MTAT.07.017
Applied Cryptography

Elliptic Curve Cryptography (ECC)

University of Tartu

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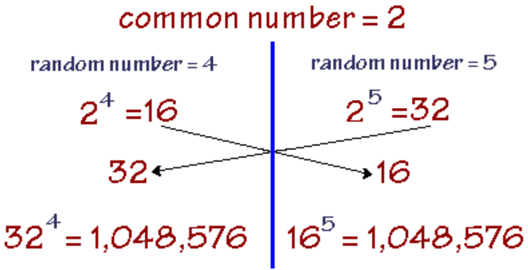
Diffie-Hellman key exchange

Ralph Merkle, Martin Hellman, Whitfield Diffie (1976)

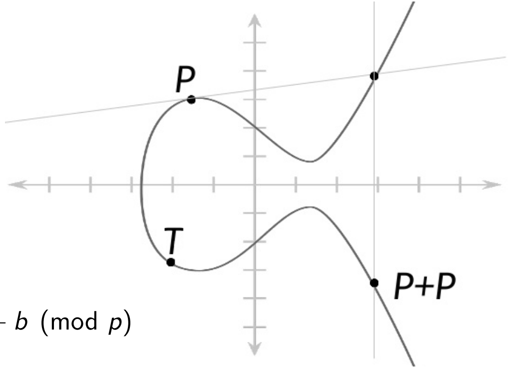
- The first public-key cryptography algorithm

Diffie-Hellman (DH) key exchange



- $(2^5)^4 = 2^{5 \cdot 4} = (2^4)^5$
- In practice: multiplicative group of integers modulo prime p is used
- Discrete logarithm problem:
  - hard to find x, given $2^x = 32 \mod p$
- ElGamal and DSA are based on DL problem
- Secure against passive eavesdropping

Elliptic Curve Cryptography



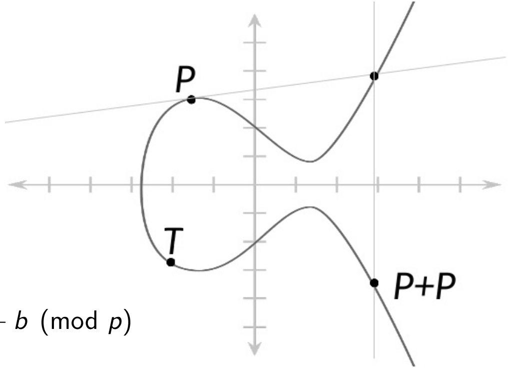
$$y^2 \pmod{p} = x^3 + ax + b \pmod{p}$$

- Formulas for point addition, doubling
- DL problem: find x, given $a^x = b \mod p$
- ECDL problem: find x, given $\underbrace{P + P + \cdots + P}_{x \ times} = x \cdot P = T$
- Security: 256-bit ECC $\approx$ 3072-bit RSA $\approx$ 128-bit security level

Standard (named) elliptic curves

- secp256k1
- NIST P-256 / secp256r1 / prime256v1
- NIST P-384 / secp384r1
- NIST P-521 / secp521r1
- brainpoolP256r1
- brainpoolP384r1
- brainpoolP512r1
- Curve25519
- Ed25519

EC domain parameters



$y^2 \pmod p = x^3 + ax + b \pmod p$

- p – field size
- a, b – equation parameters
- G – generator (base point)
- n – order of the curve (total number of points on curve)
- h – cofactor (number of cyclic subgroups)

EC domain parameters

```
$ openssl ecparam -name prime256v1 -text -param_enc explicit -no_seed --noout
Field Type: prime-field
Prime:
    00:ff:ff:ff:ff:00:00:00:01:00:00:00:00:00:00:
    00:00:00:00:00:00:ff:ff:ff:ff:ff:ff:ff:ff:ff:
    ff:ff:ff
A:
    00:ff:ff:ff:ff:00:00:00:01:00:00:00:00:00:00:
    00:00:00:00:00:00:ff:ff:ff:ff:ff:ff:ff:ff:ff:
    ff:ff:fc
B:
    5a:c6:35:d8:aa:3a:93:e7:b3:eb:bd:55:76:98:86:
    bc:65:1d:06:b0:cc:53:b0:f6:3b:ce:3c:3e:27:d2:
    60:4b
Generator (uncompressed):
    04:6b:17:d1:f2:e1:2c:42:47:f8:bc:e6:e5:63:a4:
    40:f2:77:03:7d:81:2d:eb:33:a0:f4:a1:39:45:d8:
    98:c2:96:4f:e3:42:e2:fe:1a:7f:9b:8e:e7:eb:4a:
    7c:0f:9e:16:2b:ce:33:57:6b:31:5e:ce:cb:b6:40:
    68:37:bf:51:f5
Order:
    00:ff:ff:ff:ff:00:00:00:00:ff:ff:ff:ff:ff:ff:
    ff:ff:bc:e6:fa:ad:a7:17:9e:84:f3:b9:ca:c2:fc:
    63:25:51
Cofactor:  1 (0x1)
```

EC point operations

- Point addition
  - $P + Q = R$

$$\lambda = \frac{y_q - y_p}{x_q - x_p}$$

$$x_r = \lambda^2 - x_p - x_q$$

$$y_r = \lambda(x_p - x_r) - y_p$$

- Point doubling
  - $P + P$

$$\lambda = \frac{3x_p^2 + a}{2y_p}$$

- Point multiplication
  - $k \times P$ "double-and-add"
- Point at infinity (∞)
  - $n \times P = \infty$
- Point negation
  - $P + (-P) = \infty$ $\quad -P = (x_p, p - y_p)$
- Point validation
  - Check whether the point is on the curve: $y^2 \pmod{p} \stackrel{?}{=} x^3 + ax + b \pmod{p}$

Elliptic Curve Diffie-Hellman (ECDH)

Alice		**Bob**
Key generation		
$d_a \xleftarrow{\$} [1, n-1]$		$d_b \xleftarrow{\$} [1, n-1]$
$Q_a = d_a \times G$		$Q_b = d_b \times G$
	$Q_a \longrightarrow$	
	$\longleftarrow Q_b$	
Validate Q_b		Validate Q_a
Shared secret calculation		
$Q_{ab} = d_a \times Q_b$		$Q_{ab} = d_b \times Q_a = d_b \times (d_a \times G)$

- Shared secret: x coordinate of Q_{ab}
- ECDH $\rightarrow$ shared secret $\rightarrow$ hybrid encryption

EC private key file format

```
$ openssl genpkey -algorithm EC -pkeyopt ec_paramgen_curve:prime256v1 -out priv.pem
$ cat priv.pem
-----BEGIN PRIVATE KEY-----
MIGHAgEAMBMGByqGSM49AgEGCCqGSM49AwEHBG0wawIBAQQgrFXchvnjGXn+Xmpo
7NXSNXgRPNfJRC4Ypx53X7Hn/3TbBJysAGGocPi3BWBL03m0iF0ukNAR
-----END PRIVATE KEY-----
$ openssl pkcs8 -topk8 -in priv.pem -outform der -out priv.der -nocrypt
```

```
$ dumpasn1 priv.der
  0 135: SEQUENCE {
  3   1:   INTEGER 0
  6  19:   SEQUENCE {
  8   7:     OBJECT IDENTIFIER ecPublicKey (1 2 840 10045 2 1)
 17   8:     OBJECT IDENTIFIER prime256v1 (1 2 840 10045 3 1 7)
       :     }
 27 109:   OCTET STRING, encapsulates {
 29 107:     SEQUENCE {
 31   1:       INTEGER 1
 34  32:       OCTET STRING
       :         AC 55 DC 86 F9 E3 19 79 FE 5E 6A 68 2B 57 3C 6A
       :         96 AD 31 27 53 B7 65 76 48 20 00 7E 4D 83 7B 74
 68  68:       [1] {
 70  66:         BIT STRING
       :           04 39 E0 CE 3D 25 F8 04 33 FF E6 A9 E4 5C 81 DC
       :           6E 51 A6 00 26 58 CE EC D5 D2 35 78 11 3C D7 C9
       :           44 2E 18 A7 1E 77 5F B1 E7 FF 74 DB 04 9C AC 00
       :           61 A8 70 F8 B7 05 60 4B D3 79 B4 88 53 AE 90 D0
       :           11
       :         }
       :       }
       :     }
       :   }
```

```
PrivateKeyInfo ::= SEQUENCE {
    version    Version,
    privateKeyAlgorithm AlgorithmIdentifier,
    privateKey    OCTET STRING ::= ECPrivateKey,
}

AlgorithmIdentifier  ::=  SEQUENCE  {
    algorithm   OBJECT IDENTIFIER ::= id-ecPublicKey,
    parameters  ECParameters
}

ECParameters ::= CHOICE {
    namedCurve         OBJECT IDENTIFIER
    -- implicitCurve   NULL
    -- specifiedCurve  SpecifiedECDomain
}

ECPrivateKey ::= SEQUENCE {
  version        INTEGER { ecPrivkeyVer1(1) } (ecPrivkeyVer1),
  privateKey     OCTET STRING,
  parameters [0] ECParameters {{ NamedCurve }} OPTIONAL,
  publicKey  [1] BIT STRING OPTIONAL
}
```

https://tools.ietf.org/html/rfc5915
https://tools.ietf.org/html/rfc5208

EC public key file format

```
$ openssl ec -in priv.pem -pubout -out pub.pem
$ cat pub.pem
-----BEGIN PUBLIC KEY-----
MFkwEwYHKoZIzj0CAQYIKoZIzj0DAQcDQgAEo5fWms2aid56D6q9XLD5QAQJjvG8
i0OPO28akFtaIvVSkZ9EGOo1eGlBt43dCSXSGaeMdUmgZlPrYEEPnKxi0w==
-----END PUBLIC KEY-----

$ openssl ec -in pub.pem -pubin -outform der --out pub.der
$ dumpasn1 pub.der
  0  89: SEQUENCE {
  2  19:   SEQUENCE {
  4   7:     OBJECT IDENTIFIER ecPublicKey (1 2 840 10045 2 1)
 13   8:     OBJECT IDENTIFIER prime256v1 (1 2 840 10045 3 1 7)
       :     }
 23  66:   BIT STRING
       :     04 A3 97 D6 9A CD 9A 89 DE 7A 0F AA BD 5C B0 F9
       :     40 04 09 8E F1 BC 8B 43 8F 3B 6F 1A 90 5B 5A 22
       :     F5 52 91 9F 44 18 EA 35 78 69 41 B7 8D DD 09 25
       :     D2 19 A7 8C 75 49 A0 66 53 EB 60 41 0F 9C AC 62
       :     D3
       :   }
```

```
SubjectPublicKeyInfo  ::=  SEQUENCE  {
    algorithm          AlgorithmIdentifier,
    subjectPublicKey   BIT STRING ::= ECPoint
}

ECPoint ::= OCTET STRING

AlgorithmIdentifier  ::=  SEQUENCE  {
    algorithm   OBJECT IDENTIFIER ::= id-ecPublicKey,
    parameters  ECParameters
}

id-ecPublicKey OBJECT IDENTIFIER ::= {
iso(1) member-body(2) us(840) ansi-X9-62(10045) keyType(2) 1 }

ECParameters ::= CHOICE {
    namedCurve         OBJECT IDENTIFIER
    -- implicitCurve   NULL
    -- specifiedCurve  SpecifiedECDomain
}
```

https://tools.ietf.org/html/rfc5480

EC point compression

```
$ openssl ec -in pub.pem -pubin -conv_form compressed -outform der -out pub_compressed.der
$ dumpasn1 pub_compressed.der
  0  57: SEQUENCE {
  2  19:   SEQUENCE {
  4   7:     OBJECT IDENTIFIER ecPublicKey (1 2 840 10045 2 1)
 13   8:     OBJECT IDENTIFIER prime256v1 (1 2 840 10045 3 1 7)
       :     }
 23  34:   BIT STRING
       :     03 A3 97 D6 9A CD 9A 89 DE 7A 0F AA BD 5C B0 F9
       :     40 04 09 8E F1 BC 8B 43 8F 3B 6F 1A 90 5B 5A 22
       :     F5
       :   }
```

- The first byte of `ECPoint` indicates:
  - 0x04 – uncompressed form
  - 0x03 – compressed form (y is odd)
  - 0x02 – compressed form (y is even)
  - 0x00 – point at infinity
- Patent expired in 2018

Elliptic Curve Digital Signature Algorithm (ECDSA)

Signing (given hash h, EC private key d):

1. Generate a random nonce k in the range $[1, n-1]$
2. Calculate the random point $R = k \times G$ and set r to its x coordinate: $r = R.x$
3. Calculate modular inverse k^{-1}, such that $k \cdot k^{-1} \equiv 1 \mod n$
4. Calculate $s = k^{-1} \cdot (h + r \cdot d) \mod n$ *Restart if $s = 0$*
5. Return the signature r, s

- Bit-length of h must not exceed the bit-length of the curve's order n
  - Longer hash values must be truncated to $n.bit\_length()$ most significant bits
  - Shorter hash values must be left-padded with zeros
- Nonce k must be secret and random(**!**)
  - Deterministic ECDSA (RFC 6979): nonce k derived using HMAC from h and d

Verification (given hash h, signature r, s and EC public key Q):

Validate Q

Verify r and s in $[1, n-1]$

1. Calculate $R' = (h \cdot s^{-1}) \times G + (r \cdot s^{-1}) \times Q$
2. Verification successful if $R'.x = r \mod n$

Task: ECDSA utility

Implement an ECDSA signing and verification utility.

```
$ ./ecdsa.py
Usage:
sign <private key file> <file to sign> <signature output file>
verify <public key file> <signature file> <file to verify>
```

```
$ ./ecdsa.py sign priv.pem filetosign signature
$ dumpasn1 signature
  0  69: SEQUENCE {
  2  33:   INTEGER
       :     00 E6 52 A8 B4 46 61 E9 2A 35 CD 1B 11 A7 01 5E
       :     FA 0D 2D 95 D3 56 C5 18 BC F8 72 28 4B 70 DA E3
       :     F7
 37  32:   INTEGER
       :     54 B0 D3 BF 29 77 B9 89 79 E3 64 04 24 9A C4 C3
       :     15 45 93 1C AC 20 71 A0 59 50 F4 BD E8 DE 11 A5
       :   }
```

```
ECDSA-Sig-Value ::= SEQUENCE {
    r  INTEGER,
    s  INTEGER
}
```

https://tools.ietf.org/html/rfc5480

- Use the NIST P-256 (secp256r1) curve and SHA384
- Implement ECDSA yourself using EC point operations
- Modular inverse x mod n can be calculated using `pow(x, -1, n)`

Task: Test cases

```
#!/bin/bash
echo "[+] Generating EC key pair..."
openssl genpkey -algorithm EC -pkeyopt ec_paramgen_curve:prime256v1 -out priv.pem
openssl ec -in priv.pem -pubout -out pub.pem

echo "[+] Testing ECDSA signing..."
dd if=/dev/urandom of=filetosign bs=1M count=1
./ecdsa.py sign priv.pem filetosign signature
openssl dgst -sha384 -verify pub.pem -signature signature filetosign
Verified OK

echo "[+] Testing ECDSA verification..."
openssl dgst -sha384 -sign priv.pem -out signature filetosign
./ecdsa.py verify pub.pem signature filetosign
Verified OK

echo "[+] Testing ECDSA failed verification..."
openssl dgst -sha1 -sign priv.pem -out signature filetosign
./ecdsa.py verify pub.pem signature filetosign
Verification failure
```

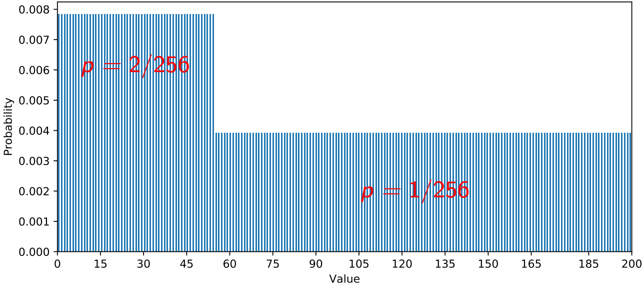
EC point operations in Python

```
$ python3
>>> from secp256r1 import curve
>>> curve.g
[48439561293906451759052585252797914202762949526041747995844080717082404635286,
      36134250956749795798585127919587881956611106672985015071877198253568414405109]
>>> curve.g[0]
48439561293906451759052585252797914202762949526041747995844080717082404635286
>>> curve.mul(curve.g, 5)
[36794669340896883012101473439538929759152396476648692591795318194054580155373,
      101659946828913883886577915207667153874746613498030835602133042203824767462820]
>>> curve.n
115792089210356248762697446949407573529996955224135760342422259061068512044369
>>> curve.mul(curve.g, curve.n)
[None, None]
>>> curve.mul(curve.g, curve.n+5)
[36794669340896883012101473439538929759152396476648692591795318194054580155373,
      101659946828913883886577915207667153874746613498030835602133042203824767462820]
>>> curve.add(curve.g, curve.g)
[56515219790691171413109057904011688695424810155802929973526481321309856242040,
      3377031843712258259223711451491452598088675519751548567112458094635497583569]
>>> curve.mul(curve.g, 2)
[56515219790691171413109057904011688695424810155802929973526481321309856242040,
      3377031843712258259223711451491452598088675519751548567112458094635497583569]
>>> curve.valid(curve.g)
True
>>> curve.valid([1,5])
False
>>> curve.compress(curve.g).hex()
'036b17d1f2e12c4247f8bce6e563a440f277037d812deb33a0f4a13945d898c296'
>>> curve.decompress(curve.compress(curve.g))
[48439561293906451759052585252797914202762949526041747995844080717082404635286,
      36134250956749795798585127919587881956611106672985015071877198253568414405109]
```

Generate random number in given range

Generate a random integer in range $[0, 200]$:

- `bi(os.urandom(1)) % 201`



- Decrease bias: `bi(os.urandom(64)) % 201`
- Rejection sampling: regenerate until in range

Questions

- What is the Elliptic Curve Discrete Logarithm Problem?
- What constitutes a private key in ECC?
- What constitutes a public key in ECC?
- What is elliptic curve point multiplication?
- What is point validation and why is it needed in ECDH?
- What does 256 bits for a 256-bit curve denote?
- Why is ECC preferred over RSA?
- How can data be encrypted using ECC?
- What is the largest value that can be signed directly using ECDSA?