$$\int \frac{dt}{\cos^2 t \sqrt{1 + \tan t}}$$

First, notice that $\frac{1}{\cos t} = \sec t$ so that

$$\int \frac{dt}{\cos^2 t \sqrt{1 + \tan t}} = \int \frac{\sec^2 t \, dt}{\sqrt{1 + \tan t}}$$

Then, notice that $\frac{d}{dx} \tan x = \sec^2 x$ and realize that you should choose

$$u = 1 + \tan t \qquad \Rightarrow \qquad du = \sec^2 t \, dt.$$

Now substituting u and du into the integral, we get

$$\begin{aligned} \int \frac{dt}{\cos^2 t \sqrt{1 + \tan t}} &= \int \frac{\sec^2 t \, dt}{\sqrt{1 + \tan t}} \\ &= \int \frac{du}{\sqrt{u}} \\ &= \int u^{-1/2} \, du \\ &= \frac{u^{1/2}}{1/2} + C \\ &= 2u^{1/2} + C \\ &= 2\sqrt{1 + \tan t} + C. \end{aligned} \tag{1}$$

$$I = \int t\,(3t+8)\ dt$$

This was a "trick" question, I guess. You could use substitution to try to solve this, but there is a much easier way:

$$\begin{aligned}
\int t\,(3t+8)\ dt &= \int 3t^2 + 8t\,dt \\
&= \int 3t^2\,dt + \int 8t\,dt \\
&= 3\int t^2\,dt + 8\int t\,dt \\
&= 3\frac{t^3}{3} + 8\frac{t^2}{2} + C \\
&= t^3 + 4t^2 + C.
\end{aligned} \tag{2}$$

$$\int \frac{t + \log t}{t}\, dt$$

First, break up the numerator and use the fact that $\int a(x) + b(x) dx = \int a(x) dx + \int b(x) dx$ to get

$$\begin{aligned}\int \frac{t + \log t}{t}\, dt &= \int 1 + \frac{\log t}{t}\, dt \\ &= \int dt + \int \frac{\log t}{t}\, dt.\end{aligned} \tag{3}$$

Then choosing

$$u = \log t \qquad \Rightarrow \qquad du = \frac{dt}{t}$$

we see that

$$\begin{aligned}\int \frac{t + \log t}{t}\, dt &= \int 1 + \frac{\log t}{t}\, dt \\ &= \int dt + \int \frac{\log t}{t}\, dt \\ &= t + \int u\, du \\ &= t + \tfrac{1}{2} u^2 + C \\ &= t + \tfrac{1}{2} (\log t)^2 + C.\end{aligned} \tag{4}$$

$$\int \sec^2 t \tan^3 t \, dt$$

First, as in the first problem above, notice that $\frac{d}{dx} \tan x = \sec^2 x$. Now choose

$$u = \tan t \qquad \Rightarrow \qquad du = \sec^2 t \, dt.$$

Substituting now gives

$$\begin{aligned}
\int \sec^2 t \tan^3 t \, dt &= \int u^3 \, du \\
&= \frac{u^4}{4} + C \qquad (5) \\
&= \tfrac{1}{4} \tan^4 t + C.
\end{aligned}$$

$$\int \sin t \sin(\cos t)\, dt$$

First, don't get sucked into the scary-looking nested trig function. Instead, realize that if we wanted du to be $\sin t\, dt$ then we would need u to be $-\cos t$. Let's try just $\cos t$ to get

$$u = \cos t \quad \Rightarrow \quad du = -\sin t\, dt \quad \Rightarrow -du = (-1)du = \sin t\, dt.$$

Substituting gives

$$\begin{aligned}
\int \sin t \sin(\cos t)\, dt &= \int \sin(\cos t) \sin t\, dt \\
&= \int \sin(u)\,(-1)du \\
&= \int -\sin(u)\, du \\
&= \cos u + C \\
&= \cos(\cos t) + C.
\end{aligned} \tag{6}$$