Observations on harmonic progressions*

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Translation notes.

I have generally tried to maintain the tone of the original, while translating the article into an idiom that can be understood by contemporary English-speaking mathematicians. To this end, the term *series* has been translated as "sequence" when appropriate, and the *etc.* commonly used by Euler to denote continuation has been replaced by the contemporary "...". I have also added emphasis to new definitions, which is common in contemporary textbooks and articles. Otherwise, Euler's notation has been kept, with the exception that some formulas are set in display mode to increase their readability. Any mathematical errors are likely to be mine.

§. 1.

By the name of *harmonic progressions* are understood all sequences of fractions in which the numerators are equal among themselves and the denominators form a true arithmetic progression. Thus the general form is given by $\frac{c}{a}, \frac{c}{a+b}, \frac{c}{a+2b}, \frac{c}{a+3b}, \ldots$. Indeed, each three consecutive terms, $\frac{c}{a+b}$, $\frac{c}{a+2b}$, $\frac{c}{a+3b}$, have the property that the differences of the outer terms and the

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middle one are proportional to the to the outer terms themselves. Namely, we have

$$\frac{c}{a+b} - \frac{c}{a+2b} : \frac{c}{a+2b} - \frac{c}{a+3b} = \frac{c}{a+b} : \frac{c}{a+3b}.$$

Further, since this is the property of *harmonic proportion*, these sequences of fractions are called harmonic progressions. They may also be called *reciprocals of the first order*, since one is subtracted from the index n in the general term $\frac{c}{a+(n-1)b}$.

§. 2.

Although in these sequences the terms continually decrease, nevertheless the sum of the terms of such a sequence continued to infinity is always infinite. To demonstrate this, there is no need for a method of summing these series; but the truth will easily be made clear from the following principle. A series which, when continued to infinity, has a finite sum, even if it is continued twice as long, will accept no increase, and that which is added after infinity will actually be infinitely small. For if this were not the case, the sum of the series, even if continued to infinity, would not be determined and therefore not finite. From this it follows that if that which arises from continuation beyond the infinite term is of finite magnitude, the sum of the series must necessarily be infinite. From this principle, we can judge whether the sum of any given series is infinite or finite.[1]

[1]This passage is more than a little cumbersome, thanks mostly to a lack of the formal idea of limits, but Euler is giving a generalization of the standard undergraduate proof of the divergence of the harmonic series; namely, that you can always group some finite number of successive terms together to get a sum greater than or equal to some finite magnitude. In proofs for the divergence of the harmonic series, it is customary to show that such groupings result in sums greater than or equal to $\frac{1}{2}$. Here, the only difference is the a in the denominator. To get rid of it, Euler will start at the "infinite" term, so that this a is completely dominated and can be disregarded, leading to a convenient cancellation, as we will see in the next subsection. In contemporary usage, we would simply take "sufficiently large values" to neutralize this a.

§. 3.

Indeed, let the series $\frac{c}{a}$, $\frac{c}{a+b}$, $\frac{c}{a+2b}$, ... be continued to infinity, and consider the term $\frac{c}{a+(i-1)b}$, where i denotes an infinite number, which is the index of this term.[2] Now let this series be further continued from the term $\frac{c}{a+ib}$ to the term $\frac{c}{a+(ni-1)b}$ whose exponent is ni. The number of these additionally appended terms is therefore $(n-1)i$; their sum, however, will be less than $\frac{(n-1)ic}{a+ib}$ but greater than $\frac{(n-1)ic}{a+(ni-1)b}$. But since i is infinitely large, a will vanish in both denominators. Hence the sum will be greater than $\frac{(n-1)c}{nb}$ but less than $\frac{(n-1)c}{b}$. From this it is clear that this sum is finite, and consequently the sum of the given series $\frac{c}{a}$, $\frac{c}{a+b}$, ..., continued to infinity, is infinitely large.

§. 4.

Further, tighter limits on the sum of the terms of the seqeunce from i to ni are derived by the properties of harmonic proportions[3]. Namely, every harmonic proportion is such that the middle term is less than one-third of the sum of all of the terms. For this reason, the middle term between $\frac{c}{a+ib}$ and $\frac{c}{a+(ni-1)b}$, which is $\frac{c}{a+\frac{ni+i-1}{2}b}$, multiplied by the number of terms $(n-1)i$, or $\frac{(n-1)ic}{a+\frac{ni+i-1}{2}b}$, will be less than the sum of the terms. Therefore, the sum of the terms will be greater than $\frac{2(n-1)c}{(n+1)b}$ because i is infinite. Furthermore, the arithmetic mean between the extreme terms is greater than one-third of the sum of the terms. From this, it follows that in a harmonic series the sum of the terms will be less than $(n-1)i$ times the arithmetic mean of the extreme terms multiplied, which is $\frac{(2a+(ni+i-1)b)c}{2(a+ib)(a+(ni-1)b)}$ or $\frac{(n+1)c}{2nib}$. Therefore, the sum will be less than $\frac{(n^2-1)c}{2nb}$, so that these two limits are $\frac{2(n-1)c}{(n+1)b}$ and $\frac{(n^2-1)c}{2nb}$,

[2] Euler starts at the "infinite" term, but really he just means that for sufficiently large indices i, the number a will be dominated in such a way that it can be disregarded. He will use this in a few sentences to gather bounds on the sum of each consecutive $(n-1)i$ terms of the sequence.

[3] Of $(n-1)i$ consecutive terms, as in subsection 1 where he examines three such consecutive terms.

and thus the sum is approximately $\frac{(n-1)c}{b\sqrt{n}}$, which is the mean between the limits.