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Macroeconometrics
Final Assignment

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1 Introduction

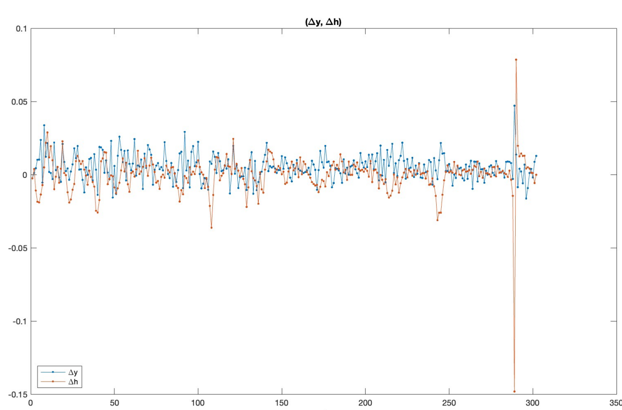
This report analyses the response of labor supply, expressed in hour worked per capita, to a technology shock, exploring two conflicting papers made by Galì(1999)[3] and Christiano(2003)[2].
The contrast between the results of the two papers is a long term descendent of the theoretical dispute that see Keynes against the Business Cycle. Galì with his work provided evidence that to a positive technology shock follows a decrease in labour supply, instead Christiano with his findings supported the school of Chicago.
In this work we will go through both the papers, borrowing at times their statistical models and techniques to solve the following task: identifying the technological shock. We will first retrieve them using both the growth rate of log labor productivity and per capita hours worked (following Galì procedures), and later we will observe what would happen substituting for hours worked the growth rate with the levels (retracing Christiano methodologies).

2 The Dataset

We begin our analysis retrieving and showing the data mentioned above which are essential to the identification of the technological shock.
From the FREDII data base , we downloaded the series of nonfarm business sector (OUTNFB), total hours (HOANBS), and population (CNP16OV). We proceed with the definition of our variable of interest. We compute the (log) labor productivity and per-capita hours worked as the log of the ratio between business sector/population and the total hours:

$$y = \log\left(\frac{\text{OUTNFB}}{\text{HOANBS}}\right) \quad \text{and} \quad h = \log\left(\frac{\text{CNP16OV}}{\text{HOANBS}}\right)$$

We then define the two bivariate time series $(\Delta\text{y}, \Delta\text{h})^T$2 and $(\Delta\text{y}, \text{h})^T$. Of the two we cause differences of scales we plot only the first. Only remark in the chart above is the anomaly present, with different



magnitudes, in both the series, but we can easily explain it as a consequence of the Covid-19 pandemic

3 Identification

3.1 Model

We think that a brief recall of the following definitions and theorems would be the best to better understand the later steps.
A time series $Y = (Y_t)_{t\in\mathbf{Z}}$ is said to be a VAR(p) process if

$$Y_t = \mu + \sum_{j=1}^{p} A_j Yt - j + \epsilon_t \tag{1}$$

where $\mu \in \mathbb{R}^n, A_1, ..., A_p \in \mathbb{R}^{n\times n}, (\epsilon_t)_{t\in\mathbb{Z}} \sim WN(0,\Omega)$ with $\Omega \in \mathbb{R}^{n\times n}$ symmetric and positive definite matrix.
Notice that 1 can be represented using its companion form:

$$\bar{Y}_t = FY_{t-1} + e_t \tag{2}$$

where

$$\bar{Y}_t = \begin{bmatrix} Y_t - \mu \\ \vdots \\ Y_{(t-p)+1} - \mu \end{bmatrix}, \quad F = \begin{bmatrix} A_1 & \dots & A_{p-1} & A_p \\ I_n & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I_n & 0 \end{bmatrix} \text{ and } \ e_t = \begin{bmatrix} \epsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

From this representation it is possible to retrieve a sufficient condition for the stationarity of the process. Indeed, Y is stable if and only if all the eigenvalues of F have modulus smaller than 1, and a stable VAR(p) process is stationary. If Y is stationary, then it admits the Wold representation:

$$Y_t = \mu + \sum_{j=0}^{\infty} C_j \epsilon_{t-j} \tag{3}$$

where $(C_j)_{j\in\mathbb{N}}$ are absolutely summable $n \times n$ matrices with $C_0 = I$. These matrices can be obtained by $C_0 = I$. These matrices can be obtained by 2; indeed substituting recursively the companion form we get

$$\bar{Y}_t = \sum_{j=0}^{+\infty} F^j e_{t-j} \tag{4}$$

considering only the first n rows:

$$Y_t - \mu = \sum_{j=0}^{+\infty} F^j_{[1:n,1:n]} \epsilon_{t-j} \tag{5}$$

from 4 and 5 is clear that C_j coincides with the upper left matrix of $F^j \quad \forall j$.
I can rewrite C_j as:

$$\begin{bmatrix} C_{j,[1,1]} & \dots & C_{j,[1,n]} \\ \vdots & \ddots & \vdots \\ C_{j,[n,1]} & \dots & C_{j,[n,n]} \end{bmatrix}$$

where $C_{j,[k,i]}$ is the effect of shock i on variable k at lag j.
In general the impulse response function obtained by 4 is not very clear to interpret from an economic point of view due to the correlation of the shocks (ϵ_t).
To provide a better interpretation, we should assume that economy is driven by exogenous orthonormal

structural shocks $(\eta_t)_{t\in\mathbb{Z}} \sim WN(0, I)$ which are dynamically propagated through the impulse response function $B(L)$, which is:

$$Y_t - \mu = \sum_{j=0}^{\infty} B_j \eta_{t-j} = B(L)\eta_t. \tag{6}$$

Wold representation uniqueness implies 4 = 6. Letting $j = 0$ follows that

$$B_j = C_j B_0 \quad \forall j \in \mathbb{N} \tag{7}$$

this implies that shocks with the IRF of previous equation van be identified choosing appropriately B_0. Following the famous paper of Blanchard and Quah(1989)[1], we define:

$$B_0^{-1} = C(1)^{-K} \tag{8}$$

where K is the Cholesky factor of $C(1)\Omega C(1)$, defined as the unique lower triangular matrix for which $KK^T = C(1)\Omega C(1)$. From the orthornomality of the shocks and 6 we can write:

$$Var(Y_j^{(i)}) = \sum_{k=1}^{n}\sum_{j=0}^{\infty} B_{j[k,i]}^2 Var(\eta_{t-j}^{(k)}) = \sum_{k=1}^{n}\sum_{j=0}^{\infty} B_{j[k,i]}^2. \tag{9}$$

and the proportion of variance $v_{i,k}$ of Y^i w.r.t η^k is :

$$v_{ik} = \frac{\sum_{j=0}^{+\infty} B_{j[k,i]}^2}{Var(Y_t^i)} \tag{10}$$

3.2 Estimation

We then proceed to show the procedure deployed for estimations with the data loaded previously. In the next steps we will only and always use as bivariate time series Y= $(\Delta y, \Delta h)^T$, underlining how the same in the code has been applied to $(\Delta y, \text{h})^T$ too.
We first estimate Y as a VAR(4) model. We would like to do this with a well-know instrument with good properties: the OLS. In order to achieve this goal we start from the compact form for a linear regression model

$$Y^* = X\beta + \epsilon \tag{11}$$

where $Y^* \in \mathbb{R}^n$ random observable target variable, $X \in \mathbb{R}^{n\times k}$ data matrix non random and deterministic with k predictors for n observations each, $\beta \in \mathbb{R}^k$ deterministic unobservable predictors and $\epsilon \in \mathbb{R}^n$ random and unobservable.
To adapt our series Y to this mode we regress it against $Y_{t-1}, ...Y_{t-p}$ for each $t \in \{p+1, ..., T\}$ with l we define the previous elements as follows:

- $Y^* = [Y_{p+1}^T ... Y_T^T]^T \in \mathbb{R}^{T-(p+1)\times 2}$
- $X = [\mathbb{1}, Y_{[p...T-1]}^T, ..., Y_{[1..T-p]}^T] \in \mathbb{R}^{T-(p+1)\times 2(p+1)}$ where $\mathbb{1} = [1, ..., 1] \in \mathbb{R}^{T-(p+1)\times 2}$ vector of ones and $Y_{[p...T-1]}^T = [Y_p^T, ..., Y_{T-1}^T] \in \mathbb{R}^{T-(p+1)\times 2}$
- $\beta = [\mu^T, \beta_1^T, ..., \beta_p^T] \in \mathbb{R}^{2(p+1)}$ where $\mu^T \in \mathbb{R}^2$ is the mean due to the presence of the constant
- $\epsilon = [\epsilon_{p+1}^T, ..., \epsilon_T^t] \in \mathbb{R}^{T-(p+1)\times 2}$

With this setting we can finally apply OLS and through them estimate $\hat{\beta}$ and $\hat{\epsilon}^T$.
Then following what we have previously described from 4 to 5 we can obtain a Wald representation. WE think is also useful to notice that, while the exact Wold representation would require an infinite number

of lags, in practice the series is truncated after a finite number of lags, 25 in this case. Indeed, since the matrices (C_j)'s are absolutely summable, their coefficients tend to zero, thus a negligible effect is lost when considering a finite horizon, thus our choice.
Then we recall 8 and 6 and to retrieve our two structural shocks, $\eta_{\{1,2\}}$ the technological and non technological, we, letting $j = 1...25$ compute the following:

$$(\widehat{B^{(cum)}}_j)_{(r,s)} = \sum_{k=0}^{i} (\widehat{B}_k)_{(r,s)} \tag{12}$$

4 Results

Here we show and analyse the results of our empirical estimation following the assignment:

4.1 Estimation of a VAR(4) for $(\Delta y, \Delta h)$

$$\hat{\mu} = \begin{bmatrix} 0.0052 \\ -0.0036 \end{bmatrix}$$

$$\hat{\beta}_1 = \begin{bmatrix} -0.0797 & -0.1210 \\ 0.3895 & 0.1186 \end{bmatrix}, \hat{\beta}_2 = \begin{bmatrix} 0.0701 & -0.0585 \\ 0.1776 & 0.1527 \end{bmatrix}, \hat{\beta}_3 = \begin{bmatrix} -0.0185 & -0.0988 \\ 0.0943 & 0.0668 \end{bmatrix}, \hat{\beta}_4 = \begin{bmatrix} 0.0323 & -0.0613 \\ 0.0301 & 0.0180 \end{bmatrix}$$

4.2 Impulse Response Function of y and h

Wold IRF Point Estimation and 68% Bootstrap bands for $(\Delta y, \Delta h)$

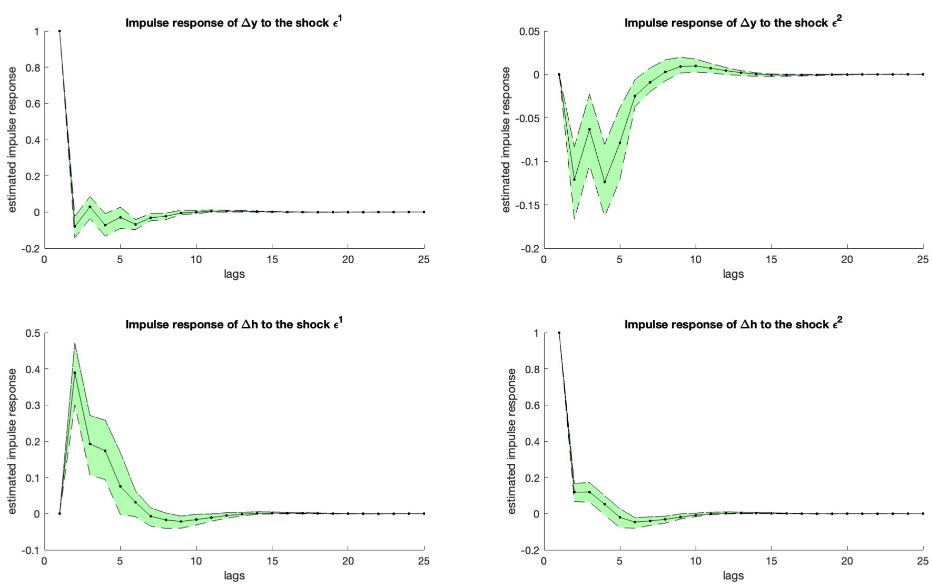


Figure 1: Point estimate of the Wold IRF and of the respective 68% approximate confidence intervals obtained from $(\Delta y, \Delta h)^T$

4.3 Structural IRF Point Estimation and 68% Bootstrap bands for $(\Delta y, \Delta h)$

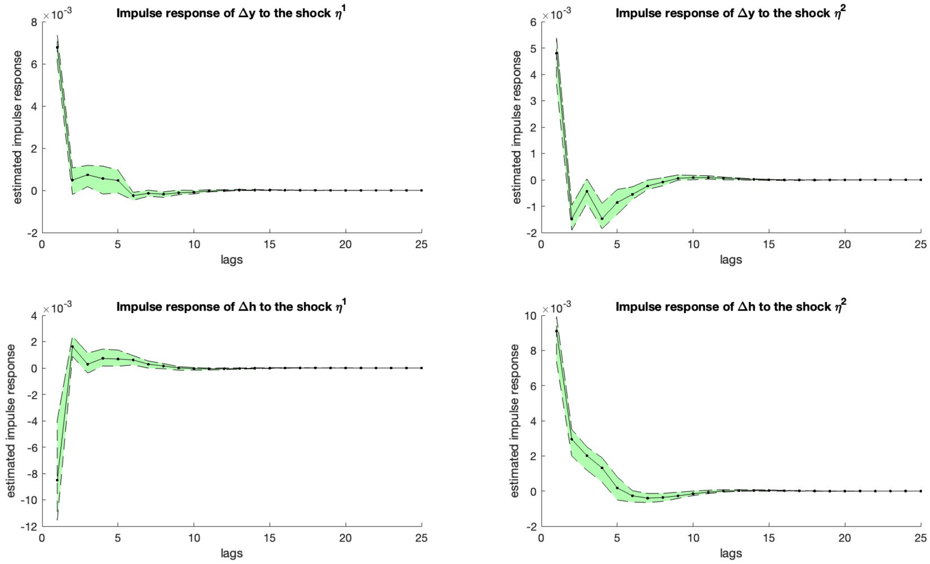


Figure 2: Point estimate of the structural IRF and of the respective 68% approximate confidence intervals obtained from $(\Delta y, \Delta h)^T$

From the plot we can clearly agree with Galì: a positive technological shock implies a decrease in per-capita hours worked, and after a stabilization takes place.

4.4 Percentage variance explained by the aggregate technological shock for $(\Delta y, \Delta h)$

From 10 we estimate:

$$v_{11} = \frac{\sum_{j=0}^{25} \hat{B}^2_{j[11]}}{Var(Y_t^1)} = 0.6224 \quad \text{and} \quad v_{21} = \frac{\sum_{j=0}^{25} \hat{B}^2_{j[21]}}{Var(Y_t^2)} = 0.4395 \tag{13}$$

4.5 Structural cumulative IRF Point Estimation and 68% Bootstrap bands for $(\Delta y, \Delta h)$ (Bonus)

Since usually an effect to the difference of a variable can be not easy to interpret and even to be consistent with what will happen later(he will no more have Δh but h only) we plot the cumulative structural IRF too:

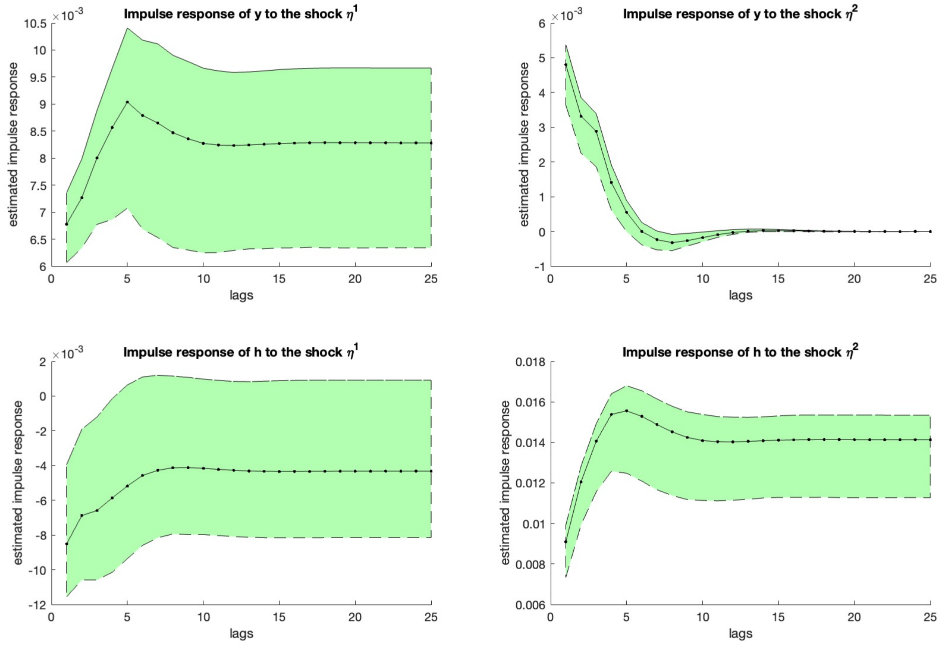


Figure 3: Structural estimate of the structural cumulative IRF and of the respective 68% approximate confidence intervals obtained from $(\Delta y, \Delta h)^T$

4.6 Estimation of a VAR(4) for $(\Delta y, h)$

$$\hat{\mu} = \begin{bmatrix} 0.0860 \\ -0.4069 \end{bmatrix}$$

$$\hat{\beta}_1 = \begin{bmatrix} -0.0678 & -0.1172 \\ 0.3878 & 1.0844 \end{bmatrix}, \hat{\beta}_2 = \begin{bmatrix} 0.0769 & 0.0560 \\ 0.1861 & 0.0457 \end{bmatrix}, \hat{\beta}_3 = \begin{bmatrix} -0.0016 & -0.0398 \\ 0.0935 & -0.0770 \end{bmatrix}, \hat{\beta}_4 = \begin{bmatrix} 0.0517 & 0.1114 \\ 0.0202 & -0.1047 \end{bmatrix}$$

4.7 Structural IRF Point Estimation and 68% Bootstrap bands for $(\Delta y, h)$

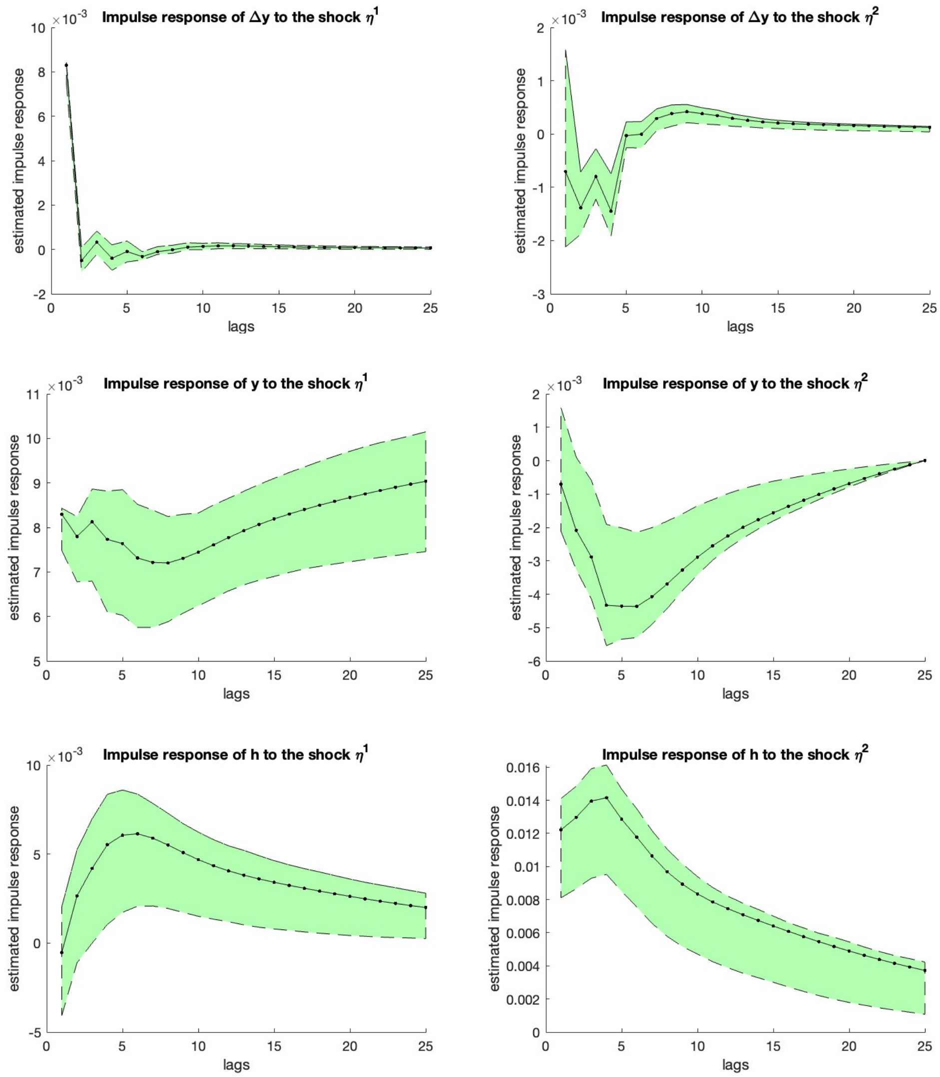


Figure 4: Point estimate of the Structural and cumulative IRF and of the respective 68% approximate confidence intervals obtained from $(\Delta y, h)^T$

Even in this case we can confirm what the theory, the paper made by Christiano, says: to a positive technological shock has a big and positive effect in per-capita hours worked.

4.8 Percentage variance explained by the aggregate technological shock for $(\Delta y, h)$

From 10 we estimate:

$$v_{11} = \frac{\sum_{j=0}^{25} \hat{B}^2_{j[11]}}{Var(Y^1_t)} = 0.9222 \quad \text{and} \quad v_{21} = \frac{\sum_{j=0}^{25} \hat{B}^2_{j[21]}}{Var(Y^2_t)} = 0.1187 \tag{14}$$

5 Conclusion

The two studies show opposite results: Galí (1999) suggests a negative labor supply response to a technology shock, in line with New Keynesian theory, while Christiano (2003) implies a positive response. Caution is needed as technical issues in unit root identification and economic modeling choices can impact results. Minor differences in identification methods lead to major discrepancies, indicating mixed evidence.

References

[1] Olivier Jean Blanchard and Danny Quah. The Dynamic Effects of Aggregate Demand and Supply Disturbances. *American Economic Review*, 79(4):655–673, September 1989.

[2] Lawrence J Christiano, Martin Eichenbaum, and Robert Vigfusson. What happens after a technology shock? Working Paper 9819, National Bureau of Economic Research, July 2003.

[3] Jordi Gali. Technology, Employment, and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations? *American Economic Review*, 89(1):249–271, March 1999.