# The Art of Realizability

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**Abstract**

This work presents the correspondance between a ultrafilter of the ground model which naturally arises from realizability algebras and the ultrafilter which generates the realizability model.

## 1 Context

Griffin [4] enables to extend Curry-Howard correspondence to classical logic. Indeed, in [4] it has been shown that Pierce's Law $(\neg A \to A) \to A$ is an admissible type for `call/cc`. This has been a major achievement in the constructivization of mathematics, since proof-programs correspondence was hitherto limited to intuitionistic logic. About this latter, in 1945 Kleene [5] exposed a method, namely realizability, which employed recursive functions as witnesses for the satisfiability of a formula for a fixed language, which eventually was a generalisation of intuitionistic semantics. Combining Griffin's result with Kleene's tradition, Krivine [6] has developed realizability for classical set theory, introducing a structure called *realizability algebra*.

**Definition 1.** A realizability algebra is a tuple $\mathcal{A} = \langle \Lambda_{\mathsf{c}}, \Pi, \succ, \perp\!\!\!\perp \rangle$, where:

- $\Lambda_{\mathsf{c}}$ is the set of terms genereted by

$$t \; := \; x \,|\, \mathsf{cc} \,|\, \lambda x.t \,|\, (t)t \,|\, k_\pi$$

$$\pi \in \Pi$$

  with $k_\pi$ *continuation* of $\pi$;

- $\Pi$ is the set of *stacks* genereted by

$$\pi \; := \; \pi_0 \,|\, t \centerdot \pi$$

$$\pi_0 \in \Pi_0, t \in \Lambda_{\mathsf{c}}$$

  for a fixed set $\Pi_0$ of *stack constants*;

- $\succ$ is a reduction relation $\langle t, \pi \rangle \succ \langle u, \rho \rangle$ between *processes* $\langle t, \pi \rangle, \langle u, \rho \rangle$, for $t, u \in \Lambda_{\mathsf{c}}, \pi, \rho \in \Pi$. For sake of readability (and history), we note $t \star \pi$ for a process $\langle t, \pi \rangle$. $\succ$ is defined as the transitive closure of following rules

$$\lambda x.t \star u \centerdot \pi \succ t[u/x] \star \pi, \tag{grab}$$
$$(t)u \star \pi \succ t \star u \centerdot \pi, \tag{push}$$
$$\mathsf{cc} \star t \centerdot \pi \succ t \star k_\pi \centerdot \pi, \tag{save}$$
$$k_\pi \star t \centerdot \rho \succ t \star \pi; \tag{restore}$$

- $\perp\!\!\!\perp$, called *pole*, is a fixed subset of processes closed for anti-reduction, i.e., for $t, u \in \Lambda_{\mathsf{c}}, \pi, \rho \in \Pi$,

$$t \star \pi \succ u \star \rho,\, u \star \rho \in \perp\!\!\!\perp \implies t \star \pi \in \perp\!\!\!\perp .$$

$\perp\!\!\!\perp$ induces a notion of orthogonality between sets of $\lambda_{\mathsf{c}}$-terms and sets of stacks. Then, for any set of stacks $X \subseteq \Pi$ we can define

$$X^{\perp\!\!\!\perp} \; := \; \{t \in \mathsf{Q} \,|\, \forall \pi \in X (t \star \pi \in \perp\!\!\!\perp)\}.$$

In order to have a consistent set of realized formulæ, fix a set $\mathrm{Q}$ of $\Lambda_{\mathsf{c}}$, such that $\mathsf{cc} \in \mathrm{Q}, \Lambda^{\circ} \subset \mathrm{Q}$ and $\forall t \in \mathrm{Q}\, \exists \pi \in \Pi(t \star \pi \notin \perp\!\!\!\perp)$[1]. $\mathrm{Q}$ is the set of *quasi-proofs.* For any set $X \subseteq \Pi$, $X$ is *realized* $X^{\perp\!\!\!\perp} \cap \mathrm{Q} \neq \emptyset$. For any formula $F$ of a language $\mathcal{L}$, a set $||F|| \subset \Pi$ is associated to it, and $F$ is said to be realized if $(\mathrm{Q} \cap ||F||^{\perp\!\!\!\perp}) \neq \emptyset$, where $||F||^{\perp\!\!\!\perp} \subseteq \Lambda_{\mathsf{c}}$ is the orthogonal of $||F||$ with the respect to $\perp\!\!\!\perp$. A term $t \in \Lambda_{\mathsf{c}}$ is a *realizer* for $F$, in symbols $t \Vdash F$, if $t \in (||F||^{\perp\!\!\!\perp} \cap \mathrm{Q})$. We denote $\Vdash F$ if there exists a realizer for $F$.

This new application has therefore produced set-theoretical models that supply programs associated to ZF-theorems.

During last twenty years, Krivine has realized relevant mathematical principles, like the Axiom of Dependant Choice ($\mathrm{DC}_{\aleph_0}$), which can be viewed as a further extension of proof-programs correspondence to (at least) real analysis. Recent developments have extended even further this correspondence, realizing choice principles on arbitrary cardinals and large cardinals axioms (in [2, 3]). Nowadays, realizability stands as a well-grounded technique, which enables to built ZF-models encompassing a “constructive behaviour". For instance, while the formulæ $\top \wedge \perp \to \perp$ and $\perp \wedge \top \to \perp$ can be considered the same one in a classical set-up, from a computational point of view these have a slightly different meaning, the former behaving has a right projection, the latter as a left one, hence they are not realized by the same program in general. In fact, assuming the existence of such a program $\pitchfork$, verifying $\pitchfork \Vdash \top \wedge \perp \to \perp, \pitchfork \Vdash \perp \wedge \top \to \perp$, introduces non-deterministic processes in the underlying calculus. Furthermore, it turns out that $\pitchfork$ transforms the realizability model in a forcing one. Forcing technique, a wide-spread tool of modern set-theory developed by Cohen [1] in 1963, can be viewed as a special case of realizability, where every formula is realized by the same program, thus it is considered as a trivialization of realizability.

## 2 Renovating realizability

We present an improved formalism for Krivine's realizability, developed by Fontanella, Geoffroy and Matthews (in [2, 3, 7]) which strengthens this framework with a forcing-like definition of the realizability model.

For a fixed model $\mathcal{M}$ of ZF, the realizability model $\mathcal{N}$ generated by a realizability algebra $\mathcal{A} \in \mathcal{M}$ is a first-order model satisfying formulæ of $\mathcal{L} = \{\not\varepsilon, \notin, \subseteq\}$. $\mathcal{L}$ is a slight modification of set theory signature, due to technical reasons, which defines a conservative extension of ZF, namely $\mathrm{ZF}_{\varepsilon}$

$$\mathrm{ZF}_{\varepsilon} := \left\{ \begin{array}{rcl|c} \in\text{-}\mathit{Extensionality} & \equiv & \forall x \forall y (x \in y \leftrightarrow \exists z \,\varepsilon\, y (x \simeq z)); & \\ \subseteq\text{-}\mathit{Extensionality} & \equiv & \forall x \forall y (x \subseteq y \leftrightarrow \forall z \,\varepsilon\, x (z \in y)); & \\ \mathit{Foundation} & \equiv & \forall \vec{x} \forall a (\forall x (\forall y \,\varepsilon\, x\, F(y, \vec{x}) \to F(x, \vec{x})) \to F(a, \vec{x})); & \\ \mathit{Comprehension} & \equiv & \forall \vec{x} \forall a \exists b \forall x (x \,\varepsilon\, b \leftrightarrow (x \,\varepsilon\, a \wedge F(x, \vec{x}))); & \\ \mathit{Paring} & \equiv & \forall a \forall b \exists c (a \,\varepsilon\, c \wedge b \,\varepsilon\, c); & F \in \mathcal{F}_{\mathcal{L}} \\ \mathit{Union} & \equiv & \forall a \exists b \forall x \,\varepsilon\, a \forall y \,\varepsilon\, x (y \,\varepsilon\, b); & \\ \mathit{Power\ Set} & \equiv & \forall a \exists b \forall x \exists y \,\varepsilon\, b \forall x (z \,\varepsilon\, y \leftrightarrow (z \,\varepsilon\, a \wedge z \,\varepsilon\, x)); & \\ \mathit{Collection} & \equiv & \forall \vec{x} \forall a \exists b \forall x \,\varepsilon\, a (\exists y\, F(x, y, \vec{x}) \to \exists y \,\varepsilon\, b\, F(x, y, \vec{x})); & \\ \mathit{Infinity} & \equiv & \forall \vec{x} \forall a \exists b (a \,\varepsilon\, b \wedge (\exists y\, F(x, y, \vec{x}) \to \exists y \,\varepsilon\, b\, F(x, y, \vec{x}))) & \end{array} \right\}.$$

$\notin$ is the negation of membership, $\subseteq$ is the subset relation. $\not\varepsilon$ is the negation of a non-extensional membership relation. Extensional equality, defined as usual by use of $\subseteq$, is denoted as $\simeq$[2]. Non-extensional equality is denoted as $=$[3].

Since the realizability relation is defined inductively on the structure of formulæ, it suffices to well-define it for atomic formulæ of the language. For this purpose, we introduce *names.*

**Definition 2.** The class of $\mathcal{A}$-names is defined inductively as

- $M_0^{\mathcal{A}} := \emptyset$;

[1] We denote $\Lambda^{\circ}$ the set of closed $\lambda$-terms. The last condition is necessary to obtain a model. Observe that for $\perp\!\!\!\perp \neq \emptyset$, $\mathrm{Q} \neq \Lambda_{\mathsf{c}}$. Indeed, if $t \star \pi \in \perp\!\!\!\perp$, for any $\rho \in \Pi$, $(k_{\pi})t \star \rho \succ t \star \pi \Longrightarrow (k_{\pi})t \star \rho \in \perp\!\!\!\perp$, thus $(k_{\pi})t \notin \mathrm{Q}$.

[2] $x \simeq y \;:=\; x \subseteq y \wedge y \subseteq x$

[3] $x = y \;:=\; \forall z (x \not\varepsilon\, z \longleftrightarrow y \not\varepsilon\, z)$

- $M^{\mathcal{A}}_{\alpha+1} := \wp(M^{\mathcal{A}}_{\alpha} \times \Pi)$, for $\alpha \in Ord$;
- $M^{\mathcal{A}}_{\lambda} := \bigcup_{\alpha\in\lambda} \wp(M^{\mathcal{A}}_{\alpha} \times \Pi)$, for $\lambda$ limit ordinal;
- $M^{\mathcal{A}} := \bigcup_{\alpha\in Ord} M^{\mathcal{A}}_{\alpha}$.

Names allow to interpret closed $\mathcal{L}$-formulæ into $\wp(\Pi)$. Indeed, we will define by induction on the structure of $F \in \mathcal{F}_{\mathcal{L}}$ the set of *falsity values* $||F|| \subseteq \wp(\Pi)$. Names play a fundamental role in the atomic-formulæ cases and the universal-quantifier case. In order to define $||F||$, a definition of *rank* for names is needed.

**Definition 3.** For every $a \in M^{\mathcal{A}}$, we define the rank of $a$ in $M^{\mathcal{A}}$ as

$$\mathrm{rank}^{\mathcal{A}}(a) = \min\{\alpha \in Ord | a \in M^{\mathcal{A}}_{\alpha+1}\}.$$

**Definition 4.** We define $||a \not\varepsilon b|| := \{\pi \in \Pi \,|\, \langle a, \pi\rangle \in b\}$ for every $a, b \in M^{\mathcal{A}}$. Moreover, by induction on $\langle \mathrm{rank}^{\mathcal{A}}(a), \mathrm{rank}^{\mathcal{A}}(b)\rangle$, we set:

- $||a \notin b|| := \bigcup_{c\in M^{\mathcal{A}}}\{t \centerdot t' \centerdot \pi \in \Pi \,|\, \langle c, \pi\rangle \in b, t \Vdash c \subseteq a, t' \Vdash a \subseteq c\}$
- $||a \subseteq b|| := \bigcup_{c\in M^{\mathcal{A}}}\{t \centerdot \pi \in \Pi \,|\, \langle c, \pi\rangle \in a, t \Vdash c \notin b\}$

The set of falsity values $||F|| \subseteq \wp(\Pi)$ for a formula $F \in \mathcal{F}_{\mathcal{L}}$ is defined by induction of the structure of $F$:

- $||\top|| := \emptyset$, $||\bot|| := \Pi$;
- atomic cases as above;
- $||G_1 \to F_2|| := \{t \centerdot \pi \in \Pi \,|\, t \in ||F_1||^{\perp\!\!\!\perp}, \pi \in ||F_2||\}$;
- $||\forall x F(x)|| := \bigcup_{a\in M^{\mathcal{A}}} ||F(a)||$.

Following Definition 1, It is possible to associate a realizer $t \in ||F||^{\perp\!\!\!\perp}$ for any formula $F$ - if it exists. As expected, ZF-axioms are realized (see [6]).

$M^{\mathcal{A}}$ allows to define basic objects of $\mathcal{N}$ in a more explicit way, consequently it represents an improvement in the comprehension of pre-existing results. A class of canonical representatives for elements of the ground model is defined, denoted as $\daleth M$[4].

**Definition 5.** By induction on $\rho(a)$ we define $\daleth(a) = \{\langle\daleth(b), \pi\rangle \,|\, b \in a, \pi \in \Pi\}$. We denote $\daleth(M) = \{\daleth(a) | a \in M\}$.

Among the elements of $\daleth M$, $\daleth 2 = \{\langle\daleth \mathrm{b}, \pi\rangle \,|\, \mathrm{b} = 0, 1; \pi \in \Pi\}$ turns out to have a relevant role, as the canonical representative of 2 in $\mathcal{N}$ may contain arbitrary copies of 0 and 1, distinguished by the relation $\varepsilon$ introduced with the language $\mathcal{L}$ above. The cardinality of $\daleth 2$ is strictly related with realizability model as $\Vdash \forall x\,\varepsilon\,\daleth 2(x = \daleth 0 \vee x = \daleth 1)$ if, and only if, $\mathcal{N}$ is a forcing model. The left-to-right implication holds if one assumes that an instruction `quote` is in $\Lambda_{\mathsf{c}}$, this term acting like an enumerator for closed $\lambda_{\mathsf{c}}$-terms. Thus, $\daleth 2$ allows to establish whether $\mathcal{A}$ produces a forcing model or not.

In order to get more information about the nature of $\mathcal{N}$, it is possible to add a boolean-algebra structure on $\daleth 2$, induced by the minimal boolean algebra $\langle 2, \leq 0, 1\rangle$ in $\mathcal{M}$, to fix a *complete* theory containing the one of realized formulæ. Let $\langle\daleth 2, \widetilde{\leq}, \daleth 0, \daleth 1\rangle$ be the induced algebra. With respect to the extensional equality $\simeq$, it is the minimal boolean-algebra of $\mathcal{N}$. $\langle\daleth 2, \widetilde{\leq}, \daleth 0, \daleth 1\rangle$ is a powerful tool to investigate the structure of the realizability model. Indeed, $\perp\!\!\!\perp$-orthogonality induces on the powerset of $\Pi$ a preorder $\leq$ defined as:

**Definition 6.** For any $X, Y \in \wp(\Pi)$, $X \leq Y$ if, and only if, $\mathsf{Q} \cap (X \to Y)^{\perp\!\!\!\perp} \neq \emptyset$ (or $\Vdash X \to Y$), where

$$X \to Y := \{t \centerdot \pi \in \Pi \,|\, t \in X^{\perp\!\!\!\perp}, \pi \in Y\}.$$

[4] It is a subclass of $M^{\mathcal{A}}$.

The induced poset has a boolean-algebra structure $\langle \wp(\Pi), \leq, \Pi, \emptyset\rangle$, which can be related via representatives $u_X \in M^{\mathcal{A}}$, for $X \in \wp(\Pi)$, to the boolean algebra on $\daleth 2$.

**Definition 7.** For $X \in \wp(\Pi)$, $u_X := \{\langle \daleth 0, \pi\rangle \,|\, \pi \in X\}$.

**Theorem 1.** *The following results hold:*

1. *For any* $X \in \wp(\Pi)$*,* $\mathcal{N} \models u_X \simeq \daleth 0 \vee u_X \simeq \daleth 1$*;*
2. *For any formula* $F$ *of* $\mathcal{L}$*,* $\Vdash F$ *if, and only if,* $\mathcal{N} \models u_{||F||} \simeq \daleth 0$*;*
3. $\mathfrak{G} := \{X \in \wp(\Pi) \,|\, \mathcal{N} \models u_X \simeq \daleth 0\}$ *is an ultrafilter of* $(\wp(\Pi), \leq)$*;*

The theorem states that (*1.*) any $u_X$ is extensionally equal either to $\daleth 0$ or to $\daleth 1$, i.e. it is contained in $\daleth 2$[5], (*2.*) formulæ whose falsity value is sent to $\daleth 0$ in $\mathcal{N}$ are precisely those that are realized, (*3.*) these formulæ generate a filter in $\wp(\Pi)$ which is contained in a ultrafilter $\mathfrak{G}$ of $\wp(\Pi)$, thus $\mathfrak{G}$ determines a complete theory containing every realized formula.

*Proof. 1.* Fix $X \in \wp(\Pi)$.
First, we show that $\lambda x.x \Vdash u_X \subseteq_\varepsilon \daleth 1$[6].
$||u_X \subseteq_\varepsilon \daleth 1|| = \bigcup_{c\in M^{\mathcal{A}}} ||c \not\varepsilon \daleth 1 \longrightarrow c \not\varepsilon u_X|| = \bigcup_{c\in M^{\mathcal{A}}} \{t \centerdot \pi \,|\, t \in ||c \not\varepsilon \daleth 1||^{\perp\!\!\!\perp}, \pi \in ||c \not\varepsilon u_X||\}$.
For $c \neq \daleth 0$, $||c \not\varepsilon u_X|| = \emptyset$, thus

$$\begin{aligned} ||u_X \subseteq_\varepsilon \daleth 1|| &= \{t \centerdot \pi \,|\, t \in ||\daleth 0 \not\varepsilon \daleth 1||^{\perp\!\!\!\perp}, \pi \in ||\daleth 0 \not\varepsilon u_X||\} \\ &= \{t \centerdot \pi \,|\, t \in ||\bot||^{\perp\!\!\!\perp}, \pi \in X\} \\ &= \Pi \to X \end{aligned}$$

which is realized by the identity $\lambda x.x$.

Next, we show that $\Vdash \daleth 0 \,\varepsilon\, c \longrightarrow \daleth 1 \subseteq c$ for any name $c \in M^{\mathcal{A}}$. Consider falsity values associated with the formula

$$||\daleth 0 \,\varepsilon\, c \longrightarrow \daleth 1 \subseteq c|| = \{t \centerdot u \centerdot \pi \,|\, t \in ||\daleth 0 \,\varepsilon\, c||^{\perp\!\!\!\perp}, u \in ||\daleth 0 \notin c||^{\perp\!\!\!\perp}, \pi \in \Pi\}.$$

Let $t \in ||\daleth 0 \,\varepsilon\, c||^{\perp\!\!\!\perp} = ||(\daleth 0 \not\varepsilon c) \to \bot||^{\perp\!\!\!\perp}$. For any $\lambda_c$-term $u \in ||\daleth 0 \notin c||^{\perp\!\!\!\perp}$, $uWW \in ||\daleth 0 \not\varepsilon c||^{\perp\!\!\!\perp}$,[7] thus $t(uWW) \in ||\bot||^{\perp\!\!\!\perp}$ for $t \in ||\daleth 0 \,\varepsilon\, c||^{\perp\!\!\!\perp}$. Then, $\lambda xy.x(yWW) \Vdash \daleth 0 \,\varepsilon\, c \longrightarrow \daleth 1 \subseteq c$, in particular for $c = u_X$.

Observe that for $\pi \in ||\daleth 0 \not\varepsilon u_X||, k_\pi \in ||\daleth 0 \,\varepsilon\, u_X||^{\perp\!\!\!\perp}$. Consider now

$$||\forall x(u_X \not\simeq \daleth 1 \longrightarrow x \not\varepsilon u_X)|| = \{t \centerdot \pi \,|\, t \in ||(u_X \subseteq \daleth 1 \wedge \daleth 1 \subseteq u_X) \longrightarrow \bot||^{\perp\!\!\!\perp}, \pi \in X\};$$

as just shown, $I \Vdash u_X \subseteq \daleth 1$; moreover, from the discussion above, $(\lambda xy.x(yWW))k_\pi \in ||\daleth 1 \subseteq u_X||^{\perp\!\!\!\perp}$ for any $\pi \in X$, thus $tI(\lambda xy.x(yWW))k_\pi \in ||\bot||^{\perp\!\!\!\perp}$. This shows that

$$\lambda x.\mathsf{cc}\lambda k.xI(\lambda xy.x(yWW))k \Vdash \forall x(u_X \not\simeq \daleth 1 \longrightarrow x \not\varepsilon u_X)$$

and implies that there exists a realizer for $u_X \not\simeq \daleth 1 \longrightarrow \forall x(x \not\varepsilon u_X)$.

Lastly, it suffices to show that $\Vdash \forall x(\forall y(y \not\varepsilon x) \longleftrightarrow x \simeq \daleth 0)$. The formula can be reduce to $\forall x(\forall y(y \not\varepsilon x) \longleftrightarrow x \subseteq \daleth 0)$.
We first show the left-to-right implication. $||\forall x(\forall y(y \not\varepsilon x) \longrightarrow x \subseteq \daleth 0)|| = \bigcup_{c\in M^{\mathcal{A}}} \{t \centerdot \pi \,|\, t \in ||\forall y(y \not\varepsilon c)||^{\perp\!\!\!\perp}, \pi \in ||c \subseteq \daleth 0||\}$. Let $t \centerdot \pi \in ||\forall x(\forall y(y \not\varepsilon x) \longrightarrow x \subseteq \daleth 0)||$ for a fixed $c \in M^{\mathcal{A}}$. Then,

$$\pi = u \centerdot \rho \in \bigcup_{d\in M^{\mathcal{A}}} \{u \centerdot \rho \,|\, u \in ||d \notin \daleth 0||^{\perp\!\!\!\perp}, \pi \in ||d \not\varepsilon c||\}.$$

[5] This is not the case from the non-extensional point of view: for $X \neq \Pi, \mathcal{N} \models (u_X \neq \daleth 0 \wedge u_X \neq \daleth 1)$.
[6] $x \subseteq_\varepsilon y := \forall z(z \not\varepsilon y \to z \not\varepsilon x)$. It is easy to show that $\mathrm{ZF}_\varepsilon \vdash \forall x, y(x \subseteq_\varepsilon y \to x \subseteq y)$
[7] $Y := (\lambda x\lambda y.(y)(x)xy)\lambda x\lambda y.(y)(x)xy$, $W := (Y)\lambda x\lambda y.(y)xx$. It is easy to see that for any $c \in M^{\mathcal{A}}$, $W \Vdash c \subseteq c$, then $\lambda x.xWW \Vdash c \simeq c$.

Since $t \star \rho \in \perp\!\!\!\perp$, $\lambda xy.x \Vdash \forall x(\forall y(y \not\varepsilon x) \longrightarrow x \subseteq \daleth 0)$. We show the left-to-right implication. $||\forall x(x \subseteq \daleth 0 \longrightarrow \forall y(y \not\varepsilon x))|| = \bigcup_{c \in M^{\mathcal{A}}}\{t \,.\, \pi \,|\, t \in ||c \subseteq \daleth 0||^{\perp\!\!\!\perp}, \pi \in ||\forall y(y \not\varepsilon c)||\}$. Fix such a falsity value $t \,.\, \pi$. $t \in ||\forall y(y \notin \daleth 0 \longrightarrow y \not\varepsilon c)||^{\perp\!\!\!\perp}$. It is easy to see that $||d \notin \daleth 0|| = \emptyset$ for any $d \in M^{\mathcal{A}}$, hence any $u \in \mathrm{Q}$ realizes $d \notin \daleth 0$, which implies $tu \in ||d \not\varepsilon c||^{\perp\!\!\!\perp}$. Without loss of generality, suppose $u = I$. Then, $tI \in ||\forall y(y \not\varepsilon c)||^{\perp\!\!\!\perp}$, which proves $\lambda x.xI \Vdash \forall x(x \subseteq \daleth 0 \longrightarrow \forall y(y \not\varepsilon x))$.

To conclude, we showed that for any $X \in \Pi$

$$\Vdash u_X \not\simeq \daleth 1 \longrightarrow \forall x(x \not\varepsilon u_X),$$

$$\Vdash \forall x(x \not\varepsilon u_X) \longleftrightarrow u_X \simeq \daleth 0,$$

which implies that

$$\Vdash u_X \not\simeq \daleth 1 \longrightarrow u_X \simeq \daleth 0.$$

*2.* For any closed formula $F \in \mathcal{F}_{\mathcal{L}}$ with parameters in $M^{\mathcal{A}}$

$$\Vdash F \text{ iff } \exists t \in \mathrm{Q}\, \forall \pi \in ||F||(t \star \pi \in \perp\!\!\!\perp) \text{ iff } \exists t \in \mathrm{Q}\,(t \Vdash \daleth 0 \not\varepsilon u_{||F||}) \text{ iff}$$
$$\text{iff } \exists t \in \mathrm{Q}\,(t \Vdash \forall x(x \not\varepsilon u_{||F||})) \text{ iff } \Vdash u_{||F||} \simeq \daleth 0$$

*3.* Consider $\mathfrak{G} = \{X \in \wp(\Pi) \,|\, \mathcal{N} \models u_X \simeq \daleth 0\}$.

- $\mathfrak{G}$ *is upward closed for* $\leq$. Let $X \in \mathfrak{G}, Y \in \wp(\Pi), X \leq Y$. By hypothesis, there exists $u \Vdash \forall x(x \not\varepsilon u_X), t \Vdash X \longrightarrow Y$, the latter equivalent to $t \Vdash \forall x(x \not\varepsilon u_X) \longrightarrow \forall x(x \not\varepsilon u_Y)$. Thus, $(t)u \Vdash \forall x(x \not\varepsilon u_Y)$.
- $\mathfrak{G}$ *is closed for meets.* Fix $X, Y \in \mathfrak{G}$. Then, $||\forall x(x \not\varepsilon u_{X \wedge Y})|| = ||\daleth 0 \not\varepsilon u_{X \wedge Y})|| = X \wedge Y$[8]. By hypothesis, there exists $t, u \in \mathrm{Q}$ such that $t \Vdash X, u \Vdash Y$, thus $\lambda x.xtu \Vdash \forall x(x \not\varepsilon u_{X \wedge Y})$.
- $\mathfrak{G}$ *is maximal.* Consider $X \in \wp(\Pi)$. We show that $\Vdash (u_X \simeq \daleth 0 \longrightarrow \perp) \longleftrightarrow u_{X \to \perp} \simeq \daleth 0$, which is equivalent to $\Vdash (\forall x(x \not\varepsilon u_X) \longrightarrow \perp) \longleftrightarrow \forall x(x \not\varepsilon u_{X \to \perp})$. Observe that

$$\begin{aligned}
||\forall x(x \not\varepsilon u_X) \longrightarrow \perp|| &= \{t \,.\, \pi \,|\, t \in ||\forall x(x \not\varepsilon u_X)||^{\perp\!\!\!\perp}, \pi \in \Pi\} \\
&= \{t \,.\, \pi \,|\, t \in ||\daleth 0 \not\varepsilon u_X||^{\perp\!\!\!\perp}, \pi \in \Pi\} \\
&= \{t \,.\, \pi \,|\, t \in X^{\perp\!\!\!\perp}, \pi \in \Pi\} \\
&= ||\daleth 0 \not\varepsilon u_{X \to \perp}|| \\
&= ||\forall x(x \not\varepsilon u_{X \to \perp})||
\end{aligned}$$

Thus, $\lambda x.xII \Vdash (\forall x(x \not\varepsilon u_X) \longrightarrow \perp) \longleftrightarrow \forall x(x \not\varepsilon u_{X \to \perp})$. □

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[8] $X \wedge Y \equiv (X \longrightarrow Y \longrightarrow \perp) \longrightarrow \perp$

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