The Art of Realizability

08/07/2024

Jacopo Furlan

LIPN - Université Sorbonne Paris Nord

Intro

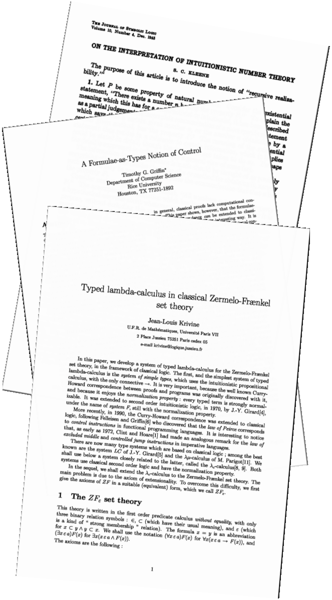
Theorem 1

Realizability Tales

1945: Kleene's Realizability
(Intuitionistic Arithmetics)

1990: Griffin's extension of Curry-Howard
correspondence to classical logic

2001: Krivine's Realizability
(Classical Set Theory)



Krivine's Realizability

- Extending Curry-Howard correspondence to Set Theory
- Extracting constructive content of mathematical proofs

- Generalization of Forcing

$\mathcal{A}$ Realizability Algebra

$$\boxed{\mathcal{A} = (\Lambda_{\mathsf{c}}, \Pi, \succ, \perp\!\!\!\perp)}$$

$$t \in \Lambda_{\mathsf{c}}$$

$$t \ := \ x \,|\, \mathsf{cc} \,|\, \lambda x.t \,|\, (t)t \,|\, k_\pi$$

$$\begin{aligned} \lambda x.t \star u \,.\, \pi &\succ t[u/x] \star \pi, && \text{(grab)} \\ (t)u \star \pi &\succ t \star u \,.\, \pi, && \text{(push)} \\ \mathsf{cc} \star t \,.\, \pi &\succ t \star k_\pi \,.\, \pi, && \text{(save)} \\ k_\pi \star t \,.\, \rho &\succ t \star \pi; && \text{(restore)} \end{aligned}$$

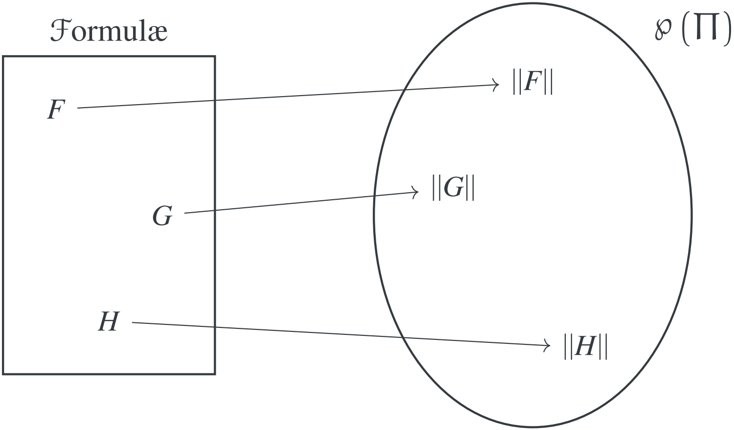
$$\pi \in \Pi$$

$$\pi \ := \ \pi_0 \,|\, t \,.\, \pi$$

$$t \star \pi \in \perp\!\!\!\perp$$

$$u \star \rho \succ t \star \pi \Rightarrow u \star \rho \in \perp\!\!\!\perp$$

Realizability Models



Realizers

Definition

For any $X \subseteq \Pi$, the orthogonal with respect to $\perp\!\!\!\perp$ is defined as

$$X^{\perp\!\!\!\perp} := \{t \in \Lambda_{\mathsf{c}} \mid \forall \pi \in X (t \star \pi \in \perp\!\!\!\perp)\}.$$

Definition

For any $X \subseteq \Pi$,

$$\Vdash X \quad \textit{iff} \quad X^{\perp\!\!\!\perp} \cap \mathrm{Q} \neq \emptyset$$

A $\mathcal{B}$oolean algebra

Definition

For any $X, Y \subseteq \Pi$,

$$X \to Y := X^{\perp\!\!\!\perp} \mathbin{.} Y := \{t \mathbin{.} \pi \mid t \in X^{\perp\!\!\!\perp}, \pi \in Y\}.$$

Definition

For any $X, Y \in \wp(\Pi)$, $X \leqslant Y$ if, and only if, $\Vdash X \to Y$.

Lemma

$\mathcal{B} = \left({}^{\wp(\Pi)}/_{\leftrightarrow}, \leqslant \right)$ induces a boolean algebra.

Realizability

$$\|d \not\varepsilon\, c\| := \quad \{\pi \mid \langle d, \pi \rangle \in c\} \quad =: X_c(d)$$

$$t \Vdash d \not\varepsilon\, c \quad \textit{iff} \quad t \Vdash X_c(d)$$

$$t \Vdash d \,\varepsilon\, c \quad \textit{iff} \quad t \Vdash X_c(d) \to \Pi$$

Definition

For any formula $F \in \mathcal{F}_{\mathrm{ZF}_\varepsilon}$, F is realized ($\Vdash F$) if, and only if,

$$\|F\|^{\perp\!\!\!\perp} \cap \mathrm{Q} \neq \emptyset$$

Zeros and Ones

$$\daleth 0 = \{\langle \daleth 0, \pi \rangle \mid \pi \in \emptyset\} = \emptyset = u_{\emptyset}$$

Definition

For any $X \subseteq \Pi$,

$$u_X := \{\langle \daleth 0, \pi \rangle \mid \pi \in X\} \in M^{\mathcal{A}}$$

$$\daleth 1 = \{\langle \daleth 0, \pi \rangle \mid \pi \in \Pi\} = u_{\Pi}$$

Theorem 1

1. $\forall X \subseteq \Pi \, (\Vdash u_X \simeq \daleth 0 \vee u_X \simeq \daleth 1)$, i.e. $\Vdash u_X \in \daleth 2$;
2. $\forall F \in \mathcal{F}_{\mathrm{ZF}_\varepsilon} \, (\Vdash F$ *iff* $\Vdash u_{||F||} \simeq \daleth 0)$;
3. $\mathfrak{G} = \{X \subseteq \Pi \mid \mathcal{N} \models u_X \simeq \daleth 0\}$ is an ultrafilter. Indeed, for any $X \subseteq \Pi$,
$$\Vdash u_X \not\simeq \daleth 0 \leftrightarrow u_{X \to \Pi} \simeq \daleth 0.$$

Proof [Sketch]

$$\Vdash u_X \simeq \daleth 0 \vee u_X \simeq \daleth 1$$

Proof.

$$\begin{tikzcd}
\boxed{\Vdash u_X \subseteq \daleth 1} \arrow[dr, Rightarrow] & & \boxed{\Vdash \daleth 1 \not\subseteq u_X \longrightarrow \daleth 0 \not\varepsilon u_X} \arrow[dl, Rightarrow, "\Vdash \forall x(x \,\varepsilon\, u_X \to x = \daleth 0)"] & \\
& \boxed{\Vdash u_X \not\simeq \daleth 1 \longrightarrow \forall x(x \not\varepsilon u_X)} \arrow[dr, Rightarrow] & & \boxed{\Vdash \forall x(x \not\varepsilon u_X) \longleftrightarrow u_X \simeq \daleth 0} \arrow[dl, Rightarrow] \\
& & \boxed{\Vdash u_X \simeq \daleth 0 \vee u_X \simeq \daleth 1} &
\end{tikzcd}$$

$$\lambda xy.(y(\lambda ab.\mathsf{cc}\lambda k.aI(\lambda cd.c(dWW))k)x)I \Vdash u_X \simeq \daleth 0 \vee u_X \simeq \daleth 1$$

Proof [Sketch]

$$\Vdash F \; \textit{iff} \; \Vdash u_{||F||} \simeq \urcorner 0$$

Proof.

$$\begin{aligned} \Vdash F \quad & \text{iff} \quad \exists t \forall \pi \in ||F|| (t \star \pi \in \perp\!\!\!\perp) \\ & \text{iff} \quad \exists t \, (t \Vdash \urcorner 0 \not\varepsilon \; u_{||F||}) \\ & \text{iff} \quad \exists t \, (t \Vdash \forall x (x \not\varepsilon \; u_{||F||})) \quad \text{iff} \quad \Vdash u_{||F||} \simeq \urcorner 0 \end{aligned}$$

Proof [Sketch]

$$\Vdash u_X \not\simeq \urcorner 0 \leftrightarrow u_{X\to\Pi} \simeq \urcorner 0$$

Proof.

$$\begin{aligned} \|\forall x(x \not\varepsilon\ u_X) \to \bot\| &= X \to \Pi \\ &= \|\forall x(x \not\varepsilon\ u_{X\to\Pi})\| \end{aligned}$$

What next?

- Using $\mathfrak{G}$ to deduce truth in $\mathcal{N}$.

Bibliography

A.S. Troelstra
"Realizability"
Handbook of proof theory, Stud. Logic Found. Math., vol. 137, North-Holland, Amsterdam (1998), pp. 407-473

R. Matthews
"A Guide to Krivine Realizability for Set Theory"
URL: https://arxiv.org/abs/2307.13563

Thank you!