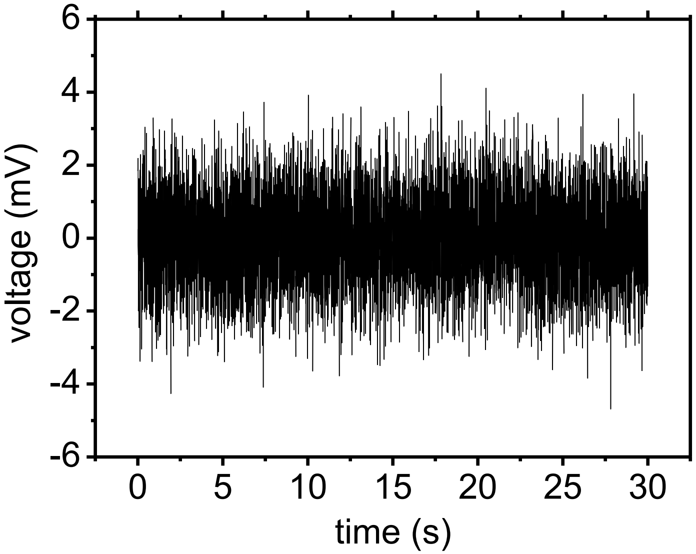
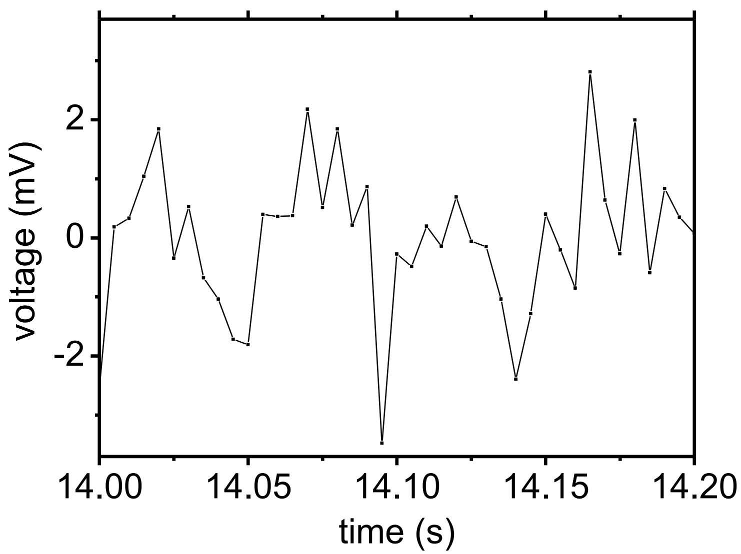
Exercise 1

Discrete Fourier Transformations and Power Spectra

The kind of problems that we have to solve:

Imagine you have this data acquired by your DAQ:



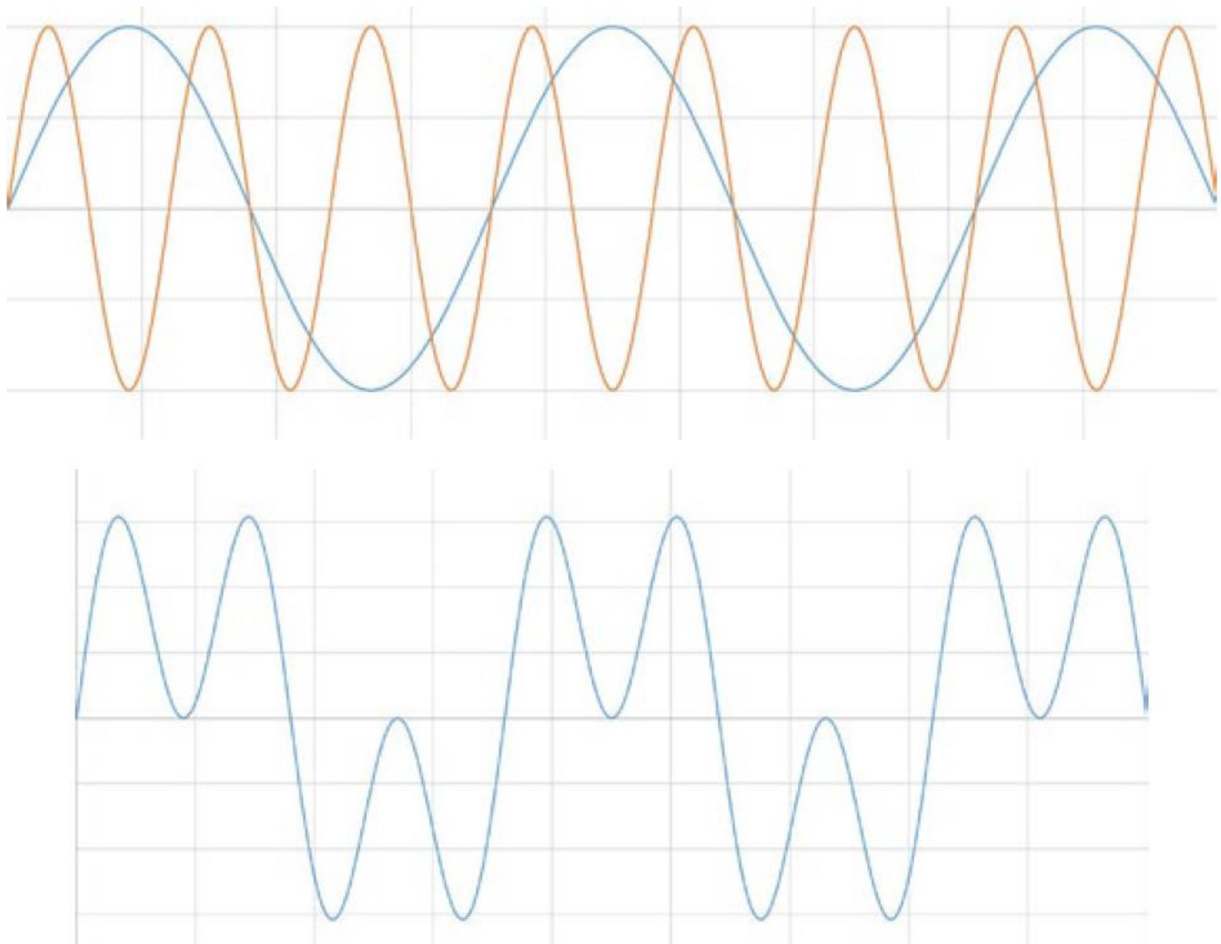


Clearly S/N < 1, but is there still some useful information contained???

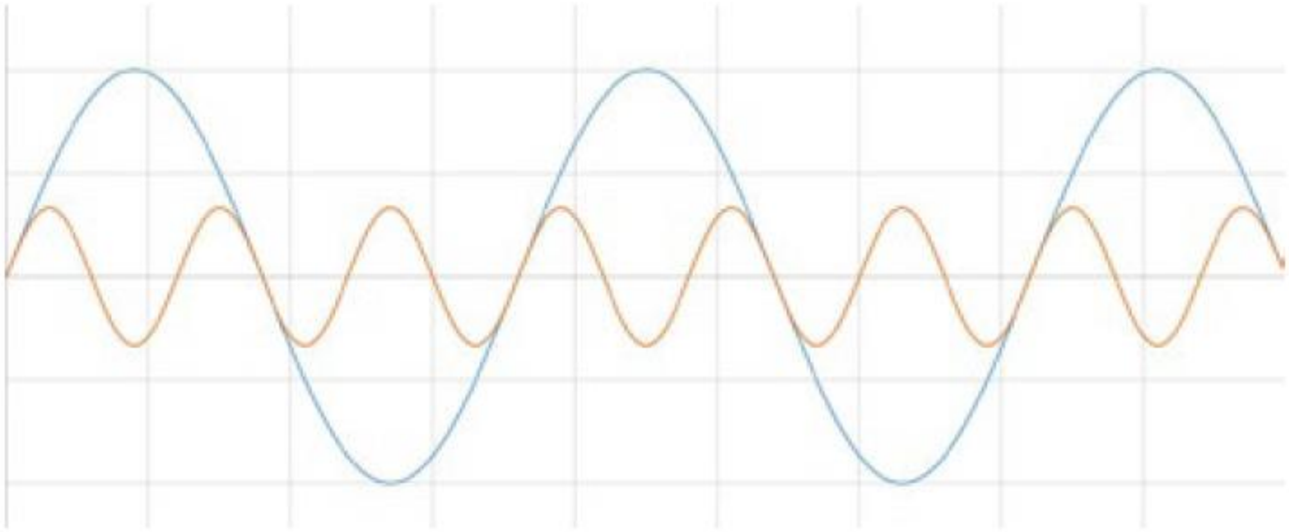
1) How to compute numbers that quantify the power of a possible signal and of the background noise?

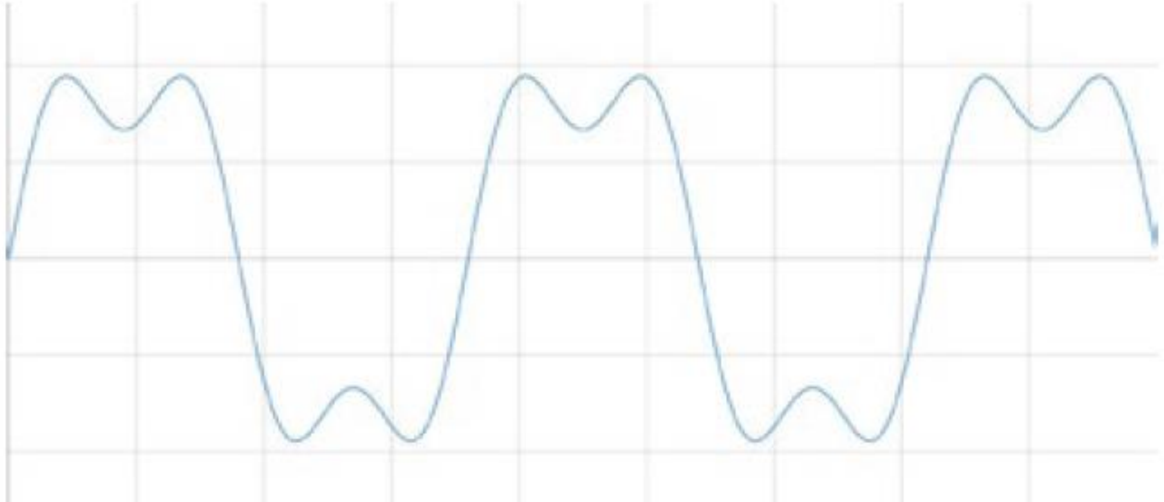
2) How to measure amplitude and phase in case there is a signal (for example for impedance analysis)?

Introduction Fourier Transformations

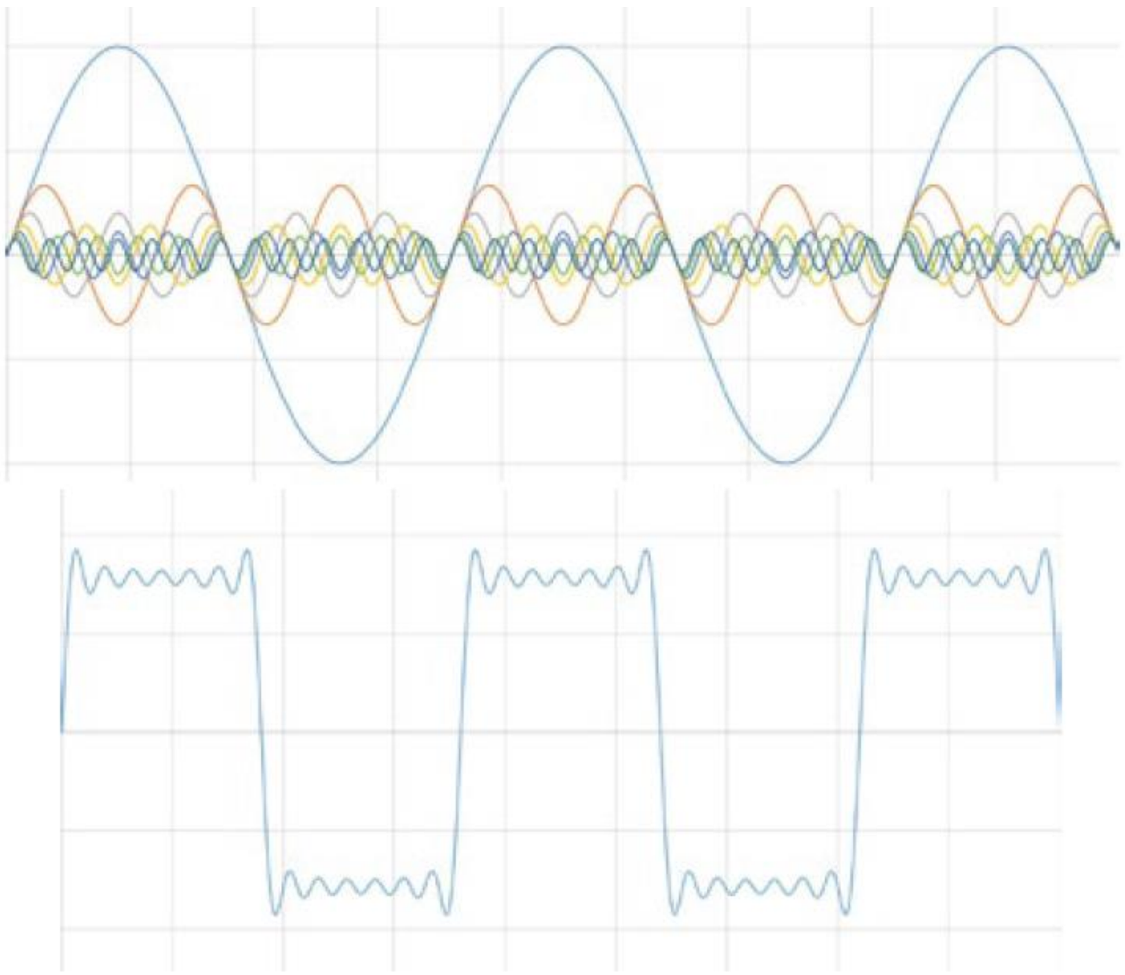


Introduction Fourier Transformations





Introduction Fourier Transformations



All signals can be represented as a sum of sine and cosine waves!

Fourier Transformations

Assume we have a signal *f(t)* in the time domain. The signal is periodic with a time interval of 1 s. Then we can represent that signal by the infinite series:

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n sin(2\pi nt) + b_n cos(2\pi nt))$$

Where a_n and b_n are unknown coefficients of the series. The integer n has units of Hertz(Hz)=1/s and corresponds to the frequency of a wave.

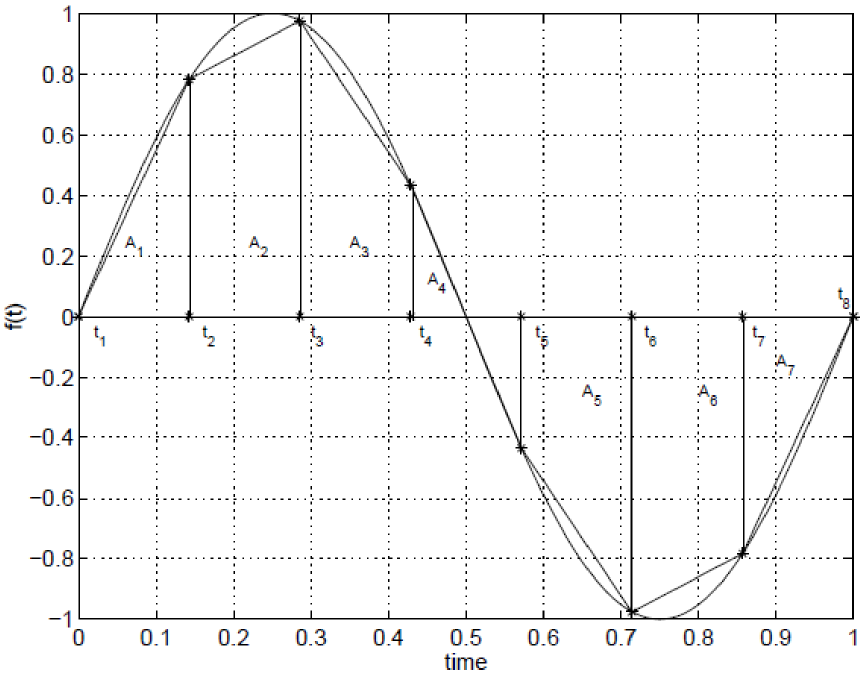
The coefficients can be calculated by employing a **Fourier Transformation** of the original data *f(t)*:

$$\int_0^1 f(t) sin(2\pi nt) dt = \frac{a_n}{2}.$$

$$\int_0^1 f(t) cos(2\pi nt) dt = \frac{b_n}{2}.$$

$$\int_0^1 f(t) = a_0.$$

Fourier Transformations of discrete data



area of trapezoidal block j

$$A_j = \frac{f(t_j) + f(t_{j+1})}{2}(t_{j+1} - t_j)$$

total area:

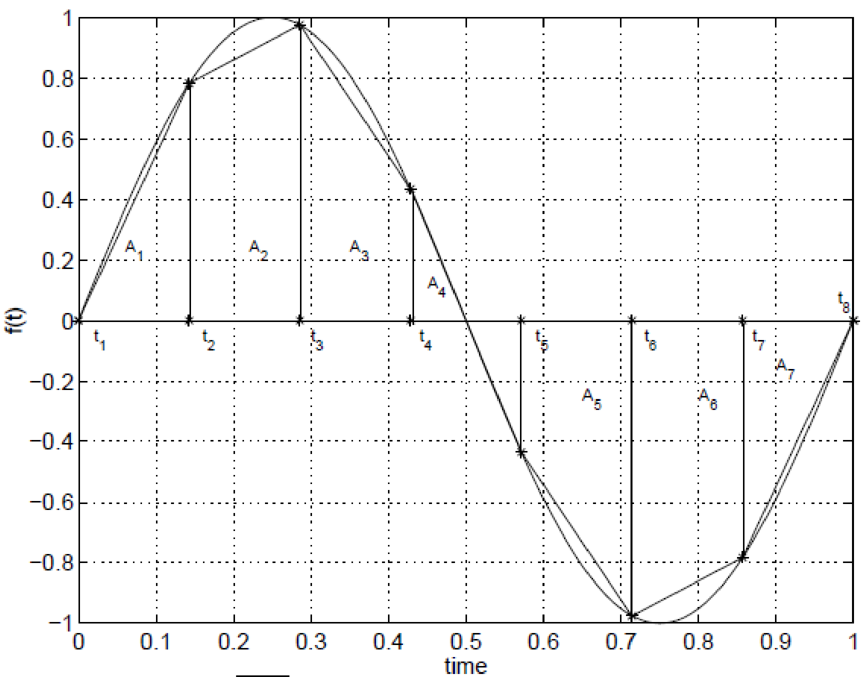
$$\int f(t)dt = \Delta t \left(f(t_1)/2 + f(t_n)/2 + \sum_{j=2}^{N-1} f(t_j) \right)$$

trapezoidal rule

Fourier coefficients for a discrete periodic data set:

$$a_n = \Delta t[sin(2\pi n t_1)f(t_1) + sin(2\pi n t_N)f(t_N) + 2\sum_{j=2}^{N-1} sin(2\pi n t_j)f(t_j)]$$

Fourier Transformations of discrete data



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trapezoidal rule

The integrals are replaced by **sums** running over all N data points

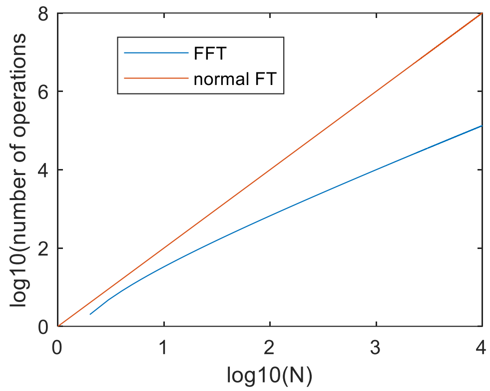
The number of frequencies to consider is limited and the frequency index runs from 0 to a maximum of N/2. In other words

f_{max} = $f_{sampling}$/2 = Nyquist frequency

(same arguments as used for explaining aliasing)

Fast Fourier Transform (FFT)

The number of operations to compute all N Fourier coefficients a_n and b_n scales with N^2 when the Algorithm based on trapezoidal rule is used. N is the number of data points contained in the interval to analyse.



A faster algorithm called Fast Fourier Transform has been developed that scales with $N\log_2 N$

The FFT algorithm is implemented in all data analysis software packages (origin, excel, matlab). It justs needs an input-vector of data and performs the transformation also on very large data sets: fft(data)

The FFT algorithm is typically written in a notation with complex numbers exploiting the Euler transformation:

$$\cos x + j\sin x = e^{jx}$$

Fast Fourier Transform (FFT)

The algorithm produces complex coefficients as output: $c_n = a + j*b$ and produces negative and positive frequencies:

$$c_n = \sum_{j=0}^{N-1} f(t_j) e^{-j2\pi n t_j} \qquad \text{for} \qquad -\frac{N}{2} < n < \frac{N}{2}$$

The real part of c_n contains the coefficients for the cosine functions. The imaginary part instead for the sine functions:

$$a_n = \frac{2}{N} \mathrm{Im}\{c_n\},\ 0 < n < \frac{N}{2}$$
$$b_n = \frac{2}{N} \mathrm{Re}\{c_n\},\ 0 < n < \frac{N}{2}$$

Negative frequencies are the complex conjugate of the positive frequency coefficients. They are meaningful when taking a FFT of a dataset of complex numbers. In our applications they can be neglected.
Remember $N/2$ is the maximum meaningful frequency of a discrete dataset.
$N/2$ is called the Nyquist frequency.

Power spectrum

Usually we are not so much interested in how the signal is distributed between sine and cosine components (or in other words in the phase angle). Instead, we would like to know the power density (Watt/Hz or V^2/Hz) at a specific frequency.

Therefore we multiply with the complex conjugate of each coefficient. Here, we want to consider only positive frequencies (single sided power spectrum). In order to maintain the full power in only the positive half of the spectrum, we multiply each coefficient with a factor of 1/2 and divide by the total number of points.

$$p_n = \frac{c_n c_n^*}{2(N/2)^2}$$

Important: In doing so all information on the phase of the signal is lost!

Power spectrum

Until now the analyzed time series had a length T of exactly 1 sec. For this situation the frequency that is assigned to the index n of the Fourier coefficient c_n is simply equal n. In general, FFT can of course be applied to timeseries of arbitrary length T. The frequency f_n that belongs to coefficient c_n is then calculated as:

$$f_n = \frac{n}{T} \quad \text{for} \quad 0 \geq n > \frac{N}{2}$$

Further we have to take care that the units of p_n are scaled correctly for time series of length T. This is resolved by dividing the smallest frequency interval $1/T$ present in the FFT:

$$p_n = \frac{c_n c_n^*}{2\,(N/2)^2} \cdot \frac{1}{1/T} \; [\mathrm{V}^2/\mathrm{Hz}]$$

We can control this result by inserting the expression for the coefficients of the sine and cosine functions:

$$p_n = \frac{1}{2}\frac{c_n c_n^*}{N^2/4} T = \frac{1}{2}\frac{\left(N/2a_n\right)^2 + \left(N/2b_n\right)^2}{N^2/4} T = \left(\frac{a_n}{\sqrt{2}}\right)^2 T + \left(\frac{b_n}{\sqrt{2}}\right)^2 T = V_{rms,\sin}^2 T + V_{rms,\cos}^2 T$$

Thus integration over the whole power spectra gives the total squared rms value:

$$P = \sum_{n=1}^{N/2-1} p_n \frac{1}{T} = V_{rms}^2$$

Phase Spectrum

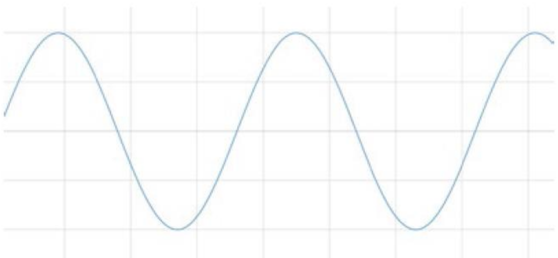
If phase is of interest the phase spectrum can be calculated from the FFT coefficients:

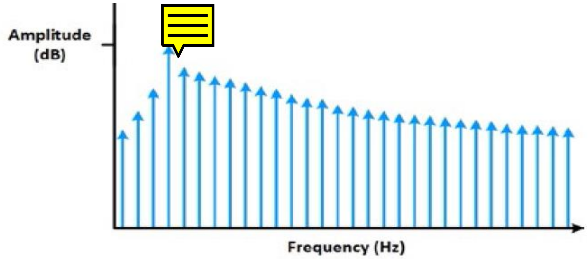
$$\Phi_n = \arctan\left(\frac{\mathrm{Im}\{c_n\}}{\mathrm{Re}\{c_n\}}\right)$$

The related frequencies f_n are calculated as for the power spectrum

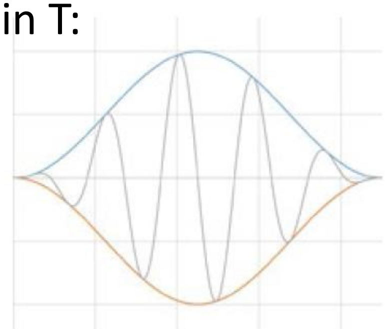
Windowing in power spectrum calculation

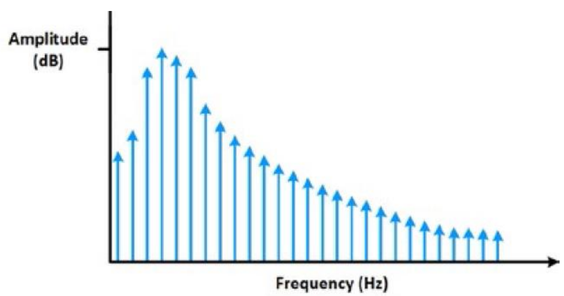
Up to now we assumed that the signal is periodic in T. Unfortunately this is most often not the case as the acquisition time T is set without knowledge of the frequency. As a consequence a phenomenon called spectral leakage occurs. Spectral leakage distorts the measurement in such a way that energy from a given frequency component is spread over adjacent frequency lines or bins.





You can use windows to minimize the effects of performing an FFT over a signal that is not periodic in T:





Windowing consists of multiplying the time record by a finite-length window with an amplitude that varies smoothly and gradually toward zero at the edges. This makes the endpoints of the waveform meet and, therefore, results in a continuous waveform without sharp transitions.

Exercises

Exercise 1.1

- Generate a dataset of 12 data points that represents a sine wave with amplitude 1 and frequency 1 Hz in the time interval 0<=t<1 s.
- Apply the fft-algorithm to calculate the complex Fourier coefficients. Calculate the amplitude and the phase of the sinewave from the Fourier coefficients.

Exercise 1.2

- Generate a signal of 2s length with sampling frequency 500 Hz that contains (i) a white background noise as produced by a resistance of 1MΩ at 300K and a bandwidth of 10 kHz, (ii) a sine wave signal at frequency 80 Hz with amplitude 12 μV, (iii) a sine wave signal at frequency 170 Hz with amplitude 6 μV.

- Calculate the power-spectrum from the signal in units of V2/Hz and dB. Test if the area of the power spectrum is equal to the rms value of the signal.
- Determine from the power spectrum the power of the combined signal and the power of each of its single constituents. Compare to the theoretical values.

- Investigate how the signal in the power-spectrum at 170 Hz depends on the sampling frequency and on the signal length. Demonstrate your results with a figure.

Exercise 1.3

- The asci-dataset “Impedance” contains 3 columns: Column 1 and 2 represent time and voltage data of the input signal that contains 5 principal frequencies. Column 3 shows the current of the output signal. Use the data to calculate a Bode Plot that shows the magnitude of impedance and phase angle of the measured circuit. Estimate R and C values from the data.

Exercise 1.4

- Create a numerical example that demonstrates how windowing can improve a power spectrum.

Yours help: Origin / Matlab examples about:

1.) Create signals by yourself

2.) Statistical Analysis of signals

3.) Apply the FFT algorithm

4.) Calculate a power spectrum

5.) Integration to quantify a peak

6.) Importing data