about them by quotient induction induction.

Given an appropriate *motive*  $M_{\overline{\text{Con}}}$ ,  $M_{\overline{\text{Sub}}}$ ,  $M_{\overline{\text{Ty}}}$ ,  $M_{\overline{\text{Tm}}}$  and appropriate *method*  $m$ , we get:

$$(\text{elim } m)_{\overline{\text{Con}}} : (\mathfrak{G} : \text{GAT}) \rightarrow M_{\overline{\text{Con}}}(\mathfrak{G})$$

$$\begin{aligned} (\text{elim } m)_{\overline{\text{Sub}}} : (\mathfrak{G} \mathfrak{H} : \overline{\text{Con}}) \rightarrow (\mathfrak{s} : \overline{\text{Sub}} \mathfrak{G} \mathfrak{H}) \rightarrow \\ M_{\overline{\text{Sub}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, (\text{elim } m)_{\overline{\text{Con}}} \mathfrak{H}, \mathfrak{s}) \end{aligned}$$

$$(\text{elim } m)_{\overline{\text{Ty}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow (\mathcal{X} : \overline{\text{Ty}} \mathfrak{G}) \rightarrow M_{\overline{\text{Ty}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, \mathcal{X})$$

$$\begin{aligned} (\text{elim } m)_{\overline{\text{Tm}}} : (\mathfrak{G} : \overline{\text{Con}}) \rightarrow (\mathcal{X} : \overline{\text{Ty}} \mathfrak{G}) \rightarrow (X : \overline{\text{Tm}}(\mathfrak{G}, \mathcal{X})) \\ \rightarrow M_{\overline{\text{Tm}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, (\text{elim } m)_{\overline{\text{Ty}}} \mathcal{X}, X) \end{aligned}$$

$\_$ -Alg:  $(\mathfrak{G} : \text{GAT}) \rightarrow \text{Set}$

# From [KKA19, Appendix A]

Syntax	Algebras
$\Gamma : \text{Con}$	$\Gamma^A : \text{Set}$
$A : \text{Ty } \Gamma$	$A^A : \Gamma^A \rightarrow \text{Set}$
$\sigma : \text{Sub } \Gamma \Delta$	$\sigma^A : \Gamma^A \rightarrow \Delta^A$
$t : \text{Tm } \Gamma A$	$t^A : (\gamma : \Gamma^A) \rightarrow A^A \gamma$
$\cdot : \text{Con}$	$\cdot^A : \equiv \top$
$\Gamma \triangleright A : \text{Con}$	$(\Gamma \triangleright A)^A : \equiv (\gamma : \Gamma^A) \times A^A \gamma$
$(A : \text{Ty } \Delta)[\sigma : \text{Sub } \Gamma \Delta] : \text{Ty } \Gamma$	$(A[\sigma])^A \gamma : \equiv A^A (\sigma^A \gamma)$
$\text{id} : \text{Sub } \Gamma \Gamma$	$\text{id}^A \gamma : \equiv \gamma$
$(\sigma : \text{Sub } \Theta \Delta) \circ (\delta : \text{Sub } \Gamma \Theta) : \text{Sub } \Gamma \Delta$	$(\sigma \circ \delta)^A \gamma : \equiv \sigma^A (\delta^A \gamma)$
$\epsilon : \text{Sub } \Gamma \cdot$	$\epsilon^A \gamma : \equiv \text{tt}$
$(\sigma : \text{Sub } \Gamma \Delta), (t : \text{Tm } \Gamma (A[\sigma])) : \text{Sub } \Gamma (\Delta \triangleright A)$	$(\sigma, t)^A \gamma : \equiv (\sigma^A \gamma, t^A \gamma)$
$\pi_1 (\sigma : \text{Sub } \Gamma (\Delta \triangleright A)) : \text{Sub } \Gamma \Delta$	$(\pi_1 \sigma)^A \gamma : \equiv \text{proj}_1 (\sigma^A \gamma)$
$\pi_2 (\sigma : \text{Sub } \Gamma (\Delta \triangleright A)) : \text{Tm } \Gamma (A[\pi_1 \sigma])$	$(\pi_2 \sigma)^A \gamma : \equiv \text{proj}_2 (\sigma^A \gamma)$
$(t : \text{Tm } \Delta A)[\sigma : \text{Sub } \Gamma \Delta] : \text{Tm } \Gamma (A[\sigma])$	$(t[\sigma])^A \gamma : \equiv t^A (\sigma^A \gamma)$
$[\text{id}] : A[\text{id}] = A$	$[\text{id}]^A : \equiv \text{refl}$
$[\circ] : A[\sigma \circ \delta] = A[\sigma][\delta]$	$[\circ]^A : \equiv \text{refl}$
$\text{ass} : (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$	$\text{ass}^A : \equiv \text{refl}$
$\text{idl} : \text{id} \circ \sigma = \sigma$	$\text{idl}^A : \equiv \text{refl}$
$\text{idr} : \sigma \circ \text{id} = \sigma$	$\text{idr}^A : \equiv \text{refl}$
$\cdot \eta : \{\sigma : \text{Sub } \Gamma \cdot\} \rightarrow \sigma = \epsilon$	$\cdot \eta^A : \equiv \text{refl}$
$\triangleright \beta_1 : \pi_1 (\sigma, t) = \sigma$	$\triangleright \beta_1^A : \equiv \text{refl}$
$\triangleright \beta_2 : \pi_2 (\sigma, t) = t$	$\triangleright \beta_2^A : \equiv \text{refl}$
	$\triangleright \eta : (\pi_1 \sigma, \pi_2 \sigma) = \sigma$
	$, \circ : (\sigma, t) \circ \delta = (\sigma \circ \delta, t[\delta])$
	$\text{U} : \text{Ty } \Gamma$
	$\text{El} (a : \text{Tm } \Gamma \text{U}) : \text{Ty } \Gamma$
	$\text{U}[] : \text{U}[\sigma] = \text{U}$
	$\text{El}[] : (\text{El } a)[\sigma] = \text{El}(a[\sigma])$
	$\Pi (a : \text{Tm } \Gamma \text{U})(B : \text{Ty } (\Gamma \triangleright \text{El } a)) : \text{Ty } \Gamma$
	$(\Pi a B)^A \gamma : \equiv (\alpha : a^A \gamma) \rightarrow B^A (\gamma, \alpha)$
	$\text{app} (t : \text{Tm } \Gamma (\Pi a B)) : \text{Tm } (\Gamma \triangleright \text{El } a) B$
	$\Pi[] : (\Pi a B)[\sigma] = \Pi(a[\sigma])(B[\sigma^\uparrow])$
	$\text{app}[] : (\text{app } t)[\sigma^\uparrow] = \text{app}(t[\sigma])$
	$\text{Id} (a : \text{Tm } \Gamma \text{U}) (t u : \text{Tm } \Gamma (\text{El } a)) : \text{Ty } \Gamma$
	$(\text{Id } a t u)^A \gamma : \equiv (t^A \gamma = u^A \gamma)$
	$\text{reflect} (e : \text{Tm } \Gamma (\text{Id } a t u)) : t = u$
	$(\text{reflect } e)^A : \equiv \text{funext } e^A$
	$\text{Id}[] : (\text{Id } a t u)[\sigma] = \text{Id}(a[\sigma])(t[\sigma])(u[\sigma])$
	$\text{Id}[]^A : \equiv \text{refl}$
	$\hat{\Pi} (T : \text{Set}) (B : T \rightarrow \text{Ty } \Gamma) : \text{Ty } \Gamma$
	$(\hat{\Pi} T B)^A \gamma : \equiv (\alpha : T) \rightarrow (B \alpha)^A \gamma$
	$(t : \text{Tm } \Gamma (\hat{\Pi} T B)) \hat{\@} (\alpha : T) : \text{Tm } \Gamma (B \alpha)$
	$(t \hat{\@} \alpha)^A \gamma : \equiv t^A \gamma \alpha$
	$\hat{\Pi}[] : (\hat{\Pi} T B)[\sigma] = \hat{\Pi} T (\lambda \alpha. (B \alpha)[\sigma])$
	$\hat{\Pi}[]^A : \equiv \text{refl}$
	$\hat{\@}[] : (t \hat{\@} \alpha)[\sigma] = (t[\sigma]) \hat{\@} \alpha$
	$\hat{\@}[]^A : \equiv \text{refl}$

$(\_)^A : (\mathfrak{G} : \text{GAT}) \rightarrow \text{Set}$

$\_$ -Alg:  $(\mathfrak{G} : \text{GAT}) \rightarrow \text{Set}$

Hom:  $(\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

# From [KKA19, Appendix A]

$\Gamma : \text{Con}$	$\Gamma^M$	$: \Gamma^A \rightarrow \Gamma^A \rightarrow \text{Set}$
$A : \text{Ty} \Gamma$	$A^M$	$: \Gamma^M \gamma^0 \gamma^I \rightarrow A^A \gamma^0 \rightarrow A^A \gamma^I \text{Set}$
$\sigma : \text{Sub} \Gamma \Delta$	$\sigma^M$	$: \Gamma^M \gamma^0 \gamma^I \rightarrow \Delta^M (\sigma^A \gamma^0) (\sigma^A \gamma^I)$
$t : \text{Tm} \Gamma A$	$t^M$	$: (\gamma^M : \Gamma^M \gamma^0 \gamma^I) \rightarrow A^M \gamma^M (t^A \gamma^0) (t^A \gamma^I)$
$\cdot : \text{Con}$	$\cdot^M \text{ tt tt}$	$\equiv \top$
$\Gamma \triangleright A : \text{Con}$	$(\Gamma \triangleright A)^M (\gamma^0, \alpha^0) (\gamma^I, \alpha^I)$	$\equiv (\gamma^M : \Gamma^M \gamma^0 \gamma^I) \times A^M \gamma^M \alpha^0 \alpha^I$
$(A : \text{Ty} \Delta) [\sigma : \text{Sub} \Gamma \Delta] : \text{Ty} \Gamma$	$(A[\sigma])^M \gamma^M \alpha^0 \alpha^I$	$\equiv A^M (\sigma^M \gamma^M) \alpha^0 \alpha^I$
$\text{id} : \text{Sub} \Gamma \Gamma$	$\text{id}^M \gamma^M$	$\equiv \gamma^m$
$(\sigma : \text{Sub} \Theta \Delta) \circ (\delta : \text{Sub} \Gamma \Theta) : \text{Sub} \Gamma \Delta$	$(\sigma \circ \delta)^M \gamma^M$	$\equiv \sigma^M (\delta^M \gamma^M)$
$\epsilon : \text{Sub} \Gamma \cdot$	$\epsilon^M \gamma^M$	$\equiv \text{tt}$
$(\sigma : \text{Sub} \Gamma \Delta), (t : \text{Tm} \Gamma (A[\sigma])) : \text{Sub} \Gamma (\Delta \triangleright A)$	$(\sigma, t)^M \gamma^M$	$\equiv (\sigma^M \gamma^M, t^M \gamma^M)$
$\pi_1 (\sigma : \text{Sub} \Gamma (\Delta \triangleright A)) : \text{Sub} \Gamma \Delta$	$(\pi_1 \sigma)^M \gamma^M$	$\equiv \text{proj}_1 (\sigma^M \gamma^M)$
$\pi_2 (\sigma : \text{Sub} \Gamma (\Delta \triangleright A)) : \text{Tm} \Gamma (A[\pi_1 \sigma])$	$(\pi_2 \sigma)^M \gamma^M$	$\equiv \text{proj}_2 (\sigma^M \gamma^M)$
$(t : \text{Tm} \Delta A) [\sigma : \text{Sub} \Gamma \Delta] : \text{Tm} \Gamma (A[\sigma])$	$(t[\sigma])^M \gamma^M$	$\equiv t^M (\sigma^M \gamma^M)$
$[\text{id}] : A[\text{id}] = A$	$[\text{id}]^M$	$\equiv \text{refl}$
$[\circ] : A[\sigma \circ \delta] = A[\sigma][\delta]$	$[\circ]^M$	$\equiv \text{refl}$
$\text{ass} : (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$	$\text{ass}^M$	$\equiv \text{refl}$
$\text{idl} : \text{id} \circ \sigma = \sigma$	$\text{idl}^M$	$\equiv \text{refl}$
$\text{idr} : \sigma \circ \text{id} = \sigma$	$\text{idr}^M$	$\equiv \text{refl}$
$\cdot \eta : \{\sigma : \text{Sub} \Gamma \cdot\} \rightarrow \sigma = \epsilon$	$\cdot \eta^M$	$\equiv \text{refl}$
$\triangleright \beta_1 : \pi_1 (\sigma, t) = \sigma$	$\triangleright \beta_1^M$	$\equiv \text{refl}$
$\triangleright \beta_2 : \pi_2 (\sigma, t) = t$	$\triangleright \beta_2^M$	$\equiv \text{refl}$
$\triangleright \eta : (\pi_1 \sigma, \pi_2 \sigma) = \sigma$	$\triangleright \eta^M$	$\equiv \text{refl}$
$, \circ : (\sigma, t) \circ \delta = (\sigma \circ \delta, t[\delta])$	$, \circ^M$	$\equiv \text{refl}$
$\text{U} : \text{Ty} \Gamma$	$\text{U}^M \gamma^M T^0 T^I$	$\equiv T^0 \rightarrow T^I$

$(\_)^M : (\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

$\text{El} (a : \text{Tm} \Gamma \text{U}) : \text{Ty} \Gamma$	$(\text{El} a)^M \gamma^M \alpha^0 \alpha^I$	$\equiv a^M \gamma^M \alpha^0 = \alpha^I$
$\text{U}[] : \text{U}[\sigma] = \text{U}$	$\text{U}[]^M$	$\equiv \text{refl}$
$\text{El}[] : (\text{El} a)[\sigma] = \text{El}(a[\sigma])$	$\text{El}[]^M$	$\equiv \text{refl}$
$\Pi (a : \text{Tm} \Gamma \text{U}) (B : \text{Ty} (\Gamma \triangleright \text{El} a)) : \text{Ty} \Gamma$	$(\Pi a B)^M \gamma^M f^0 f^I$	$\equiv (\alpha^0 : a^A \gamma^0) \rightarrow B^M (\gamma^M, \text{refl}) (f^0 \alpha^0) (f^I (a^M \gamma^M \alpha^0))$
$\text{app} (t : \text{Tm} \Gamma (\Pi a B)) : \text{Tm} (\Gamma \triangleright \text{El} a) B$	$(\text{app} t)^M (\gamma^M, \alpha^M)$	$\equiv J(t^M \gamma^M \alpha^0) \alpha^M$
$\Pi[] : (\Pi a B)[\sigma] = \Pi(a[\sigma]) (B[\sigma^\dagger])$	$\Pi[]^M$	$\equiv \text{refl}$
$\text{app}[] : (\text{app} t)[\sigma^\dagger] = \text{app}(t[\sigma])$	$\text{app}[]^M$	$\equiv \text{refl}$
$\text{id} (a : \text{Tm} \Gamma \text{U}) (t u : \text{Tm} \Gamma (\text{El} a)) : \text{Ty} \Gamma$	$(\text{id} a t u)^M \gamma^M e^0 e^I$	$\equiv \top$
$\text{reflect} (e : \text{Tm} \Gamma (\text{Id} a t u)) : t = u$	$(\text{reflect} e)^M$	$\equiv \text{UIP}$
$\text{id}[] : (\text{id} a t u)[\sigma] = \text{id}(a[\sigma])(t[\sigma])(u[\sigma])$	$\text{id}[]^M$	$\equiv \text{refl}$
$\hat{\Pi} (T : \text{Set}) (B : T \rightarrow \text{Ty} \Gamma) : \text{Ty} \Gamma$	$(\hat{\Pi} T B)^M \gamma^M f^0 f^I$	$\equiv (\alpha : T) \rightarrow (B \alpha)^M \gamma^M (f^0 \alpha) (f^I \alpha)$
$(t : \text{Tm} \Gamma (\hat{\Pi} T B)) \hat{\otimes} (\alpha : T) : \text{Tm} \Gamma (B \alpha)$	$(t \hat{\otimes} \alpha)^M \gamma^M$	$\equiv t^M \gamma^M \alpha$
$\hat{\Pi}[] : (\hat{\Pi} T B)[\sigma] = \hat{\Pi} T (\lambda \alpha. (B \alpha)[\sigma])$	$\hat{\Pi}[]^M$	$\equiv \text{refl}$
$\hat{\otimes}[] : (t \hat{\otimes} \alpha)[\sigma] = (t[\sigma]) \hat{\otimes} \alpha$	$\hat{\otimes}[]^M$	$\equiv \text{refl}$

*GAT signature*      *Category of Algs.*      *Initial Algebra*

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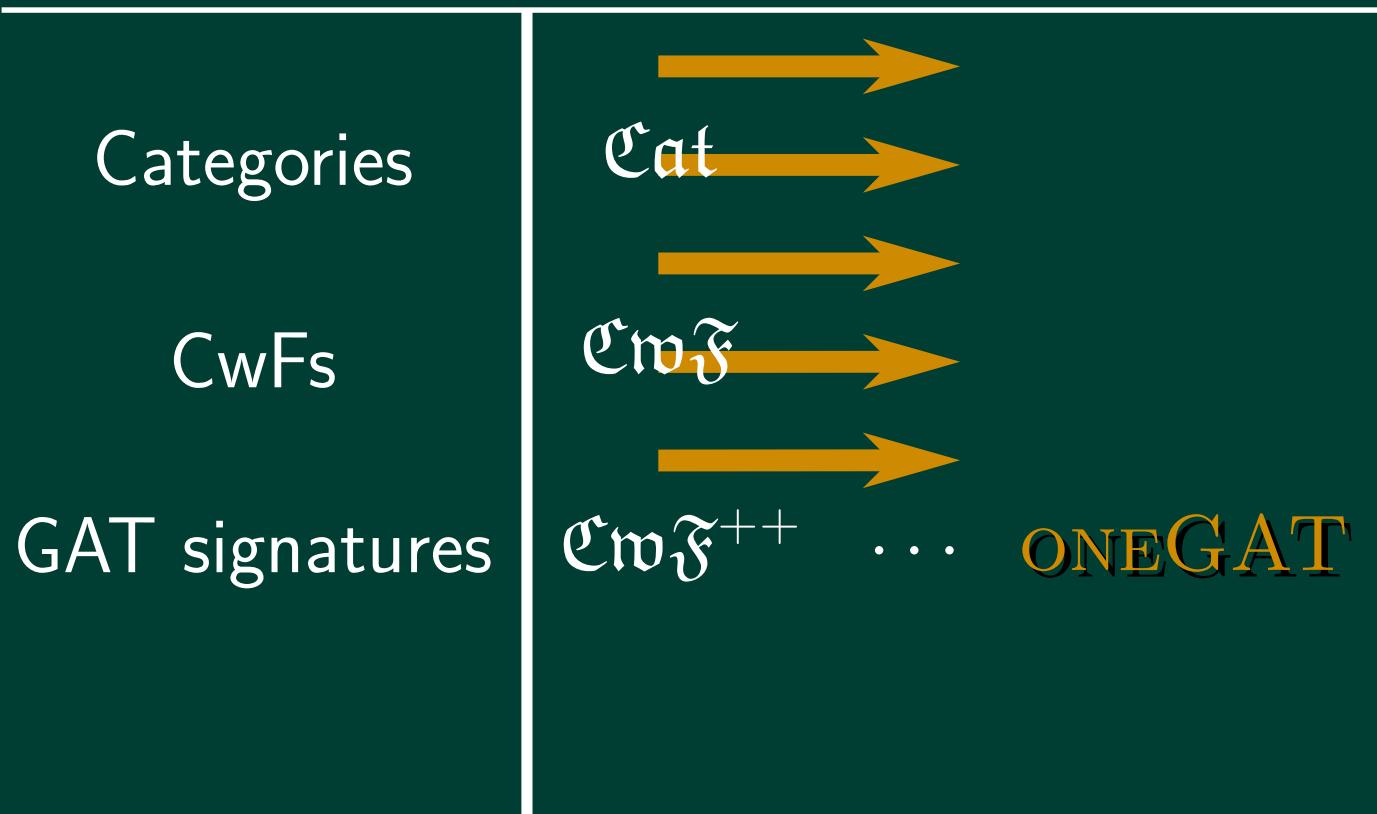
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Categories	$\mathfrak{Cat}$
CwFs	$\mathfrak{CwF}$
GAT signatures	$\mathfrak{CwF}^{++}$ ... <b>ONEGAT</b>

*GAT signature*      *Category of Algs.*      *Initial Algebra*

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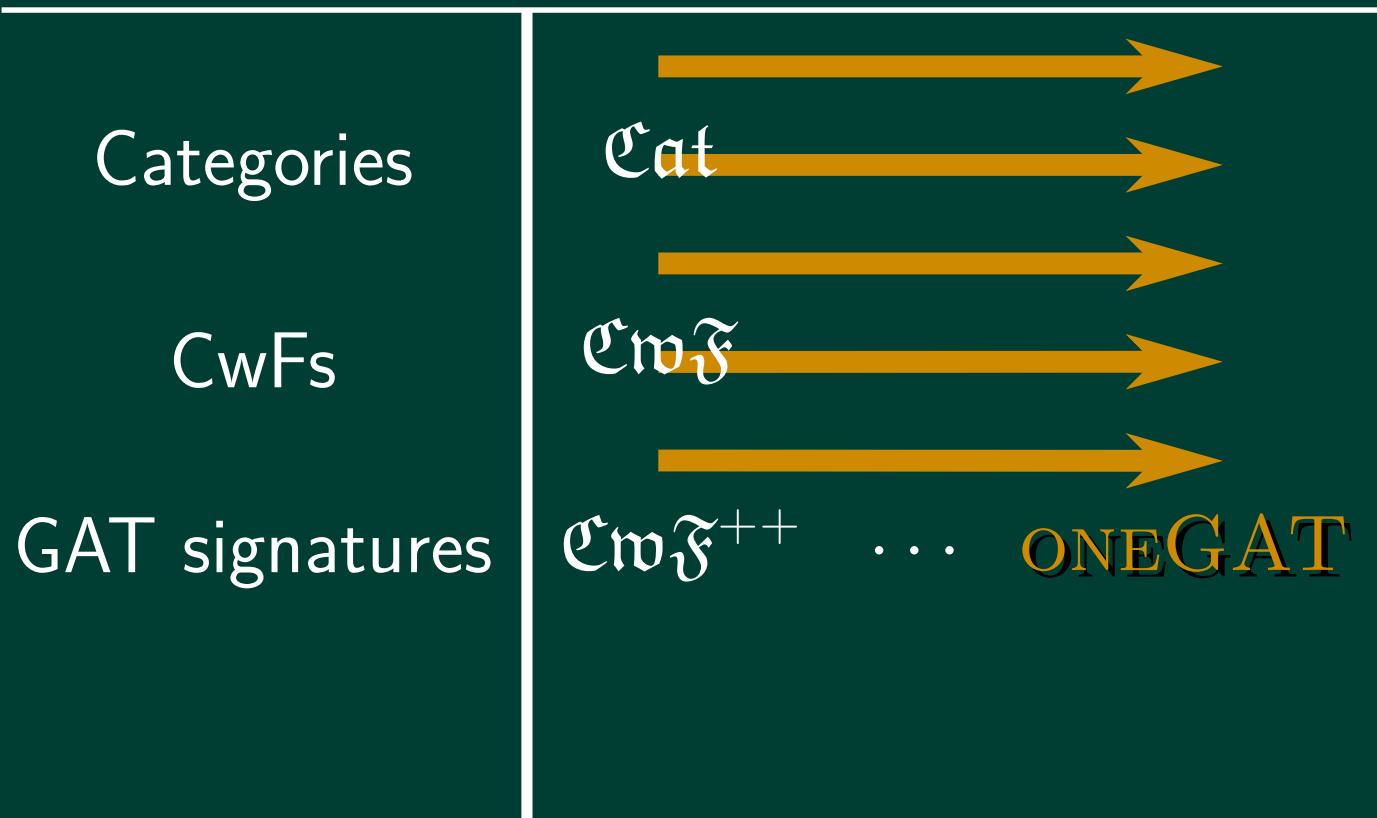
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*GAT signature*      *Category of Algs.*      *Initial Algebra*

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# Idea:

Elements of initial algebra are  
*only* those derivable from the  
*mere concept* of the structure

# Construction of initial algebras

```
def ℕ : GAT := { Nat  : U, zero : Nat, succ : Nat ⇒ Nat }
```

# Construction of initial algebras

```
def  $\mathfrak{N}$  : GAT := { Nat  : U, zero : Nat, succ : Nat  $\Rightarrow$  Nat }
```

$$\mathbb{N} := \overline{\text{Tm}}(\mathfrak{N}, \text{Nat})$$

# Construction of initial algebras

**def**  $\mathfrak{N}$  : GAT := { Nat : **U**, zero : Nat, succ : Nat  $\Rightarrow$  Nat }

$$\mathbb{N} := \overline{\text{Tm}}(\mathfrak{N}, \text{Nat})$$

Nat : U, zero : Nat, succ : Nat  $\Rightarrow$  Nat  $\vdash$  t : Nat

# Construction of initial algebras

**def**  $\mathfrak{N}$  : GAT := { Nat : U, zero : Nat, succ : Nat  $\Rightarrow$  Nat }

$$\mathbb{N} := \overline{\text{Tm}}(\mathfrak{N}, \text{Nat})$$

Nat : U, zero : Nat, succ : Nat  $\Rightarrow$  Nat  $\vdash$  t : Nat

Forms a  $\mathfrak{N}$ -algebra with zero and  $\lambda t \rightarrow \text{succ } t$ .

# Construction of initial algebras

**def**  $\mathfrak{N}$  : GAT := { Nat : **U**, zero : Nat, succ : Nat  $\Rightarrow$  Nat }

$$\mathbb{N} := \overline{\text{Tm}}(\mathfrak{N}, \text{Nat})$$

Nat : U, zero : Nat, succ : Nat  $\Rightarrow$  Nat  $\vdash$  t : Nat

Forms a  $\mathfrak{N}$ -algebra with zero and  $\lambda t \rightarrow \text{succ } t$ .

**Claim** Can do this for arbitrary GAT  $\mathfrak{G}$ , by quotient induction-induction on ONEGAT.

**Upshot:** Any GAT  
extension of  $\mathcal{C}_m\mathcal{F}$   
has an initial model



# Concrete CwFs

Ahrens and Lumsdaine [AL19] introduce the notion of **displayed categories**

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$$\text{Obj}^D : \text{Obj} \rightarrow \text{Set}$$

# Displayed categories

Ahrens and Lumsdaine [AL19] introduce the notion of **displayed categories**

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$$\text{Hom}^D : (I \ J : \text{Obj}) \rightarrow \text{Obj}^D \ I \rightarrow \text{Obj}^D \ J \rightarrow \text{Hom} \ I \ J \rightarrow \text{Set}$$

# Displayed categories

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$$\text{Obj}^D : \text{Obj} \rightarrow \text{Set}$$

$$\text{Hom}^D : (I \ J : \text{Obj}) \rightarrow \text{Obj}^D I \rightarrow \text{Obj}^D J \rightarrow \text{Hom } I \ J \rightarrow \text{Set}$$

$$\text{id}^D : (I : \text{Obj}) \rightarrow (I^D : \text{Obj}^D I) \rightarrow \text{Hom}^D I \ I^D I^D$$

# Displayed categories

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$$\text{Obj}^D : \text{Obj} \rightarrow \text{Set}$$

$$\text{Hom}^D : (I \ J : \text{Obj}) \rightarrow \text{Obj}^D I \rightarrow \text{Obj}^D J \rightarrow \text{Hom } I \ J \rightarrow \text{Set}$$

$$\text{id}^D : (I : \text{Obj}) \rightarrow (I^D : \text{Obj}^D I) \rightarrow \text{Hom}^D I \ I^D I^D$$

...

**Fact:** We can define  
“displayed  
 $\mathfrak{G}$ -algebra” for every  
GAT  $\mathfrak{G}$

- $\_\text{-Alg} : (\mathfrak{G} : \text{GAT}) \rightarrow \text{Set}$
- $\text{Hom} : (\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$
- $\_\text{-DAlg} : (\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

# From [KKA19, Appendix A]

Syntax	Displayed algebras		
$\Gamma : \text{Con}$	$\Gamma^D : \Gamma^A \rightarrow \text{Set}$	$\text{El}(a : \text{Tm} \Gamma U) : \text{Ty} \Gamma$	$(\text{El } a)^D \gamma^D \alpha \quad \cong a^D \gamma^D \alpha$
$A : \text{Ty} \Gamma$	$A^D : \Gamma^D \gamma \rightarrow A^A \gamma \rightarrow \text{Set}$	$U[] : U[\sigma] = U$	$U[]^A \quad \cong \text{refl}$
$\sigma : \text{Sub} \Gamma \Delta$	$\sigma^D : \Gamma^D \gamma \rightarrow \Delta^D (\sigma^A \gamma)$	$\text{El}[] : (\text{El } a)[\sigma] = \text{El}(a[\sigma])$	$\text{El}[]^A \quad \cong \text{refl}$
$t : \text{Tm} \Gamma A$	$t^D : (\gamma^D : \Gamma^D \gamma) \rightarrow A^D \gamma^D (t^A \gamma)$	$\Pi(a : \text{Tm} \Gamma U)(B : \text{Ty}(\Gamma \triangleright \text{El } a)) : \text{Ty} \Gamma$	$(\Pi a B)^D \gamma^D f \quad \cong (\alpha^D : a^D \gamma^D \alpha) \rightarrow B^D (\gamma^D, \alpha^D) (f \alpha)$
$\cdot : \text{Con}$	$\cdot^D \text{tt} \quad \cong \top$	$\text{app}(t : \text{Tm} \Gamma (\Pi a B)) : \text{Tm}(\Gamma \triangleright \text{El } a) B$	$(\text{app } t)^D (\gamma^D, \alpha^D) \cong t^D \gamma^D \alpha^D$
$\Gamma \triangleright A : \text{Con}$	$(\Gamma \triangleright A)^D (\gamma, \alpha) \quad \cong (\gamma^D : \Gamma^D \gamma) \times A^D \gamma^D \alpha$	$\Pi[] : (\Pi a B)[\sigma] = \Pi(a[\sigma])(B[\sigma^\dagger])$	$\Pi[]^D \quad \cong \text{refl}$
$(A : \text{Ty} \Delta)[\sigma : \text{Sub} \Gamma \Delta] : \text{Ty} \Gamma$	$(A[\sigma])^D \gamma^D \alpha \quad \cong A^D (\sigma^D \gamma^D) \alpha$	$\text{app}[] : (\text{app } t)[\sigma^\dagger] = \text{app}(t[\sigma])$	$\text{app}[]^D \quad \cong \text{refl}$
$\text{id} : \text{Sub} \Gamma \Gamma$	$\text{id}^D \gamma^D \quad \cong \gamma^D$	$\text{Id}(a : \text{Tm} \Gamma U)(t u : \text{Tm} \Gamma (\text{El } a)) : \text{Ty} \Gamma$	$(\text{Id } a t u)^D \gamma^D e \quad \cong \text{tr}_{(a^D \gamma^D)} e (t^D \gamma^D) = u^D \gamma^D$
$(\sigma : \text{Sub} \Theta \Delta) \circ (\delta : \text{Sub} \Gamma \Theta) : \text{Sub} \Gamma \Delta$	$(\sigma \circ \delta)^D \gamma^D \quad \cong \sigma^D (\delta^D \gamma^D)$	$\text{reflect}(e : \text{Tm} \Gamma (\text{Id } a t u)) : t = u$	$(\text{reflect } e)^D \quad : t^D \gamma^D \stackrel{e^D}{=} u^D \gamma^D$
$\epsilon : \text{Sub} \Gamma \cdot$	$\epsilon^D \gamma^D \quad \cong \text{tt}$	$\text{Id}[] : (\text{Id } a t u)[\sigma] = \text{Id}(a[\sigma])(t[\sigma])(u[\sigma])$	$\text{Id}[]^D \quad \cong \text{refl}$
$(\sigma : \text{Sub} \Gamma \Delta), (t : \text{Tm} \Gamma (A[\sigma])) : \text{Sub} \Gamma (\Delta \triangleright A)$	$(\sigma, t)^D \gamma^D \quad \cong (\sigma^D \gamma^D, t^D \gamma^D)$	$\hat{\Pi}(T : \text{Set})(B : T \rightarrow \text{Ty} \Gamma) : \text{Ty} \Gamma$	$(\hat{\Pi} T B)^D \gamma^D f \quad \cong (\alpha : T) \rightarrow (B \alpha)^D \gamma^D (f \alpha)$
$\pi_1(\sigma : \text{Sub} \Gamma (\Delta \triangleright A)) : \text{Sub} \Gamma \Delta$	$(\pi_1 \sigma)^D \gamma^D \quad \cong \text{proj}_1(\sigma^D \gamma^D)$	$(t : \text{Tm} \Gamma (\hat{\Pi} T B)) \hat{\wedge} (\alpha : T) : \text{Tm} \Gamma (B \alpha)$	$(t \hat{\wedge} \alpha)^D \gamma^D \quad \cong t^D \gamma^D \alpha$
$\pi_2(\sigma : \text{Sub} \Gamma (\Delta \triangleright A)) : \text{Tm} \Gamma (A[\pi_1 \sigma])$	$(\pi_2 \sigma)^D \gamma^D \quad \cong \text{proj}_2(\sigma^D \gamma^D)$	$\hat{\Pi}[] : (\hat{\Pi} T B)[\sigma] = \hat{\Pi} T (\lambda \alpha. (B \alpha)[\sigma])$	$\hat{\Pi}[]^D \quad \cong \text{refl}$
$(t : \text{Tm} \Delta A)[\sigma : \text{Sub} \Gamma \Delta] : \text{Tm} \Gamma (A[\sigma])$	$(t[\sigma])^D \gamma^D \quad \cong t^D (\sigma^D \gamma^D)$	$\hat{\wedge}[] : (t \hat{\wedge} \alpha)[\sigma] = (t[\sigma]) \hat{\wedge} \alpha$	$\hat{\wedge}[]^D \quad \cong \text{refl}$
$[\text{id}] : A[\text{id}] = A$	$[\text{id}]^D \quad \cong \text{refl}$		
$[\circ] : A[\sigma \circ \delta] = A[\sigma][\delta]$	$[\circ]^D \quad \cong \text{refl}$		
$\text{ass} : (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$	$\text{ass}^D \quad \cong \text{refl}$		
$\text{idl} : \text{id} \circ \sigma = \sigma$	$\text{idl}^D \quad \cong \text{refl}$		
$\text{idr} : \sigma \circ \text{id} = \sigma$	$\text{idr}^D \quad \cong \text{refl}$		
$\cdot \eta : \{\sigma : \text{Sub} \Gamma \cdot\} \rightarrow \sigma = \epsilon$	$\cdot \eta^D \quad \cong \text{refl}$		
$\triangleright \beta_1 : \pi_1(\sigma, t) = \sigma$	$\triangleright \beta_1^D \quad \cong \text{refl}$		
$\triangleright \beta_2 : \pi_2(\sigma, t) = t$	$\triangleright \beta_2^D \quad \cong \text{refl}$		
$\triangleright \eta : (\pi_1 \sigma, \pi_2 \sigma) = \sigma$	$\triangleright \eta^D \quad \cong \text{refl}$		
$, \circ : (\sigma, t) \circ \delta = (\sigma \circ \delta, t[\delta])$	$, \circ^D \quad \cong \text{refl}$		
$U : \text{Ty} \Gamma$	$U^D \gamma^D T \quad \cong T \rightarrow \text{Set}$		

$(\_)^D : (\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

Displayed nat-algebras are “induction data”

$$\begin{aligned}\mathfrak{N}\text{-Alg} := & (N : \text{Set}) \\ & \times (z : N) \\ & \times (s : N \rightarrow N)\end{aligned}$$

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**Question:** What kind  
of thing is the  
output of induction?

# Displayed nat-algebras are “induction data”

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$$\mathfrak{N}\text{-Sect } (N, z, s) (N^D, z^D, s^D) := (N^S : (n : N) \rightarrow N^D n)$$

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**Fact** We can define the type of **sections** of a given displayed  $\mathfrak{G}$ -algebra, for all GATs  $\mathfrak{G}$ .

$\_$ -Alg:  $(\mathfrak{G} : \text{GAT}) \rightarrow \text{Set}$

Hom:  $(\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

$\_$ -DAlg:  $(\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

$\_$ -Sect:  $(\mathfrak{G} : \text{GAT}) \rightarrow (\Gamma : \mathfrak{G}\text{-Alg}) \rightarrow \mathfrak{G}\text{-DAlg } \Gamma$   
 $\qquad \qquad \qquad \rightarrow \text{Set}$

# From [KKA19, Appendix A]

Sections			
$\Gamma^S$	$(\gamma : \Gamma^A) \rightarrow \Gamma^D \gamma \rightarrow \text{Set}$	$(\text{El } a)^S \gamma^S \alpha \alpha^D$	$\equiv a^S \gamma^S \alpha = \alpha^D$
$A^S$	$: \Gamma^S \gamma \gamma^D \rightarrow (\alpha : A^A \gamma) \rightarrow A^D \gamma^D \alpha \rightarrow \text{Set}$	$\text{U}[]^S$	$\equiv \text{refl}$
$\sigma^S$	$: \Gamma^S \gamma \gamma^D \rightarrow \Delta^S (\sigma^A \gamma) (\sigma^D \gamma^D)$	$\text{El}[]^S$	$\equiv \text{refl}$
$t^S$	$: (\gamma^S : \Gamma^S \gamma \gamma^D) \rightarrow A^S \gamma^S (t^A \gamma) (t^D \gamma^D)$	$(\Pi a B)^S \gamma^S f f^D$	$\equiv (\alpha : a^A \gamma) \rightarrow$ $B^S (\gamma^S, \text{refl}_{a^S \gamma^S \alpha}) (f \alpha) (f^D (a^S \gamma^S \alpha))$
$\cdot^S \text{ tt tt}$	$\equiv \top$	$(\text{app } t)^S (\gamma^S, \alpha^S)$	$\equiv \text{J}_{x,z} B^S (\gamma^S, z) (t^A \gamma \alpha) (t^D \gamma^D x) (t^S \gamma^S \alpha) \alpha^S$
$(\Gamma \triangleright A)^S (\gamma, \alpha) (\gamma^D, \alpha^D)$	$\equiv (\gamma^S : \Gamma^S \gamma \gamma^D) \times A^S \gamma^S \alpha \alpha^D$	$\Pi[]^S$	$\equiv \text{refl}$
$(A[\sigma])^S \gamma^S \alpha \alpha^D$	$\equiv A^S (\sigma^S \gamma^S) \alpha \alpha^D$	$\text{app}[]^S$	$\equiv \text{refl}$
$\text{id}^S \gamma^S$	$\equiv \gamma^S$	$(\text{Id } a t u)^S \gamma^S e e^D$	$\equiv \top$
$(\sigma \circ \delta)^S \gamma^S$	$\equiv \sigma^S (\delta^S \gamma^S)$	$(\text{reflect } e)^S$	$\equiv \text{UIP}$
$\epsilon^S \gamma^S$	$\equiv \text{tt}$	$\text{Id}[]^S$	$\equiv \text{refl}$
$(\sigma, t)^S \gamma^S$	$\equiv (\sigma^S \gamma^S, t^S \gamma^S)$	$(\hat{\Pi} T B)^S \gamma^S f f^D$	$\equiv (\alpha : T) \rightarrow (B \alpha)^S \gamma^S (f \alpha) (f^D \alpha)$
$(\pi_1 \sigma)^S \gamma^S$	$\equiv \text{proj}_1 (\sigma^S \gamma^S)$	$(t \hat{\@} \alpha)^S \gamma^S$	$\equiv t^S \gamma^S \alpha$
$(\pi_2 \sigma)^S \gamma^S$	$\equiv \text{proj}_2 (\sigma^S \gamma^S)$	$\hat{\Pi}[]^S$	$\equiv \text{refl}$
$(t[\sigma])^S \gamma^S$	$\equiv t^S (\sigma^S \gamma^S)$	$\hat{\@}[]^S$	$\equiv \text{refl}$
$[\text{id}]^S$	$\equiv \text{refl}$		
$[\circ]^S$	$\equiv \text{refl}$		
$\text{ass}^S$	$\equiv \text{refl}$		

$$(\underline{\quad})^S : (\mathfrak{G} : \text{GAT}) \rightarrow (\Gamma : \mathfrak{G}\text{-Alg}) \rightarrow \mathfrak{G}\text{-DAlg} \rightarrow \text{Set}$$

$\triangleright \mu_1$	$\equiv \text{refl}$
$\triangleright \beta_2^S$	$\equiv \text{refl}$
$\triangleright \eta^S$	$\equiv \text{refl}$
$, \circ^S$	$\equiv \text{refl}$
$\text{U}^S \gamma^S T T^D$	$\equiv (\alpha : T) \rightarrow T^D \alpha$

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**Theorem** Every displayed algebra over the initial  $\mathfrak{G}$ -algebra admits a section

- Principle of induction (e.g.  $\mathbb{N}$ )
- Syntax model is contextual
- Unary parametricity



$\mathfrak{G}$ -Alg: Set

$\mathfrak{G}\text{-Alg}$ : Set

$\text{Hom}_{\mathfrak{G}}$ :  $\mathfrak{G}\text{-Alg} \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

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$\text{Hom}_{\mathfrak{G}}$ :  $\mathfrak{G}\text{-Alg} \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

$\mathfrak{G}\text{-DAlg}$ :  $\mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

$\mathfrak{G}\text{-Alg} : \text{Set}$

$\text{Hom}_{\mathfrak{G}} : \mathfrak{G}\text{-Alg} \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

$\mathfrak{G}\text{-DAlg} : \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

$\mathfrak{G}\text{-Sect} : (\Gamma : \mathfrak{G}\text{-Alg}) \rightarrow \mathfrak{G}\text{-DAlg } \Gamma$   
 $\qquad\qquad\qquad \rightarrow \text{Set}$

**Observation:** Every  
GAT gives rise to a  
CwF of algebras and  
displayed algebras

## Central Dogma of Category Theory

Every notion of “structure” comes equipped with a notion of “structure-preserving morphism”, i.e. forms a category

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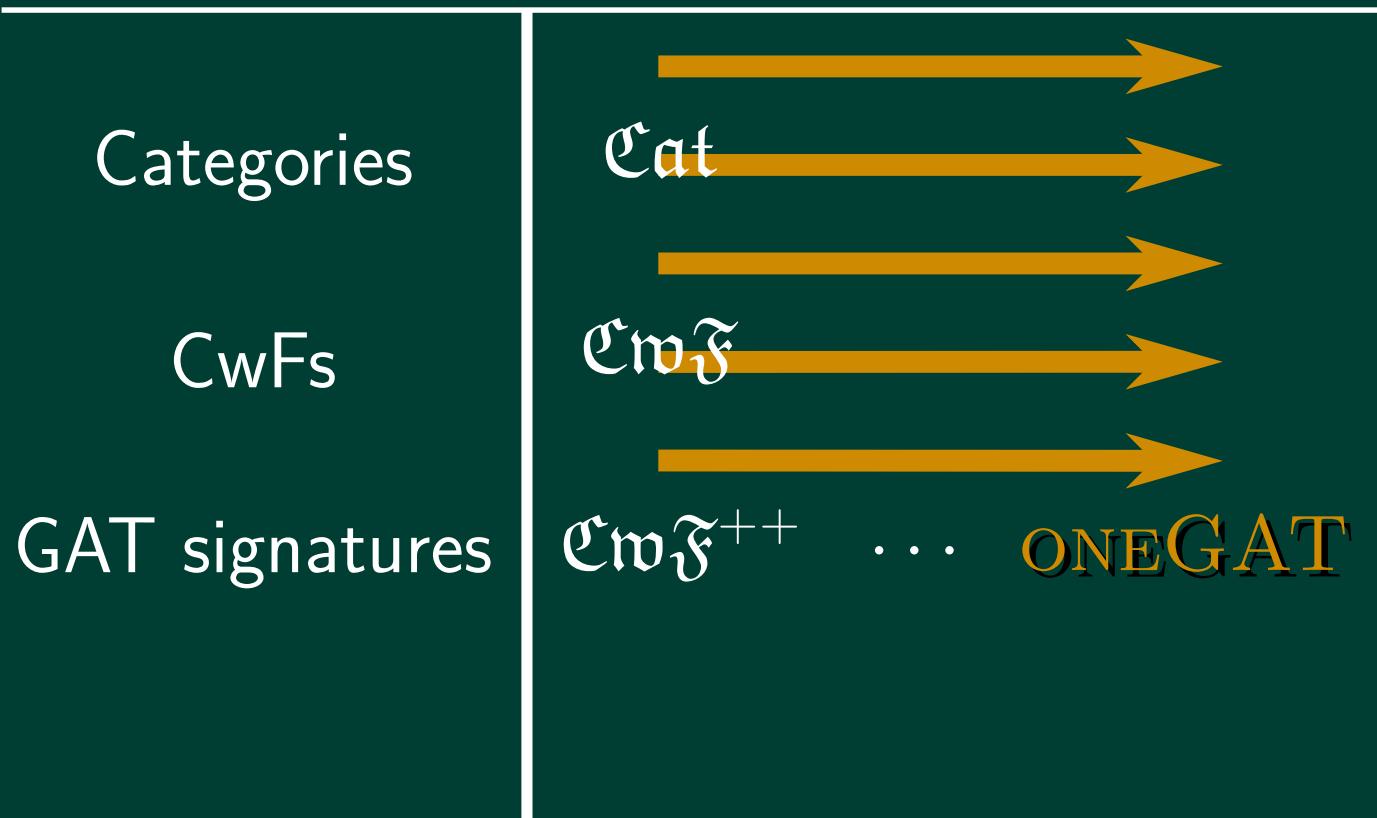
## Central Dogma of Generalized Algebra

Every notion of “structure” comes equipped with a notion of “structure-preserving morphism”, “displayed structure”, and “section”, i.e. forms a category with families

*GAT signature*      *Category of Algs.*      *Initial Algebra*

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<i>GAT signature</i>	<i>Category of Algs.</i>	<i>Initial Algebra</i>	<i>Concrete CwF</i>
Categories			
CwFs			
GAT signatures		$\mathfrak{C}\mathfrak{W}\mathfrak{F}^{++}$	$\dots$ ONEGAT



Is the **setoid model**<sup>1</sup> the same thing as the concrete CwF of setoids?

Is the **groupoid model** the same thing as the concrete CwF of groupoids?

---

<sup>1</sup>We mean the setoid model à la Altenkirch [Alt99], which uses **equivalence relations**, not the setoid model of Hofmann [Hof95a, Hof95b], which uses **partial equivalence relations**.

No

# Fibrancy data for setoid model

From [ABK<sup>+</sup>21, 7]

$$\frac{A : \text{Ty } \Gamma}{\begin{aligned} |A| : |\Gamma| &\rightarrow \mathbf{Type} \\ A^\sim : \{\gamma_0 \ \gamma_1 : |\Gamma|\} &\rightarrow \Gamma^\sim \ \gamma_0 \ \gamma_1 \rightarrow |A|\gamma_0 \rightarrow |A|\gamma_1 \rightarrow \mathbf{Prop} \\ \mathsf{refl}^* : \{\gamma : |\Gamma|\}(a : |A|\gamma) &\rightarrow A^\sim (\mathsf{refl} \ \Gamma \ \gamma) \ a \ a \\ \mathsf{sym}^* : \forall \{\gamma_0 \ \gamma_1 \ a_0 \ a_1\} \{p : \Gamma^\sim \ \gamma_0 \ \gamma_1\} &\rightarrow A^\sim p \ a_0 \ a_1 \rightarrow A^\sim (\mathsf{sym} \ \Gamma \ p) \ a_1 \ a_0 \\ \mathsf{trans}^* : A^\sim p_0 \ a_0 \ a_1 &\rightarrow A^\sim p_1 \ a_1 \ a_2 \rightarrow A^\sim (\mathsf{trans} \ \Gamma \ p_0 \ p_1) \ a_0 \ a_2 \\ \mathsf{coe} : \Gamma^\sim \ \gamma_0 \ \gamma_1 &\rightarrow |A|\gamma_0 \rightarrow |A|\gamma_1 \\ \mathsf{coh} : (p : \Gamma^\sim \ \gamma_0 \ \gamma_1)(a : |A|\gamma_0) &\rightarrow A^\sim p \ a \ (\mathsf{coe} \ A \ p \ a) \end{aligned}}$$

# Fibrancy data for setoid model

From [ABK<sup>+</sup>21, 7]

## Displayed Setoid

$A : \text{Ty } \Gamma$

$|A| : |\Gamma| \rightarrow \text{Type}$

$A^\sim : \{\gamma_0 \ \gamma_1 : |\Gamma|\} \rightarrow \Gamma^\sim \ \gamma_0 \ \gamma_1 \rightarrow |A|\gamma_0 \rightarrow |A|\gamma_1 \rightarrow \text{Prop}$

$\text{refl}^* : \{\gamma : |\Gamma|\}(a : |A|\gamma) \rightarrow A^\sim (\text{refl } \Gamma \ \gamma) \ a \ a$

$\text{sym}^* : \forall \{\gamma_0 \ \gamma_1 \ a_0 \ a_1\} \{p : \Gamma^\sim \ \gamma_0 \ \gamma_1\} \rightarrow A^\sim p \ a_0 \ a_1 \rightarrow A^\sim (\text{sym } \Gamma \ p) \ a_1 \ a_0$

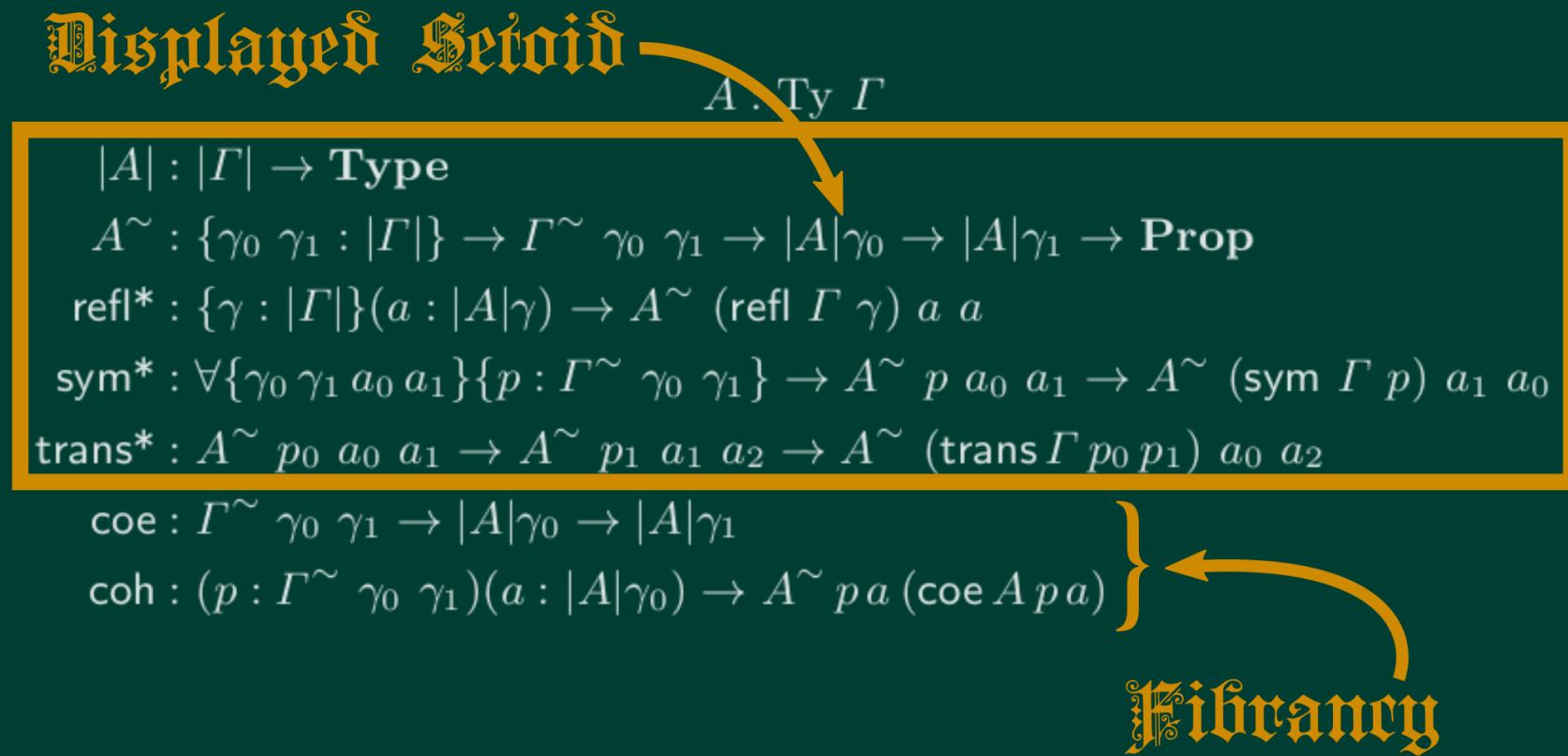
$\text{trans}^* : A^\sim p_0 \ a_0 \ a_1 \rightarrow A^\sim p_1 \ a_1 \ a_2 \rightarrow A^\sim (\text{trans } \Gamma \ p_0 \ p_1) \ a_0 \ a_2$

$\text{coe} : \Gamma^\sim \ \gamma_0 \ \gamma_1 \rightarrow |A|\gamma_0 \rightarrow |A|\gamma_1$

$\text{coh} : (p : \Gamma^\sim \ \gamma_0 \ \gamma_1)(a : |A|\gamma_0) \rightarrow A^\sim p \ a \ (\text{coe } A \ p \ a)$

# Fibrancy data for setoid model

From [ABK<sup>+</sup>21, 7]



**Idea:** Carve out  
sub-CwFs of  
concrete CwFs whose  
types are *fibrant*

<i>GAT signature</i>	<i>Category of Algs.</i>	<i>Initial Algebra</i>	<i>Concrete CwF</i>
Categories		$\mathbb{C}\mathbf{at}$	$\mathbb{C}\mathbf{at}$
CwFs		$\mathbb{C}\mathfrak{w}\mathfrak{f}$	$\mathbb{C}\mathfrak{w}\mathfrak{f}$
GAT signatures		$\mathbb{C}\mathfrak{w}\mathfrak{f}^{++}$ ... ONEGAT	

<i>GAT signature</i>	<i>Category of Algs.</i>	<i>Initial Algebra</i>	<i>Concrete CwF</i>	<i>Fibrant Model</i>
Categories				
CwFs				
GAT signatures	$\mathfrak{Cat}$ $\mathfrak{CwF}$ $\mathfrak{CwF}^{++}$	$\dots$	ONEGAT	

# GAT signature

## Category of Algs.

### Initial Algebra

### Concrete CwF

### Fibrant Model

Categories	$\mathfrak{Cat}$	
CwFs	$\mathfrak{CwF}$	
GAT signatures	$\mathfrak{CwF}^{++} \dots \text{ONEGAT}$	

# GAT signature

## Category of Algs.

### Initial Algebra

### Concrete CwF

### Fibrant Model

Categories	$\mathfrak{Cat}$		[AL19, Nor19, NA24]'s swamp
CwFs	$\mathfrak{CwF}$		
GAT signatures	$\mathfrak{CwF}^{++}$	... ONEGAT	



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- The category model provides a synthetic theory of categories?

# GAT signature

## Category of Algs.

### Initial Algebra

### Concrete CwF

### Fibrant Model

Categories	$\mathfrak{Cat}$		[AL19, Nor19, NA24]'s swamp
CwFs	$\mathfrak{CwF}$		
GAT signatures	$\mathfrak{CwF}^{++}$	... ONEGAT	



<i>GAT signature</i>	<i>Category of Algs.</i>	<i>Initial Algebra</i>	<i>Concrete CwF</i>	<i>Fibrant Model</i>	<i>Autosynthesis</i>
Categories		$\mathfrak{Cat}$			[AL19, Nor19, NA24]'s swamp
CwFs		$\mathfrak{CwF}$			
GAT signatures		$\mathfrak{CwF}^{++} \dots \text{ONEGAT}$			

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  - ▶ Yes: [Neu25]

<i>GAT signature</i>	<i>Category of Algs.</i>	<i>Initial Algebra</i>	<i>Concrete CwF</i>	<i>Fibrant Model</i>	<i>Autosynthesis</i>
Categories					
CwFs					
GAT signatures	$\mathfrak{Cat}$	$\mathfrak{Env}$	$\mathfrak{Env}^{++}$	$\dots$	ONEGAT

<i>GAT signature</i>	<i>Category of Algs.</i>	<i>Initial Algebra</i>	<i>Concrete CwF</i>	<i>Fibrant Model</i>	<i>Autosynthesis</i>
Categories					
CwFs					
GAT signatures	$\mathfrak{Cat}$ $\mathfrak{CwF}$ $\mathfrak{CwF}^{++}$	$\dots$	ONEGAT		here be dragons

- [ABK<sup>+</sup>21] Thorsten Altenkirch, Simon Boulier, Ambrus Kaposi, Christian Sattler, and Filippo Sestini.  
Constructing a universe for the setoid model.  
In *FoSSaCS*, pages 1–21, 2021.
- [AL19] Benedikt Ahrens and Peter LeFanu Lumsdaine.  
Displayed categories.  
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The End  
(thank you!)