MATHS

Undergraduate Applied Mathematics

Complete Notes

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Preliminaries

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3.1 Limits

3.1.1 Definition and Continuity

We say that a function f is continuous at the point a if a limit exists at the point a.

$$\lim_{x \to a} f(x) = L$$

3.1.2 Limit Laws

If $\lim_{x \to a} g(x) = b$ and f is continuous at b then,

$$\lim_{x \to a} (f \circ g)(x) = f(b)$$

3.1.3 The Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ over some interval except at the point a, and if $h(x) = f(x) = L$ at the point a,

$$\lim_{x \to a} g(x) = L$$

3.1.4 Trigonometric Limits

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \text{ and } \lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}$$

3.1.5 Infinite Limits

For any **periodic** function f,

$$\lim_{x \to \pm\infty} f(x) \text{ does not exist}$$

For any positive number r,

$$\lim_{x \to \pm\infty} \frac{1}{x^r} = 0$$

3.1.6 Asymptotes

The line $x = a$ is called a **vertical asymptote** of a function f if either

$$\lim_{x = a^+} f(x) = \pm\infty \text{ or } \lim_{x = a^-} f(x) = \pm\infty.$$

The line $y = a$ is called a **horizontal asymptote** of a function f if either

$$\lim_{x = -\infty} f(x) = a \text{ or } \lim_{x = \infty} f(x) = a.$$

The line $y = mx + c$ is called a **inclined asymptote** of a function f if either

$$\lim_{x = -\infty} \Big(f(x) - (mx + c) \Big) = 0 \text{ or } \lim_{x = \infty} \Big(f(x) - (mx + c) \Big) = 0.$$

3.1.7 Limit Definition of the Derivative

The **derivative** of a function f is given by

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Alternatively, this may be written as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

3.2 Derivatives

3.3 Integrals

3.3.1 Limit definition of a definite integral

The definite integral of a function f from a to b is given by

$$\int_a^b f(x) = \lim_{n\to\infty} \sum_{n=1}^{\infty} f(x_i^*)\Delta x$$

Chapter 4

MATHS 162

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MATHS 250

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5.1 Linear Algebra

5.2 Calculus

5.2.1 Limits

5.2.2 Infinite Limits of ln and e

For any non negative α,

$$\lim_{x\to\infty} \frac{\ln(x)}{x^\alpha} = 0 \text{ and } \lim_{x\to\infty} \frac{x^\alpha}{\mathrm{e}^x} = 0$$

5.2.3 The Limit Definition of e

The definition of e is given by

$$\mathrm{e}^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

5.2.4 Sequences

A sequence $f = r^n$ converges or diverges depending on the value of r,

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & |r| < 1 \\ 1, & r = 1 \\ \text{diverges}, & |r| > 1 \text{ or } r = -1 \end{cases}$$

Chapter 6

MATHS 253 Part 1: Linear Algebra

6.1 Fields and Vector Spaces

6.2 Linear Transformations and their Matrices

$$R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin^2(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$Q_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

6.3 Diagonalisation

6.4 Orthogonalisation and Inner Products

6.5 Adjoint Orthogonal Diagonolation

6.6 Quadratic Forms

A function $f : \mathbb{R}^n \to \mathbb{R}$ given below is known as a linear function and said to be of **linear form**.

$$f(\mathbf{x}) = \sum_{i=1}^{n} a_i x_i = a_1 x_1 + \ldots + a_n x_n$$

6.6.1 Quadratic Form

A function $Q : \mathbb{R}^n \to \mathbb{R}$ given by the formula below is said to be of **quadratic form**.

$$Q(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

We can place Q into symmetric presentation by requiring that because $x_i x_j = x_j x_i$, we must have $a_{ij} = a_{ji}$.

Example

$$Q_1(x) = x_1 x_2 + x_3^2 = \frac{1}{2} x_1 x_2 + \frac{1}{2} x_2 x_1 + x_3^2$$

A special identity is that any quadratic form given by its symmetric presentation can be written as

$$Q(\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x} = \mathbf{x}^T A\mathbf{x}, \quad A = (a_{ij}), \quad A \text{ is symmetric.}$$

The matrix A, is special because we can construct a change of basis matrix

$$B = P^T AP, \quad P = P_{E \leftarrow F}$$

If we orthogonally diagonalise A such that we have

$$D = P^{-1} AP = P^T AP$$

Then we will obtain the **principle axes theorem**

$$Q(\mathbf{x}) = [\mathbf{x}]_E^T A [\mathbf{x}]_E = [\mathbf{x}]_F^T P^T AP [\mathbf{x}]_F = [\mathbf{x}]_F^T D [\mathbf{x}]_F = \sum_{i=1}^{n} \lambda_i y_i^2$$

where

$$[\mathbf{x}]_F = (y_1, \, y_2, \, \ldots, \, y_n)^T$$

and λ_i are the eigenvalues of A. The corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are called the **principle axis** of Q.

6.6.2 Conic Sections

We turn our attention

6.6.3 Quadric Surfaces

We turn our attention

6.6.4 Sylvesters Criterion

A quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ given by

$$Q(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \mathbf{x}^T A \mathbf{x} = \lambda_1 y_1^2 + \ldots + \lambda_n y_n^2$$

Is positive definite iff the eigenvalues of A are positive, and negative iff the eigenvalues of A are negative. For large matrices, finding the eigenvalues can be exhaustive so we instead use **Sylvester's Criterion**.

The quadratic form is positive definite iff

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \ldots, \quad \Delta_n > 0$$

The quadratic form is negative definite iff

$$\Delta_1 > 0, \quad \Delta_2 < 0, \quad \Delta_3 > 0, \quad \ldots$$

where Δ_n is the leading minor of A.

We can also perform a **congruent diagonalisation** by diagonalisating A using elementary row operations. Although this diagonalised A does not contain eigenvalues, we can still determine the definiteness of the quadratic form using the diagonal coefficients.

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MATHS 260

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MATHS 340

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Chapter 11

MATHS 361

This is an introductory course in Partial Differential Equations (PDE's). We cover Fourier series, Fourier integrals, boundary value problems, separation of variables, Laplace transform solutions, and Green's functions, with application to the solution of second order PDE's in one, two and three dimensions.

Being an introductory course in Partial Differential Equations, it is recommended that you review your knowledge on Ordinary Differential Equations which is covered in MATHS 260.

11.1 Revision

This is a very basic revision of content in MATHS 260 that seems to reappear in MATHS 361. Practice examples for these problems so that you feel comfortable with doing them with much harder problems.

11.1.1 Seperable Variables

Given a problem of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$$

We can find a solution for $y(x)$ by solving

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

11.1.2 Integrating Factor

Given a problem of the form

$$y' + p(x)y = q$$

We must find an integrating factor $\mu(x)$ such that

$$\mu(x) = \exp\left(\int p(x)\,\mathrm{d}x\right)$$

This condition means that $\mu(x)p(x) = \mu'(x)$. This is a powerful tool since by the multiplication rule of derivatives we now have

$$\mu y' + \mu' y' = (\mu y)' = q\mu$$

It is easy to see that the solution for $y(x)$ is

$$y(x) = \frac{1}{\mu(x)}\int q(x)\mu(x)\,\mathrm{d}x, \quad \text{where } \mu = \exp\left(\int p(x)\,\mathrm{d}x\right)$$

11.1.3 Characteristic Equations

Given a homogenous differential equation of the form

$$ay'' + by' + c = 0$$

We can write it's characteristic equation in the form

$$ar^2 + br + c = 0$$

Which as you know by the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Can have real roots, repeated roots or complex roots.

If the root is real and distinct, your solution will be of the form

$$y(x) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

If the root is repeated, your solution will be of the form

$$y(x) = c_1 e^{rt} + c_2 t e^{rt}$$

If the root is complex ($r = a \pm bi$), your solution will be of the form

$$y(x) = c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt)$$

11.2 Introduction

11.2.1 What is a Partial Differential Equation

Recall in MATHS 260 differential equations contained derivatives with respect to just one independent variable. An example of such an equation is

$$y''(x) + y'(x) = \sin(x)$$

However, in MATHS 361 you will be introduced to differential equations that contain derivatives with respect to more than one independent variable. An example of such an equation is the heat equation.

$$u_t = ku_{xx}$$

Notice that u is a function of x and t. This is called a **Partial Differential Equation**. The differential equations covered in MATHS 260 will now be known as **Ordinary Differential Equations**.

11.2.2 Linear Operators

Revisiting the heat equation, instead of writing

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$$

we may instead write

$$L[\mathbf{u}] = f$$

where $f(x) = 0$ and

$$L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$$

A differential operator L is linear if

$$L[\alpha\mathbf{u} + \beta\mathbf{v}] = \alpha L[\mathbf{u}] + \beta L[\mathbf{v}],$$

for all functions u and v and all constants α and β.

Equivalently, L is linear if

$$L[\alpha\mathbf{u}] = \alpha L[\mathbf{u}] \text{ and } L[\mathbf{u} + \mathbf{v}] = L[\mathbf{u}] + L[\mathbf{v}]$$

for all functions u and v and all constants α.

11.2.3 The Equations We Shall Study

It is proposed to study three linear second-order partial differential equations (PDEs) that have applications throughout the physical sciences.

The Heat Equation

Also known as the diffusion equation, we will find $u(x,\, t)$ such that

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

where, for example, $u(x,t)$ is a temperature at position x and time t, and $\alpha^2 = \dfrac{k}{\sigma C}$ is the thermal diffusivity. k is the thermal conductivity, σ is the mass density per unit length and C is the specific heat.

The Wave Equation

Here, we will look at finding $u(x,\, t)$ such that

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

where, for example, $u(x,\, t)$ is the transverse displacement of a stretched string at position x and time t, and α is a positive constant, the wave speed.

Laplace's Equation

In this case the problem is to find $u(x,\, y)$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Notice that the Heat Equation from Section 1.3.1 reduces to the Laplace Equation if the temperature field u is independent of time.

11.3 Fourier Series

11.3.1 Convergence of Fourier Series

Let f be a periodic function with fundamental period $2L$ such that f and f' are piecewise continuous on $[-L, L]$. Then the Fourier series of f converges to $f(x)$ at each point x at which f is continuous, and to the mean value $(f(x^+) + f(x^-))/2$ at every point x at which f is discontinuous.

Rate of convergence

When using Fourier series in practical situations, we often need to truncate the series at some finite value of n. In this case, we would be interested in questions such as how good is the convergence? Also, what about the speed of the convergence? In general, the more derivatives f has, the faster the convergence. We can roughly say that if the discontinuity is in the p^{th} derivative, then a_n, b_n decay like n^{-p-1}

Gibbs Phenomenon

At a point of discontinuity, the partial sums always overshoot the limiting values. This overshoot does not tend to zero as more terms are taken, but the width of the overshooting region does tend to zero. This is known as the **Gibbs Phenomenon**.

11.3.2 Fourier Series

A periodic function f defined for $x \in [0, 2L]$ has the Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

11.3.3 Complex Fourier Series

A function f defined for $x \in [-L,\, L]$ has the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x}{L}\right)$$

where

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) \exp\left(\frac{-in\pi x}{L}\right)\,\mathrm{d}x$$

11.3.4 Fourier Series of Functions Defined on Finite Intervals

For a function f defined on $[-L,\, L]$, define f_{per}, the periodic extension of f , by

$$f_{\text{per}}(x + 2nL) = f(x)$$

You would then find the Fourier Series of this function as you would normally.

For a function f defined on $[0,\, L]$, define f_{odd} and f_{even} as the odd and even extensions of f respectively.

11.4 Sturm-Liouville Problems

A Sturm-Liouville problem is a linear homogeneous second-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y + \lambda\omega(x)y = 0, \quad x \in (a,\ b)$$

with homogeneous boundary conditions of the form

$$\alpha y(a) + \beta y'(a) = 0, \quad \gamma y(b) + \delta y'(b) = 0$$

and an eigenvalue λ which is an unspecified real number which we assume to be non-trivial.

11.4.1 Sturm Liouville Theorem

The Sturm Liouville Theorem tells us that at each point where f is continuous

$$f(x) = \sum_{n=1}^{\infty} \frac{\langle f,\ \phi_n \rangle}{\langle \phi_n,\ \phi_n \rangle}\phi_n(x) = \sum_{n=1}^{\infty} a_n\phi_n(x)$$

A consequence of this theorem is that the eigenvalues of a Sturm-Liouville problem must be real.

11.4.2 Non-negative Eigenvalues Theorem

The theorem below offers a quick method of deducing the eigenvalue λ cannot be a negative number.

If $q(x) \leq 0$ for $x \in (a,\ b)$ and $[p(x)\phi_n\phi_n']_a^b \leq 0$, then λ_n is non-negative (i.e. zero or positive).

11.4.3 Sturm-Liouville Problems in Disguise

Because a Sturm-Lioville problem can be expanded into the form

$$py'' + p'y' + qy + \lambda\omega y = 0$$

You can often use an overall multiplying factor $\sigma(x)$ to put it into the form of a Sturm-Liouville problem.

11.5 Partial Differential Equations

11.5.1 The Heat equation

Let us begin with the heat equation with homogenous Dirichlet boundary conditions

$$u_t = cu_{xx}, \quad u(0,\,t) = u(L,\,t) = 0, \quad u(x,\,0) = f(x)$$

We may use a technique known as **Separation of Variables** in order to solve this partial differential equation. Let $u(x,\,t) = X(x)T(t)$ with $X(0) = X(L) = 0$. Then

$$X(x)T'(t) = cX''(x)T(t)$$

Rearrange this expression to get

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{cT(t)} = \lambda$$

Now solve the system of ordinary differential equations

$$X''(x) = \lambda X(x)$$

Let $\lambda = -k^2$ then

$$X''(x) + k^2 X(x) = 0 \to X(x) = A\cos(kx) + B\sin(kx)$$

Now $X(0) = A = 0$ and $X(L) = B\sin(kL) = 0$ so $k = \frac{n\pi}{L}$ and

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$T'(t) = c\lambda T(t) \to T(t) = Ae^{c\lambda t} = A_n \exp\left(-c\left(\frac{n\pi}{L}\right)^2 t\right)$$

Finally,

$$u(x,\,t) = X(x)T(t) = A_n B_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-c\left(\frac{n\pi}{L}\right)^2 t\right)$$

By the superposition principle

$$u(x,\,t) = X(x)T(t) = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-c\left(\frac{n\pi}{L}\right)^2 t\right)$$

Using the boundary conditions

$$u(x,\,0) = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

We see that the above is an eigenfunction expansion and so we can write K_n in an inner product form

$$K_n = \frac{< f, \phi_n >}{< \phi_n, \phi_n >} = \frac{\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x}{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

since as we learnt in the Fourier Transforms section

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \frac{L}{2}$$

11.5.2 Steady-State solution

What if we do not have homogenous boundary conditions? We must now solve the equation $u_t = 0$ to find the steady-state solution $u_s(x)$ that satisfies our boundary conditions.

Then we must construct a new partial differential equation $U(x,\, t) = u(x,\, t) - u_s(x)$. After solving this equation we get

$$u(x,\, t) = U(x,\, t) + u_s(x)$$
$$u(x,\, t) = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-c\left(\frac{n\pi}{L}\right)^2 t\right) + u_s(x)$$

where

$$K_n = \frac{2}{L}\int_0^L \big(f(x) - u_s(x)\big) \sin\left(\frac{n\pi x}{L}\right)\, \mathrm{d}x$$

11.5.3 The wave equation

D'Alembert's solutions

Given the initial value problem of the form

$$u_{tt} = u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,\, 0) = f(x), \quad u_t(x,\, 0) = g(x), \quad -\infty < x < \infty$$

there is a solution known as d'Alembert's formula to the wave equation

$$u(x,\, t) = \frac{1}{2}\Big(f\left(x + ct\right) + f\left(x - ct\right) \Big) + \frac{1}{2c} \int\limits_{x-ct}^{x+ct} g(s)\mathrm{d}s.$$

If the boundary condition $u_t(x,\, 0) = 0$, it follows that

$$u(x,\, t) = \frac{1}{2}\Big(f\left(x + ct\right) + f\left(x - ct\right) \Big).$$

11.5.4 Laplace's equation

Laplace's equation for a rectangle

Consider the Laplace equation on the rectangle $0 < x < a$ and $0 < y < b$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

with the following boundary conditions

$$u(0,\, y), \quad u(a,\, y), \quad u(x,\, 0), \quad u(x,\, b).$$

We can solve the above equation by seperation of variables, if the boundary conditions are homogeneous (the functions are zero) on all but one edge. In other words, if three of the boundary conditions are zero, the Laplace equation can be solved.

Laplace's equation for an annulus or disk

Consider the Laplace equation for an annulus or disk in terms of polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad \pi_1 < \theta < \pi_2, \quad r < a$$

with

$$u(r,\, \pi_1) = 0, \quad u(r,\, \pi_2) = 0, \quad u(a,\, \theta) = f(\theta).$$

We can use seperation of variables to show that if $u(r,\, \theta) = R(r)A(\theta)$ satisfies $\nabla^2 u = 0$ then

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\theta^2} + \lambda A = 0, \quad r^2\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} - \lambda R = 0$$

for some constant λ.

11.6 Fourier Transforms

11.6.1 Definition

The Fourier Transform is defined as

$$\mathcal{F}\{f(t)\} = \int\limits_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t$$

Similarly, the inverse Fourier Transform is defined as

$$\mathcal{F}^{-1}\{f(t)\} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} f(t)e^{i\omega t}\,\mathrm{d}t$$

11.6.2 Dirac Delta Function

The Dirac delta function $\delta(t)$ is defined with two special properties.

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$$\int\limits_{-\infty}^{\infty} \delta(t) = \lim_{\varepsilon\to 0^+}\int\limits_{-\varepsilon}^{\varepsilon} \delta(t) = 1$$

As a mathematician, you might be confused. It is impossible to define a function over $\mathbb{R}$ that has these properties. And you're right, the dirac delta function isn't a function in the traditional sense. However, the introduction of distributions has meant that such a function has use to physicists.

A special consequence of the delta function is the shifting property

$$\int\limits_{-\infty}^{\infty} f(t)\delta(t)\,\mathrm{d}t = \lim_{\varepsilon\to 0^+}\int\limits_{-\varepsilon}^{\varepsilon} f(t)\delta(t)\,\mathrm{d}t = f(0)$$

Since the δ function is non zero only at $t = 0$, the product of $f(t)\delta(t)$ can only be non-zero at $t = 0$ and hence the infinite integral of $f(t)\delta(t)$ becomes $f(0)$.

Likewise, and by similar reasoning, if the impulse is not at the origin,

$$\int\limits_{-\infty}^{\infty} f(t)\delta(t - t_0) = f(t_0)$$

11.6.3 Orthogonality

One may derive the following property

$$\delta(t - t_0) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \exp\left(i\omega(t - t_0)\right)\,\mathrm{d}t$$

This states that two functions $\exp(it\omega)$ and $\exp(it_0\omega)$ are orthogonal unless $t = t_0$, in which case they are the same function.

11.6.4 Convolutions

The convolution $c(t)$ of two functions $f(t)$ and $g(t)$ is defined as

$$c(t) = f(t) \star g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,\mathrm{d}\tau$$

11.6.5 Time Convolution Theorem

The convolution theorem states that under suitable conditions the Fourier transform of a convolution of two signals is the pointwise product of their Fourier transforms.

$$\mathcal{F}\left\{f(t) \star g(t)\right\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}$$

$$f(t) \star g(t) = \mathcal{F}^{-1}\Big\{\mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}\Big\}$$

11.6.6 Frequency Convolution Theorem

The convolution theorem states that under suitable conditions the Fourier transform of a convolution of two signals is the pointwise product of their Fourier transforms.

$$\mathcal{F}\{f(t)g(t)\} = \frac{1}{2}\left(\mathcal{F}\{f(t)\} \star \mathcal{F}\{g(t)\}\right)$$

$$\mathcal{F}^{-1}\left\{\mathcal{F}\{f(t)\} \star \mathcal{F}\{g(t)\}\right\} = 2\pi f(t)g(t)$$

11.7 Laplace Transforms

11.7.1 Definition and Properties

Let $f(t)$ be a function that satisfies some mild conditions about its boundedness. The Laplace Transform of $f(t)$ is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty \exp(-st) f(t)\,\mathrm{d}t$$

The inverse Laplace Transform isn't as easy to define, and it is easier to reverse engineer the process of creating the function you want to find the inverse Laplace Transform of.

$f(t)$	$F(s)$	Restrictions
1	$\frac{1}{s}$	$s > 0$
$\exp(at)$	$\frac{1}{s-a}$	$s > a$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{a^2+s^2}$	$s > 0$
$\exp(at)\sin(at)$	a	$s > a$
$\exp(at)\cos(at)$	a	$s > a$
$t^n \exp(at)$	a	$s > a$
$f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	

11.7.2 Properties

The linearity property

The Laplace Transform is linear, and by consequence

$$\mathcal{L}\left\{\alpha f(t) + \beta g(t)\right\} = \alpha\mathcal{L}\left\{f(t)\right\} + \beta\mathcal{L}\left\{g(t)\right\} = \alpha F(s) + \beta G(s)$$

The s-shifting property

An important property of the Laplace Transform is

$$\mathcal{L}\left\{\exp(at)f(t)\right\} = F(s-a)$$

The t-shifting property

Another important property of the Laplace Transform is

$$e^{-sa}F(s) = \mathcal{L}\left\{f(t-a)H(t-a)\right\}$$

where $H(t)$ is the Heaviside function. You will more often use this to show that the inverse of a Laplace Transform contains the Heaviside function rather than finding the Laplace Transform of a Heaviside function.

$$\mathcal{L}^{-1}\left\{e^{-sa}F(s)\right\} = f(t-a)H(t-a)$$

The integral property

One can show using the derivative identity that

$$\mathcal{L}\left\{\int_0^x f(t)\,\mathrm{d}t\right\} = \frac{F(s)}{s}$$

11.7.3 The Inverse Laplace Transform

More often than not, you will have to expand the function $F(x,\, s)$ using **partial fraction decomposition** in order to find the inverse Laplace Transform.

11.8 Distributions

11.8.1 Test Functions

Distributions are a class of linear functionals that map a set of test functions (conventional and well-behaved functions) into the set of real numbers. In the simplest case, the set of test functions considered is $\mathcal{D}(\mathbb{R})$, which is the set of functions $\phi : \mathbb{R} \to \mathbb{R}$ having two properties:

1. ϕ is smooth
2. ϕ has compact support

11.8.2 Distributions

A distribution is a continuous linear mapping $F(\phi)$ from the set of test functions to the real numbers of the form

$$F(\phi) = < f, \, \phi > = \int\limits_{-\infty}^{\infty} f(x)\phi(x) \, \mathrm{d}x$$

One property of the distribution is

$$< f + g, \, \phi > = < f, \, \phi > \; + \; < g, \, \phi >$$

From 11.6.2 it is easy to see that

$$< \delta, \, \phi > = \int\limits_{-\infty}^{\infty} \phi(x)\delta(x) \, \mathrm{d}x = \phi(0)$$

Also, it should be trivial that

$$< H_a, \, \phi > = \int\limits_{-\infty}^{\infty} H_a(x)\phi(x) \, \mathrm{d}x = \int\limits_{a}^{\infty} \phi(x) \, \mathrm{d}x$$

11.8.3 Derivative of Distributions

The first derivative of a distribution $F(\phi)$ is

$$F'(\phi) = < f',\, \phi > = - < f,\ \phi' >= - \int\limits_{-\infty}^{\infty} f(x)\phi'(x)\,\mathrm{d}x$$

The second derivative of a distribution $F(\phi)$ is

$$F'(\phi) = < f',\, \phi > =< f,\ \phi'' >= \int\limits_{-\infty}^{\infty} f(x)\phi''(x)\,\mathrm{d}x$$

The distributional derivative of a function $f(x)$ is calculated

$$< f',\, \phi > = < g,\, \phi >$$

If the above equality holds, then $f'(x) = g(x)$ in the distributional sense.

11.8.4 The Heaviside Function

Given a function $f(x)$ where $f_1(x)$ and $f_2(x)$ are both non-zero functions and

$$f(x) = \begin{cases} f_1(x), & x < a \\ f_2(x), & x \geq a \end{cases}$$

It is often easier to rewrite the function as

$$f(x) = f_1(x) + (f_2(x) - f_1(x))H(x - a)$$

For the purpose of differentiating a distribution.

11.9 Green's Function

11.9.1 Green's Function for Homogenous Boundary Conditions

Problems of the form

$$(pu')' + qu(x) = f(x)$$

with homogenous boundary conditions

$$a_1 u(a) + a_2 u'(a) = 0$$

$$b_1 u(b) + b_2 u'(b) = 0$$

can be solved by finding a function u_1 that solves the first boundary condition and a function u_2 that solves the second boundary condition and $C = p(x)W(x)$ where W is the wronskian.

Our solution will be of the form

$$u(x) = \int_a^b G(x,\, s) f(s) \, \mathrm{d}s$$

where

$$G(x,\, s) = \begin{cases} \dfrac{u_1(s)u_2(x)}{C}, & a \leq s \leq x \leq b \\ \dfrac{u_1(x)u_2(s)}{C}, & a \leq x \leq s \leq b \end{cases}$$

11.9.2 Differential Equation For Green's Function

Consider a problem of the form

$$(pu')' + qu = f(x)$$

with Dirichlet boundary conidtions

$$u(x_1) = a, \quad u(x_2) = b.$$

This requires a bit more work to solve than the previous problem. Let $G(x,\, s)$ be the Green's function for this problem. The differential equation satisfied by $G(x,\, s)$ is

$$(pG')' + qG = \delta(x - s)$$

In order to find $G(x,\, s)$ you must solve the following problems with **boundary conditions**.

$$(pG')' + qG = 0, \quad x_1 < x < s, \quad G(x_1,\, s) = a \quad \text{(call this solution } G_1\text{)}$$
$$(pG')' + qG = 0, \quad x < s < x_2, \quad G(x_2,\, s) = b \quad \text{(call this solution } G_2\text{)}$$

Your solution will be of the form, still with some unknown constants, (we will find these later)

$$G(x,\, s) = \begin{cases} G_1 & x_1 < x < s \\ G_2 & s < x < x_2 \end{cases}$$

The **continuity condition** of G at $x = s$ is

$$G_1(x) = G_2(s)$$

From the differential equation satisfied by $G(x,\, s)$ we can obtain the jump condition.

$$\int\limits_{s+\varepsilon}^{s+\varepsilon} (pG')'\,\mathrm{d}x + \int\limits_{s+\varepsilon}^{s+\varepsilon} qG\,\mathrm{d}x = \int\limits_{s+\varepsilon}^{s+\varepsilon} \delta(x-s)\,\mathrm{d}x = 1 \quad \text{(by definition of } \delta(x)\text{)}$$

Notice that because of continuity $\int_{s+\varepsilon}^{s+\varepsilon} qG\,\mathrm{d}x = 0$ and our **jump condition** is

$$p(x)\frac{dG}{dx}\bigg|_{x=s^+} - p(x)\frac{dG}{dx}\bigg|_{x=s^-} = 1$$

For the sake of concreteness

$$p(x)G_2'(s) - p(x)G_1'(s) = 1$$

Because you already know all of the above terms (albert your G still has unknown constants), we can combine this jump condition with our continuity condition to find what those unknown constants are. Usually the solution is easy to find because you simply look at the jump condition and solve it using the identity

$$\frac{1}{\gamma}\cos^2(\gamma x) + \frac{1}{\gamma}\sin^2(\gamma x) = 1$$

Note: If your now solved boundary condition simplifies to something similar to $\sin(x) = \sin(s)$, don't fret! This is not something to worry about since we're looking 'near' $x = s$ and these are both continuous functions.

Now that you finally have a solution for $G(x,\, s)$, you can finally compute $u(x)$ as

$$u(x) = \int\limits_{x_1}^{x} G(x,\, s)f(s)\,\mathrm{d}s + \int\limits_{x}^{x_2} G(x,\, s)f(s)\,\mathrm{d}s$$

For the sake of completeness, your solution should be

$$u(x) = \int\limits_{x_1}^{x} G_1(x,\, s)f(s)\,\mathrm{d}s + \int\limits_{x}^{x_2} G_2(x,\, s)f(s)\,\mathrm{d}s$$

11.9.3 Variation of Paramaters

Given an ordinary differential equation of the form

$$a(x)y'' + b(x)y' + c(x)y = f(x).$$

We can solve this using **variation of parameters** The method is important because it solves the largest class of equations. Certain functions that don't behave nicely for f such as $\ln|x|$ or e^{x^2} can still be solved with this method.

First, you must find a solution to the homogenous equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

of the form

$$y_h(x) = c_1y_1(x) + c_2y_2(x), \quad y_1 \text{ and } y_2 \text{ are independent}$$

Then the non-homogenous differential equation

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

has a particular solution

$$y(x) = y_2(x)\left(\int \frac{y_1(x)f(x)}{a(x)W(x)}\,\mathrm{d}x\right) - y_1(x)\left(\int \frac{y_2(x)f(x)}{a(x)W(x)}\,\mathrm{d}x\right)$$

11.9.4 Non-homogenous Boundary Conditions

Given a problem of the form

$$(py')' + qy(x) = f(x)$$

with

$$B_1 = a_1 y(a) + a_2 y'(a) = \alpha$$

$$B_2 = b_1 y(b) + b_2 y'(b) = \beta$$

You can find a solution for this problem using the method below.

First, find solutions $y_1(x)$ and $y_2(x)$ to the following problem.

$$(py')' + qy(x) = 0$$

with

$$B_1 = a_1 y(a) + a_2 y'(a) = 0$$

$$B_2 = b_1 y(b) + b_2 y'(b) = 0$$

Then, find solution $y_h(x)$ to the following problem

$$(py')' + qy(x) = f(x)$$

with

$$B_1 = a_1 y(a) + a_2 y'(a) = 0$$

$$B_2 = b_1 y(b) + b_2 y'(b) = 0$$

Finally, solve for $y_p(x)$

$$(py')' + qy(x) = 0$$

with

$$B_1 = a_1 y(a) + a_2 y'(a) = \alpha$$

$$B_2 = b_1 y(b) + b_2 y'(b) = \beta$$

Your solution should be of the form

$$y(x) = \int\limits_{\alpha}^{x} G_1(x,\, s) f(s)\, \mathrm{d}s + \int\limits_{x}^{\beta} G_2(x,\, s) f(s)\, \mathrm{d}s + y_p(x)$$

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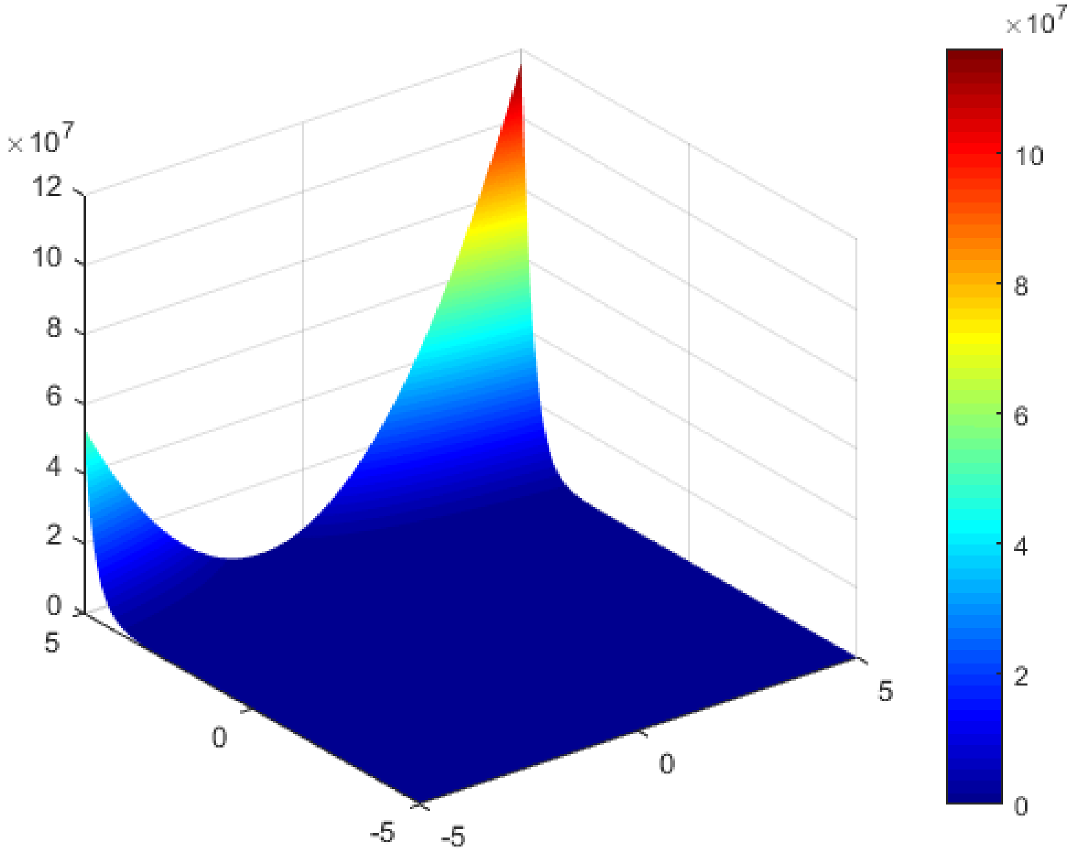
Chapter 12

MATHS 362

This course covers three distinct applications of the concepts taught in Applied Mathematics - Traffic Flow, Calculus of Variations and Asymptotic Analysis. Traffic Flow looks at modelling the flow of traffic using quasi-linear partial differential equations. Calculus of Variations looks at finding functions that maximize or minimize functionals using the Euler-Lagrange equation. Asymptotic Analysis looks at finding an approximate solution to a algebraic or differential equation involving a small parameter. All three sections assume you have an understanding of solving ordinary differential equations. If you do not, please review my MATHS 260 notes before studying these notes.



12.1 Method of Characteristics

In this course, we focus on solving first order quasi-linear partial differential equations by using the Method of Characteristics.

Recall that a linear PDE is of the form

$$a(x,\, y)\frac{\partial u}{\partial x} + b(x,\, y)\frac{\partial u}{\partial y} + c(x,\, y)u = d(x,\, y)$$

And a quasi-linear PDE is of the form

$$a(x,\, y,\, u)\frac{\partial u}{\partial x} + b(x,\, y,\, u)\frac{\partial u}{\partial y} = d(x,\, y,\, u)$$

12.1.1 Linear PDEs

Consider

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,\, 0) = f(x)$$

In this case u is a function with two independent variables, x and t. However, in the real world x is itself a function of t. We can then rewrite u as

$$u(x,\, t) = u(x(t),\, t) = u(t)$$

And by the chain rule,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial u}{\partial x}$$

Comparing with our original linear partial differential equation we can see that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c, \qquad x(0) = x_0$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0, \qquad u(x,\, 0) = f(x_0)$$

We can see clearly that

$$x_0 = x - ct \rightarrow u(x,\, t) = f(x - ct)$$

12.1.2 Quasi-Linear PDEs

Now consider a more difficult problem

$$\frac{\partial u}{\partial t} + b(x,\, t,\, u)\frac{\partial u}{\partial x} = d(x,\, t,\, u), \quad -\infty < x < \infty, \quad t > 0, \quad u(x,\, 0) = f(x)$$

Again we will solve the following differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = b(x,\, t,\, u), \qquad x(0) = x_0$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = d(x,\, t,\, u), \qquad u(x,\, 0) = f(x_0)$$

These ordinary differential equations are considerably more difficult to solve than the previous one, and methods such as seperable variables or an integrating factor may be required.

12.2 Traffic Flow

12.2.1 Density

Let us denote $N(x,\, t)$ as the number of cars from a to b. Then, the **traffic density** is given by

$$\rho(x,\, t) = \frac{\partial N}{\partial x}$$

12.2.2 Flux

Let us denote $J(x,\, t)$ as the number of cars from a to b per unit time. Then, the **traffic flow** is given by

$$J(x,\, t) = \rho(x,\, t)v(x,\, t) = \rho v(\rho)$$

12.2.3 Conservation Law

The rate of change in cars from a to b can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b \rho(x,\, t)\, \mathrm{d}x = \int_a^b \frac{\partial \rho}{\partial t}\, \mathrm{d}x = J(a,\, x) - J(b,\, t)$$

Notice by the Fundamental Theorem of Calculus that

$$\int_a^b \frac{\partial J}{\partial x}\, \mathrm{d}x = J(b,\, x) - J(a,\, t)$$

Combining the terms together it is trivial to see that

$$\int_a^b \frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x}\, \mathrm{d}x = 0$$

And thus

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

12.2.4 Elementary Traffic Model

$$v(\rho) = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$$
$$J(\rho) = v_{\max}\left(\rho - \frac{\rho^2}{\rho_{\max}}\right)$$
$$c(\rho) = v_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)$$

12.2.5 Fan-like Characteristics

12.2.6 Shock Waves

The location of the shock is determined by solving the ODE,

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \frac{J^+ - J^-}{\rho^+ - \rho^-} = \frac{[J]}{[\rho]}$$

If you are given the elementary traffic model

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \quad \text{where } c(\rho) = v_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)$$

Then the location of the shock is determined by solving the ODE,

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = v_{\max}\left(1 - \frac{\rho^+ - \rho^-}{\rho_{\max}}\right)$$

12.3 Calculus of Variations

This section has direct applications to **classical mechanics** and shares content with PHYSICS 315.

12.3.1 The Euler-Lagrange equation

The Euler-Lagrange differential equation is the fundamental equation in Calculus of Variations. It states that if I is defined by an integral of the form

$$I = \int_a^b F(y, y', t)\,\mathrm{d}t$$

then I has a stationary value if the Euler-Lagrange differential equation below is satsified,

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0$$

You will often be asked to find such a function $y(x)$ that minimises or maximises the integral I using the Euler-Lagrange Equations.

12.3.2 A classic example

It is obvious to us that the shortest distance between two points is a straight line. However, using Calculus of Variations, we can prove that a straight line is the minimised distance between two points.

Let $\mathrm{d}s$ be the length of a small arc in a plane. Then

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \text{ or } \mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

The total length of the curve is then going to be

$$I = \int_a^b \mathrm{d}s = \int F(x,\, y,\, y')\,\mathrm{d}x, \quad F(x,\, y,\, y') = \sqrt{1 + (y')^2}$$

We can use the Euler-Lagrange equation found on the next page.

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0, \quad \frac{\partial F}{\partial y} = 0 \text{ and } \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}}$$

The Euler-Lagrange equation simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0 \text{ and so } \frac{y'}{\sqrt{1 + (y')^2}} = c$$

Solving for y' you should obtain the following result,

$$y' = \frac{c}{\sqrt{1 - c^2}} = a, \quad a \in \mathbb{R}.$$

Integrating both sides with respect to x, we finally have

$$y(x) = ax + b$$

Which is the equation of a straight line between two arbitrary points on a plane. We have then proved that the equation of a straight line is an extremum for the functional I, and it is obvious that this equation is a minimum.

12.3.3 Derivation of the Euler-Lagrange equation

Given an integral of the form

$$I = \int_a^b F(y, y', t)\,\mathrm{d}t$$

Let us conveniently write

$$Y(t) = y(t) + \alpha\eta(t), \quad \eta(a) = \eta(b) = 0$$

Replacing $y(t)$ with $Y(t)$ we get

$$I(\alpha) = \int_a^b F(Y, Y', t)\,\mathrm{d}t$$

For a minimum to exist we require the first derivative to be equal to zero, i.e.

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0$$

Using the chain rule, we obtain the following identity

$$\frac{\mathrm{d}I}{\mathrm{d}\alpha} = \int_a^b \left[\frac{\partial F}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}\alpha} + \frac{\partial F}{\partial Y'}\frac{\mathrm{d}Y'}{\mathrm{d}\alpha}\right]\,\mathrm{d}t.$$

It is clear from our definition of $Y(t)$ the following identities,

$$\frac{\mathrm{d}Y}{\mathrm{d}\alpha} = \eta(t), \quad \frac{\mathrm{d}Y'}{\mathrm{d}\alpha} = \eta'(t)$$

Now setting $\alpha = 0$ we have the following identity

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_a^b \left[\frac{\partial F}{\partial y}\eta(t) + \frac{\partial F}{\partial y'}\eta'(t)\right]\,\mathrm{d}t = 0$$

It is easy to see the second term can be rewritten as

$$\int_a^b \frac{\partial F}{\partial y'}\eta'(t)\,\mathrm{d}t = \int_a^b \frac{\partial F}{\partial y'}\frac{\mathrm{d}\eta}{\mathrm{d}t}\,\mathrm{d}t = \int_a^b \frac{\partial F}{\partial y'}\mathrm{d}\eta$$

Using integration by parts we find that

$$\int_a^b \frac{\partial F}{\partial y'}\mathrm{d}\eta = \left.\frac{\partial F}{\partial y'}\eta(t)\right|_a^b - \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial y'}\right)\eta(t)\,\mathrm{d}t$$

The left term is zero since $\eta(a) = \eta(b) = 0$. We now have

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(t)\,\mathrm{d}t = 0$$

Finally, since $\eta(t)$ is an arbitrary function, we have the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

12.3.4 Special case for the Euler-Lagrange equation

In many physical problems, F_x (the partial derivative of F with respect to x) turns out to be 0, in which case a manipulation of the Euler-Lagrange differential equation reduces to the greatly simplified and partially integrated form known as the **Beltrami identity**,

$$F - y'\frac{\partial F}{\partial y'} = c$$

12.3.5 Derivation of the special case

In this case, the integrand does not depend on x explicitly, i.e.

$$\frac{\partial F}{\partial x} = 0$$

However, since $y = y(x)$, the integrand does depend on x implicitly, i.e.

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y''$$

By the multiplication rule of derivatives, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}y'\right) = \frac{\partial F}{\partial y'}y'' + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)y'$$

Subtracting both sides from each other, we have

$$\frac{\mathrm{d}F}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}y'\right) = \frac{\partial F}{\partial y}y' - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)y'$$

Factorising the terms we find that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F - y'\frac{\partial F}{\partial y'}\right) = \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right]y'$$

By the Euler-Lagrange equation, the brack on the right hand side must be equal to zero. Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$$

Integrating both sides finally gives us the Beltrami identity,

$$F - y'\frac{\partial F}{\partial y'} = c$$

12.3.6 Euler-Lagrange equations with constraints

We can use **Lagrange Multipliers** to find the stationary points of a functional I subject to the constraint functional J with integrand G.

$$K = \int_a^b F(y, y', t) + \lambda\, G(y, y', t)\, \mathrm{d}t = \int_a^b H(y, y', t)\, \mathrm{d}t$$

12.3.7 Euler-Lagrange Equations With Several Dependent Variables

Let us now consider a functional I with several dependent variables as is the case below.

$$I = \int_a^b F\left(y_1, y_2, y_1', y_2', t\right)\, \mathrm{d}t$$

The resulting Euler-Lagrange Equation is then a system of equations

$$\frac{\partial F}{\partial y_1} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_1'}\right) = 0 \text{ and } \frac{\partial F}{\partial y_2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_2'}\right) = 0$$

It can easily be shown how this can be generalised for n independent variables.

12.3.8 Euler-Lagrange Equations With Several Independent Variables

Let us now consider a functional I with several independent variables as is the case below.

$$I = \iint\limits_{\Omega} F\left(y,\, y_{t_1},\, y_{t_2},\, t_1,\, t_2\right)\, \mathrm{d}t_1\, \mathrm{d}t_2$$

The resulting Euler-Lagrange Equation is then of the form

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial t_1}\left(\frac{\partial F}{\partial z_1}\right) - \frac{\partial}{\partial t_2}\left(\frac{\partial F}{\partial z_2}\right) = 0$$

where

$$z_1 = \frac{\partial y}{\partial t_1} \text{ and } z_2 = \frac{\partial y}{\partial t_2}$$

It can easily be shown how this can be generalised for n independent variables.

12.3.9 Hamilton's Principle

Given the potential energy V and the kinetic energy T of a pendulum of length ℓ, the equation of motion $y(t)$ can be found by finding the stationary points of the following functional.

$$I = \int_a^b F\, \mathrm{d}t = \int_a^b T - V\, \mathrm{d}t$$

12.3.10 Rayleigh-Ritz Method

Given a functional

$$I = \int_a^b F(y,\, y',\, t)\,\mathrm{d}t, \quad y(a) = \alpha, \quad y(b) = \beta$$

We can approximate $y(x)$ by a function with a set of linearly independent functions u_n

$$\phi = u_0 + \sum_{n=1}^{N} a_n u_n$$

where u_0 satisfies any nonhomogenous boundary conditions.

Example.

Given a functional I we would like to find a function $y(x)$ that makes this integral stationary

$$I = \int_0^1 \left((y')^2 - 4y\right)\mathrm{d}x, \quad y(0) = y(1) = 0$$

We can approximate $y(x) = c\sin(\pi x)$ which satisfies the homogenous boundary conditions.

$$I = c^2\pi^2 \int_0^1 \cos^2(\pi x)\,\mathrm{d}x - 4c\int_0^1 \sin(\pi x)\,\mathrm{d}x$$

Observe that the function I is minimum when

$$\frac{\mathrm{d}I}{\mathrm{d}c} = 2c\pi^2 \int_0^1 \cos^2(\pi x)\,\mathrm{d}x - 4\int_0^1 \sin(\pi x)\,\mathrm{d}x = 0$$

Now we can solve for c

$$c = \frac{4\int_0^1 \sin(\pi x)\,\mathrm{d}x}{2\pi^2\int_0^1 \cos^2(\pi x)\,\mathrm{d}x} = \frac{16}{2\pi^3} = \frac{8}{\pi^3}$$

Hence, the function

$$y(x) = \frac{8}{\pi^3}\sin(\pi x)$$

is an approximation for the function that makes this integral stationary

12.4 Asymptotic Analysis

12.4.1 Asymptotic Expansion

Taylor series

Example.

We would like to approximate the function f into the form

$$f(\varepsilon) = \frac{\sin(\varepsilon)}{1-\cos(\varepsilon)} = c_1\varepsilon^\alpha + c_2\varepsilon^\beta$$

Using a taylor series expansion we can approximate f as

$$f(\varepsilon) = \frac{\varepsilon - \frac{\varepsilon^3}{3!} + O(\varepsilon^5)}{1-(1-\frac{\varepsilon^2}{2!}+\frac{\varepsilon^4}{4!}) + O(\varepsilon^6)} = \frac{\varepsilon - \frac{\varepsilon^3}{6} + O(\varepsilon^5)}{\frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{24} + O(\varepsilon^6)} = \frac{\varepsilon}{\frac{\varepsilon^2}{2}}\left(\frac{1-\frac{\varepsilon^2}{6}+O(\varepsilon^4)}{1-\frac{\varepsilon^2}{12}+O(\varepsilon^4)}\right)$$

We can simplify this down to

$$f(\varepsilon) = \frac{2}{\varepsilon}\left(1-\frac{\varepsilon^2}{6}+O(\varepsilon^4)\right)\left(1+\frac{\varepsilon^2}{12}+O(\varepsilon^4)\right) = f(\varepsilon) = \frac{2}{\varepsilon}\left(1+\frac{\varepsilon^2}{12}-\frac{\varepsilon^2}{6}+O(\varepsilon^4)\right) = \frac{2}{\varepsilon}-\frac{\varepsilon}{6}$$

Therefore our approximation is of the form

$$f(\varepsilon) = \frac{\sin(\varepsilon)}{1-\cos(\varepsilon)} = \frac{2}{\varepsilon} - \frac{\varepsilon}{6} + O(\varepsilon^3)$$

Limits Technique

The two term asymptotic expansion of a function f can be represented as

$$f = a_1\phi_1 + a_2\phi_2 + o(\phi_2)$$

where

$$a_1 = \lim_{\varepsilon\to\varepsilon_0} \frac{f}{\phi_1} \text{ and } a_2 = \lim_{\varepsilon\to\varepsilon_0} \frac{f - a_1\phi_1}{\phi_2}$$

An indirect approach to determining the asymptotic expansion is setting $x \sim x_0 + \varepsilon^\alpha x_1 + \dots$ and then equating the coefficients of the problem.

Example.

We would like to approximate the function f into the form

$$f(\varepsilon) = \frac{1}{1 - \varepsilon^{-1}\sin(\varepsilon)} = c_1\varepsilon^\alpha + c_2\varepsilon^\beta$$

Let us first taylor expand this function

$$f(\varepsilon) = \frac{1}{\frac{\varepsilon^2}{3!} - \frac{\varepsilon^4}{5!} + O(\varepsilon^4)} = \frac{6}{\varepsilon^2(1 - \frac{\varepsilon^2}{20} + O(\varepsilon^2))}$$

This form isn't too helpful to solve directly. Use the limit definitions to find c_1 and c_2

$$c_1 = \lim_{\varepsilon\to 0} \frac{f}{\varepsilon^\alpha} = \frac{6}{\varepsilon^{2+\alpha}(1 - \frac{\varepsilon^2}{20} + O(\varepsilon^2))} = \frac{6}{\varepsilon^{2+\alpha}} = 6, \quad \alpha = -2$$

Now let's calculate c_2 and β.

$$c_2 = \lim_{\varepsilon\to 0} \frac{f - 6\varepsilon^{-2}}{\varepsilon^\beta} = \lim_{\varepsilon\to 0} \varepsilon^{-\beta}\left(\frac{6}{\varepsilon^2(1 - \frac{\varepsilon^2}{20})} - \frac{6}{\varepsilon^2}\right) = 6\lim_{\varepsilon\to 0} \varepsilon^{-2-\beta}\left(\frac{1 - (1 - \frac{\varepsilon^2}{20})}{(1 - \frac{\varepsilon^2}{20})}\right) = 6\lim_{\varepsilon\to 0} \varepsilon^{-2-\beta}\left(\frac{\frac{\varepsilon^2}{20}}{1 - \frac{\varepsilon^2}{20}}\right)$$

We can simplify this down to

$$c_2 = \frac{3}{10}\lim_{\varepsilon\to 0} \varepsilon^{-\beta}\left(\frac{1}{1 - \frac{\varepsilon^2}{20}}\right) = \frac{3}{10}, \quad \beta = 0$$

Our final solution is then

$$f(\varepsilon) = \frac{1}{1 - \varepsilon^{-1}\sin(\varepsilon)} = 6\varepsilon^{-2} + \frac{3}{10}$$

12.4.2 Asymptotic Solution of Algebraic Equations

Suppose we wanted to find the asymptotic solution to an equation such as

$$x^2 + 2\varepsilon x - 1 = 0$$

The first step is to determine the approximate location of the roots. We know this equation has roots near $x = \pm 1$.

This means our asymptotic expansion must be of the form

$$x_0 + x_1\varepsilon^\alpha + \ldots, \qquad \alpha > 0$$

Since our solution cannot be zero or unbounded as $\varepsilon \to 0$
Now we expand the function

$$(x_0 + x_1\varepsilon^\alpha + \ldots)^2 + 2\varepsilon(x_0 + x_1\varepsilon^\alpha + \ldots) - 1 = 0$$

or

$$(x_0^2 + 2x_0x_1\varepsilon^\alpha + \ldots) + 2\varepsilon(x_0 + x_1\varepsilon^\alpha + \ldots) - 1 = 0$$

Solving for $O(1)$ we have $x_0 - 1 = 0$ gives $x_0 = \pm 1$. Equating the coefficients on both sides, the $O(\varepsilon)$ term must be zero and yet we have $\pm 2\varepsilon$ on the left hand side. Therefore another term must be in $O(\varepsilon)$ and hence $\alpha = 1$

$$(x_0^2 + 2x_0x_1\varepsilon + \ldots) + 2\varepsilon(x_0 + x_1\varepsilon + \ldots) - 1 = 0$$

Solving for $O(\varepsilon)$ we now have

$$2x_0x_1 + 2x_0 = 0 \to 2x_0(x_1 + 1) = 0$$

Since we already know $x_0 \neq 0$ we must have $x_1 = -1$.

Thus, the two-term asymptotic expansion of the roots of that expression is

$$x \sim x_0 + x_1\varepsilon^\alpha = \pm 1 - \varepsilon$$

12.4.3 Asymptotic Solution of Differential Equations

Consider the following differential equation with small parameter ε.

$$y'(x) + y(x) + (y(x))^2 = 0, \quad 0 < x < 1, \quad y(0) = \varepsilon$$

Assume that the solution has an asymptotic expansion of the form

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + o(\varepsilon^2)$$

Then expand into the form

$$(y_0' + \varepsilon y_1' + \varepsilon^2 y_2') + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2) + (y_0^2 + 2\varepsilon y_0 y_1 + 2\varepsilon^2 y_0 y_2 + 2\varepsilon^2 y_1) = 0$$

Solving the $O(1)$ term we get

$$y_0' + y_0 + y_0^2 = 0, \quad y_0(0) = 0$$

which has the unique soluion $y_0(x) = 0$. Simplified, this is

$$(\varepsilon y_1' + \varepsilon^2 y_2') + (\varepsilon y_1 + \varepsilon^2 y_2) + 2\varepsilon^2 y_1 = 0$$

Solving the $O(\varepsilon)$ term we get

$$y_1' + y_1 = 0, \quad y_1(0) = 1$$

Solving by seperable variables we get

$$\frac{1}{y_1}\mathrm{d}y_1 = \int -\,\mathrm{d}x \to \ln(y_1) = -x + c \to y_1(x) = Ae^{-x}$$

Including the boundary condition we get $y_1(0) = A = 1$ so

$$y_1(x) = e^{-x}$$

Solving the $O(\varepsilon^2)$ term we get

$$y_2' + y_2 + 2e^{-x} = 0$$

which using an integrating factor $\mu = e^x$ gives

$$\begin{aligned}(\mu y_2)' &= -2\\ (\mu y_2) &= -2x + c\\ y_2(x) &= \frac{-2x + c}{e^x}\end{aligned}$$

Use the boundary condition

$$y_2(0) = -2 + c = 0 \to c = 2$$

Therefore we finally have

$$y(x) \sim \varepsilon e^{-x} + 2\varepsilon^2 e^{-x}(-x + 1)$$

12.5 Matched Asymptotic Expansions

12.5.1 Problems involving Boundary Layers

Consider the following differential equation with small positive parameter ε.

$$\varepsilon y'' - y' + xy = 0, \quad 0 < x < 1, \quad y(0) = 1, \quad y(1) = 0.$$

We must find the outer solution, inner solution and composite solution.

Outer solution

In order to find the outer solution, set $\varepsilon = 0$ and solve.

$$-y' + xy = 0$$

Using seperable variables we get

$$\int \frac{1}{y}\,\mathrm{d}y = \int x\,\mathrm{d}x \to y = A\,\exp\left(\frac{1}{2}x^2\right)$$

This function can only be satisfied at $y(0) = 1$ and so we have a boundary layer at $x = 1$.

$$y_{\text{outer}} = \ \exp\left(\frac{1}{2}x^2\right)$$

Inner solution

We know we have a boundary layer at $x = 1$ and so must introduce a boundary condition coordinate $\bar{x}$

$$\bar{x} = \frac{x-1}{\varepsilon^\alpha} \text{ or } x = \varepsilon^\alpha \bar{x} + 1$$

Note by the chain rule

$$\frac{d}{dx} = \frac{d}{d\bar{x}}\frac{d\bar{x}}{dx} = \frac{1}{\varepsilon^\alpha}\frac{d}{dx}$$

Now we rewrite the differential equation with our new coordinates

$$\varepsilon^{1-2\alpha}y'' - \varepsilon^{-\alpha}y' + (\varepsilon^\alpha \bar{x} + 1)y = 0, \quad y(1) = 0$$

We must figure out which term to match now. We have already matched (2) and (3) in finding the outer solution so we do not reattempt this, only looking at (1) and (2) or (1) and (3). Setting $1 - 2\alpha = \alpha$ does not give (2) a higher order solution so this is incorrect. Set (1) = (2) gives

$$1 - 2\alpha = -\alpha \to \alpha = 1$$

We now have

$$\varepsilon^{-1}y'' - \varepsilon^{-1}y' + (\varepsilon\bar{x} + 1)y = 0, \quad y(1) = 0$$

Solving the $O(\varepsilon^{-1})$ term gives us

$$y'' - y' = 0 \to r(r-1) = 0 \to y = Ae^0 + Be^{\bar{x}} = A + B\,\exp\left(\frac{x-1}{\varepsilon}\right)$$

By the boundary condition $A + B = 0$ so we finally have the general solution

$$y_{\text{inner}}(x) = A\left(1 - \exp\left(\frac{x-1}{\varepsilon}\right)\right)$$

Matching

We now have

$$y_{\text{outer}} = \exp\left(\frac{1}{2}x^2\right)$$

and

$$y_{\text{inner}} = A\left(1 - \exp\left(\frac{x-1}{\varepsilon}\right)\right)$$

We now require

$$\lim_{\bar{x}\to\infty} y_{\text{inner}} = \lim_{x\to 0} y_{\text{outer}}$$

Since

$$\lim_{x\to 0} y_o(x) = e^{1/2}$$

we require

$$A = e^{1/2}$$

Composite Solution

And our final solution is of the form

$$y(x) \sim y_{\text{o}} + y_{\text{i}} - e^{1/2}$$

So we finally have

$$y(x) \sim \exp\left(\frac{1}{2}x^2\right) - e^{1/2}\left(\ \exp\left(\frac{x-1}{\varepsilon}\right)\right)$$

12.5.2 Problems involving Interior Layers

12.5.3 Problems involving Corner Layers

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Chapter 13

MATHS 363

This course considers some of the most common topics that applied mathematicians encounter in their work: fitting of models to observational (noisy) data, interpreting all computed solutions as (multivariate) random variables, numerical methods for partial differential equations, and the related modelling problems

It is recommended that you study STATS 125 and STATS 210 before taking this course, only so that a lot of the functions and logical statements provided already make intuitive sense for you.

13.1 Finite Difference Method

Given the Linear BVP

$$u'' + p(x)u' + q(x)u = f(x)$$

The central difference formula for the second derivative is

$$u''(x) \approx \frac{u(x-h) - 2u(x) + u(x+h)}{h^2}$$

The central difference formula for the first derivative is

$$u'(x) \approx \frac{u(x+h) - u(x-h)}{2h}$$

We can then rewrite the BVP as

$$\frac{u(x-h) - 2u(x) + u(x+h)}{h^2} + p(x)\frac{u(x+h) - u(x-h)}{2h} + q(x)u = f(x)$$

We can factor the terms to give

$$\Big[-2 + h^2 q(x)\Big]u(x) + \Big[1 - \frac{h}{2}p(x)\Big]u(x-h) + \Big[1 + \frac{h}{2}p(x)\Big]u(x+h) = h^2 f(x)$$

Simplified this is

$$a_i u_i + b_i u_{i-1} + c_i u_{i+1} = h^2 f_i$$

Now for $i = 1, 2, 3$ this is (note: u_0 and u_4 are treated as constants)

$$a_1 u_1 + c_1 u_2 = h^2 f_1 - b_1 \alpha$$
$$a_2 u_2 + b_2 u_1 + c_2 u_3 = h^2 f_2$$
$$a_3 u_3 + b_3 u_2 = h^2 f_3 - c_3 \beta$$

The 3×3 finite element equations for this are

$$\begin{bmatrix} a_1 & c_1 & 0 \\ b_2 & a_2 & c_2 \\ 0 & b_3 & a_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} h^2 f_1 - b_1 \alpha \\ h^2 f_2 \\ h^2 f_3 - c_3 \beta \end{bmatrix}$$

This is a tridiagonal matrix that can be numerically solved with an efficient method using MATLAB. This problem can be solved by hand but when working with $N \times N$ finite element equations you will need MATLAB.

13.2 Variational Formulation

Given a problem of the form

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[\kappa(x)u'(x)\right] = f(x)$$

We can introduce a test function $\nu(x)$ such that

$$-\int_0^L \nu(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[\kappa(x)u'(x)\right]\,\mathrm{d}x = \int_0^1 \nu(x)f(x)\,\mathrm{d}x$$

We must solve for the bilinear form $a(u, v)$

$$a(u,\,\nu) = -\int_0^1 \kappa u'\nu'\,\mathrm{d}x$$

and the linear functional

$$F(\nu) = \int_0^1 \nu(x)f(x)\,\mathrm{d}x$$

We can then find an approximate solution of the form

$$u(x) = \sum_{n=1}^{N} a_n\phi_n(x)$$

13.3 Parameter Estimation

13.4 Probability and Random Variables

13.4.1 Distribution and Density Functions

For a random variable X, we can define the distribution function $F(x) = \mathbb{P}(X \leq x)$. We can also define the density function $\pi(x)$ such that

$$\int_{-\infty}^{x} \pi(t)\,\mathrm{d}t = F(x)$$

An important property of the density function is

$$\int_{-\infty}^{\infty} \pi(x)\,\mathrm{d}x = 1$$

13.4.2 Mean, Variance and Standard Deviation

The mean μ of a random variable X is

$$\mu = \mathbb{E}(X) = \int_{-\infty}^{\infty} x\pi(x)\,\mathrm{d}x$$

The variance σ^2 of a random variable X is

$$\mathrm{var}(X) = \sigma^2 = \mathbb{E}\left((x-\mu)^2\right) = \int_{-\infty}^{\infty} (x-\mu)^2\pi(x)\,\mathrm{d}x$$

The standard deviation σ is defined as $\sqrt{\sigma^2}$. One can be able to show that

$$\mathrm{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \text{ and } \mathrm{var}(X) \geq 0.$$

The univariate normal distribution is of the form

$$\pi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

and is denoted as $X \sim \mathcal{N}(\mu,\ \sigma^2)$.

13.4.3 Joint Distribution, Mean and Covariance

The covariance Γ_X of a random varable X is

$$\Gamma_X = \mathbb{E}\left((x-\mu)(x-\mu)^T\right) = \int_{-\infty}^{\infty} (x-\mu)(x-\mu)^T\pi(x)\,\mathrm{d}x$$

The crossovariance Γ_{XY} of a random varables $X,\ Y$ is

$$\Gamma_{XY} = \mathbb{E}\left((x-\mu_X)(y-\mu_Y)^T\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y)^T\pi(x,\ y)\,\mathrm{d}x\,\mathrm{d}y$$

13.4.4 The Mean as an Operator

Because taking a mean amounts to computing an integral, which has the standard algebraic properties, the mean is a linear operator and acts on the probability space only. As a consequence of this we have

$$\mathbb{E}(aX+b) = \int_{-\infty}^{\infty} (ax+b)\pi(x)\,\mathrm{d}x = a\int_{-\infty}^{\infty} x\pi(x) + b\int_{-\infty}^{\infty} \pi(x) = a\mu + b$$

One can also show that

$$\Gamma = \mathbb{E}(XX^T) - \mu\mu^T$$

13.4.5 Marginal and Conditional Distributions, Conditional Mean and Variance

The marginal density is defined as

$$\pi(x_1) = \int_{-\infty}^{\infty} \pi(x)\,\mathrm{d}x_2$$

The conditional density is defined as

$$\pi(x_2|x_1) = \frac{\pi(x_1,\, x_2)}{\pi(x_1)}$$

Using the above formula, we can express **Bayes Formula**.

$$\pi(x) = \pi(x_2 \mid x_1)\pi(x_1) = \pi(x_1 \mid x_2)\pi(x_2)$$

13.5 Linear Least Squares Estimation

13.6 Optimization and Nonlinear Methods

13.6.1 Line Search Methods

Consider the problem of finding the minimizer for a functional $\Psi(x) = \Psi(x_1,\ x_2)$. We must first calculate the gradient $\Delta\Psi$ and the Hessian H.

Then, the formula for the Newton-Raphson search method of this functional is given by

$$d = -H^{-1}\Delta\Psi$$

13.6.2 Search Directions for Nonlinear Least Squares Problems

Given a nonlinear model

$$y = f(\alpha_1,\ \alpha_2,\ t) + e$$

where

$$y \in \mathbb{R}^N, \quad t \in \mathbb{R}^N, \quad e \in \mathbb{R}^N, \quad x = (\alpha_1,\ \alpha_2)^T$$

We can find the Jacobian

$$J = \begin{bmatrix} \dfrac{\partial y_1}{\partial \alpha_1} & \dfrac{\partial y_1}{\partial \alpha_2} \\ \dfrac{\partial y_2}{\partial \alpha_1} & \dfrac{\partial y_2}{\partial \alpha_2} \\ \vdots & \vdots \\ \dfrac{\partial y_N}{\partial \alpha_1} & \dfrac{\partial y_N}{\partial \alpha_2} \end{bmatrix}$$

Then, the formula for the Gauss-Newton search method is given by

$$d = -(J^T J)^{-1}\Delta\Psi = (J^T J)^{-1} J^T (y - A(x))$$

13.6.3 Covariance

The approximation for the covariance is given by

$$\Gamma = \sigma^2 (J^T J)^{-1}$$

13.7 Estimation Theory

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