LGWA Forecasting

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Forecasting for the Lunar Gravitational Wave Antenna

What will be the sensitivity of the LGWA to gravitational waves?
How many sources will it detect?
How well will it constrain their parameters?

Forecasting for the Lunar Gravitational Wave Antenna

What will be the sensitivity of the LGWA to gravitational waves?

In order to answer this, let us start from the way LGWA works.

Recap: what is LGWA?

Figure from the LGWA collaboration.

LGWA payload

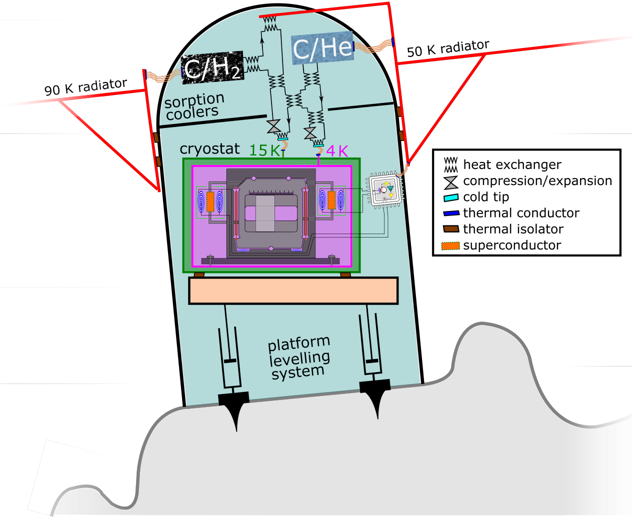


Figure from van Heijningen et al., 2023.

LGWA readout

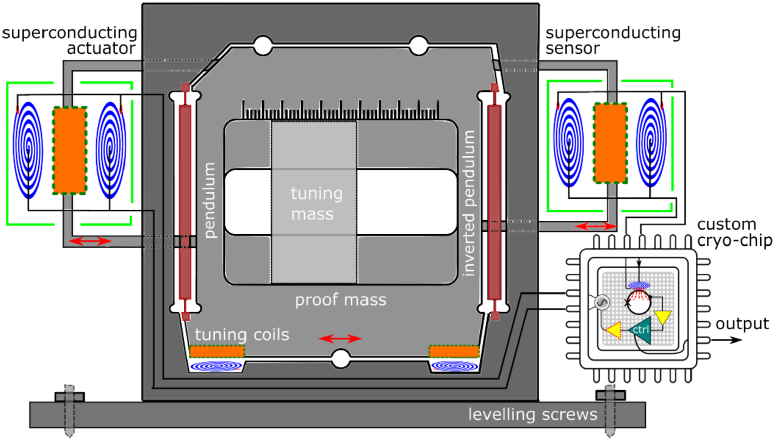


Figure from van Heijningen et al., 2023.

Colored Gaussian noise

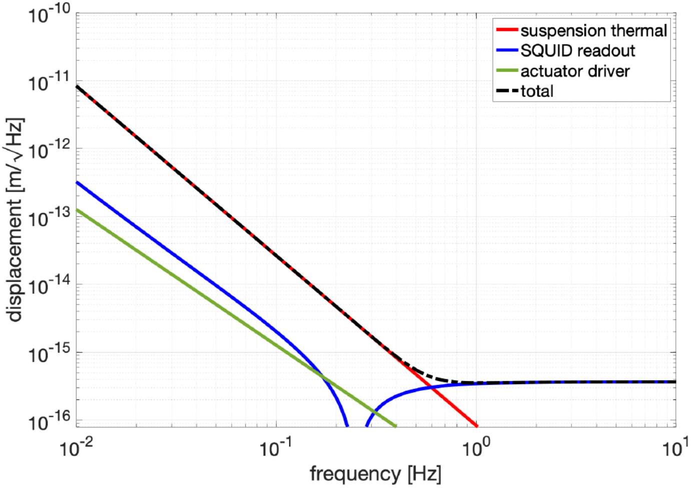
The assumption made for essentially all gravitational wave detectors is that their noise is Gaussian and colored by a certain power spectral density. Take $n(f)$ to be the Fourier transform of the noise: then

$$\langle n(f)n(f')\rangle = \frac{1}{2}S_n(f)\delta(f-f')$$

$S_n(f)$ has units of m^2/Hz if $n(t)$ is a displacement.

Sometimes we quote the ASD: $\sqrt{S_n(f)}$.

Readout noise



(b)

Figure from the LGWA collaboration

Gravitational wave response

How does this readout noise translate to noise on our measurement of gravitational waves?

We assume that the response function from the gravitational polarizations $h_{ij}(t)$ to the displacement at the sensor $s(t)$ is *linear*: in the frequency domain,

$$s(f) = L(f) D_{ij} h_{ij}(f)$$

- $L(f)$, with the dimensions of a length, is the displacement amplitude corresponding to a monochromatic unit strain;
- D_{ij} is a pattern function, which for LGWA looks like $D_{ij} = n_i b_j$, where b_j is a tangent unit vector to the surface while n_i is normal to it.

Structure of LGWA

LGWA is planned to have: - 4 stations, to reduce overall noise and perform active cancellation; - for each station, two orthogonal horizontal measurement channels.

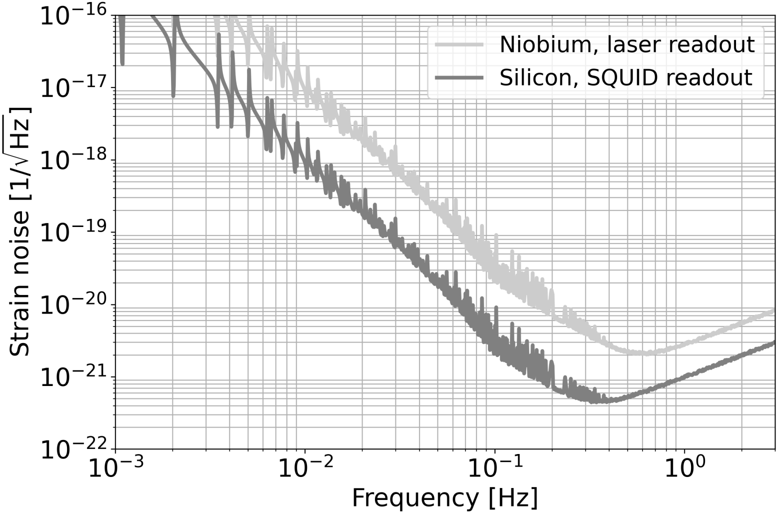
Strain noise

If we have: - an estimate for the PSD of the detector displacement noise, $S_{\mathsf{disp}}(f)$, - an estimate of the lunar response $L(f)$, - the number of stations N,

we can compute the strain noise as

$$S_{\mathsf{strain}}(f) = \frac{S_{\mathsf{disp}}(f)}{N L^2(f)}$$

LGWA estimated noise



Matched filtering

The gravitational-wave **detection** problem: we have data

$$d(t) = \underbrace{h_\theta(t)}_{\text{GW}} + \underbrace{n_{\text{Gaussian}}(t) + n_{\text{non-Gaussian}}(t)}_{\text{noise } n(t)}$$

and we want to find where $h(t)$ is, while typically $|h| \ll |n|$. The Gaussian component has a spectral density $S_n(f)$. Let's assume we want to use a **linear filter**, so that in the time domain it reads:

$$\hat{\rho}(\tau) = \int d(t+\tau) f(t) \mathrm{d}t$$

for some function $f(t)$.

This is applicable only for **modelled** signals, such as those from Compact Binary Coalescences!

We want to maximize the "**distinguishability**" of the signal: we can quantify it with the signal-to-noise ratio

$$\frac{S}{N} = \frac{\hat{\rho}(\text{a signal is present})}{\text{root-mean-square of } \hat{\rho}}$$

Ignoring the non-Gaussian part of the noise, the **optimal** solution is $\hat{\rho} \propto (d|h)$, where

$$(a|b) = 4\Re \int_0^\infty \frac{a(f)b^*(f)}{S_n(f)} \mathrm{d}f \,,$$

Optimal signal-to-noise ratio

The **signal-to-noise** ratio statistic is

$$\rho = \frac{S}{N} = \frac{(d|h)}{\sqrt{(h|h)}}$$

With the expected noise realization ($\langle n(t) \rangle = 0$):

$$\rho_{\mathsf{opt}} = \sqrt{(h|h)} = 2\sqrt{\int_0^\infty \frac{|h(f)|^2}{S_n(f)} \mathsf{d}f}\,.$$

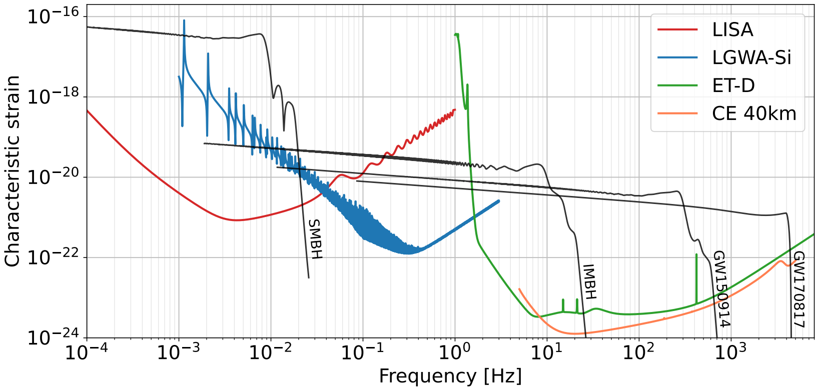
If we do not have the data, this is a good proxy. For a real detector, we will need to do injection studies and compute a False Alarm Rate (FAR).

Optimal SNR in characteristic strain

If we define: - the characteristic signal strain $h_c(f) = 2f|h(f)|^2$; - the characteristic noise strain $h_n(f) = \sqrt{fS_n(f)}$
we can rewrite the optimal SNR as

$$\rho^2_{\mathsf{opt}} = \int_0^\infty \frac{h_c^2(f)}{h_n^2(f)} \mathsf{d} \log f$$

Characteristic strain in the multiband scenario



SNR thresholds

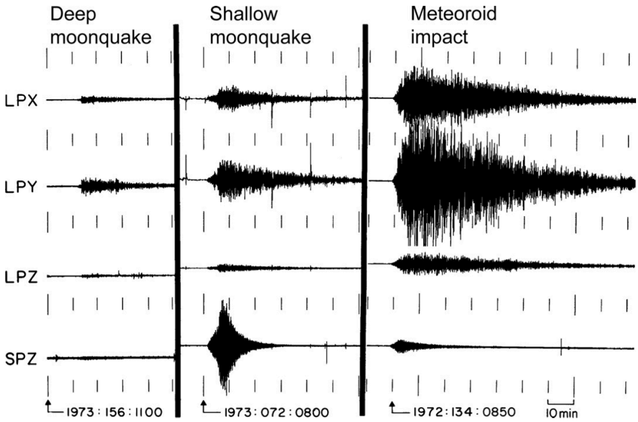
What is a "high enough" value for the SNR?

Without time shifts nor non-Gaussianities, the SNR would simply follow a χ^2 distribution with two degrees of freedom: "five σ" significance with a threshold of $\rho = 5.5$.
In real data this has to be estimated through **injections**:

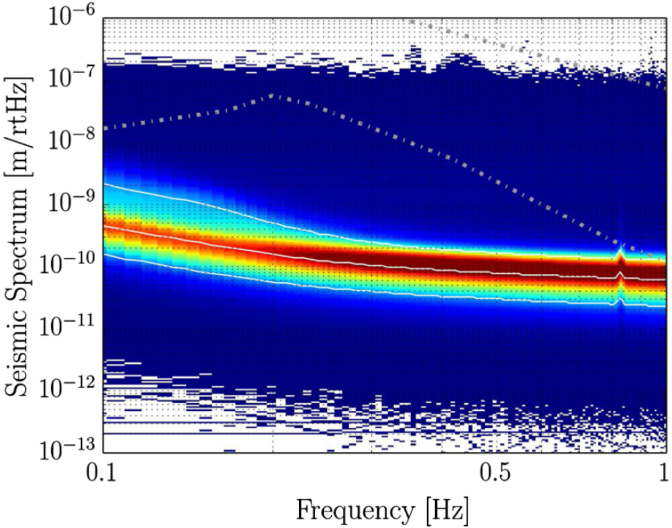
$$\mathsf{FAR} = \mathsf{FAR}_8 \exp\left(-\frac{\rho - 8}{\alpha}\right) .$$

For BNS in O1: $\alpha = 0.13$ and $\mathsf{FAR}_8 = 30000\mathsf{yr}^{-1}$.

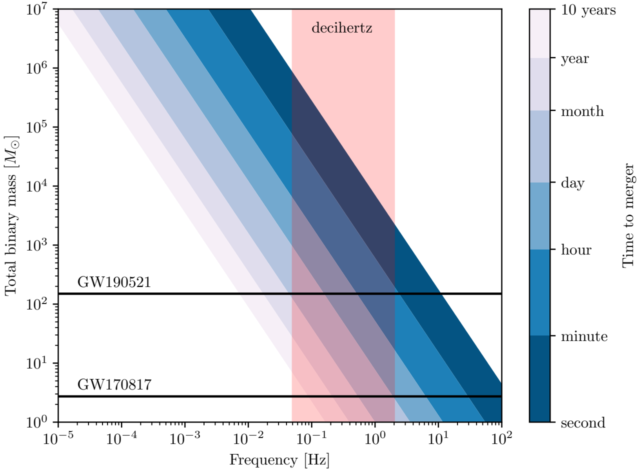
Non-Gaussian noise we are ignoring (mitigated)



Gaussian noise we are ignoring (upper bound)



Duration of CBC signals



Gravitational wave data analysis

Suppose we measure $d = h_\theta + n$, where our model for $h_\theta = h(t;\theta)$ depends on several parameters (typically, between 10 and 15).

We can estimate the parameters θ by exploring the **posterior distribution**

$$p(\theta|d) = \mathcal{L}(d|\theta)\pi(\theta) = \mathcal{N} \exp\left((d|h_\theta) - \frac{1}{2}(h_\theta|h_\theta)\right)\pi(\theta)\,,$$

where $\pi(\theta)$ is our **prior distribution** on the parameters. We are neglecting non-Gaussianities in the noise, and assuming its spectral density is known!

The posterior is explored **stochastically** (with MCMC, nested sampling...) yielding many samples θ_i distributed according to $p(\theta|d)$, with which can compute summary statistics:

- mean $\langle\theta_i\rangle$,
- variance $\sigma_i^2 = \langle(\theta_i - \langle\theta_i\rangle)^2\rangle$,
- covariance $\mathcal{C}_{ij} = \langle(\theta_i - \langle\theta_i\rangle)(\theta_j - \langle\theta_j\rangle)\rangle$.

At this stage, we are not making any approximation, and the covariance matrix is just a **summary** tool - the full posterior is still available.

Fisher matrix

In the Fisher matrix approximation, we are approximating the likelihood as

$$\mathcal{L}(d|\theta) \approx \mathcal{N} \exp\left(-\frac{1}{2}\Delta\theta^i \mathcal{F}_{ij} \Delta\theta^j\right)$$

where $\Delta\theta^i = \theta^i - \langle\theta^i\rangle$.
A **multivariate normal distribution**, with covariance matrix $\mathcal{C}_{ij} = \mathcal{F}^{-1}_{ij}$. This is a good approximation for the posterior in the high-SNR limit, since the prior matters less then.

The Fisher matrix $\mathcal{F}_{ij}$ can be computed as the scalar product of the derivatives of waveforms:

$$\mathcal{F}_{ij} = \left\langle \partial_i \partial_j \mathcal{L} \right\rangle \Big|_{\theta = \langle \theta \rangle} = (\partial_i h | \partial_j h) = 4\Re \int_0^\infty \frac{1}{S_n(f)} \frac{\partial h}{\partial \theta_i} \frac{\partial h^*}{\partial \theta_j} \mathrm{d}f \,.$$

Problems with LGWA Fisher matrices

See the talk tomorrow!

Exercises: horizons

Let us define the GW parameters as $(\theta_{\text{int}}, \theta_{\text{ext}}, z, d_L(z))$, where: - θ_{int} are the intrinsic source parameters, such as the masses; - θ_{ext} are the extrinsic parameters except for distance: inclination angles, sky position, polarization angle, phase, time of arrival; - z is the redshift and $d_L(z)$ the corresponding luminosity distance, assuming some cosmology.

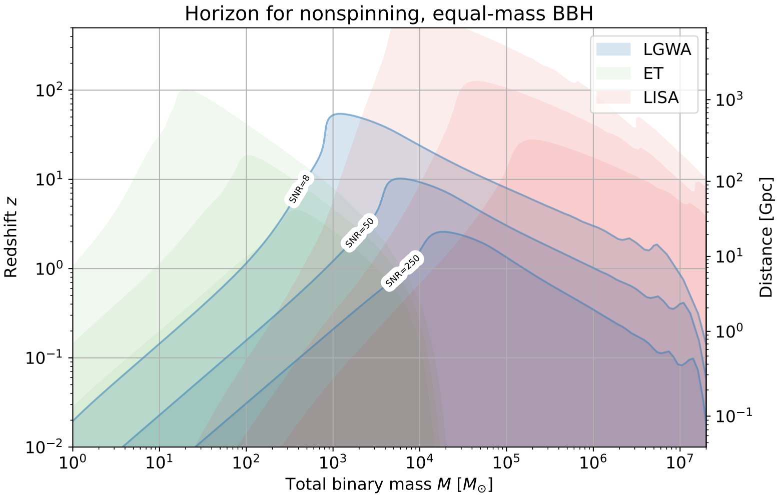
Then the horizon redshift z_{hor} for an SNR ρ^* satisfies:

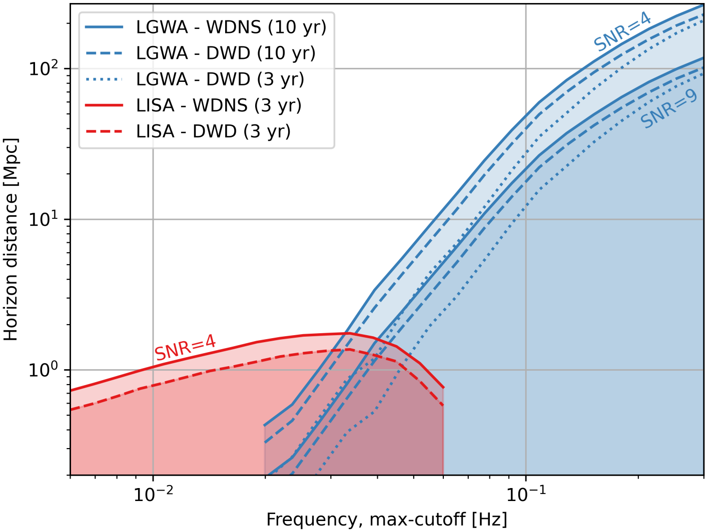
$$\rho^* = \max_{\theta_{\text{ext}}} \mathsf{SNR}(\theta_{\text{int}}, \theta_{\text{ext}}, z, d_L(z))$$

In practice, we do not need to maximize over all source parameters. The loudest inclination angle is $\iota = 0$, and at this angle both polarizations h_+ and $h_\times$ have equal amplitude, therefore we can also set $\psi = 0$ without fear of straying from the maximum.

The overall phase does not affect the SNR so we can also set it to zero, and finally the motion of the Moon is symmetric enough that we do not expect to be far from a maximum if we assume a fixed time as opposed to maximising over it.

Therefore, we just maximise over the sky position (right ascension and declination).


Horizon for nonspinning, equal-mass BBH
LGWA
ET
LISA
SNR=8
SNR=50
SNR=250
Redshift z
Distance [Gpc]
Total binary mass M [M⊙]
10^-2
10^-1
10^0
10^1
10^2
10^-1
10^0
10^1
10^2
10^3
10^4
10^5
10^6
10^7



LGWA - WDNS (10 yr)
LGWA - DWD (10 yr)
LGWA - DWD (3 yr)
LISA - WDNS (3 yr)
LISA - DWD (3 yr)
SNR=4
SNR=9
SNR=4
Horizon distance [Mpc]
10^2
10^1
10^0
10^{-2}
10^{-1}
Frequency, max-cutoff [Hz]

Exercises: IMBH population