Financial Risk Laboratory: 1. Credit Spread for Corporate Bonds

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Overview

A Corporate Bond is a Defaultable Bond

See: [Schon, 2003]

Survival Indicator Function

Default is described by the stochastic variable τ, that is the *default* time[a].
The *survival indicator function* is $I(t)$:

$$I(t) = \mathbf{1}_{\tau > t} = \begin{cases} 1 & \text{if } \tau > t \\ 0 & \text{if } \tau \leq t \end{cases}$$

[a] we call also τ *stopping time*

Probability of Survival

At time t we introduce the *probability of survival* between t and T as $P(t, T)$:

$$P(t, T) = \mathbf{E}_t \left[I(T) \right] \tag{1}$$

Pricing

Price of the *risk-free* zero coupon bond[a] that pays 1$ at time T

[a] Schonbucher 3.1

$$B(t,T) = \mathbf{E}_t\left[e^{-\int_t^T r_s ds} \cdot 1\right] \tag{2}$$

where r_t is the stochastic instantaneous IR at time t.

Price of the *defaultable* zero coupon bond[a] that pays 1$ at time T

[a] Schonbucher 3.2

$$\bar{B}(t,T) = \mathbf{E}_t\left[e^{-\int_t^T r_s ds} \cdot \mathbf{1}_{\tau > T}\right] \tag{3}$$

where we suppose that, in the event of default, the investor will loose everything (no recovery).

The Term Structure of Risk–free Rates Revisited

We usually define the term structure of the risk–free rates thanks to a vector of *spot* zero-coupon rates defined on a set of standard maturities (*tenors*).
We can, however, think at the zero–coupon curve **in continuous**.
In order to do so, we introduce the *risk–free instantaneous continuously compounded forward rates*:

$$f(t,T) = -\frac{\partial}{\partial T} lnB(t,T) \tag{4}$$

So that:

$$B(t,T) = e^{-\int_t^T f(t,s)ds} \tag{5}$$

Parallel for the Defaultable Zero Coupon Bond

We follow the same approach and introduce the *defaultable instantaneous continuously compounded forward rates*:

$$\bar{f}(t,T) := -\frac{\partial}{\partial T} ln\bar{B}(t,T) \Rightarrow \bar{B}(t,T) = e^{-\int_t^T \bar{f}(t,s)ds} \tag{6}$$

Spread

$$h(t,T) = \bar{f}(t,T) - f(t,T) > 0 \tag{7}$$

The *spread* $h(t,T)$ is positive since $\bar{B}(t,T) < B(t,T)$.

Interpretation of the Spread

Independence Assumption

The dynamics of the stochastic instantaneous IR, here *risk-free*, are independent of the default time, therefore we factor-in the terms under the expectation in (3):

$$\bar{B}(t,T) = \mathbf{E}_t\left[e^{-\int_t^T r_s ds}\right] \cdot \mathbf{E}_t\left[\mathbf{1}_{\tau > T}\right] = B(t,T) \cdot P(t,T) \tag{8}$$

Definition

Continuous Implied Hazard Rate of Default $h(t,T)$

$$h(t,T) := -\frac{\partial}{\partial T} lnP(t,T) \tag{9}$$

By comparing (7) with the definition (9) we see that, under the independence assumption, the *spread* naively introduced as the difference of two rates is indeed the hazard rate of default.

Pricing of the Defaultable Bond

Thanks to the definitions above, we recast the pricing formula of the defaultable bond (8) with zero recovery following a notation parallel to the pricing formula of the risk-free bond (5):

$$\bar{B}(t, T) = exp\left[-\int_t^T [f(t, s) + h(t, s)]\, ds\right] \tag{10}$$

Credit Spread for ZC Bonds: No Recovery Case

Zero Recovery Rate

Based on a single zero-coupon bond with maturity T, zero recovery rate ($\pi = 0$) and price $\bar{B}$:

$$\bar{B}(t,T) = B(t,T) \cdot P(t,T) = B(t,T) \cdot e^{-\Lambda(t,T)(T-t)} \tag{11}$$

where Λ was introduced as the *credit spread*

By comparison with (10):

$$\Lambda(t,T) = \frac{1}{T-t} \int_t^T h(t,s)ds \tag{12}$$

so that Λ is the time average of the hazard rate h across the residual life of the zero coupon bond

An Actively Traded Instrument: The Coupon–Bearing Defaultable Bond

Risk–neutral pricing of a defaultable bond paying N fixed coupons of amount $\bar{c}_n$ at T_n and principal at maturity T_N according to this formalism (Schonbucher 3.21 and 3.22):

$$\bar{C}(t) = \begin{array}{lcr} \sum_{n=1}^{N} \bar{c}_n \bar{B}(t, T_n) & + & [coupons] \\ \bar{B}(t, T_N) & + & [principal] \\ \pi \sum_{n=1}^{N} [P(t, T_{n-1}) - P(t, T_n)]\, B(t, T_n) & & [recovery] \end{array} \quad (13)$$

where we introduced the recovery rate π different from zero and

$$P(t, T_1) - P(t, T_2)$$

is the rate of default between T_1 and T_2 as seen in t [1] with $t < T_1 < T_2$

[1] See Definition 3.4 of Schonbucher

Discussion of Models Based on Hazard–rate

Common conditions in the fixed–income business:

- Instantaneous hazard rates are small: $h(t, T) << 1$
- Large uncertainty about the recovery rate π

Market prices of defaultable bonds set by (13) are consistent with multiple sets of (h, π) $\rightarrow$ calibration of (h, π) from market prices of multiple bonds issued by the same name is not practical.

The third term in the RHS of (13) (recovery) is smaller than the other terms (coupons and principal).

Trader's Z–spread

Fixed income traders (opposite to traders in CDS) replace (13) with:

$$\begin{aligned} \bar{C}(t) = \textstyle\sum_{n=1}^{N} \bar{c}_n \hat{B}(t, T_n) &\quad + \quad [coupons] \\ \hat{B}(t, T_N) &\quad + \quad [principal] \end{aligned} \tag{14}$$

where $\hat{B}(t, T) > \bar{B}(t, T)$ is a discount factor that, formally, is calculated like the price of a zero–coupon defaultable bond with zero recovery:

$$\hat{B}(t, T) = exp\left[-\int_t^T [f(t, s) + z(t, s)]ds\right] \tag{15}$$

where $0 < z(t, T) < h(t, T)$ is the *Z–spread*:

Definition

Z–spread

Difference between the zero–coupon rate used to discount a defaultable cash–flow and the one used to discount a risk–free cash–flow

Discussion of the Trader's Choice

(13) is derived from a rigorous model, that under independence between defaults and risk–free rates reads (Schonbucher 3.3):

$$\bar{B}(t, T) = B(t, T) \cdot P(t, T) \tag{16}$$

Therefore, (13) can be used either for pricing and hedging portfolios composed by bonds and credit derivatives

Simply a scoring

(14) is simply a *scoring* that maps a defaultable–bond price to a *level of risk*, represented by the Z–spread of the issuer for the given maturity.

Break down

If the hazard rate becomes high, calibration of (14) to bond prices gives nonsensical results; practitioners say *the bond does not trade any more at spread, rather it trades* **at recovery**

Corporate Bond: Description

Figure: Basic data of the corp. bond issued by General Electic with 4 Yr. maturity. Source: Bloomberg

Corporate Bond: Credit Spread

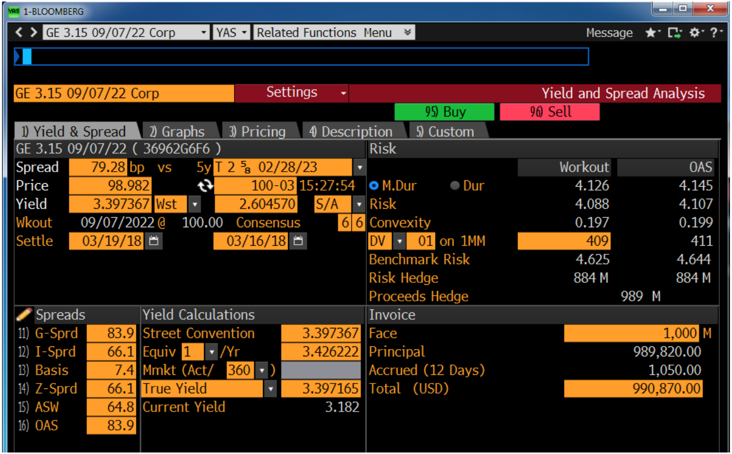


Figure: Yield and credit spread calculation of General Electic 4 Yr. corp. bond. Source: Bloomberg

Credit Spread for ZC Bonds: the Finite Recovery Rate Case

We first observe on the market the traded price of the risky zero coupon bond $\hat{B}(t, T)$ with credit spread s where:

$$\hat{B}(t, T) = B(t, T) \cdot P(t, T) = B(t, T) \cdot e^{-s(t,T)(T-t)} \qquad (17)$$

By comparison with (15):

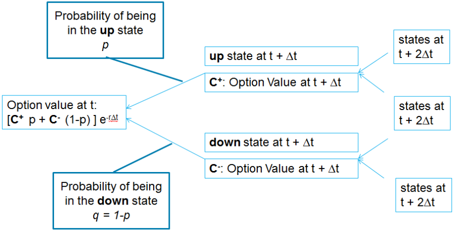
$$s(t, T) = \frac{1}{T - t} \int_t^T z(t, s) ds \qquad (18)$$

so that s is the time average of the z-spread z across the residual life of the zero coupon bond

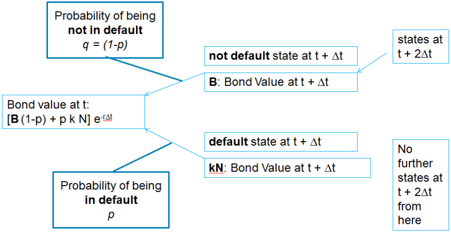
A Simple One-period Pricing Model (Jarrow 1995)

What if we price the risky zero-coupon bond with nonzero recovery rate π? Answer (Jarow 1995): pricing on a tree (the firm can only default at discrete time):

Stock Option



Corporate Bond



No-arbitrage Argument (1/2)

At time t a risk-neutral investor is offered to invest 1$ in one of the following prospects:

1. A risk-free security (e.g. a zero-coupon bond) that pays $e^{r\cdot(T-t)}$ at time $T > t$
2. A risky bond that:
   - pays $e^{(r+s)\cdot(T-t)}$ at time $T > t$ with probability $1 - p_{t,T}$
   - pays $\pi \cdot e^{(r+s)\cdot(T-t)}$ at time $T > t$ with probability $p_{t,T}$

   where s is credit spread and $p_{t,T}$ is the unconditional default probability between t and T:

$$p_{t,T} = 1 - P(t, T) \tag{19}$$

No-arbitrage Argument (2/2)

The choice of the investor will be neutral [2] if:

$$e^{r\cdot(T-t)} = [(1-p_{t,T}) + p_{t,T}\cdot\pi]\, e^{(r+s)\cdot(T-t)} = [1-p_{t,T}(1-\pi)]\, e^{(r+s)\cdot(T-t)} \tag{20}$$

Definition

Annualized risk-neutral unconditional default probability
From equation (20) we set $T-t=1$ and derive the annualized probability $p = p_{t,t+1}$:

$$p = \frac{1-e^{-s}}{1-\pi} \tag{21}$$

[2] Hence *risk-neutral*, since the investor is supposed not to take into account the uncertainty of returns if investing in the risky asset

Risk-neutral vs. Real-world

Risk-neutral unconditional default probability under arbitrary time horizon

Derivation outlined in the previous slide can be easily extended to infer at time t the unconditional default probability $p_{t,T}$ under any time horizon $T - t$:

$$p_{t,T} = \frac{1 - e^{-s(t,T)\cdot(T-t)}}{1-\pi} \tag{22}$$

Since $p_{t,T}$ is derived on the basis of a no-arbitrage argument for a risk-neutral investor, $p_{t,T}$ is also defined *risk-neutral*, opposite for a measure of the default probability based on the frequency of past defaults of firms belonging to the same class, that is *real-world*.

References

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Robert A. Jarrow, and Stuart M. Turnbull. *Pricing derivatives on financial securities subject to credit risk.*

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