4.4 The Mean Value Theorem

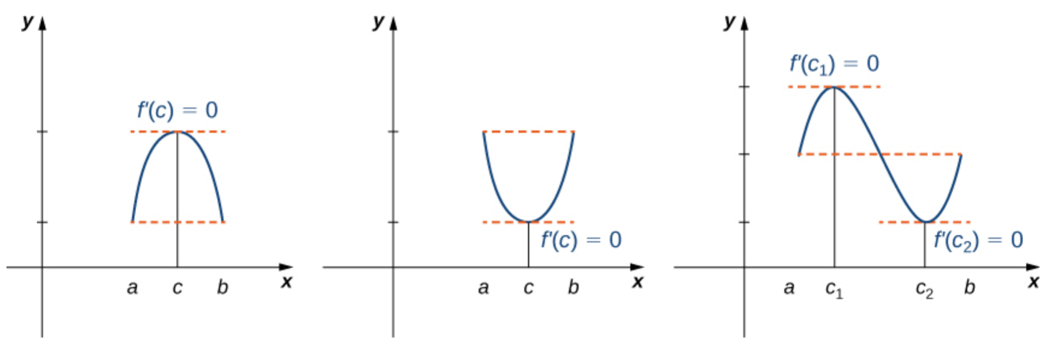
Objectives:

- Rolle's theorem.
- The Mean Value Theorem. (MVT)
- Three important consequences of the Mean Value Theorem.

Rolle's Theorem.

Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.



(a) (b) (c)

1. For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values of c in the given interval where $f'(c) = 0$.

   (a) $f(x) = x^2 + 2x$ over [-2,0]

(b) $f(x) = x^3 - 4x$ over [-2,2]

2. Given $f(x) = 2x^2 - 8x - 2$

(a) If we apply Rolle's Theorem to $f(x)$ over [0,4], how many values of c exist so that $f'(c) = 0$? What are the values of c?

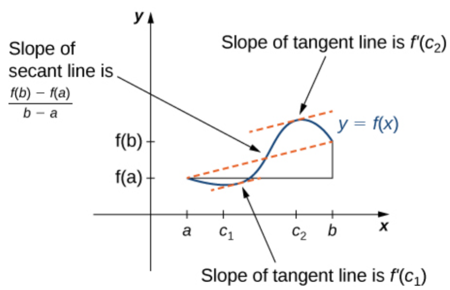
(b) If we considered $f(x)$ over [-3,11] instead, would Rolle's Theorem still apply?

Mean Value Theorem.

Mean Value Theorem

Let f be continuous over the closed interval $[a,b]$ and differentiable over the open interval (a,b). Then, there exists at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



3. Consider the function $f(x) = 10x + \dfrac{4}{x}$. Find the exact value of all numbers that satisfy the conclusion of the mean value theorem for f on the interval [2,4].

4. At 10:13 a.m., you pass a speed radar on the freeway. At 10:38 a.m., you pass a second speed radar along the same freeway. If the second radar was located 24 miles from the first radar, and the speed limit along the entire stretch of highway is 70 mph, could you be fined for speeding based on the radar data?

   (a) Find your average velocity in miles per hour (note: 25 minutes = $5/12$ hour) from 10:13 a.m. to 10:38 a.m. Round to 2 decimal places if needed.

   (b) By the MVT, since the average velocity over the relevant time interval was ______________ than the posted speed limit of 70 mph, you ______________ be fined for speeding based on the radar data.

4.5 The First and Second Derivative Tests

Objectives:

- State the first derivative test for critical points.
- The Concavity Test .
- The Second Derivative Test.

First Derivative Test.

Definitions:

- A ***partition number*** of f' is a number p for which either $f'(p) = 0$ or $f'(p)$ does not exist.
- A ***critical number*** of f is a number c for which either $f'(c) = 0$ or $f'(c)$ does not exist AND c is in the domain of f.

Note: Critical numbers are also partition numbers, but partition numbers may not be critical numbers if not in the domain.

1. Given $f(x) = \dfrac{x^2+3}{x-1}$, state the domain and find the partition nubers of f' and critical numbers of f.

What is a Sign Chart? We will use a sign chart to track where $f' > 0$ (+) or where $f' < 0$ (-).

Steps for making a Sign Chart

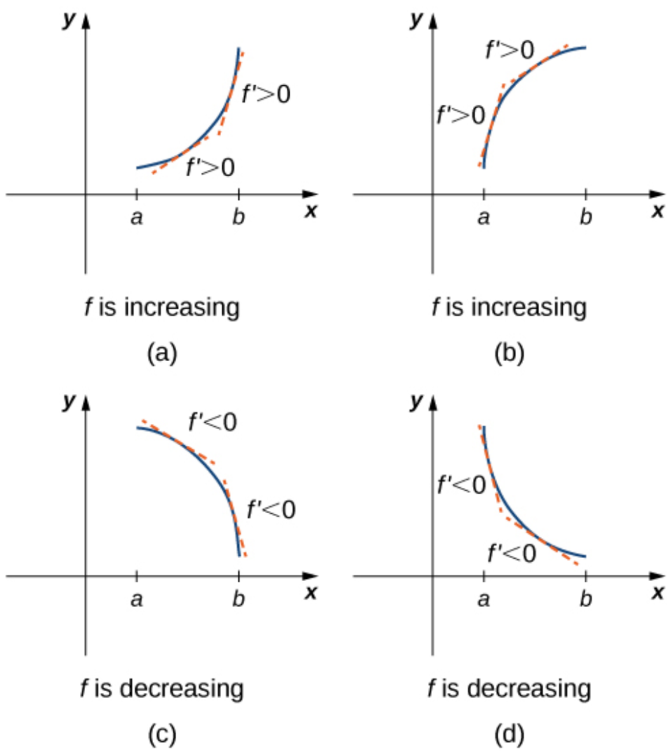
1. Plot all partition and critical numbers on a number line.
2. Choose values to test regions on the number line around the critical/partition numbers.
3. Plug test values into f' and record the sign ($\pm$).

2. Make a sign chart for the function in Problem 1. $\left(\textbf{Note: } f'(x) = \dfrac{(x-3)(x+1)}{(x-1)^2} \right)$

Increasing/Decreasing Test:

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval



3. Use the sign chart from Problem 2 to find intervals on which $f(x) = \dfrac{x^2+3}{x-1}$ is increasing/decreasing. $\left(\textbf{Note: } f'(x) = \dfrac{(x-3)(x+1)}{(x-1)^2} \right)$

Definition: Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **Relative(Local) Maximum** value of f if $f(c) \geq f(x)$ when x is near c.
- **Relative(Local) Minimum** value of f if $f(c) \leq f(x)$ when x is near c.

Note: We say that $f(c)$ is a relative(local) extremum if $f(c)$ is either a relative maximum or a relative minimum.

The First Derivative Test: Suppose that c is a critical number of a continuous function f.

(a) If $f'(x)$ changes from positive to negative around c, then f has a relative maximum at c.

(b) If $f'(x)$ changes from negative to positive around c, then f has a relative minimum at c.

(c) If $f'(x)$ does not change sign around c (for example, $f'(x)$ is positive on both sides of c or negative on both sides), then f has no relative maximum or minimum at c.

4. Use the sign chart from Problem 2 to find any x-value where $f(x) = \dfrac{x^2+3}{x-1}$ has a relative extremum. $\left(\textbf{Note: } f'(x) = \dfrac{(x-3)(x+1)}{(x-1)^2} \right)$

Steps for finding where f is Increasing/Decreasing or has any Relative Extrema

1. Find the domain of f.
2. Find all partition numbers of f'.
3. Find all critical numbers of f.
4. Make a sign chart to track where $f' > 0$ or $f' < 0$.

5. For the function below, find all intervals on which f is increasing/decreasing and the x-values of any relative extrema.

   $f(x) = 5(x^2 - 36x + 180)^{4/5}$

6. Suppose the function $f(x)$ has a domain of all real numbers and its derivative is shown below.
   $f'(x) = (x+2)^2(x-7)^4(x-8)^7$.

   (a) Find all intervals on which $f(x)$ is increasing.

   (b) Find all intervals on which $f(x)$ is decreasing.

   (c) State all x-values where f has either a relative maximum or relative minimum.

7. Suppose the function $f(x)$ has a domain of all real numbers except $x = -7$. The first derivative of $f(x)$ is shown below.
   $f'(x) = \dfrac{-5(x-1)}{(x+7)^7}$

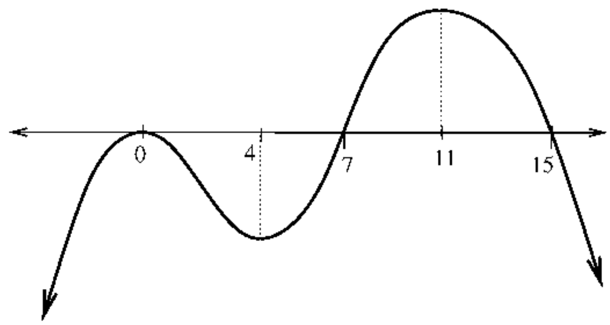
   (a) Find all intervals on which $f(x)$ is increasing.

   (b) Find all intervals on which $f(x)$ is decreasing.

   (c) Find the x-values where $f(x)$ has relative extrema.

8. Use the graph of $f'(x)$ to answer questions about the function $f(x)$.

This is the graph of $f'(x)$.



(a) Find all critical values of $f(x)$.

(b) Find all intervals on which $f(x)$ is increasing and all intervals on which $f(x)$ is decreasing.

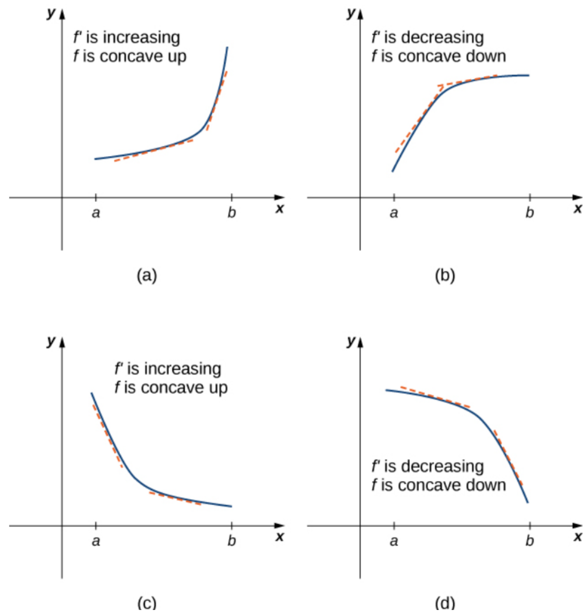
(c) Find the x-coordinates of all relative extrema on the graph of $f(x)$.

The Concavity Test

Concave Up and Down:

1. We say that the graph of f is **concave up** on an interval if f' is increasing on the interval.
2. We say that the graph of f is **concave down** on an interval if f' is decreasing on the interval.

Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.

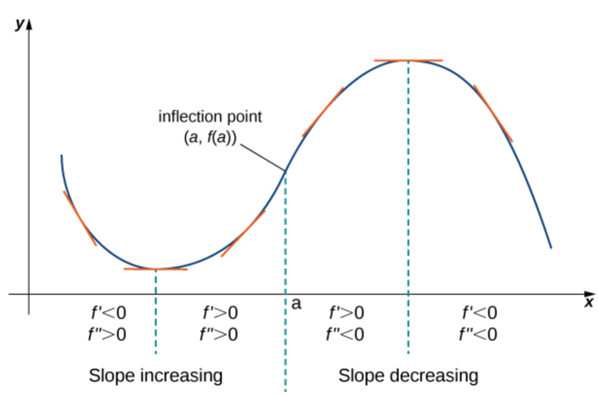


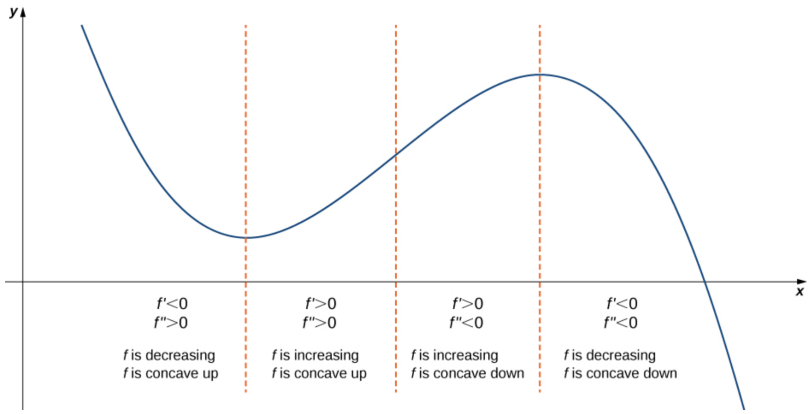
Concavity Test:

(a) If $f''(x) > 0$ for all x on an interval, then the graph of f is concave up on the interval.

(b) If $f''(x) < 0$ for all x on an interval, then the graph of f is concave down on the interval.

(c) A point $(c, f(c))$, where c is in the domain of f, where a curve changes it's direction of concavity is called an ***inflection point***.





9. Complete the following:

(a) $g'(x)$ is positive when ________________________________

(b) $g'(x)$ is negative when ________________________________

(c) What statements are equivalent to the statement that **g(x) is concave up?**

(d) What statements are equivalent to the statement that **g(x) is concave down?**

Definition:

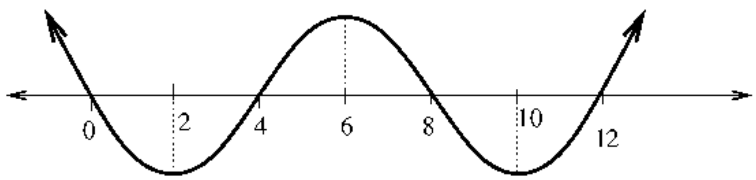
A ***partition number*** of f'' is a number p for which either $f''(p) = 0$ or $f''(p)$ does not exist.

Steps for finding where f is Concave Up/Concave Down or if f has any Inflection Poin

1. Find the domain of f.
2. Find all partition numbers of f''
3. Make a sign chart to track where $f'' > 0$ or $f'' < 0$.

10. Use the graph of $f''(x)$ to answer questions about the function $f(x)$.

This is the graph of $f''(x)$.



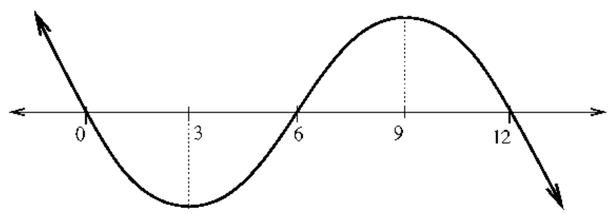
(a) Give the intervals where $f(x)$ is concave up.

(b) Give the intervals where $f(x)$ is concave down.

(c) Find the x-coordinates of the inflection points for $f(x)$.

11. Use the graph of the function $f'(x)$ to answer the following questions.

This is the graph of $f'(x)$.



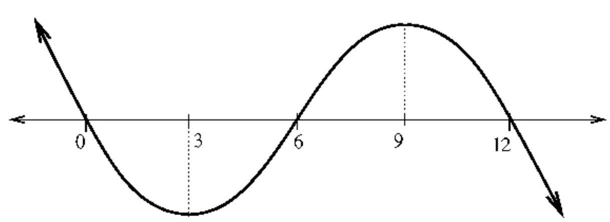
(a) Give the intervals where $f(x)$ is concave up.

(b) Give the intervals where $f(x)$ is concave down.

(c) Find the x-coordinates of the inflection points for $f(x)$.

12. Use the graph of the function $f(x)$ to answer the following questions.

This is the graph of $f(x)$.



Give the intervals where $f(x)$ is concave up or down and find the x-values of any inflection points.

13. Give the intervals where $f(x)$ is concave up or down and find the x-values of any inflection points.

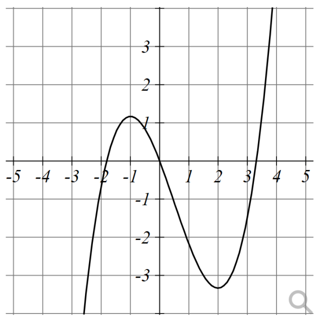
    $f(x) = -4x^3 - 12x^2 + 96x + 6$

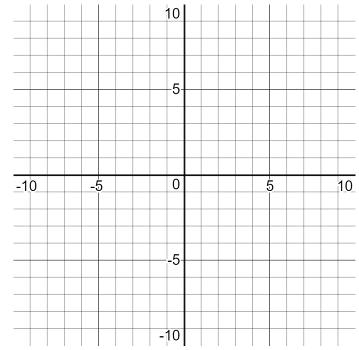
14. Suppose the function $g(x)$ has a domain of all real numbers except $x = -5$. The second derivative of $g(x)$ is shown below.

    $g''(x) = \dfrac{(x-5)(x+4)}{(x+5)^3}$

    Give the intervals where $g(x)$ is concave up or down and find the x-values of any inflection points.

15. Use the graph of $f(x)$ to sketch a graph of $f'(x)$.



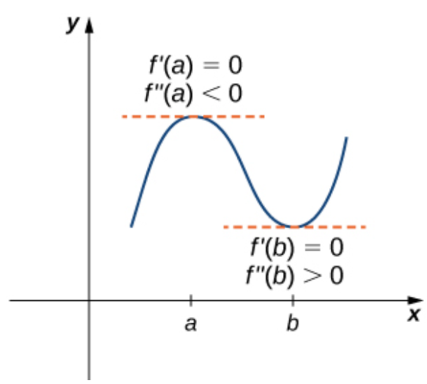


The Second Derivative Test: Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a relative minimum.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a relative maximum.

Note: If $f'(c) \neq 0$, then c is neither a maximum nor a minimum. If $f''(c) = 0$ then the test is inconclusive. It tells us nothing!



16. Assuming that the function $f(x)$ is continuous on the interval $(-\infty, \infty)$, indicate whether each of the points listed below is a relative maximum, relative minimum, neither or cannot be determined from the information given.

   (a) $(1, f(1))$ if $f'(1) = 0$ and $f''(1) = -3$

   (b) $(0, f(0))$ if $f'(0) = -3$ and $f''(0) = -3$

   (c) $(-1, f(-1))$ if $f'(-1) = 0$ and $f''(1) = 10$

   (d) $(3, f(3))$ if $f'(3) = 0$ and $f''(3) = 0$

17. Given $f(x) = x^3 - 6x^2$, use the Second Derivative Test to find the x-values of any relative extrema if possible.