Introduction — Strategic games

- A model of interacting decision-makers a.k.a *players*
- Each player has a set of possible *actions*, or *strategies*
- Typically, each player is affected by actions of all players – modelling of interaction
- Each player has *preferences* about the action *pro£le* – usually modelled as the *payoff functions*

Remark: the payoffs are assumed to have only ordinal sign£cance, i.e. the only thing that matters is the ordering, not the relative size of payoffs.

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Introduction — Strategic games

- Time is absent from the model:
  - each player chooses his/her actions once and for all
  - all players choose their actions “simultaneously”, i.e. without any knowledge of the others' choices
  - however, the players' actions can in principle involve unlimited contingencies

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Introduction — Strategic games

1 Prisoner's Dilemma...

- *Players:* the two suspects
- *Strategies:* each player's possible strategies: {*Confess, Don't confess*}
- *Preferences:*
  - Player 1: $\{C,NC\},\{NC,NC\},\{C,C\},\{NC,C\}$
  - Player 2: $\{NC,C\},\{NC,NC\},\{C,C\},\{C,NC\}$

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Introduction — Strategic games

...Prisoner's Dilemma...

- Table form:

	Do not confess	Confess
Do not confess	2,2	0,3
Confess	3,0	1,1

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Introduction — Strategic games

...Prisoner's Dilemma...

Example: *Working on a joint project*

(Two people want to decide on their level of effort)

	Work hard	Goof off
Work hard	2,2	0,3
Goof off	3,0	1,1

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Introduction — Strategic games

...Prisoner's Dilemma

Example: *Duopoly*

(Two £rms want to decide what price to charge for a product)

	High	Low
High	1500,1500	-500,2000
Low	2000,-500	1000,1000

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Introduction — Strategic games
2 Bach or Stravinsky...

- *Players:* two people
- *Strategies:* each player's possible actions: {*Bach*, *Stravinsky*}
- *Preferences:*
  - Player 1: $\{B, B\}, \{S, S\}, \{\{B, S\}$, or $\{S, B\}\}$
  - Player 2: $\{S, S\}, \{B, B\}, \{\{B, S\}$, or $\{S, B\}\}$

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Introduction — Strategic games
...Bach or Stravinsky...

Payoff table representation of BoS:

	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

a couple go to concert
one likes Bach.
another likes Stravinsky

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Introduction — Strategic games
...Bach or Stravinsky

Example: *Merging £rms*

(Two £rms want to decide whose technology to use)

	First	Second
First	2,1	0,0
Second	0,0	1,2

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Introduction — Strategic games
3 "Chicken"...

- *Players:* two drivers
- *Strategies:* each player's possible strategies: {*Continue* , *Stop*}
- *Preferences:*
  - Player 1: $\{C, S\}, \{S, S\}, \{S, C\}, \{C, C\}$
  - Player 2: $\{S, C\}, \{S, S\}, \{C, S\}, \{C, C\}$

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Introduction — Strategic games
..."Chicken"...

Payoff table representation of "Chicken":

	Continue	Stop
Continue	-10,-10	5,0
Stop	0,5	1,1

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Introduction — Strategic games
..."Chicken"

Example: *Air £ght*

(Two £ghter pilots on head-on collision course want to decide what to do)

	Continue	Turn away
Continue	0,0	0.9,0.1
Turn away	0.1,0.9	0.5,0.5

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Introduction

Strategic games

4 Matching pennies...

- Example of a zero-sum game (purely con¤ictual)
- *Players:* two people
- *Strategies:* each player's possible strategies: {*Head*, *Tail*}
- *Preferences:*
  - Player 1: $\{\{H,H\}$, or $\{T,T\}\}$, $\{\{H,T\}$, or $\{T,H\}\}$
  - Player 2: $\{\{H,T\}$, or $\{T,H\}\}$, $\{\{H,H\}$, or $\{T,T\}\}$

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Introduction

Strategic games

...Matching pennies...

Payoff table representation of Matching pennies:

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

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Introduction

Strategic games

...Matching pennies

Example: *New product*

(Two £rms (established and new) want to decide on the appearance (color) of a new product in a £xed size market)

	Yellow	Orange
Yellow	1100,900	1700,300
Orange	1700,300	1100,900

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Introduction

Strategic games

5 Stag Hunt...

- *Players:* two hunters
- *Strategies:* each player's possible strategies: {*Stag*, *Hare*}
- *Preferences:*
  - Player 1: $\{S,S\}$, $\{\{H,H\}$, or $\{H,S\}\}$, $\{S,H\}$
  - Player 2: $\{S,S\}$, $\{\{H,H\}$, or $\{S,H\}\}$, $\{H,S\}$

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Introduction

Strategic games

...Stag Hunt

Payoff table representation of Stag Hunt:

	Stag	Hare
Stag	2,2	0,1
Hare	1,0	1,1

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Introduction

Nash Equilibrium

- Assumptions:
  - rational players (always choose the best available action given their beliefs)
  - the beliefs a player has about others are correct (given their rationality)
- De£nition: The (pure) action pro£le s^* is a Nash equilibrium if for every player i,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

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Introduction — Nash Equilibrium

- Remarks:
  - Here (and later) the index "$-i$" stands for "all players except i"
  - The Nash equilibrium as de£ned above may not exist for a strategic game
  - There may be more than one Nash equilibrium

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Introduction — Nash Equilibrium — Examples...

Prisoner's Dilemma

	Do not confess	Confess
Do not confess	2,2	0,3
Confess	3,0	1,1

- The only Nash equilibrium is (Confess, Confess)
- In fact, the action "Do not Confess" is *strictly dominated*

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Introduction — Nash Equilibrium — ...Examples...

Bach or Stravinsky

	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

- There are two Nash equilibria. Which?

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Introduction — Nash Equilibrium — ...Examples...

"Chicken"

	Continue	Stop
Continue	-10,-10	5,0
Stop	0,5	1,1

- What is the Nash equilibrium?

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Introduction — Nash Equilibrium — ...Examples...

Matching Pennies

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

- What is the Nash equilibrium?

None

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Introduction — Nash Equilibrium — ...Examples

Stag Hunt

	Stag	Hare
Stag	2,2	0,1
Hare	1,0	1,1

- What is the Nash equilibrium?

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Introduction — Nash Equilibrium: Strict equilibria

De£nition: The (pure) strategy pro£le s^* is a *strict* Nash equilibrium if for every player i,

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)$$

Example:

	L	M	R
U	1,1	1,0	0,1
D	1,0	0,1	1,0

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Introduction — Nash Equilibrium: Best response functions...

- If a player has many (possibly in£nitely many) possible strategies, it may be dif£cult to examine each strategy pro£le directly
- For each strategy pro£le of all other players s_{-i}, de£ne the *best response function* as

$$B(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i\}$$

$b_2(S) = G$

$b_2(D) = B$

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Ex : basketball

P1 : offence

P2 : deffence

S_1 = {shoot, Dribble}

S_2 = {get closer, Back off}

Introduction — Nash Equilibrium: ...Best response functions...

- Best response functions are in general *set valued*, i.e. there may be more than one element in the set $B(s_{-i})$.
- We can now de£ne Nash equilibrium via best response functions:

Proposition: The strategy pro£le s^* is a Nash equilibrium of a strategic game if and only if

$$s_i^* \in B(s_{-i}^*) \text{ for every player } i$$

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Introduction — Nash Equilibrium: ...Best response functions...

Example:

	L	C	R
U	1,2*	2*,1	1*,0
M	2*,1*	0, 1*	0, 0
D	0,1	0,0	1*,2*

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Introduction — Nash Equilibrium: ...Best response functions...

- For a game with two players such that every set $B(s_{-i})$ consists of one element, we can £nd Nash equilibria by solving the system of equations:

$$s_1^* = b_1(s_2^*)$$

$$s_2^* = b_2(s_1^*)$$

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Introduction — Nash Equilibrium: ...Best response functions...

Example: *Synergistic relationship*

- *Players:* two individuals
- *Strategies:* each player's possible strategies: nonnegative numbers (effort levels)
- *Preferences:* represented by payoff function

$$u_i(a) = a_i(c + a_j - a_i), \quad i = 1, 2$$

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$$u_i'(a) = \frac{\partial u_i}{\partial a_i} = 0 \Rightarrow \underbrace{b_i(a_j)}_{\uparrow} = \frac{1}{2}(c + a_j).$$

对 a_j 的 i 的 best response

$$a_i^* = b_i(a_j)$$

Introduction
Nash Equilibrium
...Best response functions

- Best response function:

$$b_i(a_j) = \frac{1}{2}(c + a_j)$$

- System of equations:

$$a_1 = \frac{1}{2}(c + a_2)$$

$$a_2 = \frac{1}{2}(c + a_1)$$

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NE is (C, C).

Introduction
Nash Equilibrium
Dominated strategies...

Deﬁnition: In a strategic game, the player i's strategy s_i' *strictly dominates* another strategy s_i'' if

$$u_i(s_i', s_{-i}) > u_i(s_i'', s_{-i}) \text{ for all } s_{-i}$$

Example: The Prisoner's Dilemma (the strategy *Confess* strictly dominates *Do not Confess*)

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Introduction
Nash Equilibrium
...Dominated strategies...

Deﬁnition: In a strategic game, the player i's strategy s_i' *weakly dominates* another strategy s_i'' if

$$u_i(s_i', s_{-i}) \geq u_i(s_i'', s_{-i}) \text{ for all } s_{-i}$$

and there exists an strategy proﬁle s_{-i} such that

$$u_i(s_i', s_{-i}) > u_i(s_i'', s_{-i})$$

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Introduction
Nash Equilibrium
...Dominated strategies...

- Proposition: A *strictly dominated* strategy cannot be used in any Nash equilibrium
- Proposition: A *weakly dominated* strategy cannot be used in any *strict* Nash equilibrium
- However, a *weakly dominated* strategy can be used in a *nonstrict* Nash equilibrium

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Introduction
Nash Equilibrium
...Dominated strategies...

Examples:

strict NE

strict NE

	L	R
U	1,1	0,0
D	0,0	0,0

weekly dominate L

non-strict NE

	L	R
U	1,1	2,0
D	0,2	2,2

non-strict NE

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Introduction
Nash Equilibrium
...Dominated strategies

Example: *Collective decision-making*

- *Players:* n people
- *Strategies:* each person's set of possible strategies is the set of policies (real numbers)
- *Preferences:* each person prefers strategy proﬁle s to s' if and only if the median policy in s is closer to the number x_i^* than that in s'

Claim: The strategy of naming x_i^* weakly dominates all other strategies for each player i

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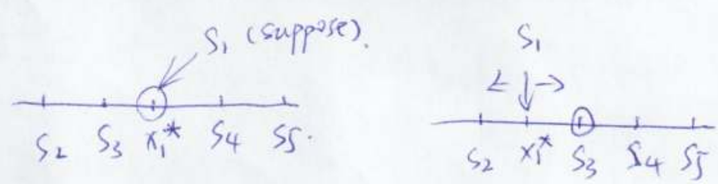
x_i^* is player i's favorite policy



one firm ← Monopoly situation

$u(q_1, q_2) = (q_1+q_2)(\alpha - q_1 - q_2) - \beta(q_1+q_2)$

08/20/2015

$= q(\alpha - q) - \beta q.$

$u'(q) = \alpha - q - q - \beta = 0$

$q = \frac{\alpha-\beta}{2}$ $\quad q_1^P = q_2^P = \frac{\alpha-\beta}{4}.$

Introduction — Nash Equilibrium — Cournot's duopoly model...

- Two £rms produce a single good
- The i-th £rm's cost of producing q_i units of the good is $c_i(q_i)$
- All the output is sold at a single market clearing price $p(q)$
- The i-th £rm's payoff function is given by the pro£t:

$$u_i(q_1, q_2) = q_i p(q_1 + q_2) - c_i(q_i)$$

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Introduction — Nash Equilibrium — ...Cournot's duopoly model...

- Assumptions: the payoff function u_i is strictly concave and the appropriate boundary conditions are satis£ed
- The best response functions are given by:

$$\frac{du_i}{dq_i} = 0$$

or

$$p(q_j + b_i(q_j)) + p'(q_j + b_i(q_j))b_i(q_j) - c_i'(b_i(q_j)) = 0$$

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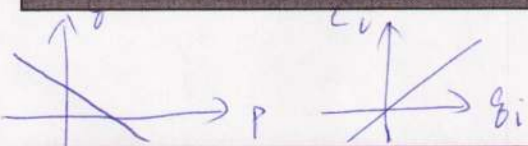

Introduction — Nash Equilibrium — ...Cournot's duopoly model...

Example: *Linear demand and cost*

- Demand function: $p(q) = \max\{0, \alpha - q\}$
- Cost function: $c_i(q_i) = \beta q_i$
- We obtain for the best response:

$$b_1(q_2) = (\alpha - q_2 - \beta)/2$$

$$b_2(q_1) = (\alpha - q_1 - \beta)/2$$

Nash equilibrium: $q_1^* = q_2^* = (\alpha - \beta)/3$

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帕累托均衡.

Introduction — Nash Equilibrium — ...Cournot's duopoly model

- The Nash equilibrium is not Pareto-optimal
- The Pareto-optimal symmetric solution has

$$q_1^P = q_2^P = (\alpha - \beta)/4$$

- This is a generic situation in games

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Introduction — Nash Equilibrium — Hotelling competition...

- A linear city of lengths 1, uniformly populated, with unit travel cost of t
- Two stores at the ends, each with unit cost c
- People buy from the store which gives them the lowest overall cost
- The stores want to decide on prices

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Introduction — Nash Equilibrium — Hotelling competition...

- If p_i is the price set by the store i than the demands of the stores 1 and 2 are equal to

$$D_1(p_1, p_2) = \frac{p_2 - p_1 + t}{2t}$$

$$D_2(p_1, p_2) = \frac{p_1 - p_2 + t}{2t}$$

- The payoffs are given by pro£ts:

$$u_1(p_1, p_2) = \frac{(p_1 - c)(p_2 - p_1 + t)}{2t}$$

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Introduction

Nash Equilibrium

...Hotelling competition...

$$u_2(p_1, p_2) = \frac{(p_2 - c)(p_1 - p_2 + t)}{2t}$$

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Introduction

Nash Equilibrium

...Hotelling competition

- Best responses:

$$b_1(p_2) = \frac{p_2 + t + c}{2}$$

$$b_2(p_1) = \frac{p_1 + t + c}{2}$$

- Solving for p_i^*, we obtain:

$$p_1^* = p_2^* = t + c$$

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Introduction

Nash Equilibrium

Symmetric games and equilibria...

De£nition: A two-player strategic game is *symmetric* if the players' sets of strategies are the same and the players' preferences are represented by payoff functions for which $u(s_1, s_2) = u_2(s_2, s_1)$ for every strategy pro£le (s_1, s_2).

Example:(*Symmetric game with two actions*)

	A	B
A	w, w	x, y
B	y, x	z, z

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Introduction

Nash Equilibrium

...Symmetric games and equilibria...

De£nition: A strategy pro£le s^* in as strategic game in which each player has the same set of strategies is a *symmetric Nash equilibrium* if it is a Nash equilibrium and a_i^* is the same for every player i.

- A symmetric game may have no symmetric equilibrium
- A nonsymmetric equilibrium in a symmetric game may not correspond to a steady state if the players are drawn from the same population

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Introduction

Nash Equilibrium

...Symmetric games and equilibria

Examples:

	A	B
A	1,1	0,0
B	0,0	1,1

	A	B
A	0,0	1,1
B	1,1	0,0

	A	B	C
A	1,1	2,1	4,1
B	1,2	5,5	3,6
C	1,4	6,3	0,0

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Introduction

Nash Equilibrium

War of attrition...

- *Players:* two parties
- *Strategies:* set of possible concession times
- *Preferences:* represented by the payoff function

$$u_i(t_1, t_2) = \begin{cases} -t_i & \text{if } t_i < t_j \\ \frac{1}{2}v_i - t_i & \text{if } t_i = t_j \\ v_i - t_j & \text{if } t_i > t_j \end{cases}$$

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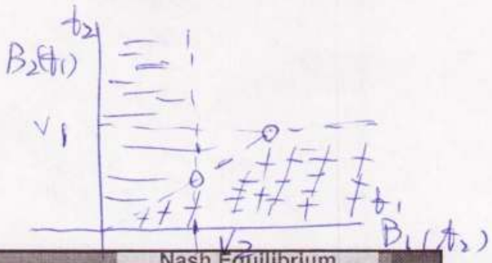
$V_1 > V_2$.



Introduction — Nash Equilibrium — ...War of attrition...

- Best response functions:

$$B_i(t_j) = \begin{cases} \{t_i \mid t_i > t_j\} & \text{if } t_j < v_i \\ \{t_i \mid t_i = 0 \text{or} t_i > t_j\} & \text{if } t_j = v_i \\ \{0\} & \text{if } t_j > v_i \end{cases}$$

- Nash equilibria:
  - $-t_1 = 0$ and $t_2 \geq v_1$
  - $-t_2 = 0$ and $t_1 \geq v_2$

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Introduction — Nash Equilibrium — ...War of attrition

- There is no £ght in an equilibrium
- The player with a higher valuation may concede £rst
- The equilibria are asymmetric: players have to be from different populations in order for us to interpret the equilibria as steady states

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Introduction — Nash Equilibrium — Second-price auction...

- n bidders, v_i is the value player i attaches to the object
- The valuations are positive and different, assume that $v_1 > v_2 > \ldots > v_n > 0$
- Each bidder submits a sealed bid, and the bidder with the highest bid gets the object at a price equal to the second highest bid
- Assume that ties are broken in favor of bidder with lower number

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Introduction — Nash Equilibrium — ...Second-price auction...

- *Players:* n bidders
- *Strategies:* each person's set of possible strategies is the set of possible bids (nonnegative numbers)
- *Preferences:* described by payoff functions:

$$u_i(b) = \begin{cases} v_i - \bar{b} & \text{if player } i \text{ gets the object} \\ 0 & \text{otherwise} \end{cases}$$

where $\bar{b}$ is the second highest bid

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Introduction — Nash Equilibrium — ...Second-price auction...

Equilibria:

- $(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$
- $(b_1, \ldots, b_n) = (v_1, 0, \ldots, 0)$
- $(b_1, \ldots, b_n) = (v_2, v_1, \ldots, 0)$ – player 2 gets the object
- Can we £nd an equilibrium in which player n gets the object?

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Introduction — Nash Equilibrium — ...Second-price auction

- Which of the above equilibria should we expect to realize in practice?
- Claim: In a second price sealed bid auction, a player's bid equal to her valuation weakly dominates all her other bids

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player k $(b_1, b_2, \cdots, b_n) = (0, 0, \cdots, 0, V_1)$.

$P_k = V_k$ 1). win object ($\Rightarrow b_k > \bar{b}$) payoff $= V_k - \bar{b} \geq 0$.

payoff can't increase by changing b_k.

2). doesn't win, payoff $= 0$.

Introduction — Nash Equilibrium — First-price auction...

- *Players:* n bidders
- *Strategies:* each person's set of possible strategies is the set of possible bids (nonnegative numbers)
- *Preferences:* described by payoff functions:

$$u_i(b) = \begin{cases} v_i - b_i & \text{if player } i \text{ gets the object} \\ 0 & \text{otherwise} \end{cases}$$

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Introduction — Nash Equilibrium — ...First-price auction...

Equilibria:

- $(b_1, \ldots, b_n) = (v_1, v_1, v_3, \ldots, v_n)$
- $(b_1, \ldots, b_n) = (v_2, v_2, v_3, \ldots, v_n)$
- Any equilibria in which player 2 gets the object? Player n?

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Introduction — Nash Equilibrium — ...First-price auction...

Claim: In a Nash equilibrium, the two highest bids are the same, one of them being submitted by player 1, and the highest bid is between v_2 and v_1 (inclusive).
Claim: A player's bid of at least her valuation is weakly dominated, and a bid of less than the valuation is not weakly dominated

Conclusion: The equilibrium $(b_1, \ldots, b_n) = (v_2, v_2, v_3, \ldots, v_n)$ in a £rst-price sealed bid auction is *distinguished* (in what sense?)

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All games can be

	complete info	incomplete info
static	NE →	BE ↓
dynamic	↓ SPE	PBE

PRS Game

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

Static Games with Complete Information — Definitions

- A *strategic game* consists of
  - a finite set $\mathcal{I} = \{1, 2, \ldots, I\}$ (set of players)
  - for each player $i \in \mathcal{I}$, a nonempty set S_i (set of strategies available to player i)
  - for each player $i \in \mathcal{I}$, a preference relation $\succeq_i$ on $S = \times_{i \in \mathcal{I}} S_i$
- Sometimes we will denote such a game by $(\mathcal{I}, (S_i), (\succeq_i))$
- A game is *finite* if all sets S_i are finite

1

Static Games with Complete Information — Definitions

- Very often the preference relation $\succeq_i$ in a strategic game can be represented by a *payoff function* u_i: $S \to R$ such that $u_i(s) \geq u_i(s')$ whenever $s \succeq_i s'$. We will denote such a game by $(\mathcal{I}, (S_i), (u_i))$
- *A Nash equilibrium of a strategic game* $(\mathcal{I}, (S_i), (\succeq_i))$ is a strategy profile $s^* \in S$ such that for each player $i \in \mathcal{I}$

$$(s_i^*, s_{-i}^*) \succeq_i (s_i, s_{-i}^*)$$

for all $s_i \in S_i$

2

Static Games with Complete Information — Definitions

- The best response function of the player i is the set-valued function defined by

$$B_i(s_{-i}) = \{s_i \in S_i : (s_i, s_{-i}) \succeq_i (s_i', s_{-i}) \; \forall s_i' \in S_i\}$$

- A Nash equilibrium is a strategy profile s^* such that

$$s_i^* \in b_i(s_{-i}^*) \; \forall i \in \mathcal{I}$$

S_i 孩子的 strategy.

3

Static Games with Complete Information — Existence of Nash equilibrium

Theorem: The strategic game $(\mathcal{I}, (S_i), (\succeq_i))$ has a Nash equilibrium if for all $i \in \mathcal{I}$

1) the set S_i for each player is a nonempty compact convex subset of a Euclidian space

2) (technical) the preference relation $\succeq_i$ is continuous and quasi-concave on S_i (e.g. is described by a concave function)

4

Static Games with Complete Information — Existence of Nash equilibrium

Consequence: A finite strategic game may not have a Nash equilibrium (Example: "Matching pennies"). To ensure all such games have NE we need to extend to probabilistic strategies (lotteries).

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Static Games with Complete Information — Mixed strategy equilibrium

- The players choices are selected probabilistically
- Need to extend the concept of preferences to lotteries on the set S
- For convenience, we use the payoff function description of the preference structure

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Example : RPS mixed strategies.

$\delta_1(R) = 0.5 \quad \delta_1'(R) = \frac{1}{3}$

$\delta_1(P) = 0.3 \quad \delta_1'(P) = \frac{1}{6}$

$\delta_1(S) = 0.2 \quad \delta_1'(S) = \frac{1}{2}$.

$\delta_1 = (0.5, 0.3, 0.2)$

$\delta' = (\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$

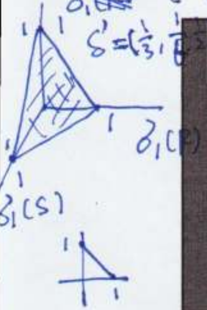
RPS game.

$\delta_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

$\delta_2 = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$.

$U_1(\delta_1, \delta_2) = -\frac{1}{9} + \frac{1}{18} + \frac{1}{6} - \frac{1}{18} - \frac{1}{6} + \frac{1}{9} = 0$

	R	P	S
R	$0,0$ $\frac{1}{6}$	$-1,1$ $\frac{1}{9}$	$1,-1$ $\frac{1}{18}$
P	$1,-1$ $\frac{1}{6}$	$0,0$ $\frac{1}{9}$	$-1,1$ $\frac{1}{18}$
S	$-1,1$ $\frac{1}{6}$	$1,-1$ $\frac{1}{9}$	$0,0$ $\frac{1}{18}$



Static Games with Complete Information — Mixed strategy equilibrium

- Let $(\mathcal{I}, (S_i), (u_i))$ be a strategic game
- Denote $\Sigma_i \equiv \Delta(S_i)$ the set of all probability distributions over S_i. We call a member σ_i of S_i a *mixed strategy* of player i. We denote $\sigma_i(s_i)$ the probability σ_i assigns to s_i

7

Static Games with Complete Information — Mixed strategy equilibrium

- Assuming independence of players' decisions, a profile $\sigma = (\sigma_i)_{i \in \mathcal{I}}$ induces the probability distribution on set S:

$$P(s) = \prod_{j \in \mathcal{I}} \sigma_j(s_j)$$

- The player i's evaluation of the mixed strategy profile σ is

$$U_i(\sigma) = \sum_{s \in S} \left(\prod_{j \in \mathcal{I}} \sigma_j(s_j) \right) u_i(s)$$

8

Static Games with Complete Information — Mixed strategy equilibrium

Definition: The *mixed extension* of the strategic game $(\mathcal{I}, (S_i), (u_i))$ is the strategic game $(\mathcal{I}, (\Sigma_i), (U_i))$ where (Σ_i) is the set of probability distributions over S_i, and U_i: $\times_{j \in \mathcal{I}} \Sigma_j \to R$ assigns to each $\sigma \in \times_{j \in \mathcal{I}} \Sigma_j$ the expected value of the lottery over S induced by σ

Definition: A *mixed strategy Nash equilibrium of a strategic game* is a Nash equilibrium of its mixed extension

9

Static Games with Complete Information — Mixed strategy equilibrium — Properties...

Lemma: A set of pure strategy Nash equilibria of a strategic game is a subset of the set of its mixed strategy equilibria

Theorem (Nash): Every finite strategic game has a mixed strategy Nash equilibrium

10

Static Games with Complete Information — Mixed strategy equilibrium — ...Properties...

Lemma: σ^* is a mixed strategy Nash equilibrium of a finite strategic game if and only if for every player $i \in \mathcal{I}$ every pure strategy in the support of σ_i^* is a best response to σ^*_{-i}

Corollary: There can be no mixed strategy strict equilibrium

11

Static Games with Complete Information — Mixed strategy equilibrium — ...Properties

Corollary: To find all Nash equilibria of a finite strategic game one needs to check all possible supports of σ_i^* for all players (e.g. for 2 players with n strategies for each, that's $(2^n - 1)^2$ combinations)

!!

12

$\delta_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

$\delta_1' = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$

$\delta_1'' = (\frac{1}{2}, 0, \frac{1}{2})$.

$\delta_1''' = (1, 0, 0)$.

$\mathrm{supp}(\delta_1) = \{R, P, S\}$.

$\mathrm{supp}(\delta_1') = \{R, P, S\}$

$\mathrm{supp}(\delta_1'') = \{R, S\}$

$\mathrm{spp}(\delta_1''') = \{R\}$.

$C_n^1 + C_n^2 + \cdots + C_n^n = 2^n - 1$

about support

support possible !! F**k.

q H $\quad$ T $1-q$ $\quad$ support(σ_1) = supp(σ_2) = $\{H, T\}$.

p H	1,-1	-1,1
$1-p$ T	-1,1	1,-1

$\sigma_1(H)=p. \quad (p,1-p)$

$\sigma_2(H)=q \quad (q,1-q).$

$$\boxed{\begin{aligned} u_1(H,\sigma_2) &= u_1(T,\sigma_2) \\ u_2(\sigma_1, H) &= u_2(\sigma_1, T). \end{aligned}}$$

→ to be Var = 0.

$1\cdot q + (-1)(1-q) = (-1)\cdot q + 1(1-q)$

$-p + 1-p = p-(1-p).$ $\Rightarrow \sigma^* = \big(\underbrace{(\tfrac{1}{2},\tfrac{1}{2})}_{\sigma_1^*}, \underbrace{(\tfrac{1}{2},\tfrac{1}{2})}_{\sigma_2^*}\big)$

Static Games with Complete Information — Mixed strategy equilibrium

Examples...

Matching pennies

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

What are the mixed equilibria?

13

Static Games with Complete Information — Mixed strategy equilibrium

...Examples...

Bach or Stravinsky

	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

Any mixed equilibria?

14

Static Games with Complete Information — Mixed strategy equilibrium

...Examples...

Inspection game (assume $v > w > g > h > 0$)

	Inspect (q)	Do not inspect ($1-q$)
p Shirk	$0,-h$	$w,-w$
$1-p$ Work	$w-g, v-w-h$	$w-g, v-w$

What are the Nash equilibria? $0q + w(1-q) = q(w-g) + (1-q)(w-g).$

$\Rightarrow (1-q)w = w-g$

$w - wq = w-g$

$q = \frac{g}{w}$

$-hp + (v-w-h)(1-p)$

$= -pw + (1-p)(v-w)$

$p = \frac{h}{w}.$

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Static Games with Complete Information — Mixed strategy equilibrium

...Examples...

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

What are the Nash equilibria?

16

Static Games with Complete Information — Mixed strategy equilibrium

...Examples

As we mentioned before, finding all equilibria of larger finite games may get tedious very quickly

	L	M	R
U	2,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

What are the Nash equilibria?

17

Static Games with Complete Information — Mixed strategy equilibrium

Dominated strategies...

Definition: In a strategic game, the player i's strategy s_i is *strictly dominated* if there exists a mixed strategy σ_i such

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \text{for all } s_{-i}$$

18

这是被动语句 F**K.

$\sigma_1=(\frac{1}{2},0\ \frac{1}{2})$. $\Rightarrow u_1(\sigma_1,L)=\frac{1}{2}>0$

$u_1(\sigma_1,R)=\frac{1}{2}>0$

Static Games with Complete Information — Mixed strategy equilibrium

...Dominated strategies...

Definition: In a strategic game, the player i's strategy s_i is *weakly dominated* if there exists a mixed strategy σ_i such

$$u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_{-i}$$

and there exists an strategy profile s_{-i} such that

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$$

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Static Games with Complete Information — Mixed strategy equilibrium

...Dominated strategies...

- A pure strategy may be dominated by a mixed strategy even if it is not dominated by any pure strategy

Example:

	L	R
U	2,0	−1,0
M	0,0	0,0
D	−1,0	2,0

20

Static Games with Complete Information — Mixed strategy equilibrium

...Dominated strategies

- A mixed strategy assigning a positive probability to any pure dominated strategy is dominated
- A mixed strategy can be strictly dominated even if it assigns positive probabilities to pure strategies that are not even weakly dominated

Example:

	L	R
U	1,3	−2,0
M	−2,0	1,3
D	0,1	0,1

21

Static Games with Complete Information — Multiple equilibria

- Need some selection mechanism in order to have prediction power
  - Focal points
  - Pareto optimality

3,3	0,0
0,0	1,1

22

Static Games with Complete Information — Infinite games

- Pure strategy equilibria may not exist if the strategy spaces are not convex or the payoff functions are not quasi-concave
- Just like in finite games, mixed equilibria may still exist

Theorem: Every infinite strategic-form game with strategy spaces that are nonempty compact subsets of a metric space and continuous payoff functions has a Nash equilibrium in mixed strategies

23

Static Games with Complete Information — Iterated Strict Dominance

Example: how can we easily find all NE of the following 3×3 game?

Answer: by repeated getting rid of strictly dominated strategies

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

24

Static Games with Complete Information — Iterated Strict Dominance

- Set $S_i^0 \equiv S_i$ and $\Sigma_i^0 \equiv \Sigma_i$
- For $n = 1, 2, \ldots$, define

$$S_i^n = \{s_i \in S_i^{n-1} | \nexists\, \sigma_i \in \Sigma_i^{n-1} \text{ s.t. } u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})\ \forall s_{-i} \in S_{-i}^{n-1}\}$$

and

$$\Sigma_i^n = \{\sigma_i \in \Sigma_i | \sigma_i(s_i) > 0 \text{ only if } s_i \in S_i^n\}$$

25

Static Games with Complete Information — Iterated Strict Dominance

- Set

$$S_i^\infty = \cap_{n=0}^\infty S_i^n$$

25

Static Games with Complete Information — Iterated Strict Dominance

Proposition: The set $S^\infty = S_1^\infty \times \cdots \times S_I^\infty$ does not depend on the order of deletion of strictly dominated strategies

Note: The above is not true for *weakly dominated* strategies

26

Static Games with Complete Information — Iterated Strict Dominance

Definition: A game is solvable by iterated strict dominance if, for each player i, S_i^∞ consists of one element

Proposition: If a game is solvable by iterated strict dominance, the resulting profile is the unique Nash equilibrium

27

Static Games with Complete Information — Iterated Strict Dominance

Example: Cournot (linear) model

- Round 1: $S_i^1 = [0, q_i^m]$, $i = 1, 2$
- Round 2: $S_i^2 = [b_i(q_j^m), q_i^m] \equiv [\underline{q}_i^2, q_i^m]$
- Round 3: $S_i^2 = [\underline{q}_i^2, b_i(\underline{q}_j^2)] \equiv [\underline{q}_i^2, \bar{q}_i^3]$
- Round n: $S_i^n = [\underline{q}_i^n, \bar{q}_i^n]$ where
  - for $n = 2k+1$, $\underline{q}_i^{2k+1} = \underline{q}_i^{2k}$ and $\bar{q}_i^{2k+1} = b_i(\underline{q}_j^{2k})$
  - for $n = 2k$, $\underline{q}_i^{2k} = b_i(\bar{q}_j^{2k-1})$ and $\bar{q}_i^{2k} = \bar{q}_i^{2k-1}$

28

Static Games with Complete Information — Rationalizability

- We want to find all strategies that a rational player could play
- Such strategies should be best responses to some beliefs
- The beliefs themselves should be rational (the opponents' strategies are their best responses to some beliefs they might have and so on...)

29

Static Games with Complete Information — Rationalizability

- Set $\tilde{\Sigma}_i^0 \equiv \Sigma_i$
- For $n = 1, 2, \ldots$, define

$$\tilde{\Sigma}_i^n = \{\sigma_i \in \tilde{\Sigma}_i^{n-1} | \exists \sigma_{-i} \in Conv(\tilde{\Sigma}_j^{n-1}) \text{ s.t.}$$
$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \; \forall \sigma_i' \in \tilde{\Sigma}_j^{n-1}\}$$

- Set

$$R_i = \cap_{n=0}^{\infty} \tilde{\Sigma}_i^n$$

30

Static Games with Complete Information — Rationalizability

Example:

	L	R
A	3	0
B	0	3
C	2	2
D	1	1

Strategies A and B are rationalizable while any nontrivial mixture of A and B is not

非平凡混合.

31

Static Games with Complete Information — Rationalizability

Theorem: The set of rationalizable strategies is nonempty and contains at least one pure strategy for each player.

Each $\sigma_i \in R_i$ is a best response to an element of $\times_{j \neq i} Conv(R_j)$

32

Static Games with Complete Information — Rationalizability and strict dominance

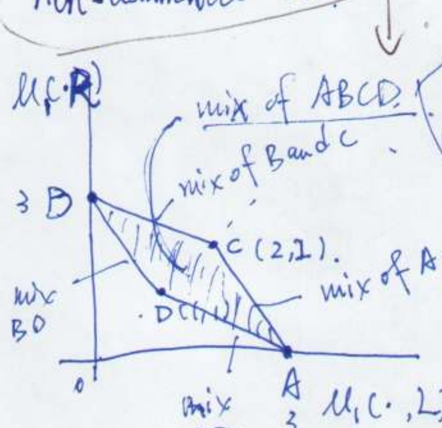
Proposition: The set of rationalizable strategies is contained in the set that survives iterated strict dominance

Theorem: Rationalizability and iterated strict dominance coincide in two-player games

Remark: The above statement is not in general true for many-player (> 2) games

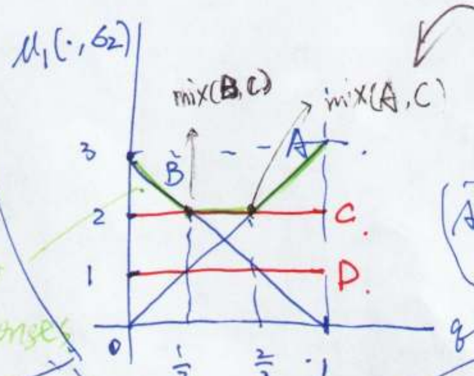
33

to find non-dominated str



B $(0, P, 1-P, 0)$
C $(P', 0, 1-P', 0)$
A best responses

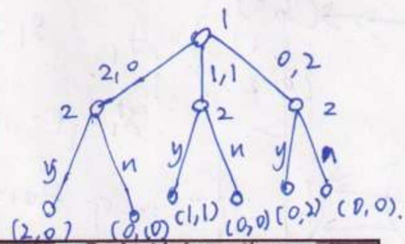
$\sigma_2 = (q, 1-q)$.



to find Rationalizable str:

$(A, B, C, (P, 0, 1-P, 0)$
$(0, P', 1-P', 0)$

the same.



Dynamic Games with Complete Information — Perfect information

Example: Two players divide two identical valuable objects as follows: the first player announces the division scheme (0+2, 1+1, 2+0), and the second player either agrees to the scheme or rejects it. In the first case the players divide the objects according to the announced scheme, in the second both receive nothing.

1

Dynamic Games with Complete Information — Perfect information

The extensive form of a game contains the following:

1. the set of players
2. the order of moves
3. the players' payoffs as a function of their moves
4. choices available to players when they move
5. information available to players when they move
6. probability distribution over exogenous events

perfect information

Note: for the first pass, we will neglect the last two items

2

Dynamic Games with Complete Information — Perfect information

Definition: An **extensive form game with** *perfect* **information** consists of:

- A set $\mathcal{I}$ of players
- A set H of sequences (histories) that satisfy the following properties
  - the empty sequence $\emptyset$ is in H
  - if $(a^k)_{k=1,\dots,K} \in H$ and $L < K$ then $(a^k)_{k=1,\dots,L} \in H$

3

Dynamic Games with Complete Information — Perfect information

- for an infinite sequence, if $(a^k)_{k=1,\dots,K} \in H$ for any positive K then $(a^k)_{k=1}^{\infty} \in H$

A history $(a^k)_{k=1,\dots,K} \in H$ is *terminal* if it is infinite or there is no a^{K+1} such that $(a^k)_{k=1,\dots,K+1} \in H$. The set of terminal histories is denoted Z

- A function P that assigns to each nonterminal history a member of $\mathcal{I}$. So that $P(h)$ is the player who moves after the history h

3

Dynamic Games with Complete Information — Perfect information

- A set of actions $A(h)$ available to player $P(h)$ after the history h: $A(h) = \{a : (h, a) \in H\}$
- For each player $i \in \mathcal{I}$, a preference relation $\succeq_i$ on Z

3

Dynamic Games with Complete Information — Perfect information

Example revisited:

- $\mathcal{I} = \{1, 2\}$
- H consists of 10 histories: $\emptyset$, (2,0), (1,1), (0,2), ((2,0),y), ((2,0),n), ((1,1),y), ((1,1),n), ((0,2),y), ((0,2),n)
- $P(\emptyset) = 1$ and $P(h) = 2$ for every other nonterminal history

4

$S_1: \ \phi_{H_1} \rightarrow \left.\begin{matrix}(2,0)\\(1,1)\\(0,2)\end{matrix}\right\} A_1. \quad s_1 = (2,0).$

$A_1 = \{(2,0),(1,1),(0,2)\}$

$A_2 = \{y, n\}.$

$S_1 = \{(2,0),(1,1),(0,2)\}.$

$S_2: \begin{matrix}(2,0)\\(1,1)\\(0,2)\end{matrix} \rightarrow \begin{matrix}y\\n\end{matrix} \quad A_2$ it's one of them.

Dynamic Games with Complete Information — Perfect information

- ((2,0),y)$\succ_1$((1,1),y)$\succ_1$((0,2),y)$\sim_1$((2,0),n)$\sim_1$((1,1),n)$\sim_1$((0,2),n) and ((0,2),y)$\succ_2$((1,1),y)$\succ_2$((2,0),y)$\sim_2$((2,0),n)$\sim_2$((1,1),n)$\sim_2$((0,2),n)

$S_2 = \{yyy, yyn, yny, ynn, nyy, nyn, nny, nnn\}$.

4

Dynamic Games with Complete Information — Perfect information — Strategies...

- Let H_i be the set of all histories h_i such that $P(h_i) = i$
- Let $A_i \equiv \cup_{h_i \in H_i} A(h_i)$ be the set of all actions available to player i

Definition: A (pure) *strategy of player* $i \in \mathcal{I}$ in an extensive form game with perfect information is a map s_i: $H_i \to A_i$ which assigns an action from A_i to any history in H_i. The set of all strategies of player i is $S_i = \times_{h_i \in H_i} A(h_i)$.

5

Dynamic Games with Complete Information — Perfect information — ...Strategies

Figure 1: Extensive form of "divide the objects" game

6

Dynamic Games with Complete Information — Perfect information — Nash equilibrium...

- For each strategy profile s, let the *outcome* $O(s) \in Z$ be the terminal history that results when each player follows s_i

Definition: A *Nash equilibrium of an extensive form game with perfect information* is a strategy profile s^* such that for every player $i \in \mathcal{I}$

$$O(s_i^*, s_{-i}^*) \succeq_i O(s_i, s_{-i}^*)$$

7

((2,0), nyy) is not because if (2,0) → (1,1). better for player!

Dynamic Games with Complete Information — Perfect information — ...Nash equilibrium...

- How many strategy profiles does the "divide the objects" game have? (24)
- How many Nash equilibria? (9: ((2,0),yyy), ((2,0),yyn), ((2,0),yny), ((2,0),ynn), ((1,1),nyy), ((1,1),nyn), ((0,2),nny), ((2,0),nny), ((2,0),nnn)) check. by
- Are all of them equally good? (No, two of the above are 'better' then the rest)

Can she get improvement based ~~one~~ on other's strategy?

8

Dynamic Games with Complete Information — Perfect information — ...Nash equilibrium...

Strategic form of a dynamic game

Definition: The strategic form of the dynamic game with perfect information $\Gamma = (N, H, P, (\succeq_i))$ is the strategic game $(N, (S_i), (\succeq_i'))$ in which for each player $i \in \mathcal{I}$

- S_i is the set of strategies of player i in Γ
- $\succeq_i'$ is defined by $s \succeq_i' s'$ iff $O(s) \succeq_i O(s')$ for every $s \in \times_{i \in \mathcal{I}} S_i$ and $s' \in \times_{i \in \mathcal{I}} S_i$

9

NO → better NE

player ~~1~~ believe player will choose ...

believable strategies so ~~that~~ to make her payoff max.

(2,0): y n; (1,1): y; (0,2): y.

~~{yyy}~~ {yyy, nyy}. of player 2.

Dynamic Games with Complete Information

Perfect information

...Nash equilibrium...

Figure 2: Extensive form

10

Dynamic Games with Complete Information

Perfect information

...Nash equilibrium...

	(L,L)	(L,R)	(R,L)	(R,R)
U	2,1	2,1	0,0	0,0
D	-1,1	3,2	-1,1	3,2

Table 1: Strategic form

11

Dynamic Games with Complete Information

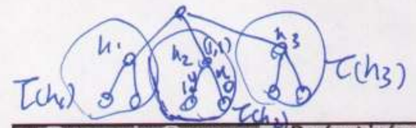
Perfect information

...Nash equilibrium

- For a strategic form of a dynamic game, for any $i \in \mathcal{I}$, define the strategies $s_i \in S_i$ and $s_i' \in S_i$ to be *equivalent* if for each $s_{-i} \in S_{-i}$, $(s_i, s_{-i}) \sim_j' (s_i, s_{-i})$ form all $j \in \mathcal{I}$

Definition: The *reduced strategic form* of Γ is the strategic game $(N, (S_i), (\succeq_i'))$ in which each set S_i' contains one member of each set of equivalent strategies in S_i and $\succeq_i''$ is the preference ordering over $\times_{j \in \mathcal{I}} S_j'$ induced by $\succeq_i'$

12



Dynamic Games with Complete Information

Perfect information

Equilibrium refinement...

Definition: The *subgame of the extensive form game with perfect information* $\Gamma = (\mathcal{I}, H, P, (\succeq)_i)$ that follows history h is the extensive form game $\Gamma(h) = (\mathcal{I}, H|_h, P|_h, (\succeq)_i|_h)$ induced by the original game Γ

Definition: *A subgame perfect equilibrium* is a strategy profile s^* such that for any nonterminal history $h \in H \setminus Z$ it is a Nash equilibrium of any subgame $\Gamma(h)$ of the original game

13

Dynamic Games with Complete Information

Perfect information

...Equilibrium refinement...

Example: Cournot's duopoly competition

Set $\alpha = 12$, $\beta = 0$ so that

$$p(q) = 12 - q_1 - q_2$$

and

$$u_i(q) = q_i(12 - q_1 - q_2)$$

14

Dynamic Games with Complete Information

Perfect information

...Equilibrium refinement...

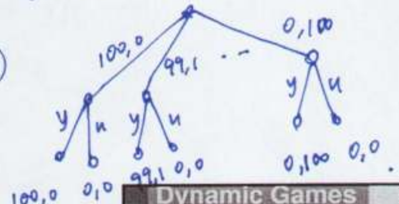
Figure 3: Two-stage Cournot's competition with discrete choices

15

SPE : (6, 4 4 3).

用这个例子理解 SPE.

Divide 100 objects.

① 

$|S_1| = 101.$
$|S_2| = 2^{101}$
$|S| = 101 \cdot 2^{101}$.

② Count NE. $((100,0), y, \cdots)$. 2^{100}.
$((100,0), n, nn\cdots \frac{y}{n})$ 2.
$((99,1), n, y, \cdots)$ 2^{99}
~~((99,1), n, nn⋯)~~
$((1,99), nn\cdots y \frac{y}{n})$ 2^1
$((0,100), nnn\cdots ny) = 1$.

NE:
$\underbrace{2^{100}+2^{99}+\cdots+2+1}_{+2} = (2^{101}-1)+2 = 2^{101}+1$

Dynamic Games with Complete Information — Perfect information — ...Equilibrium refinement...

"Divide the dollar" game: same as above with a 100 objects to be divided according to the same rules

- How many different strategy profiles?
- How many of them are Nash equilibria?
- How many of all Nash equilibria are subgame perfect?

③. SPE: $((100,0), yyy\cdots y)$
$((99,1), nyy\cdots y)$.

16

Dynamic Games with Complete Information — Perfect information — ...Equilibrium refinement...

Example: *Chess*

- Is it a perfect information game?
- How many different strategies do the whites have for the first move? 20
- How many different strategies do the whites have for the first two moves?

$(20^{20^{20}})^{20} = 20^{400}$

17

Dynamic Games with Complete Information — Perfect information — ...Equilibrium refinement...

Underlined: Theorem: Every finite extensive game with perfect information has a subgame perfect equilibrium

Proof: Constructive by backwards induction starting from the terminal nodes

Remark: The theorem does not claim uniqueness. However a finite game in which no player is indifferent between any two outcomes, has a unique subgame perfect equilibrium

18

拿H也说L.
拿L: pass

be strictly dominated

Dynamic Games with Complete Information — Perfect information — ...Equilibrium refinement

Theorem: *One deviation property.* The strategy profile of a finite horizon Γ with perfect information s^* is a subgame perfect equilibrium if and only if for any subgame $\Gamma(h)$ any player $i \in \mathcal{I}$ can't improve the payoff by using a strategy that differs from $s^*|_h$ only in the first prescribed move.

Remark: The theorem does not hold in general for infinite horizon games

19

$\boxed{\text{behavior strategies}}$ is

$b^* = (((0,1), (\frac{2}{5}, \frac{3}{5})), (\frac{3}{5}, \frac{2}{5}))$.

$\boxed{\text{mixed strategies}}$ is

$S^* = ((0,0,\frac{2}{5},\frac{3}{5}), (\frac{3}{5},\frac{2}{5}))$.

参考 P21 上面的例子理解.

	C	Ch	
LL	-1,1	-1,1	$\frac{1}{2}(-1,1)+\frac{1}{2}(1,-1)$
LH	0,0	2.5,-2.5	$\frac{1}{2}(4,1)+\frac{1}{2}(-4,4)$
HL	0,0	1.5,-1.5	$\frac{1}{2}(-4,4)+\frac{1}{2}(4,-4)$
HH	1,-1	0,0	
	q	$1-q$	

P

$1.5(1-q) = q \Rightarrow q = \frac{3}{5}$.
$-1(1-p) = -1.5p \Rightarrow p = \frac{2}{5}$.
Nash and SPE.

$\frac{1}{2}(1,-1)+\frac{1}{2}(-1,1)$
$\frac{1}{2}(4,-4)+\frac{1}{2}(-4,4)$

$b_1^*(H|h_1)=1$
$b_1^*(L|h_1)=0$
$b_1^*(L|h_2)=\frac{2}{5}$
$b_2^*(c|I_3)=\frac{3}{5}$
$b_1^*(H|h_2)=\frac{3}{5}$
$b_2^*(ch|I_3)=\frac{2}{5}$.

Dynamic Games with Complete Information — Imperfect information

In addition to the above definition of an extensive form game we introduce

- Possibilities of exogenous events
- Partition of histories into *information sets* such that a player can't distinguish between different histories in the same information set
  - The set of actions should be same for all histories in an information set: $A(h) = A(h')$ for $h, h' \in I$

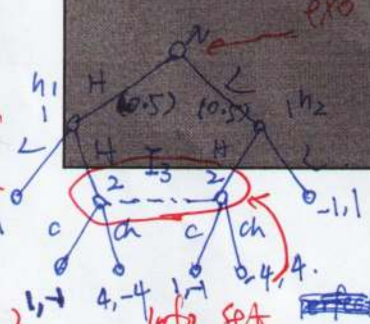
20

explain.

Dynamic Games with Complete Information — Imperfect information

Remark: We can look at information sets as generalization of histories

Info sets / exo moves.



higher
Lower
card

20

C — concede
ch — challenge.
check.

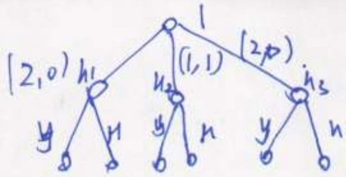
$S_1 = \{LL, LH, HL, HH\}$
$S_2 = \{c, ch\}$.

对状态1的两种
即HL的分别策略 what player 1 says

这里类似于上面的 $\{yyy\cdots n, nn\cdots y\cdots\}$

info set ~~perfect~~ ---- means to player 2, They are the same info.

即 eg: ① LH means: 对H, L; 对L 说H. ②. HH means: 对H say H, 对L say H imperfect.



$h_{1,2,3}$: ~~history~~ history.

$S_2 = \{yyy, yyn, yny, ynn, nyy, nyn, nny, nnn\}$

$\sigma_2 = (0,0,0,0,\frac{1}{3},\frac{1}{6},\frac{1}{3},\frac{1}{6})$.

Find behavior strategy equivalent to σ_2.

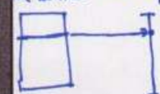
$b_2(y|h_1) = 0$. $b_2(y|h_2) = \frac{1}{2}$ $b_2(y|h_3) = \frac{2}{3}$

$b_2(n|h_1) = 1$. $b_2(n|h_2) = \frac{1}{2}$. $b_2(n|h_3) = \frac{1}{3}$

$b_2 = ((0,1), (\frac{1}{2},\frac{1}{2}), (\frac{2}{3},\frac{1}{3}))$ → both infinite sets but Σ_2 is bigger.

$\dim(\Sigma_2) = 7 = 2^3 - 1$ dependent on how many numbers; is same thing like a line → a point.

$\dim(B_2) = 3$



Dynamic Games with Complete Information — Imperfect information — Mixed and behavior strategies...

Definition: A *pure strategy* of player i in an extensive form game is a map $s_i : \mathcal{I}_i \to S_i$ from the set of information sets to the set of strategies of player i.

Definition: A it mixed strategy of player i is a probability distribution over the set player i's pure strategies. A *behavior strategy* of player i is a collection of independent probability distributions over actions sets $A(I_i)$ for all information sets of player i.

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Dynamic Games with Complete Information — Imperfect information — ...Mixed and behavior strategies...

- In a game of *perfect recall* a player never forgets any information he once knew.
- Let $X_i(h)$ be the sequence of information sets and actions he takes on the way to history h.

Definition: A game has perfect recall if for each player i, $X_i(h) = X_i(h')$ whenever h and h' belong to the same information set of player i

Imperfect recall game.

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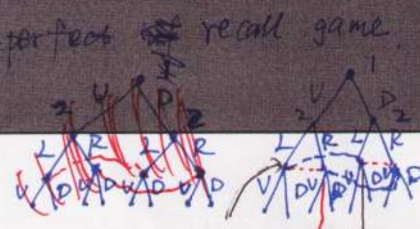

perfect recall.

imperfect recall (player 1 forget what he chose)

can't figure out UL and DL ---- he can't figure out UL and UR, it's ok. It's imperfect info.

Dynamic Games with Complete Information — Imperfect information — ...Mixed and behavior strategies...

Definition: Two strategies of player i are *equivalent* if for every pure strategy profile s_{-i} of other players they produce the same outcome.

Theorem: In a game of perfect recall every mixed strategy is equivalent to the unique behavior strategy it generates, and each behavior strategy is equivalent to each mixed strategy that generates it.

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Dynamic Games with Complete Information — Imperfect information — ...Mixed and behavior strategies...

one player

forgot

Figure 4: A game with imperfect recall

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$\boxed{\begin{array}{l} b_1^*(L|I_1) = 0,\ b_1^*(L|I_1') = \frac{1}{2} \\ b_2^*(L|I_2) = 1,\ b_2^*(L|I_2') = \frac{1}{2} \end{array}}$ behavior strategies.

Dynamic Games with Complete Information — Imperfect information — ...Mixed and behavior strategies...

- For games of perfect information, subgame perfection and backwards induction are equivalent
- For games of imperfect information, subgame perfection is more general than backwards induction

~~It's more simple and easier.~~

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Dynamic Games with Complete Information — Imperfect information — ...Mixed and behavior strategies

Figure 5: Subgame perfection is more general than backwards induction

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	L	R
L	2,-2	-2,2
R	-2,2	2,-2

$S^* = ((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$

payoff $= (0,0)$.

remove subgame

3,1 (because player 2 will choose (3,1))

$S_1 = \{LL, LR, RL, RR\} = S_2$.

SPE $\boxed{\begin{array}{l} \sigma_1^*(LL) = 0 = \sigma_1^*(LR). \\ \sigma_1^*(RL) = \frac{1}{2} = \sigma_1^*(RR) \\ \sigma_2^*(LL) = \frac{1}{2} = \sigma_2^*(LR) \end{array}}$ mixed strategies.

☆ 动态 ~~博弈~~

In a dynamic games, strategies are in the whole game, what you have to do in every ~~possibility~~ possible plays.

Dynamic Games with Complete Information — Critiques

- Example: chain store game. A chain store has branches in K cities with a single local potential competitor in each city. In each period a competitor decides whether to compete or not.
- If the local store competes, the chain store can either fight or cooperate.
- If the local store stays out, it gets 1 unit of payoff and the chain store gets 5. In case of entrance, if the chain store cooperates each gets 2, and if chain

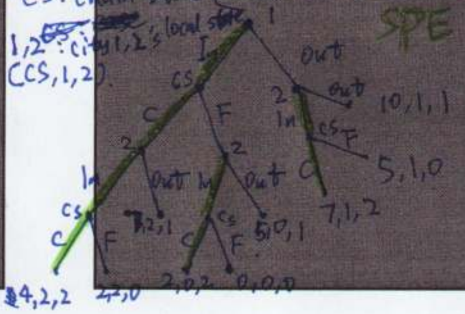
Dynamic Games with Complete Information — Critiques

store fights each gets 0.

CS: chain store
1,2: city 1,2's local store
(CS,1,2)

Chain store, K=2

SPE



Dynamic Games with Complete Information — Critiques

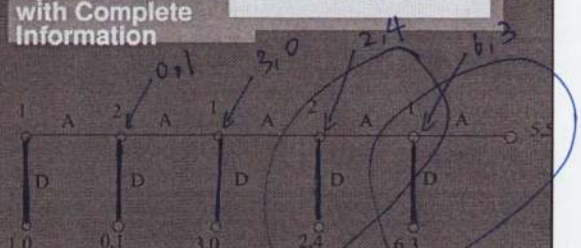


Figure 6: Centipede game

as SPE shows, player choose D (1,0). game over.
means SPE can't do anything here (in this case)

Dynamic Games with Complete Information — Repeated games — Bargaining...

- Two players want to divide a pie of size 1
- In the first period, player 1 makes an offer (a division scheme (x_1, x_2) such that $x_1 + x_2 = 1$)
- The second player either accepts the offer (game ends) or rejects it (game continues in period 2)
- In the second period, player 2 makes an offer to player 1
- Game continues in this way until some offer is accepted

Dynamic Games with Complete Information — Repeated games — ...Bargaining...

- $\delta_1 < 1$ and $\delta_2 < 1$ are discount factors for players 1 and 2
- The payoff function is

$$u_i = \begin{cases} x_i \delta_i^{t-1}, & \text{if offer is accepted in period t} \\ 0, & \text{if no offer is accepted} \end{cases}$$

strategy
pl 1. offers (x_1, x_2)
accepts $\ge x_1$
pl 2. offers (x_1, x_2)
accepts $\ge x_2$.
NE for arbitrary (x_1, x_2).

Dynamic Games with Complete Information — Repeated games — ...Bargaining...

- What are Nash equilibria of this game?
- What division schemes are possible in Nash equilibria?
- Which of them are subgame perfect?
- Is the division scheme still arbitrary?

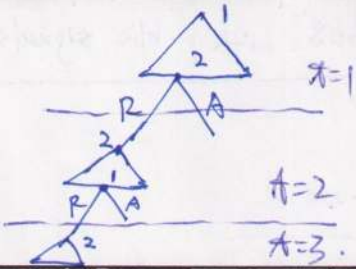
the payoff of no deviation should be better than the one of one deviation!!

player 1 responds
1). offer is (x_1, x_2) where $x_1 \ge 1-b^*$.
No deviation: payoff $= x_1 \ge 1-b^*$
one deviation: pl 1 rejects, offer $(a^*, 1-a^*)$
pl 2 accept. payoff $= \delta_1 a^*$.
2). offer is (x_1, x_2) where $x_1 < 1-b^*$
No deviation: pl 1 rejects, payoff $= \delta_1 a^*$.
one deviation: pl 1 accepts payoff $= x_1 < 1-b^*$

For any $x_1 \ge 1-b^*$, $x_1 \ge \delta_1 a^*$
For any $x_1 < 1-b^*$, $\delta_1 a^* \ge x_1$

$\Downarrow$

$1-b^* = \delta_1 a^*$.
Likewise $1-a^* = \delta_2 b^*$



player 1 makes offer
No $\boxed{\text{deviation}}$: payoff = a^*. (Continuation) $\Rightarrow$ $\boxed{(1-b^*)\delta_1 \le a^*}$

9/13/2015

One deviation : 1) $x_1 > a^*$. pl 2 reject. offer $(1-b^*, b^*)$ pl 1 accepts. payoff $= (1-b^*)\delta_1$.
2). $x_1 < a^*$. pl 2 accepts. payoff $x < a^*$

Page

likewise: $(1-a^*)\delta_2 \le b^*$.

Dynamic Games with Complete Information — Repeated games — ...Bargaining...

Unique subgame perfect equilibrium:

no deviation

- In any period player 1 makes an offer, he offers $(a^*, 1-a^*)$
- In any period player 2 makes an offer, he offers $(1-b^*, b^*)$

do sth else unexpected

where a^* and b^* satisfy

$$1-b^* = \delta_1 a^* \text{ and } 1-a^* = \delta_2 b^*$$

$$\Rightarrow a^* = \frac{1-\delta_2}{1-\delta_1\delta_2}, \quad b^* = \frac{1-\delta_1}{1-\delta_1\delta_2}$$

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Dynamic Games with Complete Information — Repeated games — ...Bargaining...

- Player 2 accepts offer (x_1, x_2) if $x_2 \ge 1-a^*$
- Player 1 accepts offer (x_1, x_2) if $x_1 \ge 1-b^*$

} no deviation.

$\delta_1(1-b^*) = \delta_1^2 a^* < a^*$.

Quick offer exchange

$\delta_i = e^{-\Delta_i t}$ $\quad a^* = \lim_{t\to 0} \frac{1-e^{-\Delta_2 t}}{1-e^{-(\Delta_1+\Delta_2)t}} = \frac{1-1+\Delta_2 t}{1-1+(\Delta_1+\Delta_2)t} = \frac{\Delta_2}{\Delta_1+\Delta_2}$.

equal.

泰勒展开?

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$\delta_1 = \delta_2 = \delta$, $a^* = \frac{1-\delta}{1-\delta^2} = \frac{1}{1+\delta} > \frac{1}{2}$. the advantage of first move.

if player 1 offer $(a^*, 1-a^*)$ { case 1: player 2 accept and payoff $= 1-\frac{1}{1+\delta} = \frac{\delta}{1+\delta}$
case 2: player 2 reject, offer $(1-b^*, b^*)$, player 1 accept. payoff $= \delta b^* = \frac{\delta}{1+\delta}$ → player 2

Dynamic Games with Complete Information — Repeated games — ...Bargaining...

Proof of uniqueness

- Denote $\bar{v}_i$ and $\underline{v}_i$ the highest and lowest continuation payoffs of player i in a subgame starting with that player's offer
- Denote $\bar{w}_i$ and $\underline{w}_i$ the highest and lowest continuation payoffs of player i in a subgame starting with the other player's offer

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Dynamic Games with Complete Information — Repeated games — ...Bargaining...

- In any subgame starting from player's 1 offer:
  - player 2 will not reject any offer such that $x_2 \ge \delta_2 \bar{v}_2$ – therefore $\underline{v}_1 \ge 1 - \delta_2 \bar{v}_2$
  - player 2 will reject any offer such that $x_2 < \delta_2 \underline{v}_2$ – therefore $\bar{v}_1 \le 1 - \delta_2 \underline{v}_2$
- Likewise, considering a subgame starting from the player's 2 offer, we obtain:
  - $\underline{v}_2 \ge 1 - \delta_1 \bar{v}_1$
  - $\bar{v}_2 \le 1 - \delta_1 \underline{v}_1$

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Dynamic Games with Complete Information — Repeated games — ...Bargaining...

- Combining above inequalities, we obtain:

$$\underline{v}_1 \ge \frac{1-\delta_2}{1-\delta_1\delta_2}$$

$$\bar{v}_1 \le \frac{1-\delta_2}{1-\delta_1\delta_2}$$

- Likewise

$$\underline{v}_2 = \bar{v}_2 = \frac{1-\delta_1}{1-\delta_1\delta_2}$$

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Dynamic Games with Complete Information — Repeated games — ...Bargaining...

- In a subgame starting from player's 2 offer, player 1 rejects any offer that would give him less than the discounted minimum payoff starting from the next period: $\underline{w}_1 \ge \delta_1 \underline{v}_1$
- In a subgame starting from player's 2 offer, player 2 is not going to give his opponent an offer that would make his (opponent's) utility more than the discounted maximum of what the opponent could ensure himself: $\bar{w}_1 \le \delta_1 \bar{v}_1$

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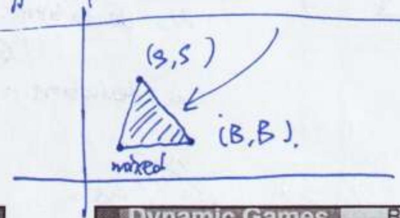
payoff possible in BoS. with public signals.



Dynamic Games with Complete Information — Repeated games — ...Bargaining...

- We conclude that

$$\underline{w}_1 = \bar{w}_1 = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$$

- In the same way,

$$\underline{w}_2 = \bar{w}_2 = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$$

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Dynamic Games with Complete Information — Repeated games — ...Bargaining...

Properties of subgame perfect equilibrium

- Strategies are stationary: in any period a player proposes the same division, and accepts according to the same criteria
- First mover has an advantage: if $\delta_1 = \delta_2 = \delta$, the share the first player gets is

$$a^* = \frac{1}{1+\delta} > \frac{1}{2}$$

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Dynamic Games with Complete Information — Repeated games — ...Bargaining

- *Comparative statics of impatience.* Take the limit of short time periods: $\delta_i = \exp(-r_i\Delta)$. Then

$$\lim_{\Delta\to 0} a^* = \frac{r_2}{r_1+r_2}$$

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Dynamic Games with Complete Information — Repeated games — Correlated equilibrium...

Example: BoS

		$\frac{1}{3}$	$\frac{2}{3}$
		Bach	Stravinsky
$\frac{2}{3}$	Bach	2,1	0,0
$\frac{1}{3}$	Stravinsky	0,0	1,2

- There are two pure and one mixed equilibrium (with payoff (2/3, 2/3))
- If there exists a public binary signal w/p 1/2, there is also a correlated equilibrium with payoff (3/2, 3/2)

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player i can show out a signal of her decision
eg. if player1 will choose B, she will tell player 2, and will not change the decision. $(p=\frac{1}{2})$.
So, player 2 will follow by B.

$payoff = \frac{1}{2}\times 2 + \frac{1}{2}\times 1 = \frac{3}{2}$.

Dynamic Games with Complete Information — Repeated games — ...Correlated equilibrium...

- If the signal used for correlation is public (common for all), then all achievable correlated payoff vectors are in the convex hull of the set of uncorrelated equilibria (pure and mixed)
- If we allow signals that are private and correlated, we can achieve payoff vectors outside of the convex hull.

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Q -

player 1 sees "U", ~~if he follows, expected payoff will be~~

$P_1(UL|U) = \frac{P(UL)}{P(UL)+P(UR)}$

$P_2(UR|U) = \frac{P(UR)}{P(UL)+P(UR)}$.

If pl 1 follows the rule: $u_1 = \frac{P_{UL}}{P_{UL}+P_{UR}} \times 7$

if .. doesn't ....: $u_1 = \frac{P_{UL}}{P_{UL}+P_{UR}} \times 6 + \frac{P_{UR}}{P_{UR}+P_{UL}} \times 2$.

$7P_{UL} \geq 6P_{UL} + 2P_{UR}$.

$P_{UL} \geq 2P_{UR}$. → This is one condition

similar to

Dynamic Games with Complete Information — Repeated games — ...Correlated equilibrium...

Example:

	L	R
U	7,2	0,0
D	6,6	2,7

- There are two pure strategy equilibria and one mixed strategy equilibrium with payoff of (14/3, 14/3)
- How can we construct a correlated equilibrium that lies outside of the convex hull of the above?

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signals: { UL, UR, DL, DR }.

player 1 sees first only

player 2 sees second only.

Rule: "play what you see".

$P_{UL}, P_{UR}, P_{DL}, P_{DR}$ probability of signals.

这里是 player 1 看到 "U" 条件下的 u_1.

→ 这里是直接全局看到的 u,

$$P_1(UL|U) = \frac{P(U|UL)\cdot P(UL)}{P(U|UL)\cdot P(UL) + P(U|UR)\cdot P(UR) + P(U|DL)\cdot P(DL) + P(U|DR)\cdot P(DR)}$$

$$= \frac{P_{UL}\cdot 1}{P_{UL}\cdot 1 + P_{UR}\cdot 1 + 0 + 0} = \frac{P_{UL}}{P(UL) + P_{UR}}.$$

$$P(B_i|A) = \frac{P(A|B_i)\cdot P(B_i)}{\sum P(A|B_i)\cdot P(B_i)}.$$

② player 1 sees "D".

$6P_{DL} + 2P_{DR} \geq 7P_{DL}.$

$2P_{DR} \geq P_{DL}.$ 2nd condition.

③ player 2 sees "L".

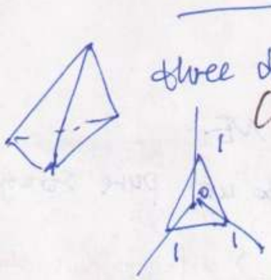
$2P_{UL} + 6P_{DL} \geq 7P_{DL}.$

$2P_{UL} \geq P_{DL}$. 3rd condition.

④ player 2 sees "R".

$0\cdot P_{UR} + 7\cdot P_{DR} \geq 2P_{UR} + 6P_{DR}.$

$P_{DR} \geq 2P_{UR}$ 4th condition.



three dimensional space of probabilities.

Correlated Equilibrium

CE w/public signal . payoff

NE : (U,L), (D,R). $(\frac{14}{3}, \frac{14}{3})$.

P_1, P_2, P_3

mixed : $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$.

E_{pr} = set of all CE with private signal.

$(U,L) \in E_{pr}$?

$P_{UL} = 1$ check right ✓.

$(\frac{14}{3}, \frac{14}{3}) \in E_{pr}$?

$P_{UL} = \frac{2}{9}$, $P_{UR} = \frac{1}{9}$. $P_{DL} = \frac{4}{9}$. $P_{RD} = \frac{2}{9}$ ✓.

Highest system payoff with public and private signal

$P_1 = P_2 = 0$. $P_3 = 1$. payoff $= (\frac{14}{3}, \frac{14}{3}) = (4\frac{2}{3}, 4\frac{2}{3})$.

$\max(7P_{UL} + 6P_{DL} + 2P_{DR})$

s.t. $P_{UL} \geq 2P_{UR}$ — worse minimize.

$2P_{DR} \geq P_{DL}$ — best one to maximize.

$2P_{UL} \geq P_{DL}$

$P_{DR} \geq 2P_{UR}$.

$7P_{UL} + 6P_{DL} + 2P_{DR} = 7P_{DR} + 6P_{DL} + 2P_{UL}$

$P_{UR} = 0$. $P_{DL} = \frac{1}{2}$. $P_{DR} = P_{UL} = \frac{1}{4}$.

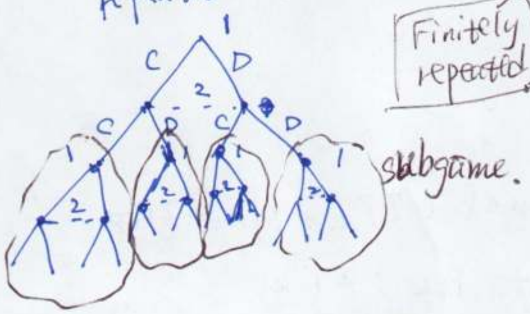
payoff $= 6\cdot\frac{1}{2} + 7\cdot\frac{1}{4} + 2\cdot\frac{1}{4} = 5\frac{1}{4} > 4\frac{2}{3}$.

private signal and public signal

the difference on set or graph?

bigger

Prisoner's dilemma. repeated twice



Repeated games

Finitely repeated

Infinitely repeated

Example

BoS repeated 100 times

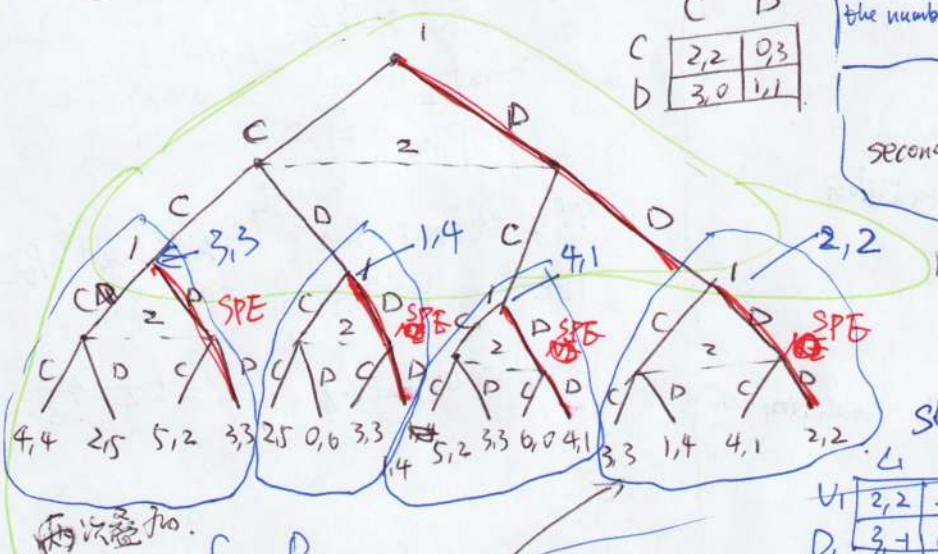
stage game

	B	S
B	2,1	0,0
S	0,0	1,2

3 NE: (B,B), (S,S) mixed.

find all NE in stage 1 and 2, then combine them.

of "trivial" SPE
the number $= 3^{100}$



	C	D
C	2,2	0,3
D	3,0	1,1

	C	D
C	3,3	1,4
D	4,1	2,2

	C	D
C	3,3	1,4
D	4,1	2,2

Example (from hw 4)

first stage

	L_1	R_1
U_1	2,2	-1,3
D_1	3,-1	0,0

second stage

		L_2	R_2
p	U_2	6,4	3,3
1-p	D_2	3,3	4,6
		q	1-q

NE: (U_2, L_2) ①
(D_2, R_2) ②.

$6q + 3(1-q) = 3q + 4(1-q)$

$3q = 1-q, \; q = \frac{1}{4}$.

likewise $p = \frac{3}{4}$.

subgame. payoff $= (\frac{15}{4}, \frac{15}{4})$. mixed ③

	L_1	R_1
U_1	2,2	-1,3
D_1	3,-1	0,0

Unique NE (D_1, R_1)

A: "Trivial" SPE.

$(0,0) \to (6,4)$ ①
$(0,0) \to (4,6)$ ②
$(0,0) \to (3\frac{3}{4}, 3\frac{3}{4})$ ③.

no matter what happened in the 1st stage, they play NE in the second stage.

B: "Non-trivial" SPE.

1. Second stage has to be NE.
2. In 1st stage they have to use pure strategies because mixed strategies are not obvious to check. | don't play NE.

1). $(2,2) \xrightarrow{\text{① try firstly}} (6,4)$ default thus, this default doesn't exist.

two potential deviate P1 → (4, 6) $4 < 6 - \frac{1}{2}$ (3-2)

"punishments" should be ~~strong~~ enough P2 (X) ~~nothing~~ $< 4 - \frac{1}{2}$ (3-2).

P1: punish player 1. ~~to~~ prevent player chooses a "deviate" ~~better~~ choice in stage 1.

② try secondly (4,6) default
(2,2) P1 → (X).
P2 → (6,4).

③ try third: $(3\frac{3}{4}, 3\frac{3}{4})$ default
but this is the lowest one.
can't ~~be~~ get punishment.

Thus, there is no SPE in this game, in which they can play (2,2) in the first period.

2). $(3,-1) \longrightarrow (4,6)$
P2: → (6,4)
or
$(3\frac{3}{4}, 3\frac{3}{4})$.

3). $(-1,3) \longrightarrow (6,4)$
P1: → (4,6)
or
$(3\frac{3}{4}, 3\frac{3}{4})$.

Finitely repeated

2 cases

Unique NE for stage

Unique SPE for repeated simple. ("trivial")

They just repeat the NE for all times No matter how long the game is.

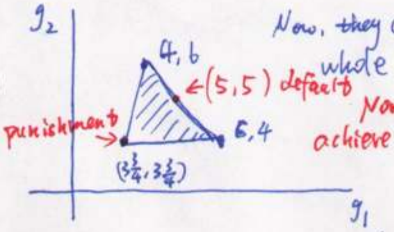
Multiple NE for stage.

"Punishment" ~~stamp~~ skims yield ~~multiple~~ multiple SPE.

+

multiple "trivial" SPE with different NE ~~with different~~ used in stages

If public signal is available (before period 2)
They can use CE in period 2.



Now, they can use the whole area.
Now, we can achieve (2,2).

Repeated Games

- Finitely rep
  - Unique NE of stage → Unique "trivial" SPE
  - Multiple NE of stage → "Gazillion" "trivial" SPE; "Nontrivial" SPE based on "punishment"
- Infinitely repeated
  - "trivial" SPE
  - "Gazillion" of non-trivial SPE (even for unique NE)

Infinitely : (P_{48})

g_i payoff at stage t.

$u_i = \sum_{t=0}^{\infty} \delta^t g_i = g_i \sum_{t=0}^{\infty} \delta^t = g_i \frac{1}{1-\delta}$ We will not talk about it.

define: $\underline{u_i} = (1-\delta) \sum_{t=0}^{\infty} \delta^t g_i = g_i$ more convinent.

Disconnted utility function 贴现效用.

$\min_{\alpha_{-i}} \left[\max_{\alpha_i} g_i(\alpha_i, \alpha_{-i}) \right]$.

best response of player i to α_{-i}.

This means ~~everyone~~ anyone else wants to hurt player i as possible as they can.

$\underline{V_i} = (\underline{V_1}, \underline{V_2}, \cdots, \underline{V_I})$.

$g_i \geq \underline{V_i}$ "individual rational" payoffs
it's about the stage.

	L_q	R_{1-q}
P_1 U	-2, 2	1, -2
P_2 M	1, -2	-2, 2
$1-P_1-P_2$ D	0, 1	0, 1

$\alpha_2 = (q, 1-q)$

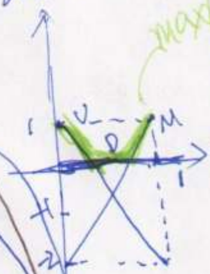
$\underline{V_1} = \min_q \left[\max_{\alpha_1} g_1(\alpha_1, \alpha_2) \right]$

$g_1(\alpha_1, q)$

$g_1(U, q) = -2q + 1 - q = 1 - 3q$.

$g_1(M, q) = q - 2(1-q) = 3q - 2$.

$g_1(D, q) = 0$.



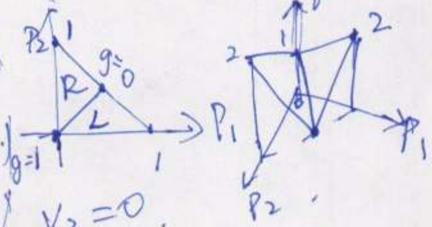
$\underline{V_1} = 0$.

$m_2^1 = (q, 1-q)$ where $\frac{1}{3} \leq q \leq \frac{2}{3}$.

$\underline{V_2} = \min_{P_1, P_2} \left[\max_{(L,R)} g_2(\alpha_1, a_2) \right]$

$g_2(\alpha_1, L) = 2P_1 - 2P_2 + (1 - P_1 - P_2)$
$= P_1 - 3P_2 + 1$.

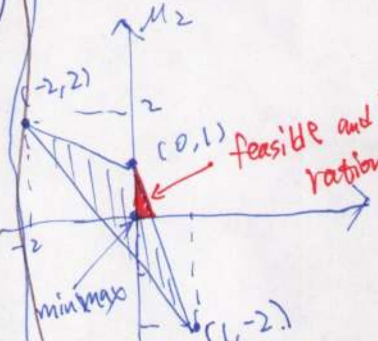
$g_2(\alpha_1, R) = -2P_1 + 2P_2 + 1 - P_1 - P_2$
$= -3P_1 + P_2 + 1$



$\underline{V_2} = 0$.

$m^2 = (0.5, 0.5, 0)$.

Set of feasible payoff.



$(0,0) \to$ any CE of the second period.

$(2,2) \to$ any CE of $\begin{cases} g_1 \geq 4.75 \\ g_2 \geq 4.75 \end{cases}$.

$(3,-1) \to$ any CE with $g_2 \geq 4.75$.

$(-1,3) \to$ any CE with $g_1 \geq 4.75$.

Dynamic Games with Complete Information

Repeated games

...Correlated equilibrium...

- It's possible to prove that one can, without loss of generality, restrict attention to "direct" private signals of the form s of which player i observes the component s_i.
- Any correlated equilibrium (including the standard "uncorrelated" ones) can then be described by specifying probabilities of different signals s.
- To see whether a given probability distribution over S represents a correlated equilibrium, one needs to

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Dynamic Games with Complete Information

Repeated games

...Correlated equilibrium

check whether each player would want to deviate from what the signal says, given the probabilities of the other players' actions.

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Summarize.

Dynamic Games with Complete Information

Repeated games

Definitions...

- *Stage game* – finite I-player simultaneous-move game
  - –finite action space of player i, A_i
  - –payoff functions g_i: $A \to R$
  - –$\mathcal{A}_i$ – set of probability distributions over A_i
- $h_t = (a^0, a^1, \ldots a^{t-1})$ the realized actions at all periods before t
- $H^t = (A)^t$ – the space of all possible period-t histories

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Dynamic Games with Complete Information

Repeated games

...Definitions...

- Pure strategy s_i of player i – sequence of maps s_i^t: $H^t \to A_i$
- Mixed (behavior) strategy σ_i – sequence of maps σ_i: $H^t \to \mathcal{A}_i$
- Each period begins a proper subgame – these are all proper subgames

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Dynamic Games with Complete Information

Repeated games

...Definitions...

- Discounted utility function (common discount factor δ):

$$u_i(\sigma) = E_\sigma[(1-\delta)\sum_{t=0}^{\infty} \delta^t g_i(\sigma^t(h^t))]$$

- Continuation payoff from time t:

$$E_\sigma[(1-\delta)\sum_{\tau=t}^{\infty} \delta^{\tau-t} g_i(\sigma^\tau(h^\tau))]$$

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Dynamic Games with Complete Information

Repeated games

...Definitions...

- Time average utility function:

$$\lim_{T\to\infty}\inf E[(1/T)\sum_{t=0}^{T} g_i(\sigma^t(h^t))]$$

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Dynamic Games with Complete Information — Repeated games — ...Definitions

Examples:

- If α^* is a NE of the stage game, then the strategy in which all players play α^* in each stage is a subgame perfect equilibrium of the repeated game
- If $\alpha^1, \ldots, \alpha^m$ are different NE of the stage game, then the strategy when all players play α^j according to any deterministic rule is a subgame perfect equilibrium of the repeated game

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Dynamic Games with Complete Information — Repeated games — Folk theorems

- *Minmax value* of player i

$$\underline{v}_i = \min_{\alpha_{-i}} \max_{\alpha_i} g_i(\alpha_i, \alpha_{-i})$$

- *Minmax profile* against player i – the opponents' strategy m^i_{-i} that attains the minimum
- Example:

about punishment

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Dynamic Games with Complete Information — Repeated games — Folk theorems...

	L	R
U	-2,2	1,-2
M	1,-2	-2,2
D	0,1	0,1

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Dynamic Games with Complete Information — Repeated games — ...Folk theorems...

- Player i's payoff is at least his minmax value in any NE of the repeated game, regardless of the level of discount factor
- The set of *feasible* payoff vectors for any discount factor is

$$V = \text{conv.hull}\{v | \exists a \in A \text{ with } g(a) = v\}$$

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Dynamic Games with Complete Information — Repeated games — ...Folk theorems...

Folk theorem

For every feasible payoff vector v with $v_i > \underline{v}_i$, there exists a $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$ there is a NE with payoff vector v

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Dynamic Games with Complete Information — Repeated games — ...Folk theorems...

Sketch of proof:

- Find (if possible) pure action profile a such that $g(a) = v$
- Strategy: Play a_i in period 0 and continue playing a_i until someone deviates. If player j deviates, play m^j_i in all later periods

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Dynamic Games with Complete Information
Repeated games
...Folk theorems...

- Player that deviates first in period t obtains no more than

$$(1-\delta^t)v_i + \delta^t(1-\delta)\max_a g_i(a) + \delta^{t+1}\underline{v}_i$$

which is less than v_i provided $\delta > \underline{\delta}_i$ given by

$$(1-\underline{\delta}_i)\max_a g_i(a) + \underline{\delta}_i \underline{v}_i = v_i$$

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Dynamic Games with Complete Information
Repeated games
...Folk theorems...

Example: Repeated prisoner's dilemma

	C	D
C	3,3	0,4
D	4,0	1,1

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Dynamic Games with Complete Information
Repeated games
...Folk theorems...

Example: Repeated G

	C	D
C	2,3	1,5
D	0,1	0,1

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Dynamic Games with Complete Information
Repeated games
...Folk theorems...

Weak perfect folk theorem

Let α^* be an NE of the stage game with payoff vector e. Then for any $v \in V$ with $v_i > e_i$ for all players i, there is a $\underline{\delta}$ such that for all $\delta > \underline{\delta}$ there is a subgame perfect equilibrium with payoffs v

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Dynamic Games with Complete Information
Repeated games
...Folk theorems...

Definition: A game is *continuous at infinity* if for each layer i the utility function u_i satisfies

$$\sup_{h,\tilde{h}|h^t=\tilde{h}^t} |u_i(h) - u_i(\tilde{h})| \to 0 \quad \text{as } t \to \infty$$

Theorem: In an infinite horizon game with observed actions that is continuous at infinity , profile s is subgame perfect if and only if there is no player i and strategy $\hat{s}_i$ that agrees with s_i except at a single t and

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Dynamic Games with Complete Information
Repeated games
...Folk theorems...

h^t, and such that $\hat{s}_i$ is a better response to s_{-i} that s_i conditional on history h^t being reached

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Dynamic Games with Complete Information

Repeated games

...Folk theorems...

Perfect folk theorem

Assume that the dimension of the set V of feasible payoff equals the number of players. Then for any $v \in V$ with $v_i > \underline{v}_i$ for all players i, there is a $\underline{\delta}$ such that for all $\delta > \underline{\delta}$ there is a subgame perfect equilibrium with payoffs v

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Dynamic Games with Complete Information

Repeated games

...Folk theorems...

Example:

The stage game is $(\{1,2,3\},(A_i),(u_i))$ where $A_i = [0,1]$ and

$$u_i = a_1 a_2 a_3 + (1-a_1)(1-a_2)(1-a_3)$$

- The set of feasible payoffs is $\{(v_1,v_2,v_3) | v_i \in [0,1]\}$
- For any δ, the payoff of any player in any subgame perfect equilibrium is at least 1/4

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Trigger strategy

1. Stick to plan while ~~nobod~~ somebody deviate.
2. As soon as somebody deviate, do something else.

Trigger strategy

weak ~~fo~~ folk theorem.

1. Stick to plan giving V_i on average.
2. As soon as player j deviate.
   minmax player j rest of time.

player j payoff.

If no deviate, $(1-\delta)\sum_{t=0}^{\infty} \delta^t$.

Deviation at time τ.

$(1-\delta)\sum_{t=0}^{\tau-1} \delta^t + \delta^\tau \max g_j(a)(1-\delta) + \underline{v}_j (1-\delta)\sum_{t=\tau+1}^{\infty} \delta^t$.

$= v_j (1-\delta)\frac{1-\delta^\tau}{1-\delta} + \delta^\tau \max g_j(a)(1-\delta) + \underline{V}_j (1-\delta)\frac{\delta^{\tau+1} \times 1}{1-\delta}$

$= v_j (1-\delta^\tau) + (1-\delta)\delta^\tau \max_a g_j(a) + \underline{v}_j (\delta^{\tau+1})$

eg.

	L	R
U	2,3	1,5
D	0,1	0,1

find minmax player2 = 1.

if player 2 does not deviate.

$$\text{payoff} = (1-\delta)\times 3\sum_{t=0}^{\infty}\delta^t = 3.$$

If player 2 deviates in period τ:

$3\times(1-\delta^\tau) + (1-\delta)\delta^\tau \times 5 + 1\times\delta^{\tau+1}$

$= 3 - 3\delta^\tau + 5\delta^\tau - 5\delta^{\tau+1} + \delta^{\tau+1}$

$= 3 + \delta^\tau(2-4\delta) = \underline{3+2\delta^\tau(1-2\delta)} = A.$

if $\delta < \frac{1}{2}$, then. $A > 3$.

Thus, the payoff of (2,3) in the infinite game.

can't be obtained if $\delta < \frac{1}{2}$.

chapter 1.

some strategic games.

1. Prisoner's Dilemma.
2. Bach or Stravinsky.
3. Chicken.
4. Matching pennies.
5. Stag Hunt.

Best response function:

$B(s_{-i}) = \{ s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i \}$.

NE $\Leftrightarrow$ $s_i^* \in B(s_{-i}^*)$.

$u_i'(a) = \frac{\partial u_i}{\partial a_i} = 0 \Rightarrow a_i^* = b_i(a_j)$

6. Cournot's duopoly 7. auction.

chapter 2.

static + Complete Info.

to find the Mixed NE find $\boxed{\text{firstly}}$ support(σ_i)

$$\begin{cases} u_1(s_1, \sigma_2) = u_1(s_2, \sigma_2) = \cdots = u_1(s_k, \sigma_2) \\ u_2(\sigma_1, s_1') = u_2(\sigma_1, s_2') = \cdots = u_2(\sigma_1, s_k') \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_1^* = (\ , \ ) \\ \sigma_2^* = (\ , \ , \ ) \end{cases} \Rightarrow \sigma^* = (\sigma_1^*, \sigma_2^*)$$

remember to check the strict domination!!

* A pure strategy may be dominated by a mixed strategy even if it's not dominated by any pure strategy

best response $\Rightarrow$ rationalizable strategies

* Rationalizability and iterated strict dominance coincide in two-player games.

Chapter 3.

Dynamic + Complete Info.

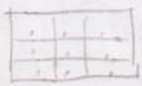
H: a set of sequences histories. including $\emptyset$

P(h): the player who moves after the history h.

$A(h) = \{a : (h, a) \in H\}$

$S_i = \times_{h_i \in H_i} A(h_i)$.

Extensive form: 

Strategic form: 

SPE: every player can't get better based on changing her strategy

better SPE: every player can't get better based on changing other's ....

$\Gamma(h)$: subgame.

eg. Divide 100 objects.

eg. H/L - C/ch

Info set, history.

behavior strategies.

mixed strategies.

eg. Divide 2 objects.

talk about player 2.

2 methods $\begin{cases} \text{subgame perfection.} \\ \text{backwards induction.} \end{cases}$

more general for imperfect info.

eg. Chain store game

中心思想:

In a dynamic games, stratgies are in the whole game, what you have to do in every possible plays.

Repeated games.

eg. twice repeated. with (δ).

$$\begin{cases} 1-b^* = \delta_1 a^* \\ 1-a^* = \delta_2 b^* \end{cases}$$

if $\delta_1 = \delta_2 = \delta$, $a^* = \frac{1}{1+\delta}$.

the one who goes firstly will get more.

eg. CE: correlated equilibrium.

private signal $\rightarrow$ highest symmetric payoff.

public signal. (using correlated equilibria)

	L	R
U		
D		

Finitely repeated.

$\begin{cases} \text{Unique NE for stage - repeat the NE for all times} \\ \text{Multiple NE for stage - "Punishment"} \end{cases}$

If public signal before period 2.

talk about all details of every cases.

Infinitely repeated.

$\underline{V}_i = \min_{\alpha_{-i}} [\max g_i(\alpha_i, \alpha_{-i})]$.

m_2^1 : player 2's strategy to ~~min~~ $[\max g_i]$

use set of feasible payoff and $\overbrace{(\underline{V}_1, \underline{V}_2)}^{\text{min max}}$

to find the feasible individually rational payoffs.

Folk theorems:

If player j deviates in period τ obtains no more than:

$$V_j(1-\delta^\tau) + (1-\delta)\delta^\tau \max_a g_j(a) + \underline{V}_j(\delta^{\tau+1})$$

原本的收益

Static Games with Incomplete Information

Example: Contributing to public good

- Two players make a 0-1 decision about contributing to a public good
- For the £xed costs c_i of contributing their payoff is given by

	C	N
C	$1-c_1, 1-c_2$	$1-c_1, 1$
N	$1, 1-c_2$	$0, 0$

1

C : contribute.
n : non-contribute.

Static Games with Incomplete Information

- Each player knows his own cost, but not the opponent's cost
- Both players believe that it is common knowledge that the costs are drawn independently from the same continuous and strictly increasing cdf $P(\cdot)$ on $[\underline{c}, \bar{c}]$ where $\underline{c} < 1 < \bar{c}$
- A pure strategy is a function $s_i(c_i)$: $[\underline{c}, \bar{c}] \to \{0,1\}$
- Payoff function is $u_i(s_i, s_j, c_i) = \max(s_1, s_2) - c_i s_i$

2

If c_i is low, → C
If c_i is high, → N

Static Games with Incomplete Information

- A *Bayesian equilibrium* is a pair of strategies (strategy pro£le) $(s_1^*(\cdot), s_2^*(\cdot))$ such that

$$s_i^*(\cdot) = \arg\max_{s_i \in S_i^{C_i}} E_{c_j} u_i(s_i, s_j^*(c_j), c_i)$$

where $S_i^{C_i}$ is the set of maps from $[\underline{c}, \bar{c}]$ to $\{0,1\}$

3

Static Games with Incomplete Information

- Let $z_j = Pr(s_j^*(c_j) = 1)$
- Then it is optimal for player i to contribute if $c_i < 1 - z_j$, i.e. every player contributes only if $c_i \le c_i^*$
- Thus the threshold value is found from $c_i^* = 1 - z_j = 1 - P(c_j^*)$
- Symmetric equilibrium if there is unique c^* such that $c^* = 1 - P(1 - P(c^*))$

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Static Games with Incomplete Information — De£nitions

- Players have associated *types* θ_i such that $\theta_i \in \Theta_i$ are drawn from a probability distribution $p(\theta_1, \dots, \theta_I)$
- Type θ_i is known to player i only
- Payoff function $u_i(s_1, \dots, s_I, \theta_1, \dots, \theta_I)$
- $S_i^{\Theta_i}$ – the space of pure strategies of player i – set of all maps from Θ_i to S_i → pure strategies
- $\Sigma_i^{\Theta_i}$ – the space of pure strategies of player i – set of all maps from Θ_i to Σ_i → mixed strategies

5

ex-ante
事前

ex-post
事后

Static Games with Incomplete Information — De£nitions

- The (pure) strategy pro£le $s(\cdot)$ is a Bayesian equilibrium if, for each player i,

$$s_i(\cdot) \in \arg\max_{s_i' \in S_i^{\Theta_i}} \sum_{\theta_i} \sum_{\theta_{-i}} p(\theta_i, \theta_{-i}) \times$$
$$\times u_i(s_i'(\theta_i), s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i}))$$

- Equivalently:

$$s_i(\theta_i) \in \arg\max_{s_i' \in S_i} \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) u_i(s_i', s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i}))$$

6

P×(1-q)之类的
P×q.
试
不一定是
无关事件?

calculate firstly

this is simpler.

Static Games with Incomplete Information — War of attrition

- Two players simultaneously choose numbers s_i in $[0, \infty)$
- The payoffs are

$$u_i(s_i, s_j) = \begin{cases} -s_i & \text{if } s_j \geq s_i \quad \text{lose} \\ \theta_i - s_j & \text{if } s_j < s_i \quad \text{win} \end{cases}$$

- The type θ_i is player i's private information and takes values in $[0, \infty)$ with cdf P and pdf p
- We are interested in pure strategy Bayesian equilibria

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Static Games with Incomplete Information — War of attrition

- $s_i(\theta_i)$ must satisfy

$$s_i(\theta_i) \in \arg\max_{s_i} \{-s_i Pr(s_j(\theta_j) \geq s_i) + \int_{\{\theta_j | s_j(\theta_j) < s_i\}} (\theta_i - s_j(\theta_j)) p_j(\theta_j)\, d\theta_j\}\}$$

win

- We can show that we desired $s_i(\cdot)$ is strictly increasing and continuous (see textbook)

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Static Games with Incomplete Information — War of attrition

- Introducing the inverse function $\Phi_i(s_i)$ we can write θ_i

$$s_i(\theta_i) \in \arg\max_{s_i} \{-s_i(1 - P_j(\Phi_j(s_i))) + \int_0^{s_i} (\theta_i - s_j) p_j(\Phi_j(s_j)) \Phi_j'(s_j)\, ds_j\}$$

- Differentiating w.r.t. s_i and equating to 0 we get:

$$\Phi_i(s_i) p_j(\Phi_j(s_i)) \Phi_j'(s_i) = 1 - P_j(\Phi_j(s_i))$$

$s_i(\theta_i) \rightarrow \theta_j(s_j)$

$\theta_j = \Phi_j(s_j)$

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Static Games with Incomplete Information — War of attrition

- Assuming $P_1 = P_2 = 1 - \exp(-\theta)$ £nd:
  - symmetric equilibria
  - asymmetric equilibria

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Static Games with Incomplete Information — Iterated strict dominance

Example: Contributing to public good:

- Assume that $\underline{c} < 1 - P(1)$ and there exists unique c^* such that $c^* = 1 - P(1 - P(c^*))$
- First round: Contributing is strictly dominated for $c_i \in (c_1, \bar{c})$ where $c_1 = 1$
- Second round: Not contributing is strictly dominated for $c_i \in [\underline{c}, c_2)$ where $c_2 = 1 - P(c_1)$

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Static Games with Incomplete Information — Iterated strict dominance

- Round $2k$: Not contributing is strictly dominated for $c_i < c^{2k}$ where $c^{2k} = 1 - P(c_{2k-1})$
- Round $2k+1$: Contributing is strictly dominated for $c_i > c^{2k+1}$ where $c^{2k+1} = 1 - P(c_{2k})$
- $c^{2k} \rightarrow c_-$ and $c^{2k+1} \rightarrow c_+$ such that $c_\pm = 1 - P(c_\mp)$ (since $P(\cdot)$ is continuous)
- Therefore $c_+ = c_- = c^*$

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$Z_j = Pr(S_j^*(c_j) = 1)$ → contribute

→ non-contribute

E $u_i(0, S_j^*(c_j)) = Z_j \cdot 1 + (1-Z_j)\cdot 0 = Z_j$.

$u_i(1, S_j^*(c_j)) = 1 - c_i$ map

$$\boxed{S_i^*(c_i) = \begin{cases} 0, & \text{if } Z_j > 1-c_i \\ 1, & \text{if } Z_j < 1-c_i \end{cases}}$$

$$= \begin{cases} 0, & c_i > 1 - Z_j \equiv c_i^* \\ 1, & c_i < 1-Z_j \equiv c_i^* \end{cases}$$. 密度分布

$c_i^* = 1 - Z_j = 1 - Pr(S_j^*(c_j) = 1)$ 累积分布

$= 1 - Pr(c_j < c_j^*) = 1 - P(c_j^*)$.

$$\begin{cases} c_1^* = 1 - P(c_2^*) \\ c_2^* = 1 - P(c_1^*) \end{cases}$$

Symmetric BE.

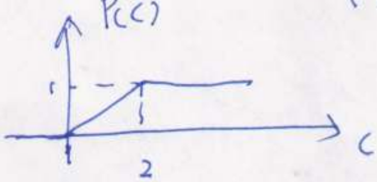
$c_1^* = c_2^* = c^*$.

$c^* = 1 - P(c^*)$.

对 argmax{ A }

A 求导 Ps.

$c \sim U[0,2]$. $P(c) = \begin{cases} 0, & c<0 \\ \frac{c}{2}, & 0 \le c \le 2 \\ 1, & c > 2 \end{cases}$



$c^* = 1 - \frac{c^*}{2}$, $c^* = \frac{2}{3}$.

$$\begin{cases} f(x) = 10 - x^2 \\ \max f(x) = 10, \\ \arg\max f(x) = 0. \end{cases}$$

$*$: $S_i(c_i): [\underline{c}, \bar{c}] \to \{0,1\}$

$$\begin{cases} \text{when } c_i < c_i^*: & S_i(c_i) = 1 \\ \text{when } c_i > c_i^*: & S_i(c_i) = 0. \end{cases}$$

→ $E(\theta) = 1$.

eg. $\underline{\lambda=1}$, $\theta_1 = 3$, $\theta_2 = 4$ $\begin{cases} S_i(\theta_i) \\ \theta_i = \varphi_i(S) \end{cases}$ means: the same

$S_1^* = 4.5$, $S_2^* = 8$.

$u_1(S_1^*) = -4.5$.

$u_2(S_2^*) = -0.5$.

$Pr(S_j(\theta_j) > S_i)$ ∵ $S(\varphi)\uparrow$ ⇓ ↑ ⇓ ∴

Φ_i → $= \theta_j = \Phi_j(S_j)$

$Pr(\Phi_j(S_j(\theta_j)) > \Phi_j(S_i))$

$= Pr(\theta_j > \Phi_j(S_i))$.

$= 1 - P_j(\Phi_j(S_i))$.

$$-(1 - P_j(\Phi_j(S_i)) + S_i\, p_j(\Phi_j(S_i))\Phi_j'(S_i)$$

$$\cancel{+(\theta_i - S_i)\Phi_j(S_i)\cdot\Phi_j'(S_i)}$$

$$+(\theta_i - S_i)\, p_j(\Phi_j(S_i))\, \Phi_j'(S_i) = 0.$$

$\Rightarrow \Phi_i(S_i) p_j(\Phi_j(S_i))\Phi_j'(S_i) = 1 - P_j(\Phi_j(S_i))$.

If $P_i(\theta) = P_j(\theta) = 1 - e^{-\lambda\theta}$.

$\theta \sim$ the same distribution. $p_j(\theta) = \lambda e^{-\lambda\theta}$.

∴ $\cancel{e^{-\lambda\Phi_j(S_i)}} = \Phi_i(S_i)\, \lambda \cancel{e^{-\lambda\Phi_j(S_i)}}\, \Phi_j'(S_i)$

$\Rightarrow \Phi_i(S_i)\Phi_j'(S_i) = \frac{1}{\lambda}$.   right hand is a number

$\Rightarrow \begin{cases} \Phi_1(S_1)\Phi_2'(S_1) = \frac{1}{\lambda} \\ \Phi_2(S_2)\Phi_1'(S_2) = \frac{1}{\lambda} \end{cases}$ thus: S_1, S_2, S don't matter 反正都被消掉了.

$\boxed{\text{asymmetric BE}}$: $\Rightarrow \boxed{\begin{cases} \Phi_1(S)\Phi_2'(S) = \frac{1}{\lambda} \\ \Phi_2(S)\Phi_1'(S) = \frac{1}{\lambda} \end{cases} \quad \text{add: } \Phi_1(0) = \Phi_2(0) = 0}$

same function.

$\boxed{\text{Symmetric BE}}$:

$\Phi_1(S) = \Phi_2(S) = \Phi(S)$.

$$\begin{cases} \Phi(S)\Phi'(S) = \frac{1}{\lambda} \\ \Phi(0) = 0. \end{cases}$$

$\Phi'(S) = \frac{1}{\lambda\Phi(S)} \Rightarrow \begin{cases} \Phi(S) = \sqrt{\frac{2}{\lambda}S} \\ \Phi'(S) = \frac{1}{\sqrt{2\lambda S}} \end{cases}$

$\Phi = \sqrt{\frac{2S}{\lambda}} \to S = \frac{\lambda\theta^2}{2}$.

$S^*(\theta) = \frac{\lambda\theta^2}{2}$.

Static Games with Incomplete Information — Iterated strict dominance

Example:

Interim iterated dominance (type of player 1 is determined)

	L	R
U	10,12	10,0
D	0,0	12,10

	L	R
U	12,0	0,10
D	10,12	10,0

No strategy can be eliminated for player 1

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Static Games with Incomplete Information — Iterated strict dominance

Example:

Ex ante iterated dominance

	L	R
UU	11,6	5,5
UD	10,12	10,0
DU	6,0	6,10
DD	5,6	11,5

Unique outcome survives

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asymmetric BE: $\phi_1'(s)\varphi_2(s) - \varphi_1(s)\varphi_2'(s) = 0$.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad \left(\frac{\varphi_1(s)}{\varphi_2(s)}\right)' = 0 \Rightarrow \frac{\varphi_1(s)}{\varphi_2(s)} = K.$$

$$\varphi_1(s) = K\varphi_2(s).$$

---

3 players public contribute game.

I am i. $z_j = \Pr(s_j^*(\theta_j) = 1)$

$z_k = \Pr(s_k^*(\theta_k) = 1)$.

$$u_i(1, s_j^*, s_k^*) = 1 - c_i.$$

$$E\, u_i(0, s_j^*, s_k^*) = 0\cdot(1-z_j)(1-z_k) + 1\cdot(1-(1-z_j)(1-z_k))$$

$$s_i^*(c_i) = \begin{cases} 1, & c_i < c_i^* \\ 0, & c_i > c_i^* \end{cases}$$

$c_i^* = (1-z_j)(1-z_k)$ cdf not pdf

$z_j = \Pr(s_j^* = 1) = \Pr(c_j < c_j^*) = P(c_j^*)$

likewise, $z_k = P(c_k^*)$.

$$c_i^* = (1 - P(c_j^*))(1 - P(c_k^*)).$$

Symmetric BE:

$$c_1^* = c_2^* = c_3^* = c^*.$$

$$c^* = (1-P(c^*))(1-P(c^*)) = (1-P(c^*))^2$$

$c \sim U[0,2]$. $P(c) = \begin{cases} 0, & c<0 \\ \frac{c}{2}, & 0 \le c \le 2 \\ 1, & c>2 \end{cases}$

$$c^* = \left(1 - \frac{c^*}{2}\right)^2$$

$$c^* = 4 - 2\sqrt{3}$$

for every player $s_i^*(c_i) = \begin{cases} 1, & c_i < c_i^* \\ 0, & c_i > c_i^* \end{cases}$

Exchange Game.

b_i is the ~~tibet~~ nuber of player i.

$b_j \sim U[0,1]$, non-exchange.

$u_i(0, s_j^*) = b_i$ exchange.

$E\,u_i(1, s_j^*) = b_i \Pr(s_j^* = 0) + \cancel{b_j \Pr(s_j^* = 1)}$

$E(b_j \mid s_j^* = 1)\cdot \Pr(s_j^* = 1)$.

$\Pr(s_j^* = 1) = z_j$.

$$b_i^* = b_i^*(1 - z_j) + E(b_j \mid s_j^* = 1)\cdot z_j.$$

$$b_i^* z_j = E(b_j \mid s_j^* = 1)\cdot z_j.$$

$$b_i^* = E(b_j \mid s_j^* = 1).$$

$$b_i^* = E(b_j \mid b_j < b_j^*) = \frac{b_j^*}{2}.$$

because $b_j \sim U[0,1]$.

$$\begin{cases} b_1^* = \frac{b_2^*}{2} \\ b_2^* = \frac{b_1^*}{2} \end{cases} \Rightarrow b^* = (0,0).$$

They will never exchange.

Two-period reputation game → 名誉 ~~2,1,0,1,~~

Separating equilibrium:

player 1 of type "S":

$\sigma_1(A|S)=1$

(δ: discount factor) from period 0 ~~to~~ to period 1.

payoff $= D_1 + \delta D_1$

If player 1 deviates in period 0

he will obtain $P_1 + \delta M_1$.

For separating equilibrium to exist.

$$D_1 + \delta D_1 \geq P_1 + \delta M_1$$

$$\underbrace{D_1 - P_1}_{\text{cost of prey}} \geq \underbrace{\delta (M_1 - D_1)}_{\text{benefit of "monopoly"}}$$

Pooling equilibrium

player 2 has to exit.

$E\, u_2(S) = p D_2 + (1-p) P_2$.

$u_2(E) = 0$.

$p D_2 + (1-p) P_2 \leq 0$.

$+\ D_1 - P_1 < \delta (M_1 - D_1)$.

$$p \leq \frac{-P_2}{D_2 - P_2} = \frac{|P_2|}{D_2 + |P_2|}$$

Find y

player 1 is indifferent between P and A.

$u_1(A) = D_1 + \delta D_1$

$E u_1(P) = P_1 + \delta[(1-y) M_1 + y D_1]$

$$y = 1 - \frac{D_1 - P_1}{\delta (M_1 - D_1)}$$

Hybrid equilibrium

player 2 belief update

$$u_2(S|P) = \frac{P(P|S)\cdot P(S)}{P(P|S)\cdot P(S) + P(P|C)\cdot P(C)}$$

$$= \frac{\sigma_1(P|S)\cdot p}{\sigma_1(P|S)\cdot p + 1\cdot(1-p)} = \frac{(1-x)p}{(1-x)p + 1 - p} = \frac{p - px}{1 - px}$$

$$u_2(C|P) = \frac{1-p}{1-xp}.$$

Find x

player 2 is indifferent between stay and exit.

$u_2(E) = 0$

$$E\, u_2(S) = D_2 \frac{p - px}{1 - px} + P_2 \frac{1-p}{1-px} = 0.$$

$$x = 1 - \frac{(1-p)(-P_2)}{p D_2}$$

Dynamic Games with Incomplete Information

- Subgame perfect equilibrium is not much use in games of incomplete information
- We need other equilibrium refinements (that take beliefs and their update into account)
  - perfect Bayesian equilibrium
  - sequential equilibrium

PBE

1

Dynamic Games with Incomplete Information

Example: Two-period reputation game

- Players: two firms
- In period 0, firm 1 takes an action a_1 from set $\{A, P\}$
- In period 1, firm 2, after observing the action, takes an action a_2 from set $\{S, E\}$
- Firm 1 can have two types: $\theta = \{s, c\}$. From point of view of firm 2, the probability that $\theta = s$ is equal to p
- Type $\theta = c$ always plays $a_1 = P$

2

accommodate 兼容.

prey 掠夺.

stay

exit.

crazy: always prey

sane: will decide.

Dynamic Games with Incomplete Information

- Payoff of player 1: in any period, $g_1(A,S) = D_1$, $g_1(P,S) = P_1$, $g_1(\cdot, E) = M_1$ where $M_1 > D_1 > P_1$
- Payoff of player 2: in any period, $g_2(A,S) = D_2$, $g_1(P,S) = P_2$, $g_1(\cdot, E) = 0$ where $D_2 > 0 > P_2$

3

for player 1. different type chooses different behavior — "message" for player 2.

Dynamic Games with Incomplete Information

Separating equilibrium

action: $\sigma_1(A|s) = 1$, $\sigma_2(S|A) = 1$, $\sigma_2(E|P) = 1$.

belief: $\mu_2(s|A) = 1$, $\mu_2(s|A) = 1$, $\mu_2(c|P) = 1$

exists when

$$D_1 - P_1 \geq \delta(M_1 - D_1)$$

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for player 1. different type has the same behavior which is "message" for player 2.

Dynamic Games with Incomplete Information

Pooling equilibrium

$\sigma_1(P|s) = 1$, $\sigma_2(E|P) = 1$, $\mu_2(s|A) = 1$, $\mu_2(s|P) = p$, $\mu_2(c|P) = 1 - p$

exists when

$$D_1 - P_1 < \delta(M_1 - D_1)$$

and

$$pD_2 + (1-p)P_2 \leq 0$$

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mixed

Dynamic Games with Incomplete Information

Hybrid equilibrium

$\sigma_1(A|s) = x$, $\sigma_1(P|s) = 1 - x$, $\sigma_2(S|P) = y$, $\sigma_2(E|P) = 1 - y$, $\mu_2(s|A) = 1$, $\mu_2(s|P) = \frac{p - px}{1 - px}$, $\mu_2(c|P) = \frac{1-p}{1-px}$ where

$$x = 1 - \frac{(1-p)(-P_2)}{pD_2}$$

and

$$y = 1 - \frac{D_1 - P_1}{\delta(M_1 - D_1)}$$

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$p = P(s)$.

player 2 behaviors based on his belief.

PBE

~~pure strategy~~

Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

- Consider the case of multi-stage games with observed actions
- Each player $i = 1, \ldots, I$ has a private type $\theta_i \in \Theta_i$ distributed according to pdf $p_i(\cdot)$
- Types are independent: $p(\theta) = \prod_{i=1}^{I} p_i(\theta_i)$
- Game is played in periods $t = 0, 1, \ldots, T$ with players moving simultaneously
- $h^t = (a_0, \ldots, a^{t-1})$ history at the beginning of period t

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Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

- Behavior strategy of player i: $\sigma_i: (H, \Theta_i) \to \Sigma_i$
- Payoff function of player i: $u_i(h^{T+1}, \theta)$
- Look for Bayesian equilibria that are also such for all continuation games starting from period $t \neq 0$
- For each continuation subgame, we need to specify updated beliefs of player i about types of opponents

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Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

Belief update:

- Posterior beliefs are independent:

$$\mu_i(\theta_{-i}|\theta_i, h^t) = \prod_{j \neq i} \mu_i(\theta_j|h^t)$$

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Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

- Bayes rule is used for updating from t to $t+1$:

$$\mu_i(\theta_j|(h^t, a^t)) = \frac{\mu_i(\theta_j|h^t)\sigma_j(a_j^t|h^t, \theta_j)}{\sum_{\theta_j'} \mu_i(\theta_j'|h^t)\sigma_j(a_j^t|h^t, \theta_j')}$$

Note: if player j's action in period t has conditional probability 0, the above rule tells us nothing about $\mu_i(\theta_j|(h^t, a^t))$

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Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

- The updating process is not influenced by actions of other players:

$$\mu_i(\theta_j|(h^t, a^t)) = \mu_i(\theta_j|(h^t, \hat{a}^t)) \text{ if } a_j^t = \hat{a}_j^t$$

- The posterior beliefs are consistent with a common joint distribution:

$$\mu_i(\theta_k|h^t) = \mu_j(\theta_k|h^t) \text{ for } i \neq j \neq k$$

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Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

Best response: For each player i, history h^t and alternative strategy σ_i'

$$u_i(\sigma|h^t, \theta_i, \mu(\cdot, h^t)) \geq u_i((\sigma_i', \sigma_{-i})|h^t, \theta_i, \mu(\cdot, h^t))$$

Definition: A perfect Bayesian equilibrium is the "pair" (σ, μ) that satisfies the above conditions

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Two - Period PG Game

Belief update.

Bayes update of the distribution. pdf.

$f(x) \to f(x|A)$.

$$f(x|A) = \frac{P(A|x) \cdot f(x)}{P(A)}.$$

$$P(A) = \int P(A|x') f(x') \, dx'.$$

Cumulative update:

$F(x) \to F(x|A)$.

$$F(x) = \int_{-\infty}^{x} f(y) \, dy.$$

$$F(x|A) = \int_{-\infty}^{\cancel{\infty}} f(y|A) \, dy,$$

$$\boxed{F(x|A) = \frac{1}{P(A)} \int_{-\infty}^{x} P(A|y) f(y) \, dy}.$$

从player i 的视角,
因为 i 知道自己的 c_i.

$$P(c_j|00) = \frac{1}{Pr(00)} \int_{-\infty}^{c_j} Pr(00|c_j') \, p(c_j') \, dc_j'.$$

$$Pr(00) = Pr(c_j > \hat{c}) = 1 - P(\hat{c}).$$

$$Pr(00|c_j) = \begin{cases} 0, & \text{if } c_j < \hat{c} \\ 1, & \text{if } c_j > \hat{c}. \end{cases}$$

$$P(c_j|00) = \frac{1}{1 - P(\hat{c})} \cdot \begin{cases} 0, & \text{if } c_j < \hat{c}. \\ \int_{\hat{c}}^{c_j} p(c_j') \, dc_j', & \text{if } c_j > \hat{c}. \end{cases} \Rightarrow P(c_j) - P(\hat{c}).$$

$0 \quad c_j \quad \hat{c}$

$1 \times P(c_j)$

$0 \quad \hat{c} \quad c_j$

$$P(c_j|00) = \begin{cases} 0, & \text{if } [\underline{c}, \hat{c}] \\ \dfrac{P(c_j) - P(\hat{c})}{1 - P(\hat{c})}, & \text{if } [\hat{c}, \bar{c}]. \end{cases}$$

what are players are going to do if period 1 if $h' = 00$.

Recall: In a single-period PG game. the symmetric BE was given by:

$$c^* = 1 - P(c^*).$$

$$\check{c} = 1 - P(\check{c}|00).$$

$$\check{c} = 1 - \frac{P(\check{c}) - P(\hat{c})}{1 - P(\hat{c})}.$$

$$= \frac{1 - P(\check{c})}{1 - P(\hat{c})}.$$

Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

Example: Two period public good game

- Two players make decisions whether to contribute to public good in two periods
- In each period, the public good (if contributed to by at least one player) gives each player a utility of 1
- The cost of contributing for player i is c_i and is known to player i only

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PBE

Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

- In the beginning of the game the knowledge of player i about the cost of player j is described by cdf $P(\cdot)$ which is public knowledge
- The utility of the second period is discounted with factor $\delta < 1$

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Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

- Strategy for player i: $\sigma_i^0(x|c_i)$, $\sigma_i^1(x|h^1, c_i)$ where $x \in \{0, 1\}$ and $h^1 \in \{00, 01, 10, 11\}$
- We can show that, in any PBE, there exists a cutoff cost $\hat{c}_i$ such that player i contributes iff $c_i < \hat{c}_i$
- We are interested in a symmetric PBE where $\hat{c}_1 = \hat{c}_2 = \hat{c}$

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Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

The belief update

1. Nobody contributed in period 0 $h^1 = (0,0)$.

$$P(c_i|00) = \begin{cases} \frac{P(c_i) - P(\hat{c})}{1 - P(\hat{c})} & \text{if } c_i \in [\hat{c}, \bar{c}] \\ 0 & \text{if } c_i \in [\underline{c}, \hat{c}] \end{cases}$$

Then in symmetric period 1 equilibrium player i contributes iff $c_i \leq \check{c}$ where

$$\check{c} = \frac{1 - P(\check{c})}{1 - P(\hat{c})}$$

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$\check{c}$ is new $\hat{c}$

4 ~~scenarios~~ scenarios.

Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

The belief update

2. Both players contributed in period 0

$$P(c_i|11) = \begin{cases} \frac{P(c_i)}{P(\hat{c})} & \text{if } c_i \in [\underline{c}, \hat{c}] \\ 1 & \text{if } c_i \in [\hat{c}, \bar{c}] \end{cases}$$

Then in symmetric period 1 equilibrium player i contributes iff $c_i \leq \tilde{c}_i$ where

$$\tilde{c} = \frac{P(\hat{c}) - P(\tilde{c})}{P(\hat{c})}$$

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$\tilde{c}$ is new $\hat{c}$

Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

The belief update

3. Only player i contributed in period 0

The opponents know that $c_i \leq \hat{c}$ and $c_j \geq \hat{c}$, respectively

In one (obvious) equilibrium, player i contributes in period 1 and player j does not

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period 0的报酬 × period 1的 可能结果 ∵ t是条件的.

P(ĉ), 是 period 0 的

the whole payoff given I contribute in period 0

∵ don't contribute in period 0.

Dynamic Games with Incomplete Information — Perfect Bayesian equilibrium

We can find $\hat{c}$ from the condition that type $\hat{c}$ be indifferent between contributing and not contributing:

$$1-\hat{c}+\delta(P(\hat{c})v^{11}(\hat{c})+[1-P(\hat{c})]v^{10}(\hat{c}))$$

$$=P(\hat{c})+\delta(P(\hat{c})v^{01}(\hat{c})+[1-P(\hat{c})]v^{00}(\hat{c}))$$

where $v^{00}(\hat{c})=1-\hat{c}$, $v_{11}(\hat{c})=\frac{P(\hat{c})}{P(\hat{c})}$, $v^{10}(\hat{c})=1-\hat{c}$, $v^{01}(\hat{c})=1$

Substituting we obtain

$$1-P(\hat{c})=\hat{c}+\delta P(\hat{c})\bar{c}$$

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history = 0,0. I'm more willing to distribute.

ĉ 是自己的 心理 阈值

I know the other one's cost is high. I will distribute again. opposite above.

Dynamic Games with Incomplete Information — Multi-period reputation game — Example: chain store game...

- A long-run player (chain store) is entering N markets in succession (labelled backwards: N to 1).
- In each market there is a potential short-run competitor (local store) that can either enter (compete with the chain store) or stay out.
- If a local store enters, the chain store can either accommodate it or fight.
- The chain store has two possible types: sane and crazy. The crazy type always fights

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Dynamic Games with Incomplete Information — Multi-period reputation game — Example: chain store game...

- If a competitor enters and the sane chain store accommodates it (the CS) gets a payoff of 0. If the chain store fights it gets -1, and if the competitor stays out it obtains $a>0$
- A competitor obtains 0 if it stays out, -1 if it enters and is fought, and b if it enters and gets accommodated
- Before the game begins, all competitors believe that the chain store is crazy with probability p_c.

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Dynamic Games with Incomplete Information — Multi-period reputation game — ...Example: chain store game...

- The competitors observe the results of the play before their move and update their beliefs.

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Dynamic Games with Incomplete Information — Multi-period reputation game — ...Example: chain store game...

- We will solve the problem by induction.
- Let $P_i(F)$ be the probability that the entrant i is fought. Entrant i will enter iff $P_i(F)<\bar{p}\equiv\frac{b}{b+1}$.
- If there is only one market (period) in the game, the sane chain store will always accommodate an entry.
- The (only) entrant 1 will enter if and only if $p_c<\bar{p}$ (since $P_1(F)=p_c$).

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∵ no other info

Dynamic Games with Incomplete Information — Multi-period reputation game — ...Example: chain store game...

- Suppose that there are two periods in the game. Entrant 2 goes first, and entrant 1 can observe the result before making decision.
- If faced with an entry in market 2, the chain store will not fight it if the cost of fighting in market 2 (1) exceeds the possible benefit of monopoly in the first market (a), i.e. if $a<1$ (separating equilibrium).
- If $a>1$, the (sane) chain store will fight an entry if, in addition, fighting causes the second competitor to

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$V''(\hat{c})$ is expected ~~[illegible]~~ payoff.

in period 1 for player with cost $\hat{c}$ if $h = 11$.

$$V''(\hat{c}) = 1 \, \underline{\Pr(\text{player } j \text{ contribute})}. \quad P(c_j|11)$$

$$= \Pr(c_j < \hat{c})$$

$\tilde{c}$ $\hat{c}$ $\check{c}$

$$= \cancel{\quad} \frac{P(\tilde{c})}{P(\hat{c})}.$$

conditional distribution

$\check{c}$ nobody contribute 00

$\tilde{c}$ ~~nobody~~ both contribute 11.

$$\begin{cases} \check{c} = \dfrac{1 - P(\check{c})}{1 - P(\hat{c})} \\ \tilde{c} = \dfrac{P(\hat{c}) - P(\tilde{c})}{P(\hat{c})} \\ 1 - P(\hat{c}) = \hat{c} + \delta P(\hat{c}) \tilde{c} \end{cases}$$

3 equilibriums

3 variables

CS game

consider competitor (local store) #i.

$$E \begin{cases} u_i(\text{out}) = 0. \\ u_i(I_n) = P_i(F)(-1) + (1 - P_i(F)) b \\ \quad = -P_i(F)(1+b) + b. \end{cases}$$

$$P_i(F) = \frac{b}{b+1} \equiv \tilde{p} \quad \text{定值}$$ 阈值 threshold

$$S_i^* = \begin{cases} I_n. & \text{if } P_i(F) < \tilde{p} \\ \text{out}. & \text{if } P_i(F) > \tilde{p}. \end{cases}$$

player i can randomize only if $P_i(F) = \tilde{p}$.

$N = 2$.

$$P_C > \tilde{p} \Rightarrow P_2(F) \geq P_C > \tilde{p}.$$

player 2 stays out

$$P_1(F) = P_C > \tilde{p}.$$

player 1 stays out.

$$\left.\begin{matrix} P_C < \tilde{p} \\ a > 1 \end{matrix}\right\} \text{Hybrid equilibrium only.}$$

CS randomizes in stage 2.

player 1 randomizes in stage 1.

$$u_1(c|F) = \tilde{p}$$

$$u_1(c|F) = \frac{\overbrace{\sigma_{cs}(F|c)}^{=1} \cdot P_C}{P_2(F)} = \tilde{p}.$$

$$\boxed{P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)}}$$

it means it will fight

$$\underline{P_2(F) = \frac{P_C}{\tilde{p}} > P_C}$$ ~~[illegible]~~ not only crazy fight, sane probably fight too. 适合实际.

$$\sigma_2(I_n) = \begin{cases} 1, & \text{if } \frac{P_C}{\tilde{p}} < \tilde{p} \\ 0, & \text{if } \frac{P_C}{\tilde{p}} < \tilde{p}. \end{cases} = \begin{cases} 1, & \text{if } P_C < \tilde{p}^2 \\ 0, & \text{if } P_C > \tilde{p}^2 \end{cases}$$

$N = 3$. ~~$P_C > \tilde{p}^2$, $\sigma_3(I_n) = 0$~~

$P_C > \tilde{p}^2$ $\sigma_3(I_n) = 0$.

~~$P_C < \tilde{p}^2$~~,

CS randomizes in stage 3.

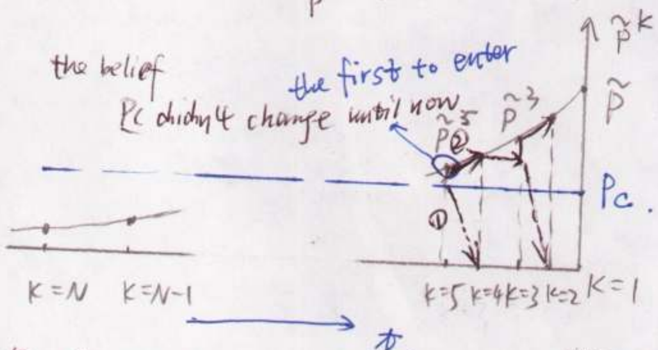
player 2 has to randomizes in stage 2.

$$u_2(c|F) = \tilde{p}_3^{\,2}$$ in city 3 (the first city)

$$u_2(c|F) = \frac{\sigma_{cs}(F|c) \cdot P_C}{P_3(F)} = \tilde{p}^2.$$

$$P_3(F) = \frac{P}{\tilde{p}^2}$$

$$\sigma_3(I_n) = \begin{cases} 1, & \text{if } \frac{P_C}{\tilde{p}^2} < \tilde{p} \\ 0, & \text{if } \frac{P_C}{\tilde{p}^2} > \tilde{p}. \end{cases} = \begin{cases} 1, & \text{if } P_C < \tilde{p}^3 \\ 0, & \text{if } P_C > \tilde{p}^3. \end{cases}$$



① at K=5, CS randomizes. if accomodate, $P_C \overset{\text{become}}{=} 0$.

② if fight. up to $\tilde{p}^4$, k=3, CS randomizes as above.

随机: K

$P_C < \tilde{p}^k$

K=5, entrant 5 是第一个进入的.

后面 k= 4, 3, 2, 1 可能都进入,

前面的 N 到 5 不进入.

PBE

Homework 6

1. There are two players, a plaintiff and a defendant in a civil suit. The plaintiff knows whether or not he will win if the case goes to trial, but the defendant does not have this information. The defendant knows that the plaintiff knows who would win, and the defendant has prior beliefs that that there is probability $\frac{1}{3}$ that the plaintiff will win; these prior beliefs are common knowledge. If the plaintiff wins, his payoff is is 3 and the defendant's payoff is -4; if the plaintiff loses, his payoff is -1 and the defendant's payoff is 0.

The plaintiff has two possible actions: he can ask either for a low settlement of $m = 1$ or a high settlement of $m = 2$. If the defendant accepts a settlement offer of m, the plaintiff's payoff is m and the defendant's is $-m$. If the defendant rejects the settlement offer, the case goes to court. List all the pure-strategy PBE strategy profiles. For each such profile, specify the posterior beliefs of the defendant as a function of m, and verify that the combination of these beliefs and the profile is in fact a PBE. Explain why the other profiles are not PBE.

2. In a multi-period chain store game with 20 markets, the payoffs of all players are just like described in class with $a = 2$ and $b = 1$. All entrants originally believe that the chain store is crazy with probability 0.1 (and the chain store knows it).

a) How many entrants will stay out in PBE?

b) What the chain store expected payoff in the whole game?

c) What would the chain store do if entrant 15 (counted from the end, like in class) entered? What would the updated beliefs of entrant 14 be?

d) What would it do if entrant 2 entered? What would the updated beliefs of entrant 1 be?

3. Suppose that in the chain store game there is one entrant, two markets (A and B) and two periods. The entrant can enter each market at most once and can enter at most one market in each period. But he can choose which market to enter first. The chain store is either sane in both markets or crazy in both. Crazy chain store always fights. Payoff in market A for both players are just like described in class for the multi-period chain store game. In market B, all payoffs are multiplied by 2. Which market should the entrant enter first?

4. For the two-period public good game discussed in class, find the threshold values for the first period and for the second period following 00 and 11 histories in the symmetric PBE. Assume that initially each player believes that the other player's cost is uniformly distributed between 0 and 2. Suppose the cost of player 1 is equal to 0.5 and the cost of player 2 is equal to 0.4. Describe how the game will be played assuming that the symmetric PBE is realized.

Homework 6.

Puxin Xu.

1. Court game

Possible seperating PBE

1). $\sigma_1(H|w)=1$, $\sigma_1(L|l)=1$, $\mu_2(w|H)=1$, $\mu_2(l|L)=1$.

$\sigma_2(A|H)=1$, $\sigma_2(R|L)=1$,

type w, No deviation $\Rightarrow$ payoff $=2$.

deviation $\Rightarrow$ payoff $=3>2$

$\because 3>2$. $\therefore$ This is no a PBE.

2). $\sigma_1(L|w)=1$, $\sigma_2(H|l)=1$, $\mu_2(l|H)=1$. $\mu_2(w|L)=1$.

$\sigma_2(A|L)=1$, $\sigma_2(R|H)=1$.

type W, No deviation. $\Rightarrow$ payoff $=1$.

deviation $\Rightarrow$ payoff $=3>1$.

$\because 3>1$. $\therefore$ This is no a PBE.

Pooling PBE.

1). $\sigma_1(H|w)=1$, $\sigma_1(H|l)=1$. $\mu_2(w|H)=p=\frac{1}{3}$. $\mu_2(l|H)=\frac{2}{3}$;

$\sigma_2(R|H)=1$, $\sigma_2=(w|L)=\frac{P(wL)}{P(L)}=\frac{0}{0}=q$

in H. $\mu_2(A)=-2$

$E\mu_2(R)=\frac{1}{3}\times(-4)=-\frac{4}{3}>-2$.

$\therefore \sigma_2(R|H)=1$.

type W. No deviation. $\Rightarrow$ payoff $=3$.

deviation $\Rightarrow$ payoff ≤ 3.

type l. No deviation. $\Rightarrow$ payoff $=-1$

deviation $\Rightarrow$ payoff $=\begin{cases}-1. & q<\frac{1}{4}\\ 1. & q>\frac{1}{4}\end{cases}$

Deviation payoff of type l, $\mu_2(A)=-1$

$E\mu_2(R)=q(-4)+(1-q)0=-4q$.

$\sigma_2(R|L)=\begin{cases}1, & \text{if } q<\frac{1}{4}\\ 0, & \text{if } q>\frac{1}{4}\end{cases}$ $\mu_1(L|l)=\begin{cases}-1. & \text{if } q<\frac{1}{4}\\ 1, & \text{if } q>\frac{1}{4}\end{cases}$

Thus, this is a PBE iff $q<\frac{1}{4}$. $\boxed{\text{ANS}}$

2). $\sigma_1(L|w)=1$, $\sigma_1(L|l)=1$. $\mu_2(w|L)=\frac{1}{3}$. $\mu_2(l|L)=\frac{2}{3}$.

$\because$ in L, $\mu_2(A)=-1$ $\therefore \mu_2(A|L)=1$.

$E\mu_2(R)=-\frac{4}{3}<-1$

type l. No deviation $\Rightarrow$ payoff $=1$

Deviation $\Rightarrow$ payoff $=\begin{cases}2 & \text{if } A\\ -1 & \text{if } R\end{cases}$

type w, No deviation $\Rightarrow$ payoff $=1$

deviation $\Rightarrow$ payoff (>1) $=\begin{cases}2, & \text{if } A\\ 3 & \text{if } R\end{cases}$

$\therefore$ This is no a PBE. for any q

There is one pooling PBE in this game.

Both types give "high" offer.

defendant rejects. $\mu_2(W|L) < \frac{1}{4}$.

2. a). $\tilde{P} = \frac{b}{b+1} = \frac{1}{2}$.

$P_c < \tilde{P}^k$, $k = 3$.

$n = 20 - 3 = 17$. 17 entrants will stay out. $\boxed{ANS}$

b). $17 \times a + 0 = 17 \times 2 = 34$.

the chain store expected payoff $= 34$ $\boxed{ANS}$

c). if entrant 15 entered, the chain store will fight.

$P_{c_{14}} = P_c$.

d). if it fighted agaist entrant 3, it will randomizes,

the $P_{c_1} = \begin{cases} 0 \\ \tilde{P} \end{cases}$. if fights agaist entrant 2.

if it accommodated entrant 3, it will accommodate.

$P_{c_1} = 0$.

3. $A: \hat{P} = \frac{b}{b+1}$; $B: \hat{P} = \frac{2b}{2b+2} = \frac{b}{b+1}$.

if $P_c > \hat{P}$, entrant will not in.

$P_c = \hat{P} = \frac{b}{b+1} \Rightarrow b = \frac{P_c}{1-P_c}$ (In Market 1 (backward)).

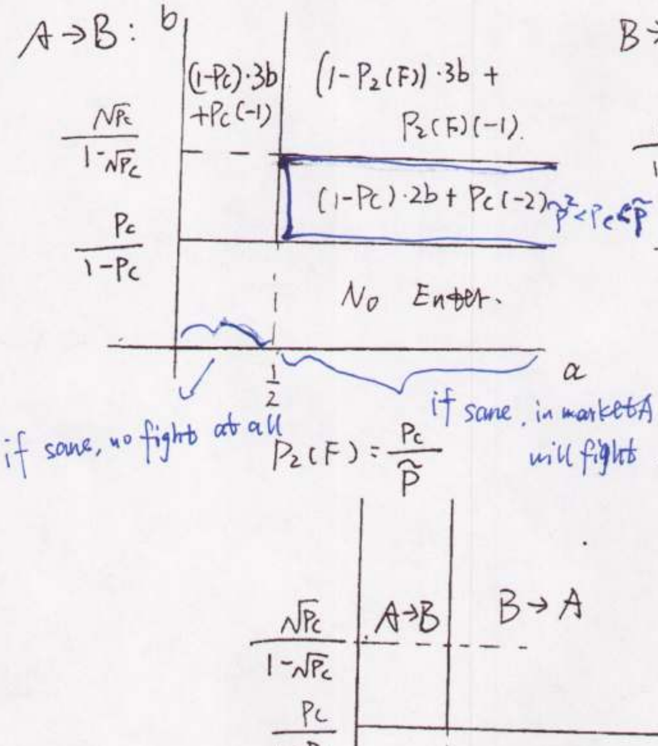
$P_c = \hat{P}^2 \Rightarrow b = \frac{\sqrt{P_c}}{1-\sqrt{P_c}}$. (In Market 2 (backward)).

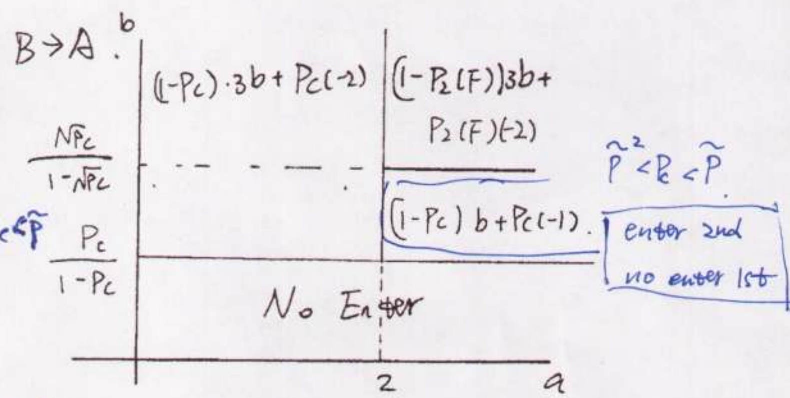
Considering the chain store:

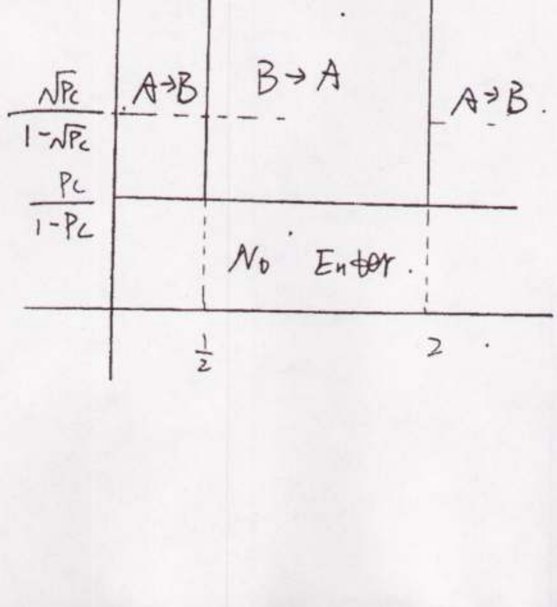
seperating equalibrium : cost of fight > benefit of monopoly.

if $A \to B$: $1 > 2a \Leftrightarrow a < \frac{1}{2}$ in A in B

if $B \to A$: $2 > a \Leftrightarrow a < 2$. in B in A.







4.

$c_i \sim U[0,2]$.

$$P(c_i) = \begin{cases} \frac{c_i}{2}, & c_i \in [0,2] \\ 0, & c_i < 0 \\ 1, & c_i > 2 \end{cases}$$

$$\begin{cases} \check{c} = \dfrac{1-P(\check{c})}{1-P(\hat{c})} \\ \tilde{c} = \dfrac{P(\hat{c}) - P(\tilde{c})}{P(\hat{c})} \\ 1 - P(\hat{c}) = \hat{c} + \delta P(\tilde{c})\, \tilde{c} \\ \delta = 1 \end{cases} \Rightarrow \begin{cases} \hat{c} = \\ \check{c} = \\ \tilde{c} = \end{cases}$$

Dynamic Games with Incomplete Information — Multi-period reputation game

...Example: chain store game...

stay out, i.e. if $p_c > \tilde{p}$ (pooling equilibrium).

Dynamic Games with Incomplete Information — Multi-period reputation game

...Example: chain store game...

- If $a > 1$ and $p_c < \tilde{p}$, the chain store can neither accommodate nor fight with probability 1 since in the former case it would be better off deviating and in the latter the second competitor would still enter thus making fighting in the first period worthless.
- Therefore the chain store has to randomize in the first period in such a way that the second entrant is indifferent between staying and exiting, i.e. so that $P_1(F) = \mu_1(c|F) = \tilde{p}$.

Dynamic Games with Incomplete Information — Multi-period reputation game

...Example: chain store game...

- For this it is required that $x = \sigma_{CS}^2(F|s) = \frac{p_c}{(1-p_c)b}$.
- But then $P_2(F) = p_c + (1-p_c)x = \frac{p_c}{\tilde{p}}$. Therefore, entrant 2 will enter if and only if $p_c < \tilde{p}^2$.

Dynamic Games with Incomplete Information — Multi-period reputation game

...Example: chain store game...

- Suppose now there are 3 markets in the game.
- If $p_c > \tilde{p}^2$ the chain store fights in market 3 and entrant 3 stays out.
- If $p_c < \tilde{p}^2$. If entrant 3 entered, it is easy to see that neither fighting nor accommodating with probability 1 can be part of PBE.
- Therefore the chain store has to randomize in market 3 so that entrant 2 is indifferent between staying out and entering, i.e. $P_2(F) = \tilde{p}^2$.

Dynamic Games with Incomplete Information — Multi-period reputation game

...Example: chain store game...

- For this we need that $x = \sigma_{CS}^3(F|s) = \frac{p_c}{1-p_c}\left[\frac{(b+1)^2}{b^2} - 1\right]$.
- Then $P_3(F) = p_c + (1-p_c)x = \frac{p_c}{\tilde{p}^2}$.
- Therefore entrant 3 enters if and only if $p_c < \tilde{p}^3$.

Dynamic Games with Incomplete Information — Multi-period reputation game

...Example: chain store game

- In general, we see that entrant k enters if and only if $\mu_k(c) < \tilde{p}^k$.
- Therefore in a game with N markets, if $\tilde{p}^{k+1} < p_c < \tilde{p}^k$, the first entrant to enter will be k, and $N - k$ entrants will stay out in PBE.
- Thus in a long game, even small amount of uncertainty about the chain store type is sufficient to make many (majority) entrants stay out.

Dynamic Games with Incomplete Information — Sequential equilibrium

- Perfect Bayesian equilibrium does not specify beliefs off the equilibrium path
- So, if a player deviates, beliefs about his/her type are largely arbitrary
- It would be useful to have an equilibrium refinement free of this drawback
- In the following, we consider a general extensive form game

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Dynamic Games with Incomplete Information — Sequential equilibrium

Notation

- Set of nodes X
- $h(x)$ is the information set containing node x
- $i(x)$ is the player at the move at node x
- Behavior strategy of player i at info set $h(x)$: $\sigma_i(\cdot|h(x))$
- Given the strategy profile σ, $P^\sigma(x)$ and $P^\sigma(h)$ are the probabilities that node x and info set $h(x)$ are reached

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Dynamic Games with Incomplete Information — Sequential equilibrium

Consistency

- Σ^0 is the set of all *completely mixed* behavior strategies so that if $\sigma \in \Sigma^0$, then $P^\sigma(x) > 0$ for all nodes x
- For any x, let $\mu(x) = \frac{P^\sigma(x)}{P^\sigma(h(x))}$ (Bayes rule)
- Let Ψ^0 be the set of all *assessments* (σ, μ) such that $\sigma \in \Sigma^0$ and μ is found from σ by the Bayes rule above

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Dynamic Games with Incomplete Information — Sequential equilibrium

Definition: An assessment (σ, μ) is *consistent* if

$$(\sigma, \mu) = \lim_{n\to\infty} (\sigma^n, \mu^n)$$

for some sequence (σ^n, μ^n) in Ψ^0

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Dynamic Games with Incomplete Information — Sequential equilibrium

Sequential rationality Definition: An assessment (σ, μ) is *sequentially rational* if, for any information set h and any alternative strategy $\sigma'_{i(h)}$,

$$u_{i(h)}(\sigma|h, \mu(h)) \geq u_{i(h)}((\sigma'_{i(h)}, \sigma_{-i(h)})|h, \mu(h))$$

where $u_{i(h)}(\sigma|h, \mu(h))$ is the expected utility of player $i(h)$ given that information set h is reached, the beliefs are given by $\mu(h)$ and the strategy profile is σ

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Dynamic Games with Incomplete Information — Sequential equilibrium

Definition: A *sequential equilibrium* is an assessment (σ, μ) that satisfies the above conditions of consistency and sequential rationality

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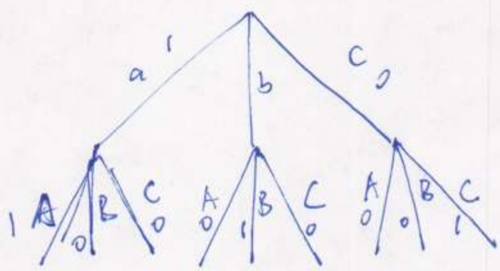

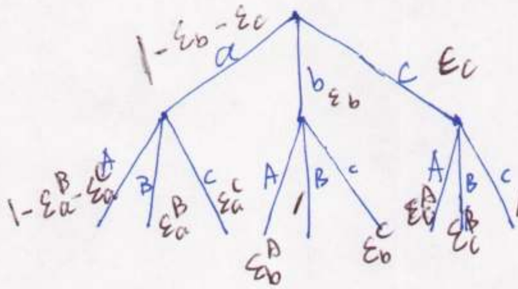

$$\boxed{\mu_2(c|B) = \frac{\varepsilon_c \varepsilon_c^B}{\varepsilon_a^B + \varepsilon_b + \varepsilon_c \varepsilon_c^B} \to 1}$$

$$\mu_2(b|C) = \frac{\varepsilon_b \varepsilon_b^C}{\varepsilon_a^C + \varepsilon_b \varepsilon_b^C + \varepsilon_c} \to 1.$$

$$\mu_2(b|B) = \frac{\varepsilon_b}{\varepsilon_a^B + \varepsilon_b + \varepsilon_c \varepsilon_c^B} \to 0$$

$$\mu_2(c|C) = \frac{\varepsilon_c}{\varepsilon_a^C + \varepsilon_b \varepsilon_b^C + \varepsilon_c} \to 0$$

$$\Downarrow$$

$$\frac{\varepsilon_b}{\varepsilon_c \varepsilon_c^B} \to 0, \quad \frac{\varepsilon_c}{\varepsilon_b \varepsilon_b^C} \to 0.$$

$$\Downarrow$$

$\frac{1}{\varepsilon_c^B \varepsilon_b^C} \to 0.$ is impossible

Dynamic Games with Incomplete Information — Sequential equilibrium

Properties of sequential equilibrium

Theorem: For any finite extensive from game there exists at least one sequential equilibrium

Theorem: For generic finite extensive form games of perfect recall, the set of sequential equilibrium probability distributions is finite

Note: the set of sequential equilibrium *strategy profiles* is in general infinite because of off path randomizations

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Dynamic Games with Incomplete Information — Sequential equilibrium

Example:

- Player 1 has three types: $\theta \in \{a, b, c\}$ and three possible period t actions: A, B and C
- In period t, the beliefs of player 2 about player 1 type were: $\mu_2(a|h^{t-1}) = 1$
- The equilibrium strategy of player 1 in period t is $\sigma_1(A|h^{t-1}, a) = \sigma_1(B|h^{t-1}, b) = \sigma_1(C|h^{t-1}, c)$
- We are interested in the updated beliefs of player 2: $\mu_2(\cdot|(h^{t-1}, a_1^t))$

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Dynamic Games with Incomplete Information — Sequential equilibrium

- The following beliefs are possible in PBE: $\mu_2(b|(h^{t-1}, C)) = 1$ and $\mu_2(c|(h^{t-1}, B)) = 1$
- These beliefs can't be part of a sequential equilibrium
- Why?
- What if we had only two possible types of player 1 or two periods?

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Dynamic Games with Incomplete Information — Sequential equilibrium

Theorem: In any multi-stage game of incomplete information with independent types, the sets of PBE and sequential equilibria coincide if either each player has at most two possible types or the game has two periods

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