CME 307 / MS&E 311 — Winter 2016-2017
Optimization — Jan 13, 2017
Prof. Yinyu Ye — Due Thursday Jan 26 5pm

Homework Assignment 1
Discuss Session Friday Jan 27 in Class

Optional Reading. Read Luenberger and Ye's *Linear and Nonlinear Programming Fourth Edition* Chapters 1, 2, 6 and Appendices A and B.

Solve the following problems (10pts each for the first 8 problems):

1. Do the followings:

   - Consider the feasible set

     $$F := \{x \in R^n : \ Ax = b, \ x \geq 0\}$$

     where data matrix $A^{m \times n}$ and vector $b \in R^m$. Prove that F is a convex set.

   - Fix data matrix A and consider the b-data set for F defined above:

     $$B := \{b \in R^m : \ F \text{ is not empty}\}.$$

     Prove that B is a convex set.

   - Fix data matrix A and consider the linearly constrained convex minimization problem

     $$\begin{aligned} z(b) := \min \quad & f(x) \\ \text{s.t.} \quad & Ax = b, \ x \geq 0 \end{aligned}$$

     where $f(x)$ is a convex function, and the minimal value function $z(b)$ is an implicit function of b. Prove that $z(b)$ is a convex function of $b \in B$.

2. Show that the the dual cone of the n-dimensional nonnegative orthant cone R^n_+ is itself, that is,

   $$(R^n_+)^* = R^n_+.$$

   (Hint: show that $R^n_+ \subset (R^n_+)^*$ and $(R^n_+)^* \subset R^n_+$.)

3. Using Theorem 5 in Lecture Note #1 to prove that the linear system

$$A^T Ax = A^T b$$

always has a solution x for any given matrix $A \in R^{m\times n}$ and vector $b \in R^m$.

4. Let $g_1, \dots, g_m$ be a collection of concave functions on R^n such that

$$S = \{x : g_i(x) > 0 \text{ for } i = 1, \dots, m\} \neq 0.$$

Show that for any positive constant μ and any convex function f on R^n, the function (called Barrier function)

$$h(x) = f(x) - \mu \sum_{i=1}^{m} \log(g_i(x))$$

is convex over S. (Hint: directly apply the convex/concave function definition or analyze the Hessian of $h(x)$.)

5. Consider the min-risk portfolio management problem in Lecture Note #2

$$\begin{array}{ll} \min & x^T V x \\ \text{s.t.} & r^T x \geq \mu, \\ & e^T x = 1, \ x \geq 0, \end{array}$$

where data vector $r \in R^n$ representing expected return of n stocks, and $V \in R^{n\times n}$ representing co-variance matrix of n stocks, μ representing the desired return of an investment portfolio, and e is the vector of all ones. The decision problem is to allocate a total 100% of asset to each stock to minimize the risk while keep the desired return. Thus, x_i, $i = 1, ..., n$, represents the percentage of the total asset invested in stock i.

- Now, suppose that, for simplicity, the company's policy is to invest in each stock at one of the three possible levels: 0.05%, 0.1% and 0.2%. How to add constraints to enforce this policy? (Hint: for each stock i, define three binary (taking values either 0 or 1) variables y_{i1}, y_{i2} and y_{i3} and let $x_i = 0.05y_{i1} + 0.1y_{i2} + 0.2y_{i3}$ and $y_{i1} + y_{i2} + y_{i3} \leq 1$.)
- Suppose that the company also does not want to invest in more than 20 stocks. How to add constraints to enforce this additional policy?

6. Consider the SOCP problem described in in Lecture Note #3:

$$\begin{array}{ll} \min & 2x_1 + x_2 + x_3 \\ \text{s.t.} & x_1 + x_2 + x_3 = 1, \\ & x_1 - \sqrt{x_2^2 + x_3^2} \geq 0. \end{array}$$

   - Show that the feasible region is a convex set.
   - Try to find a minimizer of the problem and "argue" why it is a minimizer.

7. Prove that the set $\{Ax : \ x \geq 0 \in R^n\}$ is closed and convex cone. (Hint: apply Carathéodory's theorem in Lecture Note #3 to prove the closeness .)

8. Farkas' lemma can be used to derive many other (named) theorems of the alternative. This problem concerns a few of these pairs of systems. Using Farkas's lemma, prove each of the following results.

   - Gordan's Theorem. Exactly one of the following systems has a solution:

$$\text{(i)} \quad Ax > 0$$
$$\text{(ii)} \quad y^T A = 0, \quad y \geq 0, \quad y \neq 0.$$

   - Stiemke's Theorem. Exactly one of the following systems has a solution:

$$\text{(i)} \quad Ax \geq 0, \quad Ax \neq 0$$
$$\text{(ii)} \quad y^T A = 0, \quad y > 0$$

   - Gale's Theorem. Exactly one of the following systems has a solution:

$$\text{(i)} \quad Ax \leq b$$
$$\text{(ii)} \quad y^T A = 0, \quad y^T b < 0, \quad y \geq 0$$

9. (20pts) Computation Team Work: Consider the sensor localization problem on plane R^2 with one sensor and three anchors $a_1 = (1;0)$, $a_2 = (-1;0)$ and $a_3 = (0;2)$. Suppose the Euclidean distances from the sensor to the three anchors are d_1, d_2 and d_3 respectively and known to us.Then, from the anchor and distance information, we can locate the second by finding $x \in R^2$ such that

$$\|x - a_i\|^2 = d_i^2, \ i = 1, 2, 3.$$

   Do the following numerical experimentations:

- Generate any sensor point in the convex hull of the three anchors, compute its distances to three anchors d_i, $i = 1, 2, 3$, respectively. Then solve the SOCP relaxation problem

  $$\|x - a_i\|^2 \le d_i^2, \ i = 1, 2, 3.$$

  Did you find the correct location? What about if the sensor point was in the outside of the convex hull?

- Now try the SDP relaxation

  $$(a_i; -1)(a_i; -1)^T \bullet \begin{pmatrix} I & x \\ x^T & y \end{pmatrix} = d_i^2, \ i = 1, 2, 3; \ \begin{pmatrix} I & x \\ x^T & y \end{pmatrix} \succeq 0 \in S^3,$$

  which can be written in the standard form

  $$\begin{aligned} (1; 0; 0)(1; 0; 0)^T \bullet Z &= 1, \\ (0; 1; 0)(0; 1; 0)^T \bullet Z &= 1, \\ (1; 1; 0)(1; 1; 0)^T \bullet Z &= 2, \\ (a_i; -1)(a_i; -1)^T \bullet Z &= d_i^2, \ i = 1, 2, 3, \\ Z &\succeq 0 \in S^3. \end{aligned}$$

  Did you find the correct location everywhere on the plane?

You can download CVX or MOSEK directly to solve these numerical problems.