$$\int \sin \sqrt{t}\, dt$$

The thing to see is that we would love for this to be more like $\sin t$ than $\sin \sqrt{t}$. We can make that happen by substituting, say $w = \sqrt{t}$, where I use w instead of u because I'm going to use a u later for integration by parts (IBP) and I want to keep things clear. So we have

$$w^2 = t \quad \Rightarrow \quad 2w\, dw = dt$$

Substituting these into the original integral gives

$$\int \sin \sqrt{t}\, dt = 2 \int w \sin w\, dw$$

You should be saying to yourself, "Self, this looks like a textbook IBP problem!" And you would be correct :-) Specifically, you should see that if we let $u = w$, then that term will "disappear" in the integral on the right hand side of the IBP formula. Let's watch it happen. Let

$$\begin{aligned} u = w \quad &\Rightarrow \quad du = dw \\ dv = \sin w\, dw \quad &\Rightarrow \quad v = -\cos w \end{aligned}$$

which, using the IBP formula, gives us

$$\begin{aligned} \int \sin \sqrt{t}\, dt &= 2 \int w \sin w\, dw \\ &= 2\left(w\,(-\cos w) - \int (-\cos w)\ dw\right) \\ &= -2w \cos w + 2 \int \cos w\, dw \\ &= -2w \cos w + 2 \sin w + C \\ &= -2\sqrt{t} \cos \sqrt{t} + 2 \sin \sqrt{t} + C. \end{aligned}$$

Finally, we have

$$\boxed{\int \sin \sqrt{t}\, dt = 2 \sin \sqrt{t} - 2\sqrt{t} \cos \sqrt{t} + C.}$$

$$\int \frac{\arctan x}{x^2}\,dx$$

This type of integral has a tendency to scare students into panic, but it's not too hard if you understand one key thing. You are likely scared because you don't know the *integral* of $\arctan x$, and certainly not when it's divided by some power of x, but you *do* know the *derivative* of $\arctan x$! This fact should *scream* to you to use IBP. You can set $u = \arctan x$, then you don't ever have to worry about integrating that part of it. Let's see it in action.

$$\begin{aligned} u = \arctan x \quad &\Rightarrow \quad du = \frac{dx}{1+x^2} \\ dv = \frac{dx}{x^2} \quad &\Rightarrow \quad v = \frac{-1}{x} \end{aligned}$$

Using the IBP formula gives

$$\int \frac{\arctan x}{x^2}\,dx = \frac{-1}{x}\arctan x + \int \frac{dx}{x(1+x^2)}$$

From here you have options. You could use partial fraction decomposition, but I'm going to go with trig substitution. Let

$$x = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta\,d\theta$$

Then we have

$$\begin{aligned} \int \frac{\arctan x}{x^2}\,dx &= \frac{-1}{x}\arctan x + \int \frac{dx}{x(1+x^2)} \\ &= \frac{-1}{x}\arctan x + \int \frac{1}{\tan\theta(1+\tan^2\theta)}\sec^2\theta\,d\theta \\ &= \frac{-1}{x}\arctan x + \int \frac{\sec^2\theta}{\tan\theta\sec^2\theta}\,d\theta \\ &= \frac{-1}{x}\arctan x + \int \frac{\cos\theta}{\sin\theta}\,d\theta \\ &= \frac{-1}{x}\arctan x + \log|\sin\theta| + C \\ &= \frac{-1}{x}\arctan x + \log\left|\frac{x}{\sqrt{1+x^2}}\right| + C \end{aligned}$$

Where the last few lines use u-sub and the unit triangle. So we have

$$\boxed{\int \frac{\arctan x}{x^2}\,dx = \frac{-1}{x}\arctan x + \log\left|\frac{x}{\sqrt{1+x^2}}\right| + C}$$

$$\int x\sqrt{2-\sqrt{1+x^2}}\,dx$$

Another scary-looking thing because of the nested square roots. The thing I do is to ask myself what I would want to clean those radicals up, starting from the innermost one. In this case, if $1+x^2$ were just something squared, then I would be rid of that radical. So let's do it. Let

$$w^2 = 1+x^2 \quad\Rightarrow\quad w\,dw = x\,dx$$

Substituting this in gives

$$\int x\sqrt{2-\sqrt{1+x^2}}\,dx = \int w\sqrt{2-w}\,dw$$

Well... much better, but not quite there. I'd like to get rid of that last square root. If only I had something that, when it was squared, equaled $2-w$... Let

$$z^2 = 2-w \quad\Rightarrow\quad -2z\,dz = dw$$

then we get

$$\begin{aligned}\int x\sqrt{2-\sqrt{1+x^2}}\,dx &= \int w\sqrt{2-w}\,dw\\ &= \int (2-z^2)\sqrt{z^2}(-2z\,dz)\\ &= -2\int (2-z^2)z^2\,dz\\ &= 2\int z^4\,dz - 4\int z^2\,dz.\end{aligned}$$

And now it's easy

$$\begin{aligned}\int x\sqrt{2-\sqrt{1+x^2}}\,dx &= 2\int z^4\,dz - 4\int z^2\,dz\\ &= \tfrac{2}{5}(2-w)^{5/2} - \tfrac{4}{3}(2-w)^{3/2} + C\\ &= \tfrac{2}{5}(2-1+x^2)^{5/2} - \tfrac{4}{3}(2-1+x^2)^{3/2} + C.\end{aligned}$$

Finally,

$$\boxed{\int x\sqrt{2-\sqrt{1+x^2}}\,dx = \tfrac{2}{5}(1+x^2)^{5/2} - \tfrac{4}{3}(1+x^2)^{3/2} + C.}$$

$$\int \frac{dx}{\sqrt{x} + x\sqrt{x}}$$

First, I find this visually awful, so let's rewrite it.

$$\int \frac{dx}{\sqrt{x} + x\sqrt{x}} = \int \frac{dx}{\sqrt{x}(1+x)}$$

Now taking a cue from the above solution, put

$$u^2 = x \quad \Rightarrow \quad 2u\,du = dx$$

so that

$$\int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{2u\,du}{u(1+u^2)} = 2\int \frac{du}{1+u^2}.$$

Now we are left with an easy trig-sub problem. So let

$$u = \tan\theta \quad \Rightarrow \quad du = \sec^2\theta\,d\theta$$

which gives

$$\begin{aligned}
\int \frac{dx}{\sqrt{x}(1+x)} &= 2\int \frac{du}{1+u^2} \\
&= 2\int \frac{\sec^2\theta\,d\theta}{1+\tan^2\theta} \\
&= 2\int \frac{\sec^2\theta\,d\theta}{\sec^2\theta} \\
&= 2\int d\theta \\
&= 2\theta + C \\
&= 2\arctan u + C \\
&= 2\arctan\sqrt{x} + C.
\end{aligned}$$

We have

$$\boxed{\int \frac{dx}{\sqrt{x}(1+x)} = 2\arctan\sqrt{x} + C.}$$

$$\int \frac{dx}{\sqrt{\sqrt{x}+1}}$$

Just like the two above, we want to nix the radicals. So two substitutions will do it. Let

$$u^2 = x \quad \Rightarrow \quad 2u\,du = dx$$

which gives

$$\int \frac{dx}{\sqrt{\sqrt{x}+1}} = \int \frac{2u\,du}{\sqrt{u+1}}.$$

Now put

$$w^2 = u + 1 \quad \Rightarrow \quad 2w\,dw = du$$

which gives

$$\begin{aligned}
\int \frac{dx}{\sqrt{\sqrt{x}+1}} &= \int \frac{2u\,du}{\sqrt{u+1}} \\
&= \int \frac{2(w^2-1)(2w\,dw)}{w} \\
&= 4\int (w^2-1)\,dw \\
&= 4\int w^2\,dw - 4\int dw \\
&= \tfrac{4}{3}w^3 - 4w + C \\
&= \tfrac{4}{3}(u+1)^{3/2} - 4(u+1)^{1/2} + C \\
&= \tfrac{4}{3}\left(\sqrt{x}+1\right)^{3/2} - 4\left(\sqrt{x}+1\right)^{1/2} + C.
\end{aligned}$$

Finally,

$$\boxed{\int \frac{dx}{\sqrt{\sqrt{x}+1}} = \tfrac{4}{3}\sqrt{\sqrt{x}+2}\left(\sqrt{x}-2\right) + C.}$$