Warm-up

Consider the functions defined on the set $\{1, 2, 3, 4, 5\}$ by the rules

$$f(x) = x^5 - 15x^4 + 85x^3 - 224x^2 + 268x - 111 \qquad \text{and} \qquad g(x) = x^2 - 6x + 9.$$

Write down the two sets of ordered pairs

$$\{(x, f(x)) \mid x \in \{1, 2, 3, 4, 5\}\} \qquad \text{and} \qquad \{(x, g(x)) \mid x \in \{1, 2, 3, 4, 5\}\}.$$

Are they the same function or different functions?

Relations

Looking at the two functions defined in the warm-up,

$$f(x) = x^5 - 15x^4 + 85x^3 - 224x^2 + 268x - 111 \qquad \text{and} \qquad g(x) = x^2 - 6x + 9,$$

it makes sense to think that they define different rules because f and g are different. However, the two sets of ordered pairs are the same, and so these two functions, restricted to $\{1, 2, 3, 4, 5\}$ define the same rule! Today we will be talking about *relations*. A *relation* from a set X to a set Y is a set of ordered pairs (x, y) with $x \in X$ and $y \in Y$. Viewed this way, a function $S \longrightarrow T$ is simply a relation, R, from S to T where every element in S appears as the first element of exactly one ordered pair in R.

However, there are many relations that do not define functions, so that all functions are relations, but not all relations are functions. Thus relations are more general than functions. Here are some examples.

Example 1. Define a relation, called the *neighbor relation*, on the integers where i is a neighbor of j is the absolute value of their difference is 1. The relation has infinite cardinality, but some elements include

$$(-1, 0),\ (0, -1),\ (0, 1),\ (1, 0),\ (1, 2),\ (2, 1),\ (2, 3)$$

Example 2. When we derived the formula for binomial coefficients, $\binom{n}{k}$, we saw that there were $k!$ different permutations of a k-element subset of an n-element set S. Any of these permutations is equivalent for the purposes of specifying the subset. We can define two permutations to be *set-equivalent* if they are permutations of the same subset of S. This is a relation on the set of k-element permutations of X.

Example 3. If S and T are sets, then define a relation where the ordered pair (S, T) is an element if and only if S is a subset of T. In this case, if S and T are related, then we usually write $S \subseteq T$.

Example 4. If $x, y \in \mathbb{R}$, then define a relation where the ordered pair (x, y) is an element if and only if x is less than y. In this case, if x and y are related, then we usually write $x < y$.

These last examples bring up a good point. We usually write $x < y$, not $(x, y) \in <$, and $S \subseteq T$, not $(S, T) \in \subseteq$. Likewise, given a relation R, we commonly write aRb if $(a, b) \in R$.

We could define any number of properties of relations, but there are three (well, four) that are particularly useful, namely *reflexivity*, *(anti)symmetry*, and *transitivity*.

Let R be a relation defined on a set X.

a. R is *reflexive* if for all $x \in X$ we have $(x, x) \in R$, *i.e.*, xRx.

b. R is *symmetric* if $(x_1, x_2) \in R$ if and only if $(x_2, x_1) \in R$, *i.e.*, x_1Rx_2 iff x_2Rx_1. R is *antisymmetric* if $(x_1, x_2) \in R$ and $(x_2, x_1) \in R$ only if $x_1 = x_2$, *i.e.*, x_1Rx_2 and x_2Rx_1 only if $x_1 = x_2$.

c. R is *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$, *i.e.*, aRb and $bRc \Rightarrow aRc$.

Example 5. Consider the relations from Examples 1-4. Which are reflexive? Symmetric? Antisymmetric? Transitive?

Example 6. Which properties does the *less than or equal to* relation have?

Example 7. Analyze the relations "is a sibling of", "is the mother of", and "is a descendant of".

Example 8. Write down the set of all 3-element permutations of $\{1, 2, 3, 4\}$ that are set-equivalent to

a. 123 **b.** 124 **c.** 134 **d.** 234

Do any of these sets have any permutations in common?

Writing out the set-equivalences of each permutation gives

$$\begin{array}{ll} 123: & \{123,\ 132,\ 213,\ 231,\ 312,\ 321\} \\ 124: & \{124,\ 142,\ 214,\ 241,\ 412,\ 421\} \\ 134: & \{134,\ 143,\ 314,\ 341,\ 413,\ 431\} \\ 234: & \{234,\ 243,\ 324,\ 342,\ 423,\ 432\} \end{array}$$

A simple examination shows that these sets are mutually disjoint. A careful check will also show that these sets contain every possible 3-element permutation of $\{1, 2, 3, 4\}$.

Example 9. Write down the set of all neighbors in $\mathbb{Z}$ of

a. 0 **b.** 1 **c.** 2 **d.** 3

Do any of these sets have any elements in common?

Example 10. Write down all subsets of each of the following sets.

a. $\{1, 2, 3\}$ **b.** $\{1, 2\}$ **c.** $\{1, 3\}$

Do any of these sets of subsets have any elements in common?

Example 11. Write down the set of all positive integers less than

a. 2 **b.** 3 **c.** 4

Do any of these sets have any elements in common?

Note that only Example 8 gives us a scenario that partitions the space (set) of all things (3-element permutations of $\{1,2,3,4\}$) into mutually disjoint sets. Indeed, we say that a set P containing subsets of a set S is a *partition* if its elements are mutually disjoint and their union is S. In this case we call these mutually disjoint sets *equivalence classes*, or just *classes*, and we say that the relation is an *equivalence relation*. Note also that the relation in Example 8, the set-equivalence relation, was reflective, symmetric, and transitive. Any relation that has all three properties is an equivalence relation! It is easy to prove that these two ideas are the same, but we skip the proof here. Stated as a proposition, we have

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Proposition 1. (Equivalence Relations from Properties) Let R be a relation on S. The following are equivalent:

a. R partitions S into mutually disjoint subsets (R is an equivalence relation). That is, for each pair of elements x and y of S, the sets

$$S_x = \{z \in S \mid (x,z) \in R\} \qquad \text{and} \qquad S_y = \{z \in S \mid (y,z) \in R\}$$

are either equal or disjoint. These sets are called *equivalence classes*.

b. R is reflexive, symmetric, and transitive.

We can even go a little further. Suppose we have a set S and we partition it. Then we can define a relation R on this partition to get an equivalence relation.

Proposition 2. (Equivalence Relations from Partitions) Let P be a partition of a set S. Suppose we define a relation R by

$$R = \{(x,y) \mid x \text{ and } y \text{ are in the same element of } P\}.$$

Then R is an equivalence relation whose equivalence classes are the elements of P.

Let's consider three relations, $<$, $\leq$, and $\subseteq$. We have that $<$ is not relfexive since, *e.g.*, $3 \not< 3$, is neither symmetric nor antisymmetric, but is transitive. The relations $\leq$ and $\subseteq$, on the other hand, are reflexive, antisymmetric, and transitive. Any such relation is said to be a *partial order*, and a set that admits a partial order is called a *partially ordered set* or just *poset*.

We use the word "partial" because there may be elements that cannot be ordered. For two real numbers x and y, either $x \leq y$ or $y \leq x$ (or both, if $x = y$); however, if $S = \{1, 2\}$ and $T = \{1, 3\}$, then neither $S \subseteq T$ nor $T \subseteq S$, even though $\subseteq$ is a partial order.

Now suppose that R is a partial order on S and $a, b, \in S$. if either aRb or bRa, then we say that a and b are *comparable*. If neither is true, then they are *incomparable*. If every such pair of elements in S is comparable, then we say that R is a *total order* on S and that S is a *totally ordered set*. Thus $\leq$ is a total order and $\subseteq$ is not.

Sometimes, a totally ordered set has a smallest element. The set

$$\{n \in \mathbb{Z} \mid 0 < n\}$$

is totally ordered by $\leq$ and has a smallest element, namely 0. However, the set

$$\{x \in \mathbb{R} \mid 0 < x\}$$

is likewise totally ordered by $\leq$, but it has no smallest element. A *well-ordered set* is a totally ordered set with the property that every non-empty subset has a smallest element. This is an important property with many far-reaching implications.