



# UNIVERSITÀ DEGLI STUDI DI PADOVA

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Relativistic non-ideal flows

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## Abstract

Stuff emits radiation when it falls into a black hole. I'd like to see exactly how much of it.

## 1 Notational preface

I will use greek indices ( $\mu, \nu, \rho \dots$ ) to denote 4-dimensional indices ranging from 0 to 3, and latin indices ( $i, j, k \dots$ ) to denote 3-dimensional indices ranging from 1 to 3.

I will use the “mostly plus” metric for flat Minkowski space-time,  $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$ : therefore four-velocities will have square norm  $u^\mu u_\mu = -1$ . I will use Einstein summation convention: if an index appears multiple times in the same monomial, it is meant to be summed over

Take a diffeomorphism  $x \rightarrow y$ , with Jacobian matrix  $\partial y^\mu / \partial x^\nu$ . The indices of contravariant vectors, trasforming as

$$V^\mu \rightarrow \left( \frac{\partial y^\mu}{\partial x^\nu} \right) V^\nu \quad (1.1)$$

will be denoted as upper indices, while the indices of covariant vectors, trasforming as

$$V_\mu \rightarrow \left( \frac{\partial x^\nu}{\partial y^\mu} \right) V_\nu \quad (1.2)$$

will be denoted as lower indices; the same applies to higher rank tensors.

Unless otherwise specified, I will work in geometrized units, where  $c = G = 1$ .

Take a tensor with many indices,  $T_{IJ}$ , where  $I$  is shorthand for the  $n$  indices  $\mu\nu\rho \dots$  and the same applies to  $J$ . These indices can be symmetrized and antisymmetrized, and I will use the following conventions:

$$T_{(I)J} = \frac{1}{n!} \sum_{\sigma \in S_n} T_{\sigma(I)J} \quad (1.3)$$

$$T_{[I]J} = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign } \sigma T_{\sigma(I)J} \quad (1.4)$$

where  $S_n \ni \sigma$  is the group of permutations of  $n$  elements, and  $\text{sign } \sigma$  is 1 if  $\sigma$  is an even permutation (it can be obtained in an even number of pair swaps) and -1 otherwise.

## 2 Useful formulas

### 2.1 Nobili

**Tensor calculus** The covariant derivative keeps account of the shifting of the basis vectors:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma_{\alpha\mu}^\nu A^\alpha \quad (2.1)$$

The rank-3 objects  $\Gamma$  are called Christoffel symbols. They are not tensors! they depend on the choice of basis  $e_\alpha$ , and they satisfy  $\nabla_\mu e_\alpha = \Gamma_{\mu\alpha}^\nu e_\nu$ .

If we have the metric, they can be calculated as:

$$\Gamma_{\nu\rho}^\mu = \frac{1}{2} g^{\mu\alpha} \left( \partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho} \right) \quad (2.2)$$

This also tells us that they are symmetric in the lower two indices:  $\Gamma_{\nu\rho}^\mu = \Gamma_{\rho\nu}^\mu$ .

The divergence of a vector field  $A^\mu$  can be calculated as:

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} A^\mu \right) \quad (2.3)$$

where  $g$  is the determinant of the metric.

We can also define the D'Alembertian operator  $\square = \nabla_\mu \nabla^\mu = \nabla_\mu \partial^\mu$ , which can only act on scalars, and it does so like:

$$\square A = \nabla_\mu (\partial^\mu A) = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu A) \quad (2.4)$$

If we differentiate and antisymmetrize (so, take the rotor of) an antisymmetric tensor  $F_{[\mu\nu]}$ , the Christoffel symbols cancel:

$$\nabla_{[\mu} F_{\nu\rho]} = \partial_{[\mu} F_{\nu\rho]} \quad (2.5)$$

The derivative with respect to proper time is  $\frac{d}{d\tau} = u^\mu \partial_\mu$ .

Covariant acceleration is defined as:

$$a^\nu = u^\mu \nabla_\mu u^\nu \quad (2.6)$$

**Curvature** The curvature of spacetime is fully described by the Riemann curvature tensor, which is a fourth rank tensor: for any generic vector  $V^\mu$ ,

$$R^\mu_{\nu\rho\sigma} V^\nu \stackrel{\text{def}}{=} [\nabla_\rho, \nabla_\sigma] V^\mu \quad (2.7)$$

It can be calculated using the Christoffel symbols, and while they are not tensors  $R^\mu_{\nu\rho\sigma}$  is one. This result follows by direct computation from formula (2.7).

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda} \Gamma^\lambda_{\rho\nu} \quad (2.8)$$

The Christoffel symbols can be nonzero if we choose certain coordinates even for flat spacetime, but the Riemann tensor is zero iff the spacetime is flat.

**Geodesics** If we have a path  $x^\mu(\lambda)$ , we would like to see if it is a geodesic, that is, if it is stationary with respect to path length. To do this we can stationarize the action corresponding to the lagrangian  $\mathcal{L}(x, \dot{x}) = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$  (where we use  $\dot{x} = dx/d\lambda$ ). The Lagrange equations then are:

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0 \quad (2.9)$$

Where  $\Gamma$  are the Christoffel symbols, which can be calculated by differentiating the metric, as shown in (2.2).  $\mathcal{L}$  is an integral of these Lagrange equations.

If the parameter  $\lambda$  is taken to be the proper time  $s$ , then the equation is

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0 \quad (2.10)$$

Notice that this is equivalent to the covariant acceleration (2.6) being zero.

**Fermi-Walker transport** Take a general vector field  $V^\mu(s)$  defined along a curve, with its tangent vector  $u^\mu$  whose covariant acceleration is  $a^\mu$ . Then we say that  $V^\mu$  is transported according to Fermi-Walker iff it satisfies

$$\dot{V}^\mu = u^\nu \nabla_\nu V^\mu = V^\rho (u^\mu a_\rho - a^\mu u_\rho) \quad (2.11)$$

This condition is always satisfied by  $V^\mu = u^\mu$ , since  $a^\mu u_\mu = 0$ , whether or not the curve is a geodesic. The tangent vector is *parallel* transported only for geodesics.

**Tetrads and projectors** We want to work in a reference in which the velocity  $u^\mu$  is purely timelike. This can always be found by the equivalence principle. Such a reference can be completed into what is called a tetrad, for which the metric becomes the Minkowski metric in a neighbourhood of the point we consider.

We call the velocity  $u^\mu = V_{(0)}^\mu$ , and add to it three other vectors  $V_{(i)}^\mu$  such that

$$g_{\mu\nu} V_{(\alpha)}^\mu V_{(\beta)}^\nu = \eta_{(\alpha)(\beta)} \quad (2.12)$$

where the brackets around the indices denote the fact that they label four vectors, not the components of a tensor.

We can choose the vectors  $V_{(i)}^\mu$  so that they are Fermi-Walker transported along the worldline defined by  $u^\mu$ : this allows us to find the relativistic equivalent of a nonrotating frame of reference.

It is useful to project tensors onto the space-like and time-like subspaces defined by our tetrad (and we wish to do so in a coordinate-independent manner, so just taking the 0th and  $i$ -th components in the tetrad will not suffice). We therefore define the projectors:

$$h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu} \quad \pi_{\mu\nu} = -u_\mu u_\nu \quad (2.13)$$

respectively onto the space- and time-like subspaces.

**Metrics** The simplest physically relevant one is the Schwarzschild metric. It describes a spherically symmetric object of mass  $M$ , in spherical coordinates. Defining  $\Phi = -M/r$ , we have:

$$ds^2 = -(1 + 2\Phi) dt^2 + \frac{1}{1 + 2\Phi} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (2.14)$$

or, equivalently,

$$g_{\mu\nu} = \text{diag} \left( -(1 + 2\Phi), \frac{1}{1 + 2\Phi}, r^2, r^2 \sin^2 \theta \right) \quad (2.15)$$

We can see that it approaches the flat metric  $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$  in the limit  $M \rightarrow 0$ . Its determinant is  $g = -r^4 \sin^2 \theta$ .

**Fluid mechanics** In usual relativistic single-body mechanics, we use the 4-velocity  $u^\mu$  and the corresponding 4-momentum  $p^\mu = mu^\mu$ . The 0-th component of this vector is the energy of the body, while the  $i$ -th components are its momentum: we then have  $p^\mu p_\mu = m^2 = E^2 - |p|^2$ .

When dealing with a continuum, we will have a certain density of particles per unit of volume, we call this  $n$ . The current of particles is then  $N^\mu = nu^\mu$ . If these particles have a certain rest mass  $m_0$ , we can then define the vector  $\rho_0 u^\mu = m_0 n u^\mu = m_0 N^\mu$ .

This satisfies a conservation equation:  $\nabla_\mu (\rho_0 u^\mu) = 0$ .

Particles in a fluid can have three kinds of energy we concern ourselves with: mass, kinetic energy and other forms of energy (thermal, chemical, nuclear...). We can always perform a change of coordinates to bring us to a frame in which the kinetic energy is zero. We write the sum of the other two forms of energy as  $\rho = \rho_0(1 + \epsilon)$ . So,  $\epsilon$  is the ratio of the internal non-mass energy to the mass.

Now, the vector  $\rho u^\mu$  describes the flux of energy. We can then write the equation for the conservation of momentum:

$$f^\mu = \nabla_\nu (\rho u^\mu u^\nu) \quad (2.16)$$

**Ideal fluids** They are fluids with  $\eta = \xi = \kappa = 0$ , that is, without viscosity (neither compressive nor shear) nor heat transmission. They are described by the following stress-energy tensor:

$$T^{\mu\nu} = \rho u^\mu u^\nu + p h^{\mu\nu} \quad (2.17)$$

**Spherical accretion** We work with the Schwarzschild metric (2.14); we treat a fluid with 4-velocity  $u^\mu$  in spherical coordinates, since the problem we are looking at is stationary and spherically symmetric the velocity is:

$$u^\mu = \begin{pmatrix} \gamma^2/y \\ yv \\ 0 \\ 0 \end{pmatrix} \quad (2.18)$$

where we define the Lorentz factor as usual,  $\gamma = (1 - v^2)^{-1/2}$ , and  $y = \gamma\sqrt{1 + 2\Phi}$ .

The conservation of mass holds: if  $\rho_0$  is the rest mass density of the fluid, we must have  $\nabla_\mu(\rho_0 u^\mu) = 0$ . This, using the formula for covariant divergence (2.3), yields:

$$\frac{d}{dr}(\rho_0 y v r^2) = 0 \quad (2.19)$$

In the newtonian limit both  $\gamma$  and  $y$  approach 1; also, the infalling mass rate  $\dot{M}$  at a certain radius is  $\rho_0(r)v(r)4\pi r^2$ . Then, by continuity to the newtonian limit, the quantity which is constant wrt the radius must be  $\dot{M}/(4\pi)$ .

We also have the Euler equation:

$$(p + \rho)a^\mu = -h^{\mu\nu}\partial_\nu p \quad (2.20)$$

And the equation for the variation of the total internal energy, which holds for ideal fluids at constant entropy:

$$\frac{d\rho}{d\tau} = \frac{p + \rho}{\rho_0} \frac{d\rho_0}{d\tau} \quad (2.21)$$

From these we can show that the quantity  $\gamma h \sqrt{1 + 2\Phi}$  (where  $h = (p + \rho)/\rho_0$  is the specific enthalpy), is a constant of motion. In the nonrelativistic, weak-field limit this becomes

$$\gamma h \sqrt{1 + 2\Phi} \approx \frac{p}{\rho_0} + \frac{v^2}{2} - \frac{M}{r} + \epsilon = \text{const} \quad (2.22)$$

## 2.2 Taub

This section summarizes my study of A. H. Taub's review of relativistic fluid dynamics, [Tau78].

**Nonrelativistic** Nonrelativistic fluid mechanics are described by the equations:

$$\partial_t \rho + \partial_i(\rho v^i) = 0 \quad (2.23a)$$

$$\rho(\partial_t v^i + v^j \partial_j v^i) = \partial_j T^{ij} \quad (2.23b)$$

$$\rho \partial_t E + v^i \partial_i E = \partial_i(T^{ij} v_j + \kappa \partial^i T) \quad (2.23c)$$

where  $\rho$  is the density of the fluid,  $v^i$  are the components of its velocity,  $T^{ij}$  is the stress tensor (or, equivalently, the space-like components of the energy-momentum tensor),  $E$  is the energy of the fluid,  $\kappa$  is the thermal conductivity,  $T$  is the temperature of the fluid.

The nonrelativistic stress tensor can be written as:

$$T_{ij} = -(p + \xi \partial_k v^k) \delta_{ij} + \eta \partial_{(i} v_{j)} \quad (2.24)$$

where  $p$  is the (isotropic) pressure,  $\eta$  the viscosity,  $\xi$  is the compression viscosity. We are assuming that the normal stresses are only those exerted by pressure, so the diagonal terms  $T_{ii}$  (not summed)

must just be  $-p$ . So, the term  $-\xi\partial_k v^k$  must equal  $\eta\partial_{(i}v_{i)} = 2\eta\partial_i v_i$  (not summed). Therefore, by isotropy,  $\xi = 2\eta/3$ .

Note that we are working in Euclidean 3D space, so the metric is the identity and upper and lower indices are equivalent.

The energy is a sum of kinetic and specific energy:

$$E = v^i v_i / 2 + \varepsilon \quad (2.25)$$

where  $\varepsilon$  is the specific energy (of a type that is different from kinetic) per unit mass.

**Relativistic** The dynamics of the fluid are described by the conservation of the stress-energy tensor  $\nabla_\mu T^{\mu\nu} = 0$  and the conservation of mass  $\nabla_\mu (\rho u^\mu) = 0$ .

Any stress-energy tensor can be decomposed in its space and time-like parts in the local rest frame of the fluid:

$$T_{\mu\nu} = w u_\mu u_\nu + w_\mu u_\nu + u_\mu w_\nu + w_{\mu\nu} \quad (2.26)$$

where

$$w = T_{\mu\nu} u^\mu u^\nu = \rho_0(1 + \varepsilon) \quad (2.27a)$$

$$w_\mu = T_{\nu\sigma} h_\mu^\sigma u^\nu = -\kappa h_\mu^\sigma (\partial_\sigma T + T a_\sigma) \quad (2.27b)$$

$$w_{\mu\nu} = T_{\rho\sigma} h_\mu^\rho h_\nu^\sigma = (p - \xi\theta) h_{\mu\nu} - 2\eta\sigma_{\mu\nu} \quad (2.27c)$$

with  $\theta = \nabla_\mu u^\mu$ ,  $a_\mu$  is the covariant acceleration,  $\sigma_{\sigma\tau} = 1/2(\nabla_\mu u_\nu + \nabla_\nu u_\mu)h_\sigma^\mu h_\tau^\nu - 1/3\theta h_{\sigma\tau}$ , and as in the nonrelativistic section  $\eta$  is the viscosity,  $\xi$  is the compression viscosity,  $\kappa$  is the thermal conductivity,  $T$  is the temperature field,  $p$  is the pressure,  $\rho_0$  is the rest mass density while  $\rho = \rho_0(1 + \varepsilon)$  is the rest energy.

### 3 Nobili, Turolla, Zampieri

See [NTZ91].

The cooling function  $\Lambda(T)$  is defined by the following relation, which describes the variation in the energy density by radiative processes:

$$\frac{dU}{dt} = n_b^2 (\Gamma(T) - \Lambda(T)) \quad (3.1)$$

where  $U$  is the energy density (measured in  $\text{erg cm}^{-3}$ ),  $n_b$  is the baryon density (measured in  $\text{cm}^{-3}$ ), while  $\Gamma$  and  $\Lambda$  are the heating and cooling functions, both measured in  $\text{erg cm}^3 \text{s}^{-1}$ , see [GH12, equation (1)].

The cooling function of the infalling gas is

$$\begin{aligned} \Lambda(T) = & \left( \left( 1.42 \times 10^{-27} T^{1/2} \left( 1 + 4.4 \times 10^{-10} T \right) + 6.0 \times 10^{-22} T^{-1/2} \right)^{-1} \right. \\ & \left. + 10^{25} \left( \frac{T}{1.5849 \times 10^4 \text{ K}} \right)^{-12} \right)^{-1} \text{ erg cm}^{-3} \text{ s}^{-1} \end{aligned} \quad (3.2)$$

The version of this equation in Stellingwerf and Buff is similar: the first constant is  $2.4 \times 10^{-27}$  instead of  $1.42 \times 10^{-27}$ , and the factor  $(1 + 4.4 \times 10^{-10} T)$  is just 1.

## 4 Thorne's PSTF moment formalism

Following [Tho81].

Given any tensor  $A^{\mu_1 \dots \mu_k}$  we can use the tensor  $h^{\mu\nu}$  to project it into the space-like subspace defined by the velocity  $u^\mu$ :

$$A^{\mu_1 \dots \mu_k} \rightarrow (A^{\mu_1 \dots \mu_k})^P = \left( \prod_i h^{\mu_i}_{\nu_i} \right) A^{\nu_1 \dots \nu_k} \quad (4.1)$$

Then, we can take the symmetric part of any (?) tensor as outlined in 'Notational preface' on page 2:

$$A^{\mu_1 \dots \mu_k} \rightarrow (A^{\mu_1 \dots \mu_k})^S = A^{(\mu_1 \dots \mu_k)} \quad (4.2)$$

We can select the trace-free part of a projected, symmetric tensor by

$$A^{\mu_1 \dots \mu_k} \rightarrow (A^{\mu_1 \dots \mu_k})^{TF} = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \frac{k!(2k-2i-1)!!}{(k-2i)!(2k-1)!!(2i)!!} h^{(\alpha_1 \alpha_2 \dots \alpha_{2i-1} \alpha_{2i}} A^{\alpha_{2i+1} \dots \alpha_k) \beta_1 \dots \beta_i}_{\beta_1 \dots \beta_i} \quad (4.3)$$

To see what this is doing, let us consider its action on a rank-two tensor:

$$A^{\mu\nu} \rightarrow A^{\mu\nu} - \frac{1}{3} h^{\mu\nu} A^\rho_\rho \quad (4.4)$$

Now, let us consider all the unit vectors  $n^\mu$  in the space normal to the velocity, which have  $n_\mu u^\mu = 0$  and  $n^\mu n_\mu = 1$ . They span a three-dimensional sphere.

If we have a function  $F: S^2 \rightarrow \mathbb{R}$ , we can decompose it into harmonics as such:

$$F(n) = \sum_{k=0}^{\infty} \mathcal{F}_{\alpha_1 \dots \alpha_k} \prod_{i=1}^k n^{\alpha_i} \quad (4.5)$$

Where the PTSF moments  $\mathcal{F}_{\alpha_1 \dots \alpha_k}$  can be computed as

$$\mathcal{F}_{\alpha_1 \dots \alpha_k} = \frac{(2k+1)!!}{4\pi k!} \left( \int F \prod_{i=1}^k n^{\alpha_i} d\Omega \right)^{TF} \quad (4.6)$$

Now, consider a photon, whose trajectory in spacetime is parametrized as  $\gamma(\xi)$ , with a choice of  $\xi$  such that the photon's momentum is

$$p = \frac{d}{d\xi} \quad (4.7)$$

Now, our observer has a timelike velocity  $u^\mu$ . We can find a spacelike vector  $n^\mu$  corresponding to the space-like part of the movement of the photon, or

$$p^\mu = (-u^\nu p_\nu)(u^\mu + n^\mu) \quad (4.8)$$

Now, we define a parameter  $l$  which corresponds to the space distance the photon moved through in this frame (this is *not* covariant!)

$$l = \int (-u^\nu p_\nu) d\xi \quad (4.9)$$

now,  $d/dl$  is parallel to  $p$  but it has different length, in fact since  $dl/d\xi = (-u^\nu p_\nu)$  it is  $d/dl = u + n$ .

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