$$\int \arctan t \, dt$$

Let $u = \arctan t$, then clearly (make sure you see that it *is* clear) $dv = dt$. Then

$$\begin{aligned} u = \arctan t \quad &\Rightarrow \quad du = \frac{dt}{1+t^2} \\ dv = dt \quad &\Rightarrow \quad v = t \end{aligned}$$

so that

$$\begin{aligned} \int \arctan t \, dt &= uv - \int v \, du \\ &= t \arctan t - \int \frac{t \, dt}{1+t^2} \end{aligned}$$

Now use u-substitution with

$$u = 1 + t^2 \quad \Rightarrow \quad \tfrac{1}{2} du = t \, dt$$

which gives

$$\begin{aligned} \int \arctan t \, dt &= t \arctan t - \int \frac{t \, dt}{1+t^2} \\ &= t \arctan t - \tfrac{1}{2} \int \frac{du}{u} \\ &= t \arctan t - \tfrac{1}{2} \log |u| + C \\ &= t \arctan t - \tfrac{1}{2} \log |1 + t^2| + C. \end{aligned}$$

So we have (Where did the square root come from? Where did the absolute value go?)

$$\boxed{\int \arctan t \, dt = t \arctan t - \log \sqrt{1+t^2} + C.}$$

$$\int \cos(\log t)\, dt$$

The key observation here is that, since $\frac{d \log t}{dt} = \frac{1}{t}$ and $\int dt = t$, those two terms will cancel out when we do our integration by parts (take a second to try to see it). You would have figured this out by experimentation eventually, but better if you could see it before you got started. So now let

$$\begin{aligned} u = \cos(\log t) \quad &\Rightarrow \quad du = -\tfrac{1}{t}\sin(\log t)\, dt \\ dv = dt \quad &\Rightarrow \quad v = t \end{aligned}$$

so that

$$\begin{aligned} \int \cos(\log t)\, dt &= uv - \int v\, du \\ &= t\cos(\log t) - \int t\left(-\tfrac{1}{t}\sin(\log t)\right) dt \\ &= t\cos(\log t) + \int \sin(\log t)\, dt \end{aligned}$$

Now do another integration by parts with

$$\begin{aligned} u = \sin(\log t) \quad &\Rightarrow \quad du = \tfrac{1}{t}\cos(\log t)\, dt \\ dv = dt \quad &\Rightarrow \quad v = t \end{aligned}$$

to get

$$\begin{aligned} \int \cos(\log t)\, dt &= t\cos(\log t) + \int \sin(\log t)\, dt \\ &= t\cos(\log t) + t\sin(\log t) - \int \cos(\log t)\, dt. \end{aligned}$$

Rearranging gives

$$2\int \cos(\log t)\, dt = t\cos(\log t) + t\sin(\log t)$$

so that

$$\boxed{\int \cos(\log t)\, dt = \frac{t\cos(\log t) + t\sin(\log t)}{2}}.$$

$$\int t \sin t \cos t \, dt$$

This one wouldn't be bad if we could drop the t term in the front. It would "disappear" (by turning into a 1) if we took it's derivative, so let's make that our u in our integration by parts:

$$u = t \quad \Rightarrow \quad du = dt$$
$$dv = \sin t \cos t \, dt \quad \Rightarrow \quad ???$$

Now it remains to integrate our dv term. We can do this with u-substitution, but we just used u so lets call it w-substitution instead to minimize confusion. The derivative of $\sin t$ is $\cos t$ so let

$$w = \sin t \quad \Rightarrow \quad dw = \cos t \, dt$$

Then we have

$$\begin{aligned} v &= \int dv = \int \sin t \cos t \, dt \\ &= \int w \, dw = \frac{w^2}{2} \\ &= \tfrac{1}{2} \sin^2 t. \end{aligned}$$

Putting it together, our original integration by parts gives

$$\begin{aligned} \int t \sin t \cos t \, dt = uv - \int v \, du \\ &= \frac{t}{2} \sin^2 t - \frac{1}{2} \int \sin t \, dt \\ &= \frac{t}{2} \sin^2 t + \frac{1}{2} \cos t + C. \end{aligned}$$

Our solution is

$$\boxed{\int t \sin t \cos t \, dt = \frac{t \sin^2 t + \cos t}{2} + C.}$$

$$\int \arctan \frac{1}{t}\, dt$$

We don't know the integral of this thing, but we *do* know it's derivative, so to use IBP, it would make sense to let $u = \arctan \frac{1}{t}$. In this case, looking at our dv term first we get

$$dv = dt \quad \Rightarrow \quad v = t$$

Since $u = \arctan \frac{1}{t}$, by chain rule we get

$$du = -\frac{1}{t^2}\left(\frac{1}{1+\frac{1}{t^2}}\right) dt = -\frac{1}{t^2}\left(\frac{t^2}{1+t^2}\right) dt = -\frac{dt}{1+t^2}.$$

Now our IBP gives

$$\begin{aligned}\int \arctan \frac{1}{t}\, dt &= uv - \int v\, du \\ &= t \arctan \frac{1}{t} - \int t\left(\frac{-dt}{1+t^2}\right) \\ &= t \arctan \frac{1}{t} + \int \frac{t\, dt}{1+t^2}\end{aligned}$$

Using u-substitution on this last integral we let

$$u = 1 + t^2 \quad \Rightarrow \quad \frac{du}{2} = t\, dt$$

which gives

$$\begin{aligned}\int \arctan \frac{1}{t}\, dt = t \arctan \frac{1}{t} + \int \frac{t\, dt}{1+t^2} \\ = t \arctan \frac{1}{t} + \frac{1}{2}\int \frac{du}{u} \\ = t \arctan \frac{1}{t} + \frac{1}{2}\log|1+t^2| + C.\end{aligned}$$

Finally we have

$$\boxed{\int \arctan \frac{1}{t}\, dt = t \arctan \frac{1}{t} + \log \sqrt{1+t^2} + C.}$$

$$\int 3^t \sin t \, dt$$

If you've done a few of these, you'll see that this is a "circular" IBP problem where we will end up with a term that is a factor of the original intergral–similar the ones where we get a "2I" in class then divide by 2 to get our final answer. Terms like 3^t work like e^t (*i.e.*, they don't "go away" by integration or differentiation) but with a messy coefficient. So we will use that term as our dv.

$$u = \sin t \quad \Rightarrow \quad du = \cos t \, dt$$
$$dv = 3^t \, dt \quad \Rightarrow \quad v = \tfrac{1}{\log 3} \, 3^t$$

Where $\frac{1}{\log 3}$ is just a number (it's about 0.91, and you should double check that on your calculator). So now we have

$$\int 3^t \sin t \, dt = \tfrac{1}{\log 3} \, 3^t \sin t - \tfrac{1}{\log 3} \int 3^t \cos t \, dt$$

Now we do another IBP on this right-most integral with

$$u = \cos t \quad \Rightarrow \quad -du = \sin t \, dt$$
$$dv = 3^t \, dt \quad \Rightarrow \quad v = \tfrac{1}{\log 3} \, 3^t$$

which gives

$$\begin{aligned}\int 3^t \sin t \, dt &= \tfrac{1}{\log 3} \, 3^t \sin t - \tfrac{1}{\log 3} \int 3^t \cos t \, dt \\ &= \tfrac{1}{\log 3} \, 3^t \sin t - \tfrac{1}{(\log 3)^2} \, 3^t \cos t - \tfrac{1}{(\log 3)^2} \int 3^t \sin t \, dt\end{aligned}$$

Rearranging terms gives

$$\left(1 + \tfrac{1}{(\log 3)^2}\right) \int 3^t \sin t \, dt = \tfrac{1}{\log 3} \, 3^t \sin t - \tfrac{1}{(\log 3)^2} \, 3^t \cos t$$

Finally,

$$\boxed{\int 3^t \sin t \, dt = \tfrac{\log 3}{1 + (\log 3)^2} \left(\sin t - \tfrac{1}{\log 3} \cos t\right) 3^t \approx 0.5 \, (\sin t - 0.91 \cos t) \, 3^t.}$$

Whew!