Assignment5_AY23_24
Group 14

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1 Certificate Pricing

The main objective of this case study was to determine the participation coefficient ($\boldsymbol{\alpha}$) of a certificate with a Payoff equal to

$$\alpha(S(t) - P)^{+} \tag{1}$$

where S(t) corresponds to a basket composed by two stocks: ENEL and AXA.
We computed the participation coefficient by imposing the positive discounted cash flows equal to the negative discounted cash flows in t=0. To do that we needed to compute some quantities.
First, we computed the discounts both using the Bootstrap and the Interpolation method (for the ones which did not appear in the Bootstrap). Then we computed the forward interest rates useful to simulate the dynamics of the stocks.
We ran a Montecarlo simulation with 10^5 iterations and we found the values of S(t) thanks to which we calculated the average payoff.
By numerically solving the cash flows equation we found $\boldsymbol{\alpha} = \mathbf{0.0777}$.
The 95% confidence interval is: **IC = [0.0758, 0,0795]**.

2 Pricing Digital option

In this exercise we were asked to verify the difference between the price of a digital option computed according to the Black model and the one computed with respect to the implied volatility approach.
Exploiting spline interpolation (characteristic for the Vega computation) we found the volatility smile correspondent to the strikes given in the text (model volatility).
To price a digital option via black model we used the closed formula:

$$d_{\mathrm{CB}}(K) = -\frac{\partial CB(K)}{\partial K} = B(t_0, t)N[d2] \tag{2}$$

where K is equal to S0 (since we are ATM spot) and F0 is the forward price in Garman-Kohlagen model.

In order to price the digital option via the implied volatility approach (taking into account the slope of the curve) we used the following formula:

$$d_{\mathrm{CB}}(K) = -\frac{\partial CB(K, \sigma(K))}{\partial K} - \frac{\partial \sigma(K)}{\partial K} \cdot \frac{\partial CB(K, \sigma)}{\partial \sigma} = \text{BlackTerm} - \text{SlopeImpact} \cdot \text{Vega} \tag{3}$$

where the derivative of the smile is calculated as: $\frac{\sigma_2 - \sigma_1}{k_2 - k_1}$
The result that we obtained, multiplied for the payoff (5% of the Notional) are:

	value
Black price	2.2527×10^5
Volatility price	3.0116×10^5

Table 1: Digital option price

We can observe that the price calculated taking into account the digital risk is higher than the price calculated with the Black model. This is consistent with the theory since we are considering also the digital risk.

3 Pricing

This exercise tasked us with computing the price of an EU Call using the Lewis formula. To achieve this, we implemented three methods: FFT, Quadrature, and Monte Carlo.

$$\frac{C(x)}{B(t_0, t)F_0} = 1 - e^{-\frac{x}{2}} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{-i\xi x} \Phi(-\xi - \frac{i}{2}) \frac{1}{\xi^2 + \frac{1}{4}}$$

The FFT and Quadrature methods approximate the integral in the Lewis formula, while the Monte Carlo method computes the price directly from the payoff, considering the dynamic described by the normal mean-variance mixture model used in our assumption.

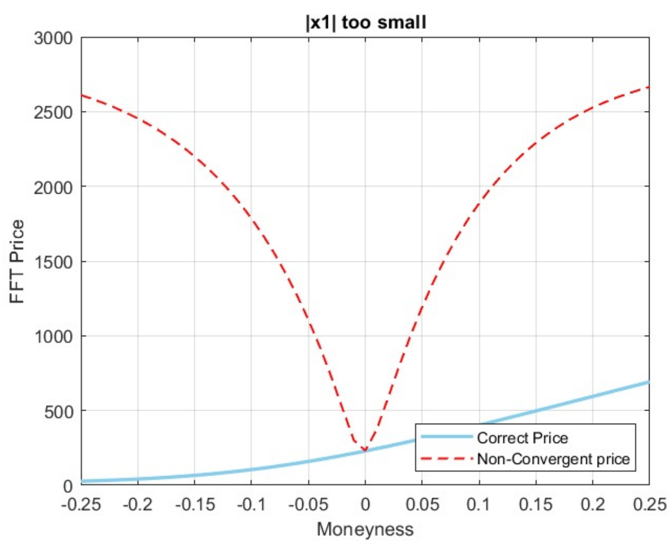
3.1 FFT

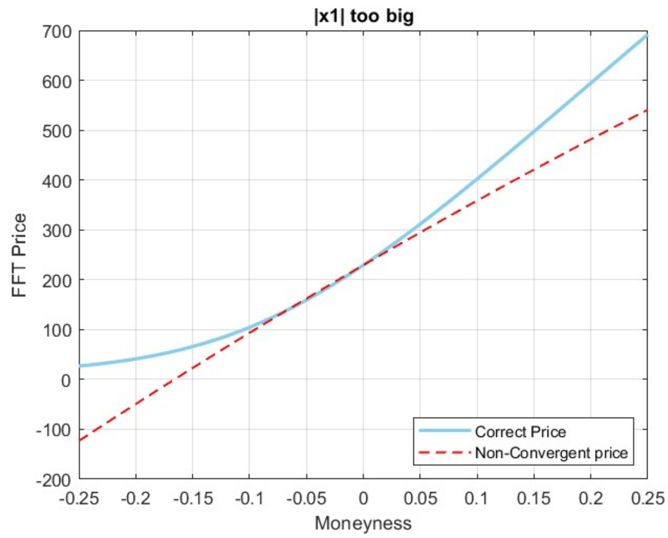
The first method we considered is the FFT (Fast Fourier Transform). This method relies on seven parameters related by five equations. The free parameters can be either M and x1 or M and dz. We experimented with various pairs to determine the ones that best approximate the price curve. We observed that when choosing x1 values that are too large (in absolute value), the method explodes, returning incorrect results. Conversely, selecting x1 values that are too small leads to poor approximation, especially near the extremes (OTM and ITM). Similarly, since "dz" is inversely proportional to x1, similar observations can be made. We used the curves obtained by the Quadrature and Monte Carlo methods as references to calibrate the free parameters for the FFT.

$$x_1 = -x_N$$
$$z_1 = -z_N$$
$$dx = \frac{x_N - x_1}{N}$$
$$dz = \frac{z_N - z_1}{N}$$
$$dx \cdot dz = \frac{2\pi}{N}$$

x1	-225
dz	0.014

Table 2: Optimal free parameters





3.2 Quadrature

The second method we explored is Quadrature integration. Here, we simply defined the entire integrand function (taking into account the term related to the Fourier transform) depending on the moneyness and csi as variables. Then, we utilized the built-in Matlab function 'quadgk' to compute the Quadrature approximation of the integral with respect to csi. Finally, we evaluated the approximation at the given moneyness values.

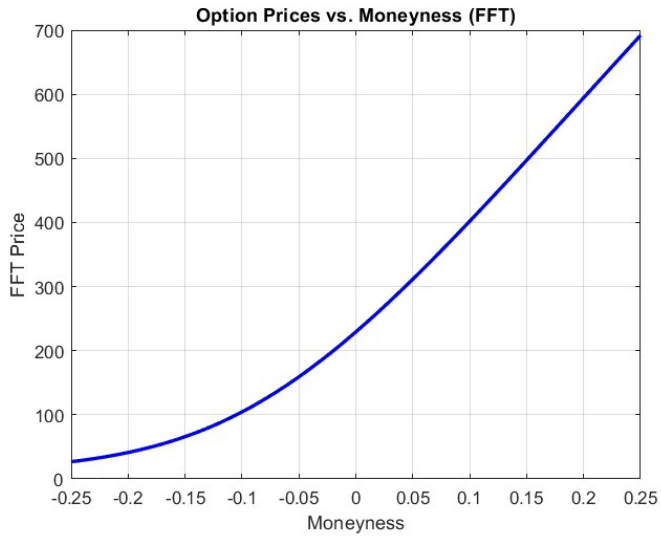
3.3 MonteCarlo

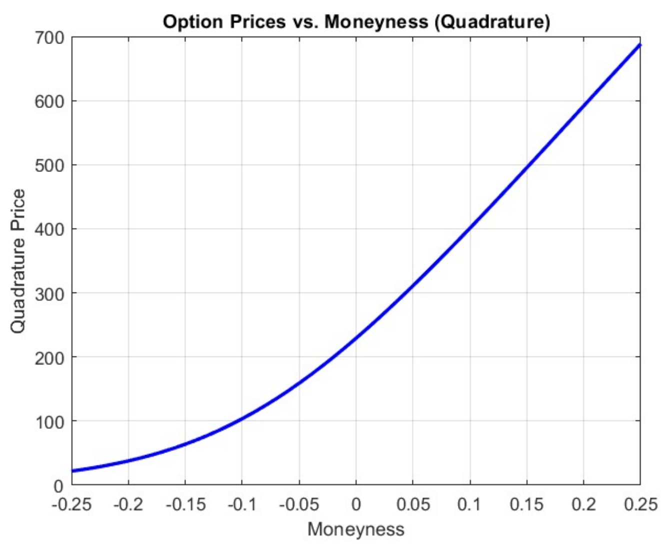
The last method required us to compute the price directly using the Monte Carlo method. We considered the dynamics of the underlying forward:

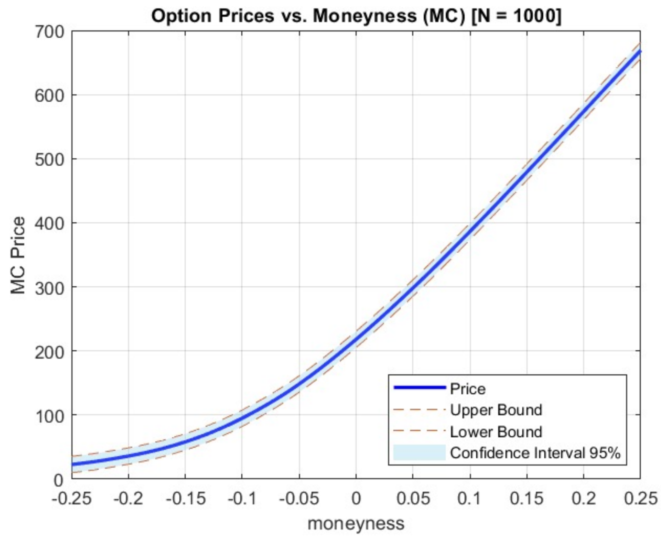
$$f_t = \ln \frac{F_t}{F_0}$$

$$f(t) = \sqrt{t-t_0} \cdot \sigma \cdot \sqrt{G} \cdot g - \left(\frac{1}{2} + \eta\right) \cdot (t-t_0) \cdot \sigma^2 \cdot G - \ln(\mathcal{L}(\eta))$$

By conducting 10^6 simulations, we obtained the required price with this method as well.

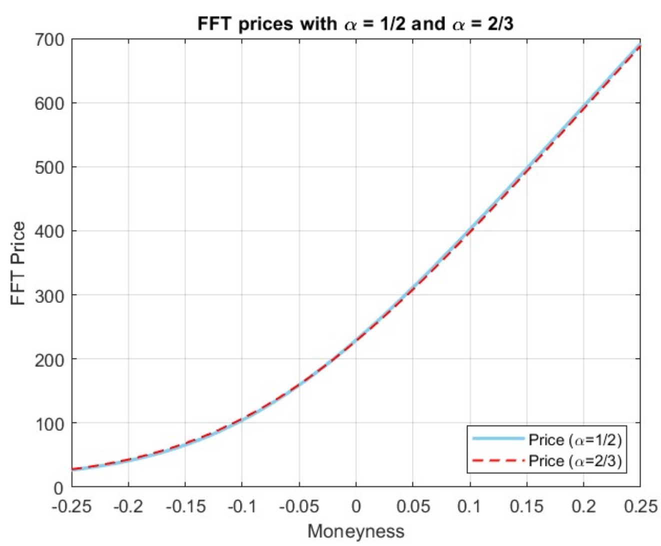

Option Prices vs. Moneyness (FFT)
FFT Price
Moneyness
0
100
200
300
400
500
600
700
-0.25
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
0.2
0.25

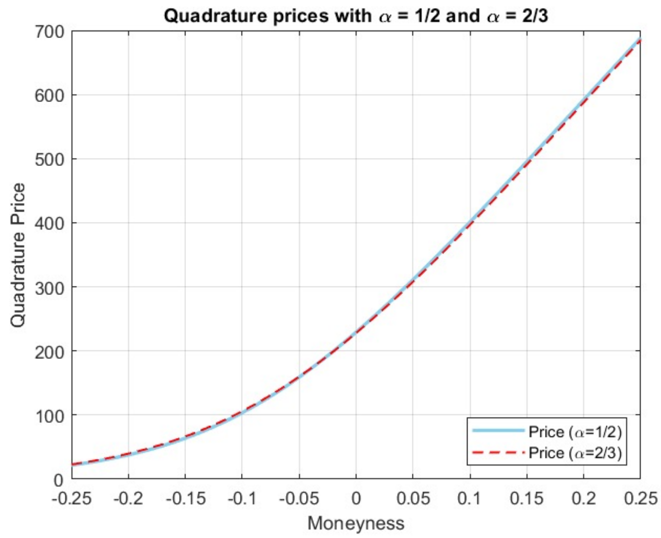

Option Prices vs. Moneyness (Quadrature)
Quadrature Price
Moneyness
0
100
200
300
400
500
600
700
-0.25
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
0.2
0.25



3.4 Optional Point

By varying the alpha parameter from $\frac{1}{2}$ to $\frac{2}{3}$ and reevaluating the prices generated by the FFT and Quadrature methods, we observed that the curves are quite similar. Only the initial parts of them exhibit slight differences





We would like to highlight a couple of significant observations that emerged during the development of Exercise 3:

1. **Complex Values from FFT and Quadrature Methods:** It's noteworthy that both the FFT and Quadrature methods yielded complex values, albeit with negligible imaginary components. Theoretically, the imaginary part of the integral should completely vanish. However, the presence of a small imaginary component can be attributed to the numerical errors introduced by the computational methods employed.

2. **Locally Linear Behavior of Price Curve:** Another interesting observation is the locally linear behavior of the price curve when the EU Call option is significantly out-of-the-money (OTM) or in-the-money (ITM). This behavior suggests a simplified trend in these extreme scenarios, potentially providing insights into the option's behavior under such conditions.

4 Volatility surface calibration

In the fourth case study we were asked to calibrate a Normal Mean-Variance Mixture model with $\boldsymbol{\alpha = \frac{1}{3}}$, considering the S&P 500 implied volatility surface via a global calibration with constant weights.

First we computed Black's prices, exploiting the volatility smile. After this, we calculated the Call prices with the Lewis formula, given the following return's characteristic function:

$$\Phi(\xi) = \exp\left(-i\xi \ln L(\eta)\right) L\left(\frac{\xi^2 + i(1+2\eta)\xi}{2}\right)$$

Then by using the built-in Matlab function `fmincon`, we found the parameters that minimize the L^2-norm difference between the prices obtained with Black and Lewis formula. The results are the following:

$\boldsymbol{\Sigma}$	0.1012
$\boldsymbol{\eta}$	13.6159
$\boldsymbol{k}$	1.1540

Table 3: Minimizing parameters

Last we computed again the Call prices with the Normal Mean-Variance Mixture model plugging in the parameters found by solving the minimization problem.
In the following figure, we can see a comparison between the implied volatility evaluated with the "new" prices and the one given by the market: