1. (a) Let $h(x)$ be defined by

$$h(x) = \begin{cases} h_1, & x < a \\ h_2, & x \geq a \end{cases}$$

where h_1, h_2 and a are constants. Find the first and second derivatives of h in the distributional sense.

(b) What is the third derivative of $|x|$ in the distributional sense.

Solution.

(a) First we find the first derivative of h in the distributional sense, which shall be denoted as $h'(x)$.

The function h can be regarded as a distribution by setting

$$H(\phi) = < h,\ \phi > = \int_{-\infty}^{\infty} h(x)\phi(x)\ dx,$$

where $\phi(x)$ is a test function. It then follows that

$$< h',\ \phi > = \ - < h,\ \phi' > = - \int_{-\infty}^{\infty} h(x)\, \phi'(x)\, \mathrm{d}x.$$

Splitting the integral into two intervals allows us to evaluate the expression,

$$-\left(\int_{-\infty}^{a} h_1\, \phi'(x)\, \mathrm{d}x + \int_{a}^{\infty} h_2\, \phi'(x)\ \ \mathrm{d}x \right) = -\left(h_1 \int_{-\infty}^{a} \phi'(x)\, \mathrm{d}x + h_2 \int_{a}^{\infty} \phi'(x)\, \mathrm{d}x \right)$$

The above expression evaluates to,

$$-\left(h_1 \left[\phi(x)\right]_{-\infty}^{a} + h_2 \left[\phi(x)\right]_{a}^{\infty} \right)$$

Since ϕ has compact support, this can be further simplified,

$$-h_1\phi(a) + h_2\phi(a) = \phi(a)\,(h_2 - h_1)$$

We can write this expression as an integral,

$$< h',\, \phi > = (h_2 - h_1) \int_{-\infty}^{\infty} \phi(x)\delta(x-a)\,\mathrm{d}x = < (h_2 - h_1)\delta(x-a),\ \phi >$$

It follows that

$$h'(x) = (h_2 - h_1)\delta(x-a)$$

Now we find the second derivative of h in the distributional sense, which shall be denoted as $h''(x)$.

Now differentiate in the distributional sense. Get

$$H''(x) = < h'',\ \phi > = - < h',\ \phi' > = - < (h_2 - h_1)\delta(x-a),\ \phi' >$$

Given $< f,\, \phi' > = - < f',\, \phi >$, it is clear that

$$H''(x) = < (h_2 - h_1)\delta'(x-a),\ \phi >$$

It follows that

$$h''(x) = (h_2 - h_1)\delta'(x-a)$$

(b) Let $|x|''' = \frac{d^3}{dx^3}|x|$. We have

$$< |x|''',\ \phi > = - < |x|, \phi''' > = - \int_{-\infty}^{\infty} |x|\phi'''(x)\ dx$$

$$= - \int_{-\infty}^{\infty} f(x)\,\mathrm{d}\phi'' = \int_{-\infty}^{0} x\,\mathrm{d}\phi'' - \int_{0}^{\infty} x\,\mathrm{d}\phi''$$

$$= \int_{0}^{\infty} \mathrm{d}\phi' - \int_{-\infty}^{0} \mathrm{d}\phi = -2\phi'(0)$$

Therefore it follows that the third derivated of $|x|$ is $-2\phi'(0)$ in the distributional sense.

2. Consider the pde

$$w_x + xw_t = 0, \quad 0 < x < \infty, \quad t > 0, \quad w(x,0) = 0, \quad w(0,t) = t$$

(a) Use separation of variables to show that

$$w(x,t) = \exp\left(k\left(t - \frac{x^2}{2}\right)\right)$$

where k is a constant.

(b) Show that the above solution does not satisfy both the initial and boundary conditions.

(c) Use Laplace Transforms to solve the above pde.

Solution.

(a) Using separation of variables, we have

$$w(x,\ t) = X(x)T(t)$$

And so our pde becomes,

$$X'(x)T(t) + xX(x)T'(t) = 0$$

Rearranging the above problem we find that

$$\frac{-X'}{xX} = \frac{T'}{T} = \lambda$$

Solving for $T(t)$ we have,

$$T'(t) = \lambda T(t) \rightarrow T(t) = A\ \exp(\lambda t)$$

Solving for $X(x)$ we have

$$\frac{dX}{dx} = -\lambda x$$

Solving the ode using seperable variables gives us

$$\int \frac{1}{X}\ dX = -\lambda \int x\ dx$$

And so

$$\ln|X| = -\frac{\lambda}{2}x^2 + c, \quad c \in \mathbb{R}$$

Finally we have,

$$X(x) = B\ \exp\left(-\frac{\lambda}{2}x^2\right)$$

Henceforth, we have

$$w(x,\ t) = AB\ \exp\left(\lambda\left(t - \frac{1}{2}x^2\right)\right)$$

Without loss of generality, take $AB = 1$ and $\lambda = k$

$$w(x, t) = \ \exp\left(k\left(t - \frac{x^2}{2}\right)\right)$$

(b) We have

$$w(x, t) = \ \exp\left(k\left(t - \frac{x^2}{2}\right)\right)$$

Notice that for all x,

$$w(x,\ 0) = \ \exp\left(-\frac{kx^2}{2}\right) \neq 0$$

Also, for many instances of t,

$$w(0,\ t) = \ \exp\left(kt\right) \neq t$$

(c) Let us define,

$$\mathcal{L}\left\{w(x,\ t)\right\} = F(x,\ s) = \int_0^\infty w(x,\ t)\ \exp(-st)\ dt$$

Then, we have,

$$\mathcal{L}\left\{w_x\right\} = \frac{dF}{dx}$$

And,

$$\mathcal{L}\left\{xw_t\right\} = sF(x,\ s) - w(x,\ 0)$$

We now have,

$$\mathcal{L}\left\{w_x + xw_t\right\} = \mathcal{L}\left\{0\right\}$$

Naturally,

$$\frac{dF}{dx} + x(sF) = 0$$

Solving this differential equation gives us

$$\frac{dF}{dx} = -xsF \rightarrow \int \frac{1}{F}\ dF = -s\int x\ dx$$

And so,

$$\ln|F| = -\frac{s}{2}x^2 + c, \quad c \in \mathbb{R}$$

$$F(x,\ s) = A\ \exp\left(-\frac{1}{2}sx^2\right)$$

By our initial condition, we must have

$$\mathcal{L}\left\{w(0,\ t)\right\} = F(0,\ s) = \mathcal{L}\left\{t\right\} = \frac{1}{s} = A$$

We now have,

$$F(x,\ s) = \frac{1}{s}\exp\left(-\frac{1}{2}sx^2\right)$$

Which we know is a shift and so it is trivial to see that,

$$w(x,\ t) = \left(t - \frac{1}{2}x^2\right)H\left(t - \frac{1}{2}x^2\right)$$

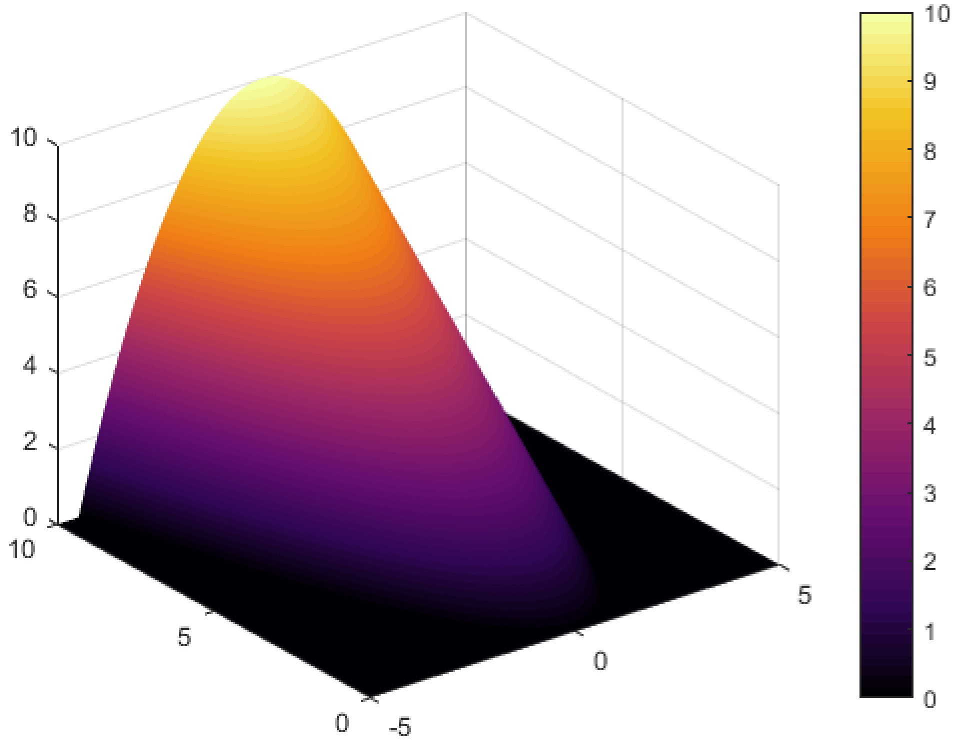

Figure 1: Surface plot of $\omega(x,\ t)$

3. Use Fourier Transforms to solve

$$u_t + u_x = 0, \quad -\infty < x < \infty, \quad t > 0, \quad u(0, x) = \sin(x).$$

Solution.

Define

$$\widehat{u}(\omega, t) = \mathcal{F}\left\{u(x,t)\right\} = \int_{-\infty}^{\infty} u(x,t)\,\exp(-i\omega x)\,dx$$

We then have

$$\mathcal{F}\left\{u_t\right\} = \frac{\partial \widehat{u}}{\partial t} = -(iw)\widehat{u} \text{ and } \mathcal{F}\left\{u_x\right\} = (i\omega)\,\widehat{u}(\omega, t)$$

As given to us in lectures. It is trivial to see that

$$\mathcal{F}\left\{u_t + u_x\right\} = \mathcal{F}\{0\}$$

We can solve,

$$\frac{\partial \widehat{u}}{\partial t} + i\omega\widehat{u} = 0 \rightarrow \widehat{u} = A\,\exp\left(-i\omega t\right)$$

The boundary condition $u(0, x) = \sin(x)$ means

$$\widehat{u}(x,t) = \widehat{\sin}(x)\,\exp(-i\omega t)$$

Since this is a product we can take advantage of the Time Convolution Theorem,

$$u(x,\ t) = \mathcal{F}^{-1}\{\hat{f}(\omega)\hat{g}(\omega)\} = f(t) \star g(t)$$

It is trivial to see that $f(x,\ t) = \sin(x)$. We need to a bit more work to find $g(x,\ t)$

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} \delta(x-t)\, \exp\left(-i\omega x\right)\, \mathrm{d}x = \mathcal{F}\left\{\delta(x-t)\right\}$$

and so $g(x,\, t) = \delta(x-t)$. Finally, we have,

$$u(x,\, t) = f \star g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} \sin(\tau)\delta(x-t-\tau)\,\mathrm{d}\tau = \sin(x-t)$$

The first order linear wave equation then has the solution $u(x,\, t) = \sin(x-t)$

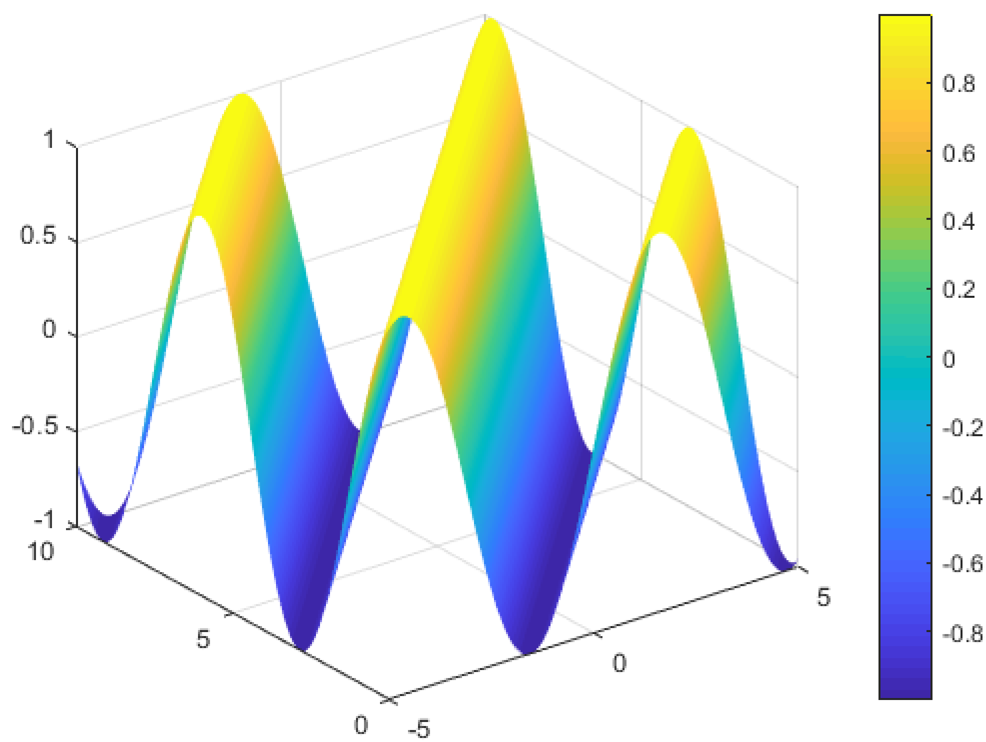

Figure 2: Surface plot of $u(x,\, t)$

4. The convolution of two functions can be defined with finite limits of integration as

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau$$

We will use this form to solve

$$f(t) = 2t^2 + \int_0^t f(t-\tau)\, \exp(-\tau)\, d\tau$$

(a) Show that

$$F(s) = \frac{4}{s^3} + \frac{1}{s+1}\, F(s)$$

where $F(s)$ is the Laplace transform of $f(t)$.

(b) Hence find $f(t)$.

Solution.

(a) First let us define

$$g(t) = \exp(-t)$$

The Laplace transform of $g(t)$ is given by,

$$G(s) = \mathcal{L}\left\{g(t)\right\} = \mathcal{L}\left\{\exp(-t)\right\} = \frac{1}{s+1}$$

We can now recognize the function $f(t)$ as follows,

$$f(t) = 2t^2 + (f \star g)(t)$$

The Laplace tranform of $f(t)$ is given by,

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \mathcal{L}\left\{2t^2\right\} + \mathcal{L}\left\{(f \star g)(t)\right\} = \frac{4}{s^3} + F(s)G(s)$$

Hence,

$$F(s) = \frac{4}{s^3} + \frac{1}{s+1}F(s)$$

(b) We have,

$$F(s) = \frac{4}{s^3} + \frac{1}{s+1}F(s)$$

Rearranging this gives us,

$$\left(1 - \frac{1}{s+1}\right)F(s) = \left(\frac{s}{s+1}\right)F(s) = \frac{4}{s^3}$$

Therefore,

$$F(s) = \left(\frac{s+1}{s}\right)\frac{4}{s^3} = 4\left(\frac{s+1}{s^4}\right) = \frac{4s}{s^4} + \frac{4}{s^4}$$

Finally,

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s^3}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s^4}\right\} = 2t^2 + \frac{2}{3}t^3$$

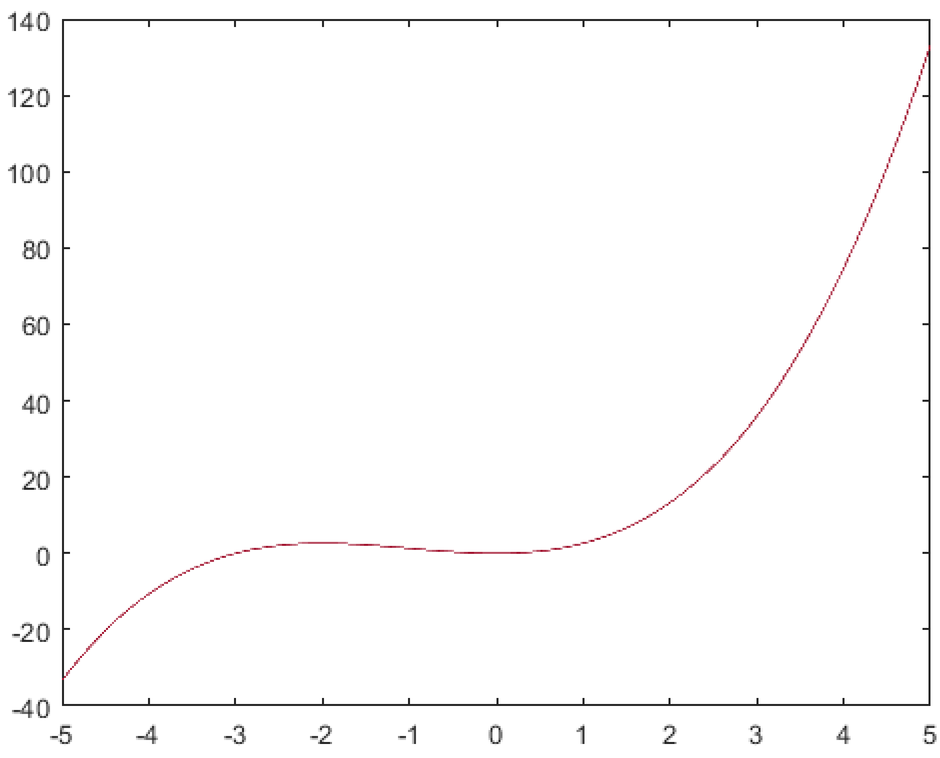

Figure 3: Line plot of $f(t)$

5. Use Green's function to solve

$$-(y'' + y) = \cos(x), \quad y(0) = y(1) = 0$$

Solution.

We must first find the homogenous solution of this differential equation.

$$-(y'' + y) = 0, \quad y(0) = y(1) = 0$$

Rearranging this gives us

$$y'' = -y$$

so we know our solution will be of the form

$$y(x) = A\cos(x) + B\sin(x)$$

We want to find a function $y_1(x)$ such that $y(0) = 0$ and a function $y_2(x)$ such that $y(1) = 0$.

$$y(0) = A = 0$$

Therefore, $y_1(x) = B\sin(x)$. Without loss of generality, take $B = 1$.

We also have to find $y_2(x)$.

$$y(1) = A\cos(1) + B\sin(1) = 0$$

The solution to this equation is given by,

$$\frac{-A}{B} = \frac{\sin(1)}{\cos(1)} = \tan(1)$$

Without loss of generality, we take $A = -\sin(1)$ and $B = \cos(1)$. And so we have $y_1(x)$ and $y_2(x)$.

$$y_1(x) = \sin(x)$$
$$y_2(x) = -\sin(1)\cos(x) + \cos(1)\sin(x)$$

Note that by the trigonometric identity

$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

We can simplify,

$$y_2(x) = \sin(x - 1)$$

Now calculate $p(x)W(x)$. For this problem $p(x) = -1$ and

$$W(x) = y_1 y_2' - y_2 y_1' = \sin(x)\cos(x-1) - \cos(x)\sin(x-1)$$

By the trigonometric identity,

$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

We can simplify,

$$W(x) = \sin(x - (x-1)) = \sin(1)$$

Hence, we have

$$C = p(x)W(x) = -\sin(1)$$

Hence,

$$G(x,s) = \begin{cases} \dfrac{\sin(s)\sin(x-1)}{-\sin(1)}, & 0 \leq s \leq x \leq 1 \\ \dfrac{\sin(x)\sin(s-1)}{-\sin(1)}, & 0 \leq x \leq s \leq 1 \end{cases}$$

Now, for all f,

$$y(x) = \int_0^1 G(x,s)\cos(s)\,\mathrm{d}s =$$

$$-\frac{\sin(x-1)}{\sin(1)}\int_0^x \sin(s)\cos(s)\,\mathrm{d}s - \frac{\sin(x)}{\sin(1)}\int_x^1 \sin(s-1)\cos(s)\,\mathrm{d}s$$

Hence, from computing the solution in MATLAB.

```
syms x s Y
f = (( -sin(x-1) / Y ) * int(sin(s) * cos(s), s, 0, x) ) - (sin(x)/Y * int(sin(s-1) * cos(s), s,x,1))
```

f =

$$\frac{\sin(x)\left(\frac{\cos(1)}{4} + \frac{\sin(1)}{2} - \frac{\cos(2\,x-1)}{4} - \frac{x\sin(1)}{2}\right)}{Y} - \frac{\sin(x-1)\sin(x)^2}{2\,Y}$$

Figure 4: Solution for $y(x)$

And hence finally,

$$y(x) = -\frac{\sin(x)}{4\sin(1)}\left[2\sin(x)\sin(x-1) - \cos(2x-1) + 2\sin(1) + \cos(1) - 2x\sin(1)\right]$$

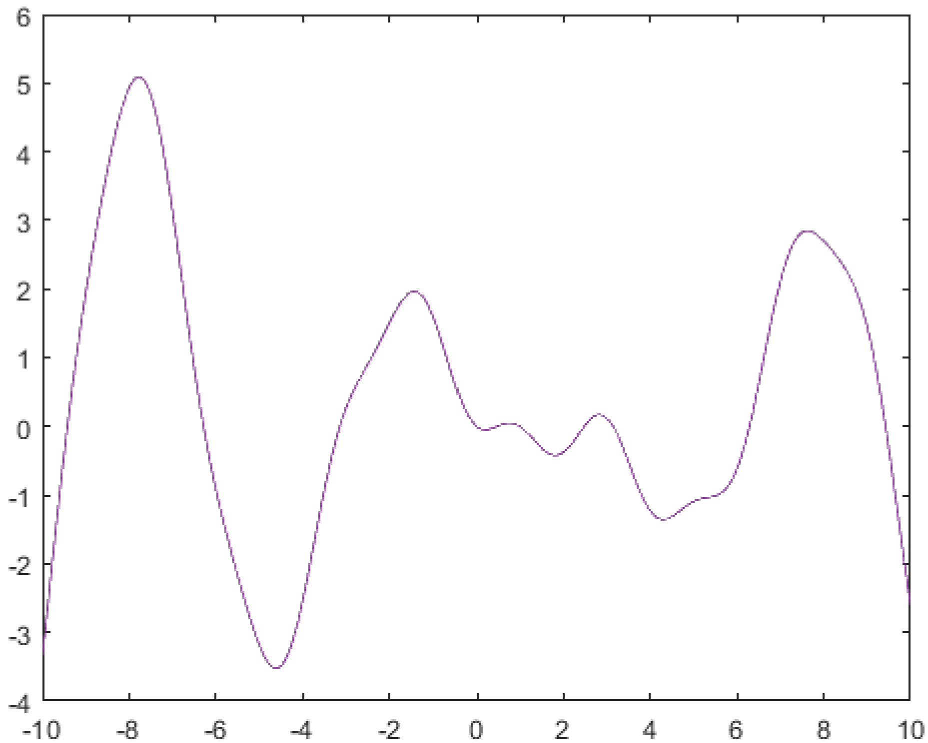

Figure 5: Line plot of $y(x)$

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