2)

AR(3) model: $y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \varepsilon_t$

<u>Unconditional Mean</u> :

$$E(y_t) = E(c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \varepsilon_t)$$

$$E(y_t) = E(c) + \phi_1 E(y_{t-1}) + \phi_2 E(y_{t-2}) + \phi_3 E(y_{t-3}) + E(\varepsilon_t)$$

$$E(y_t) = c + \phi_1 E(y_{t-1}) + \phi_2 E(y_{t-2}) + \phi_3 E(y_{t-3}) + 0$$

since $\varepsilon_t \sim N(0,1)$

$$E(y_t)(1 - \phi_1 - \phi_2 - \phi_3) = c$$

$y_t = y_{t-1} = \dots = y_{t-p}$ by stationarity

$$E(y_t) = \frac{c}{(1 - \phi_1 - \phi_2 - \phi_3)}$$

Optimal forecast for $h=1$:

$$f_{t,1} = \mu_{t+1|t}$$

$$= E(y_{t+1} | I_t)$$

$$= E(c + \phi_1 y_t + \phi_2 y_{t-1} + \phi_3 y_{t-2} + \varepsilon_{t+1} | I_t)$$

$$= E(c) + \phi_1 E(y_t) + \phi_2 E(y_{t-1}) + \phi_3 E(y_{t-2}) + E(\varepsilon_{t+1})$$

$$= c + \phi_1 y_t + \phi_2 y_{t-1} + \phi_3 y_{t-2}$$

Optimal forecast for $h=2$:

$$f_{t,2} = \mu_{t+2|t}$$

$$= E(Y_{t+2} | I_t)$$

$$= E(C + \phi_1 y_{t+1} + \phi_2 y_t + \phi_3 y_{t-1} + \varepsilon_{t+2} | I_t)$$

$$= E(C) + \phi_1 E(y_{t+1} | I_t) + \phi_2 E(y_t) + \phi_3 E(y_{t-1}) + E(\varepsilon_{t+2})$$

$$= C + \phi_1 (C + \phi_1 y_t + \phi_2 y_{t-1} + \phi_3 y_{t-2}) + \phi_2 y_t + \phi_3 y_{t-1}$$

$$= C(1+\phi_1) + \phi_1^2 y_t + \phi_1 \phi_2 y_{t-1} + \phi_1 \phi_3 y_{t-2} + \phi_2 y_t + \phi_3 y_{t-1}$$

$$= C + \phi_1 (f_{t,1}) + \phi_2 y_t + \phi_3 y_{t-1}$$

Optimal forecast for $h=3$:

$$f_{t,3} = \mu_{t+3|t}$$

$$= E(Y_{t+3} | I_t)$$

$$= E(C + \phi_1 y_{t+2} + \phi_2 y_{t+1} + \phi_3 y_t + \varepsilon_{t+3} | I_t)$$

$$= E(C) + \phi_1 E(y_{t+2} | I_t) + \phi_2 E(y_{t+1} | I_t) + \phi_3 E(y_t) + E(\varepsilon_{t+3})$$

$$= C + \phi_1 (C(1+\phi_1) + \phi_1^2 y_t + \phi_1\phi_2 y_{t-1} + \phi_1\phi_3 y_{t-2} + \phi_2 y_t + \phi_3 y_{t-1})$$
$$+ \phi_2 (C + \phi_1 y_t + \phi_2 y_{t-1} + \phi_3 y_{t-2}) + \phi_3 y_t$$

$$= C + \phi_1 (f_{t,2}) + \phi_2 (f_{t,1}) + \phi_3 y_t$$