ASEN 5044 Hw 8

Problem 2

a) Find the required CT Jacobians.

$$f_1 = x_2 \quad f_2 = -\frac{\mu x_1}{(x_1^2+x_3^2)^{3/2}} + u_1 \quad f_3 = x_4 \quad f_4 = -\frac{\mu x_3}{(x_1^2+x_3^2)^{3/2}} + u_2$$

$$\bar{x} = [x, \dot{x}, y, \dot{y}]^T = [x_1, x_2, x_3, x_4]^T$$

$$\tilde{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & & & \vdots \\ \frac{\partial f_3}{\partial x_1} & & \ddots & \\ \frac{\partial f_4}{\partial x_1} & \cdots & & \frac{\partial f_4}{\partial x_4} \end{bmatrix}$$

$$\frac{\partial f_1}{\partial x_2} = 1 \quad \frac{\partial f_1}{\partial x_1} = \frac{\partial f_1}{\partial x_3} = \frac{\partial f_1}{\partial x_4} = 0$$

$$\frac{\partial f_3}{\partial x_4} = 1 \quad \frac{\partial f_3}{\partial x_1} = \frac{\partial f_1}{\partial x_2} = \frac{\partial f_1}{\partial x_3} = 0$$

$$\frac{\partial f_2}{\partial x_2} = \frac{\partial f_2}{\partial x_4} = 0 \quad \frac{\partial f_4}{\partial x_2} = \frac{\partial f_4}{\partial x_4} = 0$$

$$\frac{\partial f_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left(-\mu x_1 (x_1^2+x_3^2)^{-3/2} + u_1\right) = -\mu (x_1^2+x_3^2)^{-3/2} + 3\mu x_1^2 (x_1^2+x_3^2)^{-5/2}$$

$$\frac{\partial f_2}{\partial x_3} = \frac{\partial}{\partial x_3}\left(-\mu x_1 (x_1^2+x_3^2)^{-3/2} + u_1\right) = 3\mu x_1 x_3 (x_1^2+x_3^2)^{-5/2}$$

$$\frac{\partial f_4}{\partial x_1} = \frac{\partial}{\partial x_1}\left(-\mu x_3 (x_1^2+x_3^2)^{-3/2} + u_2\right) = 3\mu x_1 x_3 (x_1^2+x_3^2)^{-5/2}$$

$$\frac{\partial f_4}{\partial x_3} = \frac{\partial}{\partial x_3}\left(-\mu x_3 (x_1^2+x_3^2)^{-3/2} + u_2\right) = -\mu (x_1^2+x_3^2)^{-3/2} + 3\mu x_3^2 (x_1^2+x_3^2)^{-5/2}$$

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-\mu}{r_0^3} + \frac{3\mu x_1^2}{r_0^5} & 0 & \frac{3\mu x_1 x_3}{r_0^5} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3\mu x_1 x_3}{r_0^5} & 0 & \frac{-\mu}{r_0^3} + \frac{3\mu x_3^2}{r_0^5} & 0 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{\partial f_1}{\partial u_1} = \frac{\partial f_1}{\partial u_2} = 0 \quad \frac{\partial f_2}{\partial u_1} = 1 \quad \frac{\partial f_2}{\partial u_2} = 0$$

$$\frac{\partial f_3}{\partial u_1} = \frac{\partial f_3}{\partial u_2} = 0 \quad \frac{\partial f_4}{\partial u_1} = 0 \quad \frac{\partial f_4}{\partial u_2} = 1$$

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Problem 2

a) Find the required CT Jacobians:

$$g_1 = \left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{1/2}$$

$$g_2 = \left((x_1 - x_s^i)(x_2 - \dot{x}_s^i) + (x_3 - y_s^i)(x_4 - \dot{y}_s^i)\right)\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{-1/2}$$

$$g_3 = \tan^{-1}\left(\frac{(x_3 - y_s^i)}{(x_1 - x_s^i)}\right)$$

$$\tilde{C} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} & \frac{\partial g_1}{\partial x_4} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_3}{\partial x_3} & \frac{\partial g_4}{\partial x_4} \\ \frac{\partial g_3}{\partial x_1} & \frac{\partial g_3}{\partial x_2} & \frac{\partial g_3}{\partial x_3} & \frac{\partial g_4}{\partial x_4} \end{bmatrix} \qquad \frac{\partial g_1}{\partial x_2} = \frac{\partial g_1}{\partial x_4} = 0 \qquad \frac{\partial g_3}{\partial x_2} = \frac{\partial g_3}{\partial x_4} = 0$$

$$\frac{\partial g_1}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{1/2}\right) = \frac{1}{2}\left(2(x_1 - x_s^i)\right)\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{-1/2}$$

$$* \quad \frac{\partial g_1}{\partial x_1} = \frac{x_1 - x_s^i}{\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{1/2}}$$

$$\frac{\partial g_1}{\partial x_3} = \frac{\partial}{\partial x_3}\left(\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{1/2}\right) = \frac{1}{2}\left(2(x_3 - y_s^i)\right)\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{-1/2}$$

$$* \quad \frac{\partial g_1}{\partial x_3} = \frac{x_3 - y_s^i}{\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{1/2}}$$

$$\frac{\partial g_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{(x_1 - x_s^i)(x_2 - \dot{x}_s^i) + (x_3 - y_s^i)(x_4 - \dot{y}_s^i)}{\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{1/2}}\right)$$

Product rule: $\frac{\partial}{\partial x}(uv) = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}$

$$= \left((x_1 - x_s^i)(x_2 - \dot{x}_s^i) + (x_3 - y_s^i)(x_4 - \dot{y}_s^i)\right)\left(-\frac{1}{\cancel{2}}\cdot\cancel{2}\cdot(x_1 - x_s^i)\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)\right)^{-3/2}\right)$$

$$+ \left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{-1/2}(x_2 - \dot{x}_s^i)$$

$$* \quad \frac{\partial g_2}{\partial x_1} = \frac{(x_2 - x_s^i)}{\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{1/2}} - \frac{(x_1 - x_s^i)\left((x_1 - x_s^i)(x_2 - \dot{x}_s^i) + (x_3 - y_s^i)(x_4 - \dot{y}_s^i)\right)}{\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{3/2}}$$

$$\frac{\partial g_2}{\partial x_3} = \frac{\partial}{\partial x_3}\left(\frac{(x_1 - x_s^i)(x_2 - \dot{x}_s^i) + (x_3 - y_s^i)(x_4 - \dot{y}_s^i)}{\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{1/2}}\right)$$

Product Rule: $\frac{\partial}{\partial x}(uv) = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}$

$$= \left((x_1 - x_s^i)(x_2 - \dot{x}_s^i) + (x_3 - y_s^i)(x_4 - \dot{y}_s^i)\right)\left(-\frac{1}{2}\cdot\cancel{2}\cdot(x_3 - \dot{y}_s^i)\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{-3/2}\right)$$

$$+ \left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{-1/2}(x_4 - \dot{y}_s^i)$$

$$* \quad \frac{\partial g_2}{\partial x_3} = \frac{(x_4 - y_s^i)}{\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{1/2}} - \frac{(x_3 - y_s^i)\left((x_1 - x_s^i)(x_2 - \dot{x}_s^i) + (x_3 - y_s^i)(x_4 - \dot{y}_s^i)\right)}{\left((x_1 - x_s^i)^2 + (x_3 - y_s^i)^2\right)^{3/2}}$$

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Problem 2

a) Find the required G Jacobians:

Product Rule: $\frac{\partial}{\partial x}(uv) = v\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}$

$$\frac{\partial g_2}{\partial x_2} = \frac{\partial}{\partial x_2}\left(\frac{(x_1-x_s^i)(x_2-\dot{x}_s^i)+(x_3-y_s^i)(x_4-\dot{y}_s^i)}{((x_1-x_s^i)^2+(x_3-y_s^i)^2)^{1/2}}\right)$$

$$= \left((x_1-x_s^i)(x_2-\dot{x}_s^i)+(x_3-y_s^i)(x_4-\dot{y}_s^i)\right)(0)$$

$$+ \left((x_1-x_s^i)^2+(x_3-y_s^i)^2\right)^{1/2}(x_1-x_s^i)$$

$$* \quad \frac{\partial g_2}{\partial x_2} = \frac{(x_1-x_s^i)}{((x_1-x_s^i)^2+(x_3-y_s^i)^2)^{1/2}}$$

Product Rule

$$\frac{\partial g_2}{\partial x_4} = \frac{\partial}{\partial x_4}\left(\frac{(x_1-x_s^i)(x_2-\dot{x}_s^i)+(x_3-y_s^i)(x_4-\dot{y}_s^i)}{((x_1-x_s^i)^2+(x_3-y_s^i)^2)}\right)$$

$$= \left((x_1-x_s^i)(x_2-\dot{x}_s^i)+(x_3-y_s^i)(x_4-\dot{y}_s^i)\right)(0)$$

$$+ \left((x_1-x_s^i)^2+(x_3-y_s^i)^2\right)^{1/2}(x_3-y_s^i)$$

$$* \quad \frac{\partial g_2}{\partial x_4} = \frac{(x_3-y_s^i)}{((x_1-x_s^i)^2+(x_3-y_s^i)^2)^{1/2}}$$

Chain Rule: $\frac{\partial}{\partial x}(\tan^{-1}(u)) = \frac{\partial}{\partial u}(\tan^{-1}(u))\frac{\partial u}{\partial x}$

$$u = \frac{x_3-y_s^i}{x_1-x_s^i}$$

$$\frac{\partial u}{\partial x_1} = (x_3-y_s^i)\frac{\partial}{\partial x_1}\left((x_1-x_s^i)^{-1}\right)$$

$$= -\frac{(x_3-y_s^i)}{(x_1-x_s^i)^2}$$

$$\frac{\partial g_3}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\tan^{-1}\left(\frac{(x_3-y_s^i)}{(x_1-x_s^i)}\right)\right)$$

$$= \frac{1}{1+u^2}\cdot\frac{\partial u}{\partial x_1}$$

$$= \frac{1}{1+\frac{(x_3-y_s^i)^2}{(x_1-x_s^i)^2}}\cdot -\frac{(x_3-y_s^i)}{(x_1-x_s^i)^2}$$

$$* \quad \frac{\partial g_3}{\partial x_1} = \frac{-(x_3-y_s^i)}{(x_1-x_s^i)^2+(x_3-y_s^i)^2}$$

Chain Rule: $\frac{\partial}{\partial x}(\tan^{-1}(u)) = \frac{\partial}{\partial u}(\tan^{-1}(u))\cdot\frac{\partial u}{\partial x}$

$$u = \frac{x_3-y_s^i}{x_1-x_s^i}$$

$$\frac{\partial u}{\partial x_3} = (x_1-x_s^i)^{-1}\frac{\partial}{\partial x_3}(x_3-x_s^i)$$

$$= (x_1-x_s^i)^{-1}$$

$$\frac{\partial g_3}{\partial x_3} = \frac{\partial}{\partial x_3}\left(\tan^{-1}\left(\frac{(x_3-y_s^i)}{(x_1-x_s^i)}\right)\right)$$

$$= \frac{1}{1+u^2}\cdot\frac{\partial u}{\partial x_3}$$

$$= \frac{1}{1+\frac{(x_3-y_s^i)^2}{(x_1-x_s^i)^2}}\cdot(x_1-x_s^i)^{-1}$$

$$= \frac{1}{(x_1-x_s^i)+\frac{(x_3-y_s^i)^2}{(x_1-x_s^i)}}$$

$$* \quad \frac{\partial g_3}{\partial x_3} = \frac{(x_1-x_s^i)}{(x_1-x_s^i)^2+(x_3-y_s^i)^2}$$