1. Which of the following PDEs is linear? Explain your answers and show your work.

   (a) $u_t + uu_x + u_{xx} = 0$

   (b) $u_{tt} + \sqrt{x}u_{xx} + \cos(x)u_x - e^{x^2} = 0$

Solution.

(a) This PDE is not linear as $L[\alpha u] = \alpha L[u]$ does not hold.

Proof.
Let us define $f = 0$ assume that $L[u] = u_t + uu_x + u_{xx}$ is a linear operator. By definition, the following property must hold,

$$L[\alpha u] = \alpha L[u], \text{ for any } \alpha \in \mathbb{R}.$$

Notice that, for the left hand side,

$$L[\alpha u] = (\alpha u)_t + (\alpha u)(\alpha u)_x + (\alpha u)_{xx} = \alpha u_t + \alpha^2 uu_x + \alpha u_{xx}$$

However, on the right hand side,

$$\alpha L[u] = \alpha u_t + \alpha uu_x + \alpha u_{xx}.$$

Hence, they are not equal for $\alpha \neq 0, 1$. And so, by contradiction, the above PDE cannot be linear.

(b) This PDE is linear since $L[\alpha u + \beta v] = \alpha L[u] + \beta L[v]$.

Proof.
Let us define $f = e^{x^2}$ and $L[u] = u_{tt} + \sqrt{x}u_{xx} + \cos(x)u_x$. By definition, $L[u]$ is a linear operator iff the following property holds,

$$L[\alpha u + \beta v] = (\alpha u + \beta v)_{tt} + \sqrt{x}(\alpha u + \beta v)_{xx} + \cos(x)(\alpha u + \beta v)_x$$

$$= \alpha u_{tt} + \beta v_{tt} + \alpha\sqrt{x}u_{xx} + \beta\sqrt{x}v_{xx} + \alpha\cos(x)u_x + \beta\cos(x)v_x$$

$$= \alpha L[u] + \beta L[v]$$

2. Consider the function $f(x) = x^3$ for $x \in [0, 1]$.

   (a) Construct the even extension of f and find its Fourier series.

   (b) Construct the odd extension of f and find its Fourier series.

Solution.

(a) We denote $f_e(x) = x^3$ for $x \in [0, 1]$ and $f_e(x) = -x^3$ for $x \in [-1, 0)$. We also extend this by a period of 2 such that $f_e(x + 2) = f_e(x)$. We now construct its fourier series given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x).$$

We have

$$a_0 = \frac{1}{2}\int_{-1}^{1} f_e(x)\ dx = \int_0^1 x^3\ dx = \frac{1}{4}x^4|_0^1 = \frac{1}{4}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f_e(x)\cos(n\pi x)\ dx = 2\int_0^1 x^3 \cos(n\pi x)\ dx$$

Solving $\int x^3 \cos(n\pi x)$, first with a u-substitution we have.

$$\frac{1}{n^4\pi^4}\int u^3 \cos(u)\ du$$

Then, integrating by parts with $u = u^3$, $v' = \cos(u)$ we have.

$$\frac{1}{\pi^4 n^4}\left(u^3 \sin(u) - \int 3\,u^2 \sin(u)\,du\right)$$

Now, calculating $\int u^2 \sin(u)\ du$ with integration by parts $u = u^2, v' = \sin(u)$, we have

$$\left(-u^2\cos(u) + 2\int\ u\cos(u)\,du\right)$$

Now, in one final swoop we calculate $\int u\cos(u)\ du$ using integration by parts, $u = u$, $v' = \cos(u)$.

$$\left(u\sin(u) - \int \sin(u)\,du\right)$$

Piecing it all together, what you have is

$$\frac{1}{\pi^4 n^4}\left(u^3\sin(u) - 3\left(-u^2\cos(u) + 2\left(u\sin(u) + \cos(u)\right)\right)\right)$$

$$\frac{1}{\pi^4 n^4}\left(u^3\sin(u) + 3u^2\cos(u) - 6u\sin(u) - 6\cos(u)\right)$$

And hence you have

$$\frac{1}{\pi^4 n^4}\left(\pi^3 n^3 x^3 \sin(\pi n x) - 3\left(-\pi^2 n^2 x^2 \cos(\pi n x) + 2\left(\pi n x \sin(\pi n x) + \cos(\pi n x)\right)\right)\right)$$

Now we get the definite integral,

$$\frac{1}{\pi^4 n^4}\left(\pi^3 n^3 \sin(\pi n) - 3\left(-\pi^2 n^2 \cos(\pi n) + 2\left(\pi n \sin(\pi n) + \cos(\pi n)\right)\right)\right)$$

$$-\frac{1}{\pi^4 n^4}\left(0 - 3\left(0 + 0 + \cos(0)\right)\right)$$

Simplified,

$$\frac{1}{\pi^4 n^4}\left(-3\left(-\pi^2 n^2 \cos(\pi n) + 2\left(\cos(\pi n)\right)\right)\right)$$

$$\frac{3}{\pi^4 n^4}$$

Simplified,

$$\cos(\pi n)\frac{-3}{\pi^4 n^4}\left(-\pi^2 n^2 + 2\right) + \frac{3}{\pi^4 n^4}$$

Simplified,

$$\frac{3}{\pi^4 n^4}\left(-\pi^2 n^2 + 3 + (-1)^{n+1}\right)$$

Hence,

$$a_n = \frac{6}{\pi^4 n^4}\left(3 - \pi^2 n^2 + (-1)^{n+1}\right)$$

We also have

$$b_n = \frac{1}{1}\int_{-1}^{1} f_e(x)\sin(n\pi x) = 0, \text{ since we are integrating an odd function.}$$

Hence are Fourier series is given by the following,

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x), \text{ where } a_0 \text{ and } a_n \text{ are given above.}$$

(b) We denote $f_0(x) = x^3$ for $x \in [-1, 1]$ and extend this by a period of 2 such that $f_e(x+2) = f_e(x)$. We now construct the fourier series given by,

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x).$$

We have

$$a_0 = \frac{1}{2}\int_{1}^{1} f_0(x)\ dx = 0$$

$$a_n = \frac{1}{1}\int_{1}^{1} f_0(x)\cos(n\pi x)\ dx = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f_0(x)\sin(n\pi x)\ dx = 2\int_{0}^{1} x^3 \sin(n\pi x)\ dx$$

We solve $\int x^3 \sin(n\pi x)$ and apply u-substitution $u = n\pi x$.

$$\int \frac{u^3 \sin(u)}{\pi^4 n^4} du$$

We apply Integration By Parts with $u = u^3$, $v' = \sin(u)$,

$$= \frac{1}{\pi^4 n^4}\left(-u^3 \cos(u) + 3\int\ u^2 \cos(u)\, du\right)$$

We now solve $\int\ u^2 \cos(u)\, du$ using Integration by Parts, with $u = u^2$, $v' = \cos(u)$

$$\left(u^2 \sin(u) - 2\int\ u \sin(u)\, du\right)$$

And finally, solve $\int u \sin(u)\ du$ using Integration by Parts with $u = u$

$$\left(-u\cos(u) + \int\ \cos(u)\, du\right) = (-u\cos(u) +\ \sin(u))$$

Combining this mess together, we somehow get,

$$= \frac{1}{\pi^4 n^4}\left(-u^3\cos(u) + 3\left(u^2\sin(u) - 2\left(-u\cos(u) +\ \sin(u)\right)\right)\right)$$

Plugging back $u = n\pi x$ we get,

$$= \frac{1}{\pi^4 n^4}\left(-(n\pi x)^3\cos((n\pi x)) + 3\left((n\pi x)^2\sin((n\pi x)) - 2\left(-(n\pi x)\cos((n\pi x)) +\ \sin((n\pi x))\right)\right)\right)$$

Solving $\int_0^1 x^3 \sin(n\pi x)$, we get,

$$\frac{1}{\pi^4 n^4}\left(-(n\pi)^3 \cos((n\pi)) + 3\left((n\pi)^2 \sin((n\pi)) - 2(-(n\pi)\cos((n\pi)) + \sin((n\pi)))\right)\right)$$

$$-\frac{1}{\pi^4 n^4}(0 + 3(-2(0 + 0))) = 0$$

Simplified,

$$\frac{1}{\pi^4 n^4}\left(-(n\pi)^3 \cos((n\pi)) + 3(0 - 2(-(n\pi)\cos((n\pi)) + 0))\right)$$

$$\frac{1}{\pi^3 n^3}\left(6 + (-1)^n - n^2\pi^2\right)$$

Hence,

$$b_n = \frac{2}{\pi^3 n^3}\left(6 + (-1)^n - n^2\pi^2\right)$$

Therefore we can write our Fourier series as follows,

$$F(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x), \text{ where } b_n \text{ is given above}$$

(c)

3. Consider the following problem:

   (a) Find the steady state solution $u_s(x)$ of this problem.

   (b) Write a new PDE, boundary conditions and initial conditions for $U(x,t) = u(x,t) - u_s(x)$

   (c) Use separation of variables to find a solution to the PDE, boundary conditions and initial conditions. You must justify each step of your solution carefully to get full marks.

   (d) Suppose you had tried to apply separation of variables directly to the original problem without removing the steady state solution. At what point would this approach fail? Explain.

Solution.

(a) We will use the heat equation $u_{xx} = u_t$ and the fact that the steady state solution occurs at $u_t = 0$. Therefore, $u_{xx} = 0$ and we have the boundary conditions $u_x(0) = 3, \quad u(2) + u_x(2) = 1$.

Since the steady state solution must be of the form $u_s(x) = Ax + B$, we have $(u_s)_x(0) = A = 3$ and $u_s(2) + (u_s)_x(2) = 6 + B + 3 = 1$ and so $B = -8$.

Hence, the steady state solution of this problem is given by,

$$u_s(x) = 3x - 8$$

(b) Let us define $L[u] = f$ such that $f = 0$ and the linear operator $L[u] = u_{xx} - u_t$. We then have $L[U(x,t)] = L[u(x,t) - u_s(x)] = L[u(x,t)] - L[u_s(x)] = 0 - 0 = 0$. As such, we know that U satisfies the heat equation,

$$U_{xx} = U_t$$

Our initial condition would be given by

$$U(x,0) = u(x,0) - u_s(x) = 8 - 3x, \quad 0 < x < 2.$$

Our homogeneous boundary conditions are given by,

$$U_x(0,t) = u_x(0,t) - (u_s)_x(0) = 3 - 3 = 0$$

$$U(2,t) + U_x(2,t) = u(2,t) + u_x(2,t) - u_s(2) - (u_s)_x(2) = 1 - 1 = 0$$

(c) Assume that $U(x,t) = X(x)T(t)$. Since this PDE satisfies the heat equation, we can see that
$$U_{xx} = X''(x)T(t) = X(x)T'(t) = U_t.$$
Seperating the variables, we see that
$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -\lambda$$
We now have two ODE's that can be solved,
$$X''(x) + \lambda X(x) = 0$$
$$T'(t) + \lambda T(t) = 0$$
Our boundary conditions are given by
$$U_x(0,t) = X'(0)T(t) \to X'(0) = 0, \text{ since } T(t) \neq 0$$
$$U(2,t)+U_x(2,t) = X(2)T(t)+X'(2)T(t) = 0 \to T(t)(X(2)+X'(2)) = 0 \to X(2)+X'(2) = 0$$
Also note how this is a Sturm-Liouville problem. with $q(x) = 0 \leq 0$ and
$$[p\phi_n\phi_n']_0^2 = [\phi_n\phi_n']_0^2 = \phi_n(2)\phi_n'(2) - \phi_n(0)\phi_n'(0) = -\phi_n(2)^2 \leq 0.$$
Hence we have no negative eigenvalues.
We know the $\lambda = 0$ is not an eigenvalue to the problem because we end up with $X'' = 0$ hence $X = Ax + B$. Then $X' = A = 0$ so $X = B$ and $X(2) + X'(2) = B = 0$. This is a trivial solution.
We now check for positive eigenvalues such that $\lambda = k^2$ for $k \in \mathbb{R} \setminus \{0\}$ and so $X'' + k^2X = 0$.
We use the characteristic polynomial $m^2 + k^2$ so $m = \pm ki$.
Expanding this we get
$$X(x) = A\cos(kx) + B\sin(kx)$$
$$X'(x) = -Ak\sin(kx) + Bk\cos(kx)$$
Given the Boundary Conditions we see that $X'(0) = Bk = 0 \to B = 0$. And so $X(x) = A\cos(kx)$ and $X'(x) = -Ak\sin(kx)$
So,
$$X(2) + X'(2) = A\cos(kx) - Ak\sin(kx) = 0$$
And therefore we must have
$$\cos(kx) - k\sin(kx) = 0$$

Hence we have,

$$\cos(kx) = k\sin(kx) \rightarrow \frac{\sin(kx)}{\cos(kx)} = tan(kx) = \frac{1}{k}$$

Which has no real solutions.

(d) You will have issues when writing down the Boundary Conditions and they will be contradictory.

4. Find a smooth function $f(x)$ for $x \in [-1, 1]$ that cannot be represented by

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

for any values of the constants a_0 and a_n. Explain the reasons for your answer.

Solution.

Recall that the fourier approximation is given by the following, and that the above term is a special case for when $f(x)$ is even.

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

Hence, we can easily propose an odd function such as $f(x) = x$ for $x \in [-1, 1]$ that cannot be represented by the given formula for any values of a_0 and a_n.

5. Consider the following eigenvalue problem:

$$y'' + \lambda y = 0, \ \ 0 < x < 2$$

$$y(0) = 0, \ \ y'(2) = 0$$

(a) Is this a Sturm-Liouville problem? Explain.

(b) What conclusions can you draw from application of the theorem(s) to this problem?

(c) Find all eigenvalues and eigenfunctions.

Solution.

(a) Yes, the above eigenvalue problem is a Sturm-Liouville problem. This is because a traditional Sturm-Liouville problem is of the following form.

$$\frac{d}{dx}\left(p(x)y'\right) + q(x)y + \lambda\omega(x)y = 0, \ \ p(x) > 0, \ \ \omega(x) > 0, \ \ a < x < b.$$

Which the above eigenvalue problem is, since we have $p(x) = 1, \ \ q(x) = 0, \ \ \omega(x) = 1$, and $(a, b) = (0, 2)$.

We must also have boundary conditions of the following form.

$$\alpha y(a) + \beta y'(a) = 0$$

$$\gamma y(b) + \delta y'(b) = 0$$

Which it is, since $(\alpha, \beta, \gamma, \delta) = (1, 0, 0, 1)$.

Since all the requirements of a traditional Sturm-Liouville problem are met, we can conclude that our eigenvalue problem is, indeed, a Sturm-Liouville problem.

(b) We can draw the conclusion that there are no negative eigenvalues for this problem, since $q(x) = 0$ (already given) and $[p\phi_n\phi_n']_a^b = [\phi_n\phi_n']_0^2 = \phi_n(0)\phi_n'(0) - \phi_n(0)\phi_n'(0) = 0 \leq 0$.

(c) From the previous answer, we know that all eigenvalues for this problem will be non negative (i.e. zero or positive). Let us test the case where $\lambda = 0$.

We have $y'' = 0, \;\; y(0) = 0, \;\; y'(2) = 0$. Since $y = Ax + B$, we have $y(0) = B = 0$ and $y'(2) = A = 0$, which is a trivial solution and hence $\lambda = 0$ is not an eigenvalue to this problem.

Let us look at $\lambda = k^2$, where $k \in \mathbb{R}\backslash\{0\}$. We have the CP $m^2+k^2 = 0 \to m = \pm ki$. We can then put this solution in the form $y = A\cos(kx) + B\sin(kx)$ and since $y(0) = A = 0$, we have $y = B\sin(kx)$. We now have $y'(2) = Bk\cos(2k) = 0$, where $B \neq 0$ and $k \neq 0$, since they both lead to the trivial solution $y = 0$ which we don't want.
Solving $\cos(2k) = 0$ gives us $k = \frac{\pi}{2}\left(n - \frac{1}{2}\right)$ for $n = 1, 2, \ldots$.

Hence, we have the eigenvalues $\lambda = k^2 = \frac{\pi^2}{4}\left(n - \frac{1}{2}\right)^2$ for $n = 1, 2, \ldots$.

We also have the eigenfunction $\phi_n = \sin\left(\frac{\pi}{2}\left(n - \frac{1}{2}\right)x\right)$ for $n = 1, 2, \ldots$.

$$\text{SARIMA}(1,\ 1,\ 1) \times (0,\ 1,\ 1)_4$$
$$(1-\rho_1 B)(1-B)(1-B^4)y_t = (1+\alpha_1 B)(1+A_1 B^4)\varepsilon_t$$
$$(1-\rho_1 B)(1-B-B^4+B^5)y_t = (1+\alpha_1 B + A_1 B^4 + \alpha_1 A_1 B^5)\varepsilon_t$$
$$(1-B-B^4+B^5-\rho_1 B+\rho_1 B^2+\rho_1 B^5-\rho_1 B^6)y_t = (1+\alpha_1 B + A_1 B^4 + \alpha_1 A_1 B^5)\varepsilon_t$$
$$y_t = (B+B^4-B^5+\rho_1 B-\rho_1 B^2-\rho_1 B^5+\rho_1 B^6)y_t + (1+\alpha_1 B + A_1 B^4 + \alpha_1 A_1 B^5)\varepsilon_t$$
$$y_t = (B+\rho_1 B-\rho_1 B^2+B^4-B^5-\rho_1 B^5+\rho_1 B^6)y_t + (1+\alpha_1 B + A_1 B^4 + \alpha_1 A_1 B^5)\varepsilon_t$$
$$y_t = ((1+\rho_1)B-\rho_1 B^2+B^4-(1+\rho_1)B^5+\rho_1 B^6)y_t + (1+\alpha_1 B + A_1 B^4 + \alpha_1 A_1 B^5)\varepsilon_t$$

$$y_t = (1+\rho_1)y_{t-1} - \rho_1 y_{t-2} + y_{t-4} - (1+\rho_1)y_{t-5} + \rho_1 y_{t-6} + \varepsilon_t + \alpha_1 \varepsilon_{t-1} + A_1 \varepsilon_{t-4} + \alpha_1 A_1 \varepsilon_{t-5}$$