1. Consider the following PDE:

$$u_t = u_{xx} + x, \quad 0 < x < 5, \quad t > 0$$

$$u_x(0,\ t) = 0, \quad u(5,\ t) = 1, \quad t > 0$$
$$u(x,\ 0) = 20\ \exp(-x^2), \quad 0 < x < 5.$$

(a) Solve using separation of variables. You may leave the eigenfunction expansion coefficients in inner product form.

(b) Plot the solution at t=1,3,5 and 30, along with the initial condition and steady state solution, using 15 terms in your truncated expansion.

Solution.

(a) First we must find the steady state solution (i.e. $u_t = 0$).

$$u_{xx} + x = 0$$
$$\to u_{xx} = -x$$
$$\to u_x = -\frac{1}{2}x^2 + c_1$$
$$u_x(0,t) = c_1 = 0$$
$$\to u = -\frac{1}{6}x^3 + c_2$$
$$u(5,t) = -\frac{1}{6}5^3 + c_2 = 1$$
$$\to c_2 = 1 + \frac{5^3}{6} = \frac{131}{6}$$

$$u_s(x) = -\frac{1}{6}x^3 + \frac{131}{6}$$

Let us define $L(u) = u_t - u_{xx}$ and $f = x$. Our Partial Differential Equation is $L(u) = f$.

Let us define $U(x,t) = u(x,t) - u_s(x)$. Then $L(u_s(x)) = 0 - u_{sxx} = 0 - (-x) = x$. Thus, $L(U) = L(u - u_s) = L(u) - L(u_s) = x - x = 0$.
We therefore have

$$U_t = U_{xx}$$
$$U(5,t) = u(5,t) - u_s(5) = 1 - 1 = 0$$
$$U_x(0,t) = 0$$
$$U(x,0) = u(x,0) - u_s(x) = 20\ \exp(-x^2) + \frac{1}{6}x^3 - \frac{131}{6}$$

Let us now consider the following PDE:

$$U_t = U_{xx}, \quad 0 < x < 5, \quad t > 0$$
$$U_x(0,\ t) = 0, \quad U(5,\ t) = 0, \quad t > 0$$
$$U(x,\ 0) = 20\ \exp(-x^2) + \frac{1}{6}x^3 - \frac{131}{6}, \quad 0 < x < 5.$$

We use seperation of variables,

$$U(x,t) = X(x)T(t)$$

so that our equation is now

$$X(x)T'(t) = X''(x)T(t)$$

where we can see that

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

solving for $T(t)$, we have

$$T'(t) = -\lambda T(t)$$

and thus

$$T(t) = Ae^{-\lambda t}$$

solving for $X(x)$, we have

$$X''(x) = -\lambda X(x)$$

and thus we have a Sturm Liouville Problem with $\rho = 1,\ q = 0,\ \omega = 1$.
We have non-negative eigenvalues since,

$$q = 0 \leq 0, \ \ [\rho\phi_n\phi_n']_0^5 = [\phi_n\phi_n']_0^5 = 0 \leq 0$$

Let us check $\lambda = 0$.

$$X''(x) = 0 \to X' = c_1 \to X'(0) = c_1 = 0$$
$$X'(x) = 0 \to X = c_2 \to X(5) = c_2 = 0$$

Subsequently, $\lambda = 0$ gives us the trivial solution. We now know that $\lambda > 0$ (i.e. $\lambda = k^2, k \in \mathbb{R} \setminus \{0\}$.

$$X''(x) + k^2X(x) = 0 \to X(x) = A\cos(kx) + B\sin(kx)$$

$$X'(x) = -Ak\sin(kx) + Bk\cos(kx) \to X'(0) = Bk = 0 \to B = 0$$
$$X(x) = A\cos(kx) \to X(5) = A\cos(5k) = 0 \to \cos(5k) = 0$$

Henceforth,

$$k = \frac{\pi}{5}\left(n - \frac{1}{2}\right) \text{ and } \lambda = \frac{\pi^2}{25}\left(n - \frac{1}{2}\right)^2,\ n \in \mathbb{N}$$

And thus,

$$X_n(x) = \cos\left(\frac{\pi x}{5}\left(n - \frac{1}{2}\right)\right) \text{ and } T_n(t) = A_n \exp\left(-\frac{\pi^2 t}{25}\left(n - \frac{1}{2}\right)^2\right),\quad n \in \mathbb{N}$$

And finally,

$$U_n(x,t) = X_n(x)T_n(t) = A_n \cos\left(\frac{\pi x}{5}\left(n - \frac{1}{2}\right)\right)\ \exp\left(-\frac{\pi^2 t}{25}\left(n - \frac{1}{2}\right)^2\right),\quad n \in \mathbb{N}$$

By the principle of Linear Superposition we have,

$$U(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi x}{5}\left(n - \frac{1}{2}\right)\right)\ \exp\left(-\frac{\pi^2 t}{25}\left(n - \frac{1}{2}\right)^2\right).$$

Using boundary conditions, we see that

$$U(x,0) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi x}{5}\left(n - \frac{1}{2}\right)\right) = g(x)$$

The above function is an eigenfunction expansion and so we can write A_n in inner product form,

$$A_n = \frac{< g(x),\ \phi_n(x) >}{< \phi_n(x),\ \phi_n(x) >} = \frac{\int_0^5 g(x)\ \cos\left(\frac{\pi x}{5}\left(n - \frac{1}{2}\right)\right)\ dx}{\int_0^5 \cos^2\left(\frac{\pi x}{5}\left(n - \frac{1}{2}\right)\right)\ dx}$$

Now, since $u(x,t) = U(x,t) + u_s(x)$, we have

$$u(x,\ t) = \frac{131}{6} - \frac{1}{6}x^3 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi x}{5}\left(n - \frac{1}{2}\right)\right)\ \exp\left(-\frac{\pi^2 t}{25}\left(n - \frac{1}{2}\right)^2\right),$$

$$\text{where } A_n = \frac{\int_0^5 g(x)\ \cos\left(\frac{\pi x}{5}\left(n - \frac{1}{2}\right)\right)\ dx}{\int_0^5 \cos^2\left(\frac{\pi x}{5}\left(n - \frac{1}{2}\right)\right)\ dx}$$

(b) For the MuPad plot, please see attached on the following page an inset.

We have our initial condition u(x, 0),

```
u0 := x --> 20*exp(-x^2);
```

$$x \to 20\,\mathrm{e}^{x^2}$$

We have our steady state solution $u_s(x)$,

```
us := x --> 131/6 - x^3/6
```

$$x \to \frac{131}{6} + \frac{x^3}{6}$$

Note that n is a natural number, i.e. a non-negative integer,

```
assume(n in N_)
```

We have our root(lambda) = k,

```
k := x --> cos((n-1/2)*PI*x/5)
```

$$x \to \cos\left(\frac{\pi\, x\, \left(n + \frac{1}{2}\right)}{5}\right)$$

We have our constant given by,

```
cn := simplify(int(((u0(x) - us(x))*k(x)), x=0..5)) / simplify(int((k(x))^2, x=0..5))
```

$$\frac{2 \int_0^5 \cos\left(\frac{\pi\, x\, \left(n+\frac{1}{2}\right)}{5}\right) \left(20\,\mathrm{e}^{x^2} + \frac{x^3}{6} + \frac{131}{6}\right) \mathrm{d}\,x}{5}$$

We can make c_n = c(n),

```
c := n1 -> subs(cn, n=n1);
```

$$\mathrm{n1} \to \mathrm{subs}(\mathrm{cn},\ n = \mathrm{n1})$$

Define U(x, t) = to the first 15 terms of U(x, t),

```
U := (x, t) -> sum(c(n)*k(x)*exp(-((n-1/2)*PI/5)^2*t), n=1..15)
```

$$(x,\ t) \to \sum_{n=1}^{15} c(n)\, \left(x \to \cos\left(\frac{\pi\, x\, \left(n + \frac{1}{2}\right)}{5}\right)\right)(x)\ \mathrm{e}^{\left(\frac{\left(n+\frac{1}{2}\right)\pi}{5}\right)^2 t}$$

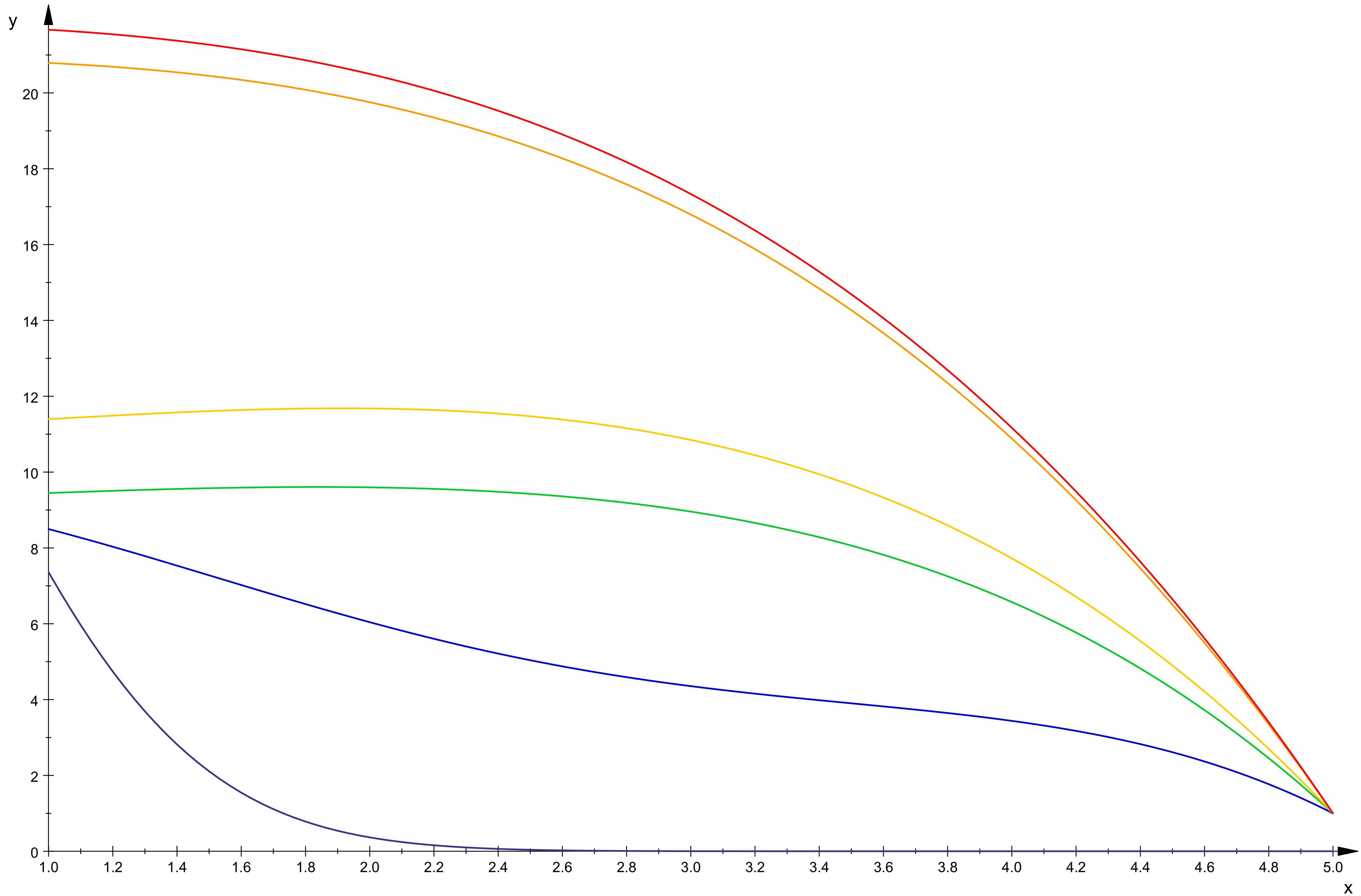
Define u(x, t) = U(x, t) + $u_s(x)$,

```
u := (x, t) -> U(x, t) + us(x)
```

$$(x,\ t) \to U(x,\ t) + \left(x \to \frac{131}{6} + \frac{x^3}{6}\right)(x)$$

Plot u(x, 0), u(x, 1), u(x, 3), u(x, 5), u(x, 30), and $u_s(x)$ on one graph,

```
plot(plot::Function2d(u0(x), x = 1..5,   LineColor = RGB::DarkViolet),
plot::Function2d(u(x, 1), x = 1..5,      LineColor = RGB::BlueMedium),
plot::Function2d(u(x, 3), x = 1..5,      LineColor = RGB::PermanentGreen),
plot::Function2d(u(x, 5), x = 1..5,      LineColor = RGB::Gold),
plot::Function2d(u(x, 30), x = 1..5,     LineColor = RGB::SafetyOrange),
plot::Function2d(us(x), x = 1..5,        LineColor = RGB::Red))
```


y
20
18
16
14
12
10
8
6
4
2
0
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0
4.2
4.4
4.6
4.8
5.0
x

2. Consider the following PDE:

$$u_{tt} = u_{xx}, \quad 0 < x < 10, \quad t > 0$$

$$u_x(0,\ t) = 0, \quad u(10,\ t) = 0, \quad t > 0$$
$$u(x,\ 0) = \exp\left(-\left(x-5\right)^2\right), \quad 0 < x < 10.$$

(a) Solve using separation of variables. You may leave the eigenfunction expansion coefficients in inner product form.

(b) Write down d'Alembert's general solution for this problem (e.g. ignoring the boundary conditions).

(c) Compare these two solutions with appropriate plots (e.g. in MuPad).

Solution.

(a) There is no need to find the steady state solution since we already have our Linear Partial Differential Equation in the form $L[u] = f$, where we define $L[u] = u_{tt} - u_{xx}$ and $f = 0$.

We solve this equation using seperation of variables, i.e. $u(x, t) = X(x)T(t)$.

Henceforth we have

$$XT'' = X''T \to \frac{X''}{X} = \frac{T''}{T} = -\lambda$$

Notice that we have the following Sturm-Liouville Problem,

$$X'' + \lambda X = 0, \ \ \rho = 1, \ \ q = 0, \ \ \omega = 1.$$

And we know we only have non-negative eigenvalues since

$$q = 0 \leq 0, \quad [\rho \phi_n \phi_n']_0^{10} = 0 \leq 0.$$

Plugging $\lambda = 0$ give us a trivial solution as you can see,

$$X'' = 0 \to X' = c_1 \to X'(0) = c_1 = 0$$

$$X = c_2 \to X(10) = c_2 = 0 \to X = 0$$

Let $\lambda = k^2,\ k \in \mathbb{R} \setminus \{0\}$. Then,

$$X'' = -k^2 X \to X(x) = A\cos(kx) + B\sin(kx)$$

$$X'(0) = B = 0 \to X(x) = A\cos(kx)$$

$$X(10) = A\cos(10k) = 0 \to \cos(10k) = 0 \to k = \frac{\pi}{10}\left(n - \frac{1}{2}\right),\ n \in \mathbb{N}$$

We now know that

$$\lambda_n = \frac{\pi^2}{100}\left(n-\frac{1}{2}\right)^2,\ n \in \mathbb{N}$$

$$\phi_n(x) = \cos\left(\frac{\pi x}{10}\left(n-\frac{1}{2}\right)\right)$$

Now, it trivially follows that,

$$T_n = A\cos\left(\frac{\pi t}{10}\left(n-\frac{1}{2}\right)\right) + B\sin\left(\frac{\pi t}{10}\left(n-\frac{1}{2}\right)\right)$$

$$U_n(x,t) = X_n(x)T_n(t) =$$
$$\left[A_n\cos\left(\frac{\pi t}{10}\left(n-\frac{1}{2}\right)\right) + B_n\sin\left(\frac{\pi t}{10}\left(n-\frac{1}{2}\right)\right)\right]\cos\left(\frac{\pi x}{10}\left(n-\frac{1}{2}\right)\right)$$

And so,

$$U(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{\pi t}{10}\left(n-\frac{1}{2}\right)\right) + B_n\sin\left(\frac{\pi t}{10}\left(n-\frac{1}{2}\right)\right)\right]\cos\left(\frac{\pi x}{10}\left(n-\frac{1}{2}\right)\right)$$

Since $U_t(x,0)$ results in the cos term going to zero, leaving $B_n\sin(\ldots)$, we can see that $B_n = 0$.

$$U(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{\pi t}{10}\left(n-\frac{1}{2}\right)\right)\right]\cos\left(\frac{\pi x}{10}\left(n-\frac{1}{2}\right)\right)$$

We now look at the initial condition,

$$U(x,0) = \sum_{n=1}^{\infty} A_n\cos\left(\frac{\pi x}{10}\left(n-\frac{1}{2}\right)\right) = \ \exp\left(-(x-5)^2\right)$$

The above function is an eigenfunction expansion and so we can write A_n in inner product form,

$$A_n = \frac{<\phi_n,\ \ \exp\left(-(x-5)^2\right)>}{<\phi_n,\ \phi_n>} = \frac{\int_0^{10}\cos\left(\frac{\pi x}{10}\left(n-\frac{1}{2}\right)\right)\ \exp\left(-(x-5)^2\right)\ dx}{\int_0^{10}\cos^2\left(\frac{\pi x}{10}\left(n-\frac{1}{2}\right)\right)\ dx}$$

And so finally, we have,

$$U(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi t}{10}\left(n - \frac{1}{2}\right)\right) \cos\left(\frac{\pi x}{10}\left(n - \frac{1}{2}\right)\right)$$

$$\text{where } A_n = \frac{\int_0^{10} \cos\left(\frac{\pi x}{10}\left(n-\frac{1}{2}\right)\right)\ \exp\left(-(x-5)^2\right)\ dx}{\int_0^{10} \cos^2\left(\frac{\pi x}{10}\left(n-\frac{1}{2}\right)\right)\ dx}$$

(b) d'Alembert's solution can be seen as the following, given a left and right component of a moving wave,

Let $\zeta(x,t) = x + at$ and $\eta(x,t) = x - at$. It is trivial to see that $u_t(x,0) = 0$ (the cos term becomes sin and $\sin(0) = 0$). Let us define $u(f(x,t(x))) = u(x,0)$. Our solution will be of the form,

$$d(x,t) = \frac{1}{2}\left(u(\zeta) + u(\eta)\right),$$

where $u(\zeta(x,t))$ describes the left-moving wave and $u(\eta(x,t))$ describes the right-moving wave.

Since the wave speed is 1, we have $a = 1$ and thus,

$$d(x,t) = \frac{1}{2}\left(u(x+t) + u(x-t)\right).$$

Which is equivalent to,

$$d(x,t) = \frac{1}{2}\left(\exp\left(-(x-5-t)^2\right) + \exp\left(-(x-5+t)^2\right)\right)$$

(c) For the MuPad plot, please see attached on the following page an inset.

We have our initial condition u(x, 0),

```
u0 := x --> exp(-(x-5)^2)
```

$$x \to e^{-(x-5)^2}$$

Note that n is a natural number, i.e. a non-negative integer,

```
assume(n in N_)
```

We have our root(lambda) = k,

```
k := x --> PI/10 * (n-1/2)
```

$$x \to \frac{\pi\left(n-\frac{1}{2}\right)}{10}$$

We have our constant given by,

```
cn := int(cos(k(x)*x)*u0(x), x=0..10) / int((cos(k(x)*x))^2, x=0..10)
```

$$\frac{\int_0^{10}\cos\left(\frac{\pi x\left(n-\frac{1}{2}\right)}{10}\right)e^{-(x-5)^2}\,dx}{\frac{5\sin\left(2\pi\left(n-\frac{1}{2}\right)\right)}{2\pi\left(n-\frac{1}{2}\right)}+5}$$

We can make c_n = c(n),

```
c := n1 -> subs(cn, n=n1)
```

$$\mathrm{n1} \to \mathrm{subs}(\mathrm{cn}, n = \mathrm{n1})$$

Define U(x, t) = to the first 15 terms of U(x, t),

```
U := (x, t) --> c(n) * cos(k(x)*t) * cos(k(x)*x)
```

$$(x, t) \to \frac{\cos\left(\frac{\pi t\left(n-\frac{1}{2}\right)}{10}\right)\sigma_1\int_0^{10}\sigma_1\, e^{-(x-5)^2}\,dx}{\frac{5\sin\left(2\pi\left(n-\frac{1}{2}\right)\right)}{2\pi\left(n-\frac{1}{2}\right)}+5}$$

where

$$\sigma_1 = \cos\left(\frac{\pi x\left(n-\frac{1}{2}\right)}{10}\right)$$

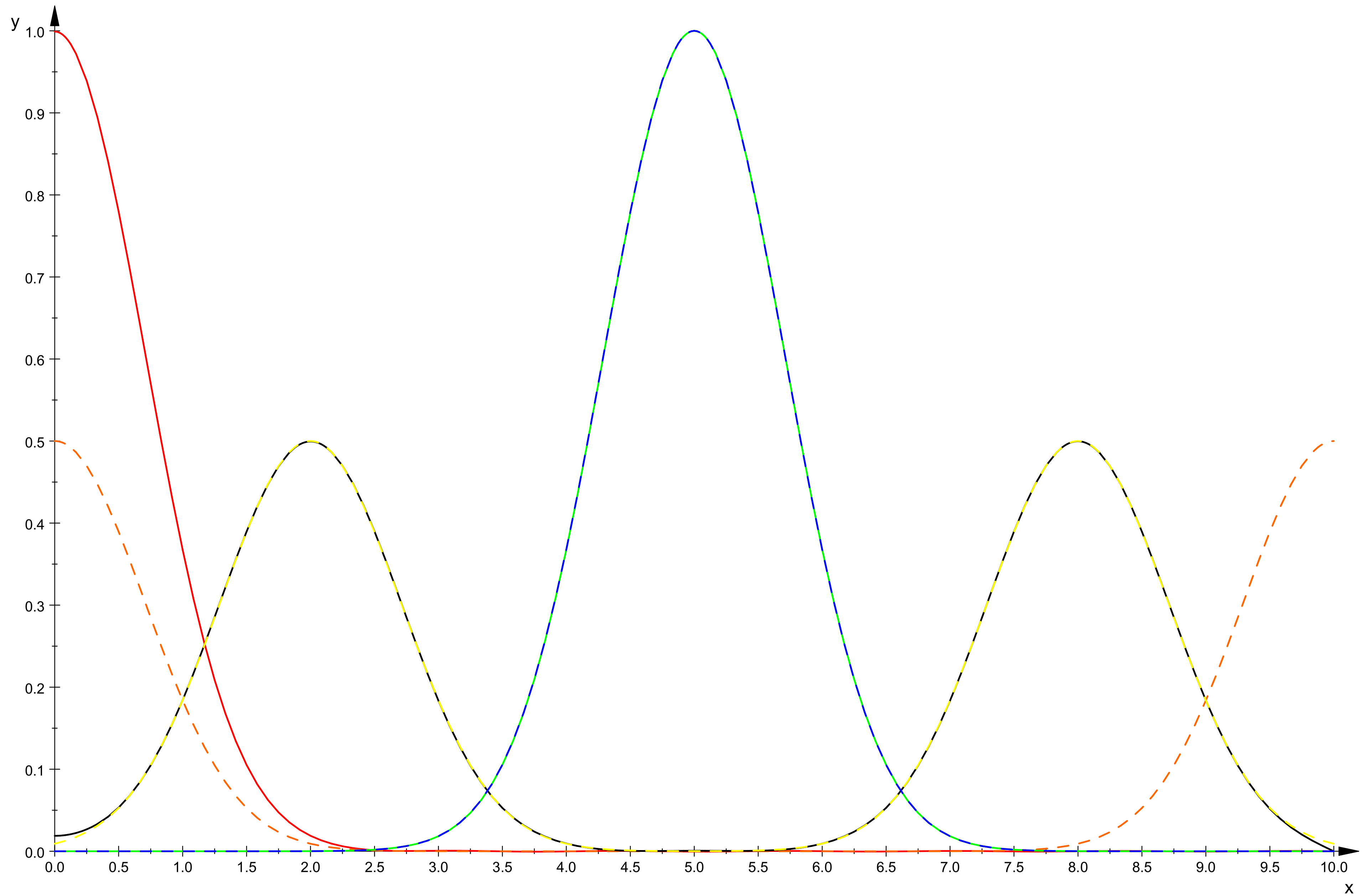
```
u := (x, t) -> sum(U(x, t), n=1..15):
```

```
d := (x, t) -> (exp(-(x-5-t)^2)/2) + (exp(-(x-5+t)^2)/2)
```

$$(x, t) \to \frac{e^{-(x-5-t)^2}}{2}+\frac{e^{-(x-5+t)^2}}{2}$$

Plot u(x, 0) and d(x, 0), u(x, 3) and d(x, 3), u(x, 5) and d(x, 5) on one graph,

```
plot(
plot::Function2d(u(x, 5), x = 0..10,     LineColor = RGB::Red),
plot::Function2d(d(x, 5), x = 0..10,     LineColor = RGB::Orange,   LineStyle = Dashed),
plot::Function2d(u(x, 3), x = 0..10,     LineColor = RGB::Black),
plot::Function2d(d(x, 3), x = 0..10,     LineColor = RGB::Yellow,   LineStyle = Dashed),
plot::Function2d(u0(x), x = 0..10,     LineColor = RGB::Green),
plot::Function2d(d(x, 0), x = 0..10,     LineColor = RGB::Blue,   LineStyle = Dashed)):
```


y
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0 4.5 5.0 5.5 6.0 6.5 7.0 7.5 8.0 8.5 9.0 9.5 10.0
x

3. Use Laplace Transforms to find the solution of the initial value problem

$$y^{(iv)} - 4y''' + 6y'' - 4y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1$$

Solution.

Let us recall the formula for the Laplace Transform of a function $f(t)$,

$$\mathcal{L}\{f(t)\} = \int_0^\infty \exp(-st)\, f(t)\, dt = F(s).$$

The Laplace Transform of the initial value problem is,

$$\mathcal{L}\{y^{(iv)} - 4y''' + 6y'' - 4y' + y\} = 0$$

Since the Laplace Transform is a linear function, we have

$$\mathcal{L}\{y^{(iv)}\} - 4\mathcal{L}\{y'''\} + 6\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 0$$

It is given in the Lookup Table that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

It is also given in the Lookup Table that

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0).$$

We can show for the rest of the cases,

$$\mathcal{L}\{f^{(n)}(t)\} = \int_0^\infty f^{(n)}(t)e^{-st}dt = \int_0^\infty e^{-st}df^{(n-1)} = \left[f^{(n-1)}e^{-st}\right]_0^\infty - \int_0^\infty f^{(n-1)}de^{-st}$$

$$= -f^{(n-1)}(0) + s\int_0^\infty f^{(n-1)}e^{-st}\, dt = -f^{(n-1)}(0) + s\mathcal{L}\{f^{(n-1)}(t)\}, \text{ where } f^{(n)} = \frac{d^n f}{dt^n}.$$

And so we have,

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2y(0) - sf'(0) - f''(0).$$

And finally,

$$\mathcal{L}\{f^{(iv)}(t)\} = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - sf'''(0).$$

Now we must determine $\mathcal{L}\left\{y^{(n)}(t)\right\}$ for $n = 0,\ 1,\ 2,\ 3,\ 4.$

$$\begin{aligned}\mathcal{L}\{y(t)\} &= F(s)\\ \mathcal{L}\{y'(t)\} &= sF(s) - y(0) = sF(s)\\ \mathcal{L}\{y''(t)\} &= s^2F(s) - sy(0) - y'(0) = s^2F(s) - 1\\ \mathcal{L}\{y'''(t)\} &= s^3F(s) - s^2y(0) - sy'(0) - y''(0) = s^3F(s) - s\\ \mathcal{L}\left\{y^{(iv)}(t)\right\} &= s^4F(s) - s^3y(0) - s^2y'(0) - sy''(0) - sy'''(0) = s^4F(s) - s^2 - s\end{aligned}$$

We can now rewrite $\mathcal{L}\{y^{(iv)}\} - 4\mathcal{L}\{y'''\} + 6\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 0$ as,

$$s^4F(s) - s^2 - 1 - 4s^3F(s) + 4s + 6s^2F(s) - 6 - 4sF(s) + F(s) = 0.$$

Or, by separating like terms,

$$F(s)\left(s^4 - 4s^3 + 6s^2 - 4s + 1\right) = F(s)(s-1)^4 = s^2 - 4s + 7$$

And so we can evaluate $F(s)$ using partial fractions as,

$$F(s) = \frac{s^2 - 4s + 7}{(s-1)^4} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{(s-1)^4}$$

where A, B, C and D are constants. Multiply through by $(s-1)^4$. Get

$$0s^3 + s^2 - 4s + 7 = A(s-1)^3 + B(s-1)^2 + C(s-1) + D$$

Since our cubic term will be $0s^3 = As^3$, it is trivial to see that $A = 0$. Because of this, our square term will be $s^2 = Bs^2$ and so $B = 1$.

$$s^2 - 4s + 7 = s^2 + (C-2)s + (1 - C + D)$$

We now know that $C - 2 = -4$ and so $C = -2$, and $D + 3 = 7$ so $D = 4$.

Hence, we have,

$$F(s) = \frac{1}{(s-1)^2} + \frac{-2}{(s-1)^3} + \frac{4}{(s-1)^4}.$$

Now go from the s variable to the t variable:

$$y(t) = \mathcal{L}^{-1}\{F(s)\} = t\,\exp(t) - t^2\,\exp(t) + \frac{2}{3}t^3\,\exp(t)$$