Lecture 8

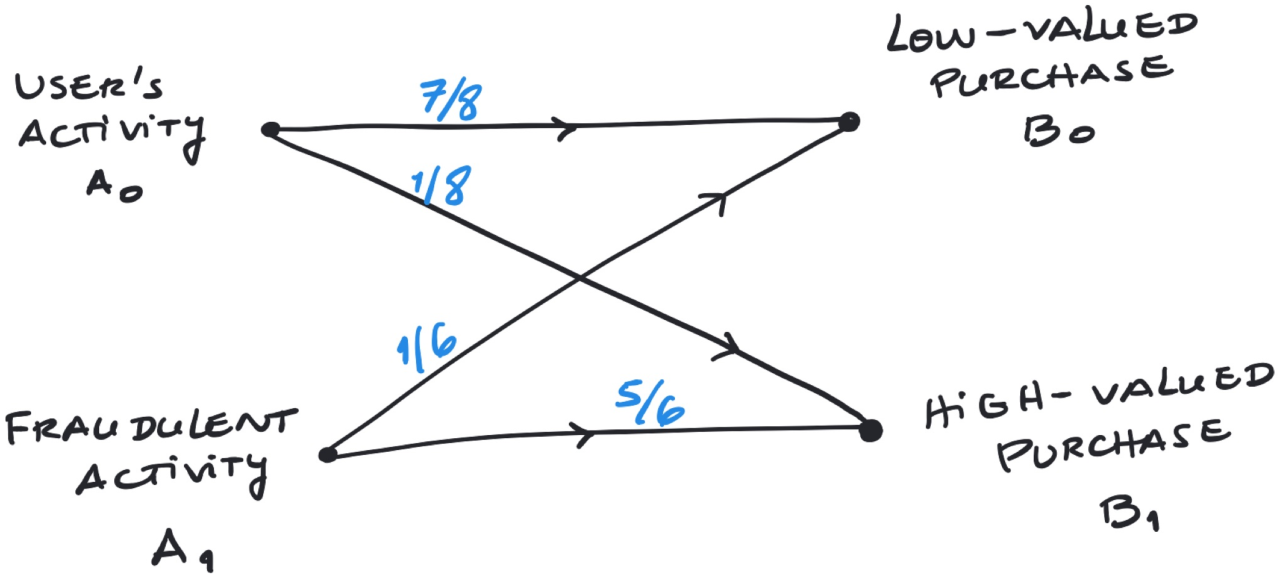
- Decision Rules
- Introduction to Random Variables
- Discrete Random Variables

Recap: Decision Rules

Let's consider the example of credit card fraud detection. Consider the following likelihoods:

In [1]:
```
from IPython.display import Image
Image('figures/FraudDetection.png',width=700)
```

Out[1]:



Consider the case that the user is a new credit card owner, so the bank will assume a prior $P(A_0) = \frac{9}{10}$.

The optimal decision rule is to decide A_0 when B_0 is received, and decide A_1 when B_1 is received.

Let's compute:

1. the MLE and MAP decision rules.

2. the probability of error.

Optimal decision rule: decide A_0 if B_0 and decide A_1 if B_1.

We are given:

- Data likelihoods:
  - $P(B_0|A_0) = \frac{7}{8}$

- $P(B_0|A_1) = \frac{1}{6}$
- $P(B_1|A_0) = \frac{1}{8}$
- $P(B_1|A_1) = \frac{5}{6}$

- Prior probabilities:
  - $P(A_0) = \frac{9}{10}$
  - $P(A_1) = \frac{1}{10}$

We can compute:

- $P(B_0) = P(B_0|A_0)P(A_0) + P(B_0|A_1)P(A|1) = \frac{7}{8} \times \frac{9}{10} + \frac{1}{6} \times \frac{1}{10} \approx 0.8042$
- $P(B_1) = P(B_1|A_0)P(A_0) + P(B_1|A_1)P(A|1) = \frac{1}{8} \times \frac{9}{10} + \frac{5}{6} \times \frac{1}{10} = 1 - P(B_0) \approx 0.19$

MLE Decision Rule

- When receiving B_0:

$$P(B_0|A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B_0|A_1)$$
$$\frac{7}{8} > \frac{1}{6} \Rightarrow \text{Decide } A_0$$

- When receiving B_1:

$$P(B_1|A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B_1|A_1)$$
$$\frac{1}{8} < \frac{5}{6} \Rightarrow \text{Decide } A_1$$

- Error for MLE:

$$\begin{aligned} P_{MLE}(E) &= P(E|B_0)P(B_0) + P(E|B_1)P(B_1) \\ &= \frac{1}{8} \times 0.8042 + \frac{1}{6} \times 0.1958 \\ &\approx 0.1332 \end{aligned}$$

MAP Decision Rule

- When receiving B_0:

$$P(A_0|B_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(A_1|B_0)$$
$$\frac{P(B_0|A_0)P(A_0)}{P(B_0)} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{P(B_0|A_1)P(A_1)}{P(B_0)}$$
$$\frac{\frac{7}{8}\times\frac{9}{10}}{0.8042} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{\frac{1}{6}\times\frac{1}{10}}{0.8042}$$
$$0.979 > 0.021 \Rightarrow \text{Decide } A_0$$

- When receiving B_1:

$$P(A_0|B_1) \underset{A_1}{\overset{A_0}{\gtrless}} P(A_1|B_1)$$
$$\frac{P(B_1|A_0)P(A_0)}{P(B_1)} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{P(B_1|A_1)P(A_1)}{P(B_1)}$$
$$\frac{\frac{1}{8}\times\frac{9}{10}}{0.1958} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{\frac{5}{6}\times\frac{9}{10}}{0.1958}$$
$$0.575 > 0.425 \Rightarrow \text{Decide } A_0$$

- Error for MAP:

$$\begin{aligned} P_{MAP}(E) &= P(E|B_0)P(B_0) + P(E|B_1)P(B_1) \\ &= 0.021 \times 0.8042 + 0.575 \times 0.1958 \\ &\approx 0.1295 \end{aligned}$$

Bayes' Theorem

The prior probability of fraudulent activity, p, can change over time. For what set of values for p will the MAP decision always decide A_0 when B_0 is received, and always decide A_1 when B_1 is received?

Let $p = P(A_1)$.

- MAP decides A_0 when B_0 is received if

$$P(A_0|B_0) > P(A_1|B_0)$$
$$\frac{P(B_0|A_0)P(A_0)}{P(B_0)} > \frac{P(B_0|A_1)P(A_1)}{P(B_0)}$$
$$P(B_0|A_0)P(A_0) > P(B_0|A_1)P(A_1)$$
$$\frac{7}{8}\times(1-p) > \frac{1}{6}\times p$$
$$p < 0.84$$

- MAP decides A_1 when B_1 is received if

$$P(A_0|B_1) < P(A_1|B_1)$$
$$\frac{P(B_1|A_0)P(A_0)}{P(B_1)} > \frac{P(B_1|A_1)P(A_1)}{P(B_1)}$$
$$P(B_1|A_0)P(A_0) > P(B_1|A_1)P(A_1)$$
$$\frac{1}{8} \times (1-p) > \frac{5}{6} \times p$$
$$p > 0.114$$

- Therefore,

$$0.114 < P(A_1) < 0.84$$

In [ ]:

Introduction to Random Variables

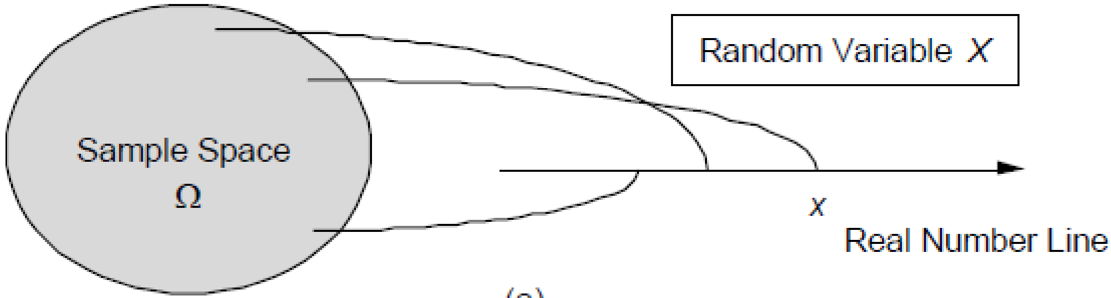
- **What is a random variable?**

Random Variable * A **random variable** (or **RV**) is a *numeric* occurrence that is random. *Formal definition:* Given an experiment and the corresponding set of possible outcomes (the sample space), a **random variable** associates a particular *number* with each outcome (see figure below). We refer to this number as the **numerical value** or simply the **value** of the RV. Mathematically, a **random variable is a real-valued function of the experimental outcome**.

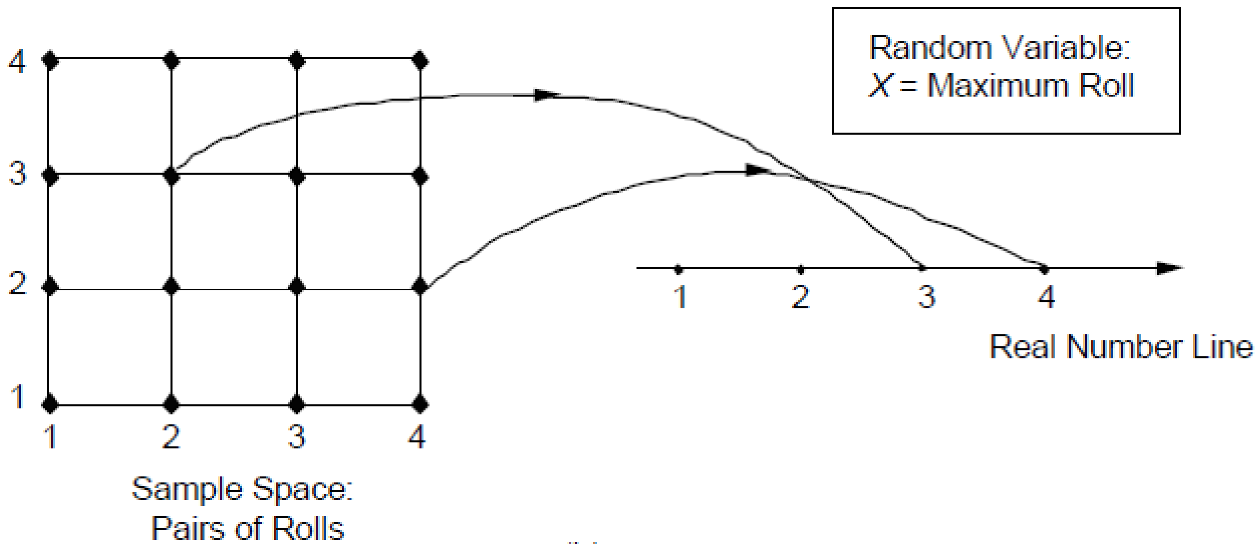
In [2]:

```
from IPython.display import Image
Image('figures/RV.png',width=800)
```

Out[2]:



(a)



(b)

- As mentioned above, we define a random variable on a probability space $(\Omega, \mathcal{F}, P)$ as a **function** from the sample space Ω to the real line $\mathbb{R}$.

Example 1 Let's create a binary RV from tossing a fair coin.

The sample space is $S = \{H, T\}$, we can represent H with value 1 in the real line and T with value 0 in the real line. Then we can define the random variable X that is going to represent the outcomes of the experiment "tossing a fair coin":

$$X(x) = \begin{cases} 1, & x = H \\ 0, & x = T \end{cases}$$

Example 2 Let's create a binary RV from tossing a fair coin twice.

The sample space is $S = \{HH, HT, TH, TT\}$. We want the RV Y to be binary so the only real value it can take are 0 or 1. Let's consider the case where we **map** the events $\{HH, HT, TH\}$ to 1 and the event $\{TT\}$ to 0. Then we can define the random variable Y as:

$$Y(y) = \begin{cases} 1, & y = \{HH, HT, TH\} \\ 0, & y = \{TT\} \end{cases}$$

Example 3 Let's create another RV from tossing a fair coin twice.

The sample space is $S = \{HH, HT, TH, TT\}$. Here we are not told Z can only be binary, then we can **map** each possible event to a different real-value number. So, we can define the random variable Z as:

$$Z(z) = \begin{cases} 1, & z = \{HH\} \\ 2, & z = \{HT\} \\ 3, & z = \{TH\} \\ 4, & z = \{TT\} \end{cases}$$

A **function of a random variable** defines another random variable. This is the case of RVs Y and Z above.

- We can associate with each RV certain "averages" of interest, suh as the **mean** and the **variance**.

- A random variable can be **conditioned** on an event or on another random variable.
  - For example, consider the random variable X of rolling a die. We can build a second RV that takes the outcome of X as the number of times we will flip a coin.
  - Another example, consider the RVs X and Z defined above. Are X and Z **independent**?
- There is a notion of **independence** of a random variable from an event or from another random variable.

Discrete Random Variables

Discrete Random Variable A random variable is called **discrete** if its range (the set of values that it can take) is either **finite** or **countably infinite**. * A discrete RV has an associate **probability mass function (PMF)**, which gives the probability of each numerical value that the random variable can take. * A **function of a discrete random variable** defines another discrete random variable, whose PMF can be obtained from the PMF of the original random variable.

Probability Mass Functions (PMFs)

The most important way to characterize an RV is through the probabilities of the values that it can take. For a discrete RV X, these are captured by a **probability mass function (PMF)** of X, denoted by p_X.

- In particular, if x is any real number, the **probability mass** of x, denoted $p_X(x)$, is the probability of the event $\{X = x\}$ consisting of all outcomes that give rise to a value X equal to x:

$$p_X(x) = P(\{X = x\})$$

- Note that

$$\sum_x p_X(x) = 1$$

as x ranges over al possible values of X, the events $\{X = x\}$ are **disjoint** and form a partition of the sample space.

- By a similar argument, for any set S of possible values of X we have

$$P(X \in S) = \sum_{x \in S} p_X(x)$$

Calculation of the PMF of a Random Variable X

For each possible value x of X:

1. Collect all the possible outcomes that give rise to the event $\{X = x\}$.
2. Add their probabilities to obtain $p_X(x)$.

Example 4 Let's create the PMF for the binary RV from tossing a fair coin.

$$p_X(x) = P(X = x) = \begin{cases} \frac{1}{2}, & x = 1 \\ \frac{1}{2}, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

Example 5 Let's create the PMF of the binary RV from tossing a fair coin twice.

$$p_Y(y) = P(Y = y) = \begin{cases} \frac{3}{4}, & y = 1 \\ \frac{1}{4}, & y = 0 \\ 0, & \text{o.w.} \end{cases}$$

Example 6 Let's create the PMF for the RV from tossing a fair coin twice.

$$p_Z(z) = P(Z = z) = \begin{cases} \frac{1}{4}, & z = 0, 1, 2, 3 \\ 0, & \text{o.w.} \end{cases}$$

Example 7 Consider the experiment "roll a fair 6-sided die". Let the RV X be the real-value function of the outcomes of this experiment, respectively $X = \{1, 2, 3, 4, 5, 6\}$ or $X \equiv$ # on top face. Let's create the PMF of RV X.

$$p_X(x) = P(X = x) = \begin{cases} \frac{1}{6}, & x = 1, 2, \ldots, 6 \\ 0, & \text{o.w.} \end{cases}$$

Example 8 Consider the experiment "flip a fair coin until heads occurs". Let the RV X be the # on flips. Let's create the PMF of RV X.

$$p_X(x) = \begin{cases} \left(\frac{1}{2}\right)^x, & x = 1, 2, \ldots \\ 0, & \text{o.w.} \end{cases}$$

Cumulative Distribution Function (CDF)

Cumulative Distribution Function If $(\Omega, \mathcal{F}, P)$ is a probability space with X a real discrete RV on Ω, the **Cumulative Distribution Function (CDF)** is denoted as $F_X(x)$ and provides the probability $P(X \leq x)$. In particular, for every x we have

$$F_X(x) = P(X \leq x) = \sum_{k \leq x} p_X(k)$$

Loosely speaking, the CDF $F_X(x)$ "accumulates" probability "up to" the value x.

The CDF $F_X(x)$ is a **probability measure**.

- Thus $F_X(x)$ inherits all the properties of a probability measure (axioms and corollaties still apply).
- The cumulative distribution function is also sometimes called the *probability distribution function (PDF)*, but I will avoid this terminology to avoid confusion with another function we will use called probability density function (pdf).

In [10]:

```
import random
import numpy as np
import numpy.random as npr
import matplotlib.pyplot as plt
%matplotlib inline
plt.style.use('bmh')
```

Arbitrary Discrete RVs in Python

The module `stats` from the library **SciPy** (pronounced "Sigh Pie") contains a large number of probability distributions as well as a growing library of statistical functions.

- `scipy` 's API: https://docs.scipy.org/doc/scipy/reference/stats.html
- If you install Python 3+ through Anaconda then you already have installed the library `scipy` .

In [11]:

```
import scipy.stats as stats
```

Example 9 Let's implement the RV in example 5:

In [12]:

```
# Experiment flipping a fair coin twice
# Binary RV Y

vals = [0,1]
probs = [1/4, 3/4]
```

In [13]:
```
Y = stats.rv_discrete(values=(vals, probs))

Y
```

Out[13]: <scipy.stats._distn_infrastructure.rv_sample at 0x7fab22174df0>

Now, we can compute and plot the **PMF** by calling different methods over the random variable Y:

In [14]:
```
?stats.rv_discrete
```

In [15]:
```
Y.pmf(vals)
```

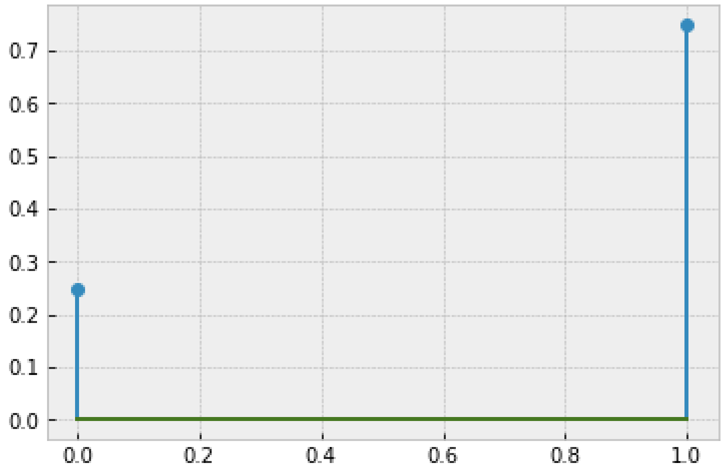
Out[15]: array([0.25, 0.75])

What is the PMF at point $x = 2$, i.e., $p_Y(2)$?

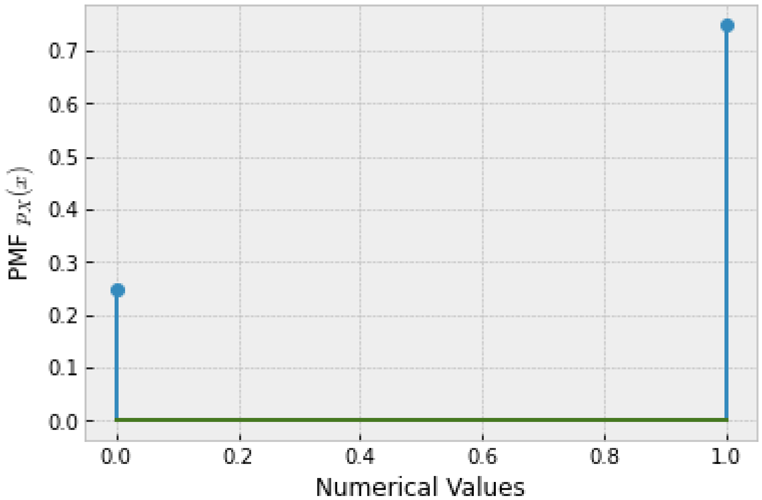
In [16]:
```
Y.pmf(2)
```

Out[16]: 0.0

In [17]:
```
plt.stem(vals, Y.pmf(vals));
```



In [18]:
```
plt.stem(vals, Y.pmf(vals))
plt.xlabel('Numerical Values')
plt.ylabel('PMF $p_X(x)$');
```



We can also *sample* random numbers from the random variable Y.

- You can think of the RV Y as a function that will reproduce certain numerical values over a probabilistic model.

In [19]:
```
sample = Y.rvs(size=20)

sample
```

Out[19]: `array([1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0])`

- What is the probability of outcome 1?

In [20]:
```
np.sum(sample==1)/len(sample)
```

Out[20]: `0.85`

- When we sample from a probabilistic model, the relative frequency of events should approach the true probabilistic model as we sample more data:

In [21]:
```
N=100_000

np.sum(Y.rvs(size=N)==1)/N
```

Out[21]: `0.74817`

Example 10 **Let's implement the RV in example 6:**

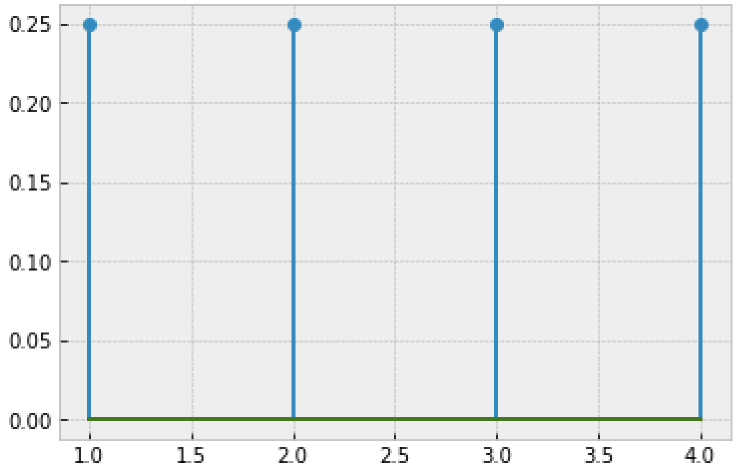
In [22]:
```
# tossing a fair coin twice
# Z not binary

vals2 = [1,2,3,4]
probs2 = [1/4]*4
```

In [23]:

```
Z = stats.rv_discrete(values=(vals2, probs2))
```

In [24]:

```
plt.stem(vals2, Z.pmf(vals2));
```



In [25]:

```
sample = Z.rvs(size=20)

sample
```

Out[25]: `array([2, 4, 3, 2, 1, 2, 3, 3, 4, 2, 4, 4, 2, 3, 1, 3, 3, 2, 2, 3])`

In [26]:

```
np.sum(sample==1)/len(sample)
```

Out[26]: `0.1`

In [27]:

```
N = 100_000

np.sum(Z.rvs(size=N)==1)/N
```

Out[27]: `0.25171`

Plotting CDF

Example 11 Let's compute and plot the CDF of RV Y:

Let's first define some range of values y:

In [28]:

```
y = np.linspace(-1, 2, 100)
y
```

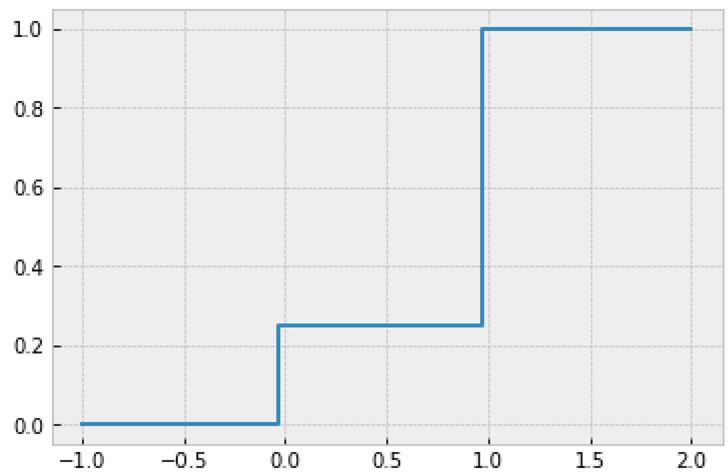
Out[28]:

```
array([-1.        , -0.96969697, -0.93939394, -0.90909091, -0.87878788,
       -0.84848485, -0.81818182, -0.78787879, -0.75757576, -0.72727273,
       -0.6969697 , -0.66666667, -0.63636364, -0.60606061, -0.57575758,
       -0.54545455, -0.51515152, -0.48484848, -0.45454545, -0.42424242,
       -0.39393939, -0.36363636, -0.33333333, -0.3030303 , -0.27272727,
       -0.24242424, -0.21212121, -0.18181818, -0.15151515, -0.12121212,
       -0.09090909, -0.06060606, -0.03030303,  0.        ,  0.03030303,
        0.06060606,  0.09090909,  0.12121212,  0.15151515,  0.18181818,
```

```
       0.21212121,  0.24242424,  0.27272727,  0.3030303 ,  0.33333333,
       0.36363636,  0.39393939,  0.42424242,  0.45454545,  0.48484848,
       0.51515152,  0.54545455,  0.57575758,  0.60606061,  0.63636364,
       0.66666667,  0.6969697 ,  0.72727273,  0.75757576,  0.78787879,
       0.81818182,  0.84848485,  0.87878788,  0.90909091,  0.93939394,
       0.96969697,  1.        ,  1.03030303,  1.06060606,  1.09090909,
       1.12121212,  1.15151515,  1.18181818,  1.21212121,  1.24242424,
       1.27272727,  1.3030303 ,  1.33333333,  1.36363636,  1.39393939,
       1.42424242,  1.45454545,  1.48484848,  1.51515152,  1.54545455,
       1.57575758,  1.60606061,  1.63636364,  1.66666667,  1.6969697 ,
       1.72727273,  1.75757576,  1.78787879,  1.81818182,  1.84848485,
       1.87878788,  1.90909091,  1.93939394,  1.96969697,  2.        ])
```

And now, plot the CDF of Y for the range of values in y:

In [29]:
```
plt.step(y, Y.cdf(y));
```

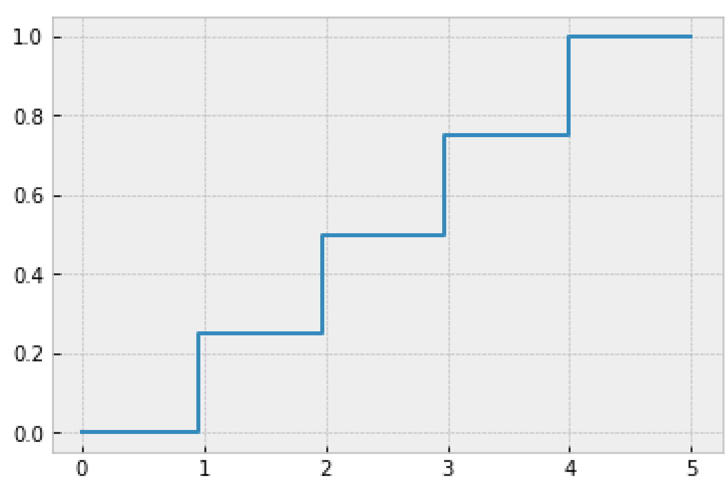


Example 6 Part 3 Let's compute and plot the CDF of RV Z:

- What range of values should we create now, to fully visualize the CDF?

In [30]:
```
z = np.linspace(0, 5, 100)
```

In [31]:
```
plt.step(z, Z.cdf(z));
```



Important Discrete RVs

The Bernoulli Random Variable

An event $A \in \mathcal{F}$ is considered a "success".

- A **Bernoulli RV** X is defined by

$$X(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

- The PMF for a Bernoulli RV X is defined by

$$p_X(x) = P(X = x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \\ 0, & \text{o.w.} \end{cases}$$

- We have seen this PMF before when we considered *data likelihood* for a coin flip. Remember for the toss of a coin, which comes up heads with probability p, and a tail with probability $1 - p$.

- We say that the "R.V. X follows a Bernoulli distribution with parameter p" and we write this as:

$$X \sim \text{Bernoulli}(p)$$

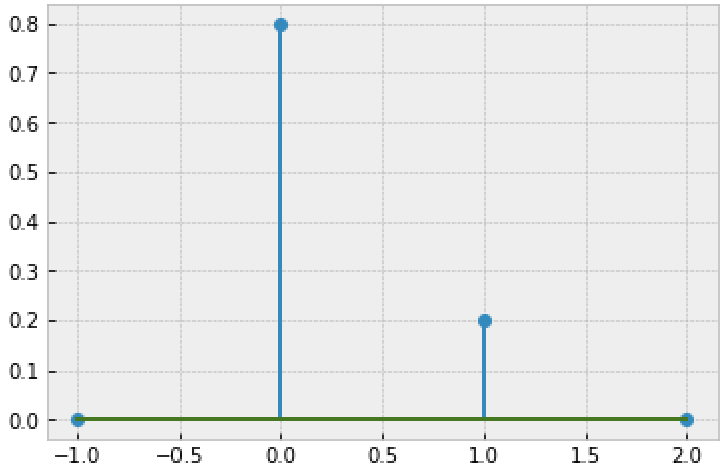
- **Engineering examples/applications:** whether a bit is 0 or 1, whether a bit is in error, whether a component has failed, whether something has been detected.

In [27]:
```
?stats.bernoulli
```

In [28]:
```
B = stats.bernoulli(0.2)
```

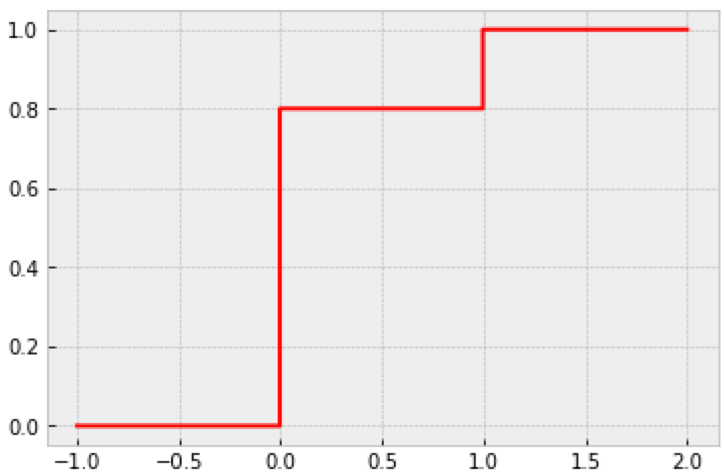
In [30]:
```
b = range(-1,3) # -1, 0, 1, 2
```

In [31]:
```
plt.stem(b, B.pmf(b));
```



In [52]:
```
x = np.linspace(-1,2,1000)
```

In [53]:
```
# plt.plot(x, B.cdf(x))
plt.step(x, B.cdf(x), 'r');

# We prefer to use the step() plotting function for plotting CDFs
```



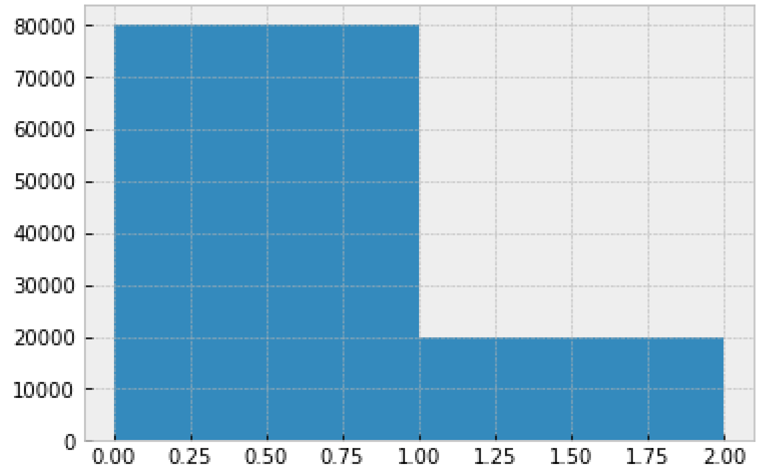
In [37]:
```
N = 100_000

sample = B.rvs(size=N)

print('Probabiity of success (b=1) is ~', np.sum(sample==1)/N)
```

```
Probabiity of success (b=1) is ~ 0.19938
```

Let's now plot the histogram of this sample. Let's start by defining the bins of the histogram:

In [38]:
```
mybins = [0,1,2]
```

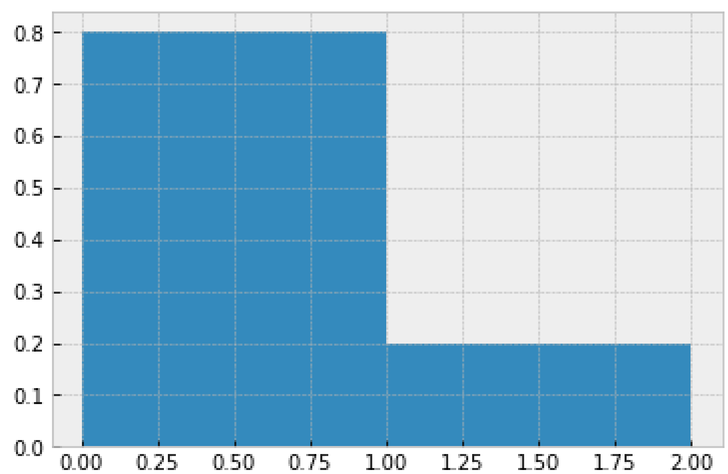
In [39]:
```
plt.hist(sample, bins=mybins);
```



We can plot the relative frequency of all the values in each bin, by changing the parameter `density` in the histogram function:

In [40]:
```
?plt.hist
```

In [41]:
```
plt.hist(sample, bins=mybins, density= True);
```



The Binomial Random Variable

- A Binomial RV represents the number of successes on n independent Bernoulli trials.
  - Example: a coin is tossed n times.
- Thus, a Binomial RV can also be defined as the sum of n independent Bernoulli RVs.
  - Example: At each toss, the coin comes up heads with probability p and a tail with probability $1 - p$, independently of prior tosses.
- Let X be the # of successes.
  - Example: X is the number of heads in the n-toss sequence.
- We refer to X as the **Binomial** RV **with parameters** n **and** p:

$$X \sim \text{Binomial}(n, p)$$

- The PMF of X is given by

$$p_X(x) = P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, \ldots, n \\ 0, & \text{o.w.} \end{cases}$$

- **Engineering examples/applications:** The number of bits in error in a packet, the number of defective items in a manufacturing run.

In [17]:
```
?stats.binom
```

In [18]:
```
Bn = stats.binom(12, 0.2)
```

Let's compute $P_X(2)$ where $X \sim \text{Binomial}(12, 0.2)$:

In [19]:
```
from scipy.special import binom
```

In [21]:
```
# probability of 2 heads in 12 flips using a coin with 20% for flipping heads is

binom(12,2)*0.2**2*(1-0.2)**(12-2)
```

Out[21]:
```
0.28346784153600024
```

Let's build a simulation, using NumPy arrays, to verify this result:

In [23]:
```
num_sims = 100_000

results = npr.choice([1,0], size=(num_sims, 12), p=[0.2, 0.8])
results
```

Out[23]:
```
array([[0, 0, 1, ..., 1, 0, 0],
       [1, 1, 1, ..., 0, 0, 0],
       [0, 0, 0, ..., 0, 0, 0],
       ...,
       [0, 0, 1, ..., 0, 0, 0],
       [1, 1, 0, ..., 0, 0, 0],
       [0, 0, 0, ..., 0, 0, 0]])
```

In [24]:
```
np.sum(results, axis=1)
```

Out[24]:
```
array([2, 4, 3, ..., 3, 5, 0])
```

In [25]:
```
np.sum(np.sum(results, axis=1)==2)
```

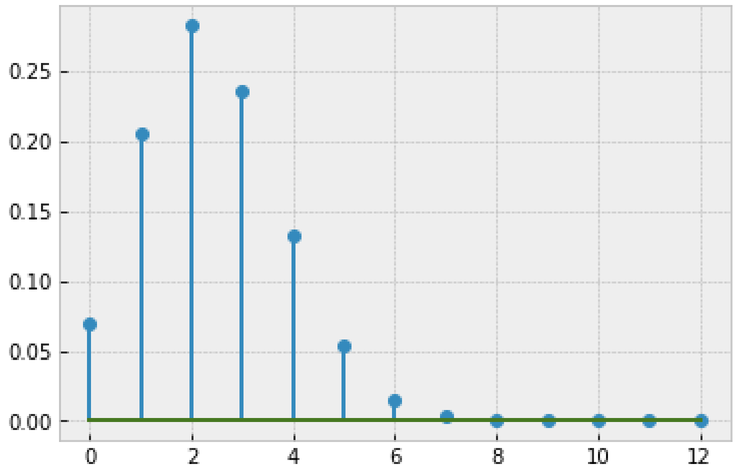
Out[25]:
```
28181
```

In [26]:
```
np.sum(np.sum(results, axis=1)==2)/num_sims
```

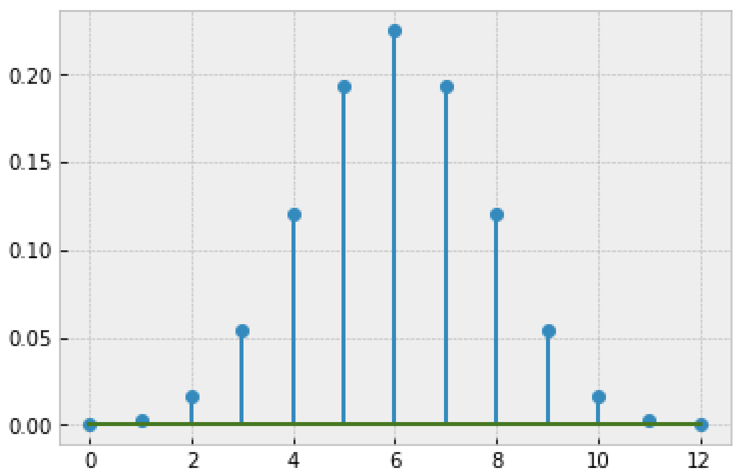
Out[26]: 0.28181

The complete PMF of this Binomial RV is:

In [27]:
```
x = range(0,13)

plt.stem(x, Bn.pmf(x));
```

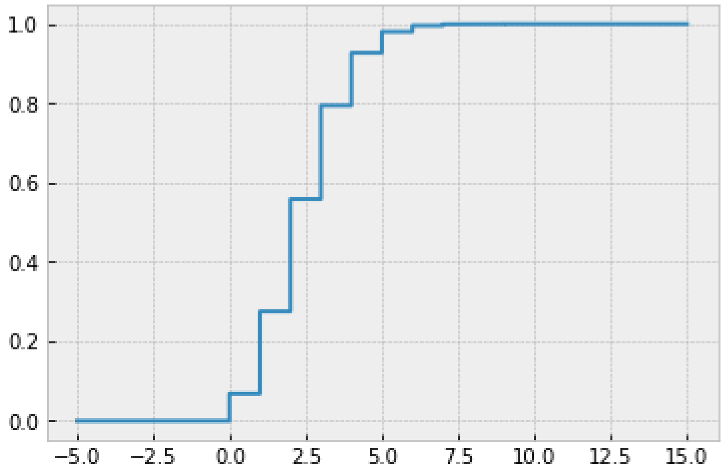


In [28]:
```
Bn2 = stats.binom(12, 0.5)

plt.stem(x, Bn2.pmf(x));
```



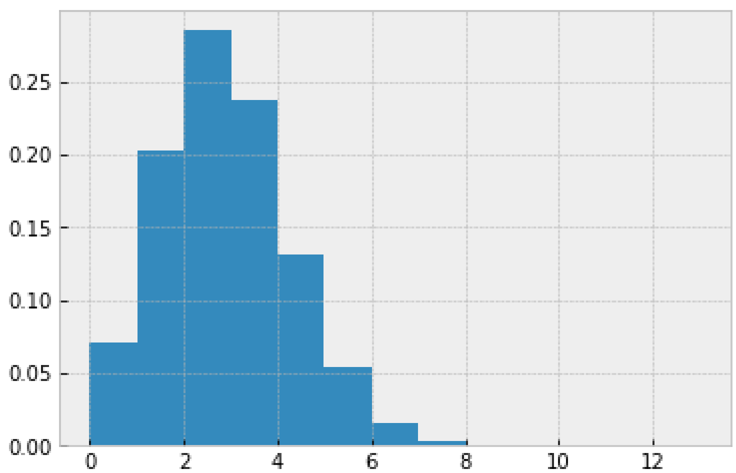
Let's plot its CDF:

In [29]:
```
x = np.linspace(-5,15,1000)

plt.step(x, Bn.cdf(x));
```

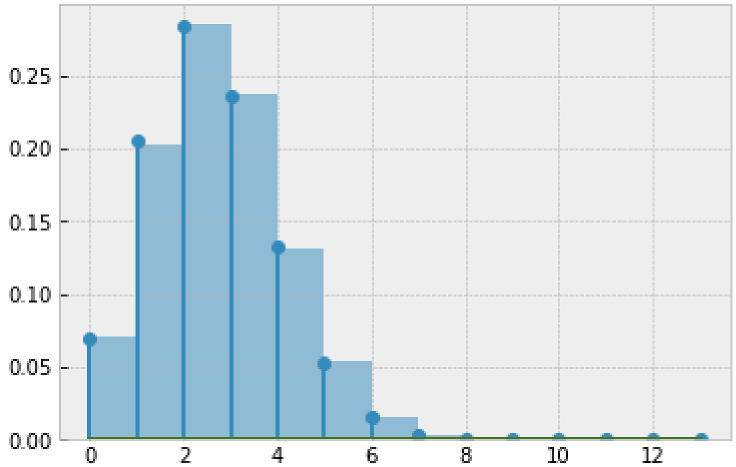


Let's generate some samples (random variables) from this distribution and plot their PMF:

In [34]:
```
N = 100_000

sample = Bn.rvs(size=N)

mybins=range(0,14)

plt.hist(sample, bins=mybins, density=True);
```

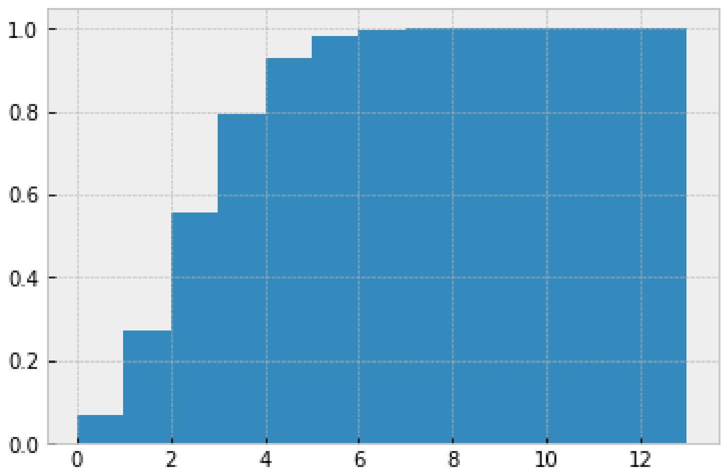


In [35]:
```
plt.hist(sample, bins=mybins, density=True, alpha=0.5)
plt.stem(mybins, Bn.pmf(mybins));
```



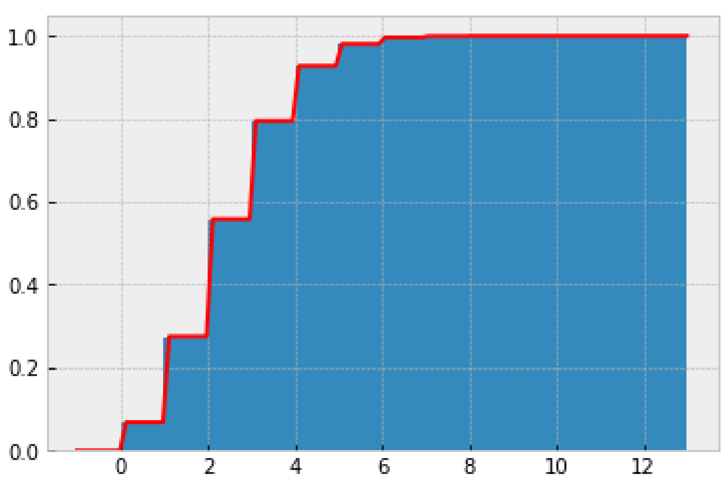
The plotting function `hist` can also plot the CDF of an RV:

In [36]:
```
plt.hist(sample, bins=mybins, density=True, cumulative=True);
```



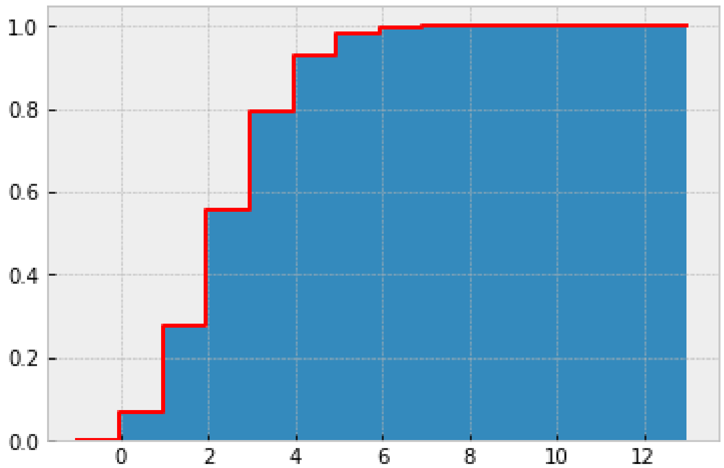
The histogram "fills" the area under the (CDF) curve. We can overlay the CDF curve on top:

In [38]:
```
plt.hist(sample, bins=mybins, density=True, cumulative=True);

x = np.linspace(-1,13,100)
plt.plot(x, Bn.cdf(x), 'r');
```



But this plotting function does not look good. The probability seems to increase in between discrete values, which is not valid. Instead, we use the `step` plotting function:

In [39]:
```
plt.hist(sample, bins=mybins, density=True, cumulative=True);

x = np.linspace(-1,13,100)
plt.step(x, Bn.cdf(x), 'r');
```



Binomial as the Sum of Bernoulli RVs

The Binomial RV can also be defined as the sum of n independent Bernoulli RVs.

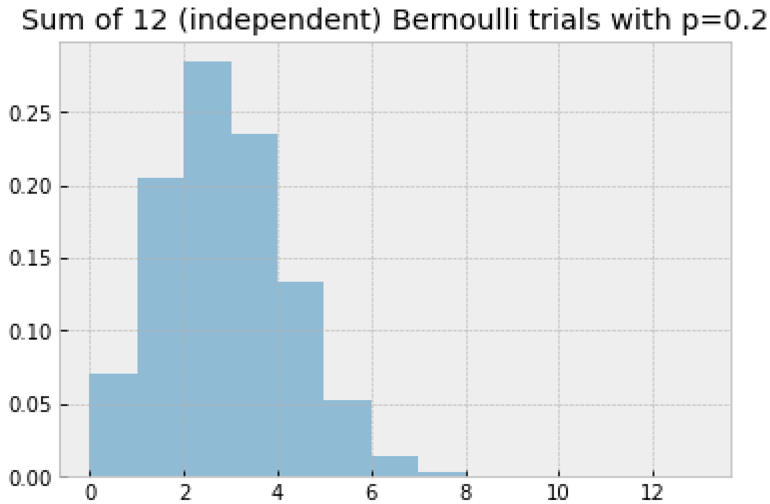
In [41]:
```
B = stats.bernoulli(0.2)

# Bernoulli(p=0.2)
```

In [42]:
```
N = 100_000

samples = B.rvs(size=(N, 12)) # generates random values from Bernoulli RV

samples
```

Out[42]:
```
array([[0, 0, 0, ..., 0, 0, 0],
       [0, 0, 0, ..., 0, 0, 0],
       [0, 0, 0, ..., 0, 1, 0],
       ...,
       [0, 0, 0, ..., 0, 0, 0],
       [0, 0, 0, ..., 0, 0, 0],
       [0, 0, 1, ..., 1, 1, 0]])
```

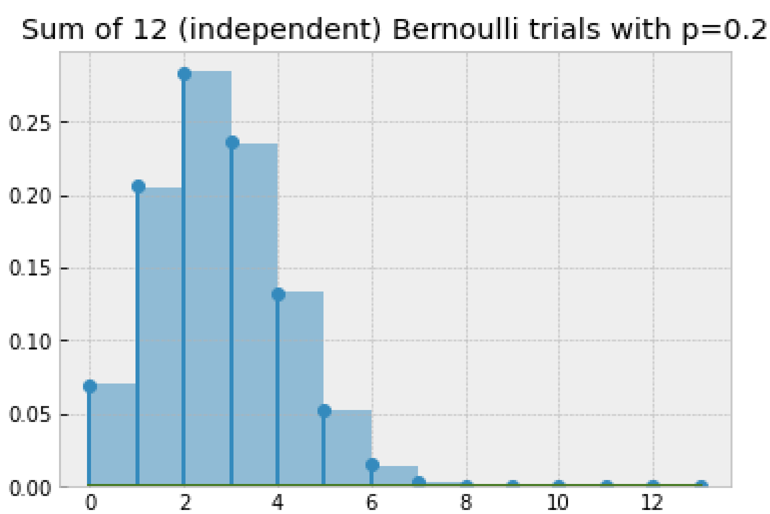
In [ ]:
```
Bsum = np.sum(samples, axis=1) # sums the columns, it gives the number of 1's pe
Bsum
```

In [44]:
```
mybins=range(0,14)
plt.hist(Bsum, bins=mybins, density=True, alpha=0.5)
plt.title('Sum of 12 (independent) Bernoulli trials with p=0.2');
```



Let's overlay the true PMF function of a Binomial RV with parameters $n = 12$ and $p = 0.2$:

In [46]:
```
mybins=range(0,14)
plt.hist(Bsum, bins=mybins, density=True, alpha=0.5)
plt.title('Sum of 12 (independent) Bernoulli trials with p=0.2')

# Bn = stats.binom(12, 0.2)
plt.stem(mybins, Bn.pmf(mybins));
```



- **Conclusion:** Adding together independent Bernoulli RVs (with the same probability p) produces a Binomial RV.

The Geometric Random Variable

- A Geometric RV occurs when independent Bernoulli trials are conducted until the first success
  - Example: repeatedly and independently toss a coin with probability of a heads equal to p, where $0 < p < 1$.
- X is the number of trials required.

- Example: The Geometric RV is the number X of tosses needed for a head to come up for the first time.

$$X \sim \text{Geometric}(p)$$

- The PMF of X is given by

$$p_X(x) = P(X = x) = \begin{cases} p(1-p)^{x-1}, & x = 1, 2, \ldots \\ 0, & \text{o.w.} \end{cases}$$

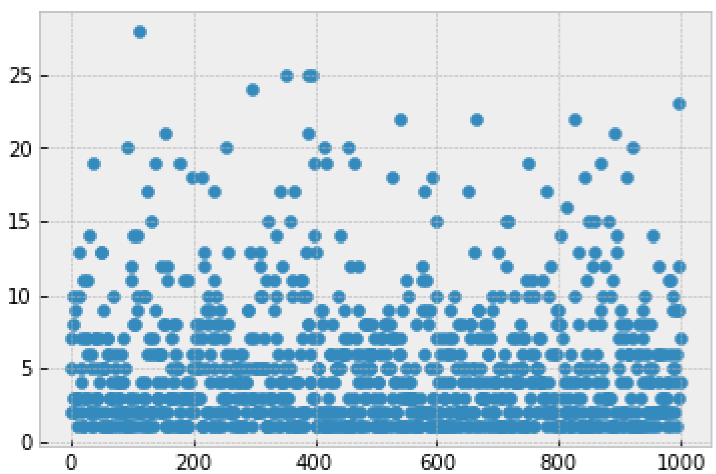
- **Engineering examples/applications:** The number of retransmissions required for a packet, number of white dots between black dots in the scan of a black and white document.

In [47]:
```
?stats.geom
```

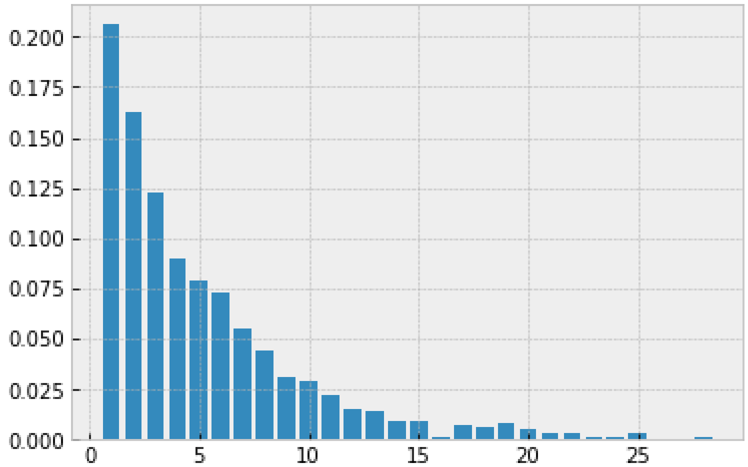
In [2]:
```
G = stats.geom(0.2)
```

In [3]:
```
N=1000
sample = G.rvs(size=N)
# randomly sample 1000 points from Geometric(0.2)
```

In [4]:
```
plt.scatter(range(len(sample)), sample);
```



In [5]:
```
vals, counts = np.unique(sample, return_counts=True)
plt.bar(vals, counts/N);
```



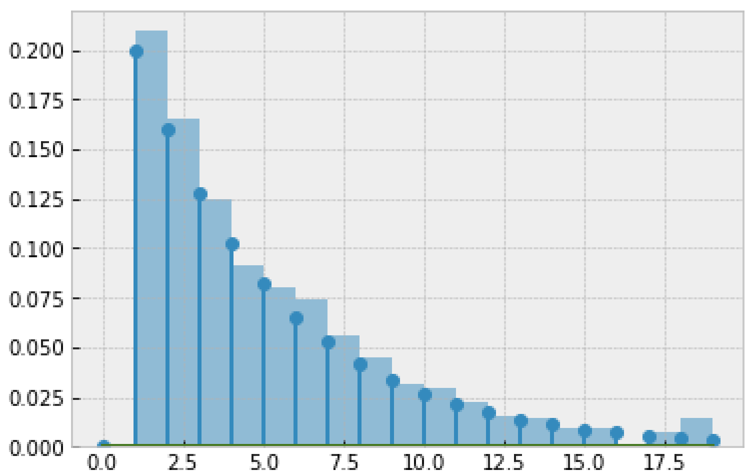
- What is the probability that the first success occurs in the 1st trial (coin flip)?

$$p_X(1) = p(1-p)^{1-1} = p = 0.2$$

- 6th trial?

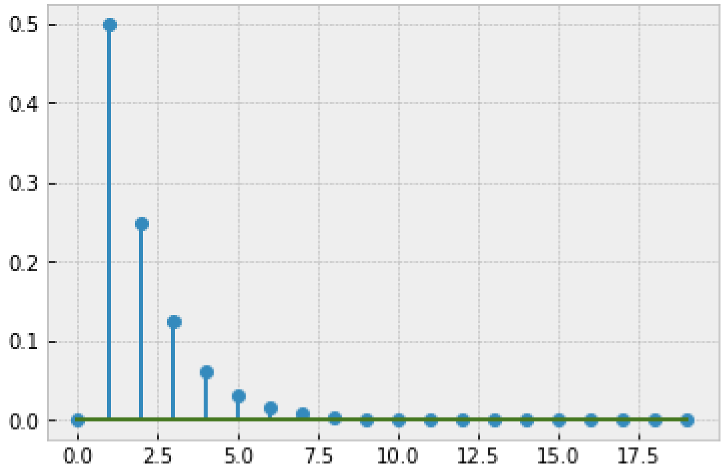
$$p_X(6) = p(1-p)^{6-1} = 0.2 \times 0.8^5 \approx 0.0655$$

In [6]:
```
mybins = range(20)
plt.hist(sample, bins=mybins, density=True, alpha=0.5)
plt.stem(mybins, G.pmf(mybins));
```



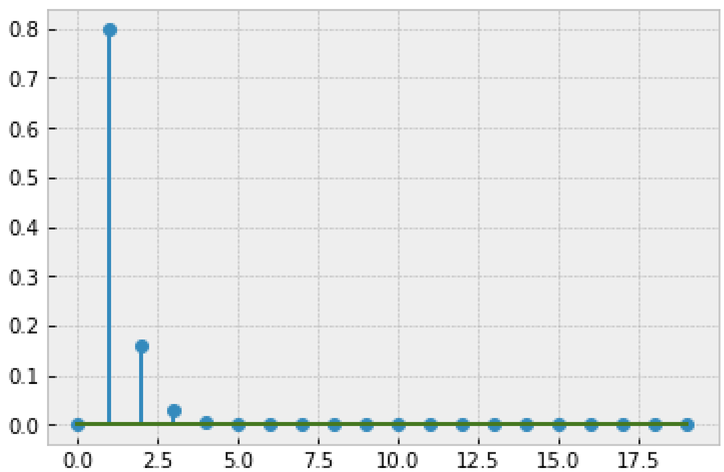
- Let's visualize the PMF for the Geometric with parameter $p = 0.5$?

In [7]:
```
G2 = stats.geom(0.5)

plt.stem(mybins, G2.pmf(mybins));
```



- Let's visualize the PMF for the Geometric with parameter $p = 0.8$?

In [8]:
```
G3 = stats.geom(0.8)

plt.stem(mybins, G3.pmf(mybins));
```



In [ ]:
```

```