Bar-induction, GCH and combinatorial properties

from an ongoing work with Laura Fontanella

Realizability Workshop - CIRM

Jacopo Furlan

LIPN - USPN & LACL - UPEC

April 8, 2025

Realizability algebra is a tuple $\mathcal{K} = (\Lambda_{\mathsf{c}}, \Pi, \succ, \perp\!\!\!\perp)$.

To build a model of ZF, we associate to any formula F a set of stacks $||F|| \subseteq \Pi$.

F is realized by t, in symbols $t \Vdash F$, if

$$\text{for any } \pi \in ||F||, t \star \pi \in \perp\!\!\!\perp + \text{technical property.}$$

If t doesn't feature such particular (and mysterious) property, but satisfies $t \star \pi \in \perp\!\!\!\perp$, we write $t \in ||F||^{\perp\!\!\!\perp}$.

Let $\kappa \in Ord$ be a cardinal, $\mathcal{K} = (\Lambda_{\mathsf{c}}, \Pi, \succ, \perp\!\!\!\perp)$ a realizability algebra such that

$$\Lambda_c = \Lambda^{\circ}(\mathsf{cc}; k_\pi; \chi; \underline{\alpha} \mid \pi \in \Pi, \alpha < \kappa)$$

satisfying

$$(\underline{s})\underline{\alpha} \succ \underline{\alpha + 1}$$

$$(\chi)\underline{\alpha}\, tu\underline{\beta} \succ \begin{cases} (t)\underline{\beta} & \beta < \alpha \\ u & \beta \geq \alpha \end{cases}$$

Observe that χ allows to encode sequences of terms. It is helpful to think at the previous term as

$$\chi\underline{\alpha}tu \sim \langle\, \underbrace{(t)\underline{0}, (t)\underline{1}, \ldots, (t)\underline{\beta}, \ldots}_{\text{first } \alpha \text{ terms}}, u, u, u, \ldots\,\rangle$$

(that's just syntactic sugar for the following).

In this setting, we can define bar-induction as

$$\Psi := \lambda gu.(Y)\lambda hkt.(u)(\chi kt)(g)\lambda z.h(\underline{s})k)(\chi)ktz$$

which has the following reduction for $H := \Psi GU$

$$H\underline{\alpha}t \succ U(\chi)\underline{\alpha}tK_{\alpha}[H,t]$$

with $K_{\alpha}[H,t] := (G)\lambda z.H(\underline{s})\underline{\alpha}\,(\chi)\underline{\alpha}tz$.

$$H\underline{\alpha}t \succ U(\chi\underline{\alpha}t)K_\alpha[H,t],$$
$$K_\alpha[H,t] \equiv (G)\lambda z.H(\underline{s})\underline{\alpha}\,(\chi)\underline{\alpha}tz$$

Ψ applies U to the sequence

$$\langle (t)\underline{0}, \ldots, (t)\underline{\beta}, \ldots, K_\alpha[H,t], \ldots \rangle$$

where the tail of the sequence is recursively extending the sequence. Indeed, in $K_\alpha[H,t]$, the sub-term $\lambda z.H(\underline{s})\underline{\alpha}\,(\chi)\underline{\alpha}tz$ is a function which returns U applied to the sequence

$$\langle (t)\underline{0}, \ldots, (t)\underline{\beta}, \ldots, z, K_{\alpha+1}[G,U,t], \ldots \rangle.$$

In order to deal with sequences of λ_{c}-terms we fix some definition.

Definition

Given a sequence $(t_\alpha)_{\alpha<\gamma}$ for some ordinal γ, we say that $v \in \Lambda_{\mathsf{c}}$ *reduces to* $(t_\alpha)_{\alpha<\gamma}$ if for any $\alpha < \gamma$ and for any $\pi \in \Pi$, $v \star \underline{\alpha} \centerdot \pi \succ t_\alpha \star \pi$.
A sequence $(v_\alpha)_{\alpha<\gamma}$ reduces to $(t_\alpha)_{\alpha<\gamma}$ if for any $\alpha < \gamma$, for any $\beta < \alpha$ and $\pi \in \Pi$, $v_\alpha \star \underline{\beta} \centerdot \pi \succ t_\beta \star \pi$.

Definition

For a fixed $u \in \Lambda_{\mathsf{c}}$, $v \in \Lambda_{\mathsf{c}}$ is *compatible* with u if there exists $\pi \in \Pi$ such that $u \star v \centerdot \pi \notin \perp\!\!\!\perp$. v is *incompatible* with u if it is not compatible.

Eventually we come to κ-*closure* as defined by Krivine.

Definition

A realizability algebra $\mathcal{K}$ is κ-*closed* for some $\kappa \in Ord$ if, and only if, for any $\gamma < \kappa$, $(t_\alpha)_{\alpha<\gamma} \subseteq \Lambda_{\mathsf{c}}$ and $u \in \Lambda_{\mathsf{c}}$, if there exists a sequence $(v_\alpha)_{\alpha<\gamma}$ reducing to $(t_\alpha)_{\alpha<\gamma}$, then there exists a term v reducing to $(t_\alpha)_{\alpha<\gamma}$ such that if for any $\alpha < \gamma$, v_α is compatible with u, then v compatible with u.

Roughly speaking, we ask any γ-sequence to have a "limit", and that compatibility is preserved by the limit.

One more definition before some results.

Definition

Define the functional $@ : M^{\mathcal{K}} \times M^{\mathcal{K}} \mapsto M^{\mathcal{K}}$ as

$$@(f, c) := \{\langle d, \pi \rangle \in M^{\mathcal{K}} \mid \langle \mathsf{op}(c, d), \pi \rangle \in f\}.$$

@ represents the set of images of c through f in the realizability model.

Definition

For an ordinal $\gamma < \kappa$, we define the $@, \gamma$-axiom of choice, $(\mathrm{AC}_{\hat{\gamma},@})$ as the following schema of formulæ

$$\forall x \exists y F(x,y) \to \exists f \forall x^{\hat{\gamma}} \exists y (F(x,y) \wedge y \mathrel{\varepsilon} @(f,x)))$$

Theorem

For any $\gamma < \kappa$, $\lambda xy.\Psi xy\underline{0}\,\underline{0} \Vdash \mathrm{AC}_{\hat{\gamma},@}$.

(Proof's idea)'s idea

We consider $G \in ||\forall x \exists y F(x,y)||^{\perp\!\!\!\perp}$, $U \in ||\forall f(\forall x^{\hat{\gamma}} \exists y(F(x,y) \wedge y \; \varepsilon \; @(f,x))) \rightarrow \bot||^{\perp\!\!\!\perp}$ and show that $\Psi G U \underline{0}\underline{0} \in ||\bot||^{\perp\!\!\!\perp}$. The proof proceeds by contradiction.

Bar-induction and G assure the existence of a sequence of terms $(u_\alpha)_{\alpha<\gamma}$ such that $u_\alpha \in ||F[\hat{\alpha}, c_\alpha]||^{\perp\!\!\!\perp}$.

We can find a sequence $(v_\alpha)_{\alpha<\gamma}$, reducing to $(u_\alpha)_{\alpha<\gamma}$ and compatible with u. Recollecting it via κ-closure generates a term $v \in ||\forall x^{\hat{\gamma}} F_{@}[x,f]||^{\perp\!\!\!\perp}$ such that $(U)v \in ||\bot||^{\perp\!\!\!\perp}$, which contradicts the preservation of compatibility.

For a fixed cardinal κ, we can define CH_κ as the statement that the powerset $\wp(\kappa)$ is in bijection with the successor cardinal κ^+; then we can denote $\mathrm{CH}_{<\kappa} := \forall\alpha < \kappa\, \mathrm{CH}_\alpha$ and $\mathrm{GCH} := \forall\kappa\, \mathrm{CH}_\kappa$.

Corollary

Suppose $\mathcal{M} \models \mathrm{GCH}$ and $\mathcal{K}$ is κ-closed. Then the realizability model $\mathcal{N}$ satisfies $\mathrm{CH}_{<\hat{\kappa}}$.

Proof's idea

@ operator allows to define for any function $f : \hat{\gamma} \overset{\varepsilon}{\longmapsto} \{\daleth 0, \daleth 1\}$ in realizability model a subset g_f contained in the submodel $\mathcal{M}_{\mathcal{D}} \subseteq \mathcal{N}$.

As $\mathcal{M}_{\mathcal{D}} \models \mathrm{GCH}$ and powersets cardinals lower than $\hat{\kappa}$ are contained in there, we get the result.

Given a boolean algebra $\mathbb{B} = (B, \mathbb{0}, \mathbb{1}, \wedge, \vee, \neg)$, we can built a realizability algebra $\mathcal{K}(\mathbb{B}) = (\Lambda_{\mathsf{c}}, \Pi, \succ_{\mathbb{B}}, \perp\!\!\!\perp_{\mathbb{B}})$, using elements in B as atomic stacks.

We fix $\Pi_0 = B$ and we define a function $\cdot^{\mathbb{B}} : \Lambda_{\mathsf{c}} \cup \Pi \mapsto \mathbb{B}$ by induction:

$$x^{\mathbb{B}} = \mathbb{1} = \mathsf{cc}^{\mathbb{B}}, k_{\pi}^{\mathbb{B}} = \pi^{\mathbb{B}},$$

$$(\lambda x.t) = t^{\mathbb{B}}, ((t)u)^{\mathbb{B}} = t^{\mathbb{B}} \wedge u^{\mathbb{B}},$$

$$\pi_0^{\mathbb{B}} = \pi_0, (t \centerdot \pi)^{\mathbb{B}} = t^{\mathbb{B}} \wedge \pi^{\mathbb{B}}.$$

The relation $\succ_{\mathbb{B}}$ is defined as

$$t \star \pi \succ_{\mathbb{B}} u \star \rho \overset{def}{\iff} (t \star \pi)^{\mathbb{B}} \leq (u \star \pi)^{\mathbb{B}}$$

for $(t \star \pi)^{\mathbb{B}} := t^{\mathbb{B}} \wedge \pi^{\mathbb{B}}$, and the pole as

$$\perp\!\!\!\perp_{\mathbb{B}} = \{t \star \pi \in \Lambda_{\mathsf{c}} \star \Pi \mid (t \star \pi)^{\mathbb{B}} = \mathbb{0}\}.$$

Definition

A boolean-algebra $\mathbb{B}$ is κ-closed if, and only if, for any $\gamma < \kappa$, and any descending sequence $(b_\alpha)_{\alpha<\gamma} \subseteq B^+$, there exists $b \in B^+$ such that $b \leq b_\alpha$.

Theorem

If $\mathbb{B}$ is κ-closed, no new function $\check{\kappa} \mapsto \check{M}$ is added in the forcing model.

Theorem

$\mathbb{B}$ is κ-closed if, and only if, $\mathcal{K}(\mathbb{B})$ is κ-closed.

In forcing setting there exists a weaker version of κ-closure, whose effects in the forcing extension resemble what we obtained before.

Definition

A boolean-algebra $\mathbb{B}$ is κ-distributive if, and only if, for any γ-family of open denses $(D_\alpha)_{\alpha<\gamma}$, $\bigcap_{\alpha<\gamma} D_\alpha$ is open dense, for any $\gamma < \kappa$.

Theorem

If $\mathbb{B}$ is κ-closed, then it is κ-distributive. The reverse holds if, and only if, $\mathrm{DC}_{<\kappa}$ holds.

Theorem

If a forcing condition is κ-distributive, then any γ-sequence of element of the ground model is preserved.

Perspective

Finding a suitable translation of κ-distributivity.

S. Berardi, M. Bezem, T. Coquand
"On the Computational Content of the Axiom of Choice"
The Journal of Symbolic Logic, vol. 63, no. 2, 1998, pp. 600–22.

J.-L. Krivine
"Bar Recursion in Classical Realisability: Dependent Choice and Continuum Hypothesis"
Proceedings CSL 2016 LIPIcs vol. 62, 2016, pp. 25:1–25:11

Thank you!