’t Hooft-Polyakov Monopoles

Francesco Gentile, Giorgio Mentasti, Jacopo Tissino

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Abstract

In this work we show the fundamental steps of the approaches by Dirac and by ’t Hooft and Polyakov to the theory of magnetic monopoles. In the former, the invariance under duality of Maxwell’s equations is postulated and the quantization of electric and magnetic charge is deduced; in the latter an *ansatz* for the Euler-Lagrange equations of the Georgi-Glashow model is found to provide a regular solution which has magnetic charge quantized according to Dirac’s condition.

Contents

1 Introduction

Even though they have never been experimentally observed, magnetic monopoles have been a subject of study ever since they were first hypothesized by Pierre Curie in 1894 [1]. The theoretical importance of such an idea became evident after the work of Dirac in 1931 [2]: he showed that, postulating the existence of the magnetic monopole and assuming that Maxwell's equations are invariant under a duality transformation, it was necessary to assume the quantization of electric and magnetic charge. The quantization of electric charge had been experimentally proved by Millikan in 1909, but couldn't be theoretically explained in the framework of classical electromagnetism. In section 2, we will show how the Dirac quantization condition can be deduced and how it can be generalised to the Zwanziger-Schwinger condition for particles carrying both electric and magnetic charge: dyons.

Many years later, in 1975, Gerardus 't Hooft [3] and Aleksandr Polyakov [4], shed new light on the theory of magnetic monopoles. They separately showed that monopoles emerge as regular solutions of the field equations of the Georgi-Glashow model, which is an SU(2) gauge theory with a Higgs potential. Their reasoning can be used in any non Abelian gauge theory with compact covering group. With this approach, monopoles are intrinsic to the theory and fulfill the same quantization condition proved by Dirac. In section 3 we will expose the derivation of the 't Hooft Polyakov ansatz for the solutions to the Euler-Lagrange equations of the Georgi-Glashow model. Then, we will discuss the existence of a charge quantization condition for dyons in the framework of the Georgi-Glashow model, following the work of B. Julia and A. Zee [5]. Lastly, we will derive the Bogomol'nyi (lower) bound for the mass of a dyon, discuss its saturation and apply it to the 't Hooft-Polyakov monopole case.

2 Dirac Monopoles

In 1931 Dirac showed that postulating the existence of a magnetic charge, we can provide a theoretical explanation for the quantization of the electric charge [2]. In this section we will recover the basic ideas exposed in Dirac's paper and show how they were generalised to a particle with both charges, called *dyon*, by Schwinger in 1966 [6] and Zwanziger in 1968 [7].

2.1 Magnetic Monopole and Electromagnetic Duality

The electromagnetic tensor $F^{\mu\nu}$, where both electric and magnetic fields are encoded:

$$E^i = F^{0i} \qquad\qquad B^i = \epsilon^{ijk} F^{jk} \,, \tag{2.1}$$

is an antisymmetric tensor and can be written as a 2-form (see appendix A):

$$F = \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ . \tag{2.2}$$

It obeys Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{2.3a}$$

$$\partial_{[\mu} F_{\nu\rho]} = 0 \,. \tag{2.3b}$$

We can define a dual electromagnetic tensor (see appendix A) by *Hodge duality*:

$$\tilde{F}^{\mu\nu} = *F^{\mu\nu} = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{2.4}$$

which allows us to write Maxwell's equations in this way:

$$\mathrm{d}F = 0 \qquad\qquad \mathrm{d} * F = *j \,, \tag{2.5}$$

where we introduced the current form $j = j_\mu \,\mathrm{d}x^\mu$. If $j = 0$, equations (2.5) are invariant with respect to the duality transformation $F \to *F$, since $*^2 = -\mathbb{1}$ for 2-forms. Such symmetry is broken when $j \neq 0$: however, even in this case, we can preserve invariance under duality if we introduce a "magnetic current" j_m such that $\mathrm{d}F = *j_m$, and complement the duality transformation as such:

$$\begin{cases} j \to j_m \\ j_m \to -j \end{cases} \ . \tag{2.6}$$

The electromagnetic duality transformation corresponds to the formal substitution of the fields $(\vec{E}, \vec{B}) \to (\vec{B}, -\vec{E})$:

$$\tilde{E}^i = \tilde{F}^{i0} = \frac{1}{2}\epsilon^{i0jk} F_{jk} = -\frac{1}{2}\epsilon^{ijk} F^{ik} = B^i \tag{2.7a}$$

$$\tilde{B}^i = -\frac{1}{2}\epsilon^{ijk}\tilde{F}^{jk} = -\frac{1}{2}\epsilon^{ijk}\epsilon^{jkl0} F_{l0} = -\frac{1}{2}\epsilon^{ijk}\epsilon^{ljk} E^l = -E^i \tag{2.7b}$$

If we consider the configuration where an electrically charged point-like particle lies in the origin, the electric and magnetic field are:

$$E^i = \frac{e}{4\pi r^3} x^i, \tag{2.8a}$$

$$B^i = 0, \tag{2.8b}$$

and satisfy:

$$\partial_i E^i = e\delta^3(\vec{r}), \tag{2.9a}$$

$$e = \int_{S^2} E_i dS^i. \tag{2.9b}$$

Now, we might apply a duality transformation to this configuration and obtain a magnetic point-like monopole, which has charge g and is placed in the origin. The fields become:

$$B^i = \frac{g}{4\pi r^3} x^i \tag{2.10a}$$

$$E^i = 0, \tag{2.10b}$$

and correspond to this electromagnetic tensor:

$$F_{ij} = \frac{g}{4\pi r^3} \varepsilon_{ijk} x^k \qquad F_{i0} = 0. \tag{2.11}$$

We now compute the 2-form associated to the tensor in equation (2.11):

$$F = F_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j = \frac{g}{4\pi r^3} \varepsilon_{ijk} x_k\, \mathrm{d}x^i\, \mathrm{d}x^j = \frac{g}{4\pi} \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi\ , \tag{2.12}$$

and deduce from it the value of the magnetic charge in the origin:

$$\int_{\mathbb{R}^3} \mathrm{d}F = \int_{S^2_\infty} F = \frac{g}{4\pi} \int_{S^2_\infty} \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi = g. \tag{2.13}$$

In the first equivalence of equation (2.13), we applied Stokes' theorem to an arbitrarily large sphere, since the integrand function has no radial dependence.

2.2 Dirac Quantization for the Monopole

We cannot find for the magnetic monopole fields, represented by the 2-form F, a global vector potential A such that $F = \mathrm{d}A$. If it existed, we would have the following paradox:

$$g = \int_{\mathbb{R}^3} \mathrm{d}F = \int_{S^2_\infty} F = \int_{S^2_\infty} \mathrm{d}A = \int_{\partial S^2_\infty} A = 0 \neq g\,, \tag{2.14}$$

Even though the form F is closed ($\mathrm{d}F = 0$), it is not exact. This happens beacause the manifold we are considering — the punctured space $\mathbb{R}^3 \setminus \{0\}$ — is not contractible and therefore Poincaré's lemma cannot be applied.

We can, however, find potentials which are defined *locally*: a possible choice of A is, in cylindric coordinates:

$$A = \frac{g}{4\pi} (c - \cos\theta)\, \mathrm{d}\phi\,, \tag{2.15}$$

where $c = \pm 1$. Differentiating this equation we have

$$\mathrm{d}A = \frac{g}{4\pi} (c - \cos\theta)\, \mathrm{d}^2\phi - \frac{g}{4\pi} \mathrm{d}\cos\theta \wedge \mathrm{d}\phi = \frac{g}{4\pi} \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi\ , \tag{2.16}$$

where the first term vanishes because $\mathrm{d}^2 = 0$. We will show that for both choices of c the potential is singular somewhere:

- for $c = 1$ the potential A is singular along the $z < 0$ axis;
- for $c = -1$, instead, A is singular along the $z > 0$ axis.

The $z \lessgtr 0$ ray is called a *Dirac string*. In both cases, we see that removing the string makes the topology trivial: the manifold becomes contractible, and the local potential is defined on the whole of either open set $\mathbb{R}^3 \setminus \{z \lessgtr 0\}$.

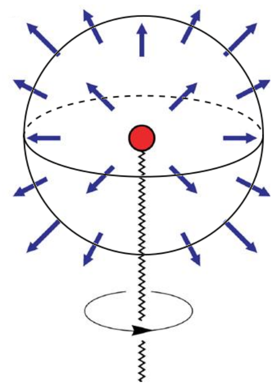

Figure 1: A schematization of Dirac string.

We call the two choices of A respectively A^+ and A^-. We say that $U_+ = \mathbb{R}^3 \setminus \{z < 0\}$ is the chart where we define A^+ and $U_- = \mathbb{R}^3 \setminus \{z > 0\}$ is the one where we define A^-.

Let us now show that the potentials indeed diverge near their Dirac string. We wish to integrate in a similar domain to the one of equation (2.14), however now we must exclude the Dirac string: the resulting domain is a 2D surface, which we denote by $S_2 \setminus D_\epsilon$; its boundary is a small circle C_ϵ around the Dirac string. As in (2.13) we have:

$$g = \int_{S_2} F = \lim_{\epsilon \to 0} \int_{S_2 \setminus D_\varepsilon} F = \lim_{\epsilon \to 0} \int_{S_2 \setminus D_\varepsilon} \mathrm{d}A = \lim_{\epsilon \to 0} \int_{C_\varepsilon} A, \tag{2.17}$$

which implies the divergence of A, since its integral along an arbitrarily small circle is requested to be fixed and finite.

We now want to calculate the transiton function between the two potentials in the region $U_+ \cap U_-$ where their domains overlap:

$$A^+ - A^- = \frac{g}{2\pi} \mathrm{d}\phi = \mathrm{d}\left(\frac{g\phi}{2\pi}\right) \equiv \mathrm{d}\lambda(x) \ . \tag{2.18}$$

This transformation can be seen as a $U(1)$ gauge transformation: let h be the map $h : U_+ \cap U_- \to G = U(1)$, such that $x \to \exp\big(ie\lambda(x)\big)$, where e is the electric charge. The corresponding transformation of the potential is:

$$A^+ = A^- + \mathrm{d}\lambda = h^{-1} A^- h - \frac{i}{e} h \, \mathrm{d}h \ . \tag{2.19}$$

In order to have the function h well defined, we must have $h(g\phi) = h(g(\phi + 2\pi))$. This condition is satisfied if we assume that:

$$ge \in 2\pi\mathbb{Z}. \tag{2.20}$$

Expression (2.20) is the so-called *Dirac Quantization Condition* and guarantees that:

$$h(g(\phi + 2\pi)) = \exp\big(2\pi ge\big)h(g\phi) = h(g\phi)\,. \tag{2.21}$$

We now want to show that the integer eg has a clear geometrical meaning: it is the winding number of the map h, i.e. the number of times h wraps around the circle S^1 as x goes around the center once. To prove this, we can split the integral in (2.17) into two contributions from both the potentials and integrate them over two hemispheres, whose borders can both be made to coincide with the equator S^1, but will have opposite orientations, whence the minus sign before A_-:

$$g = \int_{S^2} F = \int_{S^2_+} \mathrm{d}A^+ + \int_{S^2_-} \mathrm{d}A^- = \int_{S^1} \left(A^+ - A^-\right) = \int_{S^1} \mathrm{d}\lambda = \Delta\lambda, \tag{2.22}$$

Multiplying $\Delta\lambda$ by the charge e, we can deduce how many radiants the argument of the exponent has elapsed. Therefore, the winding number S_1 is:

$$S_1 = \frac{e\Delta\lambda}{2\pi} = \frac{ge}{2\pi}\,. \tag{2.23}$$

2.3 Zwanziger-Schwinger Quantization for the Dyon

We want now to consider the hypothetical case of a particle carrying both electric and magnetic charge, respectively e_1 and g_1, which is called a dyon. If this particle moves into an electromagnetic field it experiences the force

$$\vec{F} = e_1\left[\vec{E} - \vec{v}\times\vec{B}\right] - g_1\left[\vec{B} - \vec{v}\times\vec{E}\right]. \tag{2.24}$$

At the same time we suppose that electric and magnetic fields are origined by another dyon in the origin, with electric charge e_2 and magnetic charge g_2, which generates the fields:

$$\begin{aligned}\vec{E} &= \frac{e_2}{4\pi}\frac{\vec{r}}{r^3},\\ \vec{B} &= \frac{g_2}{4\pi}\frac{\vec{r}}{r^3}.\end{aligned} \tag{2.25}$$

Now, let $\vec{L} = \vec{r}\times m\vec{v}$ (where m denotes the mass) be the orbital angular momentum of the first dyon: in this case we have

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \frac{1}{4\pi}(e_1g_2 - g_1e_2)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\vec{r}}{r}, \tag{2.26}$$

which implies that a dyon in this field does not conserve the regular angular momentum $\vec{L}$, however it conserves another quantity, which we denote by $\vec{J}$:

$$\vec{J} = \vec{r} \times m\vec{v} - \frac{1}{4\pi}(e_1 g_2 - g_1 e_2)\frac{\vec{r}}{r}. \tag{2.27}$$

Consider now the projection of $\vec{J}$ on the versor $\hat{r}$, which is $J_r = (e_1 g_2 - g_1 e_2)/4\pi$: from the quantization of J_r: $J_r \in \mathbb{Z}/2$, the Zwanziger-Schwinger quantization condition follows:

$$e_1 g_2 - g_1 e_2 \in 2\pi\mathbb{Z}. \tag{2.28}$$

We notice that if we set $g_1 = 0$, $e_2 = 0$ and $e_1 = e$, that is, we consider an electrically charged particle with unit charge moving in the field generated by a pure magnetic monopole, we get Dirac's quantization condition:

$$e g_2 \in 2\pi\mathbb{Z}. \tag{2.29}$$

3 't Hooft-Polyakov Monopoles

In this section, it will be shown that free magnetic monopoles in the physical vacuum are possible as regular solutions of the field equations of the Georgi-Glashow model, as first proved in the notorious works published by Gerard 't Hoft [3] and Aleksander Polyakov [4] in 1974. We were also inspired by the lectures given by J. M. Figueroa in 1998 [8].

3.1 An Intuitive Example

A simple and very clarifying example, taken from [3], will be discussed in order to give an intuitive idea of how it is possible to have monopoles in quantum field theory.

We consider a two-dimensional spherical surface S^2 in the usual 3D space, with some magnetic flux Φ entering only in a spot. If the spot is surrounded by a contour C_0 along which the fields are null, the potential around C_0 should be equal to:

$$A_i = \partial_i \Lambda\,, \tag{3.1}$$

where Λ is a gauge transformation, which maps the wavefunction ψ into $e^{i\Lambda}\psi$. While Λ is multivalued, Φ, being a physical field, is required to be a single-valued function. Since

$$\Phi = \int_{C_0} \partial_i \Lambda \, \mathrm{d}l^i \tag{3.2}$$

should be equal to an integer multiplied by 2π, a complete gauge rotation along the contour. In an Abelian gauge theory, there must be another spot from

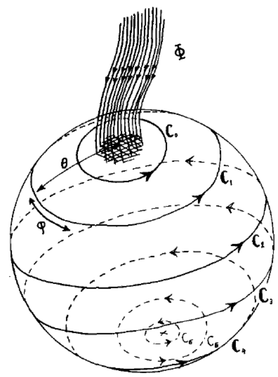

Figure 2: 't Hooft schematization of the spherical surface and the spot

which the flux comes out, because the $2k\pi$ rotation cannot be continuously changed into a constant by lowering the contour C_0 over the sphere.
Instead, in a non Abelian gauge theory with a compact covering group such as O(3), a $4k\pi$ rotation can be shifted into a constant without singularity. This implies that, if we have a non Abelian gauge theory with a compact covering group and such compact covering group has the electromagnetic group U(1) as a subgroup, the existence of the magnetic monopole would be allowed without supposing the existence of any singularity anywhere on the sphere. A theory that satisifies these requirements is the Georgi-Glashow model for the electroweak interaction.

3.2 The bosonic part of the Georgi-Glashow Model

The Georgi-Glashow model was proposed as a theory for the electroweak interaction and, in its bosonic part, is based on the gauge group SU(2) which is two-to-one homomorphic to SO(3).
The lagrangian density $\mathcal{L}$, whose associated action is invariant under gauge transformations belonging to SO(3), contains the Higgs field $\vec{\phi} = (\phi^1, \phi^2, \phi^3)$, which is a vector in the adjoint representation of SO(3) and three gauge potentials $\vec{W}^\mu$, with values in the Lie algebra of SO(3). We use superscript arrows to denote objects which take values in the adjoint representation of SO(3). $\mathcal{L}$ has the following form:

$$\mathcal{L} = \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - \frac{1}{4} G^a_{\mu\nu} G^{a\ \mu\nu} - \underbrace{\frac{1}{4}\lambda(\phi^2 - a^2)}_{V(\phi)}, \tag{3.3}$$

where $\vec{G_{\mu\nu}}$ is the gauge field-strength, defined as follows:

$$\vec{G_{\mu\nu}} = \partial_\mu \vec{W}^\nu - \partial_\nu \vec{W}^\mu - e\vec{W}^\nu \times \vec{W}^\mu, \tag{3.4}$$

and the operator D_μ is the gauge covariant derivative:

$$D_\mu \vec{\phi} = \partial_\mu \vec{\phi} - e\vec{W}_\mu \times \vec{\phi}. \tag{3.5}$$

Let us look at the physical meaning of each term in the lagrangian density:

- the term $\frac{1}{2} D_\mu \phi^a D^\mu \phi^a$ is the free field term;
- the term $-\frac{1}{4} G^a_{\mu\nu} G^{a\ \mu\nu}$ describes the dynamics of the potential $\vec{W}_\mu$;
- the term $V(\phi)$ describes the self interaction of the Higgs field and contains a non negative constant λ.

Spectrum of the Model Now we are interested in the spectrum of the model. It can be obtained through perturbation theory, considering a Higgs potential with a little fluctuation $\vec{\chi}$ around the minimum $\vec{a}$:

$$\vec{\phi} = \vec{a} + \vec{\chi}. \tag{3.6}$$

Four bosons figure in our model: the photon $A_\mu = \frac{1}{a}\vec{a} \cdot \vec{W}_\mu$, the Higgs Boson associated to the field $\vec{\phi}$, and the bosons $\vec{W}^\pm_\mu$. Their properties are shown in table 1.

Field	Definition	Mass	Charge
A_μ	Photon	0	0
ϕ	Higgs Boson	$a\sqrt{2\lambda}\hbar$	0
$W^\pm_\mu$	Two Massive Bosons	$ae\hbar$	$\pm e\hbar$

Table 1: Bosonic Spectrum of the Georgi-Glashow model.

For the sake of simplicity, we choose as a minimum the vector $\vec{a} = (0, 0, a)$. Then, working in the so-called unitary gauge, the fluctuation vector $\vec{\chi}$ can be written in such a way that $\vec{\chi} = (0, 0, \chi)$. At this point, we should expand the Higgs field around $\vec{a}$ in order to read the masses of the bosons from the quadratic term coefficients:

$$V(\phi) \simeq \frac{1}{2}\left(\sqrt{2\lambda}a\right)^2 \chi^2 = \left(\frac{M_\chi}{\hbar}\right)^2 \chi^2 \tag{3.7a}$$

$$\frac{1}{2} D_\mu \vec{\phi} \cdot D^\mu \vec{\phi} \simeq \frac{1}{2} D_\mu \vec{a} \cdot D^\mu \vec{a} + \frac{1}{2}(ea)^2 = \frac{1}{2} D_\mu \vec{a} \cdot D^\mu \vec{a} + \frac{1}{2}\left(\frac{M_W}{\hbar}\right)^2 \tag{3.7b}$$

To deduce the charges of the bosons, instead, we should see how the covariant derivative, which describese the minimal coupling for the photon A_μ:

$$\nabla_\mu = \partial_\mu + i\frac{Q}{\hbar} A_\mu \tag{3.8}$$

is embedded in the SO(3) covariant derivative. Then, the value given to Q for each boson represents the charge of the bosons.

Vacuum Configurations Another aspect of the Georgi-Glashow model which is relevant in order to understand the 't Hooft-Polyakov monopoles is the study of the vacuum configurations.
Using Noether's theorem, the stress energy tensor can be calculated. To be more specific, we are interested in its component T^{00}, which describes the energy density:

$$T^{00} = \frac{1}{2}\vec{E}^i \cdot \vec{E}^i + \frac{1}{2}\vec{B}^i \cdot \vec{B}^i + \frac{1}{2}D_0\vec{\phi} \cdot D_0\vec{\phi} + \frac{1}{2}D_i\vec{\phi} \cdot D_i\vec{\phi} + V(\phi), \tag{3.9}$$

where $\vec{E}^i = -\vec{G}^{0i}$ and $\vec{B}^i$ is defined through this relation: $\vec{G}_{ij} = \epsilon_{ijk}\vec{B}_k$.
Imposing that the energy density T_{00}, contained in expression (3.9), is null, we get the conditions for the vacuum configurations:

$$\vec{G}_{\mu\nu} = 0 \qquad D^\mu\vec{\phi} = 0 \qquad V(\phi) = 0 \tag{3.10}$$

The last two of those conditions give the so-called *Higgs Vacuum* configurations. In the Higgs vacuum, the Higgs field is such that $\vec{\phi}^2 = \vec{a}^2$. It follows that such vacuum configurations are not invariant under the whole SO(3) but under the subgroup SO(2) $\simeq$ U(1), which is the gauge group of electromagnetism. Hence, the Georgi-Glashow model undergoes a spontaneous symmetry breaking process.

3.3 The 't Hooft-Polyakov Ansatz

Hypotheses We are now interested in solutions of the Euler-Lagrange equations of the Georgi-Glashow model, which we require to have the following properties:

1. finite energy;
2. stability;
3. spherical symmetry;
4. static nature.

In order to satisfy the first condition, we require that the following integral has a finite value:

$$E = \int_{\mathbb{R}^3} T_{00}\, \mathrm{d}^3x\,. \tag{3.11}$$

This is equivalent to asking that at spatial infinity the Higgs Field $\vec{\phi}$ approaches the Higgs Vacuum, which is represented by the set $\mathcal{M}_0 = \{\vec{\phi}(\vec{r}) \mid \vec{\phi}^2 = \vec{a}^2\}$.

Hence, denoting as Σ_∞ the spherical surface in $\mathbb{R}^3$ with an infinite radius, we require the existence of a continuous function,

$$\phi_\infty \colon \Sigma_\infty \to \mathcal{M}_0, \tag{3.12}$$

such that:

$$\lim_{|\vec{r}|\to\infty} \vec{\phi}(|\vec{r}|\ \hat{r}) = \vec{\phi}_\infty(\hat{r}). \tag{3.13}$$

Those functions belong to the second homotopy group $\Pi_2(\mathbb{R}^3)$ and can be classified according to their *winding number*, which is a topological invariant specifying the number of times the map wraps around $\mathcal{M}_0$ (see appendix B).

The second condition can be read as the request for non-dissipative solutions, that will never evolve in time to a configuration where $\vec{\phi}_\infty$ is constant. In mathematical terms, we are asking for the *winding number* of the map $\vec{\phi}_\infty$ to be different from 0.

The third and fourth conditions are imposed in order to simplify the problem: we will discuss a more general result, obtained without them, in section 3.4. By static nature of the solutions, we mean that that the fields have to be time-indipendent and that $\vec{W}^{\mu=0} = 0$ at any time. The latter is more than a simple gauge-fixing, because we require the time component of the gauge field to be null *at any time*, which equates to saying that the gauge-fixing transformation, which brings us to $\vec{W}^{\mu=0} = 0$, has to be time-indipendent. Taking the expression (3.9) for the energy density and imposing the static nature condition, we get that:

$$E = \int_{\mathbb{R}^3} T_{00} = -\int_{\mathbb{R}^3} \mathcal{L} = -L. \tag{3.14}$$

Search for the solutions The spherical symmetry condition, instead, allows us to claim that there might exist two real functions of the paramer $\xi \equiv aer$, where a is the module of the Higgs minimum $\vec{a}$ and e is the electric charge: $H(\xi)$ and $K(\xi)$. We rescale the radial variable as ξ for convenience in our future calculations. Hence, the ansatz for the solution to the Euler-Lagrange equations of the Georgi-Glashow model might have this form:

$$\vec{\phi}(\vec{r}) = \frac{\vec{r}}{er^2} H(\xi) \tag{3.15a}$$

$$W_a^{\mu=i} = -\epsilon_{aij} \frac{r^j}{er^2}(1 - K(\xi)) \tag{3.15b}$$

$$W_a^{\mu=0} = 0. \tag{3.15c}$$

Plugging these fields into the Euler-Lagrange equations of the model, we obtain two differential equations describing the dynamics of $H(\xi)$ and $K(\xi)$:

$$\xi^2 \frac{\mathrm{d}^2 K}{\mathrm{d}\xi^2} = KH^2 + K(K^2 - 1) \tag{3.16a}$$

$$\xi^2 \frac{\mathrm{d}^2 H}{\mathrm{d}\xi^2} = 2K^2 H + \frac{\lambda}{e^2} H(H^2 - \xi^2), \tag{3.16b}$$

In order to find the boundary conditions for the differential equations above, we should write the energy of the system as a function of $H(\xi)$ and $K(\xi)$:

$$\begin{aligned} E = \frac{4\pi a}{e} \int_0^\infty \frac{\mathrm{d}\xi}{\xi^2} \Big(\xi^2 \frac{\mathrm{d}H}{\mathrm{d}\xi} + \frac{1}{2} \left(\xi \frac{\mathrm{d}H}{\mathrm{d}\xi} - H \right)^2 + \frac{1}{2} \left(K^2 - 1 \right)^2 + \\ + K^2 H^2 + \frac{\lambda}{4e^2} \left(H^2 - \xi^2 \right)^2 \Big)\,; \end{aligned} \tag{3.17}$$

and require that such an integral has a finite value. It follows that:

$$\xi \to \infty: \qquad \frac{H}{\xi} \to 1 \qquad K \to 0 \tag{3.18a}$$

$$\xi \to 0: \qquad H \sim O(\xi) \qquad K \sim \ 1 + O(\xi)\,. \tag{3.18b}$$

We should note that the condition that $H(\xi)$ approaches spatial infinity linearly in ξ implies that asymptotically the Higgs field has the following form:

$$\vec{\phi}_\infty(\vec{\hat{r}}) = \lim_{|\vec{r}| \to \infty} \frac{\vec{r}}{er^2} H(aer) = a\hat{r}. \tag{3.19}$$

This means that $\vec{\phi}_\infty$ is homotopic to the identity and has a winding number equal to 1. Therefore, those boundary conditions also ensure the non-dissipative nature of the 't Hooft-Polyakov solution.

Another aspect that deserves to be discussed is the existence of a solution for the system of equations (3.16a) and (3.16b), given the boundary conditions (3.18a) and (3.18b). It has been proved that a solution actually exists [9].

Asymptotic form for the 't Hooft-Polyakov Solutions We wish now to study the behaviour of the solutions of equation (3.16a) and (3.16b) in the limits for $r \to 0$ and $r \to \infty$ and then finally show that they describe an object of finite size that at big distances behaves analogously to the Dirac Monopole.

In the $r \to 0$ limit, we observe that the solutions for the fields, given the boundary conditions (3.18a) and (3.18b), show no singularity:

$$\begin{cases} H \sim \xi^2 \\ K \sim 1 + C\xi^2 \end{cases} \implies \begin{cases} \vec{\phi} \sim \vec{r} \\ W_a^i \sim \epsilon_{aij} x^j\,. \end{cases} \tag{3.20}$$

This implies that also $\vec{G}_{\mu\nu}$ is smooth and regular in the origin. Such evidence is a relevant difference between the Dirac Monopole, where the fields show a singularity at the origin, and the 't Hooft-Polyakov Monopole, where no singularity in the origin occurs.

In the limit $r \to \infty$, equations (3.16a) and (3.16b), provided the boundary conditions (3.18a) and (3.18b), have such form:

$$\begin{cases} \frac{\mathrm{d}^2 K}{\mathrm{d}\xi^2} = K \\ \frac{\mathrm{d}^2 h}{\mathrm{d}\xi^2} = \frac{2\lambda}{e^2} h \end{cases} \implies \begin{cases} K \sim e^{-\xi} = e^{-\frac{M_W r}{\hbar}} \\ h \sim e^{-\xi\sqrt{\frac{2\lambda}{e^2}}} = e^{-\frac{M_H r}{\hbar}} \end{cases}, \tag{3.21}$$

where $h := H - \xi$. Hence, the object we are looking at has finite dimensions, comparable with the Compton wave lengths $\frac{\hbar}{M_H}$ and $\frac{\hbar}{M_W}$.

In order to prove that such an object at spatial infinity behaves like a monopole, we should look at the electromagnetic field, which, in the Georgi-Glashow model is described by the following expression:

$$F_{\mu\nu} = \frac{1}{a}\vec{\phi} \cdot \vec{G}_{\mu\nu}, \tag{3.22}$$

which in the limit $r \to \infty$ gives:

$$F_{0i} = 0 \qquad F_{ij} = \epsilon_{ijk} \frac{r^k}{er^3}. \tag{3.23}$$

Hence, the asymptotic magnetic and electric fields are:

$$\vec{E} = 0 \qquad \vec{B} = -\frac{1}{e}\frac{\vec{r}}{r^3}; \tag{3.24}$$

they are compatible with a magnetic charge of:

$$m = \frac{-4\pi}{e}, \tag{3.25}$$

which is admitted by the Dirac quantization condition.

3.4 Extension to non-static and non spherically symmetric solutions

In this section we will try to extend the 't Hooft-Polyakov ansatz to generic finite-energy and non-dissipative configurations. Now, unlike section 3.3, we do not require the solution to be static and spherically symmetric.
In the static and spherically symmetric case, we proved that the Higgs field $\vec{\phi}$ approaches the Higgs vacuum at spatial infinity up to terms of the order

of $O(\exp(-r/R))$, where R is the dimension of the monopole. Therefore, it seems reasonable to suppose that in a generic finite-energy configuration the solutions to the Georgi-Glashow model satisfy the Higgs vacuum condition everywhere except for a finite numer of localised and compact regions, whose dimension is of the order of R.

We now consider a closed surface Σ that lies in the Higgs vacuum region and try to estimate the amount of magnetic charge contained within it. First, we should notice that in the Higgs vacuum, since $\vec{\phi} \times \vec{W}_\mu = -\frac{1}{e}\partial_\mu \vec{\phi}$, the field $\vec{W}_\mu$ has the component orthogonal to $\vec{\phi}$ that can be deduced from $\vec{\phi}$, whereas the one parallel to $\vec{\phi}$ is known to be the electromagnetic potential A_μ. Hence, the field $\vec{W}_\mu$ has this form:

$$\vec{W}_\mu = \frac{1}{a^2 e}\vec{\phi} \times \partial_\mu \vec{\phi} + \frac{1}{a}\vec{\phi} A_\mu. \tag{3.26}$$

Such an expression allows us to calculate the field-strength $\vec{G}_{\mu\nu}$ and the electromagnetic tensor $F_{\mu\nu} = \frac{1}{a}\vec{\phi} \cdot \vec{G}_{\mu\nu}$ in the Higgs vacuum, as a function of $\vec{\phi}$ and A_μ:

$$F_{\mu\nu} = \frac{1}{a^3 e}\vec{\phi} \cdot \left(\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi}\right) + \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{3.27}$$

Since Σ lies in the Higgs vacuum, the form of the magnetic field on Σ can be deduced from expression (3.27):

$$B_i = \epsilon_{ijk} F_{jk} = \epsilon_{ijk} \left[\frac{1}{a^3 e}\vec{\phi} \cdot \left(\partial_j \vec{\phi} \times \partial_k \vec{\phi}\right) + \partial_j A_k - \partial_k A_j \right] . \tag{3.28}$$

Therefore, the total magnetic charge g_Σ contained in Σ can be calculated as:

$$\begin{aligned} g_\Sigma &= \int_\Sigma \vec{B} \cdot \mathrm{d}\vec{S} \\ &= -\frac{1}{2ea^2} \int_\Sigma \epsilon_{ijk} \vec{\phi} \cdot \left(\partial_j \vec{\phi} \times \partial_k \vec{\phi}\right) \mathrm{d}S_i \ . \end{aligned} \tag{3.29}$$

It can be proved that the quantity:

$$N_\Sigma := \frac{1}{8\pi a^3} \int_\Sigma \epsilon_{ijk} \vec{\phi} \cdot \left(\partial_j \vec{\phi} \times \partial_k \vec{\phi}\right) \mathrm{d}S_i \tag{3.30}$$

is an integer and corresponds to the winding number of the map $\vec{\phi} \colon \Sigma \to \mathcal{M}_0 = \{\vec{\phi} : \vec{\phi}^2 = a^2\}$. Hence we have just derived a quantisation condition for the magnetic charge analogous to Dirac's up to a $\frac{1}{2}$ coefficient:

$$g_\Sigma = -\frac{4\pi}{e} N_\Sigma. \tag{3.31}$$

Furthermore, we have that the magnetic charge in Σ depends only on a topological property of the map $\vec{\phi}$ on Σ and is invariant under continuous deformations of $\vec{\phi}$ which preserve the Higgs vacuum, such as:

- time Evolution of $\vec{\phi}$;
- gauge Transformations of $\vec{\phi}$;
- changes of Σ in the Higgs vacuum.

In order to obtain the Dirac Quantization condition, we should consider that the lagrangian from which we started (expression (3.3)) is invariant under SO(3) even if the model is based on the gauge group SU(2), thanks to the fact that SO(3) $\simeq$ SU(2)/ $\mathbb{Z}_2$. However, this implies that the minimum charge allowed using SO(3) as a gauge field instead of SU(2) is $e_{min} = \frac{e}{2}$. It follows that the 't Hooft-Polyakov monopole is characterised by the same quantisation condition as the Dirac one:

$$g = -\frac{4\pi}{e} N_\Sigma = -\frac{2\pi}{e_{min}} N_\Sigma \,. \tag{3.32}$$

3.5 Relation with the Dirac Monopole

In conclusion, the Dirac Monopole and the 't Hooft-Polyakov Monopole share the same quantization condition:

$$g = -\frac{2\pi}{e} N \qquad N \in \mathbb{Z} \tag{3.33}$$

However, there are many relevant differences between the two, which are summarized below:

- **Postulate** *vs* **Necessity**

  The Dirac Monopole was introduced in the framework given by electromagnetism and its existence was assumed by Dirac as a "postulate". Claiming that nature should respect duality symmetry and assuming the monopole's existence, he showed that we can derive the well observed and otherwise theoretically unexplained quantization of the electric charge. The 't Hooft-Polyakov Monopole, instead, is intrinsic to the Georgi-Glashow model and can be deduced for any non-Abelian gauge theory with a compact covering group. It arises from a spontaneous symmetry breaking and fulfills the same quantization condition of the Dirac Monopole.

- **Singularites** *vs* **Regularites**

  The existence of the Dirac Monopole causes the existence of an origin-located singularity for the field-strength of electromagnetism $F_{\mu\nu}$. The 't Hooft-Polyakov ansatz for the solutions of the Georgi-Glashow model does not require the existence of any singularity for the field strength $\vec{G}_{\mu\nu}$.

- **Different derivations for the quantization condition**

  The quantisation of the Dirac Monopole, is obtained defining two different potentials up to a gauge transformation, and imposing the particle wavefunction to be single-valued. The quantisation of the 't Hooft-Polyakov monopole charge is due to a topological property: the winding number of the map $\vec{\phi}_\infty$, which goes from a compact closed surface containing the monopoles to a 2-dimensional sphere. For such a reason it is often addressed as a *topological charge*.

3.6 Generalisation of the Ansatz to Dyons

The 't Hooft-Polyakov Ansatz can be extended to dyons relaxing the hypotheses imposed at the beginning of section 3.3, as it was done in the 1975 work of B.Julia and A.Zee. [5].

In fact we may take the Lagrangian of the Georgi-Glashow model, which can be found in expression (3.3), and find solutions satisfying the following hyptheses:

1. finite energy;
2. stability;
3. spherical symmetry;
4. time independence.

It should be noted that we have only changed condition 4, relaxing our request on the static nature of the solutions: we want them not to depend on time, but $\vec{W}^{\mu=0}$ can be different from zero. Hence we may think that, given three radial real functions $J(\xi)$, $H(\xi)$ and $K(\xi)$, the ansatz for the solutions to the Euler-Lagrange equations of the system has the form:

$$\vec{\phi}(\vec{r}) = \frac{\vec{r}}{er^2}H(\xi) \tag{3.34}$$

$$W_a^{\mu=0} = \frac{\vec{r}}{er^2}J(\xi) \tag{3.35}$$

$$W_a^{\mu=i} = -\epsilon_{aij}\frac{r^j}{er^2}(1-K(\xi))\,. \tag{3.36}$$

Plugging them into the Euler-Lagrange equations we obtain a system of three differential equations describing the dynamics of the functions $J(\xi)$, $H(\xi)$

and $K(\xi)$:

$$\xi^2 \frac{\mathrm{d}^2 J}{\mathrm{d}\xi^2} = 2JK^2 \tag{3.37a}$$

$$\xi^2 \frac{\mathrm{d}^2 H}{\mathrm{d}\xi^2} = 2K^2 H + \frac{\lambda}{e^2} H(H^2 - \xi^2), \tag{3.37b}$$

$$\xi^2 \frac{\mathrm{d}^2 K}{\mathrm{d}\xi^2} = K\left(K^2 - J^2 + H^2 - 1\right). \tag{3.37c}$$

It should be noticed that the equations (3.16a) and (3.16b), found by 't Hooft and Polyakov, can be obtained from those above computing the limit $J \to 0$. The boundary conditions on $H(\xi)$ and $K(\xi)$ are analogous to expressions (3.18a) and (3.18b), whereas the boundary conditions on $J(\xi)$ are the following ones:

$$\xi \to \infty: \;\; J \to 0 \tag{3.38a}$$

$$\xi \to 0: \;\; J \sim \; 1 + O(\xi). \tag{3.38b}$$

With a procedure similar to the one used in section 3.3, we can calculate the electromagnetic tensor:

$$F_{\mu\nu} = \frac{1}{a}\vec{\phi} \cdot \vec{G}_{\mu\nu}, \tag{3.39}$$

and notice that at spatial infinity the magnetic field behaves as that produced by the 't Hooft-Polyakov Monopole:

$$F_{ij} = \epsilon_{ija} \frac{r_a}{r}\left(-\frac{1}{er^2}\right). \tag{3.40}$$

However, what is new is that a non-zero electric field appears:

$$E^i = F^{0i} = -\frac{r^i}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{J(r)}{r}\right], \tag{3.41}$$

and gives a total charge, which can be written as:

$$\begin{aligned} Q &= \int \mathrm{d}S_i \, F_{0i} = \\ &= \int \mathrm{d}^3 x \, \partial_i F_{0i} \\ &= -\frac{8\pi}{e}\int_0^\infty \mathrm{d}r \, \frac{JK^2}{r}. \end{aligned} \tag{3.42}$$

In conclusion, we get an object which has a quantised magnetic charge but has an electric charge with no quantisation condition.

3.7 The Monopole Mass

We wish to estimate the mass (or, equivalently, energy) of the 't Hooft-Polyakov monopole: we can actually derive a more general bound for a dyon, and then get the monopole case by setting the electric charge to be equal to zero. The energy of a dyon is generically given by the integral

$$E = \int_{\mathbb{R}^3} \frac{1}{2}\Big(E_i^a E_i^a + B_i^a B_i^a + D_\mu \phi^a D_\mu \phi^a \Big) + V(\phi)\,, \tag{3.43}$$

where $G^a_{0i} = E^a_i$ and $G^a_{ij} = \varepsilon_{ijk} B^a_k$.

We will derive a lower bound for this mass [10]; then, we will explicitly show that this bound can be reached by a specific solution: the BPS monopole.

3.7.1 The Bogomol'nyi Bound

First of all we drop the (always positive) terms $D_0\phi^a D_0\phi^a$ and $V(\phi)$. Then, we apply the famous "squaring trick": we add and subtract some terms depending on a generic angle θ:

$$\begin{aligned} E \geq \frac{1}{2}\int_{\mathbb{R}^3} E_i^a E_i^a + B_i^a B_i^a + D_i\phi^a D_i\phi^a + 2E_i^a D_i\phi^a \sin\theta - 2E_i^a D_i\phi^a \sin\theta \\ + 2B_i^a D_i\phi^a \cos\theta - 2B_i^a D_i\phi^a \cos\theta\,, \end{aligned} \tag{3.44}$$

and then we make them into squares, which we can discard while still maintaining the bound:

$$\begin{aligned} E \geq \frac{1}{2}\int_{\mathbb{R}^3} & \overbrace{\Big(E_i^a E_i^a - 2E_i^a D_i\phi^a \sin\theta + D_i\phi^a D_i\phi^a \sin^2\theta \Big)}^{(E_i^a - D_i\phi^a \sin\theta)^2} + \\ & + \overbrace{\Big(B_i^a B_i^a - 2B_i^a D_i\phi^a \cos\theta + D_i\phi^a D_i\phi^a \cos^2\theta \Big)}^{(B_i^a - D_i\phi^a \cos\theta)^2} + \\ & + 2E_i^a D_i\phi^a \sin\theta + 2B_i^a D_i\phi^a \cos\theta\,. \end{aligned} \tag{3.45}$$

The squares are positive definite, therefore the monopole mass is just bounded by

$$E \geq \int_{\mathbb{R}^3} E_i^a D_i\phi^a \sin\theta + B_i^a D_i\phi^a \cos\theta\,, \tag{3.46}$$

for *any* θ. Now, we use the following facts: by the Bianchi identities $D_\mu {}^*G^{\mu\nu,a} = 0$ we have $D_i B_i^a = 0$, therefore $B_i^a D_i\phi^a = D_i(B_i^a \phi^a)$. Also, the equations of motion read $D_\nu G^{\mu\nu,a} = e\varepsilon_{abc}\phi^c D^\mu \phi^b$ and $D_i G_{0i,a} = D_i E_i^a$. Therefore:

$$D_i\phi^a E_i^a = D_i(\phi^a E_i^a) - \phi^a D_i E_i^a = D_i(\phi^a E_i^a) - \phi^a e\varepsilon_{abc}\phi^c D^\mu \phi^b = D_i(\phi^a E_i^a)\,. \tag{3.47}$$

With these two facts we can frame all of the integrand as a divergence,

$$E \geq \int_{\mathbb{R}^3} D_i \left(E_i^a \phi^a \sin\theta + B_i^a \phi^a \cos\theta \right) \mathrm{d}^3 x \ , \tag{3.48}$$

and apply Stokes' theorem:

$$E \geq \int_{\Sigma_\infty} \phi^a \left(E_i^a \sin\theta + B_i^a \cos\theta \right) \mathrm{d}S_i \ . \tag{3.49}$$

Now, by the boundary conditions on ϕ, it will be radially outward and with absolute value a at infinity: therefore its product with the electric and magnetic fields (which are also radial at infinity) reduces to

$$E \geq a(q \sin\theta + g\cos\theta) \,, \tag{3.50}$$

where q and g are the electric and magnetic charges. The sharpest bound occurs when the (q, g) and $(\sin\theta, \cos\theta)$ 2D vectors are aligned, and corresponds to their euclidean scalar product: the final bound we get is thus found by the saturation of the Cauchy-Schwarz inequality:

$$E \geq a\sqrt{q^2 + g^2}. \tag{3.51}$$

In the 't Hooft-Polyakov monopole case, this simplifies to $E \geq a|g|$.

3.7.2 Saturation of the Bogomol'nyi Bound

A state which attains the lower bound is called a Bogomol'nyi – Prasad – Sommerfield — for short BPS — state.

In our derivation we discarded always-positive quantities in the expression of the energy integrals; hence, in order to saturate the bound, they must be assumed to vanish everywhere. In addition we assume our solution to be static: this implies that electric charge will be zero. This yields the conditions:

$$V(\vec{\phi}) = 0 \tag{3.52a}$$
$$D_0 \phi^a = 0 \tag{3.52b}$$
$$E_i^a = 0 \tag{3.52c}$$
$$B_i^a = \pm D_i \phi^a \,. \tag{3.52d}$$

Now, we make an interesting observation: if $\lambda \neq 0$ in condition (3.52a) we must have that $\phi^a \phi^a = a^2$ everywhere: therefore $D_i(\phi^a \phi^a) = 2\phi^a D_i \phi^a = 0$; an analogous equation holds with $D_i \to \partial_i$. However, we also have equation (3.52d): using it, we can also derive $\phi^a D_i \phi^a \propto \phi^a B_i^a = 0$: the solution's magnetic field along ϕ is trivial. Therefore, in order to saturate the bound for a nontrivial solution we must assume $\lambda = 0$.

If we wish to study this saturation despite this issue, we can insert the 't Hooft-Polyakov *ansatz* in the Bogomol'nyi condition (3.52d): we get the following set of differential equations:

$$\xi \frac{\mathrm{d}K}{\mathrm{d}\xi} = -KH \tag{3.53a}$$

$$\xi \frac{\mathrm{d}H}{\mathrm{d}\xi} = -K^2 + H + 1\,; \tag{3.53b}$$

these can be solved analytically, when coupled to the regular monopole equations of motion: one gets $H(\xi) = \xi \coth(\xi) - 1$ and $K(\xi) = \xi \operatorname{sech}(\xi)$.

3.7.3 Numerical estimation

We wish to give numerical values to our estimates of monopole masses. We start from the 't Hooft-Polyakov monopole: from the Bogomol'nyi bound it follows that $E \geq a|g|$, while from the quantization condition we have that the least value g can attain is $4\pi/e$, where e is the elementary electric charge.

In (Lorentz-Heaviside) natural units, we can express the vacuum expectation value of the Higgs field as $a = M_W/e$, where M_W is the mass of either W boson.

Therefore we have $E \geq 4\pi a/e = 4\pi M_W/e^2 = M_W/\alpha$.

Since M_W has been experimentally measured to be around 90 GeV, this means that $E \gtrsim 12\,\mathrm{TeV}$.

Some numerical simulations by Julia and Zee [5] for equations (3.16) have shown that numerical solutions to the monopole differential equations exist close to this bound, at $E = 1.18 M_W/\alpha$.

They also made similar simulations for dyons according to equations (3.37): they found $E \approx 1.25 M_W/\alpha$ for a dyon with electric charge of $Q \approx 44e$ and $E \approx 1.85 M_W/\alpha$ for $Q \approx 169e$.

This shows that, even while relaxing the condition of trivial potential $V(\vec{\phi})$ ($\lambda = 0$) one can get rather close to the BPS bound for the monopole mass. For dyons we still have the BPS bound, but the reasoning from before cannot be directly applied since we do not have a quantization condition for the electric charge as we do for the magnetic one.

A Differential forms and Poincaré's lemma

In this appendix we present a brief overview of differential forms, following [12].

Differential forms are a useful tool when working with completely antisymmetric tensors: in a space of dimension d one can define tensors of any rank n, but if we restrict ourselves to the antisymmetric ones, which satisfy:

$$F_{\mu_1 \mu_2 \ldots \mu_n} = F_{[\mu_1 \mu_2 \ldots \mu_n]} \tag{A.1}$$

we can see that they only have $\binom{d}{n}$ independent components: they can be nontrivial only if $n \leq d$.

The differential form F associated with the tensor $F_{\mu_1 \ldots \mu_n}$ is defined as:

$$F = \frac{1}{n!} F_{\mu_1 \ldots \mu_n} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \ , \tag{A.2}$$

where $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$ is the canonical basis of the exterior algebra $\bigwedge^n(V)$, where V is the vector space over which the tensors are constructed. This basis is antisymmetric: $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} = \mathrm{d}x^{[\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n]}$.

The wedge product is an antisymmetrized tensor product: it maps a p-form A and a q-form B to a $(p+q)$-form $A \wedge B$, defined by:

$$A \wedge B = \frac{1}{p!q!} A_{[\mu_1 \ldots \mu_p} B_{\nu_1 \ldots \nu_q]} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_q} \ . \tag{A.3}$$

It is *graded* antisymmetric: $A \wedge B = (-)^{pq} B \wedge A$.

The number of independent components being given by the binomial coefficient tells us that the spaces of n-forms and $(d-n)$-forms have the same dimension: it is natural to seek a connection between them. This is realized by *Hodge duality*: an n-form F is mapped to the $(d-n)$-form $*F$, defined by:

$$* F = \frac{1}{n!(d-n)!} \varepsilon_{\mu_1 \ldots \mu_d} F^{\mu_1 \cdots \mu_n} \, \mathrm{d}x^{\mu_{n+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_d} \ . \tag{A.4}$$

The exterior differential maps an n-form F to an $(n+1)$-form $\mathrm{d}F$, and it is defined as follows: consider the derivative operator form $\partial \equiv \mathrm{d}x^\mu \, \partial_\mu$. Then, $\mathrm{d}F \equiv \partial \wedge F$.

The functions usually considered are sufficiently regular, therefore by Schwarz's theorem $\mathrm{d}^2 = 0$.

An n-form F is called *closed* if $\mathrm{d}F = 0$ and *exact* if there exists an $(n-1)$-form A such that $F = \mathrm{d}A$. By $\mathrm{d}^2 = 0$ we have that exact implies closed, while the inverse is not generally true.

A *contractible* subset of a topological space is one which can be continuously deformed to a point. *Poincaré's lemma* states that, if a form is defined on a contractible set and it is closed, then it is also exact.

The non-contractibiliy of spaces is caused by topological nontriviality: the quotient of the closed n-forms modulo the exact n-forms characterizes this and is called the n-th de Rham cohomology group.

B Homotopy

Consider two topological spaces X and Y: two functions $f, g\colon X \to Y$ are said to be *homotopic* if there exists a continuous deformation $H\colon X \times [0,1] \to Y$ such that for any $x \in X$ we have $H(x,0) = f(x)$ while $H(x,1) = g(x)$.

Now, consider a topological space X. We denote as $\mathcal{M}(x_0)$ the set of all maps f from the n-cube $[0,1]^n \to X$ such that $f(t) \equiv x_0 \in X$ if t is on the boundary of the n-cube ($t \in \partial[0,1]^n$).

The composition of two maps $f, g \in \mathcal{M}(x_0)$ is defined as:

$$(f+g)(\vec{x}) = \begin{cases} f(2x_1, x_2, \ldots, x_n) & x_1 \in [0,1/2] \\ g(2x_1 - 1, x_2, \ldots, x_n) & x_1 \in [1/2,1]\,. \end{cases} \tag{B.1}$$

Then, the quotient of $\mathcal{M}(x_0)$ with respect to the homotopy equivalence relation (denoted by $\sim$) considered with the operation of composition $+$, formally $(\mathcal{M}(x_0)/\sim, +)$, is called the n-th homotopy group $\Pi_n(X; x_0)$. The dependence on x_0 is actually trivial: it only matters to which connected component of X it belongs. If the space is arcwise connected, there is no meaningful dependence on x_0 whatsoever.

The first homotopy group classifies loops: for example, for a torus $\mathbb{T}^2$ we have $\Pi_1(\mathbb{T}^2) = (\mathbb{Z}^2, +)$ since we have two independent non-homotopic ways of going around it, and we can go around the torus any amount of times along either.

Some notable cases for higher homotopy groups are: $\Pi_n(S^m) = 0$ for $n < m$ and $\Pi_n(S^n) = (\mathbb{Z}, +)$.

References

[1] M. P. Curie, *Sur la possibilité d'existence de la conductibilité magnétique et du magnétisme libre*, Seances de la Societé Francaise de Physique (Paris), p. 76 (1894), http://www.archive.org/stream/sancesdelasocit19physgoog#page/n82/mode/2up

[2] P. A. M. Dirac, *Quantised singularities in the electromagnetic field*, Proc. R. Soc. A133 (1931), 60-72.

[3] G. 't Hooft, *Magnetic monopoles in unified gauge theories*, Nuc.Phys. B79 (1974), 276-284.

[4] A. M. Polyakov, *Particle spectrum in quantum Field theory*, JETP Letters 20 (1974), 194-195.

[5] B. Julia and A. Zee, *Poles with both magnetic and electric charges in non-abelian gauge theories*, Phys. Rev. D11 (1975), 2227-2232.

[6] J. S. Schwinger, *Magnetic charge and quantum field theory*, Phys. Rev. 144 1087, (1966).

[7] Daniel Zwanziger,*Quantum Field Theory of Particles with Both Electric and Magnetic Charges* Phys. Rev. 176, 1489 (1968).

[8] Figueroa J.M. O'Farrill, *Electromagnetic Duality for Children*, ntcatlab.org (1998).

[9] C. H. Taubes, *Stability in Yang Mills theories*, Comm. Math. Phys. 91 (1983), 473-540.

[10] E. B. Bogomolny, *Stability of Classical Solutions*, Sov. J. Nucl. Phys. 24 (1976), 449; Yad. Fiz. 24 (1976), 861.

[11] Prasad, M. K.; Sommerfield, Charles M., *An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon*, Physical Review Letters (22 September 1975), American Physical Society.

[12] K. Lechner, *Elettrodinamica classica*, Springer 2014