Automatic Control course project

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1 INTRODUCTION

This project aims to stabilize the rotary inverted pendulum system at its zero position and zero velocity configuration utilizing linear control techniques. The process involves analyzing and linearizing the system's dynamics, followed by designing and evaluating controllers through simulation to assess their performance.

2 SYSTEM

The rotary inverted pendulum consists of a pendulum and a rotary arm connected by a revolute joint, and a servo motor, which controls the angular position of the rotary arm.
The system is described using the following equation:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + f_v(\dot{q}) + G(q) = \tau, \tag{1}$$

where $q = [\theta \;\; \alpha]^T$, and the terms $M(q)$, $C(q,\dot{q})$, $f_v(\dot{q})$, $G(q)$, and τ are defined as follows:

$$M(q) = \begin{bmatrix} J_r + m_p(L_r^2 + l_p^2(1-\cos^2(\alpha))) & m_p l_p L_r \cos(\alpha) \\ m_p l_p L_r \cos(\alpha) & J_p + m_p l_p^2 \end{bmatrix}$$

$$C(q,\dot{q}) = \begin{bmatrix} 2m_p l_p^2 \dot{\alpha} \sin(\alpha)\cos(\alpha) & -m_p l_p L_r \dot{\alpha} \sin(\alpha) \\ -m_p l_p^2 \dot{\theta} \sin(\alpha)\cos(\alpha) & 0 \end{bmatrix}$$

$$f_v(\dot{q}) = \begin{bmatrix} B_r \dot{\theta} \\ B_p \dot{\alpha} \end{bmatrix}$$

$$G(q) = \begin{bmatrix} 0 \\ -m_p l_p g \sin(\alpha) \end{bmatrix}$$

$$\tau = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

3 QUESTION 1

3.1 Linearization

As a first step, I want to linearise the equation of dynamics around the origin $x = [0\ 0\ 0\ 0]^T$, and then I will write the linearized system in the state space representation $\dot{x} = Ax + Bu$.
The linearized matrices A and B obtained are defined, using in matlab the fuction ***jacobian*** that requires the state and the equilibria value of the state. The result obtained is:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5.5880 & -17.7215 & 15.1899 \\ 0 & 30.4234 & 7.5949 & -82.7004 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 88.6076 \\ -37.9747 \end{bmatrix}$$

And the lynearized system in the space representation is:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5.5880 & -17.7215 & 15.1899 \\ 0 & 30.4234 & 7.5949 & -82.7004 \end{bmatrix} \begin{bmatrix} \theta \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 88.6076 \\ -37.9747 \end{bmatrix} \mathrm{u}$$

3.2 Stability Analysis

Once the representation of the linearised system has been obtained, its stability is assessed. This is done by analysing the eigenvalues of the matrix A. These have the following values:

$$\begin{bmatrix} \lambda_1 = 0 & \lambda_2 = -84.78690 & \lambda_3 = 0.3661 & \lambda_4 = -16.0011 \end{bmatrix}$$

Since not all eigenvalues have negative real parts, the system is unstable.

3.3 Controllability

The controllability matrix C defined as follows was used to study controllability:

$$C = \begin{bmatrix} B & BA & BA^2 & .. & BA^{(n-1)} \end{bmatrix}$$

To have a fully controllable system, the rank of the C-matrix must be maximum. In my case the resulting matrix is full rank so the system is **CONTROLLABLE**.

4 QUESTION 2

In order to stabilize my system, it was therefore decided to design a feedback loops in two ways. The first one with a gain matrix such that the closed-loop system $\dot{x} = A + BK_1$ has a convergence rate $\alpha = 2$. The second gain matrix (K_2) still holding the same convergence rate but with a minimum K norm.

4.1 K1 with convergence rate $\alpha = 2$

To find the first value, we must first set the following constraints, which are necessary to find the gain matrix.

$$\begin{cases} W \geq 0 \\ \mathrm{He}(AW + Bx) \leq -2\alpha W \end{cases}$$

With He(.) the Hermian operator defined as:$He(A) = A + A^T$
After setting the settings for the solver *opts = sdpsettings('solver', 'mosek', 'verbose', 0)*, we can use the **solvesdp** command to solve the constraints and find the gain matrix. This matrix ensure that the system become stable, this can be seen from the eigenvalues of the matrix $A + BK_1$, which are all negative.

$$K_1 = \begin{bmatrix} 8.6535 & 5344.9781 & 29.6749 & 66.9883 \end{bmatrix}$$

4.2 K2 with minimum norm and convergence rate $\alpha = 2$

In order to find the second gain matrix, i.e. with the minimum norm, a further constraint on the bound of K must be added, i.e. the third constraint that can be seen below.

$$\begin{cases} W \geq I_n \\ \mathrm{He}(AW + Bx) \leq -2\alpha W \\ \begin{bmatrix} kI_n & X^\top \\ X & kI_p \end{bmatrix} > 0 \end{cases}$$

With He(.) the Hermian operator. The gain matrix K2 that satisfy all the constraints and make the system stable is:

$$K_2 = \begin{bmatrix} 5.2566 & 3060.3077 & 17.0774 & 39.9657 \end{bmatrix}$$

4.3 SIMULATION

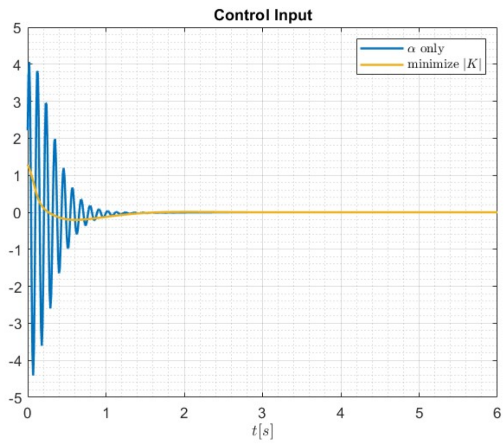
After creating a function for non-linear dynamics and setting the simulation parameters, which we recall are:
- simulation of 6 seconds defined with **simT = 0:0.01:6;**;

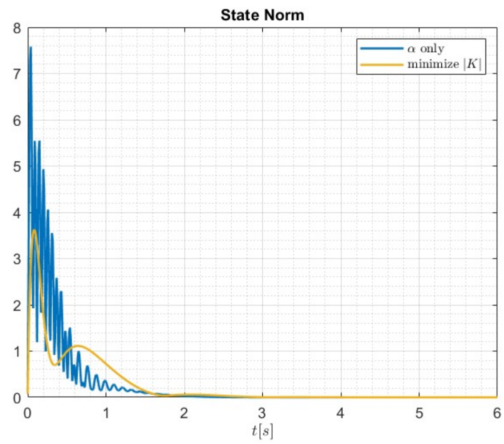
- solver **ode45** that solve nonstiff differential equations;
- initial condition

$$x(0) = \begin{bmatrix} \theta(0) \\ \alpha(0) \\ \dot{\theta}(0) \\ \dot{\alpha}(0) \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0 \\ 0.06 \\ 0 \end{bmatrix};$$

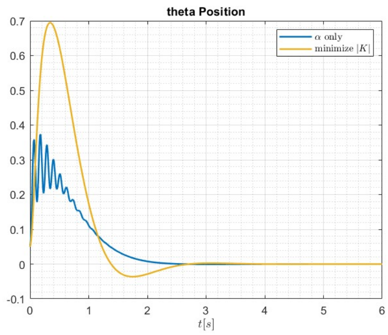
- the input of the simulation is given by **u = Kix, with i = 1,2**;
we can now simulate.
Below is the line of simulation code for the first gain matrix and the plots:

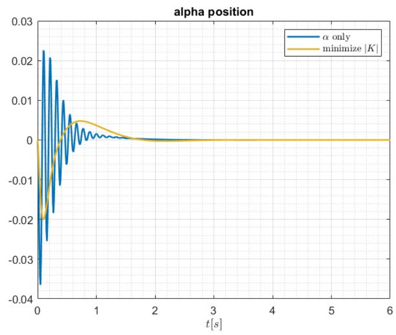
$$[\text{simT\_c}, \text{simQ\_c}] = \text{ode45}(@(t,q) f\_\text{NLDyna}(q, J_r, m_p, L_r, l_p, J_p, B_r, B_p, g, K1 \cdot q), \text{simT}, q_0);$$





After collecting the data from the simulation, I will plot the state norm and control input. It is clear from the following graphs that the K2 control is better than the K1 control. K2 ensure that the system has fewer oscillations, which for a mechanism is ideal as oscillations can damage the functioning of the system. We can notice also that the response of K2 has smoother convergence than K1. If I had to choose which of the two controls to implement I would choose K2.





Plotted above are the α and θ values obtained during the 6-second simulation.

5 QUESTION 3

5.1 OVERSHOOT

In order to calculate the overshoot M corresponding to the various closed loop matrices for K1 and K2, an optimization is used which uses the following constraints to find an minimised value of $\bar{M} = \sqrt{\bar{K}}$.

$$\bar{K} = \min_{K} \quad \text{s.t.} \quad \begin{cases} I \leq P \leq KI \\ A^T P + PA \leq -2\alpha P \end{cases}$$

$$\text{with} \quad K > 0, \quad P = P^T$$

Using this opitimaztion results in a **infeasible** solution.

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Estimating overshoot M
The problem is infeasible
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To plot an exponential bound, I therefore decided to find initial values of M, which as pointed out in class are not overshoots, this is only done to show a graphical solution. The value of M for the first control K_1

$$M_b = \sqrt{\frac{\lambda_M}{\lambda_m}} = 36173.30$$

With λ_M the largest eigenvalue and λ_m the smallest of the matrix P_b the inverse of the matrix W, that appears in the constraints of the first control

formulation.
The value of M for the second control K_2

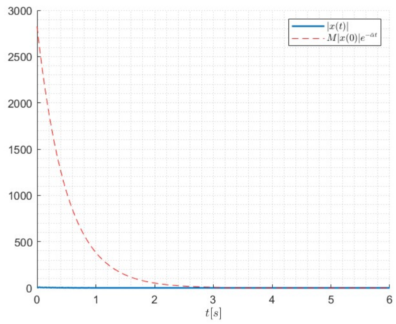
$$M_c = \sqrt{\bar{k}} = 4083.9$$

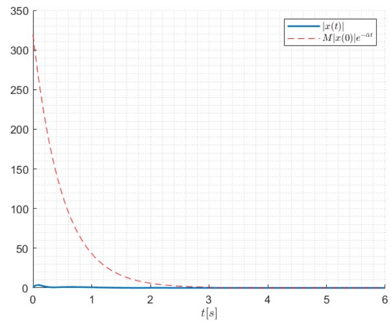
The value of $\bar{k}$ is obtained from the k that appears in the set of constraints of the second control formulation.

5.2 EXPONENTIAL BOUND

After plotting the trajectories, using the value of M obtained from the formula below, the exponential bound is shown in the same plot.

$$Exp.Bound = M|x(0)|e^{-\bar{\alpha}t}$$





Respectively, the first plot refers to the value of M found with K1, the second to the value of M found with K2.
In general, the bound obtained with the linearized model typically does not apply to non-linear systems. However, in this particular case, my system is operating in a condition close to the equilibrium point where linearization was performed. Given that my linearization includes sine and cosine functions, the assumption of small angle values ensures that the exponential bounds derived from the linearized system are likely to be valid for the actual non-linear system under these operating conditions.