Realizability and her brothers

Now!

Jacopo Furlan

Set Theory

Forcing

Realizability

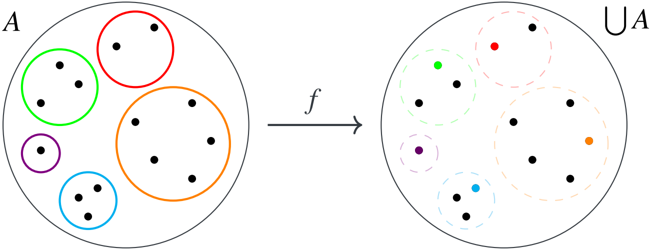
Good ol' ZF

~1930 axiomatization

$$\left\{\begin{array}{rl} \textit{Extensionality:} & \forall x, y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y) \\ \textit{Foundation:} & \exists y(y \in x) \rightarrow \exists y(y \in x \wedge \forall z(z \in x \rightarrow z \notin y) \\ \textit{Paring:} & \forall x, y \exists z(x \in z \wedge y \in z) \\ \textit{Union:} & \forall x \exists y(\forall z \in x \rightarrow \forall w \in z(w \in y)) \\ \textit{Power Set:} & \forall x \exists y \forall z(z \subseteq x \rightarrow z \in y) \\ \textit{Infinity:} & \exists x(\emptyset \in x \wedge \forall y \in x(y \cup \{y\} \in x)) \\ \textit{Replacement:} & \forall a(\forall x \in a \exists! y\, F(x, y)) \rightarrow \exists b \forall x \in a \exists y \in b\, F(x, y)) \end{array}\right\}$$

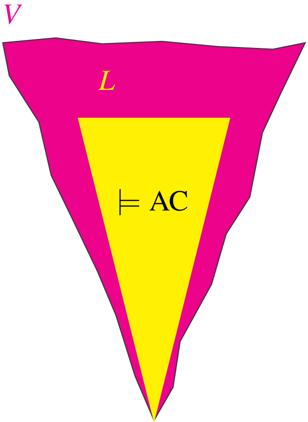
The Unbearable Lightness of the Axiom of Choice

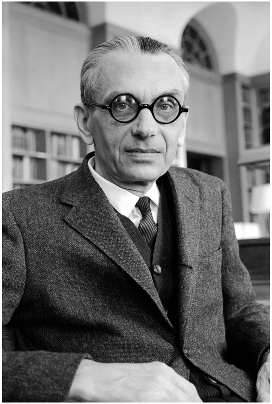
$$\mathrm{AC} \equiv \ \forall A \left(\forall x \in A \left(x \neq \emptyset \right) \ \rightarrow \ \exists f : A \longmapsto \bigcup A \ \forall x \in A \left(f(x) \in x \right) \right)$$



$$\boxed{\mathrm{ZF} \vdash \mathrm{AC}? \quad \mathrm{ZF} \nvdash \mathrm{AC}?}$$

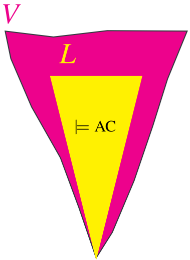
What did Gödel do?





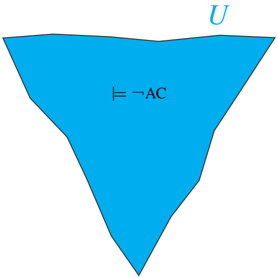
Kurt Gödel

Independence results



$$\Downarrow$$

$$\mathrm{ZF} \nvdash \neg\mathrm{AC}$$



$$\Downarrow$$

$$\mathrm{ZF} \nvdash \mathrm{AC}$$

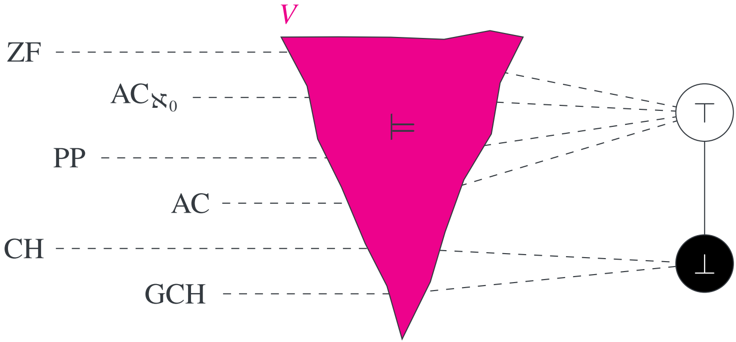
US - 1963

1 Mar: first comic issue of
The Amazing Spider-Man

22 Nov: Kennedy's assassination

15 Dec: Paul J. Cohen invents ***forcing***

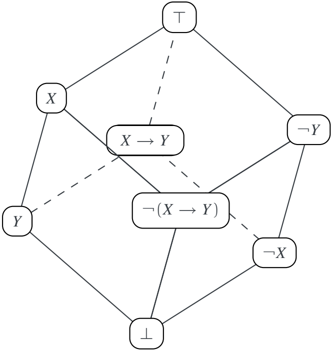
Models: people wearing theorems



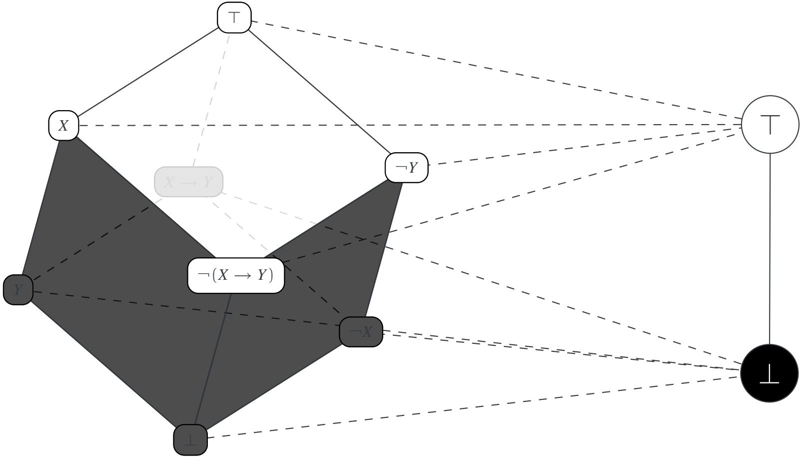
Boolean Algebras

A boolean algebra is a (partially) order set such that:

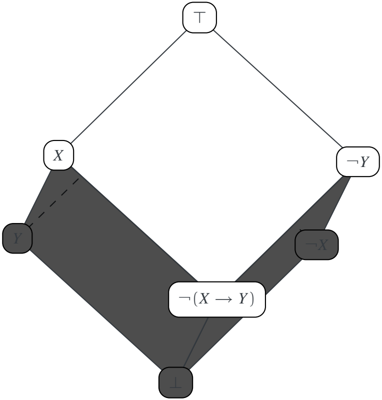
- for any X, there exists the complementation $\neg X$
- there exists a supremum $\top$
- *there exists a minimum* $\bot$
- for any X, Y, there exists $X \vee Y$
- *for any* X, Y, *there exists* $X \wedge Y$



Boolean Algebras

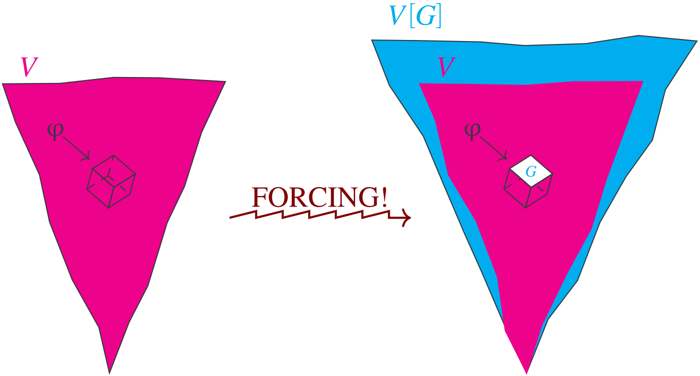


Ultrafilters



- Closed for $\leqslant$
- Closed for $\wedge$
- Maximality: for any X, X or $\neg X$ is in it

Forcing via boolean-valued models



Good ol' λ-calculus

$$t \; := \; x \mid (t)t \mid \lambda x.t$$

$$(\lambda x.t)u \to_\beta t[u/x]$$

Alonzo **Church**
(Cattedrale di Santa Maria del Fiore)
(1296-1436)
Firenze, Italy

Realizability algebras

David
by Michelangelo Buonarroti
(1501-1504)
Galleria dell'Accademia, Firenze, Italy

$$\boxed{\mathcal{A} = (\Lambda_{\mathsf{c}}, \Pi, \succ, \perp\!\!\!\perp)}$$

$$t \in \Lambda_{\mathsf{c}}$$

$$t \;:=\; x \,|\, \mathsf{cc} \,|\, \lambda x.t \,|\, (t)t \,|\, k_\pi$$

$$\pi \in \Pi$$

$$\pi \;:=\; \pi_0 \,|\, t \,.\, \pi$$

Realizability algebras

$$\boxed{\mathcal{A} = (\Lambda_{\mathsf{c}}, \Pi, \succ, \perp\!\!\!\perp)}$$

$$\begin{aligned}
\lambda x.t \star u \,.\, \pi &\succ t[u/x] \star \pi, && \text{(grab)}\\
(t)u \star \pi &\succ t \star u \,.\, \pi, && \text{(push)}\\
\mathsf{cc} \star t \,.\, \pi &\succ t \star k_\pi \,.\, \pi, && \text{(save)}\\
k_\pi \star t \,.\, \rho &\succ t \star \pi; && \text{(restore)}
\end{aligned}$$

$$t \star \pi \in \perp\!\!\!\perp, u \star \rho \succ t \star \pi$$

$$\Downarrow$$

$$u \star \rho \in \perp\!\!\!\perp$$

Vergine delle Rocce
by Leonardo da Vinci
(1483-1486)
Musée du Louvre, Paris, France

$\perp\!\!\!\perp$ and orthogonality

$$X \subseteq \Pi \rightsquigarrow X^{\perp\!\!\!\perp} := \{t \in \Lambda_{\mathsf{c}} \mid \forall \pi \in X (t \star \pi \in \perp\!\!\!\perp)\}$$

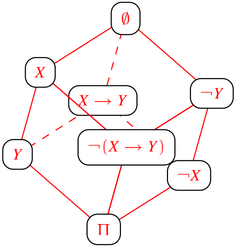
$$\Vdash X \quad \textit{iff} \quad X^{\perp\!\!\!\perp} \neq \emptyset$$

$$X, Y \subseteq \Pi \rightsquigarrow X \to Y = X^{\perp\!\!\!\perp} \,.\, Y := \{t \,.\, \pi \mid t \in X^{\perp\!\!\!\perp}, \pi \in Y\}$$

The boolean algebra of interest

$$\boxed{(\mathcal{P}(\Pi), \leqslant)}$$

$$X \leqslant Y \quad \textit{iff} \quad \Vdash X \to Y \quad \textit{iff} \quad \exists t(X^{\perp\!\!\!\perp} \xhookrightarrow{t} Y^{\perp\!\!\!\perp})$$



Realizability models

$$F \text{ formula}$$

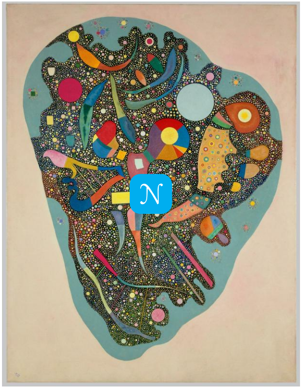
$$\Downarrow$$

$$||F|| \subseteq \Pi$$

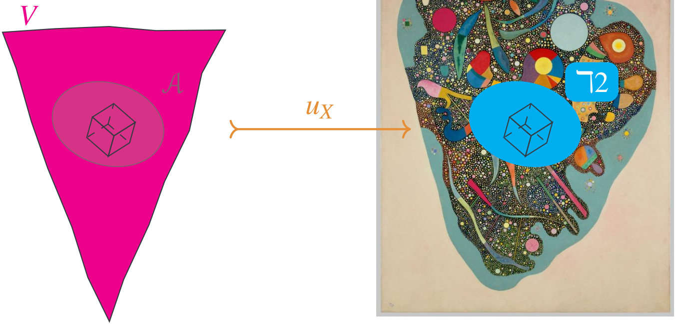
$$[||F|| \in \mathcal{P}(\Pi)]$$

$$t \Vdash F \;\; \textit{iff} \;\; t \in ||F||^{\perp\!\!\!\perp}$$

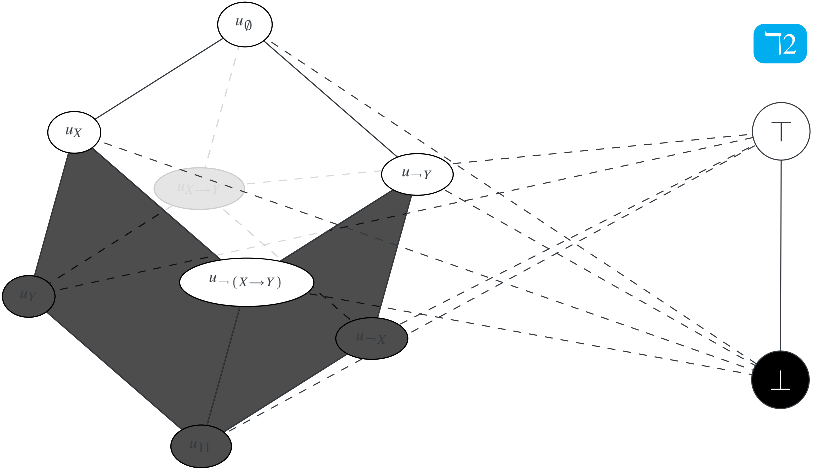
$$\Vdash F \;\; \Rightarrow \;\; \mathcal{N} \models F$$



Representatives

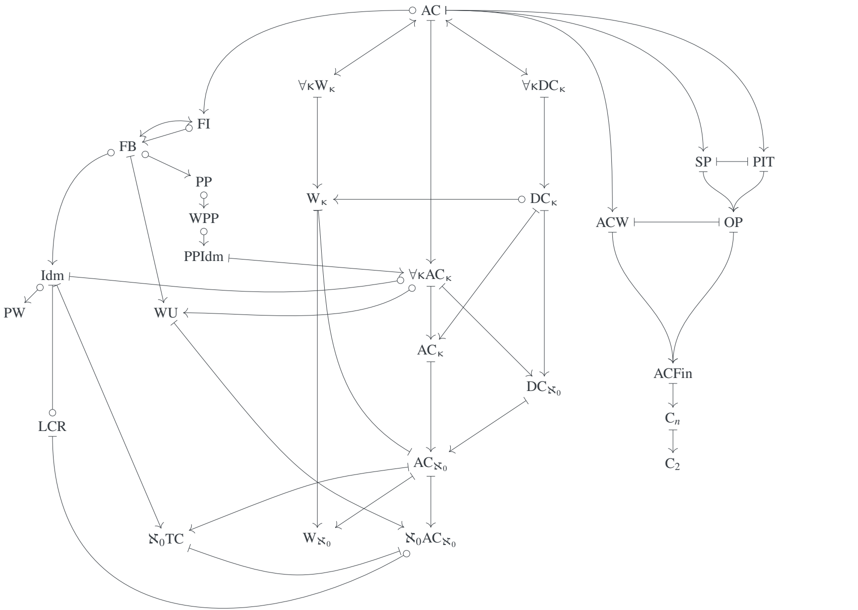


Final plot twist



ℸ2

Coda: AC strikes back



Grazie!