# Applications of Optimality Condition and Duality Theory

Yinyu Ye

Department of Management Science and Engineering

Stanford University

Stanford, CA 94305, U.S.A.

http://www.stanford.edu/˜yyye

## Arrow-Debreu's Exchange Market with Linear Economy

Each trader $i$, equipped with a good bundle vector $\mathbf{w}_i$, trade with others to maximize its individual utility function. The equilibrium price is an assignment of prices to goods so as when every producer sells his/her own good bundle and buys a maximal bundle of goods then the market clears. Thus, trader $i$'s optimization problem, for given prices $p_j$, $j \in G$, is

$$\begin{array}{ll} \text{maximize} & \mathbf{u}_i^T \mathbf{x}_i := \sum_{j \in P} u_{ij} x_{ij} \\ \text{subject to} & \mathbf{p}^T \mathbf{x}_i := \sum_{j \in P} p_j x_{ij} \le \mathbf{p}^T \mathbf{w}_i, \\ & x_{ij} \ge 0, \quad \forall j, \end{array}$$

Then, the equilibrium price vector is the one such that there are maximizers $\mathbf{x}(\mathbf{p})_i$s

$$\sum_i x(\mathbf{p})_{ij} = \sum_i w_{ij}, \ \forall j.$$

## Example of Arrow-Debreu's Model

Traders $1, 2$ have good bundle

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Their optimization problems for given prices $p_x, p_y$ are:

$$\begin{array}{rl} \max & 2x_1 + y_1 \\ \text{s.t.} & p_x \cdot x_1 + p_y \cdot y_1 \leq p_x, \\ & x_1, y_1 \geq 0 \end{array} \qquad \begin{array}{rl} \max & 3x_2 + y_2 \\ \text{s.t.} & p_x \cdot x_2 + p_y \cdot y_2 \leq p_y \\ & x_2, y_2 \geq 0. \end{array}$$

One can normalize the prices $\mathbf{p}$ such that one of them equals $1$. This would be one of the problems in HW2.

## Equilibrium conditions of the Arrow-Debreu market

Similarly, the necessary and sufficient equilibrium conditions of the Arrow-Debreu market are

$$\begin{array}{rl} \frac{\mathbf{u}_i^T \mathbf{x}_i}{\mathbf{p}^T \mathbf{w}_i} \cdot p_j \geq u_{ij}, & \forall i, j, \\ \sum_i x_{ij} = \sum_i w_{ij} & \forall j, \\ p_j > 0, \ \mathbf{x}_i \geq \mathbf{0}, & \forall i, j; \end{array}$$

where the budget for trader $i$ is replaced by $\mathbf{p}^T \mathbf{w}_i$. Again, the nonlinear inequality can be rewritten as

$$\log(\mathbf{u}_i^T \mathbf{x}_i) + \log(p_j) - \log(\mathbf{p}^T \mathbf{w}_i) \geq \log(u_{ij}), \ \forall i, j, \ u_{ij} > 0.$$

Let $y_j = \log(p_j)$ or $p_j = e^{y_j}$ for all $j$. Then, these inequalities become

$$\log(\mathbf{u}_i^T \mathbf{x}_i) + y_j - \log(\sum_j w_{ij} e^{y_j}) \geq \log(u_{ij}), \ \forall i, j, \ u_{ij} > 0.$$

Note that the function on the left is concave in $\mathbf{x}_i$ and $y_j$.

**Theorem 1** *The equilibrium set of the Arrow-Debreu Market is convex in allocations and the logarithmic of prices.*

## Exchange Markets with Other Economies

Cobb-Douglas Utility:

$$u_i(\mathbf{x}_i) = \prod_{j \in G} x_{ij}^{u_{ij}}, \; x_{ij} \geq 0.$$

Leontief Utility:

$$u_i(\mathbf{x}_i) = \min_{j \in G}\{\frac{x_{ij}}{u_{ij}}, \; x_{ij} \geq 0.\}.$$

Again, the equilibrium price vector is the one such that there are maximizers to clear the market.

## Recall Sensor Network Localization

Given $\mathbf{a}_k \in \mathbf{R}^d$, $d_{ij} \in N_x$, and $\hat{d}_{kj} \in N_a$, find $\mathbf{x}_i \in \mathbf{R}^d$ such that

$$\begin{aligned} \|\mathbf{x}_i - \mathbf{x}_j\|^2 &= d_{ij}^2, \ \forall\,(i,j) \in N_x,\ i<j, \\ \|\mathbf{a}_k - \mathbf{x}_j\|^2 &= \hat{d}_{kj}^2, \ \forall\,(k,j) \in N_a, \end{aligned}$$

$(ij)$ ($(kj)$) connects points $\mathbf{x}_i$ and $\mathbf{x}_j$ ($\mathbf{a}_k$ and $\mathbf{x}_j$) with an edge whose Euclidean length is $d_{ij}$ ($\hat{d}_{kj}$).

Does the system have a localization or realization of all $\mathbf{x}_j$'s? Is the localization unique? Is there a certification for the solution to make it reliable or trustworthy? Is the system partially localizable with certification?

The SNL problem is closely related to Data Dimension Reduction, Molecular Confirmation, Graph Realization/Embedding, etc.. and it is one of the major topics in Dada Sciences.

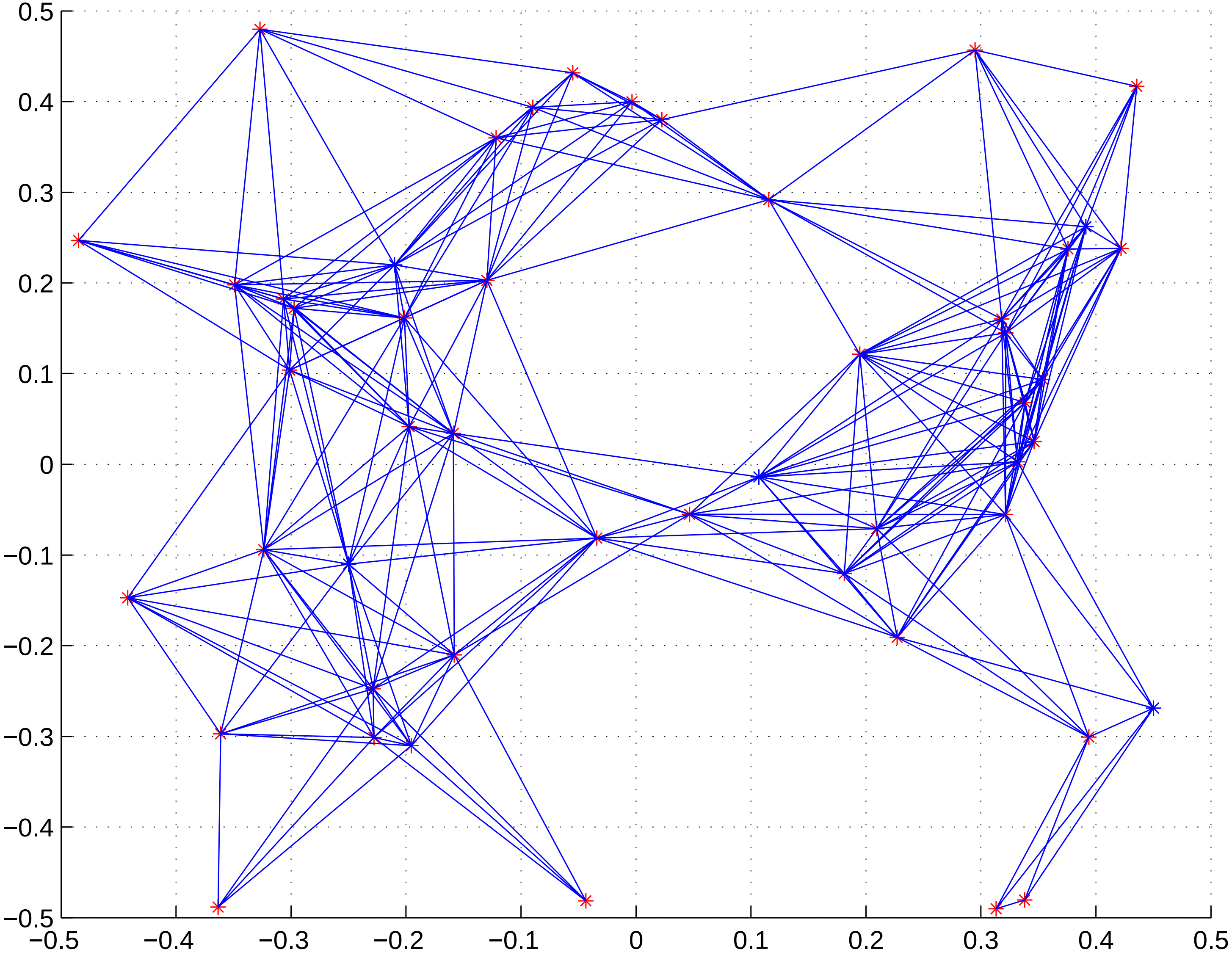


Figure 1: 50-node 2-D Sensor Localization

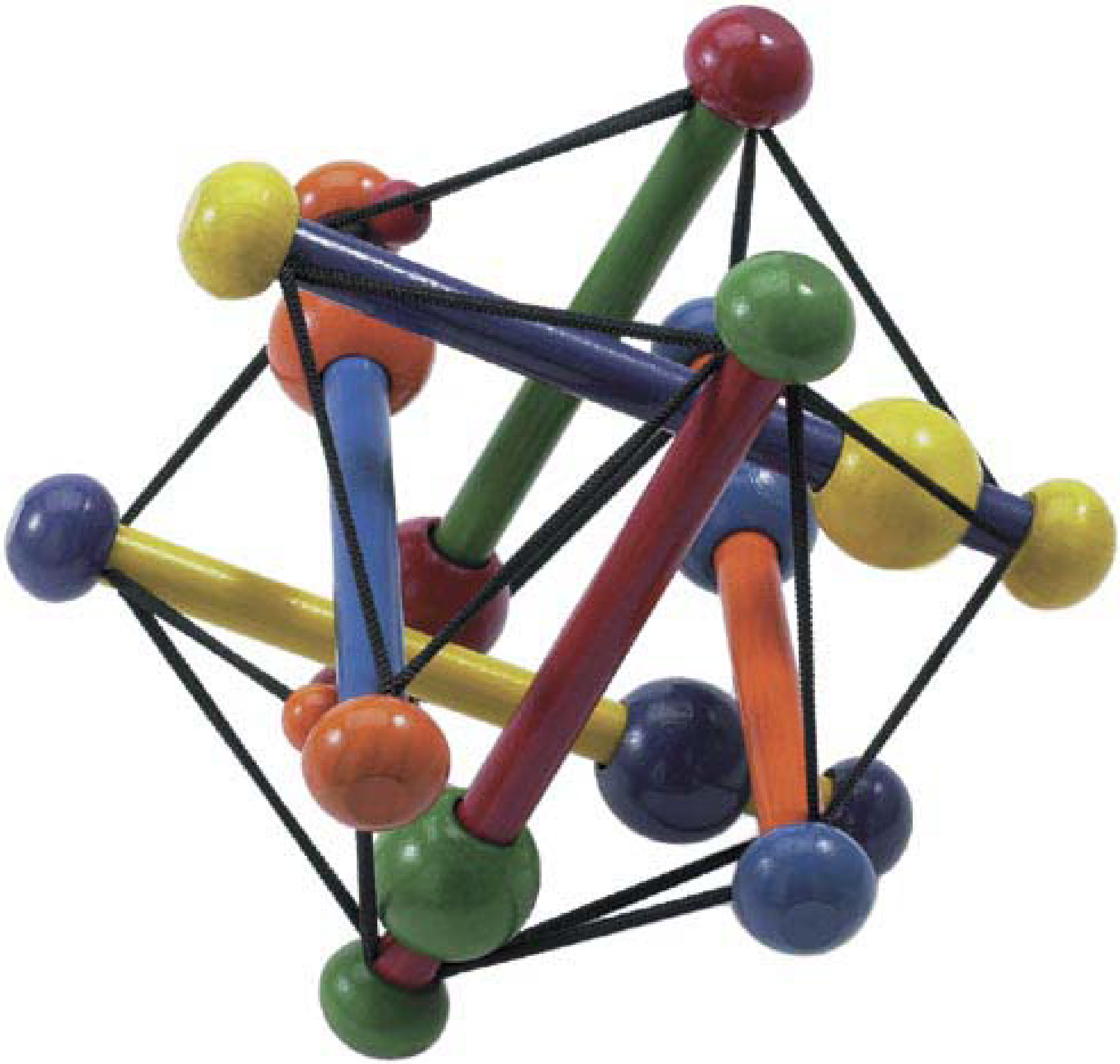

Figure 2: A 3-D Tensegrity Graph Toy; provided by Anstreicher

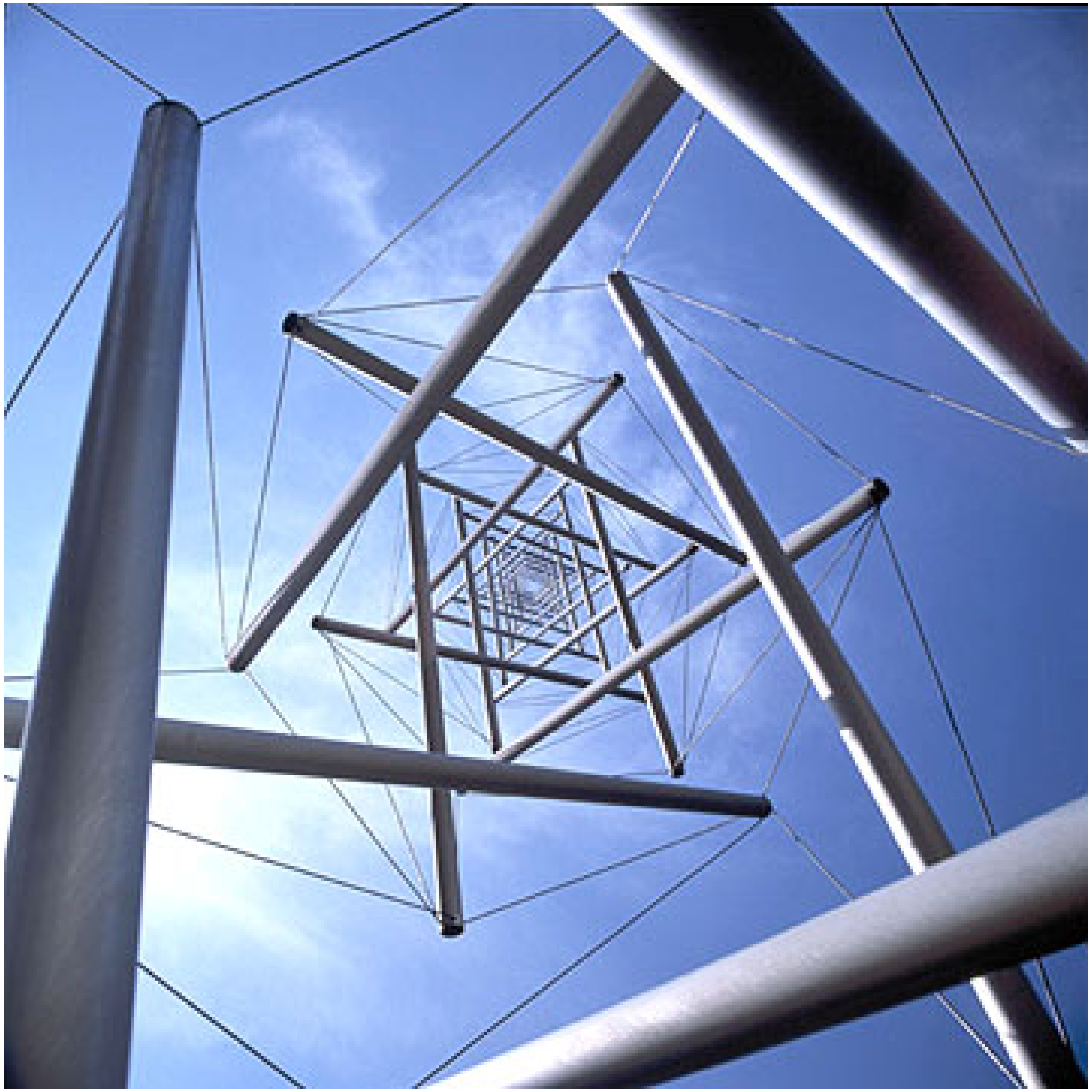

Figure 3: A 3-D Tensegrity Graph Tower; provided by Anstreicher

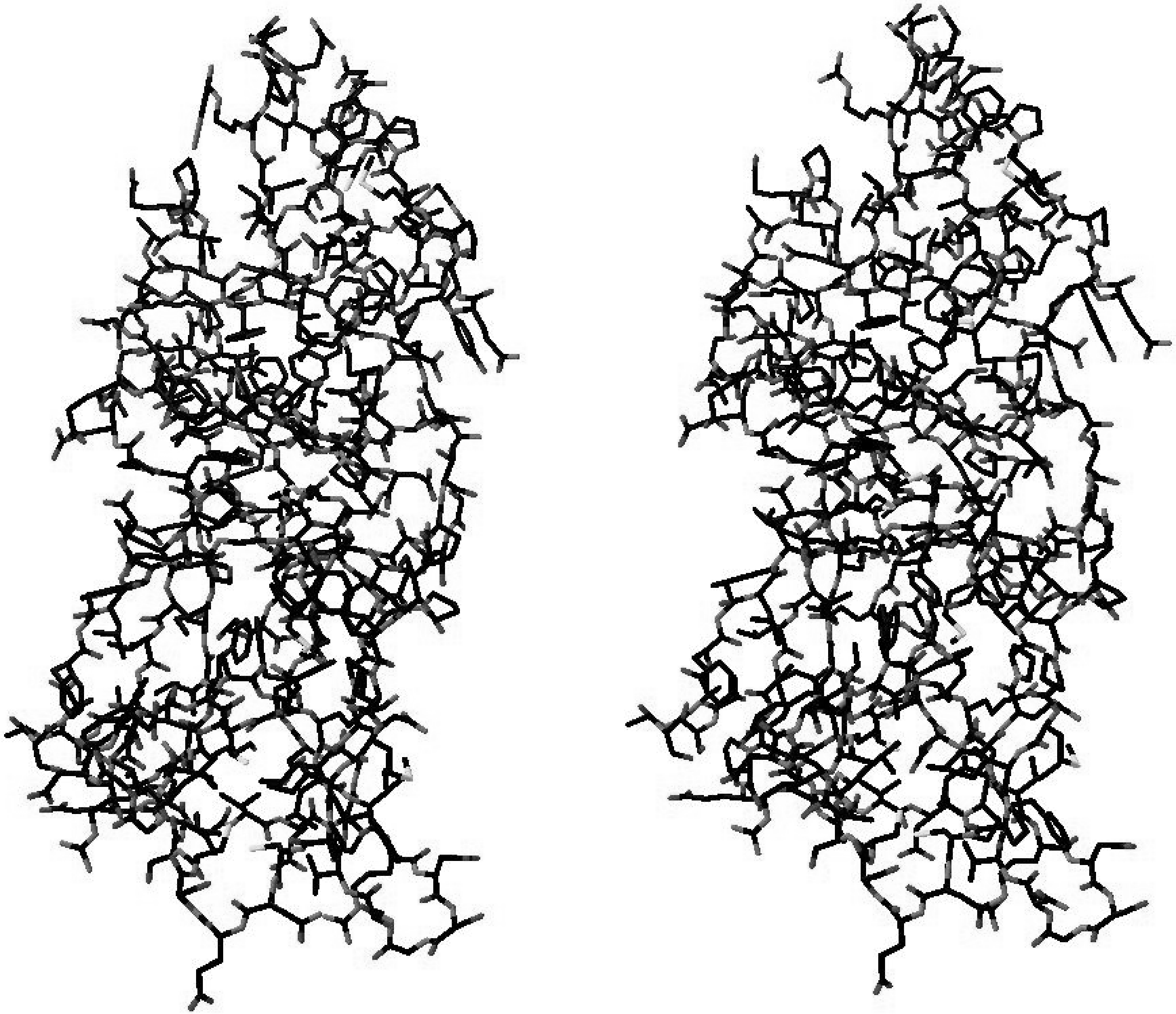

Figure 4: Molecular Conformation: 1F39(1534 atoms) with 85% of distances below $6\mathbf{r}$A and 10% noise on upper and lower bounds

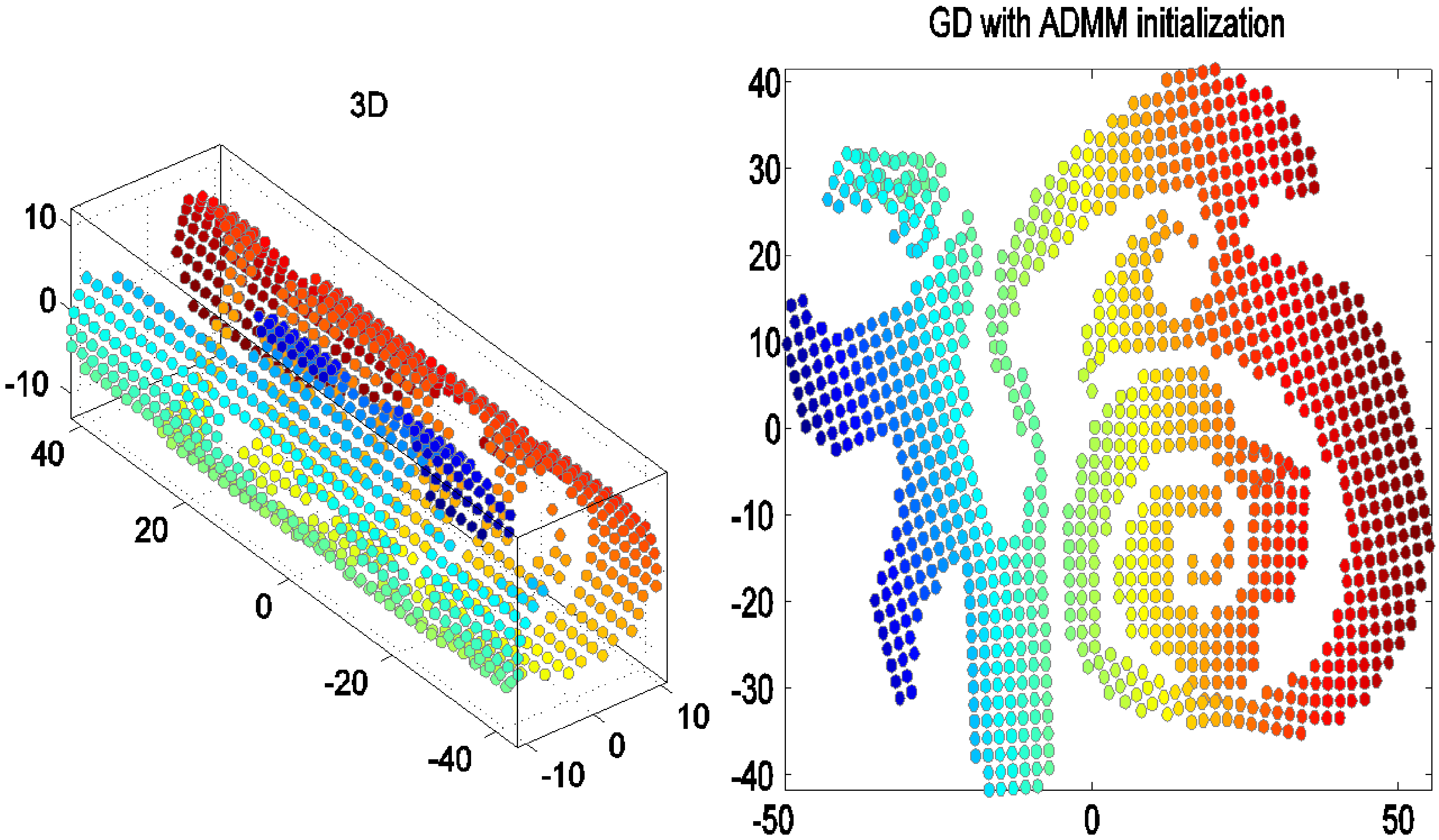


Figure 5: Dimension Reduction: Unfolding Scroll

## Variable Matrix Representation

Let $X = [\mathbf{x}_1\ \mathbf{x}_2\ ...\ \mathbf{x}_n]$ be the $d \times n$ matrix that needs to be determined and $\mathbf{e}_j$ be the vector of all zero except $1$ at the $j$th position. Then

$$\mathbf{x}_i - \mathbf{x}_j = X(\mathbf{e}_i - \mathbf{e}_j) \quad \text{and} \quad \mathbf{a}_k - \mathbf{x}_j = [I\ \ X](\mathbf{a}_k; -\mathbf{e}_j);$$

$$\begin{aligned} d_{ij}^2 = \|\mathbf{x}_i - \mathbf{x}_j\|^2 &= (\mathbf{e}_i - \mathbf{e}_j)^T X^T X (\mathbf{e}_i - \mathbf{e}_j), \\ \hat{d}_{kj}^2 = \|\mathbf{a}_k - \mathbf{x}_j\|^2 &= (\mathbf{a}_k; -\mathbf{e}_j)^T [I\ \ X]^T [I\ \ X] (\mathbf{a}_k; -\mathbf{e}_j) \\ &= (\mathbf{a}_k; -\mathbf{e}_j)^T \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix} (\mathbf{a}_k; -\mathbf{e}_j). \end{aligned}$$

Or, equivalently,

$$\begin{aligned} (\mathbf{e}_i - \mathbf{e}_j)^T Y (\mathbf{e}_i - \mathbf{e}_j) &= d_{ij}^2, \ \forall\, i, j \in N_x,\ i < j, \\ (\mathbf{a}_k; -\mathbf{e}_j)^T \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} (\mathbf{a}_k; -\mathbf{e}_j) &= \hat{d}_{kj}^2, \ \forall\, k, j \in N_a, \\ Y &= X^T X. \end{aligned}$$

## SDP Relaxation and SDP Standard Form

Relax $Y = X^T X$ to $Y \succeq X^T X$. The matrix inequality is equivalent to

$$Z := \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq \mathbf{0}.$$

Matrix $Z$ has rank at least $d$; if it's $d$, then $Y = X^T X$, and the converse is also true.

The SDP relaxation becomes: Find a symmetric matrix $Z \in \mathbf{R}^{(d+n)\times(d+n)}$ such that

$$\begin{aligned}
&Z_{1:d,1:d} = I \\
&(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T \bullet Z = d_{ij}^2, \ \forall\, i, j \in N_x,\ i < j, \\
&(\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \bullet Z = \hat{d}_{kj}^2, \ \forall\, k, j \in N_a, \\
&Z \succeq \mathbf{0}.
\end{aligned}$$

If every sensor point is connected, directly or indirectly, to an anchor point, then the solution set must be bounded.

## Sensor Localization SDP Relaxation in 2D

$$
\begin{aligned}
&(1;0;\mathbf{0})(1;0;\mathbf{0})^T \bullet Z = 1, \ (w_1) \\
&(0;1;\mathbf{0})(0;1;\mathbf{0})^T \bullet Z = 1, \ (w_2) \\
&(1;1;\mathbf{0})(1;1;\mathbf{0})^T \bullet Z = 2, \ (w_3) \\
&(\mathbf{0};\mathbf{e}_i - \mathbf{e}_j)(\mathbf{0};\mathbf{e}_i - \mathbf{e}_j)^T \bullet Z = d_{ij}^2, \ \forall\, i,j \in N_x, \ i<j, \ (w_{ij}) \\
&(\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \bullet Z = \hat{d}_{kj}^2, \ \forall\, k,j \in N_a, \ (\hat{w}_{kj}) \\
&Z \succeq \mathbf{0}.
\end{aligned}
$$

$$
\bar{Z} = \begin{pmatrix} I & \bar{X} \\ \bar{X}^T & \bar{X}^T\bar{X} \end{pmatrix} = (I,\ \bar{X})^T(I,\ \bar{X}) \in S^{n+2}
$$

is a feasible rank-2 solution for the relaxation, where $\bar{X} = [\bar{\mathbf{x}}_1\ \bar{\mathbf{x}}_2\ \ldots\ \bar{\mathbf{x}}_n]$ and $\bar{\mathbf{x}}_j$ is the true location of sensor $j$.

## The Dual of the SDP Relaxation in 2D

$$
\begin{array}{ll}
\text{min} & w_1 + w_2 + 2w_3 + \sum_{i<j\in N_x} w_{ij} d_{ij}^2 + \sum_{k,j\in N_a} \hat{w}_{kj} \hat{d}_{kj}^2 \\
\text{s.t.} & w_1(1;0;\mathbf{0})(1;0;\mathbf{0})^T + w_2(0;1;\mathbf{0})(0;1;\mathbf{0})^T + w_3(1;1;\mathbf{0})(1;1;\mathbf{0})^T + \\
& \sum_{i<j\in N_x} w_{ij}(\mathbf{0};\mathbf{e}_i - \mathbf{e}_j)(\mathbf{0};\mathbf{e}_i - \mathbf{e}_j)^T + \sum_{k,j\in N_a} \hat{w}_{kj}(\mathbf{a}_k;-\mathbf{e}_j)(\mathbf{a}_k;-\mathbf{e}_j)^T \succeq \mathbf{0}
\end{array}
$$

Variable $\hat{w}_{kj}$: internal/tensional force on edge $ij$; dual objective is the potential energy of the network.

The left-hand matrix, also in $S^{n+2}$, is called the stress matrix.

Since the primal is feasible, the minimal value of the dual is not less than $0$. Note that all $\mathbf{0}$ is an minimal solution for the dual. Thus, there is no duality gap.

## Duality Theorem for SNL

**Theorem 2** *Let $\bar{Z}$ be a feasible solution for SDP and $\bar{U}$ be an optimal stress matrix of the dual. Then,*

1. *complementarity condition holds: $\bar{Z} \bullet \bar{U} = 0$ or $\bar{Z}\bar{U} = \mathbf{0}$;*
2. *Rank$(\bar{Z})$ + Rank$(\bar{U}) \leq 2 + n$;*
3. *Rank$(\bar{Z}) \geq 2$ and Rank$(\bar{U}) \leq n$.*

An immediate result from the theorem is the following:

**Corollary 1** *If an optimal dual stress matrix has rank $n$, then every solution of the SDP has rank $2$, that is, the SDP relaxation solves the original problem exactly. Such a sensor network with distance information is called Strongly Localizable (SL).*

Physical interpretation: All stresses or internal forces are balanced at every sensor point.

## Theoretical Analyses on Sensor Network Localization

A sensor network is $2$-Universally-Localizable (UL), weaker than SL, if there is a unique localization in $\mathbf{R}^2$ and there is no $x_j \in \mathbf{R}^h,\ j = 1, ..., n$, where $h > 2$, such that

$$\|x_i - x_j\|^2 = d_{ij}^2,\ \forall\, i, j \in N_x,\ i < j,$$
$$\|(a_k; \mathbf{0}) - x_j\|^2 = \hat{d}_{kj}^2,\ \forall\, k, j \in N_a.$$

The latter says that the problem cannot be localized in a higher dimension space where anchor points are simply augmented to $(a_k; \mathbf{0}) \in \mathbf{R}^h$, $k = 1, ..., m$.

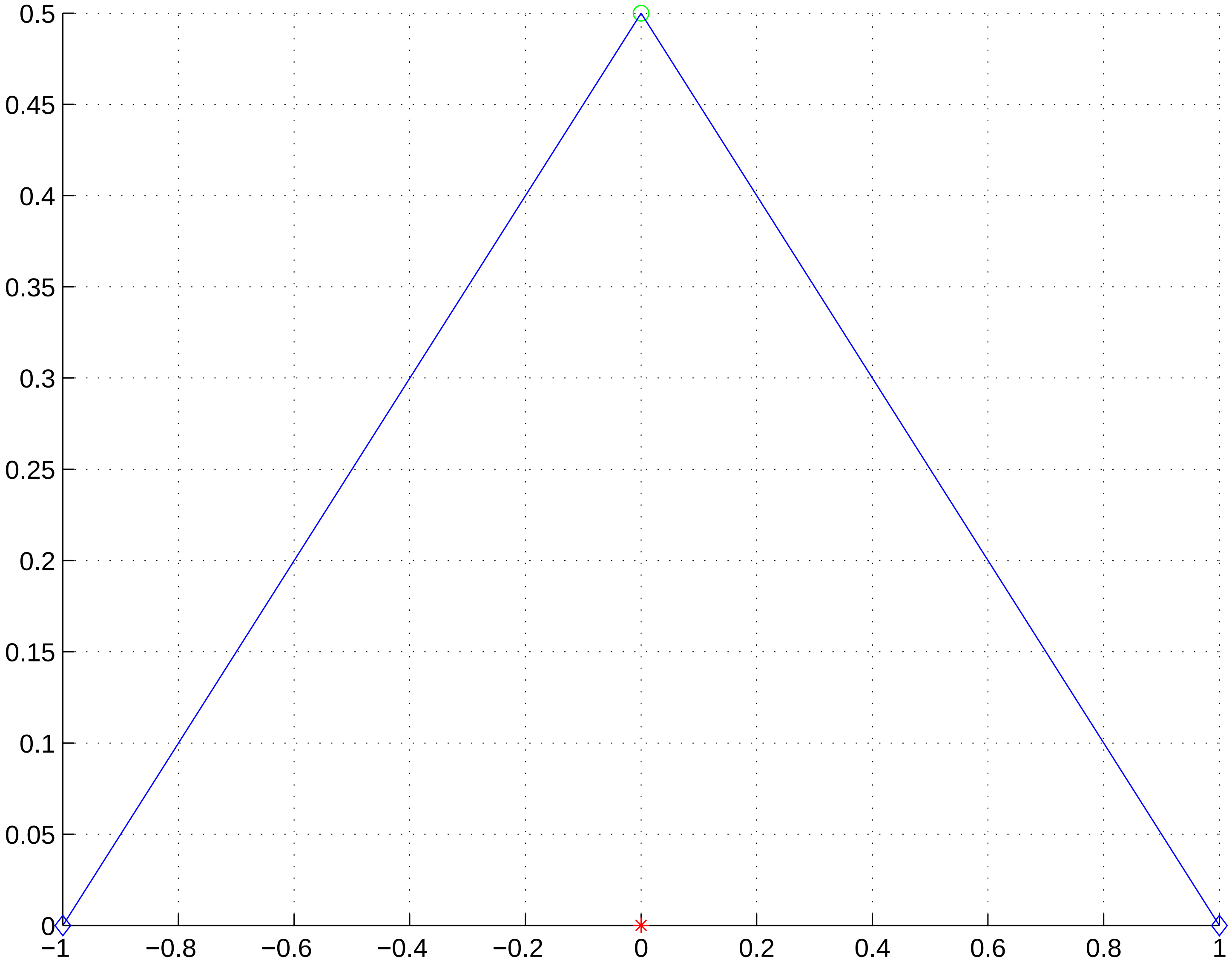


Figure 6: One sensor-Two anchors: Not localizable

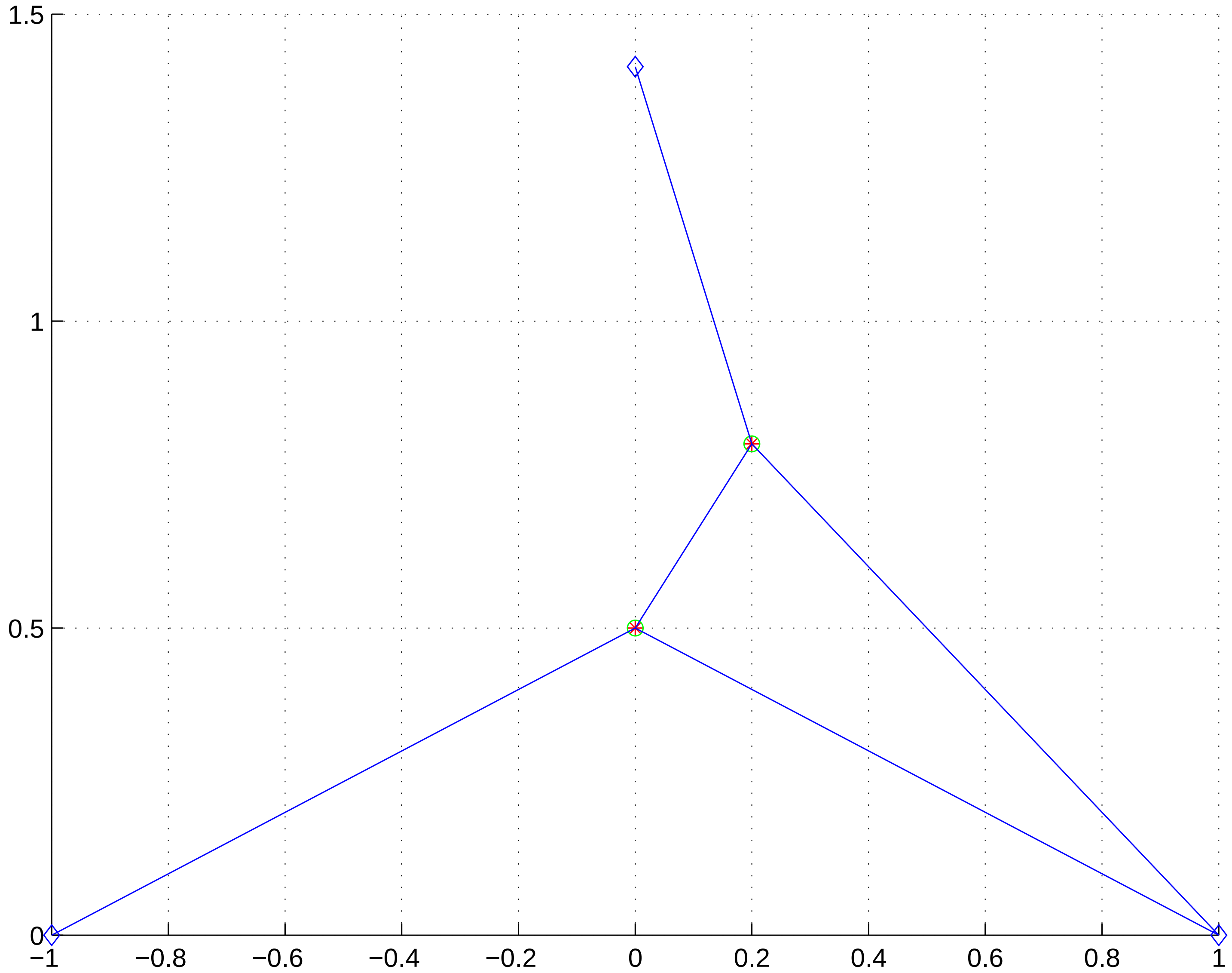


Figure 7: Two sensor-Three anchors: Strongly Localizable

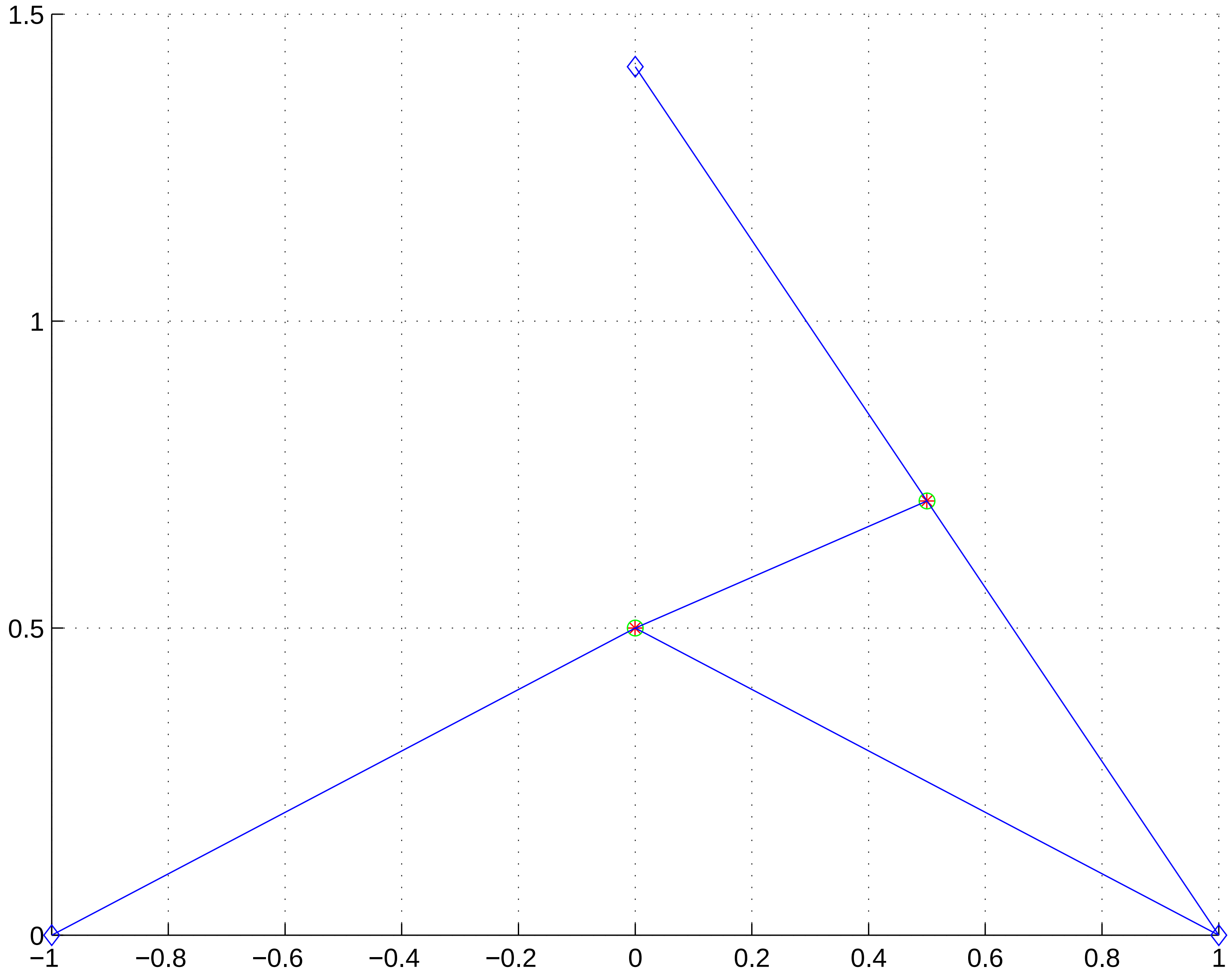


Figure 8: Two sensor-Three anchors: Localizable but not Strongly

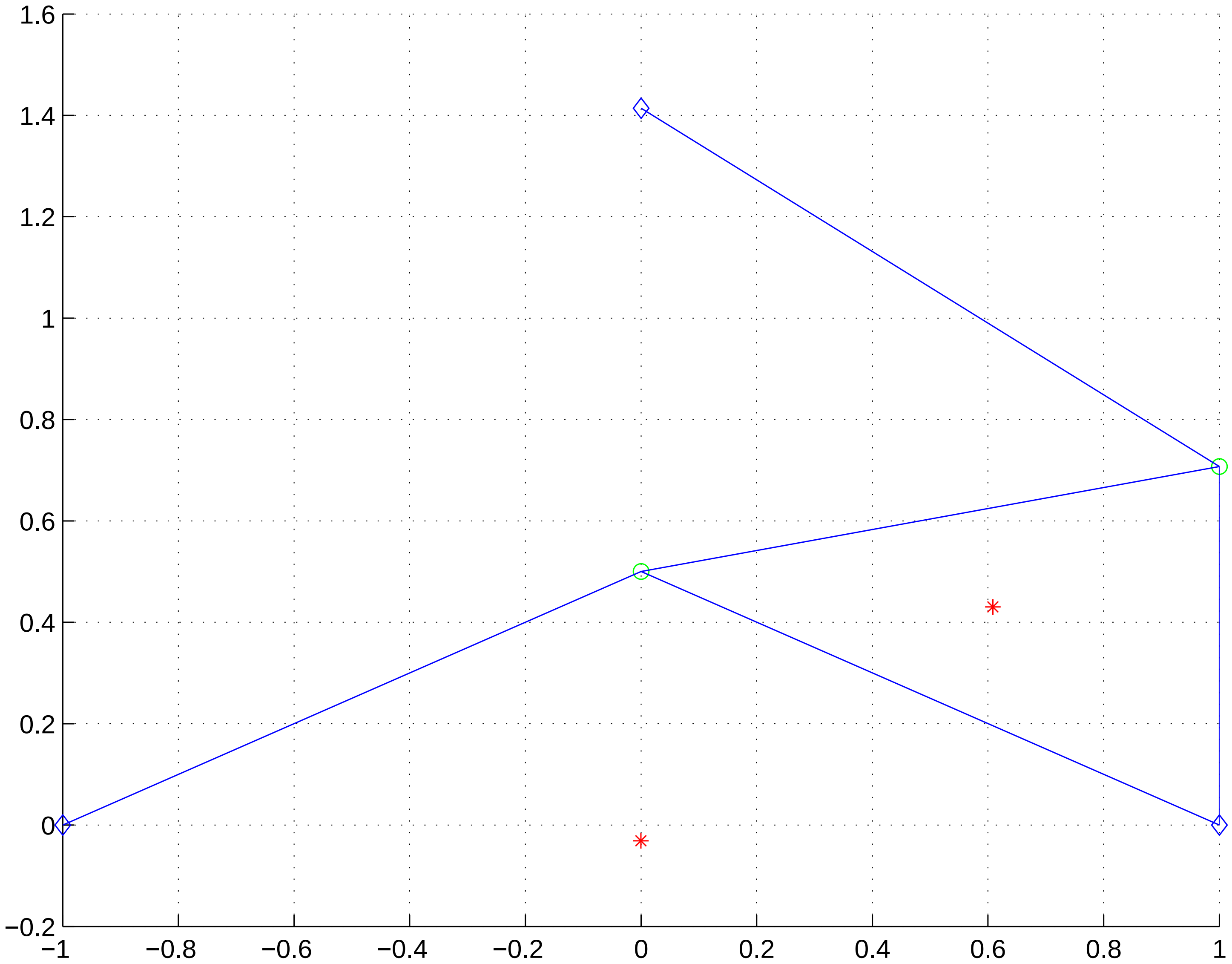


Figure 9: Two sensor-Three anchors: Not Localizable

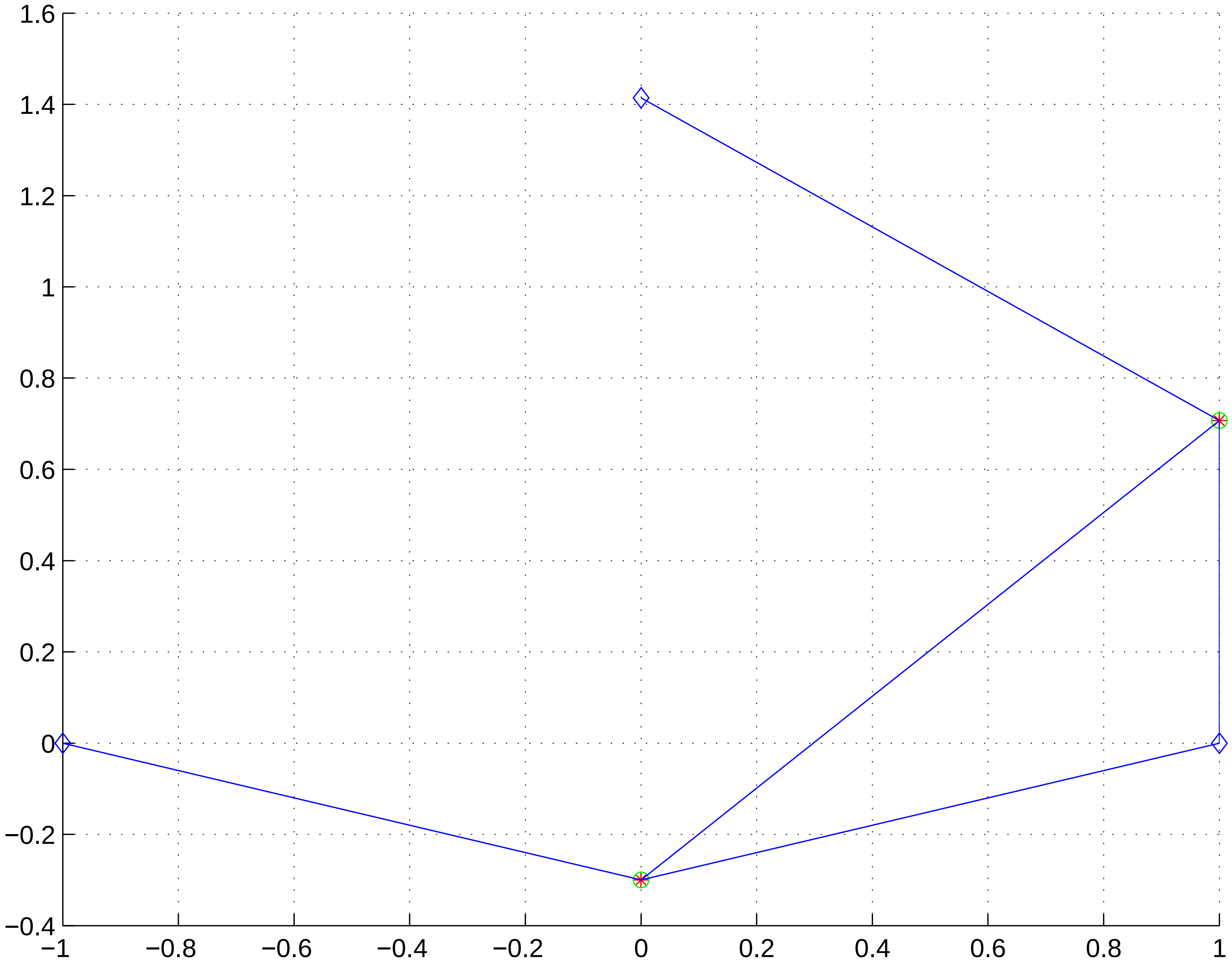


Figure 10: Two sensor-Three anchors: Strongly Localizable

## UL Problems can be Localized by the SDP Relaxation

**Theorem 3** *The following statements are equivalent:*

1. *The sensor network is $2$-universally-localizable;*
2. *The max-rank solution of the SDP relaxation has rank $2$;*
3. *The solution matrix has $Y = X^T X$ or Tr$(Y - X^T X) = 0$ .*

For the following SNL problems:

- If every edge length is specified, then the sensor network is $2$-universally-localizable (Schoenberg 1942);
- there is a sensor network (trilateral graph), with only $O(n)$ edge lengths specified, that is $2$-universally-localizable (So 2007);
- if one sensor with its edge lengths to three anchors (in general positions) are specified, then it is $2$-strongly-localizable (one of problems in HW2).

# One Sensor and Three Anchors

Find $\mathbf{x}_1 \in \mathbf{R}^2$ such that

$$\|\mathbf{a}_k - \mathbf{x}_1\|^2 = \hat{d}_{kj}^2, \text{ for } k = 1, 2, 3,$$

Let $\bar{\mathbf{x}}_1$ be the true position of the sensor.

$$\begin{aligned}
&(1; 0; 0)(1; 0; 0)^T \bullet Z = 1, \\
&(0; 1; 0)(0; 1; 0)^T \bullet Z = 1, \\
&(1; 1; 0)(1; 1; 0)^T \bullet Z = 2, \\
&(\mathbf{a}_k; -1)(\mathbf{a}_k; -1)^T \bullet Z = \hat{d}_{k1}^2, \text{ for } k = 1, 2, 3, \\
&Z \succeq \mathbf{0}.
\end{aligned}$$

$$\bar{Z} = \begin{pmatrix} I & \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_1^T & \bar{\mathbf{x}}_1^T \bar{\mathbf{x}}_1 \end{pmatrix} = (I, \ \bar{\mathbf{x}}_1)^T (I, \ \bar{\mathbf{x}}_1) \in S^3$$

is a feasible rank-2 solution for the relaxation.

## The Dual and Dual Stress Matrix

Does an optimal stress matrix $U \in S^3$ have rank $1$? More specifically, are there dual stress variables such that the stress matrix

$$\bar{U} = (-\bar{\mathbf{x}}_1; 1)(-\bar{\mathbf{x}}_1; 1)^T?$$

If true, then

$$\bar{Z}\bar{U} = (I,\ \bar{\mathbf{x}}_1)^T(I,\ \bar{\mathbf{x}}_1)(-\bar{\mathbf{x}}_1; 1)(-\bar{\mathbf{x}}_1; 1)^T = (I,\ \bar{\mathbf{x}}_1)^T\mathbf{0}(-\bar{\mathbf{x}}_1; 1)^T = \mathbf{0},$$

so that the SDP relaxation problem solves the original SNP probem exactly!