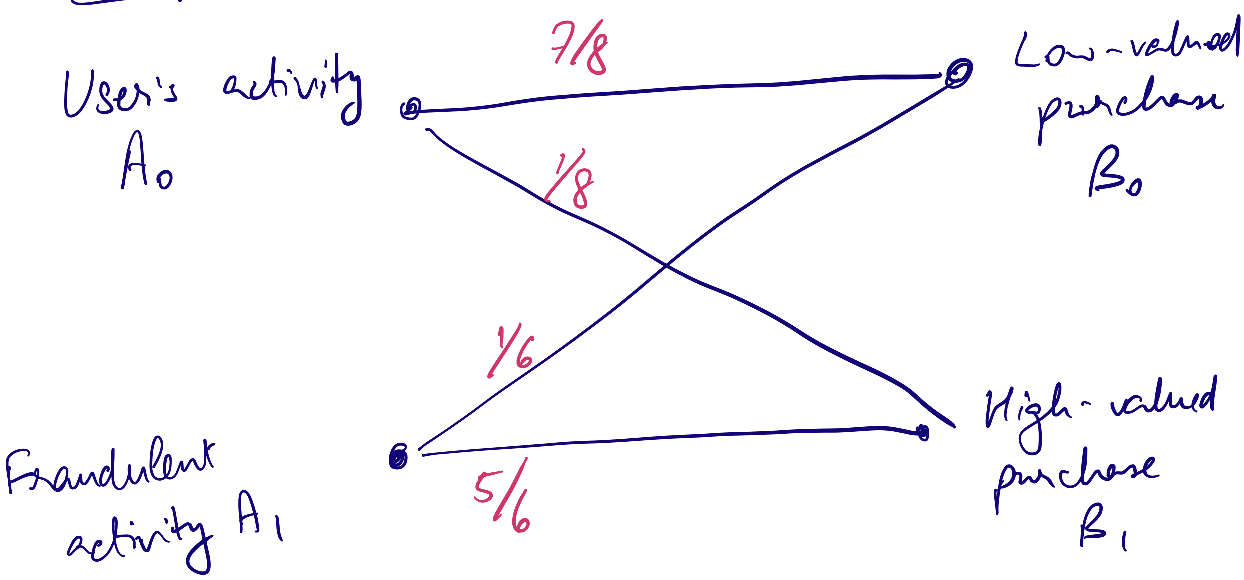
Lecture 8

Decision Rules :

MLE	MAP	Arbitrary
$P(B\|A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B\|A_1)$	$P(A_0\|B) \underset{A_1}{\overset{A_0}{\gtrless}} P(A_1\|B)$	Always pick A_i (either A_0 or A_1)

Example :



Let's say that the user is a new credit card owner, so the bank will assume a prior $P(A_0) = 9/10$.

Given:

Data likelihoods:

- $P(B_0 | A_0) = 7/8$
- $P(B_0 | A_1) = 1/6$
- $P(B_1 | A_0) = 1/8$
- $P(B_1 | A_1) = 5/6$

Prior probabilities:

- $P(A_0) = 9/10$

- $P(A_1) = 1/10$

We can compute:

- $P(B_0) = P(B_0 | A_0) P(A_0) + P(B_0 | A_1) P(A_1)$
$$= 7/8 \cdot 9/10 + 1/6 \cdot 1/10$$
$$\approx 0.8042$$

- $P(B_1) = P(B_1 | A_0) P(A_0) + P(B_1 | A_1) P(A_1)$
$$= 1/8 \cdot 9/10 + 5/6 \cdot 1/10$$
$$\approx 0.1958 = 1 - P(B_0)$$

a) MLE Decision Rule:

$$P(B|A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B|A_1)$$

$\Leftrightarrow$ If $P(B|A_0) > P(B|A_1)$, then we "decide" A_0

- When receiving B_0:

$$P(B_0|A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B_0|A_1)$$

$$7/8 > 1/6 \Rightarrow \text{Decide } A_0!$$

- When receiving B_1:

$$P(B_1|A_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(B_1|A_1)$$

$$1/8 < 5/6 \Rightarrow \text{Decide } A_1!$$

b) MAP Decision Rule:

$$P(A_0|B_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(A_1|B_0)$$

- When receiving B_0:

$$P(A_0 | B_0) \underset{A_1}{\overset{A_0}{\gtrless}} P(A_1 | B_0)$$

$$\frac{P(B_0|A_0)\,P(A_0)}{P(B_0)} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{P(B_0|A_1)\;P(A_1)}{P(B_0)}$$

$$\frac{7/8 \cdot 9/10}{0.8042} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{1/6 \cdot 1/10}{0.8042}$$

$$0.979 > 0.021$$ $\Rightarrow$ Decide A_0!

- When receiving B_1:

$$P(A_0 | B_1) \underset{A_1}{\overset{A_0}{\gtrless}} P(A_1 | B_1)$$

$$\frac{P(B_1 | A_0)\;P(A_0)}{P(B_1)} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{P(B_1 | A_1)\,P(A_1)}{P(B_1)}$$

$$\frac{1/8 \cdot 9/10}{0.1958} \underset{A_1}{\overset{A_0}{\gtrless}} \frac{5/6 \cdot 9/10}{0.1958}$$

$0.575 > 0.425 \Rightarrow$ Decide A_0!

c) Arbitrary decision rule:

Always decide A_1

- When receiving B_0:

  $\Rightarrow$ Decide A_1

- When receiving B_1:

  $\Rightarrow$ Decide A_1

Receive	MLE	MAP	Arbitrary
B_0	Decide A_0	Decide A_0	Decide A_1
B_1	Decide A_1	Decide A_0	Decide A_1
$P(E)$	0.129	0.100	0.900
		↳ Always lowest!	

Three different decision rules! How do you decide which decision rule?!

Probability of errors :

MLE

$$P(E) = P(E|B_0)\,P(B_0) + P(E|B_1)\,P(B_1)$$

$$= (1 - P(A_0|B_0))\,P(B_0) + (1 - P(A_1|B_1))\,P(B_1)$$

$$= (1 - 0.979)\cdot 0.8042 + (1 - 0.425)\cdot 0.1958$$

$$\approx 0.129$$

MAP

$$P(E) = P(E|B_0)\,P(B_0) + P(E|B_1)\,P(B_1)$$

$$= (1 - P(A_0|B_0))\,P(B_0) + (1 - P(A_0|B_1))\,P(B_1)$$

$$= (1 - 0.979)\cdot 0.8042 + (1 - 0.575)\cdot 0.1958$$

$$\approx 0.100$$

Arbitrary : Always A_1

$$P(E) = P(E|B_0)\,P(B_0) + P(E|B_1)\,P(B_1)$$

$$= (1 - P(A_1|B_0))\,P(B_0) + (1 - P(A_1|B_1))\,P(B_1)$$

$$= (1 - 0.021)\cdot 0.8042 + (1 - 0.425)\cdot 0.1958$$

$$= 0.900$$

- The prior probability of fraudulent activity, p can change over time. For what set of values for p will the MAP decision always decide A_0 if B_0, and A_1 if B_1 is received?

$\Rightarrow$ Let $P(A_1) = p$

- MAP decides A_0 when B_0 is received if

$$P(A_0 | B_0) > P(A_1 | B_0)$$

$$\frac{P(B_0|A_0)\,P(A_0)}{\cancel{P(B_0)}} > \frac{P(B_0|A_1)\,P(A_1)}{\cancel{P(B_0)}}$$

$$7/8 \cdot (1-p) > 1/6 \cdot p$$

$$\Rightarrow \quad p < 0.84$$

- MAP decides A_1 when B_1 is received if:

$$P(A_0 | B_1) < P(A_1 | B_1)$$

$$\frac{P(B_1|A_0)\,P(A_0)}{P(B_1)} < \frac{P(B_1|A_1)\,P(A_1)}{P(B_1)}$$

$$1/8 \cdot (1-p) < 5/6 \cdot p \quad \Rightarrow p > 0.14$$

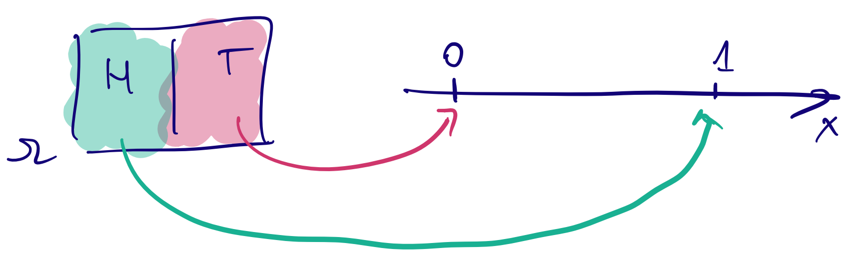
$$\Rightarrow \quad 0.114 < P(A_1) < 0.84$$

Random Variables

Example 1:

Create a binary RV from tossing a fair coin:

$\Omega = \{H, T\}$

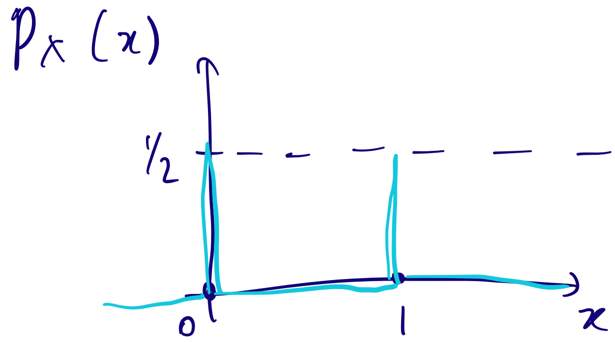


$X \equiv$ R.V. for experiment of flipping a fair coin.

"defined as"

$$X(x) = \begin{cases} 0, & x = H \\ 1, & x = T \end{cases}$$

come back



P.M.F.

$$P_X(x) = \begin{cases} P(X=1), & x=1 \\ P(X=0), & x=0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/2, & x=1 \\ 1/2, & x=0 \\ 0, & \text{o.w.} \end{cases}$$

Example 2:

Experiment flipping a fair coin twice; design a binary RV:

$$\Omega = \{HH, HT, TH, TT\}$$

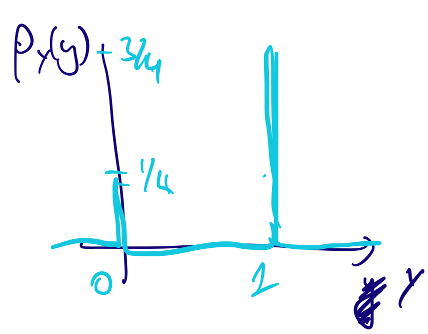
$Y \equiv$ R.V. for this experiment

$$Y(y) = \begin{cases} 1, & y = \{HH, HT, TH\} \\ 0, & y = \{TT\} \end{cases}$$

come back

P.M.F.

$$P_Y(y) = \begin{cases} 3/4, & Y=1 \\ 1/4, & Y=0 \\ 0, & \text{o.w.} \end{cases}$$



Example 3:

Design an R.V. for an experiment where you flip a fair coin twice:

$$\Omega = \{HH, HT, TH, TT\}$$

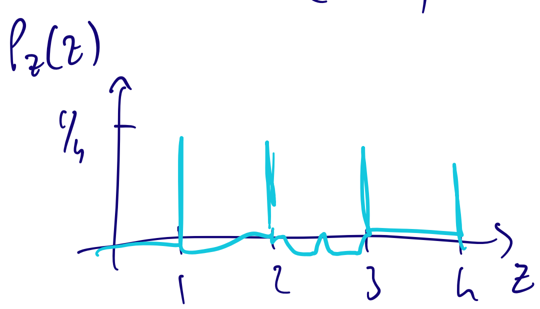
$Z \equiv$ R.V. for this experiment.

$$Z(z) = \begin{cases} 1, & z = HH \\ 2, & z = HT \\ 3, & z = TH \\ 4, & z = TT \end{cases}$$

come back

P.M.F.

$$P_Z(z) = \begin{cases} 1/4, & z = \{1, 2, 3, 4\} \\ 0, & \text{o.w.} \end{cases}$$



Probability Mass Function:

Let X be a discrete R.V.

P.M.F. $p_X(x) = P(X = x)$

Note that: $\sum_x p_X(x) = 1$

Using the same argument, for any set S of possible values of R.V. X, we have:

go back

$$P(X \in S) = \sum_{x \in S} p_X(x)$$

Cumulative Density Function:

Let X be a discrete R.V.:

CDF: $F_X(x) = P(X \leq x)$ (accumulated probability)

If X is discrete:

$$F_X(x) = P(X \leq x) = \sum_{k \leq x} p_X(k)$$

CDF is a probability measure

↳ all axioms and corollaries of prob. are satisfied.

Example 7: Rolling a fair 6-sided die

$X \equiv \#$ on top face

$$P_X(x) = \begin{cases} 1/6, & x = \{1, 2, \ldots, 6\} \\ 0, & \text{o.w.} \end{cases}$$

$$F_X(x) = P(X \leq x)$$

$$F_X(3) = P(X \leq 3) = P(X=1) + P(X=2) + P(X=3)$$

$$= 3 \cdot 1/6 = 1/2$$