Assignment 2 - Group 14

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1 Exercise 1

In the initial phase of our task, we developed a function to implement the bootstrap method for constructing the discount factors curve, employing a single-curve model. Firstly we used the mid deposit rates up to 3 months to get the discount factors required for the interpolation on the zero rate of the one corresponding to the first future's settlement date. Subsequently, by applying the same interpolation/extrapolation techniques for each future's settlement day discount and then computing the corresponding expiry date iteratively, we generated discount factors up to the expiry date of the seventh future.
For the swap rates, we initially executed an interpolation procedure to derive the one-year discount factor. We then constructed a complete set of swap rates and associated dates, and finally computed iteratively the discounts of the 2nd up to the 50th swap using past discounts.

We then plotted both the discount factors and the zero rates for the dates considered:

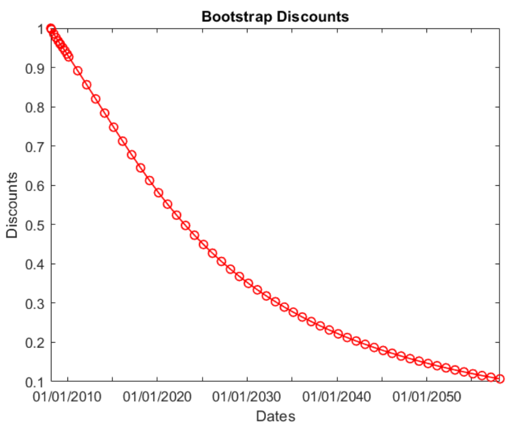

Figure 1: Discounts

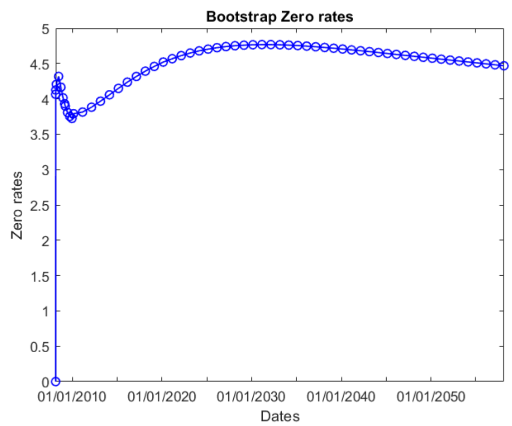

Figure 2: Zero Rates

1.1 Q

Bootstrap holds particular relevance due to several factors:

Consistency with Market Data: Bootstrap ensures that the derived discount factors are consistent with the observed market rates. By iteratively fitting the curve to market rates, it ensures that the resulting curve accurately reflects the current market conditions.

Flexibility: Bootstrap allows for the incorporation of various types of market instruments, such as swaps, bonds, or futures contracts. This flexibility allows to construct discount curves tailored to specific needs.

Accurate Pricing of Financial Instruments: The accuracy of pricing financial instruments depends on the accuracy of the discount curve used. Bootstrap helps generate reliable discount curves, which lead to more accurate pricing of derivatives, bonds, and other financial products, and are crucial for risk management activities such as valuing portfolios or calculating counterparty credit exposures.

Forward Curve Construction: Bootstrap can also be used to construct forward rate curves from discount curves, which are essential for pricing interest rate derivatives and other financial products.

Overall, bootstrap is relevant in finance because it provides a robust and flexible method for deriving discount curves from market data, which is fundamental for accurate pricing.

2 Exercise 2

With the discount curve obtained above, for a portfolio composed only by one single swap, a 6y plain vanilla IR swap vs Euribor 3m with a fixed rate 2.817% and a Notional of 10 Mln e we computed the following quantities:

- DV01-parallel shift;
- DV01Z-parallel shift;
- BPV of the 6y IRS;

The results, respectevely unitary and with Notional 10Mln, are the following:

Table 1:

Sensitivity	Unitary	Notional 10Mln
DV01	5.0142e-04	5014.2
BPV	5.2398e-04	5239.8
$DV01^z$	5.2002e-04	5200.2

We can observe that DV01 and BPV values are quite similar for a plain vanilla par swap at trade date. Indeed, both quantify the same type of interest rate sensitivity on slightly different, but essentially equivalent, scales. In particular, DV01 was calculated by determining the variation in net present values resulting from the increase of 1 basis point in the rates used for bootstrap. Conversely, for BPV, the difference in net present values was computed by shifting the fixed rate by 1 basis point. Lastly, DV01Z was computed as the variation in NPVs caused by a parallel shift in the zero rates.

Then, we computed the Macaulay Duration and got $MacD = 5.5849$ for an IB coupon bond with same fixed rate, expiry and reset dates of the IR swap, and face value equal to IRS notional. Moreover, from the theory we know that $DV01^z = MacD \cdot 1bp \cdot BondPrice$, that is consistent with our results, since we get $BondPrice = 1$.

N.B.: we wanted to specify that in the functions sensCouponBond, sensSwap we did not use the input "dates" to do our calculations. This is because we had difficulties in "writing in code" the correspondence of the indexes of the dates of our interest. To overcome this problem, we entered them "by hand" (line 24, lines 30,31) observing the excel file.

3 Coupon Bond pricing

We now proceed to price a 6y InterBank coupon bond issued on the 15th Feb 2008 with coupon rate equal to the corresponding mid-market 7y swap rate, that is 2.8173. We have assumed a 30/360 European day count for the coupons. We set a face value of 1 and a discount factor up to time i equal to $B(t_0, t_i)$. Consequently, the pricing of a coupon bond can be evaluated computing the sum of the discounted coupons and the discounted face value.

$P = \sum_{i=1}^{6} \left(S_{IR}(t_0, t_7) \cdot \delta(t_{i-1}, t_i) \cdot B(t_0, t_i)\right) + 1 \cdot B(t_0, t_6)$ where $S_{IR}(t_0, t_7) \cdot \delta(t_{i-1}, t_i) = c_i$

From the swap rates we know: $\sum_{i=1}^{7} \left(S_{IR}(t_0, t_7) \cdot \delta(t_{i-1}, t_i) \cdot B(t_0, t_i)\right) = 1 - B(t_0, t_7) \iff$

$\iff \sum_{i=1}^{6} \left(S_{IR}(t_0, t_7) \cdot \delta(t_{i-1}, t_i) \cdot B(t_0, t_i)\right) + S_{IR}(t_0, t_7) \cdot \delta(t_6, t_7) \cdot B(t_0, t_7) = 1 - B(t_0, t_7) \iff$

$\iff P - B(t_0, t_6) + S_{IR}(t_0, t_7) \cdot \delta(t_6, t_7) \cdot B(t_0, t_7) = 1 - B(t_0, t_7)$

Thus we obtain: $P = B(t_0, t_6) + 1 - B(t_0, t_7) \cdot (1 + S_{IR}(t_0, t_7) \cdot \delta(t_6, t_7))$

4 Garman - Kohlhagen formula

The dynamic of the underlying in described by a Geometric Brownian motion: $dSt = St\left[(r(t) - d(t))\,dt + \sigma(t)dWt\right]$ with initial condition S_0

Taking the derivative of the natural logarithm of S_t: $d\ln(S_t) = \frac{1}{S_t}dS_t - \frac{1}{2}\frac{1}{S_t^2}\langle dS_t^2\rangle$

Now substituting the expression of dS_t and computing $\langle dS_t^2\rangle$ we get:

$d\ln(S_t) = \left((r(t) - d(t)) - \frac{1}{2}\sigma(t)^2\right)dt + \sigma(t)dW_t$

Now integrating both sides between t and t_0:

$\int_{t_0}^{t} d\ln(S_u) = \int_{t_0}^{t}\left((r(u) - d(u)) - \frac{1}{2}\sigma(u)^2\right)du + \int_{t_0}^{t}\sigma(u)dW_u$

and defining the average values over the time to maturity, instead of the constant values in the standard formula: $D(t) = \frac{1}{t-t_0}\int_{t_0}^{t} d(u)\,du$, $R(t) = \frac{1}{t-t_0}\int_{t_0}^{t} r(u)\,du$ and $\Sigma(t)^2 = \frac{1}{t-t_0}\int_{t_0}^{t}\sigma(u)^2du$

we obtain: $\ln(S_t) - \ln(S_0) = (R(t) - D(t) - \frac{1}{2}\Sigma(t)^2)(t - t_0) + \int_{t_0}^{t}\sigma(u)\,dW(u)$

The last term is a stochastic integral with a deterministic function as integrand, so it is a centered Gaussian process, hence we can use Ito's isometry:

$Var\left(\int_{t_0}^{t}\sigma(u)\,dW(u)\right) = \mathbb{E}_0\left[\left(\int_{t_0}^{t}\sigma(u)\,dW(u)\right)^2\right] = \mathbb{E}_0\left[\int_{t_0}^{t}\sigma(u)^2\,du\right] = (t - t_0)\Sigma(t)^2$

So we can observe that: $\int_{t_0}^{t}\sigma(u)\,dW(u) \sim \mathcal{N}\left(0, (t - t_0)\Sigma(t)^2\right)$

By expliciting S_t we obtain: $S_t = S_0\exp\left[(t - t_0)\left(R(t) - D(t) - \frac{\Sigma(t)^2}{2}\right) + \int_{t_0}^{t}\sigma(u)\,dW(u)\right]$

and $S_t \sim S_0\exp\left[(t - t_0)\left(R(t) - D(t) - \frac{\Sigma(t)^2}{2}\right) \pm \sqrt{t - t_0}\cdot\Sigma(t)\cdot g\right]$ with $g \sim \mathcal{N}(0, 1)$

Now, we compute the value of the call:

$C(t, t_0) = B(t, t_0)\cdot\mathbb{E}_0\left[(S_t - k)^+\right] = B(t, t_0)\cdot\left(\mathbb{E}_0\left[S_t\cdot\mathbb{1}_{(S_t\geq k)}\right] - k\cdot\mathbb{E}_0\left[\mathbb{1}_{(S_t\geq k)}\right]\right)$

We evaluate separately the two expected values:

$S_t \geq k \iff S_0\exp\left[(t - t_0)\left(R(t) - D(t) - \frac{\Sigma(t)^2}{2}\right) - \sqrt{t - t_0}\cdot\Sigma(t)\cdot g\right] \geq k \iff$

$\iff \left[(t - t_0)\left(R(t) - D(t) - \frac{\Sigma(t)^2}{2}\right) - \sqrt{t - t_0}\cdot\Sigma(t)\cdot g\right] \geq \ln(\frac{k}{S_0}) \iff$

$g \leq \frac{\ln\left(\frac{S_0}{K}\right) - \frac{1}{2}\Sigma(t)^2(t-t_0) + (R(t)-D(t))(t-t_0)}{\Sigma(t)\sqrt{t-t_0}} =: d_2$

Hence $\mathbb{E}_0\left[\mathbb{1}_{(S_t\geq k)}\right] = \int_{-\infty}^{+\infty} dg\frac{\exp\left(-\frac{g^2}{2}\right)}{\sqrt{2\pi}}\cdot\mathbb{1}_{(g\leq d_2)} = N(d_2)$

While for the other one: $\mathbb{E}_0\left[S_t\cdot\mathbb{1}_{(S_t\geq k)}\right] = S_0\mathbb{E}_0\left[\exp\left[(t - t_0)\left(R(t) - D(t) - \frac{\Sigma(t)^2}{2}\right) - \sqrt{t - t_0}\cdot\Sigma(t)\cdot g\right]\mathbb{1}_{(g\leq d_2)}\right] =$

$= S_0\exp\left[(t - t_0)\left(R(t) - D(t)\right)\right]\int_{-\infty}^{d_2}\frac{dg}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(g + \Sigma(t)\sqrt{t - t_0}\right)^2\right]$

Substituting $g' = g + \Sigma(t)\sqrt{t - t_0}$ we get

$= S_0\exp\left[(t - t_0)\left(R(t) - D(t)\right)\right]\int_{-\infty}^{d_2+\Sigma(t)\sqrt{t-t_0}}\frac{dg'}{\sqrt{2\pi}}\exp\left(-\frac{g'^2}{2}\right) = S_0\exp\left[(t - t_0)\left(R(t) - D(t)\right)\right]N(d_1)$

Getting back to the call we compute its value:

$C(t, t_0) = B(t_0, t)\left[S_0e^{(R(t)-D(t))(t-t_0)}N(d_1) - KN(d_2)\right]$

with $d_{1,2} = \frac{1}{\Sigma(t)\sqrt{t-t_0}}\left[\ln\left(\frac{S_0}{K}\right) + (R(t) - D(t))(t - t_0) \pm \frac{1}{2}\Sigma(t)^2(t - t_0)\right]$