# Machine learning for data science I

12 June 2023

Surname, name (all caps) ______________________________

Student ID: ______________________________

This is a closed book exam.

Write clearly and justify your answers.

Time limit: 105 min.

| Question: | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|
| Points: | 20 | 20 | 20 | 20 | 20 | 100 |
| Score: | | | | | | |

1. Consider the absolute loss function for the discrete case $l(p, y) = \sum_{i=1}^{C} |p_i - y_i|$. $C$ is the number of class values, $p = [p_1, \ldots, p_C]$ is the predicted probability distribution of class values and $y = [y_i, \ldots, y_C]$ is the outcome, where all $y_i$ except one are zero. Use $r = [r_1, \ldots, r_C]$ as the true distribution of the data generative process.

[10] (a) This is an improper scoring rule. Explain what this means and describe other types of scoring rules.

[10] (b) Which distribution minimizes the risk with this loss function? Take into account multiple class values $C$ (not just 2). Prove your answer.

a)

$q = [0.7, 0.1, 0.2]$

$r = [0.5, 0.1, 0.4]$

$y = [0, 0, 1]$

b) $R(q) = \sum_i r_i \underbrace{l(q, y)}_{\sum |p_i - y_i|} = \sum r_i^2 (1 - p_i) \xrightarrow{\frac{dl}{dr}} = \underbrace{2 \sum r_i}_{2} - \underbrace{2 \sum r_i p_i}_{\sim 1x}$

$y_i = 1$: $1 - p_i$

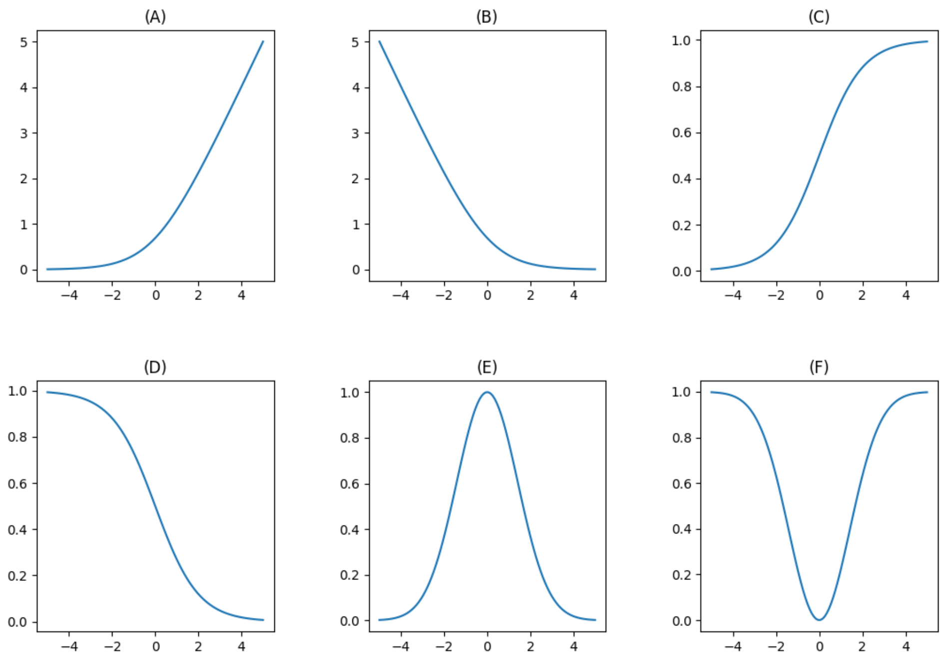
$y_i = 0$: $1 - p_i$ ?

2. Consider a family of classification models with decision function $H(x) = \text{sign}(F(x))$, $F : \mathbb{R}^d \to \mathbb{R}$, $H : \mathbb{R}^d \to \{-1, 1\}$. We will optimize the parameters $\beta$ of function $F$ by minimizing the expression $\sum_i L(y_i \cdot F(x_i))$. Explain your answers to the following questions.

[6] (a) Which of the functions illustrated below can be used as a loss function $L$ for the described classifier?

[4] (b) Which loss function is the most rebust to outliers?

[10] (c) Logistic regression is part of this familiy. What are the functions $L$ and $F$ which result in logistic regression? Which of the graphs corresponds to this loss function $L$?
Help: $p(y = 1|\beta) = \sigma(z)$, $\quad p(y = -1|\beta) = \sigma(-z)$.




a) A IS PENALIZING GOOD PREDICTIONS BECAUSE IT HAS HIGH PENALTY FOR POSITIVE PREDICTIONS

B AND D ARE OK

b) D IS MORE ROBUST

c) $p(y|\beta) = \sigma(yz)$

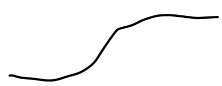
$F(x_i)$

$\underset{\beta}{\arg\max} \prod p(y_i|\beta) = \underset{\beta}{\arg\max} \sum \log p(y_i|\beta) = \arg\min \sum -\log \sigma(y_i z_i)$

$F(x) = \beta^T x$

$L(z) = -\log(\sigma(z))$

$\sigma = \frac{1}{1+e^{\alpha}}$



3. Derive an upper bound of the error rate ($\epsilon_{NN}$) of a 1-nearest neighbor classifier in terms of the error rate ($\epsilon_{OPT}$) of a Bayes optimal classifier as the number of instances approaches infinity ($n \to \infty$). The class variable has $m > 1$ possible values.

[4] (a) What is the prediction of a Bayes optimal classifier? What is its error rate?

[8] (b) Show that $\epsilon_{NN} \leq 2\epsilon_{OPT}$.

[8] (c) Prove a tighter bound $\epsilon_{NN} \leq \epsilon_{OPT}(2 - \frac{m}{m-1}\epsilon_{OPT})$.

Help: you might find the following form of the Cauchy–Schwarz inequality useful.

$$(\sum_{i=1}^{m} p_i)^2 \leq m \sum_{i=1}^{m} p_i^2$$

a) $p = [\underbrace{0.8}_{p_k}, 0.1, 0.1]$

$\varepsilon_{OPT} = 1 - p_k$

$$\varepsilon_{NN} = \sum_{i=1}^{n} p_i (1 - p_i) = \overbrace{\cancel{\sum} p_i}^{1} - \sum_i p_i^2 = 1 - p_k^2 - \sum_{i \neq k} p_i^2$$

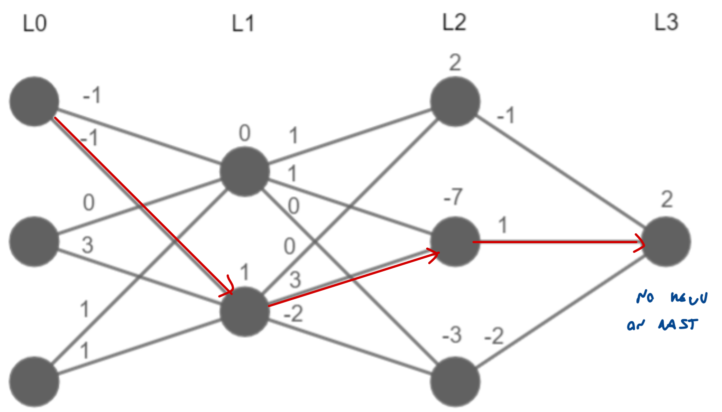
$$= (1 - p_k)(1 + p_k) - \sum_{i \neq k} p_i^2 \leqslant 2 \cdot \varepsilon_{OPT}$$

4. Given is a feed-forward neural network with two hidden layers ($L1$ and $L2$), input layer ($L0$), and output layer $L3$ (see figure). Let us denote the weighted sum of the inputs at layer $i$ with vector $z^{[i]}$ and activations with vector $a^{[i]}$. Let $W^{[i]}_{jk}$ denote the weight connecting $j$-th unit of layer $i-1$ and $k$-th unit of layer $i$ (illustrated above the lines). $B^{[i]}_{j}$ denotes the bias of the $j$-th unit in layer $i$ (illustrated above the nodes). All units use $ReLU$ activation functions except for the output layer, which doesn't use an activation function.

[6] (a) What is the output $\hat{y}$ of the neural network for a data instance $x = [2, 2, -1]$?

[14] (b) We aim to minimize the loss function $J = \frac{1}{2}\sum_{i=1}^{m}(y_i - \hat{y}_i)^2$, where $m$ indicates the number of data instances. Suppose we make one step of training (with learning rate 0.5) the illustrated network with a single data instance $x = [2, 2, -1]$ that has a correct output $y = 2$. What is the new value of $W^{[1]}_{0,1}$ (weight between the top unit of level 0 and the bottom unit of level 1) after this training step? Explain your calculation.

Hint: Think about which parts of the computation you can skip but don't forget to explain why.




ONLY AFFECTIVE PATH OF CHANGING $w_1$

(REST HAS DERIVATIVE OF RELU = 0)

a) JUST PUT THEM IN THE NN

b) $$\frac{dJ}{dw_1} = \underbrace{\frac{dz_1}{dw_1}}_{2} \cdot \underbrace{\frac{da_1}{dz_1}}_{1} \cdot \underbrace{\frac{dz_2}{da_1}}_{3} \cdot \underbrace{\frac{da_2}{dz_2}}_{1} \cdot \underbrace{\frac{dJ}{da_2}}_{3}$$

$$= 2 \quad 1 \quad 3 \quad 1 \quad 3$$

$$J = \frac{1}{2}(y - \underbrace{\hat{y}}_{a_2})^2$$

$$\frac{dJ}{da_2} = -1 \cdot (y - \hat{y}) = -(2-5) = 3$$

$$w_1' = w_1 - 0.5 \cdot 18 = -10$$

5. Bayesian statistics basics. Some help with distributions:

Bernoulli (pmf): $P(x=k) = \begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases}$ , $\quad E[x] = p$

Beta (pdf): $P(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$ , $\quad E[x] = \frac{\alpha}{\alpha+\beta}$

[4] (a) Write Bayes' Theorem for the posterior $p(\theta|y)$, given likelihood $p(y|\theta)$ and prior $p(\theta)$.

[5] (b) Derive the posterior for $\theta$ given $n$ observations $y = [y_1, \ldots, y_n]$ if the likelihood is Bernoulli($\theta$) and the prior is Beta($\alpha, \beta$).

[5] (c) Derive the definition of the posterior predictive $p(y'|y)$ of a new observation $y'$ given a set of $n$ observations $y$. Express it with the likelihood and posterior only, explain the steps.

[6] (d) Derive the posterior predictive for the Bernoulli-Beta case from above.