CME 307 / MS&E 311 — Winter 2016-2017
Optimization — Jan 27, 2017
Prof. Yinyu Ye — Due Thursday Feb 9 5pm

Homework Assignment 2
Discuss Session Friday Feb 10 in Class

Optional Reading. Read Luenberger and Ye's *Linear and Nonlinear Programming 4th Edition* Chapters 4, 6, 11.

Solve the following problems:

1. Consider problem 6) of Homework Assignment 1 where the second-order cone is replaced by the p-order cone for $p \geq 1$:

$$\begin{array}{ll} \min & 2x_1 + x_2 + x_3 \\ \text{s.t.} & x_1 + x_2 + x_3 = 1, \\ & x_1 - \|(x_2,\ x_3)\|_p \geq 0. \end{array}$$

   (a) Write out the conic dual problem.

   (b) Compute the dual optimal solution (y^*, s^*).

   (c) Using the zero duality condition to compute the primal optimal solution x^*.

2. Consider the SOCP relaxation in problem 9) of Homework Assignment 1:

$$\begin{array}{ll} \min & 0^T x \\ \text{s.t.} & \|x - a_i\|^2 \leq d_i^2,\ i = 1, 2, 3. \end{array}$$

   (a) Write down the KKT optimality conditions.

   (b) Then explain/interpret the three optimal multipliers when the true position of the sensor is inside the convex hull of the three anchors.

   (c) What would be the multiplier values if the true position of the sensor is outside the convex hull of the three anchors?

3. Consider the SDP relaxation in problem 9) of Homework Assignment 1:

$$\begin{array}{lll} \max & 0 \bullet Z & \\ \text{s.t.} & (1;0;0)(1;0;0)^T \bullet Z & = 1, \\ & (0;1;0)(0;1;0)^T \bullet Z & = 1, \\ & (1;1;0)(1;1;0)^T \bullet Z & = 2, \\ & (a_i;-1)(a_i;-1)^T \bullet Z & = d_i^2, \ i = 1,2,3, \\ & Z & \succeq 0. \end{array}$$

(a) Write out the SDP dual problem, especially the dual slack matrix $U \in S^3$.

(b) Suppose the true position of the sensor is $\bar{x} \in R^2$. Show that if

$$\bar{U} = (-\bar{x};1)(-\bar{x};1)^T,$$

then it is an optimal slack matrix.

(c) When is $\bar{U} = (-\bar{x};1)(-\bar{x};1)^T$ possible and the solution is unique?

(d) Then explain/interpret the three optimal multipliers corresponding to the three distance equality constraints.

4. Consider convex cone (Lecture Note 4, slide 18)

$$C = \{(t;x) : \ t > 0, \ tc(x/t) \le 0, \ x \in R^2\},$$

where $c(x) \in R$ is a convex function. Construct the dual cone of C for each of the following $c(x)$:

(a) $c(x) = x_1^2 + 2x_2^2 - 2x_1x_2 - 1$ (ellipsoidal cone).

(b) $c(x) = e^{x_1} + e^{x_2} - 1$ (exponential cone).

(c) (Optional) Prove that $tc(x/t)$ is a convex function if $c(.)$ is.

5. Consider the following parametric QCQP problem for a parameter $\kappa > 0$:

$$\begin{array}{ll} \min & (x_1 - 1)^2 + x_2^2 \\ \text{s.t.} & -x_1 + \frac{x_2^2}{\kappa} \ge 0 \end{array} \tag{1}$$

(a) Is $x = 0$ a first-order KKT solution?

(b) Is $x = 0$ a second-order KKT solution for some value of κ?

6. Consider the SVM problem described in Lecture Note #2:

$$\begin{array}{ll} \min & \beta + \mu\|x\|^2 \\ \text{s.t.} & a_i^T x + x_0 + \beta \geq 1, \ \forall i, \\ & b_j^T x + x_0 - \beta \leq -1, \ \forall j, \\ & \beta \geq 0. \end{array}$$

(a) Write out the Lagrangian dual of the SVM problem.

(b) Suppose we have 6 training solution in R^2: $a_1 = (0;0)$, $a_2 = (1;0)$, $a_3 = (0;1)$ and $b_1 = (0;0)$, $b_2 = (-1;0)$, $b_3 = (0;-1)$. Using the optimality conditions to find optimal solutions for $\mu = 0$ and $\mu = 10^{-5}$, respectively. Are the two optimal solutions unique for the given μ?

7. Find the Lagrangian dual of the barrier optimization problem where given parameter $\mu > 0$:

$$\begin{array}{lcl} \min & c^T x - \mu \sum_{j=1}^n \ln(x_j), & \\ \text{s.t.} & Ax & = b \\ & x & > 0, \end{array}$$

where we assume that the problem has an interior-point solution, and there exists a vector $y \in R^m$ such that $c - A^T y > 0$.

What are the first-order KKT optimality conditions?

8. Consider a generalized Arrow–Debreu equilibrium problem in which the market has n agents and m goods. Agent i, $i = 1, ..., n$, has a bundle amount of $w_i = (w_{i1}, w_{i2}, \ldots, w_{im}) \in R^m$ goods initially and has a linear utility function whose coefficients are $u_i = (u_{i1}, u_{i2}, \ldots, u_{im}) > 0 \in R^m$. The goal is to price each good so that the market clears. Note that, given the price vector $p = (p_1, p_2, \ldots, p_m) > 0$, agent i's utility maximization problem is:

$$\begin{array}{ll} \text{maximize} & u_i^T x_i \\ \text{subject to} & p^T x_i \leq p^T w_i \\ & x_i \geq 0 \end{array}$$

(a) For a given $p \in R^m$, write down the optimality conditions for agent i's utility maximization problem. Without loss of generality, you may fixed $p_m = 1$ since the budget constraint are homogeneous in p.

(b) (5pts.) Suppose that $p \in R^m$ and $x_i \in R^m$ satisfy the constraints:

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} w_i$$

$$\frac{u_i^T x_i}{p^T w_i} p_j \geq u_{ij} \qquad \forall i, j$$

$$x_{ij} \geq 0,\ p_j \geq 0 \qquad \forall i, j$$

for $i = 1, \ldots, n$. Show that p is then an equilibrium price vector.

(c) For simplicity, assume that all u_{ij} are positive so that all p_j are positive. By introducing new variables $y_j = \log(p_j)$ for $j = 1, ..., m$, the conditions can be written as follows:

$$\begin{array}{lll} \min & 0 & \\ \text{s.t.} & \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} w_i & \\ & \log(u_i^T x_i) - \log\left(\sum_{k=1}^{m} w_{ik} e^{y_k}\right) + y_j \geq \log(u_{ij}) & \forall i, j \\ & x_{ij} \geq 0, & \forall i, j \end{array}$$

Show that this problem is convex in x_{ij} and y_j. (Hint: Use the fact that $\log\left(\sum_{k=1}^{m} w_{ik} e^{y_k}\right)$ is a convex function in the y_k's.)

(d) Consider the Fisher example on slide 23 of Lecture Note #5 with two agents and two goods, where there is no fixed budgets. Rather, let

$$w_1 = (1;\ 0) \quad \text{and} \quad w_2 = (0;\ 1)$$

that is, agent 1 brings in one unit good x and agent brings in one unit of good y. Find the Arrow–Debreu equilibrium prices, where you may assume $p_y = 1$.

9. (20pts) Computation Team Work: Now consider the sensor localization problem on plane R^2 with two sensors x_1 and x_2 and three anchors $a_1 = (1; 0)$, $a_2 = (-1; 0)$ and $a_3 = (0; 2)$. Suppose we know the (Euclidean) distances from one sensor to a_1 and a_2, denoted by d_{11} and d_{12}; distances of the other to a_2 and a_3, denoted by d_{22} and d_{23}; and the distance between the two sensors, denoted by $\hat{d}_{12}$. Then, from the anchor and distance information we like locate the sensor positions $x_1, x_2 \in R^2$?

Do the following numerical experimentations:

- Generate two sensor points anywhere and try the SOCP relaxation model

$$
\begin{aligned}
\|x_1 - a_i\|^2 &\le d_{1i}^2,\ i = 1, 2 \\
\|x_2 - a_i\|^2 &\le d_{1i}^2,\ i = 2, 3 \\
\|x_1 - x_2\|^2 &\le \hat{d}_{12}^2.
\end{aligned}
$$

  Did you find the correct locations? What have you observed?

- Now try the SDP relaxation: find $X = [x_1,\ x_2] \in R^{2\times 2}$ and

$$
Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \in S^4
$$

  to meet the constraints in the standard form:

$$
\begin{aligned}
(1; 0; 0; 0)(1; 0; 0; 0)^T \bullet Z &= 1, \\
(0; 1; 0; 0)(0; 1; 0; 0)^T \bullet Z &= 1, \\
(1; 1; 0; 0)(1; 1; 0; 0)^T \bullet Z &= 2, \\
(a_i; -1; 0)(a_i; -1; 0)^T \bullet Z &= d_{1i}^2,\ i = 1, 2, \\
(a_i; 0; -1)(a_i; 0; -1)^T \bullet Z &= d_{2i}^2,\ i = 2, 3, \\
(0; 0; 1; -1)(0; 0; 1; -1)^T \bullet Z &= \hat{d}_{12}^2, \\
Z &\succeq 0 \in S^4.
\end{aligned}
$$

  Did you find the correct locations? What have you observed? Can you conclude something?