Statistical Analysis of Networks

Projection & Weighted Graphs

Motivating Example *Revisited*

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Diffusion of Ideas and Technology: The Role of Networks in Influencing the Endorsement and Use of On-Officer Video Cameras

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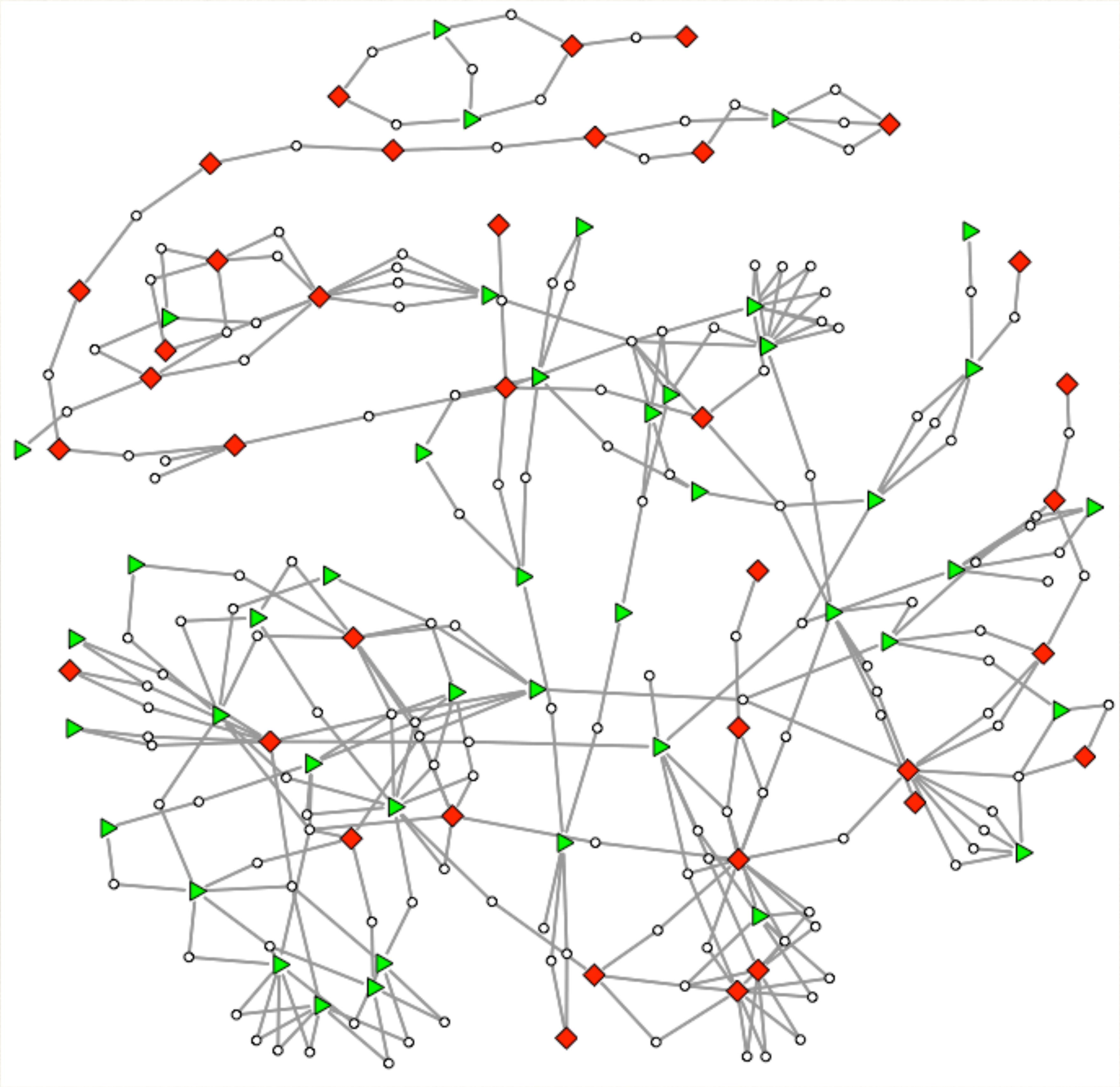
- <u>Questions</u>:
  - How do police officers "frame" body-worn cameras?
  - Is the meaning officers attribute to cameras created and transmitted in groups?

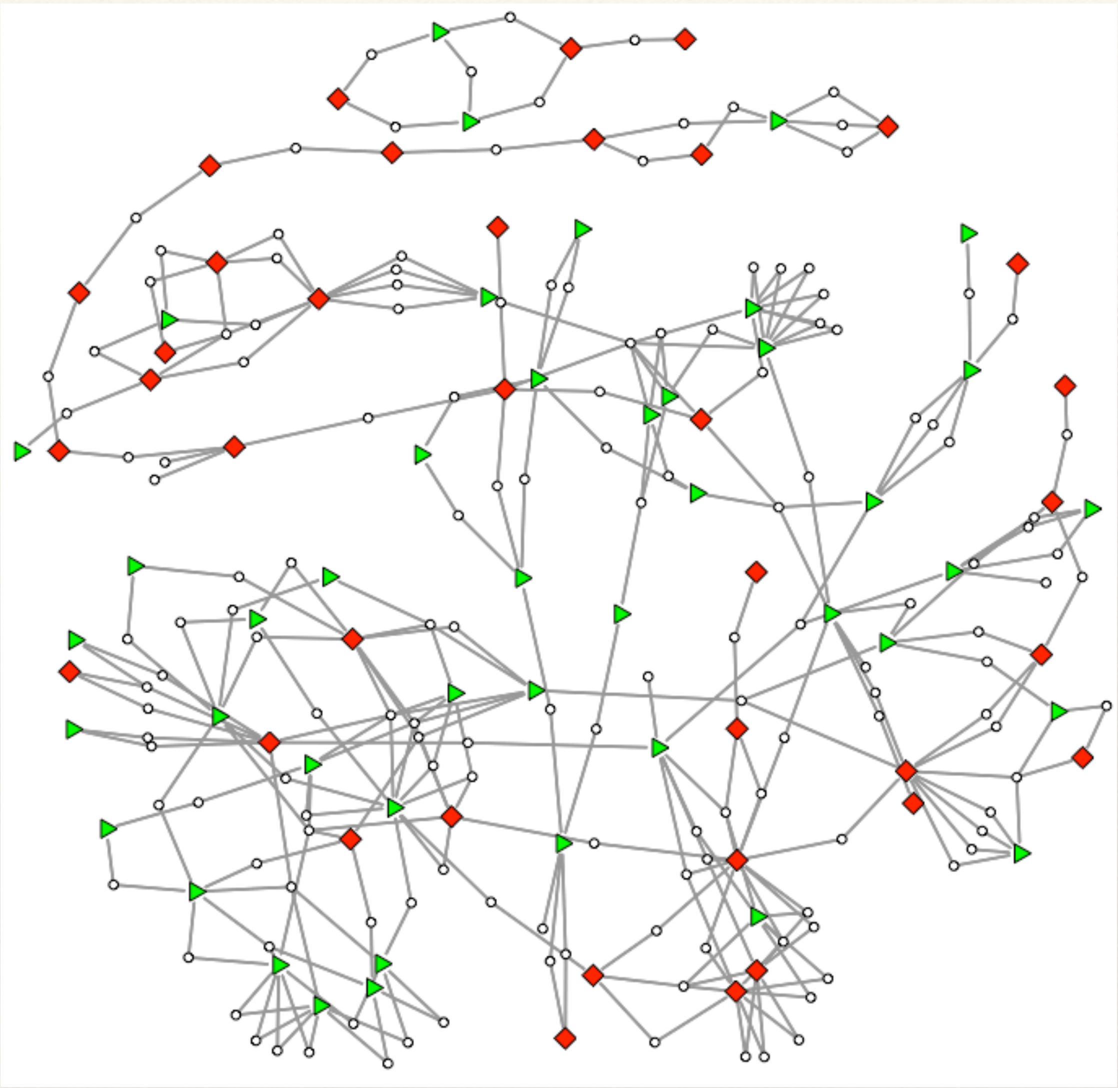
What is the **concept** of interest?

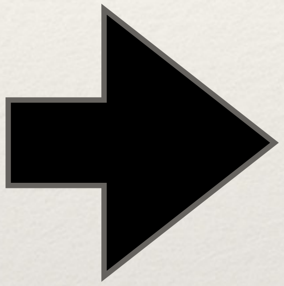
How is it **conceptualized**?

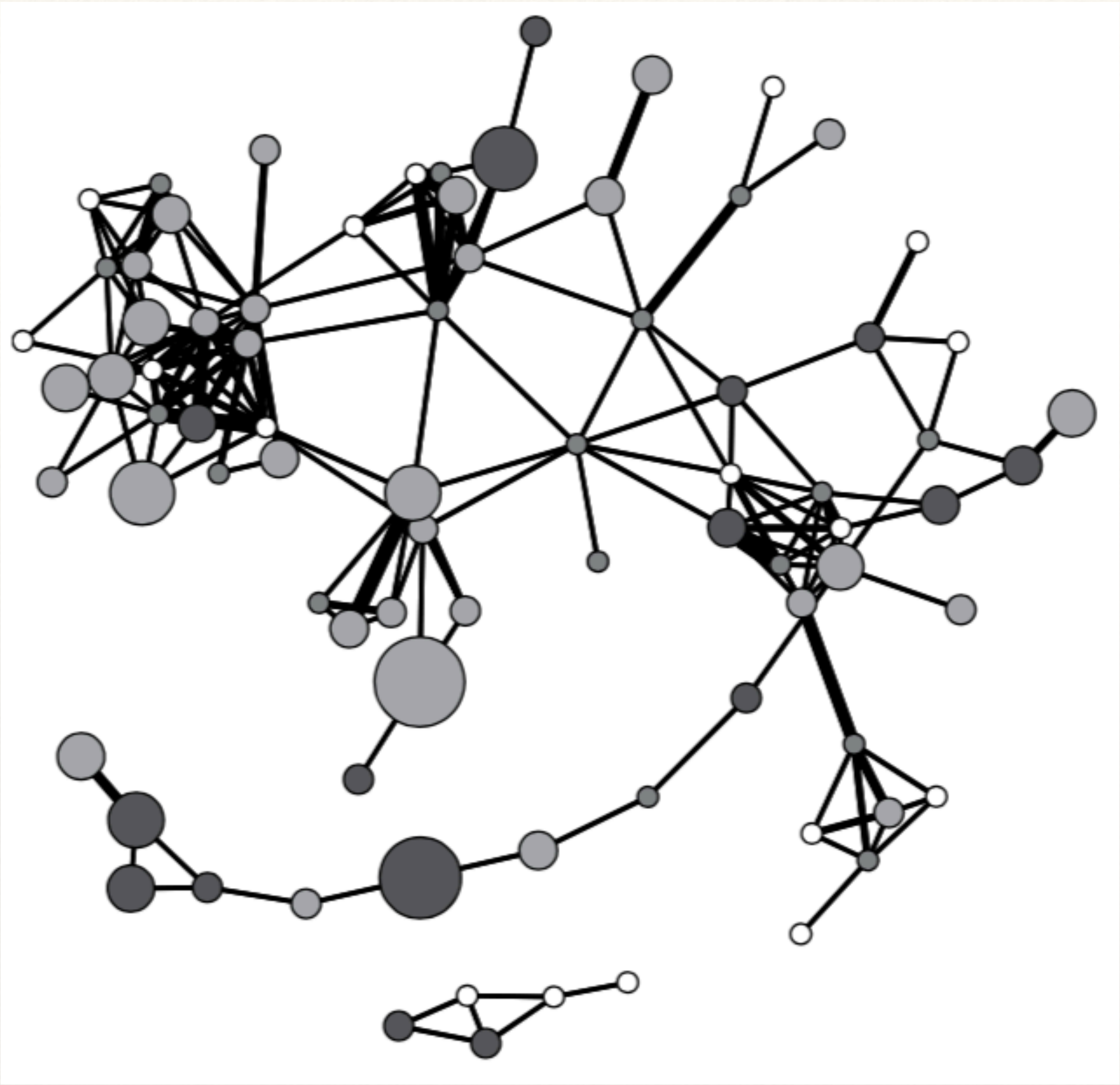
How is it **operationalized**?



Two-mode network of officers connected by incidents







One-mode network of officers connected to officers

What do the connections represent in this network?

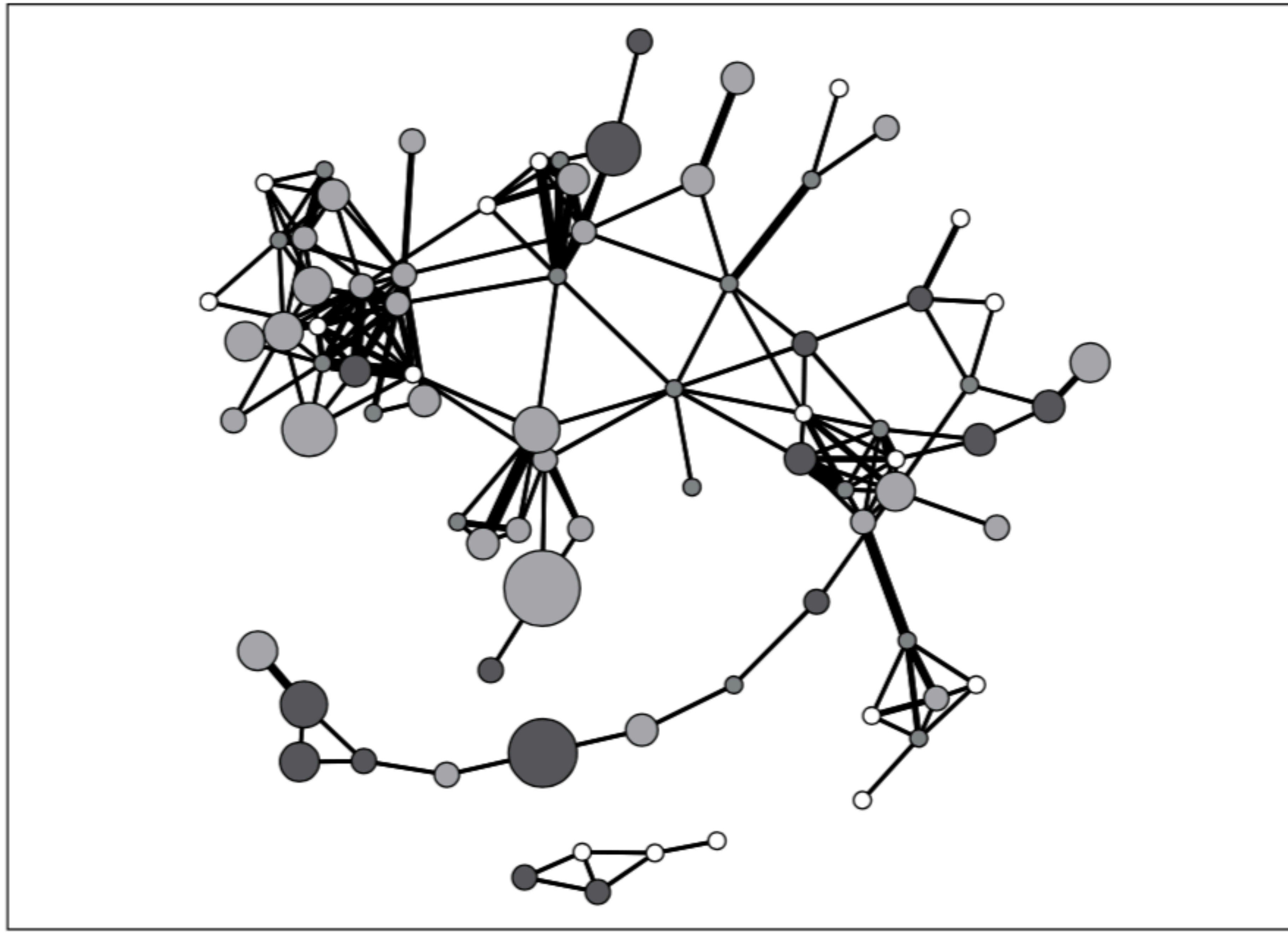


Figure 5. One-mode network of officers.
Note. Node size is proportional to change in legitimacy: darker = more negative; white = no change. Line size is proportional to number of shared incidents.

Findings: Officers views of cameras changed based on who they interacted with through the network

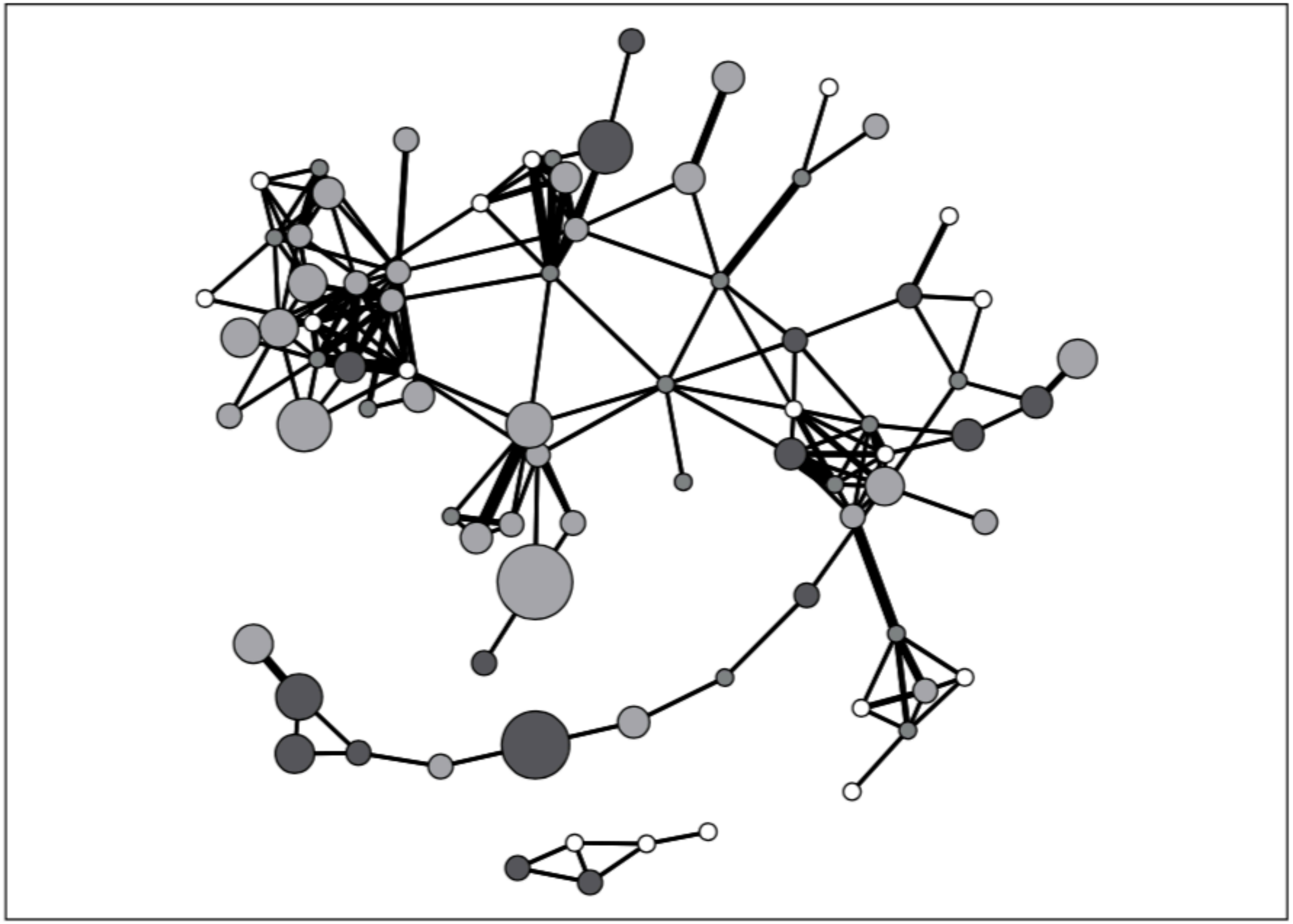


Figure 5. One-mode network of officers.
Note. Node size is proportional to change in legitimacy: darker = more negative; white = no change. Line size is proportional to number of shared incidents.

Statistical Analysis of Networks

Projection & Weighted Graphs

Learning Goals

- At the end of the lecture, you should be able to answer these questions:
  - How can we create **unipartite** graphs from **bipartite** graphs?
  - What is the difference between **dichotomized** projections and **summation** projections?

Projection

- The process by which we map the connectivity between modes to a single mode.
  - <u>Example</u>
    - Two-mode network is people in groups.
      - By projecting, we get:
        - One-mode network of people connected to people *by* groups.
        - One-mode network of groups connected *by* people.

Projection

- Breiger (1974)
  - We can build the adjacency matrix for each projected network through matrix algebra.
    - Specifically, multiplying an adjacency matrix by it's **transpose**.
      - The transpose of a matrix simply reverses the columns and rows:
        - $A^{T}_{ij} = A_{ji}$

Projection

- Breiger (1974)
    - The two-mode, *NxM*, adjacency matrix, when multiplied by it's **transpose**, produces either:
        - An *MxM* matrix (ties among *M* nodes via *N*).
        - An *NxN* matrix (ties among *N* nodes via *M*).

Transposition

Matrix **A**

	1	2	3	4	5
A	1	1	0	0	0
B	1	0	0	0	0
C	1	1	0	0	0
D	0	1	1	1	1
E	0	0	1	0	0
F	0	0	1	0	1

order is 6x5

Matrix $\mathbf{A}^\mathbf{T}$

	A	B	C	D	E	F
1	1	1	1	0	0	0
2	1	0	1	1	0	0
3	0	0	0	1	1	1
4	0	0	0	1	0	0
5	0	0	0	1	0	1

order is 5x6

Projection

- Matrix Multiplication Rules
  - To multiply two matrices, the number of columns in the first matrix must match the number of rows in the second matrix.
    - Example: 5x6 X 6x5 works, but not 5x6 X 5x6
  - The **product** matrix has the number of rows equal to the first matrix and the number of columns equal to the second matrix.
    - Example: 5x6 X 6x5 = 5x5

Projection

- Product Matrix
  - The product matrix is the projected graph.
    - Recall that there are two:
      - $\mathbf{A} \times \mathbf{A}^t$ (the "people" matrix **P**)
      - And the $\mathbf{A}^t \times \mathbf{A}$ (the "group" matrix **G**)
        - *What does each one mean?*

Matrix Multiplication

Matrix **A**

	1	2	3	4	5
A	1	1	0	0	0
B	1	0	0	0	0
C	1	1	0	0	0
D	0	1	1	1	1
E	0	0	1	0	0
F	0	0	1	0	1

order is 6x5

Matrix Multiplication

Matrix **A**

	1	2	3	4	5
A	1	1	0	0	0
B	1	0	0	0	0
C	1	1	0	0	0
D	0	1	1	1	1
E	0	0	1	0	0
F	0	0	1	0	1

order is 6x5

X

Matrix $\mathbf{A}^{\mathbf{T}}$

	A	B	C	D	E	F
1	1	1	1	0	0	0
2	1	0	1	1	0	0
3	0	0	0	1	1	1
4	0	0	0	1	0	0
5	0	0	0	1	0	1

order is 5x6

Matrix Multiplication

Matrix **A**

	1	2	3	4	5
A	1	1	0	0	0
B	1	0	0	0	0
C	1	1	0	0	0
D	0	1	1	1	1
E	0	0	1	0	0
F	0	0	1	0	1

order is 6x5

X

Matrix $\mathbf{A}^{\mathbf{T}}$

	A	B	C	D	E	F
1	1	1	1	0	0	0
2	1	0	1	1	0	0
3	0	0	0	1	1	1
4	0	0	0	1	0	0
5	0	0	0	1	0	1

order is 5x6

The product matrix is
6x6

Projection by Multiplication

$$\mathbf{A} \times \mathbf{A}^{\mathbf{T}} = \mathbf{P}$$

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

6x5 X 5x6 = 6x6

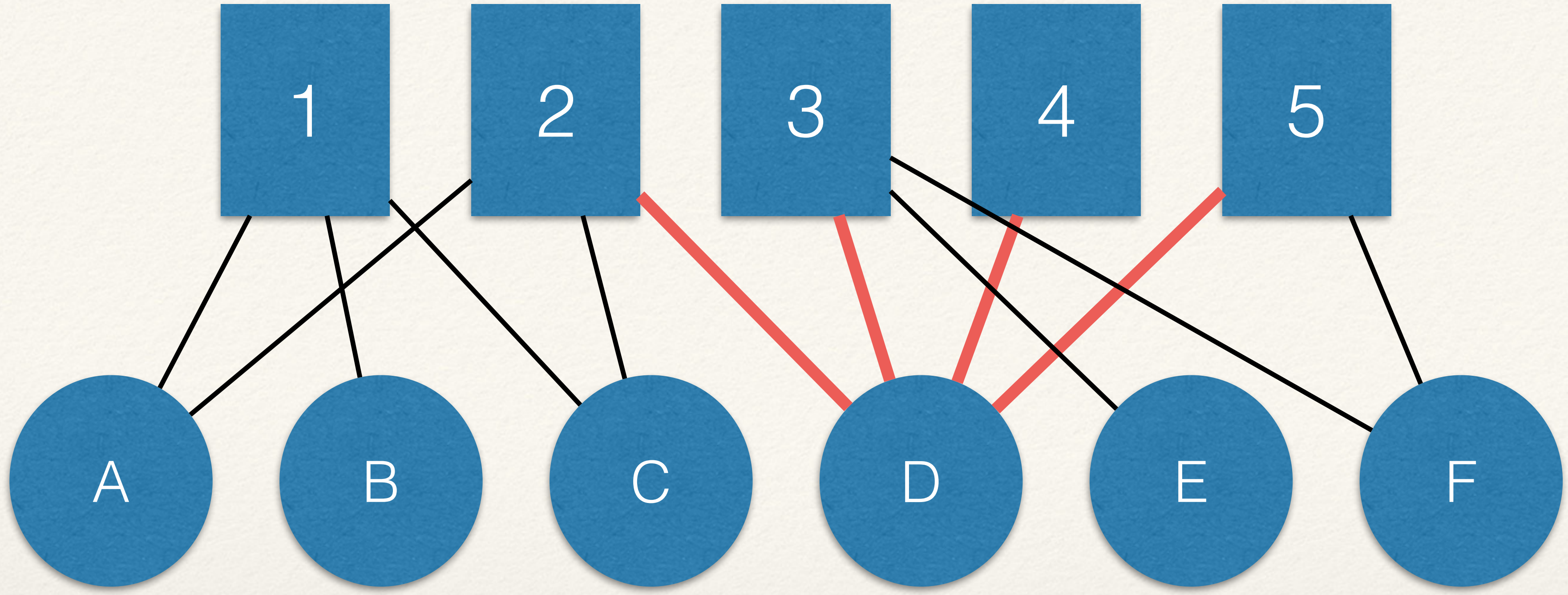
Projection by Multiplication

We want to know how people are connected by groups (i.e. the rows of our two-mode adjacency matrix)

$$\mathbf{A} \times \mathbf{A}^{\mathbf{T}} = \mathbf{P}$$

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

6x5 X 5x6 = 6x6



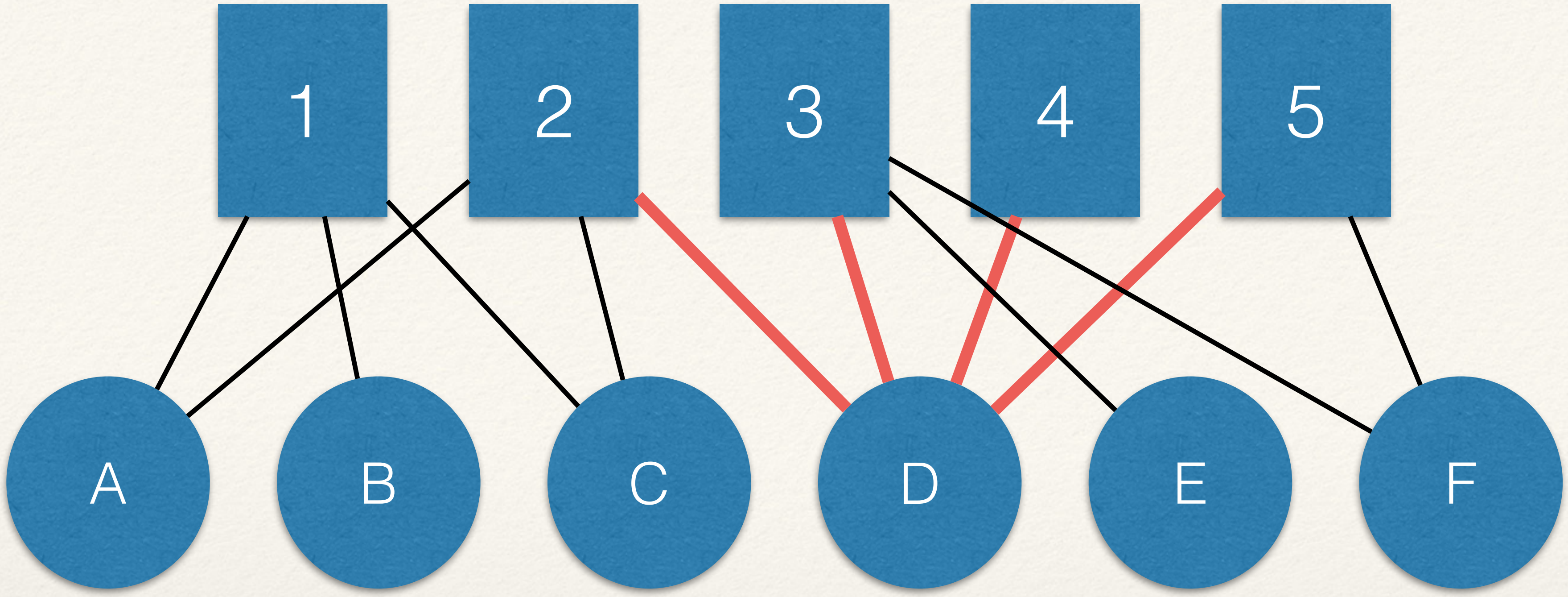
$\mathbf{A} \times \mathbf{A}^{\mathsf{T}} = \mathbf{P}$

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

The diagonal is the count of ties **the person** has with two-mode vertices

For example, D is in 4 groups

6x5 X 5x6 = 6x6

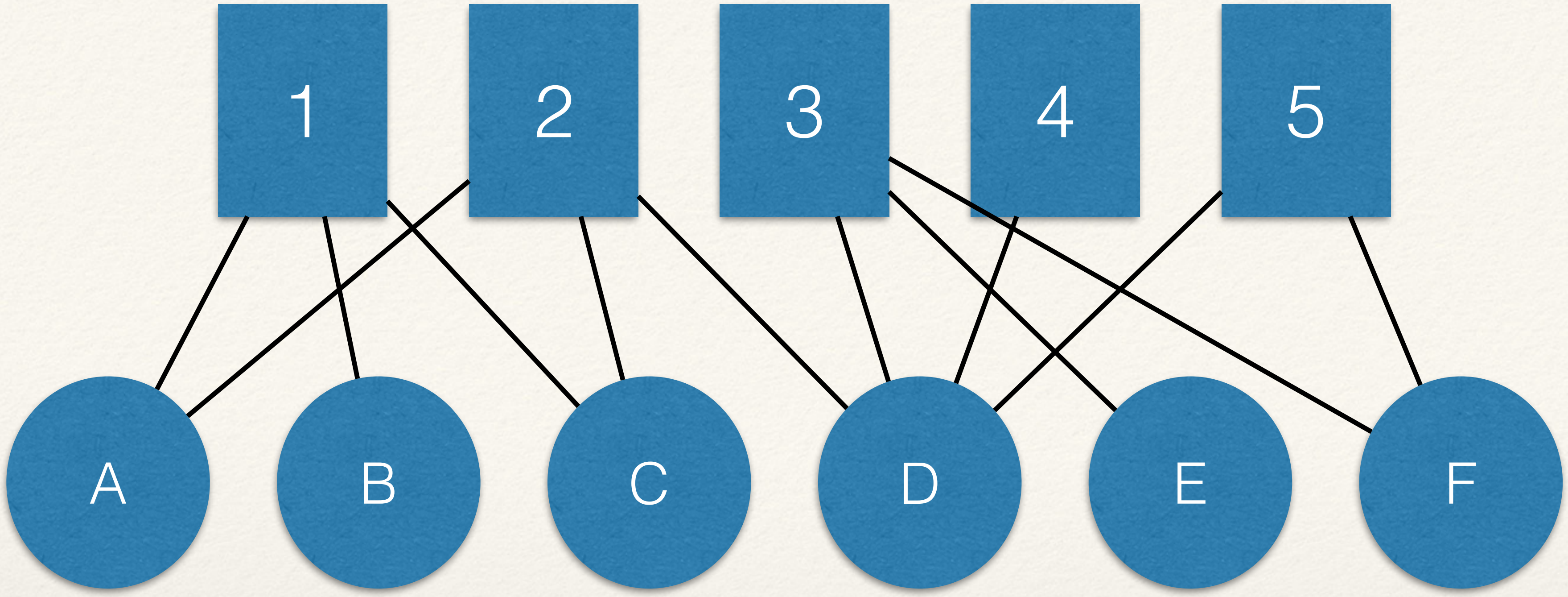


$$\mathbf{A} \times \mathbf{A}^{\mathbf{T}} = \mathbf{P}$$

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

What statistic does the diagonal give us?

6x5 X 5x6 = 6x6

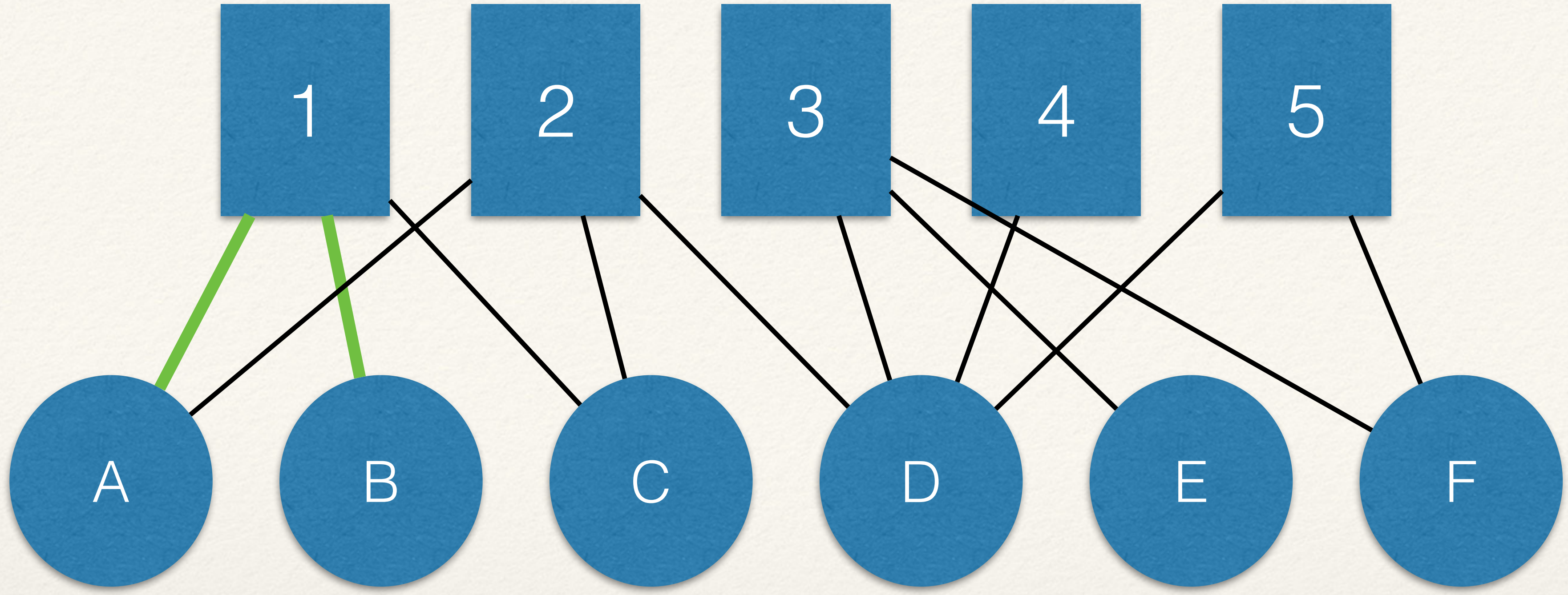


$$\mathbf{A} \times \mathbf{A}^{\mathbf{T}} = \mathbf{P}$$

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

Note, that the projection forces the product matrix to be symmetric (i.e. undirected graph)

6x5 X 5x6 = 6x6



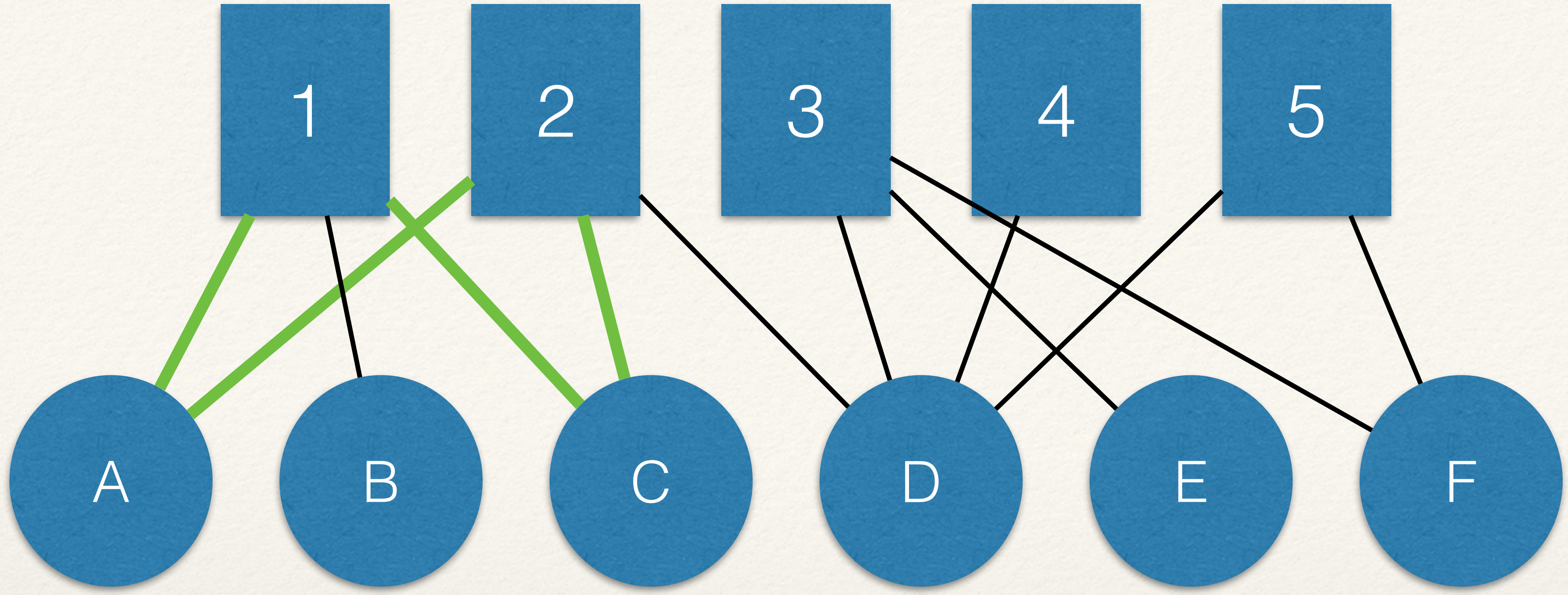
$$\mathbf{A} \times \mathbf{A}^{\mathbf{T}} = \mathbf{P}$$

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

6x5 X 5x6 = 6x6

The off-diagonal entries indicate the number of ways that vertices in the first mode are connected by vertices in the second mode

A and B are linked through a single vertex, 1



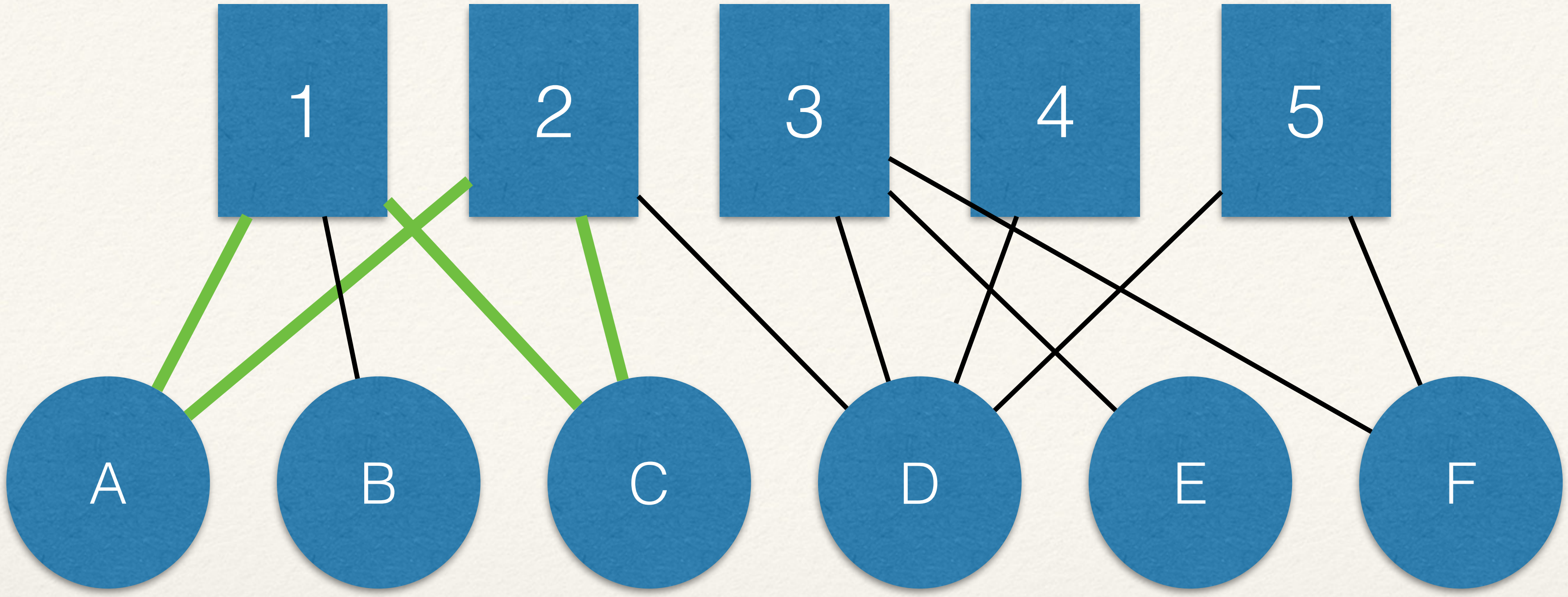
$$\mathbf{A} \times \mathbf{A}^{\mathbf{T}} = \mathbf{P}$$

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

A and C are linked through two vertices,1 and 2

So, if these are groups, A and C are members of 2 of the same groups

6x5 X 5x6 = 6x6



$$\mathbf{A} \times \mathbf{A}^{\mathbf{T}} = \mathbf{P}$$

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

NOTE: these are counts of shared vertices, not edge counts

$6x5 \text{ X } 5x6 = 6x6$

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

→

	A	B	C	D	E	F
A	0	1	1	1	0	0
B	1	0	1	0	0	0
C	1	1	0	1	0	0
D	1	0	1	0	1	1
E	0	0	0	1	0	1
F	0	0	0	1	1	0

If we treat any tie greater than 0 as binary, *called* ***dichotomizing***, and recode the diagonal as 0, we get an undirected, one-mode network

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

	A	B	C	D	E	F
A	0	1	1	1	0	0
B	1	0	1	0	0	0
C	1	1	0	1	0	0
D	1	0	1	0	1	1
E	0	0	0	1	0	1
F	0	0	0	1	1	0

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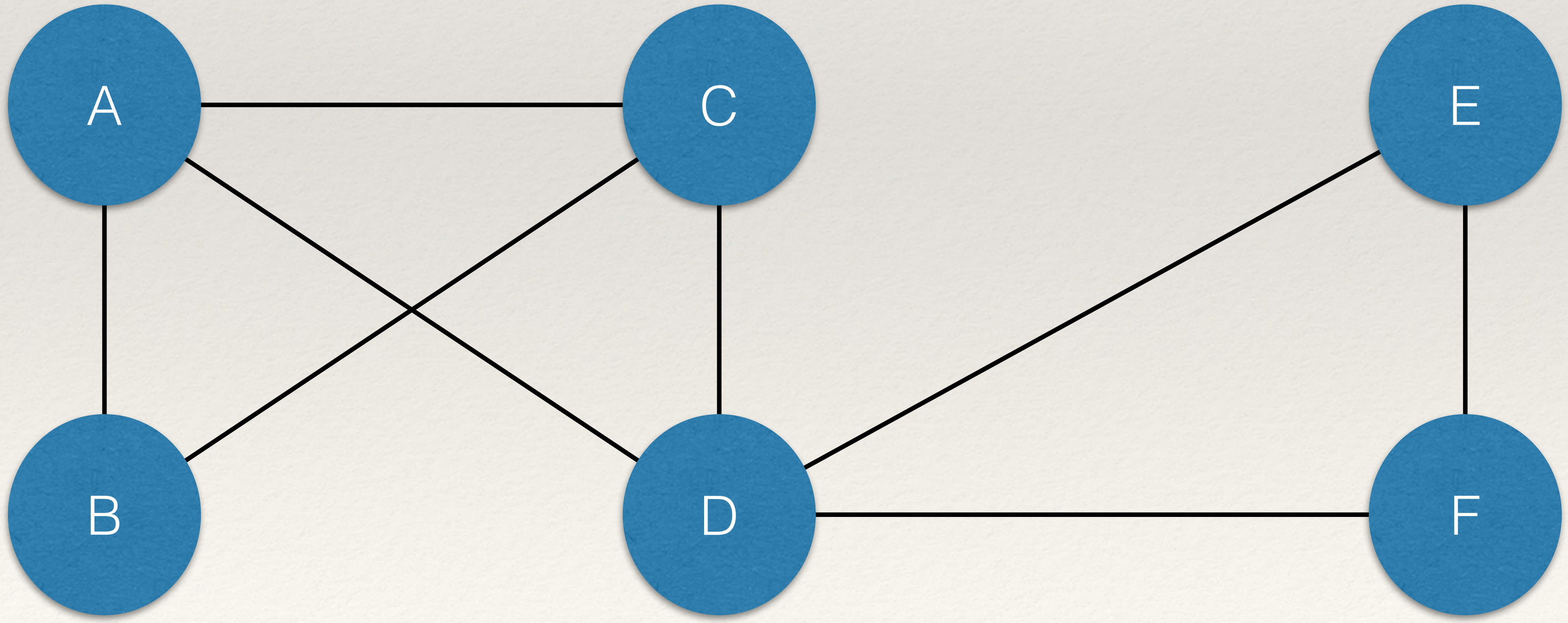


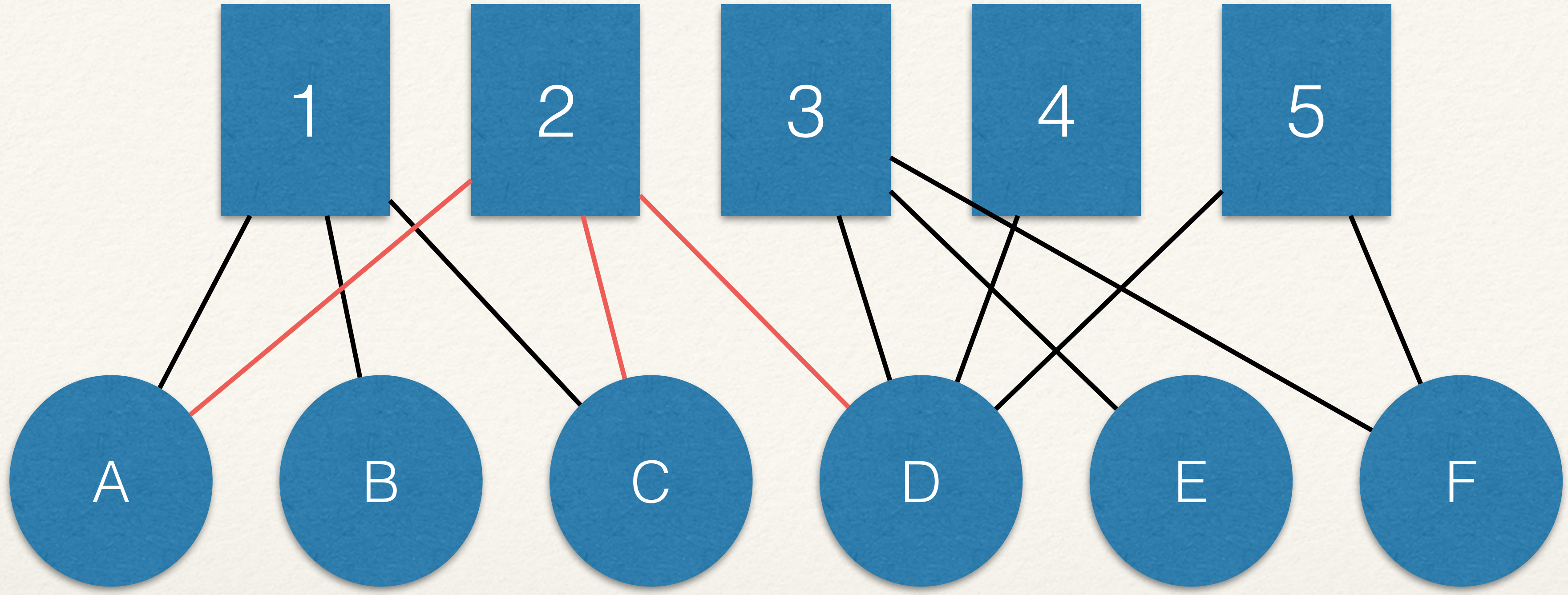
Projection by Multiplication

We want to know how <u>groups are connected</u> by people (i.e. the columns of our two-mode adjacency matrix)

$$\mathbf{A^T} \times \mathbf{A} = \mathbf{G}$$

	1	2	3	4	5
1	3	2	0	0	0
2	2	3	1	1	1
3	0	1	3	1	2
4	0	1	1	1	1
5	0	1	2	1	2

5x6 X 6x5 = 5x5



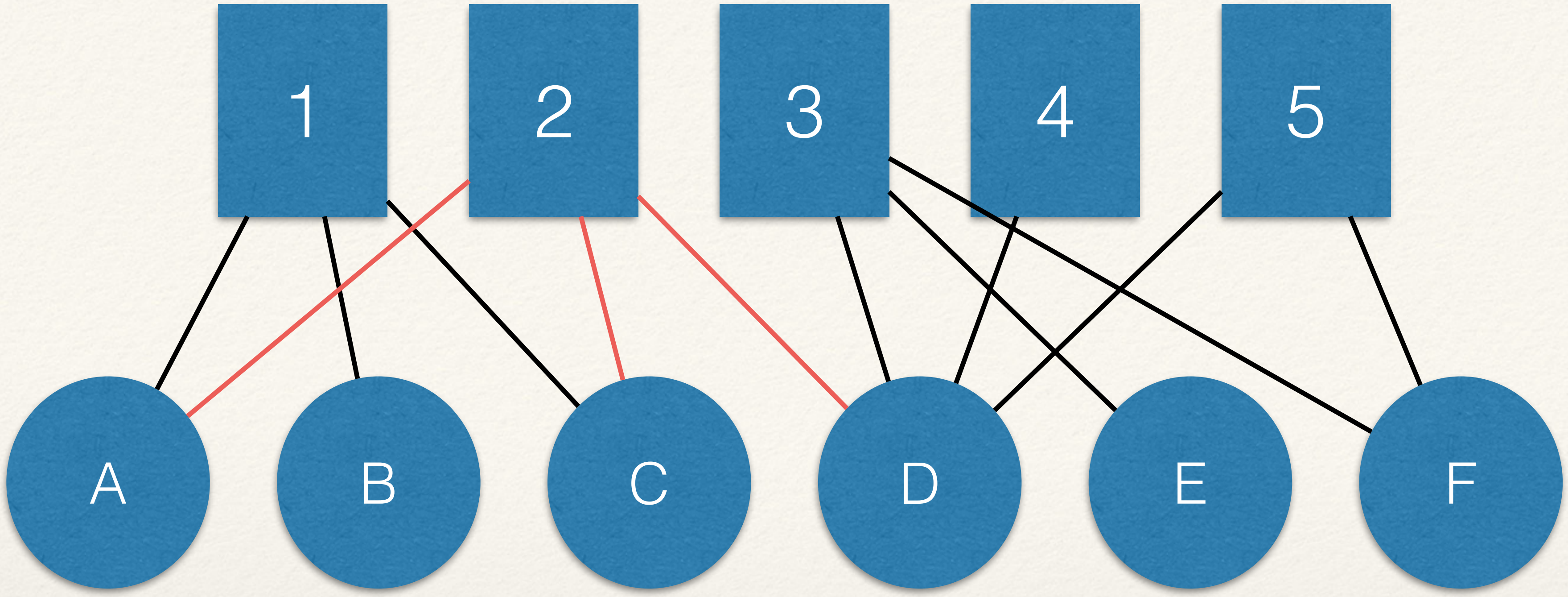
$\mathbf{A}^{\mathbf{T}} \times \mathbf{A} = \mathbf{G}$

	1	2	3	4	5
1	3	2	0	0	0
2	2	3	1	1	1
3	0	1	3	1	2
4	0	1	1	1	1
5	0	1	2	1	2

5x6 X 6x5 = 5x5

The diagonal is the count of ties the **group** has with two-mode vertices

For example, 2 has 3 people

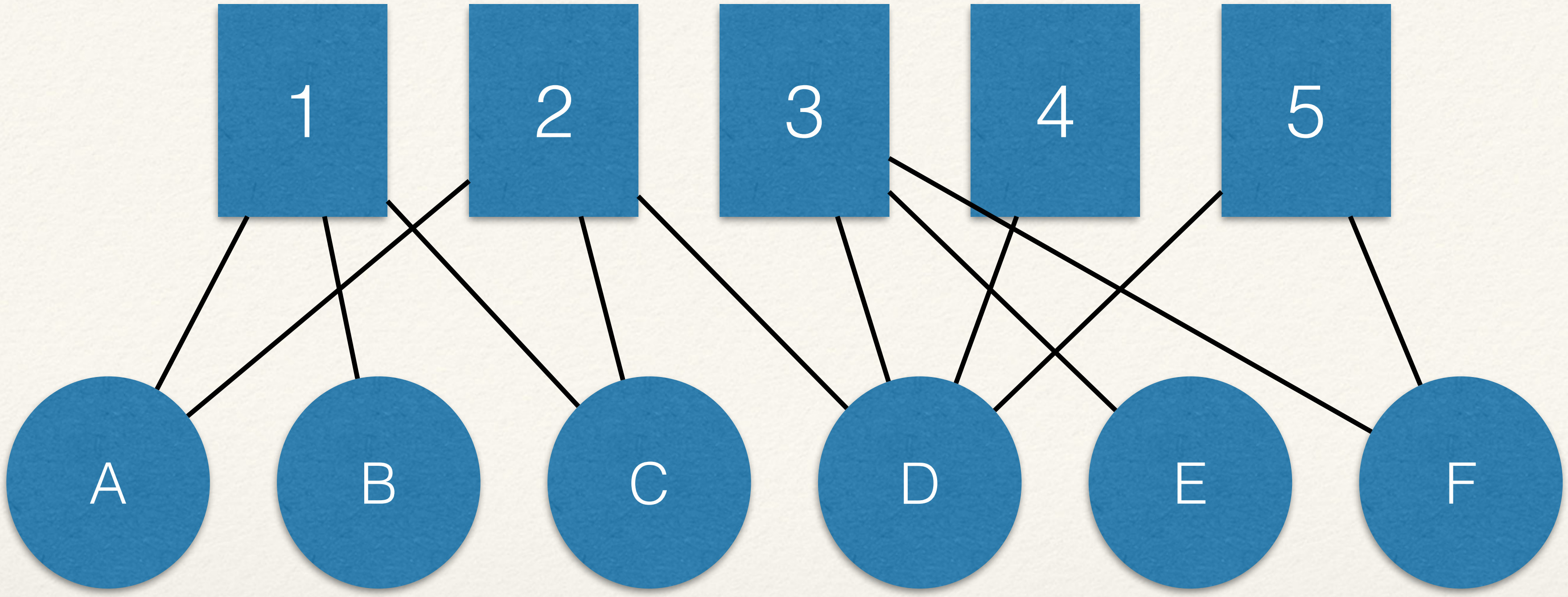


$$\mathbf{A^T} \times \mathbf{A} = \mathbf{G}$$

	1	2	3	4	5
1	3	2	0	0	0
2	2	3	1	1	1
3	0	1	3	1	2
4	0	1	1	1	1
5	0	1	2	1	2

What statistic does the diagonal give us?

5x6 X 6x5 = 5x5



$$\mathbf{A}^{\mathbf{T}} \times \mathbf{A} = \mathbf{G}$$

	1	2	3	4	5
1	3	2	0	0	0
2	2	3	1	1	1
3	0	1	3	1	2
4	0	1	1	1	1
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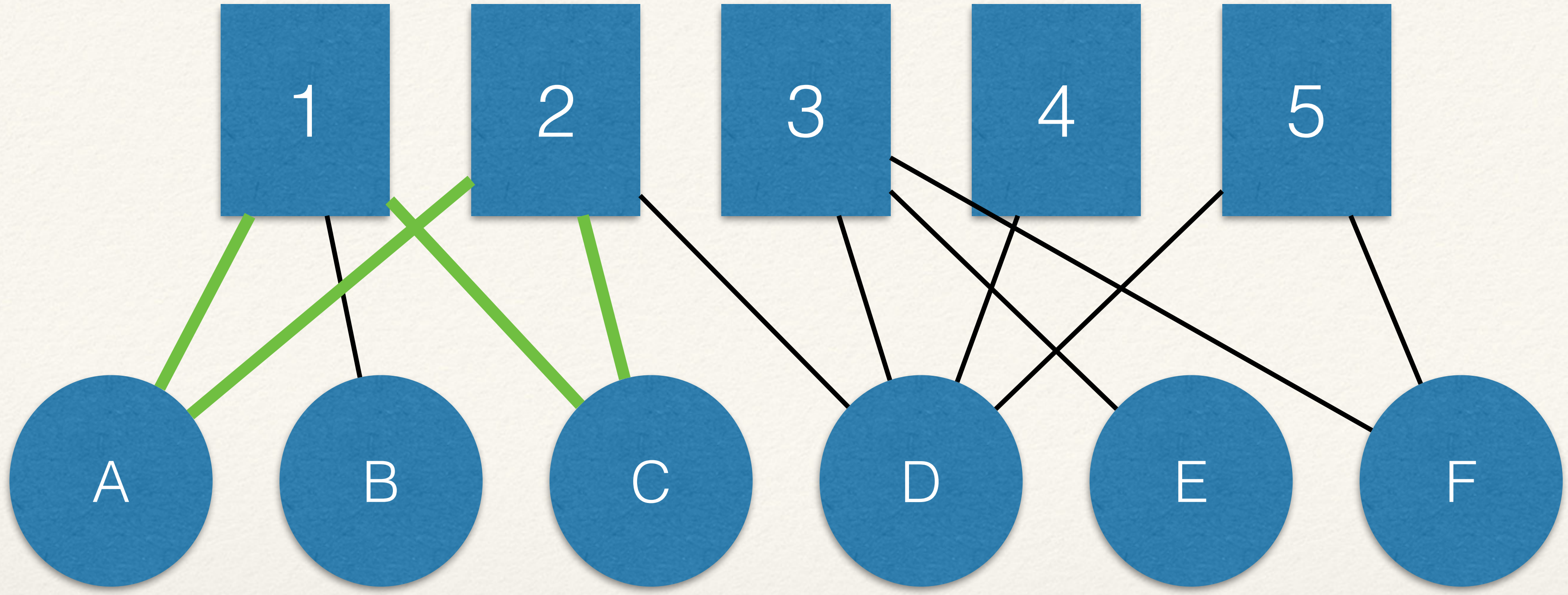
Note, that the projection forces the product matrix to be symmetric (i.e. undirected graph)

5x6 X 6x5 = 5x5



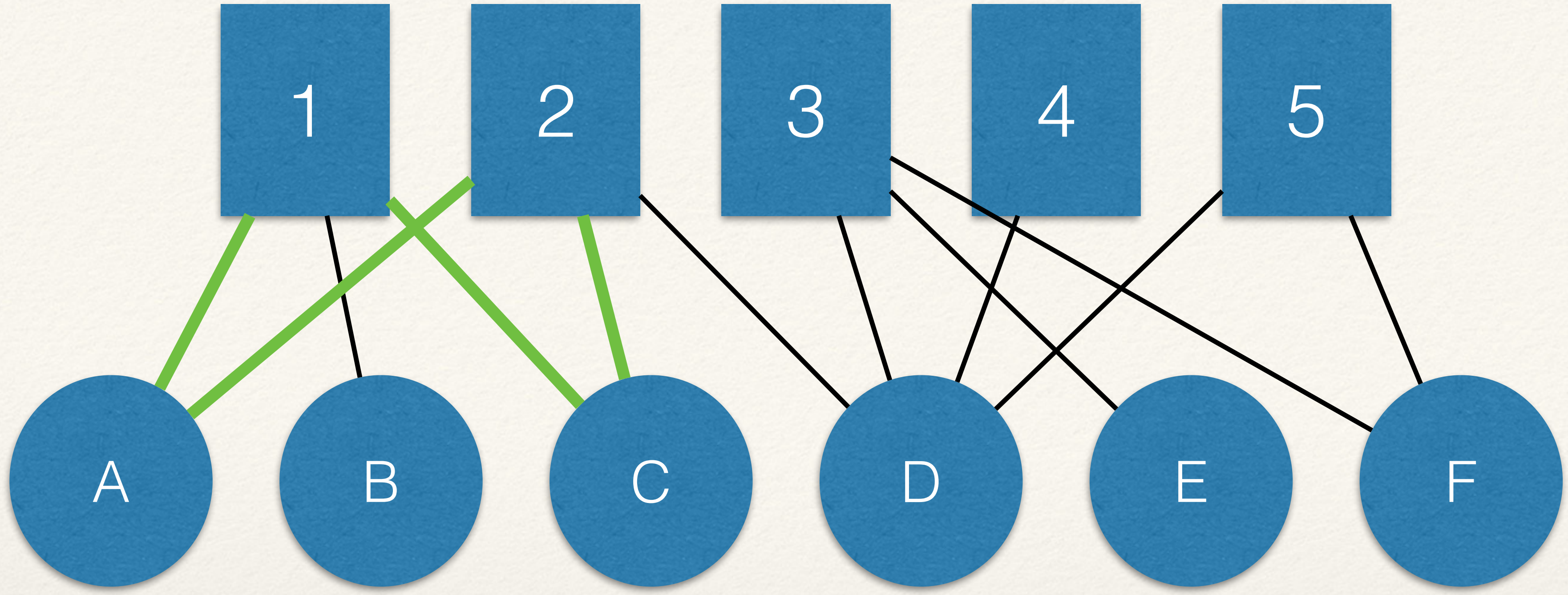
$\mathbf{A}^{\mathbf{T}} \times \mathbf{A} = \mathbf{G}$

	1	2	3	4	5
1	3	2	0	0	0
2	2	3	1	1	1
3	0	1	3	1	2
4	0	1	1	1	1
5	0	1	2	1	2

The off-diagonal entries indicate the number of ways that vertices in the second mode are connected by vertices in the first mode

1 and 2 are connected by 2 vertices, A and C

5x6 X 6x5 = 5x5



$$\mathbf{A}^{\mathbf{T}} \times \mathbf{A} = \mathbf{G}$$

	1	2	3	4	5
1	3	2	0	0	0
2	2	3	1	1	1
3	0	1	3	1	2
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NOTE: these are counts of shared vertices, not edge counts

5x6 X 6x5 = 5x5

	1	2	3	4	5
1	3	2	0	0	0
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5	0	1	2	1	2

	1	2	3	4	5
1	3	2	0	0	0
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3	0	1	3	1	2
4	0	1	1	1	1
5	0	1	2	1	2

	1	2	3	4	5
1	0	1	0	0	0
2	1	0	1	1	1
3	0	1	0	1	1
4	0	1	1	0	1
5	0	1	1	1	0

If we treat any tie greater than 0 as binary, *called **dichotomizing***, and recode the diagonal as 0, we get an undirected, one-mode network

	1	2	3	4	5
1	3	2	0	0	0
2	2	3	1	1	1
3	0	1	3	1	2
4	0	1	1	1	1
5	0	1	2	1	2

	1	2	3	4	5
1	0	1	0	0	0
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3	0	1	0	1	1
4	0	1	1	0	1
5	0	1	1	1	0

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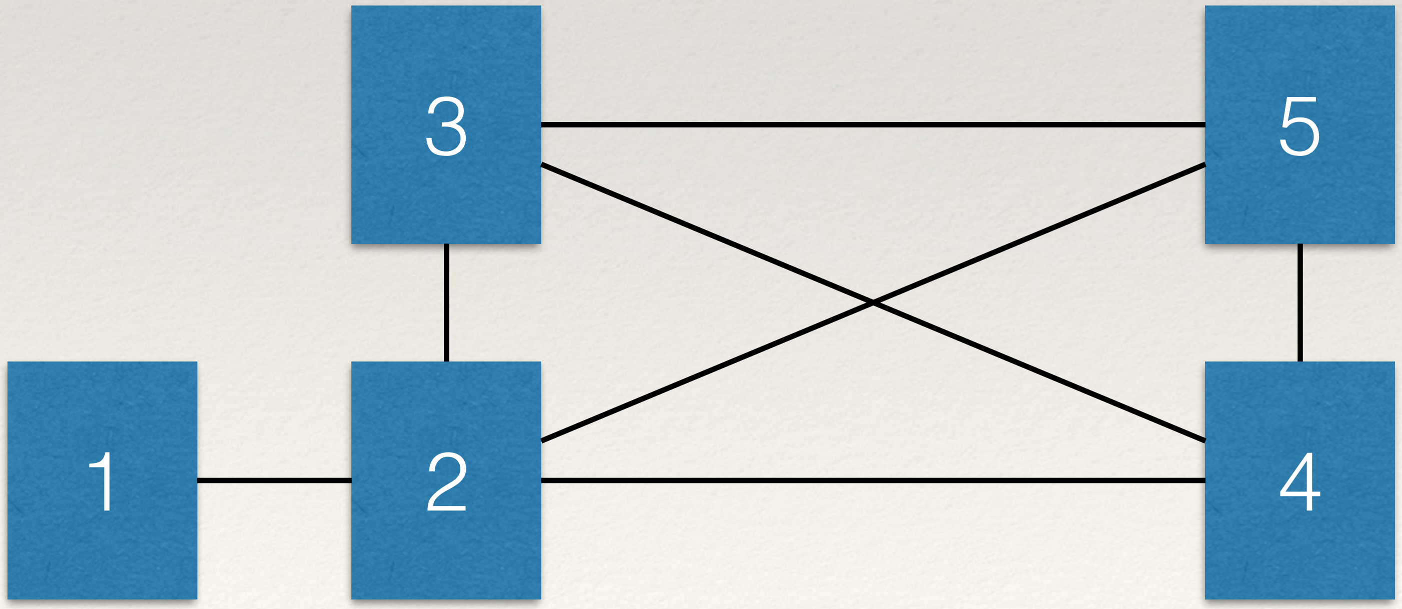
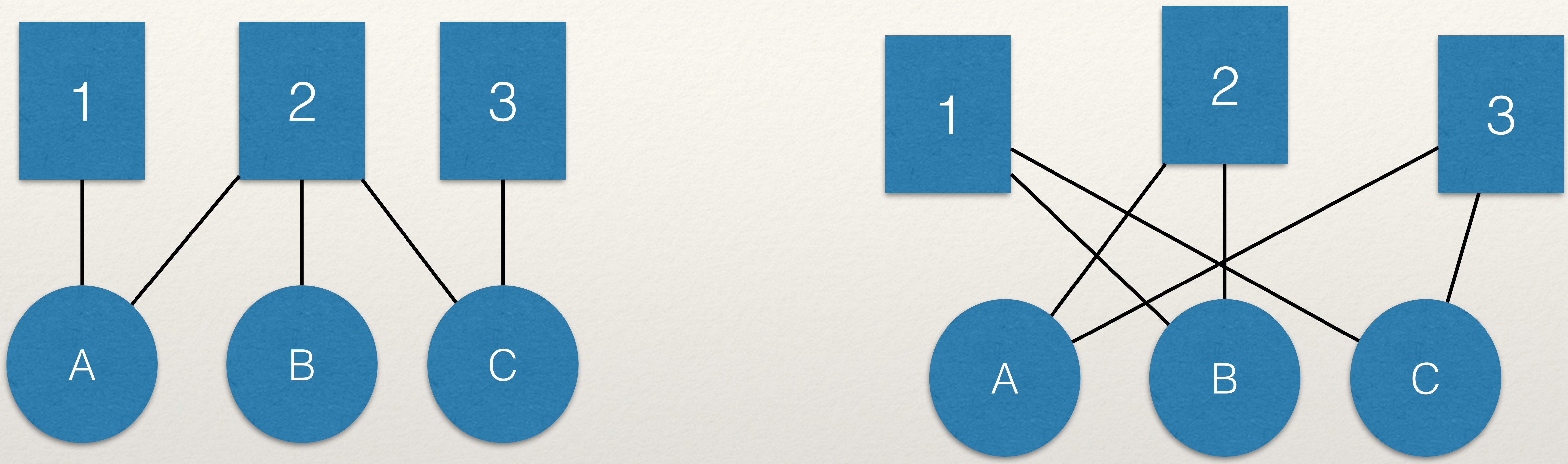


Projection

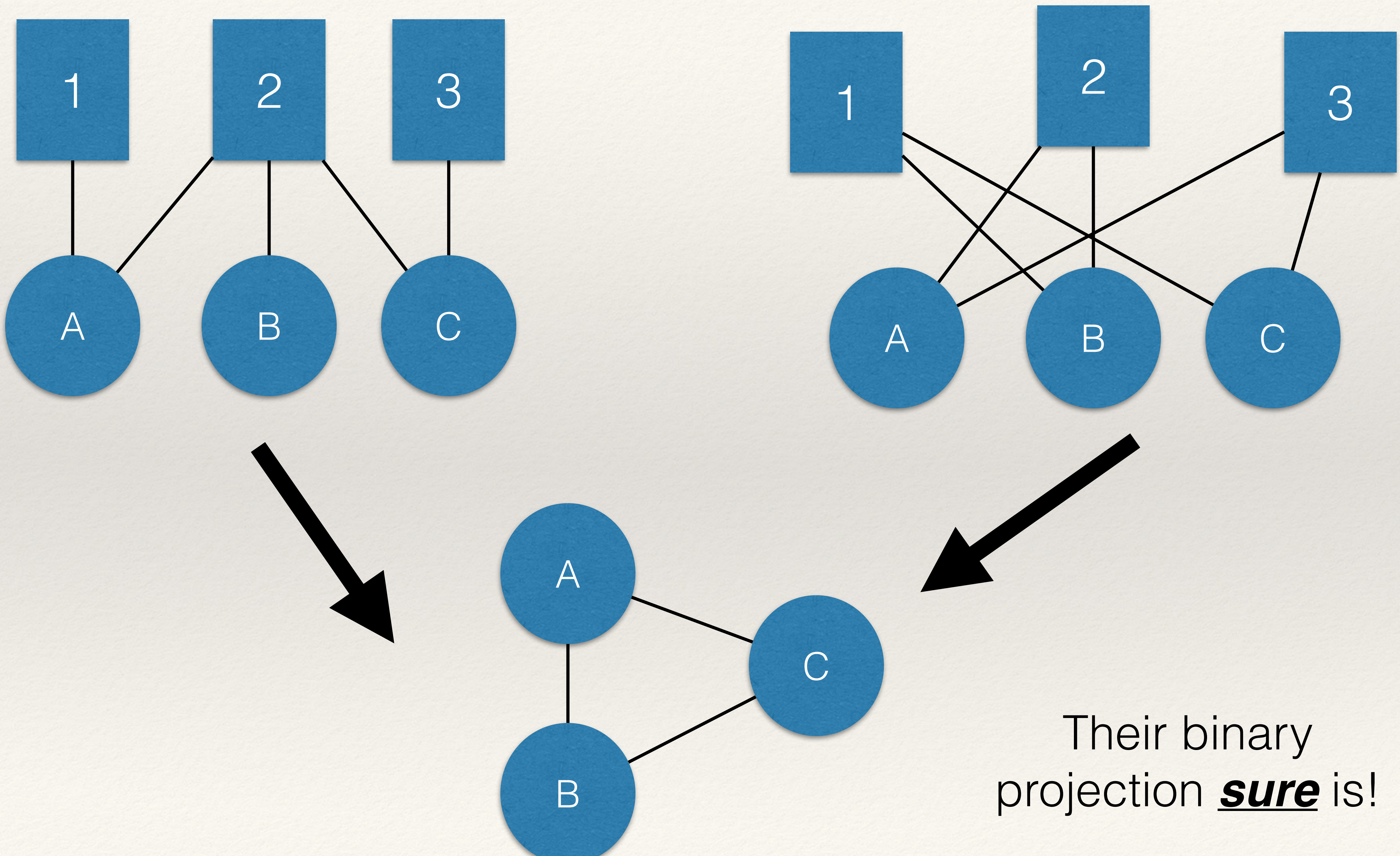
- To project, or not to project?
  - As noted by many scholars, there is data loss when we project and binarize the data.
    - Sometimes, this can be misleading.

Projection and Data Loss



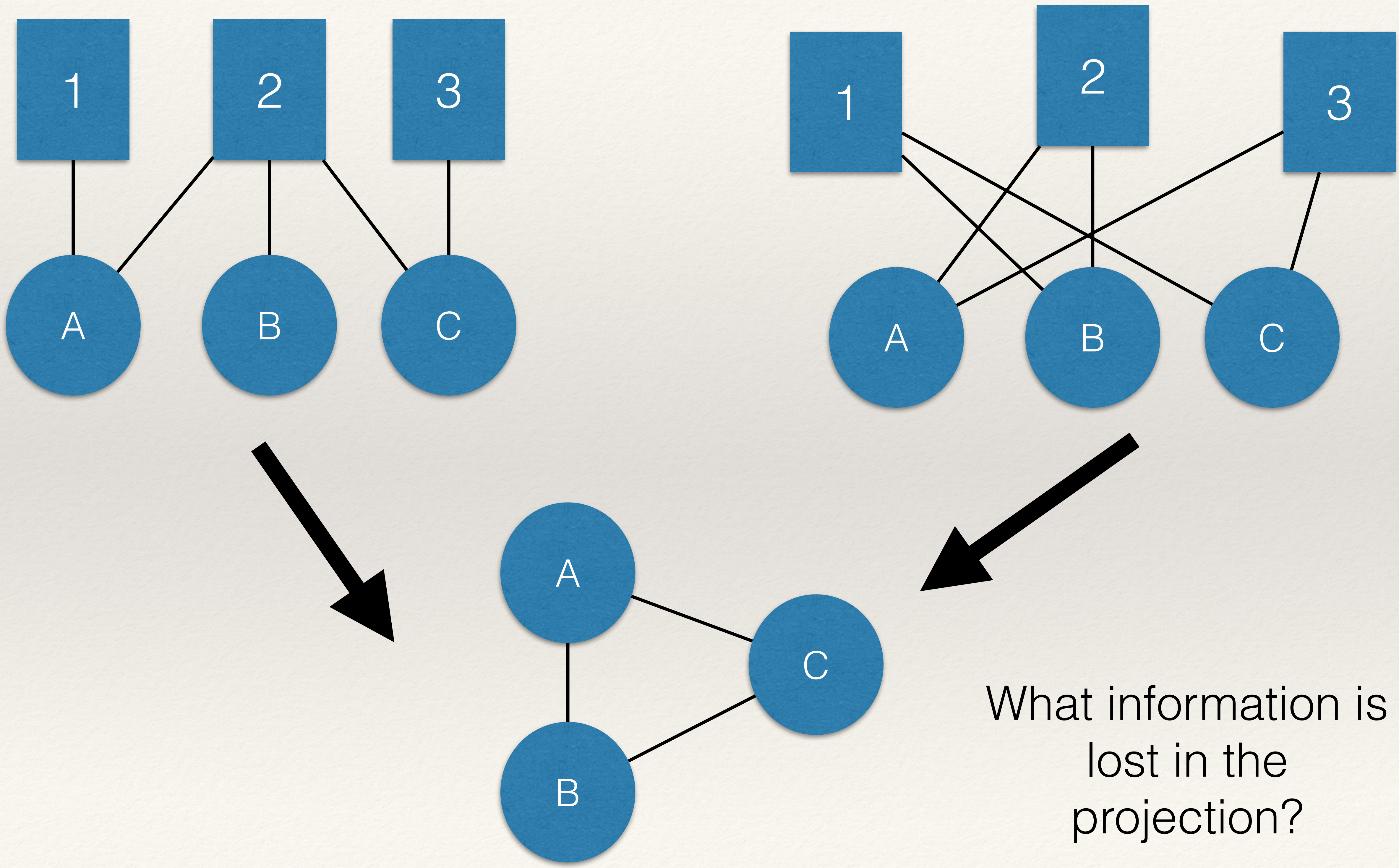
Are these bipartite graphs the same?

Projection and Data Loss



Their binary projection ***sure*** is!

Projection and Data Loss



What information is lost in the projection?

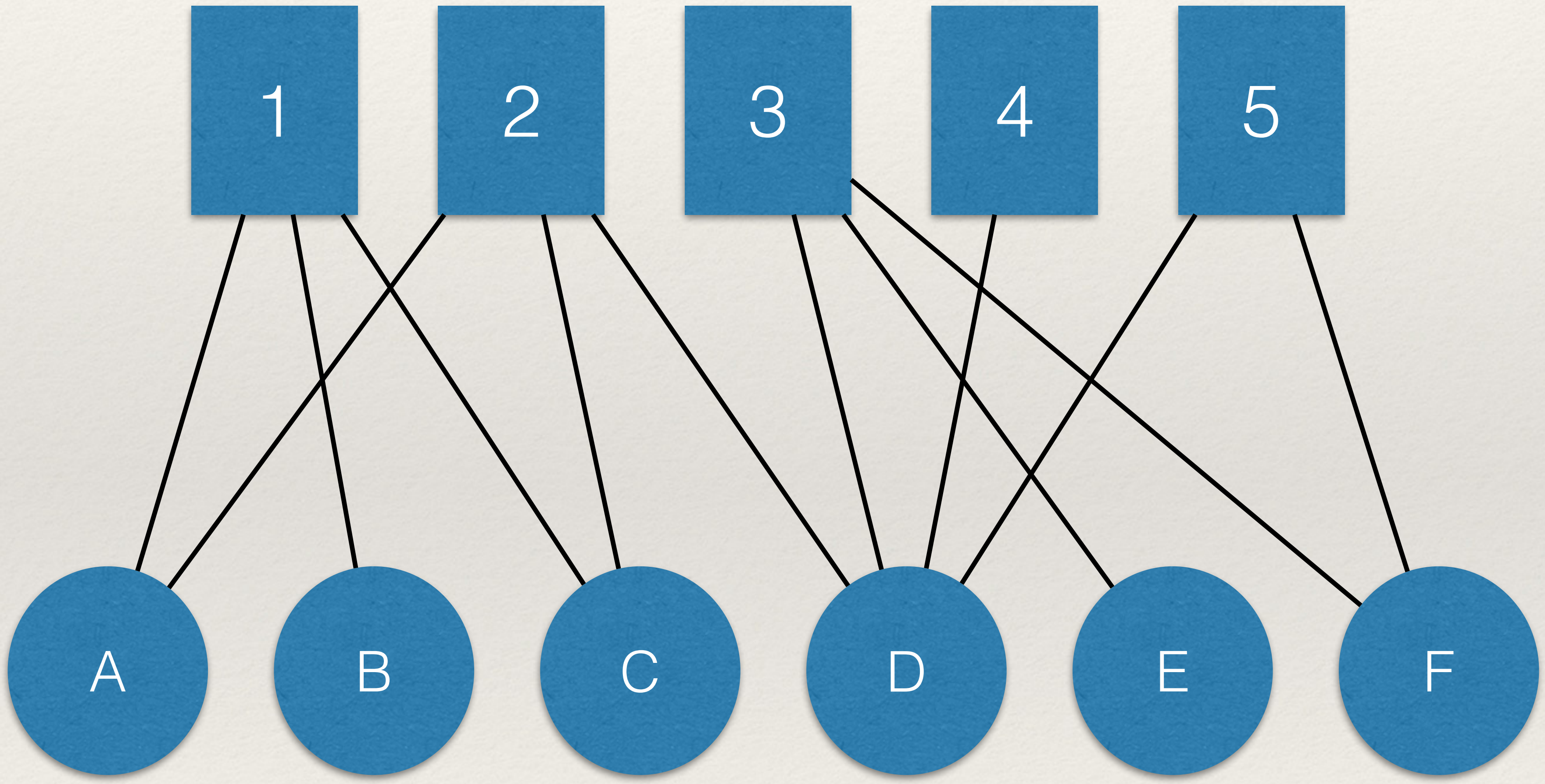
Projection

- So what do we do?
  - When you can, “keep it real” by keeping it two-mode.
  - If you must project, minimize data loss by weighting edges.

Weighted Edges

- We can use the information from the bipartite graph to weight the edges in the network.
- These weights can be used in a plot and/or in the analysis.
  - The most common method is to sum the ties between two actors (i.e. *summation method*).

Projection



	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

	A	B	C	D	E	F
A	0	1	1	1	0	0
B	1	0	1	0	0	0
C	1	1	0	1	0	0
D	1	0	1	0	1	1
E	0	0	0	1	0	1
F	0	0	0	1	1	0

If we treat any tie greater than 0 as binary, *called* ***dichotomizing***, and recode the diagonal as 0, we get an undirected, one-mode network

	A	B	C	D	E	F
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E	0	0	0	1	1	1
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$$\mathbf{A} \times \mathbf{A}^{\mathbf{T}} = \mathbf{P}$$

	A	B	C	D	E	F
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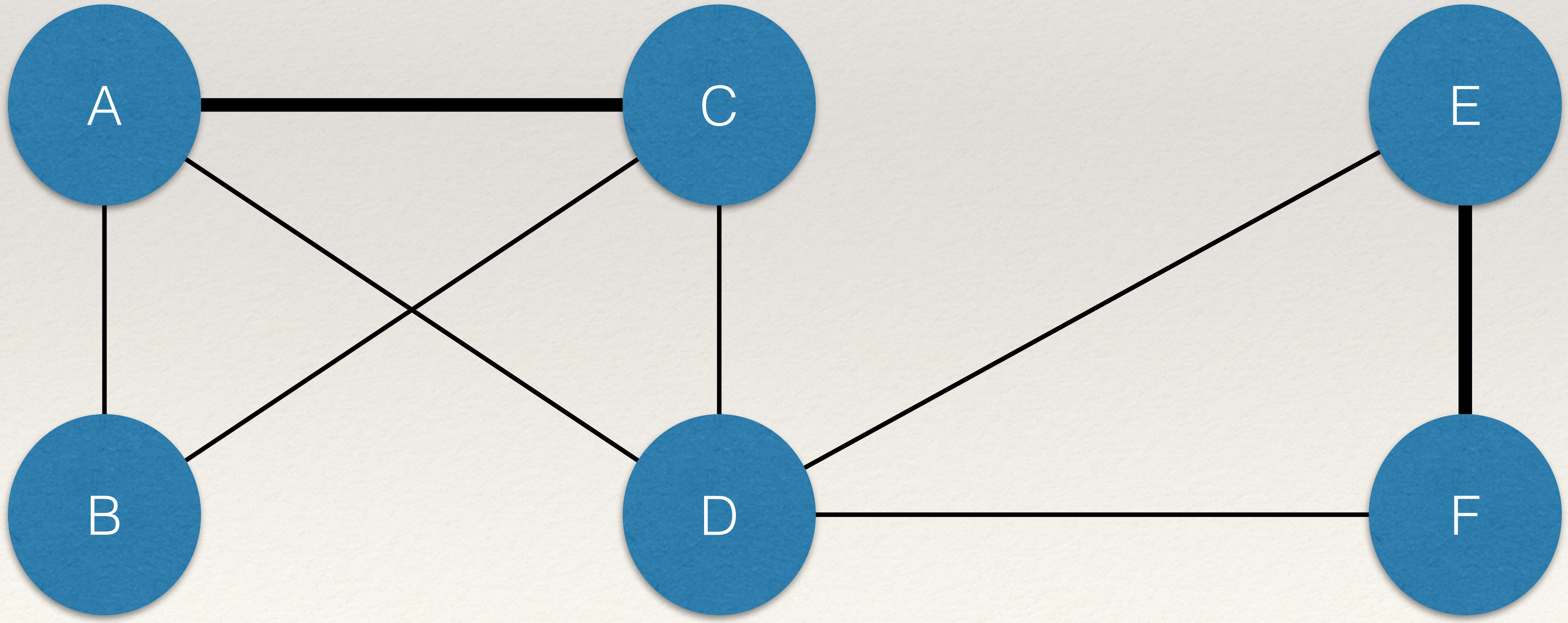
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

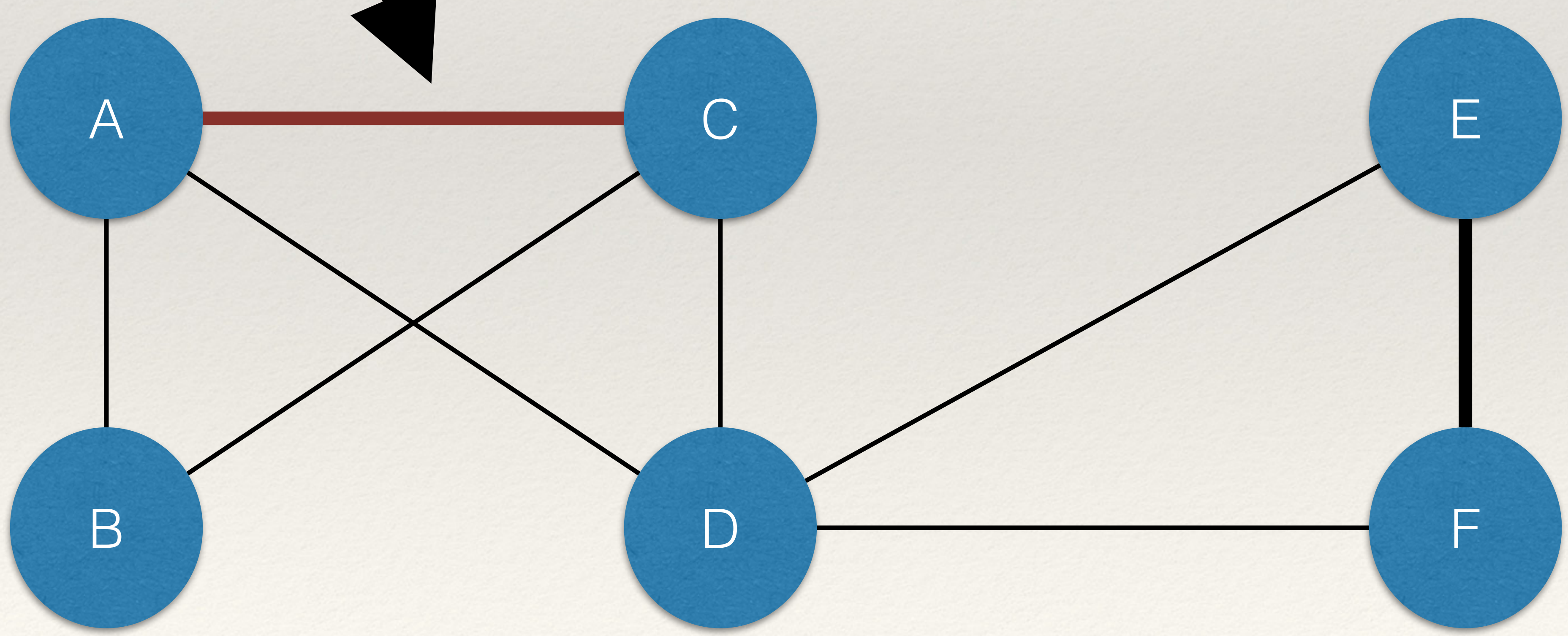
The off-diagonal entries are the tie weights



$\mathbf{A} \times \mathbf{A}^{\mathbf{T}} = \mathbf{P}$

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	[illegible]	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

The weight corresponds to the number of nodes in the second mode that the two actors share in the bipartite graph.



$\mathbf{A} \times \mathbf{A}^\mathbf{T} = \mathbf{P}$

	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	[illegible]	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
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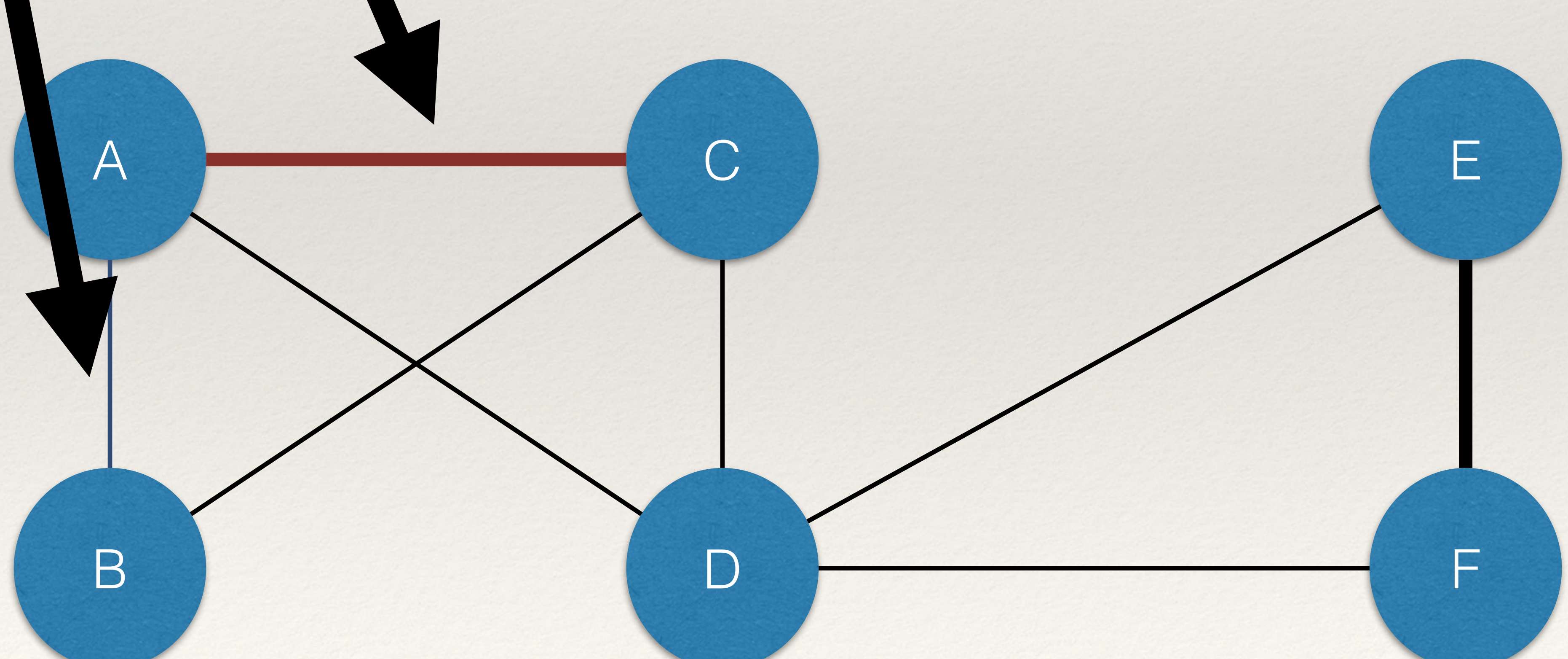
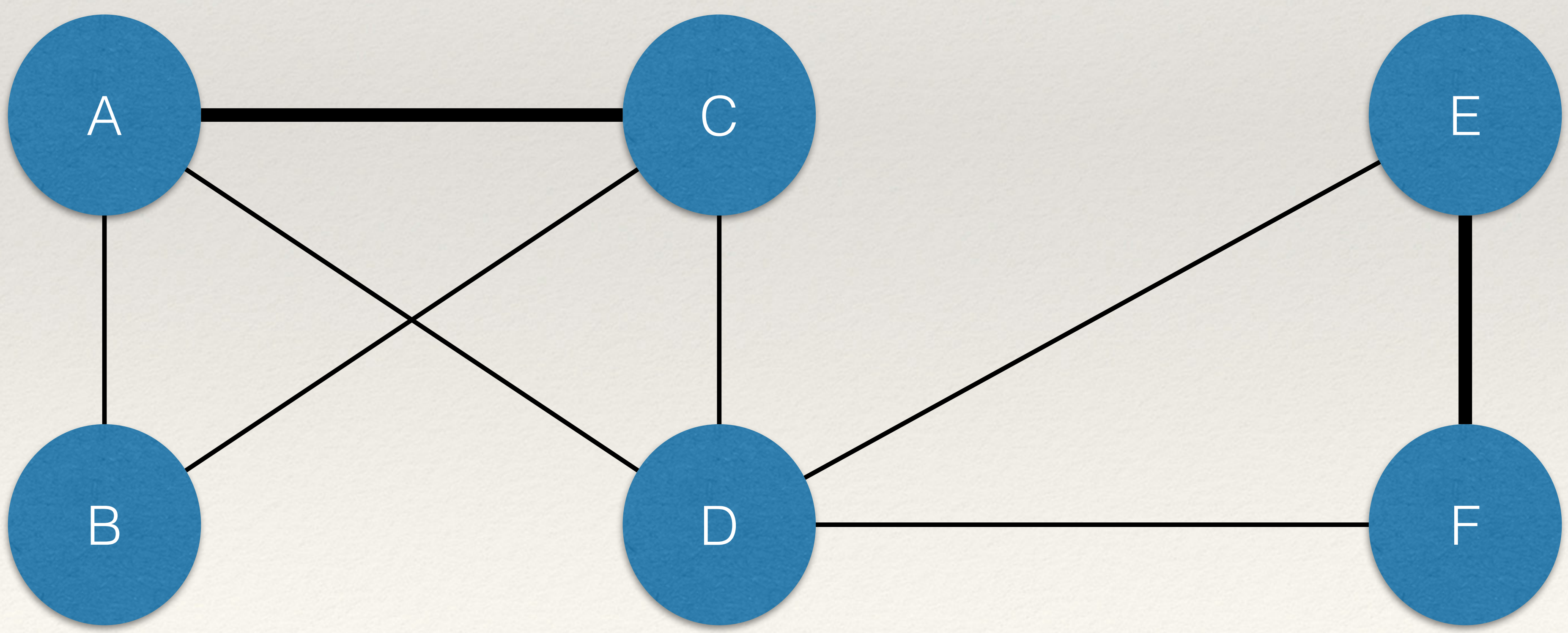
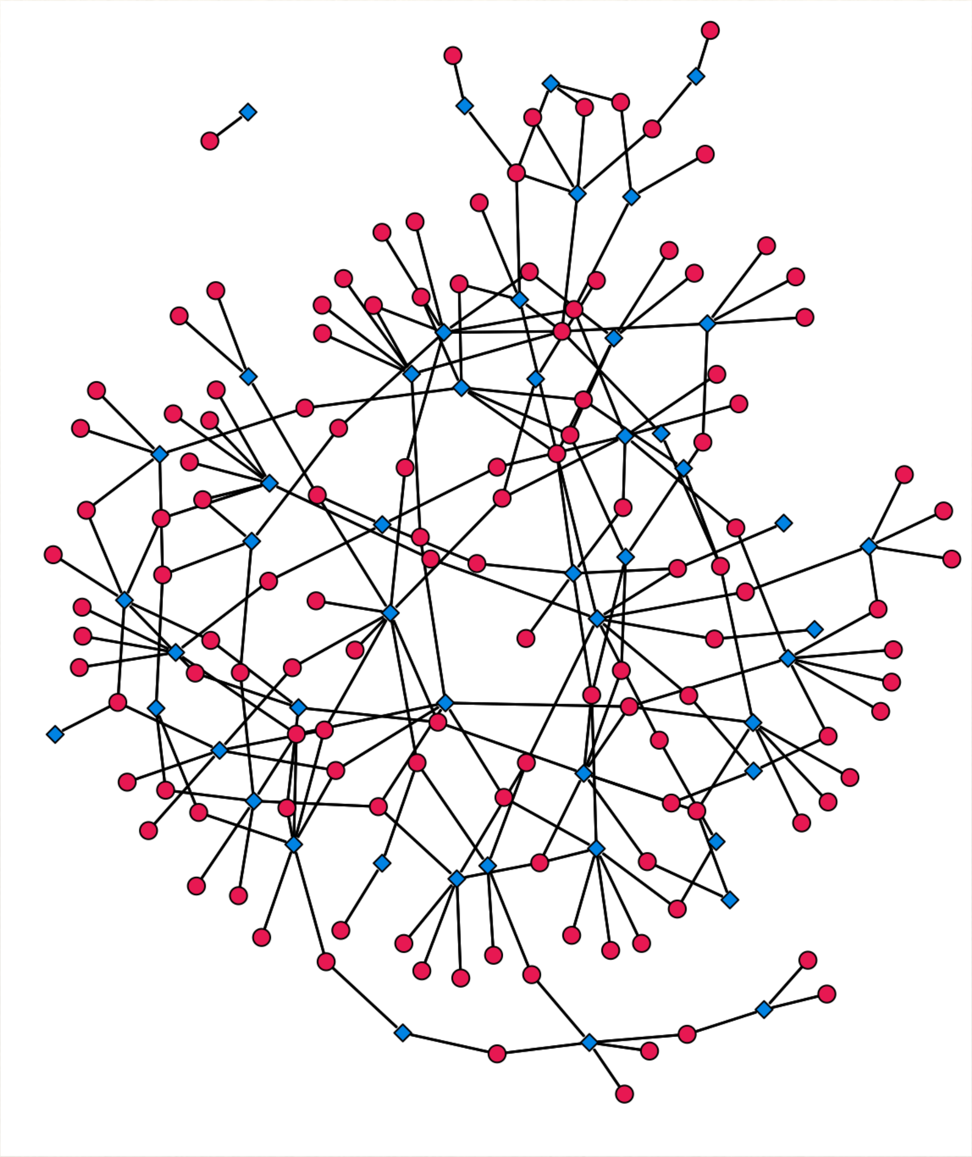


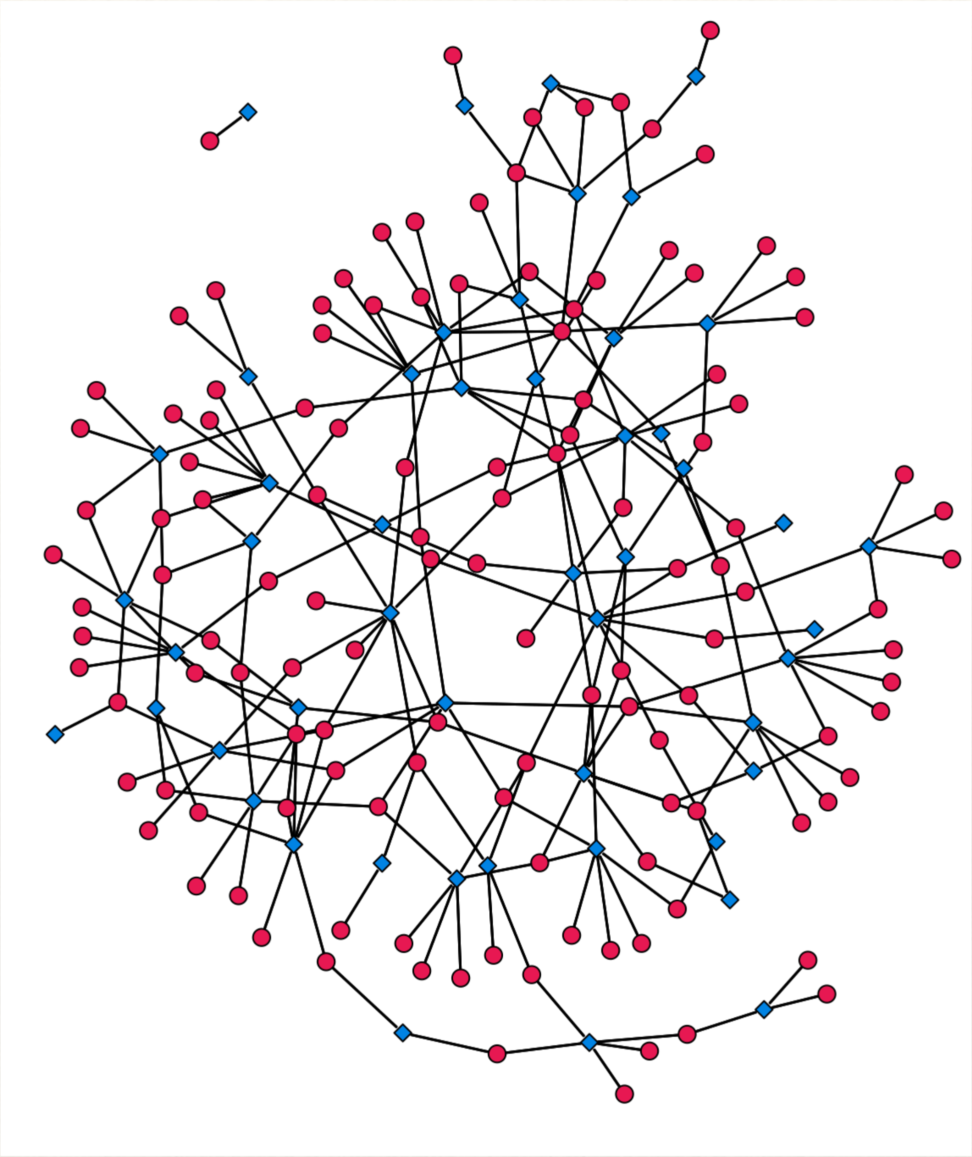
$\mathbf{A} \times \mathbf{A}^{\mathbf{T}} = \mathbf{P}$

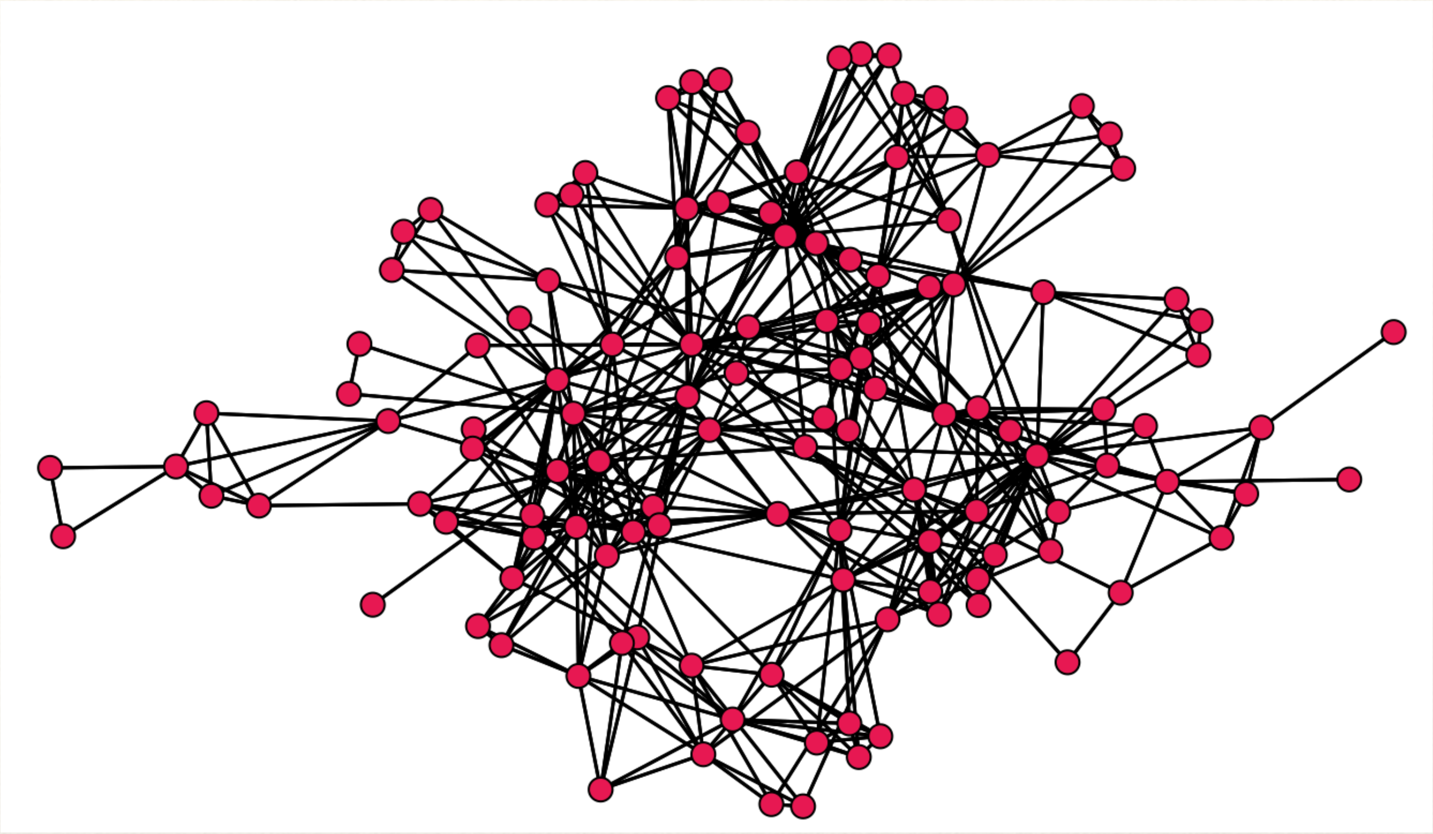
	A	B	C	D	E	F
A	2	1	2	1	0	0
B	1	1	1	0	0	0
C	2	1	2	1	0	0
D	1	0	1	4	1	2
E	0	0	0	1	1	1
F	0	0	0	2	1	2

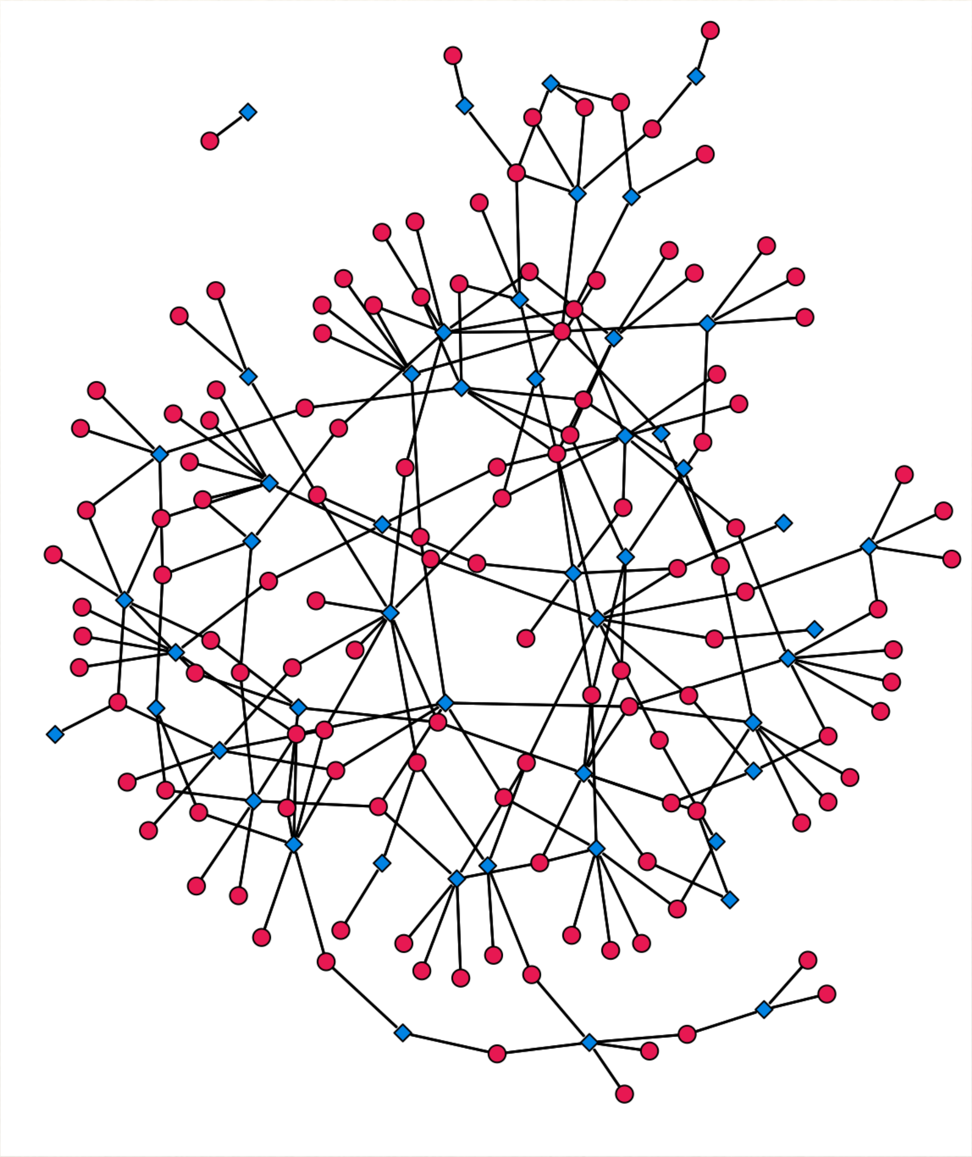
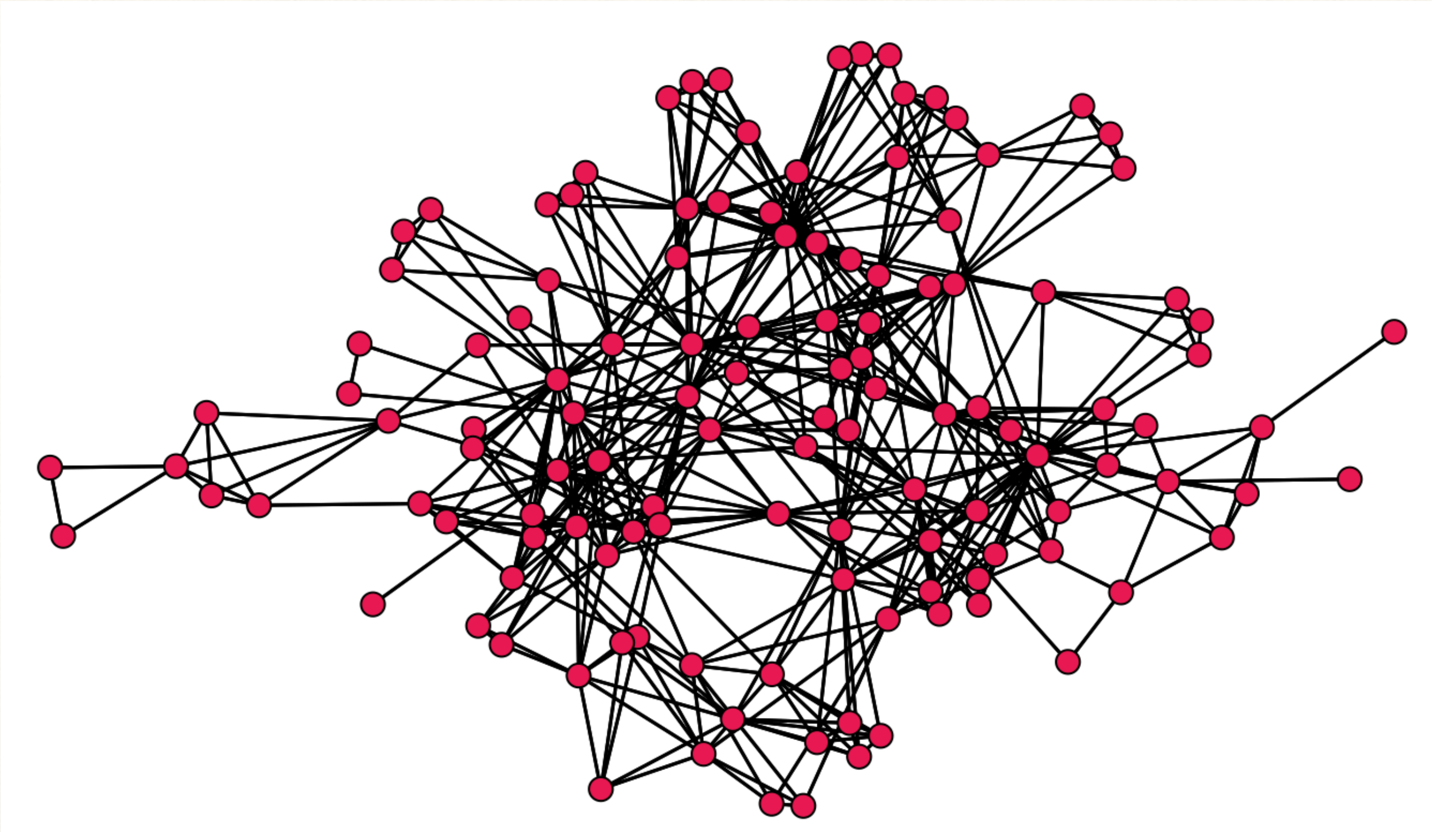
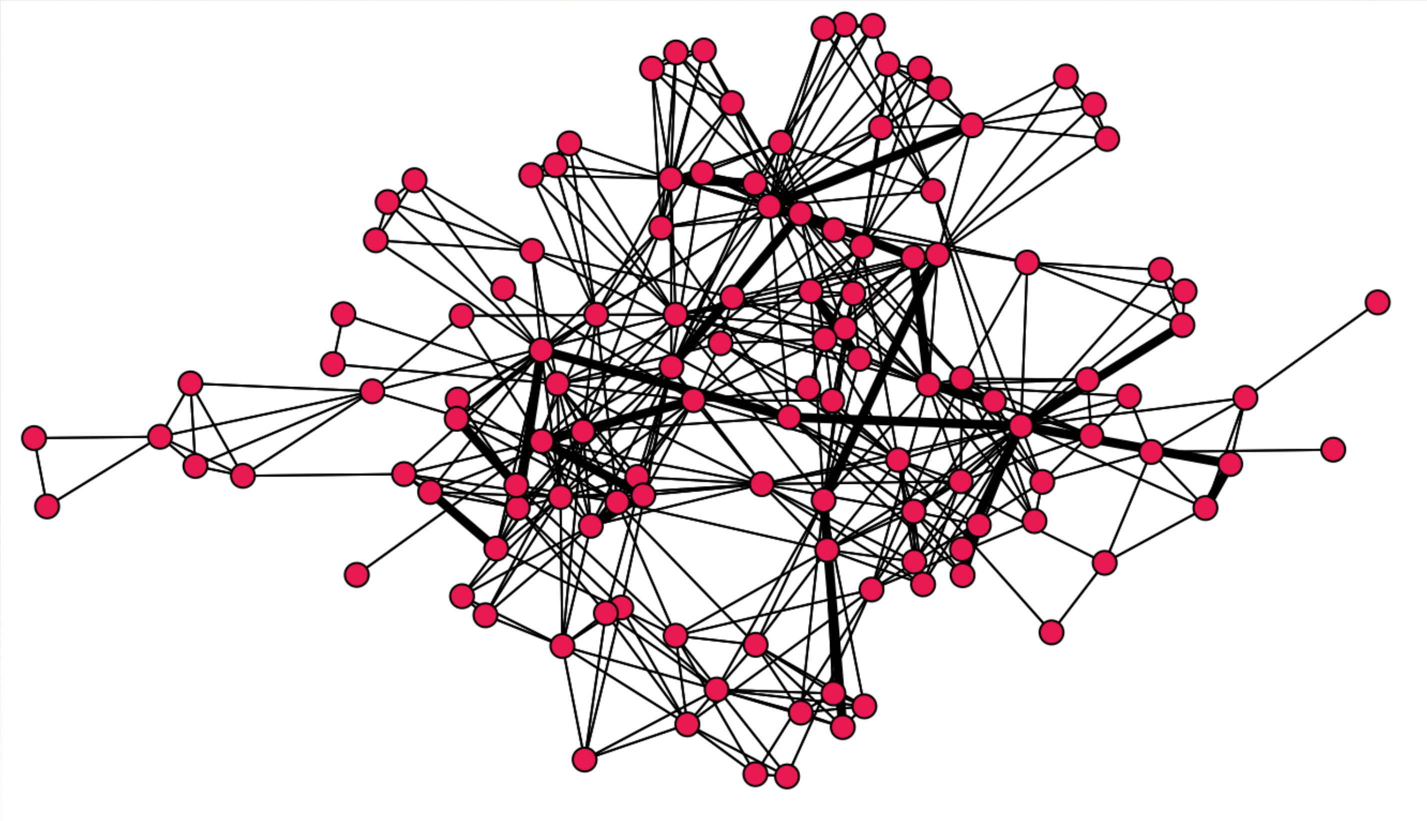
These weights are returned as the product matrix.

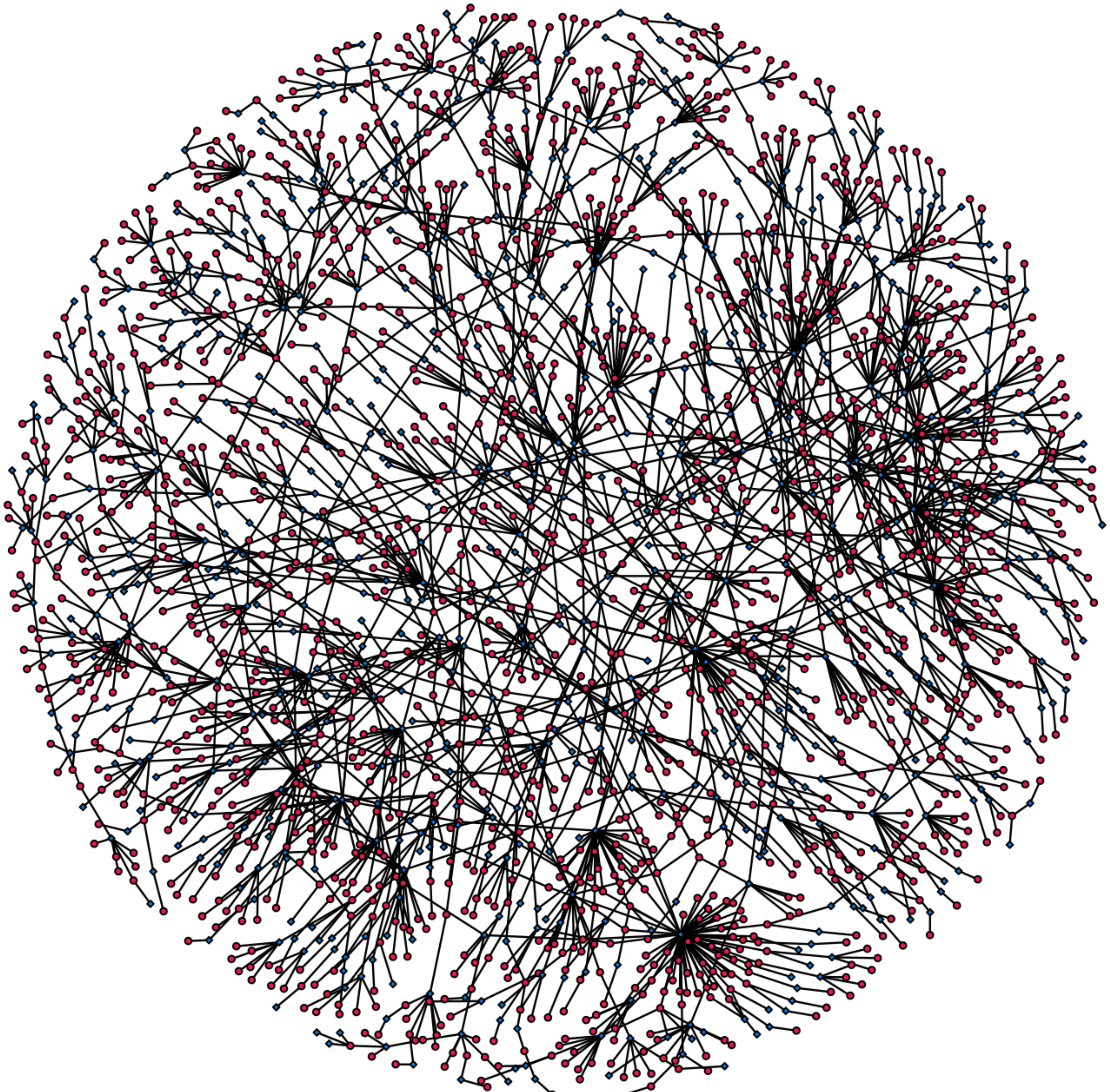


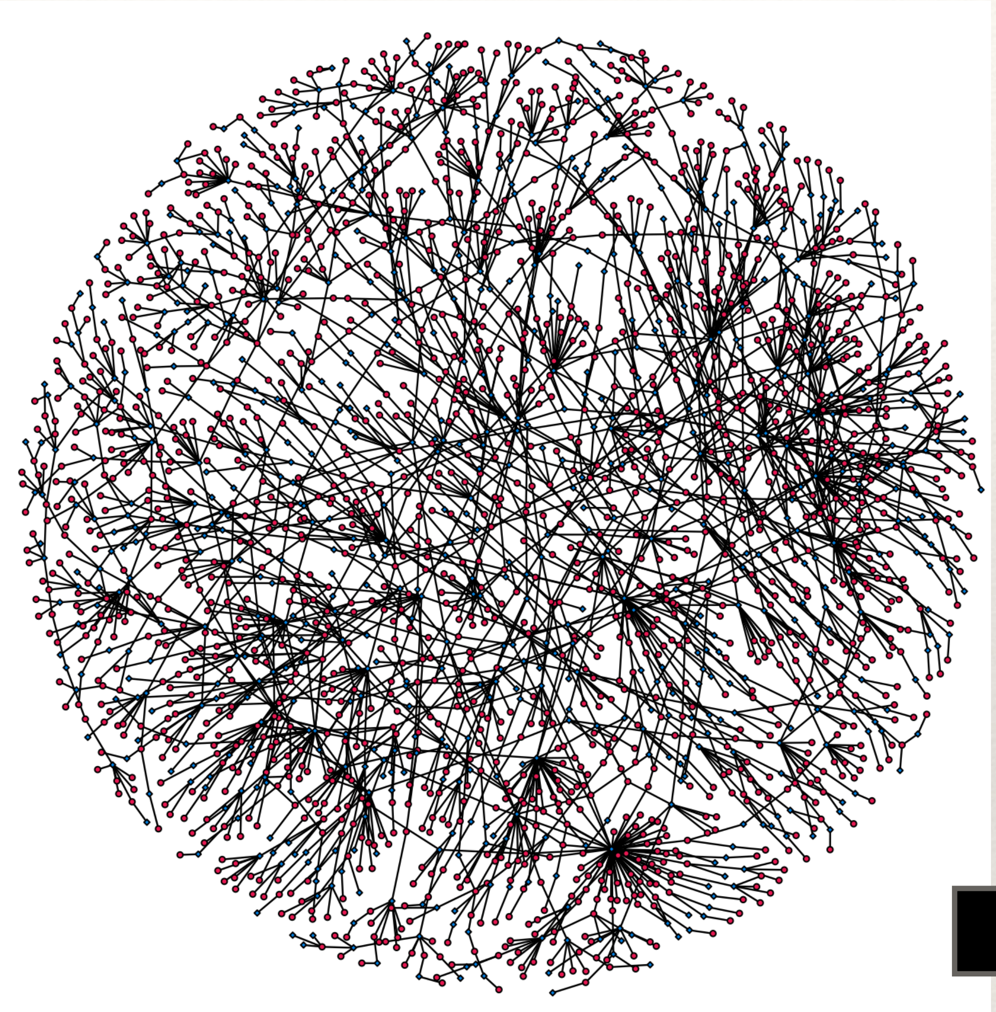
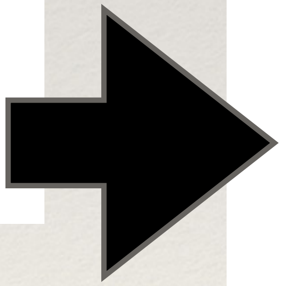
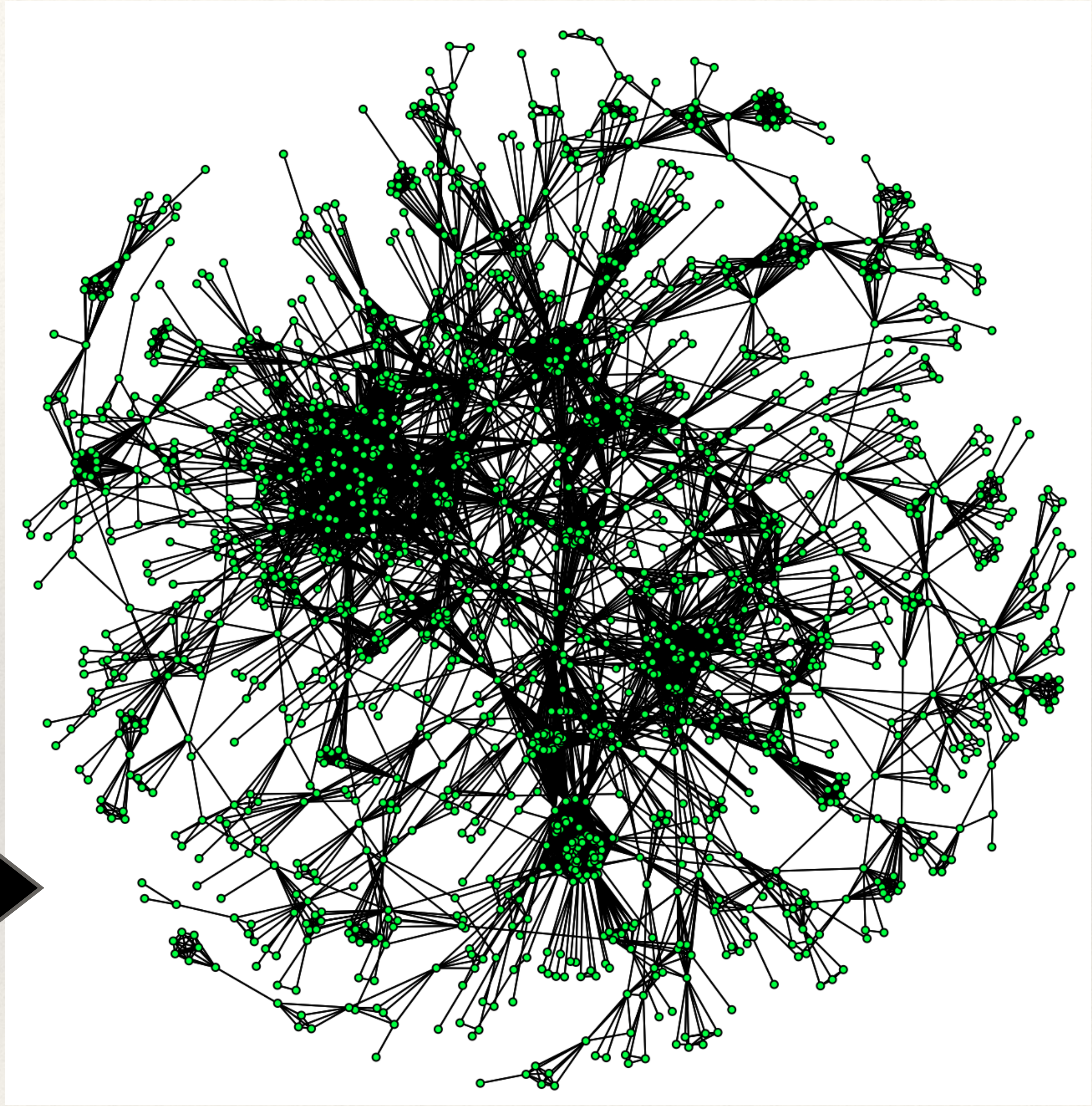


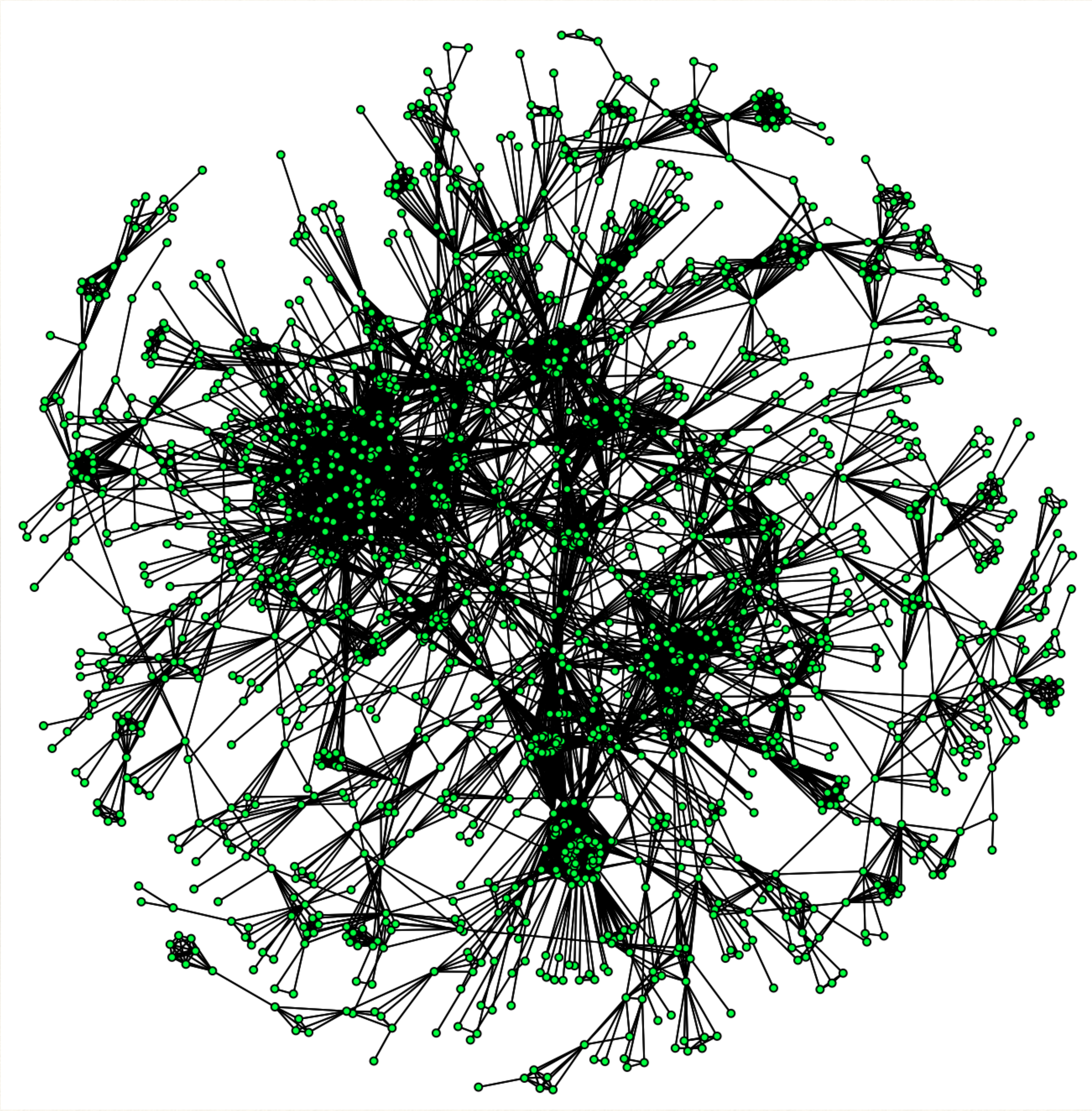








Learning Goals

- At the end of the lecture, you should be able to answer these questions:
  - How can we create **unipartite** graphs from **bipartite** graphs?
  - What is the difference between **dichotomized** projections and **summation** projections?

Questions?