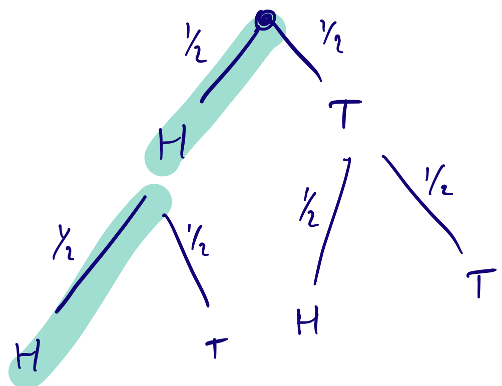
Example 1 : flip a fair coin 2 times.

$$\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$$



Fair Experiment: outcome probabilities are equal to each other

$$P((H,H)) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P((H,T)) = P((T,H)) = P((T,T)) = \frac{1}{4}$$

Example 2 : roll a 6-sided fair die 2 times

E = 1 or 2 on <u>either</u> roll

Roll 1 \ Roll 2	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	.			
4	(4,1)	(4,2)		.		
5	(5,1)	(5,2)			.	
6	(6,1)	(6,2)			.	(6,6)

$$P(E) = \frac{|E|}{|\Omega|} = \boxed{\frac{20}{36}}$$

OR

E_1 ≡ observe a 1 or 2 on roll 1

E_2 ≡ " " " " " " " 2

$E_1 \cap E_2$

Roll 2

Roll 1	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	.			
4	(4,1)	(4,2)		.		
5	(5,1)	(5,2)			.	
6	(6,1)	(6,2)			.	(6,6)

E_1

E_2

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{|E_1|}{|\Omega|} + \frac{|E_2|}{|\Omega|} - \frac{|E_1 \cap E_2|}{|\Omega|}$$

$$= \frac{12}{36} + \frac{12}{36} - \frac{4}{36}$$

$$= \boxed{\frac{20}{36}}$$

Example 3 : do by yourselves !

$E \equiv$ obs. a 4 in at least 1 roll when you roll a fair die 2 times.

Roll 2

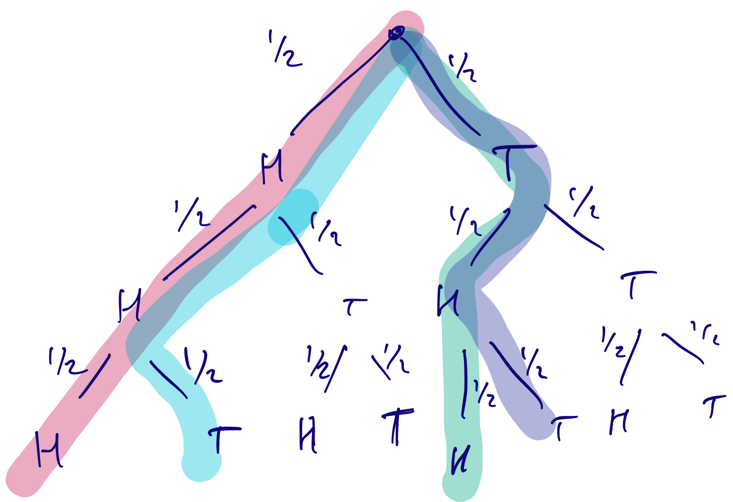
		1	2	3	4	5	6
Roll 1	1				(1,4)		
	2				(2,4)		
	3				(3,4)		
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5				(5,4)		
	6				(6,4)		

E

$$P(E) = \frac{|E|}{|\Omega|} = \frac{11}{36}$$

Example 4 : Flip a fair coin 3 times

$E \equiv$ observing heads in the 2^{nd} flip.



1st flip

2nd flip

3rd flip

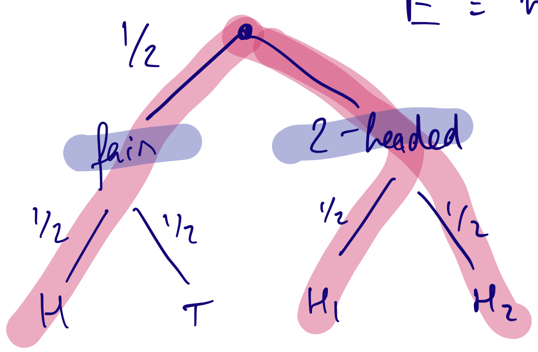
$E = \{(H,H,H), (H,H,T), (T,H,H), (T,H,T)\}$

$P(E) = \frac{|E|}{|\Omega|} = \frac{4}{8} = \frac{1}{2}$

Example 6: 1 fair coin, 1 two-headed coin

chooses 1 at random & flips.

$E \equiv$ heads



select coin at random

$\Omega = \{(\text{fair}, H), (\text{fair}, T), (\text{2-headed}, H_1), (\text{2-headed}, H_2)\}$

$P(E) = P(H) = \frac{3}{4}$

NOT a fair experiment

$P(T) = \frac{1}{4}$

Example 7: $P(H_2 | H_1)$

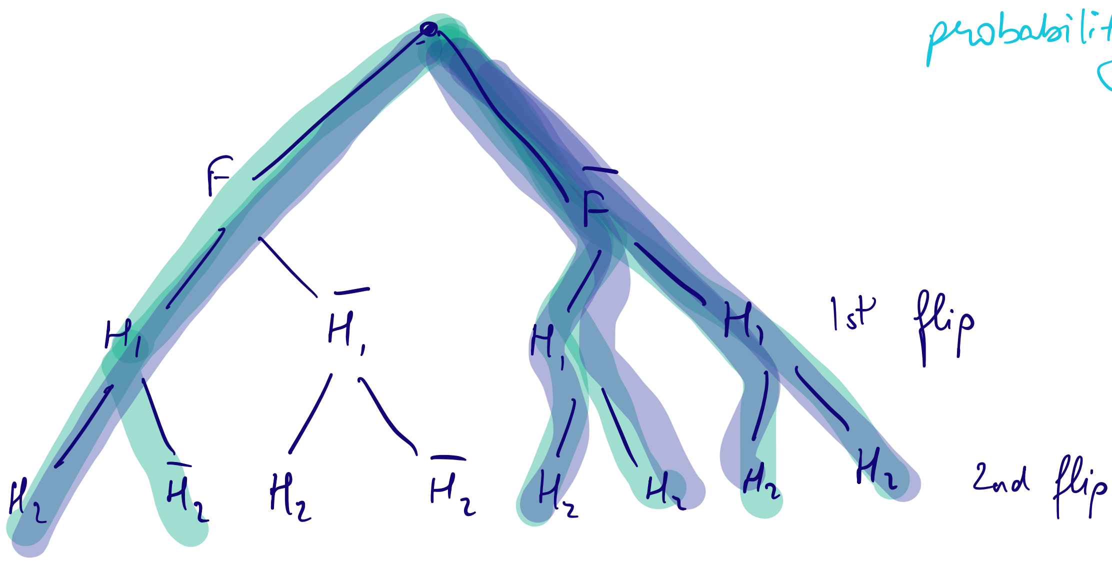
$F \equiv$ coin selected is fair

$\bar{F} \equiv$ coin " " 2-headed

$H_i \equiv$ obs. heads in flip i.

$P(H_2$ given that $H_1) = P(H_2 | H_1) = 5/6$

conditional probability



Conditional Probability

$A, B \in \mathcal{F}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$P(B) > 0$

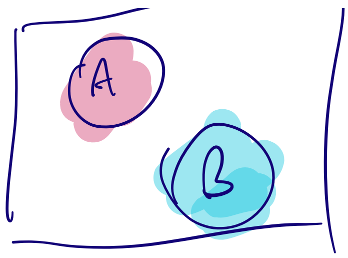
Note: if $P(B) > 0$, then conditional probability is a well-defined probability measure, which means it will obey all axioms & corollaries.

Which is it?

1. $P(A|B) \geq P(A)$
2. $P(A|B) \leq P(A)$
3. Not necessarily 1 or 2 ✓

Case 1

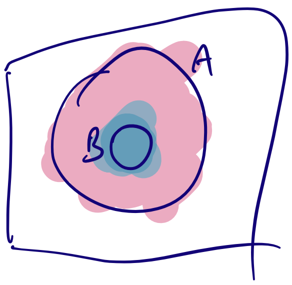
A and B are M.E., $A \cap B = \emptyset$



$$P(A|B) \triangleq \frac{P(A\cap B)}{P(B)} = 0 \le P(A)$$

Case 2

$B \subset A$

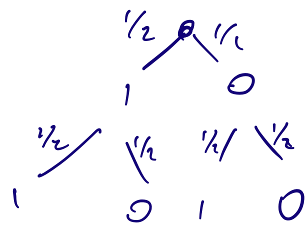


$$P(A|B) \triangleq \frac{P(A\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1 \geq P(A)$$

"By definition"

Example 4:

XOR of 2 indep. bin. values



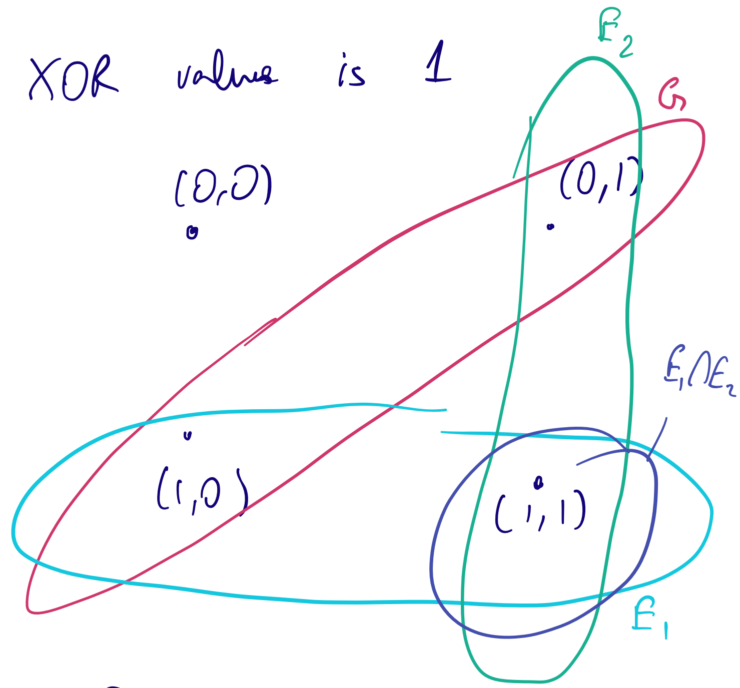
input 1	input 2	XOR ("exclusive OR")	Prob
0	0	0	1/4
0	1	1	1/4
1	0	1	1/4
1	1	0	1/4

Flipping a fair coin 2x, labeling 0 - tails
1 - heads

$E_i \equiv$ flipping heads (1) on flip i

$F_i \equiv$ " tails (0) " " "

$G \equiv$ event that XOR value is 1



$P(E_1) = \frac{1}{2}$

$P(E_2) = 1/2$

$P(G) = 1/2$

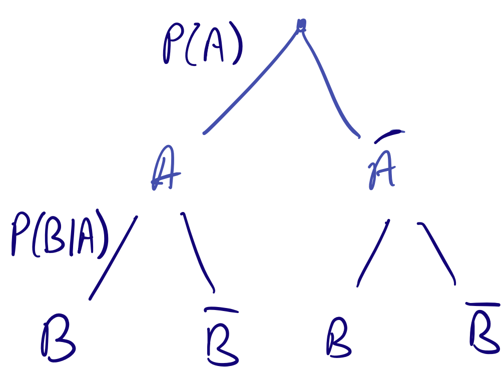
$P(E_1 \mid E_2) = \frac{1}{2}$

$P(E_2 \mid E_1) = 1/2$

$P(\underbrace{G}_{\text{M.E.}} \mid \underbrace{E_1 \cap E_2}_{\checkmark}) = \frac{0}{1} = 0$

Chain rule (probability)

$A, B \in \mathcal{F}$



CHAIN RULE

$$P(B|A) \triangleq \frac{P(B\cap A)}{P(A)} \iff \boxed{P(B\cap A) = P(B|A)\,P(A)}$$

$$P(A|B) \triangleq \frac{P(A\cap B)}{P(B)} \iff \boxed{P(A\cap B) = P(A|B)\,P(B)}$$

$$\boxed{P(A\cap B) = P(B\cap A)}$$

$$\Rightarrow P(B|A)\,P(A) = P(A|B)\,P(B)$$

$$P(A\cap B\cap C) = P(A)\,P(B|A)\,P(C|A\cap B)$$

$$= P(A)\,\frac{P(B\cap A)}{P(A)}\,\frac{P(C\cap A\cap B)}{P(A\cap B)}$$

<u>OR</u>

$$P(A\cap B\cap C) = P(A|B\cap C)\,P(B|C)\,P(C)$$

$$= \frac{P(A\cap B\cap C)}{P(B\cap C)}\,\frac{P(B\cap C)}{P(C)}\,P(C)$$

<u>Generalization</u> : Multiplication Rule

$$P\left(\bigcap_{i=1}^{N} A_i\right) = P(A_1)\,P(A_2|A_1)\,P(A_3|(A_1\cap A_2))\ldots$$

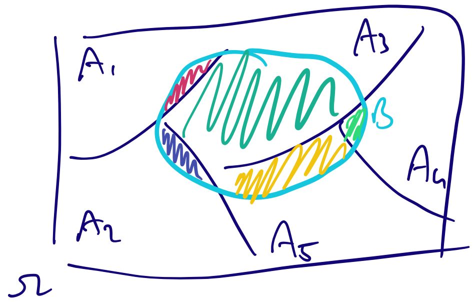
$$\ldots P\left(A_N \middle| \bigcap_{i=1}^{N-1} A_i\right)$$

Law of Total Probability

Let $A_1, A_2, \ldots, A_n$ partitions of sample space.

$\Omega = \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$

$A_i \cap A_j = \emptyset \quad \forall i \neq j$



$B = (B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3) \cup (B \cap A_4) \cup (B \cap A_5)$

Note that $B \cap A_i$ and $B \cap A_j$ are M.E., $\forall i \neq j$

That means :

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_5)$$

$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots$$

chain rule!

Law of Total Probability

If $\{A_i\}_{i=1}^{n}$ is a set of partitions of sample space Ω

then $\boxed{P(B) = \sum_{i=1}^{n} P(B|A_i)\,P(A_i)}$