UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Corso di Laurea in Fisica

Tesi di Laurea

Relativistic non-ideal flows

Relatore

Prof. Roberto Turolla

Laureando

Jacopo Tissino

Anno Accademico 2018/2019

Abstract

Stuff emits radiation when it falls into a black hole. I'd like to see exactly how much of it.

1 Notational preface

I will use greek indices (μ, ν, $\rho\ldots$) to denote 4-dimensional indices ranging from 0 to 3, and latin indices (i, j, $k\ldots$) to denote 3-dimensional indices ranging from 1 to 3.

I will use the "mostly plus" metric for flat Minkowski space-time, $\eta_{\mu\nu} = \operatorname{diag} -, +, +, +$: therefore four-velocities will have square norm $u^\mu u_\mu = -1$. I will use Einstein summation convention: if an index appears multiple times in the same monomial, it is meant to be summed over

Take a diffeomorphism $x \to y$, with Jacobian matrix $\partial y^\mu / \partial x^\nu$. The indices of contravariant vectors, trasforming as

$$V^\mu \to \left(\frac{\partial y^\mu}{\partial x^\nu}\right) V^\nu \tag{1.1}$$

will be denoted as upper indices, while the indices of covariant vectors, trasforming as

$$V_\mu \to \left(\frac{\partial x^\nu}{\partial y^\mu}\right) V_\nu \tag{1.2}$$

will be denoted as lower indices; the same applies to higher rank tensors.

Unless otherwise specified, I will work in geometrized units, where $c = G = 1$.

Take a tensor with many indices, T_{IJ}, where I is shorthand for the n indices $\mu\nu\rho\ldots$ and the same applies to J. These indices can be symmetrized and antisymmetrized, and I will use the following conventions:

$$T_{(I)J} = \sum_{\sigma \in S_n} T_{\sigma(I)J} \tag{1.3}$$

$$T_{[I]J} = \sum_{\sigma \in S_n} \operatorname{sign} \sigma T_{\sigma(I)J} \tag{1.4}$$

where $S_n \ni \sigma$ is the group of permutations of n elements, and $\operatorname{sign} \sigma$ is 1 if σ is an even permutation (it can be obtained in an even number of pair swaps) and -1 otherwise.

2 Useful formulas

2.1 Nobili

Tensor calculus The covariant derivative keeps account of the shifting of the basis vectors:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\alpha\mu} A^\alpha \tag{2.1}$$

The rank-3 objects Γ are called Christoffel symbols. They are not tensors! they depend on the choice of basis e_α, and they satisfy $\nabla_\mu e_\alpha = \Gamma^\nu_{\mu\alpha} e_\nu$.

If we have the metric, they can be calculated as:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \Big(\partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho}\Big) \tag{2.2}$$

This also tells us that they are symmetric in the lower two indices: $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$.

The divergence of a vector field A^μ can be calculated as:

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \Big(\sqrt{-g} A^\mu\Big) \tag{2.3}$$

where g is the determinant of the metric.

We can also define the Dalambertian operator $\square = \nabla_\mu \nabla^\mu = \nabla_\mu \partial^\mu$, which can only act on scalars, and it does so like:

$$\square A = \nabla_\mu (\partial^\mu A) = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu A \right) \tag{2.4}$$

If we differentiate and antisymmetryze (so, take the rotor of) an antisymmetric tensor $F_{[\mu\nu]}$, the Christoffel symbols cancel:

$$\nabla_{[\mu} F_{\nu\rho]} = \partial_{[\mu} F_{\nu\rho]} \tag{2.5}$$

The derivative with respect to proper time is $\frac{\mathrm{d}}{\mathrm{d}\tau} = u^\mu \partial_\mu$.

Covariant acceleration is defined as:

$$a^\nu = u^\mu \nabla_\mu u^\nu \tag{2.6}$$

Curvature The curvature of spacetime is fully described by the Riemann curvature tensor, which is a fourth rank tensor: for any generic vector V^μ,

$$R^\mu_{\nu\rho\sigma} V^\nu \stackrel{\text{def}}{=} [\nabla_\rho, \nabla_\sigma] V^\mu \tag{2.7}$$

It can be calculated using the Christoffel symbols, and while they are not tensors $R^\mu_{\nu\rho\sigma}$ is one. This result follows by direct computation from formula (2.7).

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda} \Gamma^\lambda_{\rho\nu} \tag{2.8}$$

The Christoffel symbols can be nonzero if we choose certain coordinates even for flat spacetime, but the Riemann tensor is zero iff the spacetime is flat.

Geodesics If we have a path $x^\mu(\lambda)$, we would like to see if it is a geodesic, that is, if it is stationary with respect to path length. To do this we can stationarize the action corresponding to the lagrangian $\mathcal{L}(x, \dot{x}) = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ (where we use $\dot{x} = \mathrm{d}x / \mathrm{d}\lambda$). The Lagrange equations then are:

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0 \tag{2.9}$$

Where Γ are the Christoffel symbols, which can be calculated by differentiating the metric, as shown in (2.2). $\mathcal{L}$ is an integral of these Lagrange equations.

If the parameter λ is taken to be the proper time s, then the equation is

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}s} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0 \tag{2.10}$$

Notice that this is equivalent to the covariant acceleration (2.6) being zero.

Fermi-Walker transport Take a general vector field $V^\mu(s)$ defined along a curve, with its tangent vector u^μ whose covariant acceleration is a^μ. Then we say that V^μ is transported according to Fermi-Walker iff it satisfies

$$\dot{V}^\mu = u^\nu \nabla_\nu V^\nu = V_\rho (u^\mu a^\rho - a^\mu u^\rho) \tag{2.11}$$

This condition is always satisfied by $V^\mu = u^\mu$, since $a^\mu u_\mu = 0$, whether or not the curve is a geodesic. The tangent vector is *parallel* transported only for geodesics.

Tetrads and projectors We want to work in a reference in which the velocity u^μ is purely timelike. This can always be found by the equivalence principle. Such a reference can be completed into what is called a tetrad, for which the metric becomes the Minkowski metric in a neighbourhood of the point we consider.

We call the velocity $u^\mu = V^\mu_{(0)}$, and add to it three other vectors $V^\mu_{(i)}$ such that

$$g_{\mu\nu} V^\mu_{(\alpha)} V^\nu_{(\beta)} = \eta_{(\alpha)(\beta)} \tag{2.12}$$

where the brackets around the indices denote the fact that they label four vectors, not the components of a tensor.

We can choose the vectors $V^\mu_{(i)}$ so that they are Fermi-Walker transported along the worldline defined by u^μ: this allows us to find the relativistic equivalent of a nonrotating frame of reference.

It is useful to project tensors onto the space-like and time-like subspaces defined by our tetrad (and we wish to do so in a coordinate-independent manner, so just taking the 0th and i-th components in the tetrad will not suffice). We therefore define the projectors:

$$h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu} \qquad \pi_{\mu\nu} = -u_\mu u_\nu \tag{2.13}$$

respectively onto the space- and time-like subspaces.

Metrics The simplest physically relevant one is the Schwarzschild metric. It describes a spherically symmetric object of mass M, in spherical coordinates. Defining $\Phi = -M/r$, we have:

$$ds^2 = -(1+2\Phi)\,dt^2 + \frac{1}{1+2\Phi}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi\right) \tag{2.14}$$

or, equivalently,

$$g_{\mu\nu} = \operatorname{diag} -(1+2\Phi), \frac{1}{1+2\Phi}, r^2, r^2\sin^2\theta \tag{2.15}$$

We can see that it approaches the flat metric $\eta_{\mu\nu} = \operatorname{diag} -,+,+,+$ in the limit $M \to 0$. Its determinant is $g = -r^4\sin^2\theta$.

Fluid mechanics In usual relativistic single-body mechanics, we use the 4-velocity u^μ and the corresponding 4-momentum $p^\mu = mu^\mu$. The 0-th component of this vector is the energy of the body, while the i-th components are its momentum: we then have $p^\mu p_\mu = m^2 = E^2 - |p|^2$.

When dealing with a continuum, we will have a certain density of particles per unit of volume, we call this n. The current of particles is then $N^\mu = nu^\mu$. If these particles have a certain rest mass m_0, we can then define the vector $\rho_0 u^\mu = m_0 n u^\mu = m_0 N^\mu$.

This satisfies a conservation equation: $\nabla_\mu(\rho_0 u^\mu) = 0$.

Particles in a fluid can have three kinds of energy we concern ourselves with: mass, kinetic energy and other forms of energy (thermal, chemical, nuclear…). We can always perform a change of coordinates to bring us to a frame in which the kinetic energy is zero. We write the sum of the other two forms of energy as $\rho = \rho_0(1+\epsilon)$. So, ϵ is the ratio of the internal non-mass energy to the mass.

Now, the vector ρu^μ describes the flux of energy. We can then write the equation for the conservation of momentum:

$$f^\mu = \nabla_\nu(\rho u^\mu u^\nu) \tag{2.16}$$

Spherical accretion We work with the Schwarzschild metric (2.14); we treat a fluid with 4-velocity u^μ in spherical coordinates, since the problem we are looking at is stationary and spherically symmetric the velocity is:

$$u^\mu = \begin{pmatrix} \gamma^2/y \\ yv \\ 0 \\ 0 \end{pmatrix} \tag{2.17}$$

where we define $\gamma = \left(1 - v^2\right)^{-1/2}$, $y = \gamma\sqrt{1+2\Phi}$.

The conservation of mass holds: if ρ_0 is the rest mass density of the fluid, we must have $\nabla_\mu\left(\rho_0 u^\mu\right) = 0$. This, using the formula for covariant divergence (2.3), yields:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho_0 y v r^2\right) = 0 \tag{2.18}$$

In the newtonian limit both γ and y approach 1; also, the infalling mass rate $\dot{M}$ at a certain radius is $\rho_0(r)v(r)4\pi r^2$. Then, by continuity to the newtonian limit, the quantity which is constant wrt the radius must be $\dot{M}/(4\pi)$.

We also have the Euler equation:

$$(p+\rho)a^\mu = -h^{\mu\nu}\partial_\nu p \tag{2.19}$$

And the equation for the variation of the total internal energy:

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \frac{p+\rho}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} \tag{2.20}$$

From these we can show that the quantity $\gamma h\sqrt{1+2\Phi}$ (where $h = (p+\rho)/\rho_0$ is the specific enthalpy), is a constant of motion. In the nonrelativistic, weak-field limit this becomes

$$\gamma h\sqrt{1+2\Phi} \approx \frac{p}{\rho_0} + \frac{v^2}{2} - \frac{M}{r} + \epsilon = \text{const} \tag{2.21}$$

2.2 Taub

This section summarizes my study of A. H. Taub's review of relativistic fluid dynamics, [Tau78].

Nonrelativistic fluid mechanics are described by the equations:

$$\partial_t\rho + \partial_i(\rho v^i) = 0 \tag{2.22a}$$

$$\rho\left(\partial_t v^i + v^j\partial_j v^i\right) = \partial_j T^{ij} \tag{2.22b}$$

$$\rho\partial_t E + v^i\partial_i E = \partial_i\left(T^{ij}v_j + \lambda\partial^i T\right) \tag{2.22c}$$

where ρ is the density of the fluid, v^i are the components of its velocity, T^{ij} is the stress tensor (or, equivalently, the space-like components of the energy-momentum tensor), E is the energy of the fluid, λ is the thermal conductivity, T is the temperature of the fluid.

The nonrelativistic stress tensor can be written as:

$$T_{ij} = -(p + \xi\partial_k v^k)\delta_{ij} + \eta\partial_{(i}v_{j)} \tag{2.23}$$

where p is the (isotropic) pressure, η the viscosity, ξ is the compression viscosity. We are assuming that the normal stresses are only those exerted by pressure, so the diagonal terms T_{ii} (not summed) must just be $-p$. So, the term $-\xi\partial_k v^k$ must equal $\eta\partial_{(i}v_{i)} = 2\eta\partial_i v_i$ (not summed). Therefore, by isotropy, $\xi = 2\eta/3$.

Note that we are working in Euclidean 3D space, so the metric is the identity and upper and lower indices are equivalent.

The energy is a sum of kinetic and specific energy:

$$E = v^i v_i / 2 + \varepsilon \tag{2.24}$$

where ε is the specific energy (of a type that is different from kinetic) per unit mass.

Bibliography

[Tau78] A H Taub. "Relativistic Fluid Mechanics". In: *Annual Review of Fluid Mechanics* 10.1 (1978), pp. 301–332. URL: https://doi.org/10.1146/annurev.fl.10.010178.001505.