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Relativistic non-ideal flows

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Abstract

Stuff emits radiation when it falls into a black hole. I'd like to see exactly how much of it.

1 Notational preface

I will use greek indices (μ, ν, $\rho\dots$) to denote 4-dimensional indices ranging from 0 to 3, and latin indices (i, j, $k\dots$) to denote 3-dimensional indices ranging from 1 to 3.

I will use the "mostly plus" metric for flat Minkowski space-time, $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$: therefore four-velocities will have square norm $u^\mu u_\mu = -1$. I will use Einstein summation convention: if an index appears multiple times in the same monomial, it is meant to be summed over

Take a diffeomorphism $x \to y$, with Jacobian matrix $\partial y^\mu / \partial x^\nu$. The indices of contravariant vectors, trasforming as

$$V^\mu \to \left(\frac{\partial y^\mu}{\partial x^\nu}\right) V^\nu \tag{1.1}$$

will be denoted as upper indices, while the indices of covariant vectors, trasforming as

$$V_\mu \to \left(\frac{\partial x^\nu}{\partial y^\mu}\right) V_\nu \tag{1.2}$$

will be denoted as lower indices; the same applies to higher rank tensors.

Unless otherwise specified, I will work in geometrized units, where $c = G = 1$.

Take a tensor with many indices, T_{IJ}, where I is shorthand for the n indices $\mu\nu\rho\dots$ and the same applies to J. These indices can be symmetrized and antisymmetrized, and I will use the following conventions:

$$T_{(I)J} = \frac{1}{n!} \sum_{\sigma \in S_n} T_{\sigma(I)J} \tag{1.3}$$

$$T_{[I]J} = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}\,\sigma T_{\sigma(I)J} \tag{1.4}$$

where $S_n \ni \sigma$ is the group of permutations of n elements, and $\mathrm{sign}\,\sigma$ is 1 if σ is an even permutation (it can be obtained in an even number of pair swaps) and -1 otherwise.

2 Useful formulas

2.1 Nobili

Tensor calculus The covariant derivative keeps account of the shifting of the basis vectors:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\alpha\mu} A^\alpha \tag{2.1}$$

The rank-3 objects Γ are called Christoffel symbols. They are not tensors! they depend on the choice of basis e_α, and they satisfy $\nabla_\mu e_\alpha = \Gamma^\nu_{\mu\alpha} e_\nu$.

If we have the metric, they can be calculated as:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho}\right) \tag{2.2}$$

This also tells us that they are symmetric in the lower two indices: $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$.

The divergence of a vector field A^μ can be calculated as:

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} A^\mu\right) \tag{2.3}$$

where g is the determinant of the metric.

We can also define the Dalambertian operator $\Box = \nabla_\mu \nabla^\mu = \nabla_\mu \partial^\mu$, which can only act on scalars, and it does so like:

$$\Box A = \nabla_\mu (\partial^\mu A) = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu A \right) \tag{2.4}$$

If we differentiate and antisymmetryze (so, take the rotor of) an antisymmetric tensor $F_{[\mu\nu]}$, the Christoffel symbols cancel:

$$\nabla_{[\mu} F_{\nu\rho]} = \partial_{[\mu} F_{\nu\rho]} \tag{2.5}$$

The derivative with respect to proper time is $\frac{\mathrm{d}}{\mathrm{d}\tau} = u^\mu \partial_\mu$.

Covariant acceleration is defined as:

$$a^\nu = u^\mu \nabla_\mu u^\nu \tag{2.6}$$

Curvature The curvature of spacetime is fully described by the Riemann curvature tensor, which is a fourth rank tensor: for any generic vector V^μ,

$$R^\mu_{\nu\rho\sigma} V^\nu \stackrel{\text{def}}{=} [\nabla_\rho, \nabla_\sigma] V^\mu \tag{2.7}$$

It can be calculated using the Christoffel symbols, and while they are not tensors $R^\mu_{\nu\rho\sigma}$ is one. This result follows by direct computation from formula (2.7).

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda} \Gamma^\lambda_{\rho\nu} \tag{2.8}$$

The Christoffel symbols can be nonzero if we choose certain coordinates even for flat spacetime, but the Riemann tensor is zero iff the spacetime is flat.

Geodesics If we have a path $x^\mu(\lambda)$, we would like to see if it is a geodesic, that is, if it is stationary with respect to path length. To do this we can stationarize the action corresponding to the lagrangian $\mathcal{L}(x, \dot{x}) = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ (where we use $\dot{x} = \mathrm{d}x/\mathrm{d}\lambda$). The Lagrange equations then are:

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0 \tag{2.9}$$

Where Γ are the Christoffel symbols, which can be calculated by differentiating the metric, as shown in (2.2). $\mathcal{L}$ is an integral of these Lagrange equations.

If the parameter λ is taken to be the proper time s, then the equation is

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}s} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0 \tag{2.10}$$

Notice that this is equivalent to the covariant acceleration (2.6) being zero.

Fermi-Walker transport Take a general vector field $V^\mu(s)$ defined along a curve, with its tangent vector u^μ whose covariant acceleration is a^μ. Then we say that V^μ is transported according to Fermi-Walker iff it satisfies

$$\dot{V}^\mu = u^\nu \nabla_\nu V^\nu = V_\rho (u^\mu a^\rho - a^\mu u^\rho) \tag{2.11}$$

This condition is always satisfied by $V^\mu = u^\mu$, since $a^\mu u_\mu = 0$, whether or not the curve is a geodesic. The tangent vector is *parallel* transported only for geodesics.

Tetrads and projectors We want to work in a reference in which the velocity u^μ is purely timelike. This can always be found by the equivalence principle. Such a reference can be completed into what is called a tetrad, for which the metric becomes the Minkowski metric in a neighbourhood of the point we consider.

We call the velocity $u^\mu = V^\mu_{(0)}$, and add to it three other vectors $V^\mu_{(i)}$ such that

$$g_{\mu\nu} V^\mu_{(\alpha)} V^\nu_{(\beta)} = \eta_{(\alpha)(\beta)} \tag{2.12}$$

where the brackets around the indices denote the fact that they label four vectors, not the components of a tensor.

We can choose the vectors $V^\mu_{(i)}$ so that they are Fermi-Walker transported along the worldline defined by u^μ: this allows us to find the relativistic equivalent of a nonrotating frame of reference.

It is useful to project tensors onto the space-like and time-like subspaces defined by our tetrad (and we wish to do so in a coordinate-independent manner, so just taking the 0th and i-th components in the tetrad will not suffice). We therefore define the projectors:

$$h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu} \qquad \pi_{\mu\nu} = -u_\mu u_\nu \tag{2.13}$$

respectively onto the space- and time-like subspaces.

Metrics The simplest physically relevant one is the Schwarzschild metric. It describes a spherically symmetric object of mass M, in spherical coordinates. Defining $\Phi = -M/r$, we have:

$$\mathrm{d}s^2 = -(1+2\Phi)\,\mathrm{d}t^2 + \frac{1}{1+2\Phi}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi\right) \tag{2.14}$$

or, equivalently,

$$g_{\mu\nu} = \operatorname{diag}\left(-(1+2\Phi),\, \frac{1}{1+2\Phi},\, r^2,\, r^2\sin^2\theta\right) \tag{2.15}$$

We can see that it approaches the flat metric $\eta_{\mu\nu} = \operatorname{diag}(-,+,+,+)$ in the limit $M \to 0$. Its determinant is $g = -r^4\sin^2\theta$.

Fluid mechanics In usual relativistic single-body mechanics, we use the 4-velocity u^μ and the corresponding 4-momentum $p^\mu = mu^\mu$. The 0-th component of this vector is the energy of the body, while the i-th components are its momentum: we then have $p^\mu p_\mu = m^2 = E^2 - |p|^2$.

When dealing with a continuum, we will have a certain density of particles per unit of volume, we call this n. The current of particles is then $N^\mu = nu^\mu$. If these particles have a certain rest mass m_0, we can then define the vector $\rho_0 u^\mu = m_0 n u^\mu = m_0 N^\mu$.

This satisfies a conservation equation: $\nabla_\mu(\rho_0 u^\mu) = 0$.

Particles in a fluid can have three kinds of energy we concern ourselves with: mass, kinetic energy and other forms of energy (thermal, chemical, nuclear…). We can always perform a change of coordinates to bring us to a frame in which the kinetic energy is zero. We write the sum of the other two forms of energy as $\rho = \rho_0(1+\epsilon)$. So, ϵ is the ratio of the internal non-mass energy to the mass.

Now, the vector ρu^μ describes the flux of energy. We can then write the equation for the conservation of momentum:

$$f^\mu = \nabla_\nu(\rho u^\mu u^\nu) \tag{2.16}$$

Ideal fluids They are fluids with $\eta = \xi = \kappa = 0$, that is, without viscosity (neither compressive nor shear) nor heat transmission. They are described by the following stress-energy tensor:

$$T^{\mu\nu} = \rho u^\mu u^\nu + p h^{\mu\nu} \tag{2.17}$$

Spherical accretion We work with the Schwarzschild metric (2.14); we treat a fluid with 4-velocity u^μ in spherical coordinates, since the problem we are looking at is stationary and spherically symmetric the velocity is:

$$u^\mu = \begin{pmatrix} \gamma^2/y \\ yv \\ 0 \\ 0 \end{pmatrix} \tag{2.18}$$

where we define the Lorentz factor as usual, $\gamma = \left(1 - v^2\right)^{-1/2}$, and $y = \gamma\sqrt{1 + 2\Phi}$.

The conservation of mass holds: if ρ_0 is the rest mass density of the fluid, we must have $\nabla_\mu\left(\rho_0 u^\mu\right) = 0$. This, using the formula for covariant divergence (2.3), yields:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho_0 y v r^2\right) = 0 \tag{2.19}$$

In the newtonian limit both γ and y approach 1; also, the infalling mass rate $\dot{M}$ at a certain radius is $\rho_0(r)v(r)4\pi r^2$. Then, by continuity to the newtonian limit, the quantity which is constant wrt the radius must be $\dot{M}/(4\pi)$.

We also have the Euler equation:

$$(p + \rho)a^\mu = -h^{\mu\nu}\partial_\nu p \tag{2.20}$$

The only interesting component of this is the radial one, so we need to calculate $a^1 = u^\mu \nabla_\mu u^1 = \mathrm{d}u^1 \Big/ \mathrm{d}\tau + \Gamma^1_{\mu\nu} u^\mu u^\nu$. To do this we will need the Schwarzschild Christoffel coefficients:

$$\Gamma^1_{\mu\nu} = \begin{bmatrix} -\frac{2M\left(\frac{M}{r} - \frac{1}{2}\right)}{r^2} & 0 & 0 & 0 \\ 0 & \frac{2M\left(\frac{M}{r} - \frac{1}{2}\right)}{r^2\left(-\frac{2M}{r} + 1\right)^2} & 0 & 0 \\ 0 & 0 & 2r\left(\frac{M}{r} - \frac{1}{2}\right) & 0 \\ 0 & 0 & 0 & 2r\left(\frac{M}{r} - \frac{1}{2}\right)\sin^2(\theta) \end{bmatrix} \tag{2.21}$$

while the proper-time derivative is $\mathrm{d}/\mathrm{d}\tau = u^\mu \partial_\mu = yv\partial_1$.

And the equation for the variation of the total internal energy, which holds for ideal fluids at constant entropy:

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \frac{p + \rho}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} \tag{2.22}$$

From these we can show that the quantity $\gamma h\sqrt{1 + 2\Phi}$ (where $h = (p + \rho)/\rho_0$ is the specific enthalpy), is a constant of motion. In the nonrelativistic, weak-field limit this becomes

$$\gamma h\sqrt{1 + 2\Phi} \approx \frac{p}{\rho_0} + \frac{v^2}{2} - \frac{M}{r} + \epsilon = \text{const} \tag{2.23}$$

2.2 Taub

This section summarizes my study of A. H. Taub's review of relativistic fluid dynamics, [Tau78].

Nonrelativistic Nonrelativistic fluid mechanics are described by the equations:

$$\partial_t \rho + \partial_i(\rho v^i) = 0 \tag{2.24a}$$

$$\rho\left(\partial_t v^i + v^j \partial_j v^i\right) = \partial_j T^{ij} \tag{2.24b}$$

$$\rho \partial_t E + v^i \partial_i E = \partial_i \Big(T^{ij} v_j + \kappa \partial^i T \Big) \tag{2.24c}$$

where ρ is the density of the fluid, v^i are the components of its velocity, T^{ij} is the stress tensor (or, equivalently, the space-like components of the energy-momentum tensor), E is the energy of the fluid, κ is the thermal conductivity, T is the temperature of the fluid.

The nonrelativistic stress tensor can be written as:

$$T_{ij} = -(p + \xi \partial_k v^k) \delta_{ij} + \eta \partial_{(i} v_{j)} \tag{2.25}$$

where p is the (isotropic) pressure, η the viscosity, ξ is the compression viscosity. We are assuming that the normal stresses are only those exerted by pressure, so the diagonal terms T_{ii} (not summed) must just be $-p$. So, the term $-\xi \partial_k v^k$ must equal $\eta \partial_{(i} v_{i)} = 2\eta \partial_i v_i$ (not summed). Therefore, by isotropy, $\xi = 2\eta/3$.

Note that we are working in Euclidean 3D space, so the metric is the identity and upper and lower indices are equivalent.

The energy is a sum of kinetic and specific energy:

$$E = v^i v_i / 2 + \varepsilon \tag{2.26}$$

where ε is the specific energy (of a type that is different from kinetic) per unit mass.

Relativistic The dynamics of the fluid are described by the conservation of the stress-energy tensor $\nabla_\mu T^{\mu\nu} = 0$ and the conservation of mass $\nabla_\mu(\rho u^\mu) = 0$.

Any stress-energy tensor can be decomposed in its space and time-like parts in the local rest frame of the fluid:

$$T_{\mu\nu} = w u_\mu u_\nu + w_\mu u_\nu + u_\mu w_\nu + w_{\mu\nu} \tag{2.27}$$

where

$$w = T_{\mu\nu} u^\mu u^\nu = \rho_0 (1 + \varepsilon) \tag{2.28a}$$
$$w_\mu = T_{\nu\sigma} h^\sigma_\mu u^\nu = -\kappa h^\sigma_\mu (\partial_\sigma T + T a_\sigma) \tag{2.28b}$$
$$w_{\mu\nu} = T_{\rho\sigma} h^\rho_\mu h^\sigma_\nu = \big(p - \xi\theta \big) h_{\mu\nu} - 2\eta \sigma_{\mu\nu} \tag{2.28c}$$

with $\theta = \nabla_\mu u^\mu$, a_μ is the covariant acceleration, $\sigma_{\sigma\tau} = 1/2 \Big(\nabla_\mu u_\nu + \nabla_\mu u_\nu \Big) h^\mu_\sigma h^\nu_\tau - 1/3 \theta h_{\sigma\tau}$, and as in the nonrelativistic section η is the viscosity, ξ is the compression viscosity, κ is the thermal conductivity, T is the temperature field, p is the pressure, ρ_0 is the rest mass density while $\rho = \rho_0(1 + \varepsilon)$ is the rest energy.

3 Nobili, Turolla, Zampieri

See [NTZ91].

The cooling function $\Lambda(T)$ is defined by the following relation, which describes the variation in the energy density by radiative processes:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = n_b^2 \big(\Gamma(T) - \Lambda(T) \big) \tag{3.1}$$

where U is the energy density (measured in $\mathrm{erg\,cm^{-3}}$), n_b is the baryon density (measured in $\mathrm{cm^{-3}}$), while Γ and Λ are the heating and cooling functions, both measured in $\mathrm{erg\,cm^3\,s^{-1}}$, see [GH12, equation 1].

The cooling function of the infalling gas is

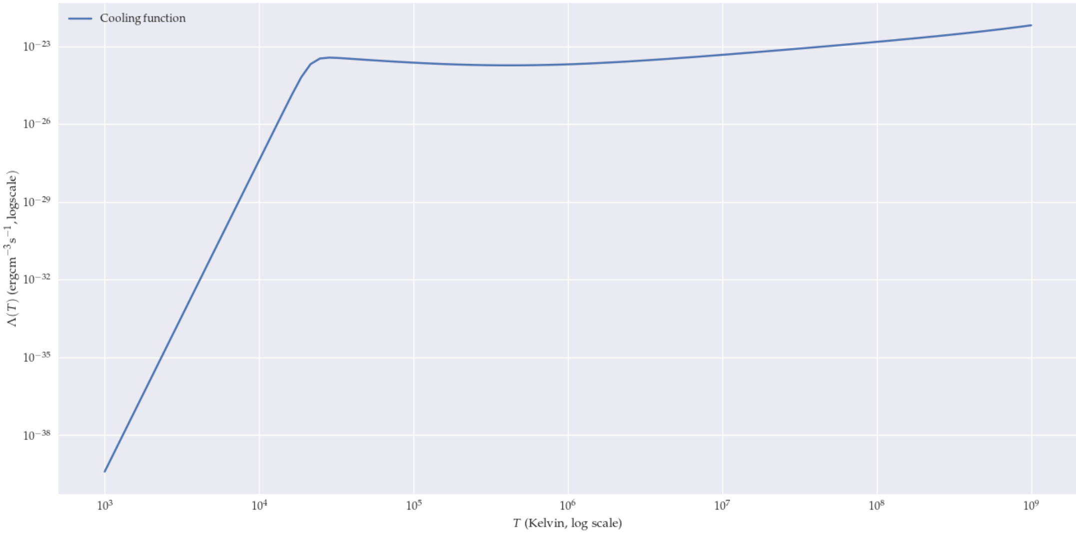

Figure 1: Cooling function graph.

$$\Lambda(T) = \left(\left(1.42 \times 10^{-27} T^{1/2} \left(1 + 4.4 \times 10^{-10} T \right) + 6.0 \times 10^{-22} T^{-1/2} \right)^{-1} \right. \\ \left. + 10^{25} \left(\frac{T}{1.5849 \times 10^4\,\mathrm{K}} \right)^{-12} \right)^{-1} \mathrm{erg\,cm^{-3}\,s^{-1}} \tag{3.2}$$

The version of this equation in Stellingwerf and Buff is similar: the first constant is 2.4×10^{-27} instead of 1.42×10^{-27}, and the factor $\left(1 + 4.4 \times 10^{-10} T\right)$ is just 1.

4 Thorne's PSTF moment formalism

Following [Tho81].

Given any tensor $A^{\mu_1 \dots \mu_k}$ we can use the tensor $h^{\mu\nu}$ to project it into the space-like subspace defined by the velocity u^μ:

$$A^{\mu_1 \dots \mu_k} \to \left(A^{\mu_1 \dots \mu_k}\right)^P = \left(\prod_i h^{\mu_i}_{\nu_i} \right) A^{\nu_1 \dots \nu_k} \tag{4.1}$$

Then, we can take the symmetric part of any (?) tensor as outlined in 'Notational preface' on page 2:

$$A^{\mu_1 \dots \mu_k} \to \left(A^{\mu_1 \dots \mu_k}\right)^S = A^{(\mu_1 \dots \mu_k)} \tag{4.2}$$

We can select the trace-free part of a projected, symmetric tensor by

$$A^{\mu_1 \dots \mu_k} \to \left(A^{\mu_1 \dots \mu_k}\right)^{TF} = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \frac{k!(2k-2i-1)!!}{(k-2i)!(2k-1)!!(2i)!!} h^{(\alpha_1\alpha_2} \dots h^{\alpha_{2i-1}\alpha_{2i}} A^{\alpha_{2i+1} \dots \alpha_k)\beta_1 \dots \beta_i}{}_{\beta_1 \dots \beta_i} \tag{4.3}$$

To see what this is doing, let us consider its action on a rank-two tensor:

$$A^{\mu\nu} \to A^{\mu\nu} - \frac{1}{3} h^{\mu\nu} A^\rho_\rho \tag{4.4}$$

Now, let us consider all the unit vectors n^μ in the space normal to the velocity, which have $n_\mu u^\mu = 0$ and $n^\mu n_\mu = 1$. They span a three-dimensional sphere.

If we have a function $F\colon S^2 \to \mathbb{R}$, we can decompose it into harmonics as such:

$$F(n) = \sum_{k=0}^{\infty} \mathscr{F}_{\alpha_1 \dots \alpha_k} \prod_{i=0}^{k} n^{\alpha_i} \tag{4.5}$$

Where the PTSF moments $\mathscr{F}_{\alpha_1 \dots \alpha_k}$ can be computed as

$$\mathscr{F}_{\alpha_1 \dots \alpha_k} = \frac{(2k+1)!!}{4\pi k!} \left(\int F \prod_{i=0}^{k} n^{\alpha_i} \, \mathrm{d}\Omega \right)^{TF} \tag{4.6}$$

Now, consider a photon, whose trajectory in spacetime is parametrized as $\gamma(\xi)$, with a choice of ξ such that the photon's momentum is

$$p = \frac{\mathrm{d}}{\mathrm{d}\xi} \tag{4.7}$$

Now, our observer has a timelike velocity u^μ. We can find a spacelike vector n^μ corresponding to the space-like part of the movement of the photon, or

$$p^\mu = (-u^\nu p_\nu)(u^\mu + n^\mu) \tag{4.8}$$

Now, we define a parameter l which corresponds to the space distance the photon moved through in this frame (this is *not* covariant!)

$$l = \int (-u^\nu p_\nu) \, \mathrm{d}\xi \tag{4.9}$$

now, $\mathrm{d}/\mathrm{d}l$ is parallel to p but it has different length, in fact since $\mathrm{d}l/\mathrm{d}\xi = (-u^\nu p_\nu)$ it is $\mathrm{d}/\mathrm{d}l = u + n$.

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