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Tesi di Laurea

Relativistic non-ideal flows

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Abstract

Stuff emits radiation when it falls into a black hole. I'd like to see exactly how much of it,

1 Notational preface

I will use greek indices $(\mu, \nu, \rho \dots)$ to denote 4-dimensional indices ranging from 0 to 3, and latin indices $(i, j, k \dots)$ to denote 3-dimensional indices ranging from 1 to 3.

I will use the “mostly plus” metric for flat Minkowski space-time, $\eta_{\mu\nu} = \text{diag } -, +, +, +$. I will use Einstein summation convention: if an index appears multiple times in the same monomial, it is meant to be summed over

Take a diffeomorphism $x \rightarrow y$, with Jacobian matrix $\partial y^\mu / \partial x^\nu$. The indices of contravariant vectors, trasforming as

$$V^\mu \rightarrow \left(\frac{\partial y^\mu}{\partial x^\nu} \right) V^\nu \quad (1.1)$$

will be denoted as upper indices, while the indices of covariant vectors, trasforming as

$$V_\mu \rightarrow \left(\frac{\partial x^\nu}{\partial y^\mu} \right) V_\nu \quad (1.2)$$

will be denoted as lower indices; the same applies to higher rank tensors.

Unless otherwise specified, I will work in geometrized units, where $c = G = 1$.

Take a tensor with many indices, T_{IJ} , where I is shorthand for the n indices $\mu\nu\rho \dots$ and the same applies to J . These indices can be symmetrized and antisymmetrized, and I will use the following conventions:

$$T_{(I)J} = \sum_{\sigma \in S_n} T_{\sigma(I)J} \quad (1.3)$$

$$T_{[I]J} = \sum_{\sigma \in S_n} \text{sign } \sigma T_{\sigma(I)J} \quad (1.4)$$

where S_n is the group of permutations of n elements, and $\text{sign } \sigma$ is 1 if σ is an even permutation (it can be obtained in an even number of pair swaps) and -1 otherwise.

2 Useful formulas

2.1 Nobili

Tensor calculus The covariant derivative keeps account of the shifting of the basis vectors:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma_{\alpha\mu}^\nu A^\alpha \quad (2.1)$$

The rank-3 objects Γ are called Christoffel symbols. They are not tensors! they depend on the choice of basis e_α , and they satisfy $\nabla_\mu e_\alpha = \Gamma_{\mu\alpha}^\nu e_\nu$.

If we have the metric, they can be calculated as:

$$\Gamma_{\nu\rho}^\mu = \frac{1}{2} g^{\mu\alpha} (\partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho}) \quad (2.2)$$

This also tells us that they are symmetric in the lower two indices: $\Gamma_{\nu\rho}^\mu = \Gamma_{(\nu\rho)}^\mu$.

The divergence of a vector field A^μ can be calculated as:

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} A^\mu) \quad (2.3)$$

where g is the determinant of the metric.

We can also define the D'Alembertian operator $\square = \nabla_\mu \nabla^\mu = \nabla_\mu \partial^\mu$, which can only act on scalars, and it does so like:

$$\square A = \nabla_\mu (\partial^\mu A) = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu A) \quad (2.4)$$

If we differentiate and antisymmetrize (so, take the rotor of) an antisymmetric tensor $F_{[\mu\nu]}$, the Christoffel symbols cancel:

$$\nabla_{[\mu} F_{\nu\rho]} = \partial_{[\mu} F_{\nu\rho]} \quad (2.5)$$

The derivative with respect to proper time is $\frac{d}{d\tau} = u^\mu \partial_\mu$.

Covariant acceleration is defined as:

$$a^\nu = u^\mu \nabla_\mu u^\nu \quad (2.6)$$

Geodesics If we have a path $x^\mu(\lambda)$, we would like to see if it is a geodesic, that is, if it is stationary with respect to path length. To do this we can stationarize the action corresponding to the lagrangian $\mathcal{L}(x, \dot{x}) = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ (where we use $\dot{x} = dx/d\lambda$). The Lagrange equations then are:

$$\ddot{x}^\mu + \Gamma_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho = 0 \quad (2.7)$$

Where Γ are the Christoffel symbols, which can be calculated by differentiating the metric, as shown in (2.2). \mathcal{L} is an integral of these Lagrange equations.

If the parameter λ is taken to be the proper time s , then the equation is

$$\frac{du^\mu}{ds} + \Gamma_{\nu\rho}^\mu u^\nu u^\rho = 0 \quad (2.8)$$

Tetrads and projectors We want to work in a reference in which the velocity u^μ is purely timelike. This can always be found by the equivalence principle. Such a reference is called a tetrad. It also allows the metric to become the Minkowski metric in a neighbourhood of the point we consider.

It is useful to project tensors onto the and space-like and time-like subspaces defined by our tetrad (and we wish to do so in a coordinate-independent manner, so just taking the 0th and i -th components in the tetrad will not suffice). We therefore define the projectors:

$$h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu} \quad \pi_{\mu\nu} = -u_\mu u_\nu \quad (2.9)$$

respectively onto the space- and time-like subspaces.

Metrics The simplest one is the Schwarzschild metric. It describes a spherically symmetric object of mass M , in spherical coordinates. Defining $\Phi = -M/r$, we have:

$$ds^2 = -(1 + 2\Phi) dt^2 + \frac{1}{1 + 2\Phi} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (2.10)$$

or, equivalently,

$$g_{\mu\nu} = \text{diag} \left(-(1 + 2\Phi), \frac{1}{1 + 2\Phi}, r^2, r^2 \sin^2 \theta \right) \quad (2.11)$$

We can see that it approaches the flat metric $\eta_{\mu\nu} = \text{diag} -, +, +, +$ in the limit $M \rightarrow 0$. Its determinant is $g = -r^4 \sin^2 \theta$.

Fluid mechanics In usual relativistic single-body mechanics, we use the 4-velocity u^μ and the corresponding 4-momentum $p^\mu = mu^\mu$. The 0-th component of this vector is the energy of the body, while the i -th components are its momentum: we then have $p^\mu p_\mu = m^2 = E^2 - |p|^2$.

When dealing with a continuum, we will have a certain density of particle per unit of volume, we call this n . The current of particles is then $N^\mu = nu^\mu$. If these particles have a certain rest mass m_0 , we can then define the vector $\rho_0 u^\mu = m_0 n u^\mu = m_0 N^\mu$.

This satisfies a conservation equation: $\nabla_\mu(\rho_0 u^\mu) = 0$.

Particles in a fluid can have three kinds of energy we concern ourselves with: mass, kinetic energy and other forms of energy (thermal, chemical, nuclear...). We can always perform a change of coordinates to bring us to a frame in which the kinetic energy is zero. We write the sum of the other two forms of energy as $\rho = \rho_0(1 + \epsilon)$. So, ϵ is the ratio of the internal non-mass energy to the mass.

Now, the vector ρu^μ describes the flux of energy. We can then write the equation for the conservation of momentum:

$$f^\mu = \nabla_\nu(\rho u^\mu u^\nu) \quad (2.12)$$

Spherical accretion We work with the Schwarzschild metric (2.10); we treat a fluid with 4-velocity u^μ in spherical coordinates, since the problem we are looking at is stationary and spherically symmetric the velocity is:

$$u^\mu = \begin{pmatrix} \gamma^2/y \\ yv \\ 0 \\ 0 \end{pmatrix} \quad (2.13)$$

where we define $\gamma = (1 - v^2)^{-1/2}$, $y = \gamma\sqrt{1 + 2\Phi}$.

The conservation of mass holds: if ρ_0 is the rest mass density of the fluid, we must have $\nabla_\mu(\rho_0 u^\mu) = 0$. This, using the formula for covariant divergence (2.3), yields:

$$\frac{d}{dr}(\rho_0 y v r^2) = 0 \quad (2.14)$$

In the newtonian limit both γ and y approach 1; also, the infalling mass rate \dot{M} at a certain radius is $\rho_0(r)v(r)4\pi r^2$. Then, by continuity to the newtonian limit, the quantity which is constant wrt the radius must be $\dot{M}/(4\pi)$.

We also have the Euler equation:

$$(p + \rho)a^\mu = -h^{\mu\nu}\partial_\nu p \quad (2.15)$$

And the equation for ???

$$\frac{d\rho}{d\tau} = \frac{p + \rho}{\rho_0} \frac{d\rho_0}{d\tau} \quad (2.16)$$

From these we can show that the quantity $\gamma h \sqrt{1 + 2\Phi}$ (where $h = (p + \rho)/\rho_0$ is the specific enthalpy), is a constant of motion. In the nonrelativistic, weak-field limit this becomes

$$\gamma h \sqrt{1 + 2\Phi} \approx \frac{p}{\rho_0} + \frac{v^2}{2} - \frac{M}{r} + \epsilon = \text{const} \quad (2.17)$$

2.2 Taub

This section summarizes my study of A. H. Taub's review of relativistic fluid dynamics, [Tau78].

Nonrelativistic fluid mechanics are described by the equations:

$$\partial_t \rho + \partial_i(\rho v^i) = 0 \quad (2.18a)$$

$$\rho(\partial_t v^i + v^j \partial_j v^i) = \partial_j T^{ij} \quad (2.18b)$$

$$\rho \partial_t E + v^i \partial_i E = \partial_i (T^{ij} v_j + \lambda \partial^i T) \quad (2.18c)$$

where ρ is the density of the fluid, v^i are the components of its velocity, T^{ij} is the stress tensor (or, equivalently, the space-like components of the energy-momentum tensor), E is the energy of the fluid, λ is the thermal conductivity, T is the temperature of the fluid.

The nonrelativistic stress tensor can be written as:

$$T_{ij} = -(p + \xi \partial_k v^k) \delta_{ij} + \eta \partial_{(i} v_{j)} \quad (2.19)$$

where p is the (isotropic) pressure, η the viscosity, ξ is the compression viscosity. We are assuming that the normal stresses are only those exerted by pressure, so the diagonal terms T_{ii} (not summed) must just be $-p$. So, the term $-\xi \partial_k v^k$ must equal $\eta \partial_{(i} v_{i)} = 2\eta \partial_i v_i$ (not summed). Therefore, by isotropy, $\xi = 2\eta/3$.

Note that we are working in Euclidean 3D space, so the metric is the identity and upper and lower indices are equivalent.

The energy is a sum of kinetic and specific energy:

$$E = v^i v_i / 2 + \varepsilon \quad (2.20)$$

where ε is the specific energy (of a type that is different from kinetic) per unit mass.



Figure 1: Test

Bibliography

[Tau78] A H Taub. "Relativistic Fluid Mechanics". In: *Annual Review of Fluid Mechanics* 10.1 (1978), pp. 301–332. URL: <https://doi.org/10.1146/annurev.fl.10.010178.001505>.