# UNIVERSITÀ DEGLI STUDI DI PADOVA

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## Relativistic non-ideal flows


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**Abstract**

Stuff emits radiation when it falls into a black hole. I'd like to see exactly how much of it.

# 1 Notational preface

I will use greek indices ($\mu$, $\nu$, $\rho\dots$) to denote 4-dimensional indices ranging from 0 to 3, and latin indices ($i$, $j$, $k\dots$) to denote 3-dimensional indices ranging from 1 to 3.

I will use the "mostly plus" metric for flat Minkowski space-time, $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$: therefore four-velocities will have square norm $u^\mu u_\mu = -1$. I will use Einstein summation convention: if an index appears multiple times in the same monomial, it is meant to be summed over

Take a diffeomorphism $x \to y$, with Jacobian matrix $\partial y^\mu / \partial x^\nu$. The indices of contravariant vectors, trasforming as

$$V^\mu \to \left(\frac{\partial y^\mu}{\partial x^\nu}\right) V^\nu \tag{1.1}$$

will be denoted as upper indices, while the indices of covariant vectors, trasforming as

$$V_\mu \to \left(\frac{\partial x^\nu}{\partial y^\mu}\right) V_\nu \tag{1.2}$$

will be denoted as lower indices; the same applies to higher rank tensors.

Unless otherwise specified, I will work in geometrized units, where $c = G = 1$.

Take a tensor with many indices, $T_{IJ}$, where $I$ is shorthand for the $n$ indices $\mu\nu\rho\dots$ and the same applies to $J$. These indices can be symmetrized and antisymmetrized, and I will use the following conventions:

$$T_{(I)J} = \frac{1}{n!} \sum_{\sigma \in S_n} T_{\sigma(I)J} \tag{1.3}$$

$$T_{[I]J} = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}\,\sigma T_{\sigma(I)J} \tag{1.4}$$

where $S_n \ni \sigma$ is the group of permutations of $n$ elements, and $\mathrm{sign}\,\sigma$ is 1 if $\sigma$ is an even permutation (it can be obtained in an even number of pair swaps) and -1 otherwise.

# 2 Useful formulas

## 2.1 Nobili

**Tensor calculus** The covariant derivative keeps account of the shifting of the basis vectors:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\alpha\mu} A^\alpha \tag{2.1}$$

The rank-3 objects $\Gamma$ are called Christoffel symbols. They are not tensors! they depend on the choice of basis $e_\alpha$, and they satisfy $\nabla_\mu e_\alpha = \Gamma^\nu_{\mu\alpha} e_\nu$.

If we have the metric, they can be calculated as:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho}\right) \tag{2.2}$$

This also tells us that they are symmetric in the lower two indices: $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$.

The divergence of a vector field $A^\mu$ can be calculated as:

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} A^\mu\right) \tag{2.3}$$

where $g$ is the determinant of the metric.

We can also define the Dalambertian operator $\Box = \nabla_\mu \nabla^\mu = \nabla_\mu \partial^\mu$, which can only act on scalars, and it does so like:

$$\Box A = \nabla_\mu(\partial^\mu A) = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \partial^\mu A \right) \tag{2.4}$$

If we differentiate and antisymmetryze (so, take the rotor of) an antisymmetric tensor $F_{[\mu\nu]}$, the Christoffel symbols cancel:

$$\nabla_{[\mu} F_{\nu\rho]} = \partial_{[\mu} F_{\nu\rho]} \tag{2.5}$$

The derivative with respect to proper time is $\frac{\mathrm{d}}{\mathrm{d}\tau} = u^\mu \partial_\mu$.

Covariant acceleration is defined as:

$$a^\nu = u^\mu \nabla_\mu u^\nu \tag{2.6}$$

**Curvature** The curvature of spacetime is fully described by the Riemann curvature tensor, which is a fourth rank tensor: for any generic vector $V^\mu$,

$$R^\mu_{\nu\rho\sigma} V^\nu \stackrel{\text{def}}{=} [\nabla_\rho, \nabla_\sigma] V^\mu \tag{2.7}$$

It can be calculated using the Christoffel symbols, and while they are not tensors $R^\mu_{\nu\rho\sigma}$ is one. This result follows by direct computation from formula (2.7).

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda} \Gamma^\lambda_{\rho\nu} \tag{2.8}$$

The Christoffel symbols can be nonzero if we choose certain coordinates even for flat spacetime, but the Riemann tensor is zero iff the spacetime is flat.

The Riemann tensor satisfies the following identities [MTW73, eqs. 8.45 and 8.76]:

$$\nabla_{[\lambda} R_{\mu\nu]\rho\sigma} = 0 \tag{2.9a}$$

$$R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]} = R_{[\rho\sigma][\mu\nu]} \tag{2.9b}$$

$$R_{[\mu\nu\rho\sigma]} = 0 = R_{\mu[\nu\rho\sigma]} \tag{2.9c}$$

If we define the Ricci tensor $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$ and the curvature scalar $R = R_{\mu\nu} g^{\mu\nu}$, we can rewrite (2.9a) as $\nabla_\mu R = 2\nabla_\nu R^\nu_\mu$.

**Geodesics** If we have a path $x^\mu(\lambda)$, we would like to see if it is a geodesic, that is, if it is stationary with respect to path length. To do this we can stationarize the action corresponding to the lagrangian $\mathcal{L}(x, \dot{x}) = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ (where we use $\dot{x} = \mathrm{d}x/\mathrm{d}\lambda$). The Lagrange equations then are:

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0 \tag{2.10}$$

Where $\Gamma$ are the Christoffel symbols, which can be calculated by differentiating the metric, as shown in (2.2). $\mathcal{L}$ is an integral of these Lagrange equations.

If the parameter $\lambda$ is taken to be the proper time $s$, then the equation is

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}s} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0 \tag{2.11}$$

Notice that this is equivalent to the covariant acceleration (2.6) being zero.

**Fermi-Walker transport** Take a general vector field $V^\mu(s)$ defined along a curve, with its tangent vector $u^\mu$ whose covariant acceleration is $a^\mu$. Then we say that $V^\mu$ is transported according to Fermi-Walker iff it satisfies

$$\dot{V}^\mu = u^\nu \nabla_\nu V^\nu = V_\rho (u^\mu a^\rho - a^\mu u^\rho) \tag{2.12}$$

This condition is always satisfied by $V^\mu = u^\mu$, since $a^\mu u_\mu = 0$, whether or not the curve is a geodesic. The tangent vector is *parallel* transported only for geodesics.

**Tetrads and projectors** We want to work in a reference in which the velocity $u^\mu$ is purely timelike. This can always be found by the equivalence principle. Such a reference can be completed into what is called a tetrad, for which the metric becomes the Minkowski metric in a neighbourhood of the point we consider.

We call the velocity $u^\mu = V^\mu_{(0)}$, and add to it three other vectors $V^\mu_{(i)}$ such that

$$g_{\mu\nu} V^\mu_{(\alpha)} V^\nu_{(\beta)} = \eta_{(\alpha)(\beta)} \tag{2.13}$$

where the brackets around the indices denote the fact that they label four vectors, not the components of a tensor.

We can choose the vectors $V^\mu_{(i)}$ so that they are Fermi-Walker transported along the worldline defined by $u^\mu$: this allows us to find the relativistic equivalent of a nonrotating frame of reference.

It is useful to project tensors onto the space-like and time-like subspaces defined by our tetrad (and we wish to do so in a coordinate-independent manner, so just taking the 0th and $i$-th components in the tetrad will not suffice). We therefore define the projectors:

$$h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu} \qquad \pi_{\mu\nu} = -u_\mu u_\nu \tag{2.14}$$

respectively onto the space- and time-like subspaces.

**Metrics** The simplest physically relevant one is the Schwarzschild metric. It describes a spherically symmetric object of mass $M$, in spherical coordinates. Defining $\Phi = -M/r$, we have:

$$\mathrm{d}s^2 = -(1+2\Phi)\,\mathrm{d}t^2 + \frac{1}{1+2\Phi}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi\right) \tag{2.15}$$

or, equivalently,

$$g_{\mu\nu} = \mathrm{diag}\left(-(1+2\Phi), \frac{1}{1+2\Phi}, r^2, r^2\sin^2\theta\right) \tag{2.16}$$

We can see that it approaches the flat metric $\eta_{\mu\nu} = \mathrm{diag}\,(-,+,+,+)$ in the limit $M \to 0$. Its determinant is $g = -r^4 \sin^2\theta$.

**Fluid mechanics** In usual relativistic single-body mechanics, we use the 4-velocity $u^\mu$ and the corresponding 4-momentum $p^\mu = mu^\mu$. The 0-th component of this vector is the energy of the body, while the $i$-th components are its momentum: we then have $p^\mu p_\mu = m^2 = E^2 - |p|^2$.

When dealing with a continuum, we will have a certain density of particles per unit of volume, we call this $n$. The current of particles is then $N^\mu = nu^\mu$. If these particles have a certain rest mass $m_0$, we can then define the vector $\rho_0 u^\mu = m_0 n u^\mu = m_0 N^\mu$.

This satisfies a conservation equation: $\nabla_\mu(\rho_0 u^\mu) = 0$.

Particles in a fluid can have three kinds of energy we concern ourselves with: mass, kinetic energy and other forms of energy (thermal, chemical, nuclear…). We can always perform a change of coordinates to bring us to a frame in which the kinetic energy is zero. We write the sum of the other two forms of energy as $\rho = \rho_0(1+\epsilon)$. So, $\epsilon$ is the ratio of the internal non-mass energy to the mass.

Now, the vector $\rho u^\mu$ describes the flux of energy. We can then write the equation for the conservation of momentum:

$$f^\mu = \nabla_\nu(\rho u^\mu u^\nu) \tag{2.17}$$

**Ideal fluids** They are fluids with $\eta = \xi = \kappa = 0$, that is, without viscosity (neither compressive nor shear) nor heat transmission. They are described by the following stress-energy tensor:

$$T^{\mu\nu} = \rho u^\mu u^\nu + p h^{\mu\nu} \tag{2.18}$$

**Spherical accretion** We work with the Schwarzschild metric (2.15); we treat a fluid with 4-velocity $u^\mu$ in spherical coordinates, since the problem we are looking at is stationary and spherically symmetric the velocity is:

$$u^\mu = \begin{pmatrix} \gamma^2/y \\ yv \\ 0 \\ 0 \end{pmatrix} \tag{2.19}$$

where we define the Lorentz factor as usual, $\gamma = \left(1 - v^2\right)^{-1/2}$, and $y = \gamma\sqrt{1 + 2\Phi}$.

The conservation of mass holds: if $\rho_0$ is the rest mass density of the fluid, we must have $\nabla_\mu\left(\rho_0 u^\mu\right) = 0$. This, using the formula for covariant divergence (2.3), yields:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho_0 y v r^2\right) = 0 \tag{2.20}$$

In the newtonian limit both $\gamma$ and $y$ approach 1; also, the infalling mass rate $\dot{M}$ at a certain radius is $\rho_0(r)v(r)4\pi r^2$. Then, by continuity to the newtonian limit, the quantity which is constant wrt the radius must be $\dot{M}/(4\pi)$: therefore

$$\dot{M} = 4\pi\rho_0 y v r^2 \tag{2.21}$$

We also have the Euler equation:

$$(p + \rho)a^\mu = -h^{\mu\nu}\partial_\nu p \tag{2.22}$$

The only interesting component of this is the radial one, so we need to calculate $a^1 = u^\mu \nabla_\mu u^1 = \mathrm{d}u^1 \big/ \mathrm{d}\tau + \Gamma^1_{\mu\nu} u^\mu u^\nu$. To do this we will need the radial Schwarzschild Christoffel coefficients:

$$\Gamma^1_{\mu\nu} = \begin{bmatrix} \frac{M(-2M+r)}{r^3} & 0 & 0 & 0 \\ 0 & \frac{M}{r(2M-r)} & 0 & 0 \\ 0 & 0 & 2M - r & 0 \\ 0 & 0 & 0 & (2M - r)\sin^2(\theta) \end{bmatrix} \tag{2.23}$$

while the proper-time derivative is $\mathrm{d}/\mathrm{d}\tau = u^\mu\partial_\mu = yv\partial_1$. Plugging in the expression for the only relevant component of $h^{\mu\nu}$, $h^{11} = g^{11} + u^1u^1 = (1 + 2\Phi)(1 + v^2\gamma^2) = y^2$ we get, after lengthy computation,

$$a^1 = y^2\left(\gamma^2 v\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1 + 2\Phi)r^2}\right) \tag{2.24}$$

Substituting this into the (radial component of the) Euler equation (2.22) we get

$$(p + \rho)y^2\left(\gamma^2 v\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1 + 2\Phi)r^2}\right) = -h^{11}\partial_1 p = -y^2\partial_1 p \tag{2.25a}$$

$$\gamma^2 v \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2} + \frac{1}{p+\rho}\frac{\mathrm{d}p}{\mathrm{d}r} = 0 \tag{2.25b}$$

In equations 3.12.7, 8 in [Nob00] there is most definitely a sign error.

We also have the equation for the variation of the total internal energy, which holds for ideal fluids at constant entropy:

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \frac{p+\rho}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} \qquad \text{or} \qquad \frac{\mathrm{d}\rho}{\mathrm{d}\rho_0} = \frac{p+\rho}{\rho_0} \tag{2.26}$$

From these we can show that

**Claim 2.1.** *The quantity $\gamma h\sqrt{1+2\Phi} = yh$ (where $h = (p+\rho)/\rho_0$ is the specific enthalpy), is a constant of motion.*

*Proof.* First of all, by direct computation it can be shown that

$$\gamma^2 v \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2} = \frac{\mathrm{d}\log y}{\mathrm{d}r} \tag{2.27}$$

Then, following Gourgoulhon [Gou06, section 6.3] we find that $\mathrm{d}p = \rho_0\,\mathrm{d}h$ in the isentropic case, therefore

$$\frac{1}{\rho+p}\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\mathrm{d}\log h}{\mathrm{d}r} \tag{2.28}$$

we can substitute the results in (2.27) and (2.28) into (2.25b):

$$\frac{\mathrm{d}\log h}{\mathrm{d}r} + \frac{\mathrm{d}\log y}{\mathrm{d}r} = \frac{\mathrm{d}\log(hy)}{\mathrm{d}r} = 0 \tag{2.29}$$

□

In the nonrelativistic, weak-field limit this becomes the classical conservation of density of energy:

$$\gamma h\sqrt{1+2\Phi} \approx \frac{p}{\rho_0} + \frac{v^2}{2} - \frac{M}{r} + \epsilon = \text{const} \tag{2.30}$$

Also, we can rewrite the RHS of (2.25b) as:

$$-\frac{\partial_1 P}{p+\rho} = -\frac{\rho_0}{\rho_0}\frac{\partial_1 P}{p+\rho} = -\frac{v_s^2}{\rho_0}\partial_1\rho_0 \tag{2.31}$$

where we define the speed of sound $v_s^2 = (\partial P/\partial\rho)_s$ (the index $s$ means the derivative is to be taken at constant entropy, and for an adiabatic process). Therefore we get:

$$\gamma^2 v \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2} + \frac{v_s^2\partial_1\rho_0}{\rho_0} = 0 \tag{2.32}$$

We can replace every occcurence of $(\partial x)/x$ with $\partial\log x$:

$$\gamma^2 v^2 \frac{\mathrm{d}\log v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2} + v_s^2\frac{\mathrm{d}\log\rho_0}{\mathrm{d}r} = 0 \tag{2.33}$$

Are the eqrefs at page 174 in [Nob00] wrong? What's up with equations 2.7.3, 5?
Somehow this equation comes up:

$$\frac{\mathrm{d}\log\rho_0}{\mathrm{d}r} + \gamma^2 v^2\frac{\mathrm{d}\log v}{\mathrm{d}r} + \frac{2v^2}{r} + \frac{M}{(1+2\Phi)r^2} = 0 \tag{2.34}$$

## 2.2 Taub

This section summarizes my study of A. H. Taub's review of relativistic fluid dynamics, [Tau78].

**Nonrelativistic** Nonrelativistic fluid mechanics are described by the equations:

$$\partial_t \rho + \partial_i (\rho v^i) = 0 \tag{2.35a}$$
$$\rho \left( \partial_t v^i + v^j \partial_j v^i \right) = \partial_j T^{ij} \tag{2.35b}$$
$$\rho \partial_t E + v^i \partial_i E = \partial_i \left( T^{ij} v_j + \kappa \partial^i T \right) \tag{2.35c}$$

where $\rho$ is the density of the fluid, $v^i$ are the components of its velocity, $T^{ij}$ is the stress tensor (or, equivalently, the space-like components of the energy-momentum tensor), $E$ is the energy of the fluid, $\kappa$ is the thermal conductivity, $T$ is the temperature of the fluid.

The nonrelativistic stress tensor can be written as:

$$T_{ij} = -(p + \xi \partial_k v^k) \delta_{ij} + \eta \partial_{(i} v_{j)} \tag{2.36}$$

where $p$ is the (isotropic) pressure, $\eta$ the viscosity, $\xi$ is the compression viscosity. We are assuming that the normal stresses are only those exerted by pressure, so the diagonal terms $T_{ii}$ (not summed) must just be $-p$. So, the term $-\xi \partial_k v^k$ must equal $\eta \partial_{(i} v_{i)} = 2\eta \partial_i v_i$ (not summed). Therefore, by isotropy, $\xi = 2\eta/3$.

Note that we are working in Euclidean 3D space, so the metric is the identity and upper and lower indices are equivalent.

The energy is a sum of kinetic and specific energy:

$$E = v^i v_i / 2 + \varepsilon \tag{2.37}$$

where $\varepsilon$ is the specific energy (of a type that is different from kinetic) per unit mass.

**Relativistic** The dynamics of the fluid are described by the conservation of the stress-energy tensor $\nabla_\mu T^{\mu\nu} = 0$ and the conservation of mass $\nabla_\mu (\rho u^\mu) = 0$.

Any stress-energy tensor can be decomposed in its space and time-like parts in the local rest frame of the fluid:

$$T_{\mu\nu} = w u_\mu u_\nu + w_\mu u_\nu + u_\mu w_\nu + w_{\mu\nu} \tag{2.38}$$

where

$$w = T_{\mu\nu} u^\mu u^\nu = \rho_0 (1 + \varepsilon) \tag{2.39a}$$
$$w_\mu = T_{\nu\sigma} h^\sigma_\mu u^\nu = -\kappa h^\sigma_\mu (\partial_\sigma T + T a_\sigma) \tag{2.39b}$$
$$w_{\mu\nu} = T_{\rho\sigma} h^\rho_\mu h^\sigma_\nu = \left(p - \xi\theta\right) h_{\mu\nu} - 2\eta \sigma_{\mu\nu} \tag{2.39c}$$

with $\theta = \nabla_\mu u^\mu$, $a_\mu$ is the covariant acceleration, $\sigma_{\sigma\tau} = 1/2 \left( \nabla_\mu u_\nu + \nabla_\mu u_\nu \right) h^\mu_\sigma h^\nu_\tau - 1/3 \theta h_{\sigma\tau}$, and as in the nonrelativistic section $\eta$ is the viscosity, $\xi$ is the compression viscosity, $\kappa$ is the thermal conductivity, $T$ is the temperature field, $p$ is the pressure, $\rho_0$ is the rest mass density while $\rho = \rho_0(1 + \varepsilon)$ is the rest energy.

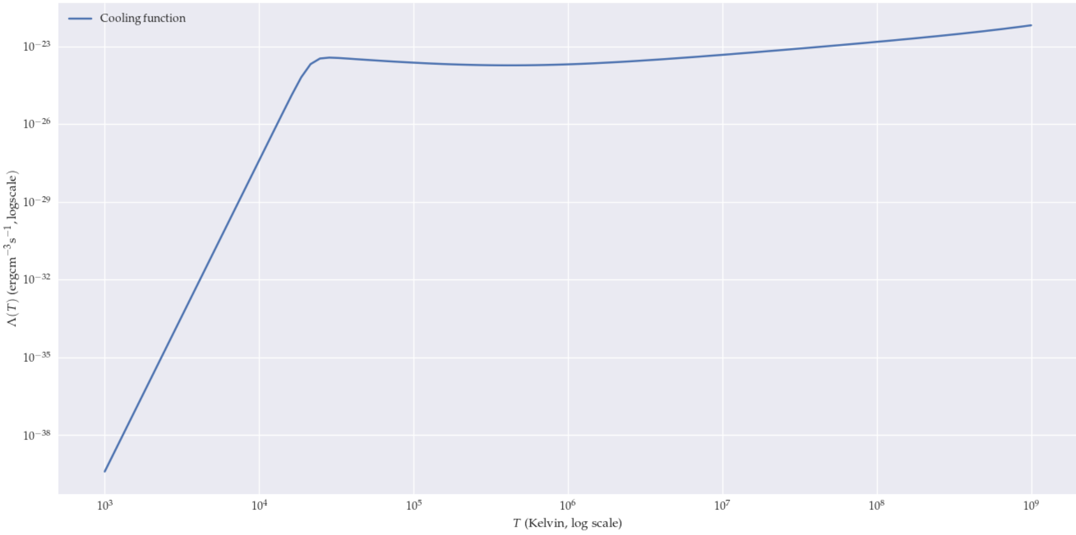


Figure 1: Cooling function graph.

## 3 Nobili, Turolla, Zampieri

See [NTZ91].

The cooling function $\Lambda(T)$ is defined by the following relation, which describes the variation in the energy density by radiative processes:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = n_b^2(\Gamma(T) - \Lambda(T)) \tag{3.1}$$

where $U$ is the energy density (measured in $\mathrm{erg\,cm^{-3}}$), $n_b$ is the baryon density (measured in $\mathrm{cm^{-3}}$), while $\Gamma$ and $\Lambda$ are the heating and cooling functions, both measured in $\mathrm{erg\,cm^3\,s^{-1}}$, see [GH12, equation 1].

The cooling function of the infalling gas is

$$\begin{aligned}\Lambda(T) = &\left(\left(1.42\times10^{-27}T^{1/2}\left(1+4.4\times10^{-10}T\right)+6.0\times10^{-22}T^{-1/2}\right)^{-1}\right.\\ &\left.+10^{25}\left(\frac{T}{1.5849\times10^4\,\mathrm{K}}\right)^{-12}\right)^{-1}\mathrm{erg\,cm^{-3}\,s^{-1}}\end{aligned} \tag{3.2}$$

The version of this equation in Stellingwerf and Buff is similar: the first constant is $2.4\times10^{-27}$ instead of $1.42\times10^{-27}$, and the factor $\left(1+4.4\times10^{-10}T\right)$ is just 1.

## 4 Thorne's PSTF moment formalism

Following [Tho81].

Given any tensor $A^{\mu_1\cdots\mu_k}$ we can use the tensor $h^{\mu\nu}$ to project it into the space-like subspace defined by the velocity $u^\mu$:

$$A^{\mu_1\cdots\mu_k} \to \left(A^{\mu_1\cdots\mu_k}\right)^P = \left(\prod_i h^{\mu_i}_{\nu_i}\right) A^{\nu_1\cdots\nu_k} \tag{4.1}$$

Then, we can take the symmetric part of any (?) tensor as outlined in 'Notational preface' on page 2:

$$A^{\mu_1\cdots\mu_k} \to (A^{\mu_1\cdots\mu_k})^S = A^{(\mu_1\cdots\mu_k)} \tag{4.2}$$

We can select the trace-free part of a projected, symmetric tensor by

$$A^{\mu_1\cdots\mu_k} \to (A^{\mu_1\cdots\mu_k})^{TF} = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \frac{k!(2k-2i-1)!!}{(k-2i)!(2k-1)!!(2i)!!} h^{(\alpha_1\alpha_2} \dots h^{\alpha_{2i-1}\alpha_{2i}} A^{\alpha_{2i+1}\dots\alpha_k)\beta_1\dots\beta_i}{}_{\beta_1\dots\beta_i} \tag{4.3}$$

To see what this is doing, let us consider its action on a rank-two tensor:

$$A^{\mu\nu} \to A^{\mu\nu} - \frac{1}{3} h^{\mu\nu} A^\rho_\rho \tag{4.4}$$

Now, let us consider all the unit vectors $n^\mu$ in the space normal to the velocity, which have $n_\mu u^\mu = 0$ and $n^\mu n_\mu = 1$. They span a three-dimensional sphere.

If we have a function $F \colon S^2 \to \mathbb{R}$, we can decompose it into harmonics as such:

$$F(n) = \sum_{k=0}^{\infty} \mathscr{F}_{\alpha_1\dots\alpha_k} \prod_{i=0}^{k} n^{\alpha_i} \tag{4.5}$$

Where the PTSF moments $\mathscr{F}_{\alpha_1\dots\alpha_k}$ can be computed as

$$\mathscr{F}_{\alpha_1\dots\alpha_k} = \frac{(2k+1)!!}{4\pi k!} \left( \int F \prod_{i=0}^{k} n^{\alpha_i} \, \mathrm{d}\Omega \right)^{TF} \tag{4.6}$$

Now, consider a photon, whose trajectory in spacetime is parametrized as $\gamma(\xi)$, with a choice of $\xi$ such that the photon's momentum is

$$p = \frac{\mathrm{d}}{\mathrm{d}\xi} \tag{4.7}$$

Now, our observer has a timelike velocity $u^\mu$. We can find a spacelike vector $n^\mu$ corresponding to the space-like part of the movement of the photon, or

$$p^\mu = (-u^\nu p_\nu)(u^\mu + n^\mu) \tag{4.8}$$

Now, we define a parameter $l$ which corresponds to the space distance the photon moved through in this frame (this is *not* covariant!)

$$l = \int (-u^\nu p_\nu) \, \mathrm{d}\xi \tag{4.9}$$

now, $\mathrm{d}/\mathrm{d}l$ is parallel to $p$ but it has different length, in fact since $\mathrm{d}l/\mathrm{d}\xi = (-u^\nu p_\nu)$ it is $\mathrm{d}/\mathrm{d}l = u + n$.

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