Lecture 14

- Moments continued
- KDE on a dataset
- Statistical Inference
- Hypothesis tests

Feedback Evaluation Form

Fill this out! Note: you need to be signed into Chrome or web browser through your UF account. The form, however, is anonymous.

Last class

- We learned about different **moments** of PDFs.

Central Moments The **central moments** of a random variable (or of its distribution) are *expected values of mean-centered powers* or related functions of the random variable. The n-th central moment of RV X is $E[(X-\mu_X)^n]$, in general, $E[(X-E[X])^n]$. If X is a **continuous** RV, the n-th central moment is

$$E[(X-\mu_X)^n] = \int_x (t-\mu_X)^n f_X(t)\, dt$$

If X is a **discrete** RV, the n-th central moment is

$$E[(X-\mu_X)^n] = \sum_x (x-\mu_X)^n p_X(x)$$

- Moments of a random variable are expected values of the random variable raised to some power.

- For a *central moment*, the mean is subtracted from the random variable before it is raised to a power.

Because different powers spread the values of the random variable in different ways, **moments can provide additional information about a random variable other than the mean value**:

In mathematics, a moment is a specific quantitative measure of the shape of a function. The most important ones are:

1. **Mean**, the 1st moment

2. **Variance**, the 2nd central moment

3. **Skewness**, the 3rd central moment

4. **kurtosis**, the 4th central moment

- We learnt how to think about **decision-making** and MAP in the case of distributions.

- We learnt about **KDE** (kernel density estimation)

Today

- We will learn about how to infer an entire PDF from data using KDE

- We will learn about Statistical Inference - how do we directly estimate the parameters, here, the moments of a PDF

- We will learn about hypothesis tests: how do we prove or disprove hypotheses when faced with data, that go beyond the difference in means / medians.

In [2]:

```
import scipy.stats as stats
import numpy as np
import numpy.random as npr
import matplotlib.pyplot as plt
import pandas as pd

%matplotlib inline
plt.style.use('bmh')
```

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4. **kurtosis**, the 4th central moment

We saw that we can compute the most common moments of a random variable using the `stats` module, but let us go through what these moments actually describe: the shape of the distribution.

Let's go over what the moments describe, on the whiteboard.

Kernel Density Estimation (KDE)

Kernel density estimation (KDE) is a non-parametric estimator of the probability density function (PDF) of a random variable. It is a fundamental data smoothing problem where inferences about the population are made, based on a finite data sample. It uses a mixture consisting of a Kernel component centered at each data point.

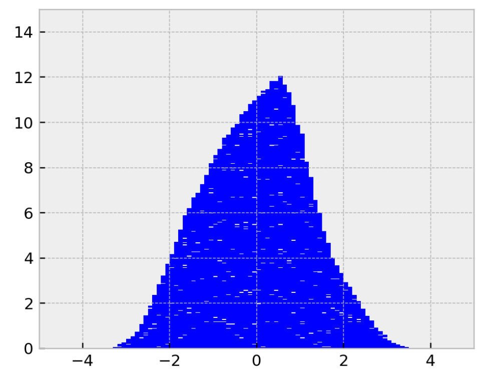
- A density estimator is an algorithm which seeks to model the probability distribution that generated a data set. For one dimensional data, you are already familiar with one simple density estimator: the histogram. A histogram divides the data into discrete bins, counts the number of points that fall in each bin, and then visualizes the results in an intuitive manner.
- One of the issues with using a *histogram* as a density estimator is that the choice of bin size and location can lead to representations that have qualitatively different features.
- In order to smooth them out, we might decide to replace the blocks at each location with a smooth function, like a Gaussian.
- Look at Lecture13-supp.ipynb for more information on KDE!

In [3]:
```
G=stats.norm()
Gvals=G.rvs(size=100)
G25=stats.norm.rvs(size=25)
```

In [4]:

```
from matplotlib import animation
#This will do an animated histogram
%matplotlib notebook

fig = plt.figure()
fig.set_dpi(100)
fig.set_size_inches(5, 4)

ax = plt.axes(xlim=(-5, 5), ylim=(0, 15))
blocks={}
floors={}
patchesPerBlock=25

def init():
    blocks['activeBlock']=0
    for num,var in enumerate(G25):
        center=round(var,1)
        blocks[num]=[]
        for i in range(patchesPerBlock):
            binedge=round(center+(i-patchesPerBlock//2)*0.1,1)
            patch = plt.Rectangle((binedge, 16), 0.1,
                                  stats.norm.pdf((i-patchesPerBlock//2)*0.1,scal
            ax.add_patch(patch)
            blocks[num]+=[patch]
            floors[binedge]=0.01
        #print(num,var, patches)


    return []

def animatePatch(i, patch,update):
    return patch,

def animate(i, blocks):
    activeBlock=blocks['activeBlock']
    #print(activeBlock,blocks)
    numFloored=0
    for patch in blocks[activeBlock]:
        x,y=patch.xy
        if y<=floors[x]:
            #floors[x]=floors[x]+0.6
            #print("floors[",x,"]=",floors[x])
            patch.xy=(x,floors[x])
            numFloored+=1
        else:
            patch.xy=(x,y-0.2)
    if numFloored>=patchesPerBlock:
        for patch in blocks[activeBlock]:
            x,y=patch.xy
            floors[x]=round(floors[x]+patch.get_height(),1)
        blocks['activeBlock']=activeBlock+1
        activeBlock+=1
        if activeBlock not in blocks.keys():
            return []
        #print(patches[activePatch])
    for i in blocks:
        #print(patchnum)
        if i!='activeBlock':
```

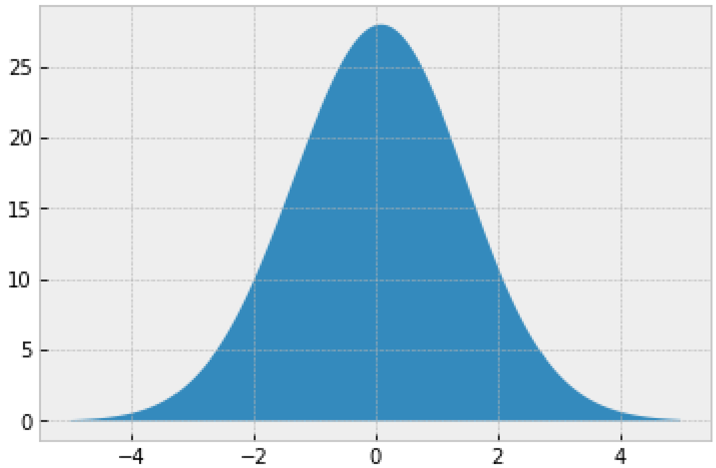
```
            for patch in blocks[i]:
                animatePatch(i, patch,activeBlock==i)

    return[]

anim = animation.FuncAnimation(fig, animate,
                               init_func=init,
                               frames=100*len(G25),
                               fargs=(blocks,),
                               interval=5,
                               blit=True,repeat=False)
anim
plt.show()
```



Here is the same type of graph created with a wider, more finely quantized Gaussian shape and 100 Gaussian random variables:

In [5]:

```
%matplotlib inline
x = np.linspace(-5,5, 1000)
density=np.zeros(x.size)
for xi in Gvals:
    ## Create new Gaussian RVs centered on the observation
    Gi=stats.norm(xi)
    ## Use the density of the RV as the shape:
    density+=Gi.pdf(x)

plt.fill_between(x, density);
```



Magic! This looks a lot like our density, even though we only had 100 observations!

The shape we use to build this new density estimate is called a **"kernel"**.

This approach is called **kernel density estimation (KDE)**.

- The free parameter of kernel density estimation is the **kernel** function, which specifies the shape of the distribution placed at each point, and the kernel **bandwidth**, which controls the size of the kernel at each point. In practice, there are many kernels you might use for a kernel density estimation: in particular, the Scikit-Learn KDE implementation supports six kernel functions, which you can read about in Scikit-Learn's Density Estimation documentation.

Applying KDE to a dataset

In [6]:

```
df = pd.read_csv('firearms-urban.csv')

df
```

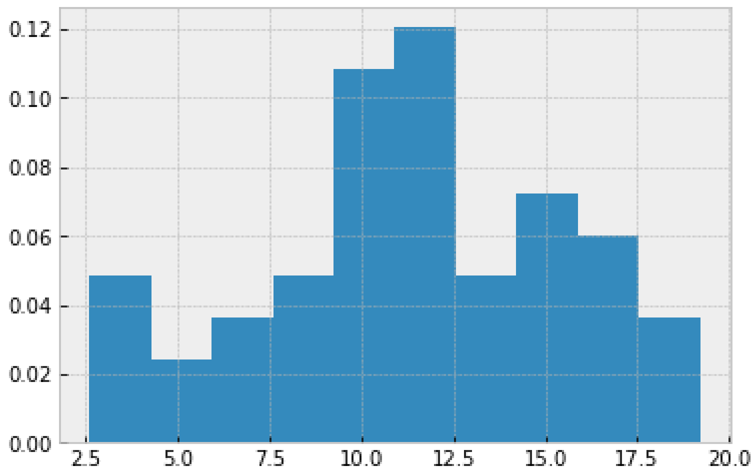
Out[6]:

	STATE	RATE-2014	Percent Urban
0	AL	16.9	59.0
1	AK	19.2	66.0
2	AZ	13.5	89.8
3	AR	16.6	56.2
4	CA	7.4	95.0
5	CO	12.2	86.2
6	CT	5.0	88.0
7	DE	11.1	83.3
8	FL	11.5	91.2
9	GA	13.7	75.1

	STATE	RATE-2014	Percent Urban
10	HI	2.6	91.9
11	ID	13.2	70.6
12	IL	9.0	88.5
13	IN	12.4	72.4
14	IA	7.5	64.0
15	KS	11.3	74.2
16	KY	13.9	58.4
17	LA	19.0	73.2
18	ME	9.4	38.7
19	MD	9.0	87.2
20	MA	3.2	92.0
21	MI	11.1	74.6
22	MN	6.6	73.3
23	MS	18.3	49.4
24	MO	15.3	70.4
25	MT	16.1	55.9
26	NE	9.5	73.1
27	NV	14.8	94.2
28	NH	8.7	60.3
29	NJ	5.3	94.7
30	NM	16.0	77.4
31	NY	4.2	87.9
32	NC	11.8	66.1
33	ND	12.3	59.9
34	OH	10.3	77.9
35	OK	15.7	66.2
36	OR	11.7	81.0
37	PA	10.5	78.7
38	RI	3.0	90.7
39	SC	15.5	66.3
40	SD	10.3	56.7
41	TN	15.1	66.4
42	TX	10.7	84.7
43	UT	12.3	90.6
44	VT	10.3	38.9

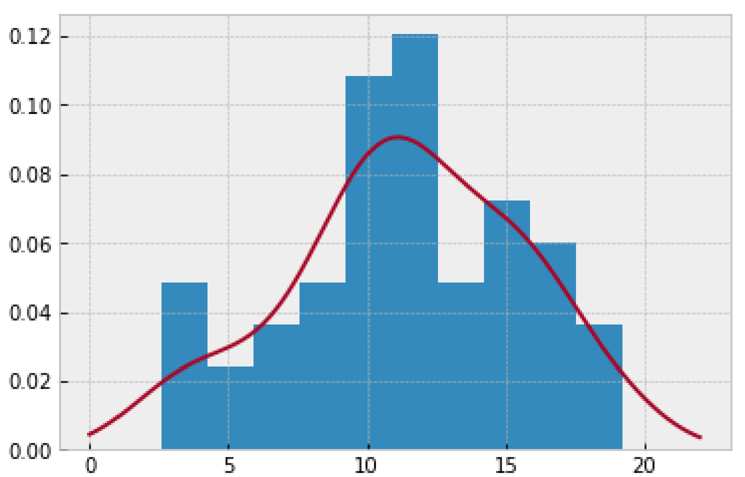
	STATE	RATE-2014	Percent Urban
45	VA	10.3	75.5
46	WA	9.7	84.1
47	WV	14.6	48.7
48	WI	8.2	70.2
49	WY	16.2	64.8

In [8]:
```
r14 = df['RATE-2014'].to_numpy()

plt.hist(r14, density = True);
```



In [17]:
```
G = stats.gaussian_kde(r14) # This uses KDE with a Gaussian kernel
```

In [18]:
```
x = np.linspace(0,22,1000)

plt.hist(r14,density=True)

plt.plot(x, G.pdf(x));
```

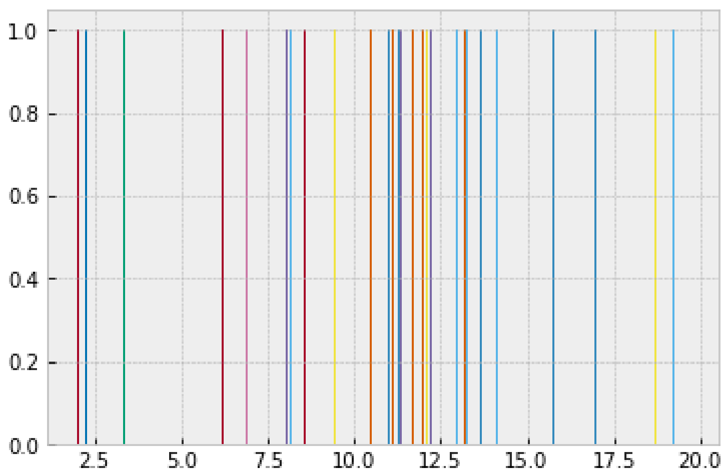


In [19]:

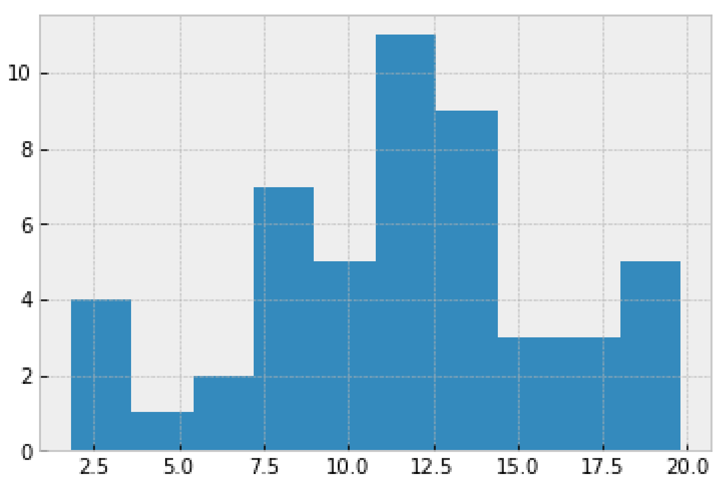
```
# Draw random samples from this estimated density function

values = G.resample(size=50)
values[0]
```

Out[19]:
```
array([12.17407336,  2.98300407,  4.68185045,  9.51943706, 11.19056936,
       19.80212086, 13.69465219,  7.59567056, 10.25305441,  1.80845309,
       11.90591407, 13.4931039 , 11.00684084,  7.68262871, 13.7636831 ,
        7.30735711, 12.88323426, 17.22610829, 18.09690921, 13.55600179,
       17.01789185,  5.92854774,  8.10621286, 13.39962589, 12.2443811 ,
       18.42532774,  7.58203363, 17.22715009,  3.55964296, 10.54177565,
       13.34408325, 15.92072999,  8.11604747, 11.98325097, 11.16012345,
       19.29532905, 18.92795329, 13.49526099, 10.85439057, 11.23107836,
       14.50061642,  7.86860863, 11.54471371, 15.88588627, 10.12161903,
        5.85170848, 14.17827307,  2.87766312,  9.04083911, 11.92653265])
```

In [20]:
```
plt.hist(values);
```

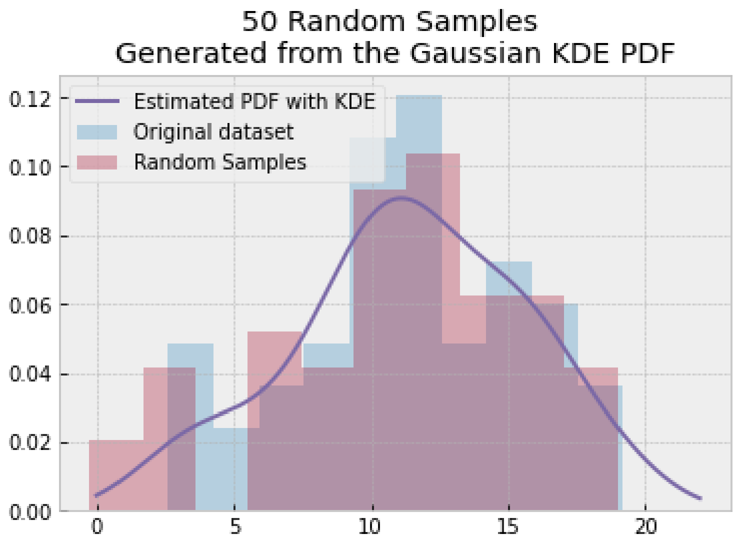


In [21]:
```
plt.hist(values[0]);
```



In [27]:
```
N=50

values = G.resample(size=N)

plt.hist(r14,density = True, alpha = 0.3, label = 'Original dataset')
plt.hist(values[0], density = True, alpha = 0.3, label='Random Samples')
```

```
plt.plot(x, G.pdf(x), label = 'Estimated PDF with KDE')


plt.legend()
plt.title(str(N)+' Random Samples \nGenerated from the Gaussian KDE PDF');
```



Statistical Inference

We have developed the Bayesian approach to inference, where unknown parameters are modeled as random variables. In all cases we worked within a single, fully-specified probabilistic model, and we based most of our derivations and calculation on judicious application of Bayes's rule.

By contrast, we can adopt a fundamentally different philosophy: we can view the unknown probabilistic parameter θ of the probability function as a *deterministic* quantity (not random) but, nevertheless, unknown quantity.

- The observational data $X = \{x_i\}_{i=1}^N$ is random and its distribution $p_X(x;\theta)$ (if X is discrete) or $f_X(x;\theta)$ (if X is continuous) depends on the value of θ (the parameters of the distribution).
- Thus, instead of working with a single probabilistic model, we will be dealing simultaneously with *multiple candidate (probabilistic) models*, one model for each possible value of θ.
- In this context, a *good* hypothesis testing or estimation procedure will be one that possesses certain desirable properties *under every candidate model*.

Classical Parameter Estimation

Given observations $X = \{x_1, x_2, \ldots, x_N\}$, an **estimator** is a random variable of the form $\hat{\theta} = g(X)$, for some function g. Note that since the distribution of X depends on θ, the same is true for the distribution of $\hat{\theta}$.

Some Terminology

Let $\hat{\theta}$ be an **estimator** of an unknown parameter θ, that is, a function of N observations $X = \{x_i\}_{i=1}^N$ whose distribution depends on θ.

Error of Estimator The **estimated error** is denoted by $\epsilon_\theta(\hat{\theta})$, is defined as

$$\epsilon_\theta = \hat{\theta} - \theta$$

Bias of Estimator The **bias** of an estimator, denoted by $b_\theta(\hat{\theta})$ is defined as

$$b_\theta(\hat{\theta}) = E[\hat{\theta}] - \theta$$

Variance of Estimator The **variance** of an estimator, denoted by $\mathrm{Var}_\theta[\hat{\theta}]$ is defined as

$$\mathrm{Var}_\theta[\hat{\theta}] = E\left[\left(\hat{\theta} - E[\hat{\theta}]\right)^2\right]$$

Mean-Square Error of Estimator The **Mean-Square Error** of an estimator, denoted by $E\left[(\hat{\theta} - \theta)^2\right]$ is defined as

$$E\left[(\hat{\theta} - \theta)^2\right] = b_\theta^2\left[\hat{\theta}\right] + \mathrm{Var}_\theta[\hat{\theta}]$$

The expected value, the variance, and the bias of $\hat{\theta}$ depend on θ, while the estimation error depends in addition on the observations $x_1, \cdots, x_N$.

Unbiased Estimator We call $\hat{\theta}$ **unbiased** if $E[\hat{\theta}] = \theta$, for every possible value of θ.

Asymptotically Unbiased Estimator We call $\hat{\theta}$ **asymptotically unbiased** if $\lim_{N\rightarrow\infty} E[\hat{\theta}] = \theta$, for every possible value of θ.

The Bias-Variance Trade-Off

Besides the bias $b_\theta(\hat{\theta})$, we are usually interested in the size of the estimation error. This is captured by the **mean squared error**, $E\left[(\hat{\theta} - \theta)^2\right]$, which is related to the bias and the variance of $\hat{\theta}$ according to the following formula:

$$E\left[(\hat{\theta}-\theta)^2\right] = b_\theta^2\left[\hat{\theta}\right] + \mathrm{Var}_\theta[\hat{\theta}]$$

This formula is important because in many statistical problems there is a **trade-off** between the two terms on the right-hand-side. Often a reduction in the variance is accompanied by an increase in the bias. Of course, a good estimator is one that manages to keep both terms small. This is also known as **The Bias-Variance Trade-off**.

Properties of Sum of Independent Gaussian RVs

Suppose now that we have two independent data samples, $X = \{x_1, x_2, \cdots, x_N\}$ and $Y = \{y_1, y_2, \cdots, y_N\}$, that we think can be modeled as coming from Gaussian distributions.

- If we observe a difference in the sample means for the two data sets, how can we determine analytically if the (true) means are different?
- Let's assume that distributions for the two data sets have a common variance and we know the variance, σ^2.

We need to know a few more facts about **sums of independent Gaussian random variables**:

1. If X and Y are independent RVs, then

$$Var[X+Y] = Var[X] + Var[Y]$$

1. If X and Y are independent Gaussian random variables such that

$$X \sim \mathrm{Gaussian}(\mu_X, \sigma_X^2)$$

and

$$Y \sim \mathrm{Gaussian}(\mu_Y, \sigma_Y^2)$$

Then

$$Z = X + Y \sim \mathrm{Gaussian}(\mu_Z, \sigma_Z^2)$$

By linearity $\mu_z = \mu_X + \mu_Y$, and by the previous property $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$.

1. If Z is a Gaussian random variable, $aZ + b$ is also a Gaussian random variable.

Let's start by considering the statistic of a single sample mean,

$$\hat{\mu}_X = \frac{1}{N}\sum_{i=1}^{N} X_i$$

By the properties above, we can see:

- $\hat{\mu}_X$ is a Gaussian random variable
- The mean of $\hat{\mu}_X$ is $E[\hat{\mu}_X] = \mu_X$
- The variance of $\hat{\mu}_X$ is

$$\begin{aligned} Var\left[\frac{1}{N}\sum_{i=1}^{N} X_i\right] &= \frac{1}{N^2} Var\left[\sum_{i=1}^{N} X_i\right] \\ &= \frac{1}{N^2}\sum_{i=1}^{N} Var\left[X_i\right] \\ &= \frac{1}{N^2}\sum_{i=1}^{N} \sigma_X^2 \\ &= \frac{\sigma_X^2}{N} \end{aligned}$$

Note that the variance of the sample mean decreases linearly as the sample size increases.

This can be used to show that the sample mean converges to the true mean if the variance of the original random variable is finite.

Z-Test: Binary Hypothesis Tests involving Sample Mean *with Known and Equal Variances*

Suppose we have two populations characterized by RVs X and Y, and the following samples $\{x_i\}_{i=1}^{M}$ and $\{y_j\}_{j=1}^{N}$, where x_i and y_j are observed values of RVs X and Y, which are assumed to have common variance σ^2.

- Let the averages of the data samples be

$$\bar{x} = \frac{1}{M}\sum_{i=1}^{M} x_i, \text{ and } \bar{y} = \frac{1}{N}\sum_{j=1}^{N} y_j$$

and denote the true means of the distributions μ_X and μ_Y, respectively.

- Note that if the number of samples from each population is relatively large (≥ 10), then even if the original population does not have a Gaussian distribution, the averages will still be approximately Gaussian - Central Limit Theorem (CLT)

If $\bar{x} \neq \bar{y}$, how can we conduct a binary hypothesis test on whether the two populations have different means?

- What is the null hypothesis?
  - H_0: **the means are the same,** $\mu_X = \mu_Y$

- H_1: **the means are not the same,** $\mu_X \neq \mu_Y$
- We will conduct this test only using the sample observations $\{x_i\}_{i=1}^{M}$ and $\{y_j\}_{j=1}^{N}$

Under the null hypothesis, we compute the difference in the sample averages and determine the probability that a difference that large would be observed under the null hypothesis.

Thus, our test statistic is the difference in averages

$$t = \bar{x} - \bar{y}$$

- Let $\hat{\mu}_X$ and $\hat{\mu}_Y$ be the sample means of random samples of sizes M and N from X and Y RVs, respectively. We can view t as an instantiation of

$$T = \hat{\mu}_X - \hat{\mu}_Y$$

If $\mu_X = \mu_Y = \mu$, then $E[\hat{\mu}_X] = E[\hat{\mu}_Y] = \mu$. Then, by linearity

$$\mu_T = E[T] = E[\hat{\mu}_X - \hat{\mu}_Y] = E[\hat{\mu}_X] - E[\hat{\mu}_Y] = \mu - \mu = 0$$

- We can compute the variance of T under the null hypothesis as:

$$\begin{aligned}
\sigma_T^2 &= Var[T] \\
&= Var[\hat{\mu}_X - \hat{\mu}_Y] \\
&= Var[\hat{\mu}_X + (-\hat{\mu}_Y)] \\
&= Var[\hat{\mu}_X] + Var[-\hat{\mu}_Y] \\
&= Var[\hat{\mu}_X] + (-1)^2 Var[\hat{\mu}_Y] \\
&= \frac{\sigma^2}{M} + \frac{\sigma^2}{N} \\
&= \left(\frac{1}{M} + \frac{1}{N}\right)\sigma^2
\end{aligned}$$

Finally, we can compute the probability of observing a difference in means as large as $t = \bar{x} - \bar{y}$. For convenience of discussion, assume $\bar{x} > \bar{y}$:

Let t be the observed difference $\bar{x} - \bar{y} > 0$.

Hypothesis test:

- What is $P(\text{see result as extreme under } H_0)$
  - One-sided Hypothesis test:

$$P(T \geq t | H_0) = Q\left(\frac{t - \mu_T}{\sigma_T}\right) = Q\left(\frac{t}{\sigma\sqrt{\frac{1}{M} + \frac{1}{N}}}\right)$$

```
* Two-sided Hypothesis test:
```

$$P(|T| \geq t|H_0) = 2Q\left(\frac{t}{\sigma\sqrt{\frac{1}{M}+\frac{1}{N}}}\right)$$

Z-Test A Z-test is any statistical test for which the distribution of the test statistic under the null hypothesis can be approximated by a normal distribution. Z-tests test the mean of a distribution. * Let $\hat{\mu}_X$ and $\hat{\mu}_Y$ be the sample means of random samples of sizes M and N from two RVs X and Y, respectively, with common variance σ^2. We can build the statistic:

$$T = \hat{\mu}_X - \hat{\mu}_Y$$

where $E[T] = 0$, $\mathrm{Var}[X] = \sigma^2\left(\frac{1}{N}+\frac{1}{M}\right)$ and

$$T \sim G\left(0, \sigma^2\left(\frac{1}{N}+\frac{1}{M}\right)\right)$$

In [ ]:

Example 1 The city of Gainesville claims the mean commute time on SW 24th Ave from I-75 to UF is 23 minutes with a variance of 50. You traveled that route 10 times over the last two weeks and had an average commute time of 27 minutes. Conduct a hypothesis test to determine whether the City of Gainesville's model is reasonable. Reject the null hypothesis if $p < 0.01$.

1. What is the null hypothesis? Define the density under H_0.

1. Compute the sample mean, $\hat{\mu}$. Compute the bias and variance of the estimator $\hat{\mu}$.

1. What is the probability that observe a result this extreme, i.e., $P(\hat{\mu} \geq 27)$? Compute the one-sided and the two-sided hypothesis test probabilities.

In [ ]:

In [ ]:

In [ ]:

Conclusion:
