Lecture 07

Suppose that our sample space contains 2 partitions $\{A_0, A_1\}$

e.g. A_0: fair coin
A_1: 2-headed coin
B: event 8 heads in 8 flips

Optimal Decision Rules

Maximum Likelihood Estimation (MLE)

$$P(B \mid A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B \mid A_1)$$

data likelihood

↳ If $P(B \mid A_0) > P(B \mid A_1)$, then we "choose" /decide A_0.

Maximum A Posteriori (MAP)

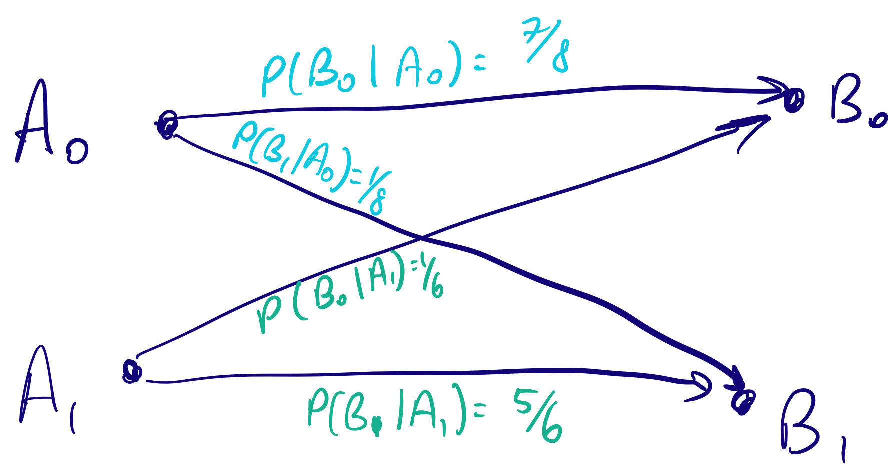
$$P(A_0 | B) \underset{A_1}{\overset{A_0}{\gtrless}} P(A_1 | B)$$

$$\iff \frac{P(B|A_0)\,P(A_0)}{P(B)} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{P(B|A_1)\,P(A_1)}{P(B)}$$

Bayes

<u>Example</u>: Binary Communication System.

- Transmitter sends A_0, A_1
- Receiver receives B_0, B_1



Goal: choose transmitter that maximizes some probability: "Probability of transmitter A_j given that we received B_i":

MAP!

$$P(A_j \mid B_i), \ \forall i,j = \{0,1\}$$

e.g. if we receive B_0 & observe

$P(A_0 \mid B_0) > P(A_1 \mid B_0)$, we

will decide A_0.

But! We were not given $P(A_j \mid B_i)$!

Bayes:
$$P(A_j \mid B_i) = \frac{P(A_j \cap B_i)}{P(B_i)}$$
$$= \frac{P(B_i \mid A_j)\, P(A_j)}{P(B_i)}$$
$$= \frac{P(B_i \mid A_j)\, P(A_j)}{\sum_k P(B_i \mid A_k)\, P(A_k)}$$

Scenario 1:

$P(A_0) = 2/5$, $P(A_1) = 3/5$:

$$P(A_0 \mid B_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(A_1 \mid B_0)$$

$$\frac{P(B_0 \mid A_0)\, P(A_0)}{P(B_0)} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{P(B_0 \mid A_1)\, P(A_1)}{P(B_0)}$$

$$\frac{7/8 \cdot 2/5}{9/20} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{1/6 \cdot 3/5}{9/20}$$

$$\frac{7}{9} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{2}{9} \Rightarrow \text{Decide } A_0!$$

So, when B_0 is received, MAP decision rule is to decide A_0

Similarly, when B_1 is received, MAP rule is to decide A_1.

If the decision rule is to always decide A_1, then we will have an error with some probability:

$$P(E) = P(E|B_0)P(B_0) + P(E|B_1)P(B_1)$$

Law of total prob.

$$P(B_0) = P(B_0|A_0)P(A_0) + P(B_0|A_1)P(A_1)$$

$$= \frac{7}{8} \cdot \frac{2}{5} + \frac{1}{6} \cdot \frac{3}{5} = \frac{9}{20}$$

$$P(B_1) = P(B_1|A_0)P(A_0) + P(B_1|A_1)P(A_1)$$

$$= \frac{1}{8} \cdot \frac{2}{5} + \frac{5}{6} \cdot \frac{3}{5} = 1 - P(B_0) = \frac{11}{20}$$

For this decision rule, always H_1 we have:

$$P(E) = \frac{7}{9} \cdot \frac{9}{20} + \left(1 - \frac{10}{11}\right) \cdot \frac{11}{20} = 0.4$$

vs.

MAP Decision rule:

$$\frac{2}{9} \cdot \frac{9}{20} + \left(1 - \frac{10}{11}\right) \cdot \frac{11}{20} = 0.15$$