UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Corso di Laurea in Fisica

Tesi di Laurea

Relativistic non-ideal flows

Relatore

Prof. Roberto Turolla

Laureando

Jacopo Tissino

Anno Accademico 2018/2019

Abstract

After reviewing the basic concepts of general-relativistic fluid mechanics, I will focus on the treatment of non-ideal (viscous, thermo-conducting) flows. An application of non-ideal relativistic flows to spherical accretion onto black holes (generalized Bondi accretion) will be also discussed.

Contents

1 Notational preface

I will use Greek indices (μ, ν, $\rho\dots$) to denote 4-dimensional indices ranging from 0 to 3, and Latin indices (i, j, $k\dots$) to denote 3-dimensional indices ranging from 1 to 3.

I will use the "mostly plus" metric for flat Minkowski space-time, $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$: therefore four-velocities will have square norm $u^\mu u_\mu = -1$. I will use Einstein summation convention: if an index appears multiple times in the same monomial, it is meant to be summed over.

I will always use the abuse of notation in which a vector is denoted by its components, and a free index meaning all of the possible values it can take, such as x^μ denoting a point in spacetime.

Take a diffeomorphism $x \to y$, with Jacobian matrix $\partial y^\mu / \partial x^\nu$. The indices of contravariant vectors, transforming as

$$V^\mu \to \left(\frac{\partial y^\mu}{\partial x^\nu}\right) V^\nu \tag{1.1}$$

will be denoted as upper indices, while the indices of covariant vectors, transforming as

$$V_\mu \to \left(\frac{\partial x^\nu}{\partial y^\mu}\right) V_\nu \tag{1.2}$$

will be denoted as lower indices; the same applies to higher rank tensors.

Unless otherwise specified, I will work in geometrized units, where $c = G = 1$.

In section 4 I will use the notation from Thorne [Tho81]: A_k will represent a sequence of k indices labelled as α_i, for i between 1 and k. The same will hold for $B_k \to \{\beta_i\}$ etc.

Take a tensor with many indices, $T_{A_k B_j}$. These indices can be symmetrized and antisymmetrized, and I will use the following conventions:

$$T_{(A_k)B_j} = \frac{1}{k!} \sum_{\sigma \in S_k} T_{\sigma(A_k)J} \tag{1.3}$$

$$T_{[A_k]B_j} = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}\,\sigma T_{\sigma(A_k)B_j} \tag{1.4}$$

where $S_k \ni \sigma$ is the group of permutations of k elements, and $\mathrm{sign}\,\sigma$ is 1 if σ is an even permutation (it can be obtained in an even number of pair swaps) and -1 otherwise.

In the case a set of indices that are not nearby need to be (anti)symmetrized, I will use vertical bars: for example, $R_{(\mu|\nu|\rho\sigma)}$ means that we take the permutations of the indices μ, ρ and σ.

2 General relativity

2.1 Special relativity

It is a theory which satisfies the following axioms [Lec14]:

1. space and time are homogeneous (i. e. shift-invariant), space is isotropic (i.e. rotation-invariant);
2. the speed of light is the same in every inertial reference frame;
3. all the laws of physics are written in the same way in every inertial reference frame.

In special relativity we model spacetime as an intrinsically flat semi-Riemannian manifold with metric signature $(-,+,+,+)$: instead of having 3D space and a time scalar coordinate, we denote events as points in 4D spacetime, with coordinates such as $x^\mu = (t, x, y, z)$. This is called Minkowski flat spacetime.

This is not just semantics: it is needed because it turns out (see for example [Lec14, section 1]) that the axioms are equivalent to the conservation of the spacetime interval $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, and the only transformations between inertial reference frames which leave it invariant turn out to regularly *mix* time and space: they are conveniently represented in 4D spacetime as $x^\mu \to \Lambda^\mu{}_\alpha x^\alpha + a^\mu$, with the $\Lambda_\mu{}^\alpha$ being $(1,1)$ tensors which satisfy $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$, and a^μ being a 4-vector.

The flat metric allows us to compute the lengths of vectors, and since it is indefinite there are nonzero vectors of positive, negative and zero spacetime length. These are respectively called spacelike, timelike and null-like.

We define the proper time τ by $\mathrm{d}\tau \stackrel{\text{def}}{=} \sqrt{-\mathrm{d}s^2}$. Unlike coordinate time $x^0 = t$ this has the advantage of being Lorentz-invariant.

We can define a tensorial velocity by differentiating the position with respect to proper time: $u^\mu \stackrel{\text{def}}{=} \mathrm{d}x^\mu/\mathrm{d}\tau$. Defined this way, the so-called 4-velocity trasforms like a tensor. If $\vec{v}$ is the regular three-velocity and v is its magnitude, we define $\gamma = 1/\sqrt{1-v^2}$ and then: $u^\mu = (\gamma, \gamma\vec{v})$.

Once we have this, we can define the 4-acceleration $a^\mu = \mathrm{d}u^\mu/\mathrm{d}\tau$ and the 4-momentum $p^\mu = mu^\mu$, where m is the rest mass of the body at hand.

The 4-velocity is a unit vector: $u^\mu u_\mu = -1$, and by differentiating this relation we get $u^\mu a_\mu = 0$.

The 0-th component of the 4-momentum vector is the energy of the body, while the i-th components define a new relativistic momentum $p^i = \gamma m v^i$: we then have $p^\mu p_\mu = m^2 = E^2 - |p|^2$.

2.2 Differential geometry and tensor calculus

Metric The metric tensor $g_{\mu\nu}$ is a symmetric $(0,2)$ tensor which defines a scalar product at every point in our manifold: $x \cdot y = g_{\mu\nu} x^\mu y^\nu$. It is not intrinsic to the manifold. By integrating the velocity vector we can find the lengths of curves $x^\mu(\lambda)$:

$$L = \int \sqrt{g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda \tag{2.1}$$

For a flat spacetime we use the Minkowski metric $\eta_{\mu\nu}$. In general, in the presence of matter the manifold will be curved, so there will not be a coordinate transformation to cast $g_{\mu\nu}$ in the form $\eta_{\mu\nu}$. If we choose a certain point P, however, it is possible to find a transformation in order to impose the conditions $g_{\mu\nu}(P) = \eta_{\mu\nu}(P)$, $\partial_\rho g_{\mu\nu}(P) = 0$ [Car97, pages 49–50].

The metric defines an invariant called the *spacetime interval*:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{2.2}$$

Tensor calculus An object such as $\partial_\mu A^\nu$ for some vector A^ν does not in general transform as a tensor, since the computation we should actually do is $\partial_\mu(A^\nu e_\nu)$ where e_ν are the basis vectors: we evaluate this with the product rule and are left with $(\partial_\mu A^\nu)e_\nu + A^\nu \partial_\mu e_\nu$. Only the first half is covariant, in flat spacetime the second half is zero but in general it is not: the way we usually confuse the derivatives of the components with the derivatives of the vector does not work here.

The covariant derivative keeps account of the shifting of the basis vectors:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\alpha\mu} A^\alpha \tag{2.3}$$

The rank-3 objects Γ are called Christoffel symbols. They are not tensors! they depend on the choice of basis e_α, and they satisfy $\nabla_\mu e_\alpha = \Gamma^\nu_{\mu\alpha} e_\nu$. They are not intrinsic to the manifold.

The definition is posed in such a way that the objects $\nabla_\mu A^\nu$ are tensorial. This allows us to see that the covariant derivative is the same as the coordinate derivative for scalars.

We can define the covariant derivative of higher order tensor analogously; adding a Christoffel symbol for every new index. The symbols corresponding to lower indices have a minus sign: this can be seen by differentiating a scalar such as $\nabla_\nu(A_\mu B^\mu) \stackrel{!}{=} \partial_\nu(A_\mu B^\mu)$ and matching the Christoffel terms.

If we have the metric (and make reasonable assumptions of the connection being torsion-free), they can be calculated as:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho} \right) \tag{2.4}$$

This also tells us that they are symmetric in the lower two indices: $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$.

The divergence of a vector field A^μ can be calculated as:

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} A^\mu \right) \tag{2.5}$$

where g is the determinant of the metric.

The derivative with respect to proper time is $\frac{\mathrm{d}}{\mathrm{d}\tau} = u^\mu \nabla_\mu$.

Covariant acceleration is defined as:

$$a^\nu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = u^\mu \nabla_\mu u^\nu \tag{2.6}$$

Stokes' theorem Following [Ung16]. If we have a manifold of dimension n, and an n-dimensional region V of this manifold equipped with coordinates x^μ a metric $g_{\mu\nu}$, with a submanifold boundary ∂V equipped with coordinates y^μ and the induced metric $h_{\alpha\beta} = \left(\partial x^\mu / \partial y^\alpha \right) \left(\partial x^\nu / \partial y^\beta \right) g_{\mu\nu}$, and for which we have a properly oriented normal vector $n^\mu(y)$; then for any vector f^μ we have:

$$\int_V \nabla_\mu f^\mu \sqrt{|\det g|} \, \mathrm{d}^n x = \int_{\partial V} f^\mu n_\mu \sqrt{|\det h|} \, \mathrm{d}^{n-1} y \tag{2.7}$$

Geodesics If we have a path $x^\mu(\lambda)$, we would like to see if it is a geodesic, that is, if it is stationary with respect to path length. To do this we can stationarize the action corresponding to the Lagrangian $\mathcal{L}(x, \dot{x}) = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ (where we use $\dot{x} = \mathrm{d}x / \mathrm{d}\lambda$). The Lagrange equations then are:

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0 \tag{2.8}$$

where Γ are the Christoffel symbols, which can be calculated by differentiating the metric, as shown in (2.4). $\mathcal{L}$ is an integral of these Lagrange equations.

If the parameter λ is taken to be the proper time s, then the equation is

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}s} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0 \tag{2.9}$$

Notice that this is equivalent to the covariant acceleration (2.6) being zero.

Fermi-Walker transport Take a general vector field $V^\mu(s)$ defined along a curve, with the curve's tangent vector u^μ whose covariant acceleration is a^μ. Then we say that V^μ is transported according to Fermi-Walker iff it satisfies

$$\dot{V}^\mu = u^\nu \nabla_\nu V^\mu = 2 V_\rho u^{[\mu} a^{\rho]} \tag{2.10}$$

This condition is always satisfied by $V^\mu = u^\mu$, since $a^\mu u_\mu = 0$, whether or not the curve is a geodesic. The tangent vector is *parallel* transported only for geodesics.

The justification of this definition is the fact that we want the transformations of our our tetrad to be infinitesimal Lorentz boosts, which are generated by antisymmetric tensors, and we want to prohibit any rotations in the plane orthogonal to a^μ and u^μ.

Tetrads and projectors We want to work in a reference in which the velocity u^μ is purely timelike. This can always be found by the equivalence principle. Such a reference can be completed into what is called a tetrad, for which the metric becomes the Minkowski metric in a neighbourhood of the point we consider.

We call the velocity $u^\mu = V^\mu_{(0)}$, and add to it three other vectors $V^\mu_{(i)}$ such that

$$g_{\mu\nu} V^\mu_{(\alpha)} V^\nu_{(\beta)} = \eta_{(\alpha)(\beta)} \tag{2.11}$$

where the brackets around the indices denote the fact that they label four vectors, not the components of a tensor.

We can choose the vectors $V^\mu_{(i)}$ so that they are Fermi-Walker transported along the worldline defined by u^μ: this allows us to find the relativistic equivalent of a non-rotating frame of reference.

It is useful to project tensors onto the space-like and time-like subspaces defined by our tetrad (and we wish to do so in a coordinate-independent manner, so just taking the 0th and i-th components in the tetrad will not suffice). We therefore define the projectors:

$$h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu} \qquad \pi_{\mu\nu} = -u_\mu u_\nu \tag{2.12}$$

respectively onto the space- and time-like subspaces.

Killing vector fields Following [MTW73, section 25.2, page 650]. Say there is a certain direction (for simplicity, along one of our coordinate axes) along which the metric is preserved: an $\widetilde{\alpha}$ such that $\partial_{\widetilde{\alpha} g_{\mu\nu}}$.

Then the metric properties of curves along the manifold are unchanged if we shift their coordinate representation by a constant along the $\widetilde{\alpha}$ coordinate axis.

Let us call the direction of this translation $\xi^\mu = \delta^\mu_{\widetilde{\alpha}}$ if we use this coordinate system. It can be shown by direct computation that

$$\nabla_\nu \xi_\mu = \frac{1}{2}\left(\partial_{\widetilde{\alpha}} g_{\mu\nu} + \partial_\nu g_{\mu\widetilde{\alpha}} - \partial_\mu g_{\nu\widetilde{\alpha}}\right) \tag{2.13}$$

but by hypothesis the first term on the RHS of (2.13) is zero, therefore we have shown that $\nabla_\nu \xi_\mu = \nabla_{[\nu} \xi_{\mu]}$ in this coordinate frame, but since this is a covariant equation it extends to every other one.

It can also be seen this way that $\nabla_{(\nu} \xi_{\mu)} = 0$: this is called *Killing's equation*. This is useful since: given a geodesic $x^\mu(\lambda)$, for which we define $u^\mu = \mathrm{d}x^\mu / \mathrm{d}\lambda$, it must be the case that $u^\nu \nabla_\nu u^\mu = 0$. Then, the component of u^μ along ξ^μ ($u^{\widetilde{\alpha}} = u^\mu \xi_\mu$) is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(u^\mu \xi_\mu\right) = u^\nu \nabla_\nu \left(u^\mu \xi_\mu\right) = \cancelto{0}{\xi^\mu u^\nu \nabla_\nu u_\mu} + \cancelto{0}{u^\nu u^\mu \nabla_\nu \xi_\mu} \equiv 0 \tag{2.14}$$

Surfaces in space-time and acceleration decomposition Following [Tau78, section 4] We look at 4D space time in 3D space-like slices: if we have a fluid moving with velocity u^μ, we can look at the solutions of the associated differential equation: $x^\mu(\xi^i, s)$, where ξ^i are the 3D coordinates of the starting position and s is the time at which we look at the solution. Then we can look at the "starting" hypersurface $\Sigma = \left\{ x^\mu(\xi^i, 0) \right\}$.

Say we have a curve $\xi^i(\tau)$ in Σ. Then we can look at the two-dimensional surface defined by the evolution of $\xi^i(\tau)$: $x^\mu(\xi^i(\tau), s) = x^\mu(\tau, s)$. If we define the "spatial" tangent vector $\lambda^\mu = \mathrm{d}x^\mu / \mathrm{d}\tau$, it follows from Schwarz's theorem that:

$$\frac{\partial^2 x^\mu}{\partial\tau \partial s} = \frac{\partial^2 x^\mu}{\partial s \partial \tau} \implies \frac{\partial u^\mu}{\partial \tau} = \frac{\partial \lambda^\mu}{\partial s} \tag{2.15}$$

Now let us take the spatial vectors of an orthonormal Fermi-Walker transported tetrad $V^{\mu}_{(a)}$ as described in 'Tetrads and projectors' on page 6, and express λ^{μ} in this frame: its covariant components will be

$$X_{(a)} = V_{(a)\mu}\lambda^{\mu} \tag{2.16}$$

If we differentiate (2.16) with respect to s, and use (2.15) with the fact that $\frac{\mathrm{d}}{\mathrm{d}\tau} = \lambda^{\mu}\nabla_{\mu}$, we get:

$$\begin{aligned}
\frac{\mathrm{d}X_{(a)}}{\mathrm{d}s} &= \frac{\mathrm{d}V_{(a)\mu}}{\mathrm{d}s}\lambda^{\mu} + V_{(a)\mu}\frac{\mathrm{d}\lambda^{\mu}}{\mathrm{d}s} &(2.17\text{a})\\
&= V^{\rho}_{(a)}\cancelto{0}{\lambda^{\mu}u_{\mu}}a_{\rho} - \cancelto{0}{V^{\rho}_{(a)}u_{\rho}}\lambda^{\mu}a_{\mu} + V^{\nu}_{(a)}\nabla_{\mu}u_{\nu} &(2.17\text{b})\\
&= V^{\rho}_{(a)}\lambda^{\mu}\nabla_{\mu}u_{\rho} &(2.17\text{c})\\
&= \left(\nabla_{\mu}u_{\rho}\right)V^{\rho}_{(a)}V^{\mu}_{(b)}X^{(b)} &(2.17\text{d})
\end{aligned}$$

where in the last step we expressed everything with respect to the tetrad coordinate system. Therefore, in those coordinates, the evolution of the components $X^{(a)}$ is linear, and defined by the tetrad components of the two-form $\nabla_{\mu}u_{\nu}$. So, we want to decompose this tensor:

$$\nabla_{\sigma}u_{\tau} = \underbrace{\omega_{\sigma\tau}}_{\substack{\text{spatial}\\\text{rotation}}} + \underbrace{\sigma_{\sigma\tau}}_{\substack{\text{spatial}\\\text{shear}}} + \underbrace{\frac{1}{3}\theta h_{\sigma\tau}}_{\substack{\text{spatial}\\\text{compression}}} - \underbrace{a_{\tau}u_{\sigma}}_{\text{acceleration}} \tag{2.18}$$

1. $\theta = \nabla_{\mu}u^{\mu}$ is the bare trace of the tensor, corresponding to the expansion velocity;
2. $a_{\mu} = u^{\nu}\nabla_{\nu}u^{\mu}$ is the covariant acceleration;
3. $\sigma_{\sigma\tau} = \left(\nabla_{(\mu}u_{\nu)}\right)h^{\mu}_{\sigma}h^{\nu}_{\tau} - 1/3\theta h_{\sigma\tau} = \nabla_{(\sigma}u_{\tau)} + a_{(\sigma}u_{\tau)} - 1/3\theta h_{\sigma\tau}$ is the spatial symmetric trace-free part of the tensor, that is, the shear stress; [1]
4. $\omega_{\sigma\tau} = h^{\nu}_{\sigma}h^{\mu}_{\tau}\nabla_{[\nu}u_{\mu]} = \partial_{[\tau}u_{\sigma]} + a_{[\tau}u_{\sigma]}$ is the spatial rotation tensor.

We can describe the rotation with a "vorticity vector":

$$\omega^{\mu} = \frac{1}{2\sqrt{-g}}\varepsilon^{\mu\nu\sigma\tau}u_{\nu}\partial_{\tau}u_{\sigma} = \frac{1}{2}\eta^{\mu\nu\sigma\tau}u_{\nu}\partial_{\tau}u_{\sigma} \tag{2.19}$$

where we define the fully antisymmetric, covariant tensor $\eta^{\mu\nu\sigma\tau} = \varepsilon^{\mu\nu\sigma\tau}/\sqrt{-g}$. With its indices lowered, it is $\eta_{\mu\nu\sigma\tau} = -\varepsilon_{\mu\nu\sigma\tau}\sqrt{-g}$.

Note, from [Car97, pages 51–52]: this is the volume form of the manifold, and it is defined this way since g is a *tensor density* of weight -2, while the bare Levi-Civita symbol is a density of weight $+1$.

The signs in [Tau78] and [Car97] seem to disagree though!

By the antisymmetry in the definition we can immediately see that $\omega^{\mu}u_{\mu} = 0$. It holds that $\omega^{\mu} \equiv 0 \iff u_{\mu} = \rho\partial_{\mu}f$ (locally!) for scalar ρ, f, since this is equivalent to u_{μ} being a closed form.

ω_{μ} and $\omega_{\mu\nu}$ are "dual" in the sense that

$$\omega_{\sigma\tau} = u^{\mu}\omega^{\nu}\eta_{\mu\nu\sigma\tau} \tag{2.20}$$

and some other useful identities can be found in equations 7.5 through 7.7 of [Tau78].

[1] A trick for this computation: why would a term like $u^{\mu}u_{\sigma}\nabla_{\tau}u_{\mu}$ be zero? We can just multiply by $1 = -u^{\tau}u_{\tau}$ to get $-u^{\mu}u_{\sigma}u_{\tau}u^{\tau}\nabla_{\tau}u_{\mu} = \sum_{\tau}u_{\sigma}u_{\tau}a_{\mu}u^{\mu} = 0$.

Is this right? It seems like we shouldn't be able to do this... The index τ in the starting formula was not summed and then it is so this does not sound convincing. Is there another way to see that $u^{\mu}u_{(\sigma}\nabla_{\tau)}u_{\mu} = 0$?

2.3 General Relativity

Curvature The curvature of spacetime is fully described by the Riemann curvature tensor, which is a fourth rank tensor: for any generic vector V^μ,

$$R^\mu_{\nu\rho\sigma} V^\nu \stackrel{\text{def}}{=} [\nabla_\rho, \nabla_\sigma] V^\mu \tag{2.21}$$

It can be calculated using the Christoffel symbols, and while they are not tensors $R^\mu_{\nu\rho\sigma}$ is one. This result follows by direct computation from formula (2.21).

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu} \tag{2.22}$$

The Christoffel symbols can be nonzero if we choose certain coordinates even for flat spacetime, but the Riemann tensor is zero iff the spacetime is flat.

The Riemann tensor satisfies the following identities [MTW73, eqs. 8.45 and 8.76]:

$$\nabla_{[\lambda} R_{\mu\nu]\rho\sigma} = 0 \tag{2.23a}$$

$$R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]} = R_{[\rho\sigma][\mu\nu]} \tag{2.23b}$$

$$R_{[\mu\nu\rho\sigma]} = 0 = R_{\mu[\nu\rho\sigma]} \tag{2.23c}$$

If we define the Ricci tensor $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$ and the curvature scalar $R = R_{\mu\nu} g^{\mu\nu}$, we can rewrite (2.23a) as $\nabla_\mu R = 2\nabla_\nu R^\nu_\mu$, which means that $\nabla_\mu \left(R^{\mu\nu} - 1/2 R g^{\mu\nu}\right) = 0$.

The Einstein Field Equations They describe the way the presence of matter changes the geometry of spacetime. They involve the *stress-energy tensor* $T^{\mu\nu}$ which is defined in 'Stress-energy tensor' on page 10 and the Einstein tensor $G^{\mu\nu} = R^{\mu\nu} - 1/2 R g^{\mu\nu}$, which is the only independent tensor which: can be constructed from only the Riemann tensor and the metric, which vanishes for flat spacetime and which identically satisfies the conservation laws $\nabla_\mu G^{\mu\nu} = 0$.

The equations are:

$$G^{\mu\nu} = 8\pi T^{\mu\nu} \tag{2.24}$$

The constant comes by imposing continuity with the newtonian limit, for which we have the Poisson equation $\nabla^2 \Phi = 4\pi\rho$, where the classical ρ is substituted by T_{00} while Φ is substituted by h_{00}, with $g_{00} = -1 + h_{00}$ (see [Car97, eq. 4.46]).

They can be written in a more general way by removing the condition that the LHS vanish for flat spacetime, and thus including there a term $\Lambda g^{\mu\nu}$ with constant Λ. It is unclear whether this term should appear, and what the value of Λ should be.

The Schwarzschild solution Following [Car97, section 7].

The EFE are generally very difficult to solve, but in certain special cases they can be dealt with. The simplest nontrivial one is that of a central mass M described with spherical coordinates (t, r, θ, φ) and in the presence of spherical symmetry.

One imposes the condition that the stress energy tensor be identically zero for radii greater than a certain (arbitrarily small) radius, $r > r_c$.

One can write down the most general possible spherically symmetric metric, which turns out to be [Car97, eq. 7.13]:

$$ds^2 = -e^{2\alpha(r,t)}\, dt^2 + e^{2\beta(r,t)}\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi\right) \tag{2.25}$$

And plug this into the equations $G_{\mu\nu} = 0 \implies R_{\mu\nu} = 0$ for all $r > r_c$. One gets the result that the metric possesses a timelike Killing vector field which is orthogonal to a family of hypersurfaces

$t = \text{const}$: therefore it is *static*, unchanging with time. α and β only depend on r, and one gets the result that

$$e^{2\alpha(r)} = e^{-2\beta(r)} = \left(1 + \frac{C}{r}\right) \tag{2.26}$$

for some C. By continuity with the weak-field limit, for which one has the newtonian gravitational field $\Phi = -M/r$ and $g_{00} = -(1 + 2\Phi)$, one sets $C = -2M$. Keeping the notation $\Phi = -M/r$ we have:

$$ds^2 = -(1 + 2\Phi)\, dt^2 + \frac{1}{1 + 2\Phi}\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi\right) \tag{2.27}$$

or, equivalently,

$$g_{\mu\nu} = \text{diag}\left(-(1 + 2\Phi), \frac{1}{1 + 2\Phi}, r^2, r^2 \sin^2\theta\right) \tag{2.28}$$

We can see that it approaches the flat metric $\eta_{\mu\nu} = \text{diag}\,(-, +, +, +)$ in the limit $M \to 0$. Its determinant is $g = -r^4 \sin^2\theta$.

If we wanted to analyze the accretion problem without *any* approximation we would need to consider the stress-energy tensor of the fluid's contribution to the metric's curvature; however this would make the already hard problem completely intractable, therefore we assume the stress-energy tensor contributions are very small compared to the massive object's.

3 Fluid dynamics

3.1 Nonrelativistic fluid mechanics

Nonrelativistic fluid mechanics are described by the equations:

$$\partial_t \rho + \partial_i(\rho v^i) = 0 \qquad \text{conservation of mass} \tag{3.1a}$$

$$\rho\left(\partial_t v^i + v^j \partial_j v^i\right) = \partial_j \sigma^{ij} \qquad \text{conservation of momentum} \tag{3.1b}$$

$$\rho \partial_t E + v^i \partial_i E = \partial_i \left(\sigma^{ij} v_j + \kappa \partial^i T\right) \qquad \text{conservation of energy} \tag{3.1c}$$

where ρ is the density of the fluid, v^i are the components of its velocity, σ^{ij} is the classical stress tensor (or, equivalently, the *negative* of the space-like components of the energy-momentum tensor), E is the energy of the fluid, κ is the thermal conductivity, T is the temperature of the fluid.

The nonrelativistic stress tensor can be written as:

$$\sigma_{ij} = -(p - \xi \partial_k v^k)\delta_{ij} + 2\eta \partial_{(i} v_{j)} \tag{3.2}$$

I flipped the sign of the term $\xi \partial_k v^k$ since I think it is a typo in [Tau78, page 301]: the pressure and compression viscosity terms should have opposite signs like in (3.6), right?

where p is the (isotropic) pressure, η the viscosity, ξ is the compression viscosity. We are assuming that the normal stresses are only those exerted by pressure, so the diagonal terms σ_{ii} (not summed) must just be $-p$. So, the term $-\xi \partial_k v^k$ must equal $\eta \partial_{(i} v_{i)} = 2\eta \partial_i v_i$ (not summed). Therefore, by isotropy, $\xi = 2\eta/3$.

Note that we are working in Euclidean 3D space, so the metric is the identity and upper and lower indices are equivalent.

The energy is a sum of kinetic and specific energy:

$$E = v^i v_i / 2 + \varepsilon \tag{3.3}$$

where ε is the specific energy (of a type that is different from kinetic) per unit mass.

3.2 The relativistic fluid

When dealing with a continuum, we will have a certain density of particles per unit of volume, which we call n. The current of particles is then $N^\mu = nu^\mu$, where u^μ is the 4-velocity field of the fluid. If these particles have a certain rest mass m_0, we can then define the vector $\rho_0 u^\mu = m_0 n u^\mu = m_0 N^\mu$, which is conserved: $\nabla_\mu(\rho_0 u^\mu) = 0$, that is, particles do not spontaneously appear or disappear nor change their mass.

Particles in a fluid can have three kinds of energy we concern ourselves with: mass, kinetic energy and other forms of energy (thermal, chemical, nuclear...). We can always perform a change of coordinates to bring us to a frame in which the kinetic energy is zero. We write the sum of mass-energy and internal energy as $\rho = \rho_0(1+\epsilon)$, the *energy* density of the fluid in its Local Rest Frame, while ρ_0 is the *mass* density in the LRF. So, ϵ is the ratio of the internal non-mass energy to the mass.

Stress-energy tensor The stress-energy tensor $T^{\mu\nu}$ is a $(2,0)$ tensor, whose μ, ν component is the flow of the μ-th component of four-momentum p^μ through a surface of constant coordinate x^ν.

Because of our choice of metric signature, the spatial part of the tensor corresponds to the *negative* of the classical continuum-mechanics stress tensor: $T^{ij} = -\sigma^{ij}$, sice that tensor describes the stresses *on* the "box" of fluid [Mor16].

For a gas of non-interacting particles, the stress-energy tensor is very simple: the momentum density is ρu^μ, and then to look at the flow through a surface of constant x^ν we just need to multiply by u^ν, so in the Local Rest Frame we have:

$$T^{\mu\nu} = \rho u^\mu u^\nu = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.4}$$

Relativistic non-ideal fluid dynamics The dynamics of the fluid are described by the conservation of the stress-energy tensor $\nabla_\mu T^{\mu\nu} = 0$ and the conservation of mass $\nabla_\mu(\rho_0 u^\mu) = 0$.

Any stress-energy tensor can be decomposed in its space and time-like parts in the local rest frame of the fluid:

$$T_{\mu\nu} = w u_\mu u_\nu + w_\mu u_\nu + u_\mu w_\nu + w_{\mu\nu} \tag{3.5}$$

where

$$w = T_{\mu\nu} u^\mu u^\nu = \rho_0(1+\varepsilon) = \rho \qquad \text{rest mass} \tag{3.6a}$$

$$w_\mu = T_{\nu\sigma} h^\sigma_\mu u^\nu = -\kappa h^\sigma_\mu (\partial_\sigma T + T a_\sigma) \qquad \text{heat conduction} \tag{3.6b}$$

$$w_{\mu\nu} = T_{\rho\sigma} h^\rho_\mu h^\sigma_\nu = (p - \xi\theta) h_{\mu\nu} - 2\eta\sigma_{\mu\nu} \qquad \text{pressure and viscous stresses} \tag{3.6c}$$

For the definition of the acceleration, vorticity etc. see equation (2.18). As in the nonrelativistic section η is the viscosity, ξ is the compression viscosity, κ is the thermal conductivity, T is the temperature field, p is the pressure, ρ_0 is the rest mass density while $\rho = \rho_0(1+\varepsilon)$ is the rest energy.

The fact that the decomposition can be written in this way is not proven in [Tau78], but it seems like a proof can be found in [Eck40], which is locked behind a paywall.

Equivalently, we can write

$$T_{\mu\nu} = T^p_{\mu\nu} \qquad - T^V_{\mu\nu} \qquad + T^h_{\mu\nu} \tag{3.7a}$$

$$= w u_\mu u_\nu + p h_{\mu\nu} \qquad - \xi\theta h_{\mu\nu} - 2\eta\sigma_{\mu\nu} \qquad + 2 w_{(\mu} u_{\nu)} \tag{3.7b}$$

perfect fluid viscous stresses heat conduction (3.7c)

It holds [Tau48] that:

$$0 \leq 3p \leq \frac{3}{2}p + \sqrt{\left(\frac{3}{2}p\right)^2 + \rho_0^2} \leq \rho = \rho_0(1+\varepsilon) \leq \rho_0 + 3p \tag{3.8}$$

Forces If we apply $h^\sigma_\mu \nabla_\nu$ to the formulation of the stress-energy tensor given in (3.7), that is, look at the spatial components of the conservation equations, we get:

$$h^\sigma_\mu \nabla_\nu \left((p+\rho)u^\mu u^\nu + p g^{\mu\nu}\right) = h^\sigma_\mu \nabla_\nu \left(T^{\mu\nu}_V - T^{\mu\nu}_h\right) \tag{3.9a}$$

$$(\rho + p)a^\sigma + h^\sigma_\nu \partial^\nu p = h^\sigma_\mu \nabla_\nu \left(+\xi\theta h^{\mu\nu} + 2\eta\sigma^{\mu\nu} - 2w^{(\mu}u^{\nu)}\right) \tag{3.9b}$$

$$= \underbrace{\nabla_\nu T^{\mu\nu}_V - u^\sigma \left(\frac{\xi\theta^2}{3} + 2\sigma_{\mu\nu}\sigma^{\mu\nu}\right)}_{\mathscr{F}^\sigma_V} \underbrace{- \nabla_\nu(w^\sigma u^\nu) - w^\nu \nabla_\nu u^\sigma + a_\mu w^\mu u^\sigma}_{\mathscr{F}^\sigma_h} \tag{3.9c}$$

I have not done the whole computation yet. Shouldn't there be an η in front of σ^2? Is it a typo in [Tau78]? The heat part does not seem to make sense, I'm missing how $-w^\mu u^\nu \nabla_\nu u_\mu = u_\mu(u^\nu \nabla_\nu w^\mu + w^\nu \nabla_\nu u^\mu)$: my results are

$$h^\sigma_\mu \nabla_\nu \left(2w^{(\mu}u^{\nu)}\right) = (\delta^\sigma_\mu + u^\sigma u_\mu)\nabla_\nu(w^\mu u^\nu + u^\mu w^\nu) \tag{3.10a}$$

$$= \nabla_\nu\left(w^\sigma u^\nu\right) + \nabla_\nu\left(u^\sigma w^\nu\right) + u^\sigma u_\mu \nabla_\nu(w^\mu u^\nu + u^\mu w^\nu) \tag{3.10b}$$

$$= \nabla_\nu\left(w^\sigma u^\nu\right) + \nabla_\nu\left(u^\sigma w^\nu\right) + u^\sigma u_\mu \left(\cancelto{0}{w^\mu \nabla_\nu u^\nu} + u^\nu \nabla_\nu w^\mu + w^\nu \nabla_\nu u^\mu + u^\mu \nabla_\nu w^\nu\right) \tag{3.10c}$$

$$= \nabla_\nu\left(w^\sigma u^\nu\right) + \nabla_\nu\left(u^\sigma w^\nu\right) + u^\sigma u_\mu(u^\nu \nabla_\nu w^\mu + w^\nu \nabla_\nu u^\mu) - u^\sigma \nabla_\nu w^\nu \tag{3.10d}$$

$$= \nabla_\nu\left(w^\sigma u^\nu\right) + w^\nu \nabla_\nu u^\sigma + u^\sigma u_\mu(u^\nu \nabla_\nu w^\mu + w^\nu \nabla_\nu u^\mu) \tag{3.10e}$$

$$\stackrel{?}{=} \nabla_\nu\left(w^\sigma u^\nu\right) + w^\nu \nabla_\nu u^\sigma - u^\sigma w_\mu a^\mu \tag{3.10f}$$

so the first two terms are there, the rest does not look right.

The vectors $\mathscr{F}^\sigma_{h,V}$ are relativistic forces on the fluid due respectively to heat flow and viscosity. In the perfect fluid case the RHS is zero we simply get Euler's equation from (3.9c).

The Second Principle in GR Because of the conservation of the stress-energy tensor, we have:

$$\nabla_\nu\left(u_\mu T^{\mu\nu}\right) = T^{\mu\nu}\nabla_\nu u_\mu \tag{3.11}$$

Let us also consider the expression of the differential per-unit-rest-mass entropy:

$$T\,\mathrm{d}S = \mathrm{d}\varepsilon + p\,\mathrm{d}\frac{1}{\rho_0} \tag{3.12}$$

Claim 3.1. *We can mold equation (3.11) into a version of the second principle of thermodynamics*

$$T\nabla_\mu S^\mu = \xi\theta^2 + 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{w^\mu w_\mu}{\kappa T} \geq 0 \tag{3.13}$$

where we define $S^\mu = \rho_0 S u^\mu + w^\mu / T$.

Proof. We will need the decompositions of the derivative of velocity (2.18), of the stress-energy tensor (3.5), (3.6), the expression of differential entropy (3.12) and the conservation of mass $\nabla_\mu(\rho_0 u^\mu) = 0$.

First of all, the LHS of (3.11) can be greatly simplified by noticing that $u^\mu w_\mu = u^\mu w_{\mu\nu} = 0$, so it becomes [1]

$$\nabla_\nu \left(u_\mu T^{\mu\nu} \right) = \nabla_\nu \left(w \underbrace{u_\mu u^\mu}_{-1} u^\nu + \underbrace{u_\mu w^\mu}_{0} u^\nu + \underbrace{u_\mu u^\mu}_{-1} w^\nu + \underbrace{u_\mu w^{\mu\nu}}_{0} \right) \tag{3.14a}$$

$$= \nabla_\nu \left(-u^\nu \rho_0 (1+\varepsilon) - w^\nu \right) \tag{3.14b}$$

$$= -\rho_0 u^\nu \partial_\nu \varepsilon - \nabla_\nu w^\nu \tag{3.14c}$$

In the RHS of (3.11) as well many terms are cancelled because they contain contractions of space and timelike indices: we get

$$\left(\nabla_\nu u_\mu \right) T^{\mu\nu} = \left(\omega_{\nu\mu} + \sigma_{\nu\mu} + \frac{1}{3}\theta h_{\nu\mu} - a_\mu u_\nu \right) \left(w u^\mu u^\nu + w^\mu u^\nu + u^\mu w^\nu + w^{\mu\nu} \right) \tag{3.15a}$$

$$= w^{\mu\nu} \left(\omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{\theta h_{\mu\nu}}{3} \right) + a_\mu w^\mu \tag{3.15b}$$

$$= \left((p - \xi\theta) h^{\mu\nu} - 2\eta\sigma^{\mu\nu} \right) \left(\omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{\theta h_{\mu\nu}}{3} \right) + a_\mu \left(-\kappa h^\mu_\sigma (\partial^\sigma T + T a^\sigma) \right) \tag{3.15c}$$

$$= (p - \xi\theta)\theta - 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} - \kappa a_\mu \partial^\mu T - \kappa T a_\mu a^\mu \tag{3.15d}$$

So far, we have:

$$-\rho_0 u^\nu \partial_\nu \varepsilon - \nabla_\nu w^\nu = (p - \xi\theta)\theta - 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} - \kappa a_\mu \partial^\mu T - \kappa T a_\mu a^\mu \tag{3.16}$$

Let us rearrange (3.16) in a convenient way:

$$+\rho_0 u^\nu \partial_\nu \varepsilon + p\theta = -\nabla_\nu w^\nu + \xi\theta^2 + 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \kappa a_\mu \partial^\mu T + \kappa T a_\mu a^\mu \tag{3.17}$$

Now let us consider a quantity we wish to obtain from these manipulations: $T\nabla_\mu S^\mu$. It can be expanded using the continuity equation into:

$$T\nabla_\mu S^\mu = T\nabla_\mu \left(\rho_0 S u^\mu + \frac{1}{T} w^\mu \right) = T\rho_0 u^\mu \partial_\mu S + \nabla_\mu w^\mu - w^\mu \frac{\nabla_\mu T}{T} \tag{3.18}$$

Let us turn the differentials in (3.12) into proper-time derivatives: $\mathrm{d} \to \mathrm{d}/\mathrm{d}\tau = u^\mu \partial_\mu$. Also, we can use the continuity equation to see that $u^\mu \partial_\mu \rho_0 = -\rho_0 \theta$. Then (3.12) becomes:

$$T\frac{\mathrm{d}S}{\mathrm{d}\tau} = \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} - \frac{p}{\rho_0^2}\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} = \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} + \frac{p\theta}{\rho_0} \tag{3.19}$$

Then we can write the LHS of (3.17), using the identities in equation (3.18) and (3.19):

$$\rho_0 \left(u^\nu \partial_\nu \varepsilon + \frac{p\theta}{\rho_0} \right) = \rho_0 T u^\nu \partial_\nu S = T\nabla_\mu S^\mu - \nabla_\mu w^\mu + \frac{1}{T} w^\mu \nabla_\mu T \tag{3.20}$$

Let us substitute (3.20) into (3.17), and then subtract the desired result (3.13) from the equation: this way, if we get an identity the proof will be complete (this may seem circular, but it is done just for convenience in the algebraic manipulations: to get a more rigorous argument one may just reverse the steps, using the identity (3.21b) in equation (3.21a) to get equation (3.13)).

$$T\nabla_\mu S^\mu - \nabla_\mu w^\mu + \frac{1}{T} w^\mu \nabla_\mu T = -\nabla_\nu w^\nu + \xi\theta^2 + 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \kappa a_\mu \partial^\mu T + \kappa T a_\mu a^\mu \tag{3.21a}$$

[1] One may think to expand $\nabla_\nu w^\nu$ and I had, bringing along many useless terms, when actually it can be kept this way and will just cancel later on.

$$-\nabla_\mu w^\mu + \frac{1}{T} w^\mu \nabla_\mu T = -\nabla_\nu w^\nu + \kappa a_\mu \partial^\mu T + \kappa T a_\mu a^\mu - \frac{w^\mu w_\mu}{\kappa T} \tag{3.21b}$$

$$+\frac{1}{T} w^\mu \nabla_\mu T = +\kappa a_\mu \partial^\mu T + \kappa T a_\mu a^\mu - \frac{w^\mu w_\mu}{\kappa T} \tag{3.21c}$$

The last term in (3.21c) looks like:

$$\frac{w^\mu w_\mu}{\kappa T} = \frac{1}{\kappa T} \kappa^2 h^\sigma_\mu h^{\mu\nu} (\partial_\sigma T + T a_\sigma)(\partial_\nu T + T a_\nu) = \kappa \left(\frac{h^{\mu\nu}}{T} \partial_\mu T \partial_\nu T + 2 a^\mu \partial_\mu T + T a_\mu a^\mu \right) \tag{3.22}$$

Inserting the identity in (3.22) and making the last w^μ explicit in (3.21c) we get:

$$-\frac{1}{T} \kappa h^\mu_\sigma (\partial^\sigma T + T a^\sigma) \partial_\mu T = +\kappa a_\mu \partial^\mu T + \kappa T a_\mu a^\mu - \kappa \left(\frac{h^{\mu\nu}}{T} \partial_\mu T \partial_\nu T + 2 a^\mu \partial_\mu T + T a_\mu a^\mu \right) \tag{3.23a}$$

$$+\frac{1}{T} h^{\mu\nu} \partial_\nu T \partial_\mu T + h^{\mu\nu} a_\nu \partial_\mu T = -a_\mu \partial^\mu T - T a_\mu a^\mu + \left(\frac{h^{\mu\nu}}{T} \partial_\mu T \partial_\nu T + 2 a^\mu \partial_\mu T + T a_\mu a^\mu \right) \tag{3.23b}$$

$$0 = -a_\mu \partial^\mu T - T a_\mu a^\mu + \left(a^\mu \partial_\mu T + T a_\mu a^\mu \right) \tag{3.23c}$$

Thus we have proved the equation in (3.13), the inequality follows directly from the fact that we are considering square moduli of spacelike vectors, and the coefficients such as ξ are assumed to be positive. □

If we assume that the fluid is in equilibrium ($\nabla_\mu S^\mu = 0$) then we must have $\theta = 0$ (no compression), $\sigma_{\mu\nu} = 0$ (no shear stresses), $w_\mu = 0$ (no heat conduction which is not equal to $-\kappa T a_\mu$... doesn't have the same ring to it).

Ideal fluids They are fluids with $\eta = \xi = \kappa = 0$, that is, without viscosity (neither compressive nor shear) nor heat transmission. They are described by the following stress-energy tensor:

$$T^{\mu\nu} = \rho u^\mu u^\nu + p h^{\mu\nu} = \rho_0 h u^\mu u^\nu + p g^{\mu\nu} \tag{3.24}$$

where $h = (p + \rho)/\rho_0$ is the enthalpy. If our fluid is ideal then the RHS of (3.13) is zero and so is w^μ, therefore $T \nabla_\mu S^\mu = T \nabla_\mu (\rho_0 S u^\mu) = T \rho_0 u^\mu \partial_\mu S$ by the continuity equation. So, unless ρ_0 and T are not zero, S is conserved along the world-lines of the fluid.

Also, the RHS of (3.9c) is zero, therefore we get the Euler equation:

$$(p + \rho) a^\mu = -h^{\mu\nu} \partial_\nu p \tag{3.25}$$

3.3 Perfect fluids' spherical accretion

We work with the Schwarzschild metric (2.27); we treat a fluid with 4-velocity u^μ in spherical coordinates, since the problem we are looking at is stationary and spherically symmetric the velocity is:

$$u^\mu = \begin{pmatrix} \gamma^2 / y \\ y v \\ 0 \\ 0 \end{pmatrix} \tag{3.26}$$

where we define the Lorentz factor as usual, $\gamma = \left(1 - v^2\right)^{-1/2}$, and $y = \gamma \sqrt{1 + 2\Phi}$ is the "energy-at-infinity per unit rest mass" (see [TFŻ81, equation 3]).

The conservation of mass holds: if ρ_0 is the rest mass density of the fluid, we must have $\nabla_\mu (\rho_0 u^\mu) = 0$. This, using the formula for covariant divergence (2.5), yields:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho_0 y v r^2\right) = 0 \tag{3.27}$$

$\mathrm{d}Q/\mathrm{d}r = 0$ is equivalent to $\mathrm{d}\log Q/\mathrm{d}\log r = 0$, therefore we can recast (3.27) into

$$\frac{\mathrm{d}\log\rho_0}{\mathrm{d}\log r} + \frac{\mathrm{d}\log yv}{\mathrm{d}\log r} + 2 = 0 \tag{3.28}$$

In the newtonian limit both γ and y approach 1; also, the infalling mass rate $\dot{M}$ at a certain radius is $\rho_0(r)v(r)4\pi r^2$. Then, by continuity to the newtonian limit, the quantity which is constant wrt the radius must be $\dot{M}/(4\pi)$: therefore

$$\dot{M} = 4\pi\rho_0 y v r^2 \tag{3.29}$$

We also have the Euler equation (3.25).

The only interesting component of this is the radial one, so we need to calculate $a^1 = u^\mu\nabla_\mu u^1 = \mathrm{d}u^1 \big/ \mathrm{d}\tau + \Gamma^1_{\mu\nu}u^\mu u^\nu$. To do this we will need the radial Schwarzschild Christoffel coefficients:

$$\Gamma^1_{\mu\nu} = \begin{bmatrix} \frac{M(-2M+r)}{r^3} & 0 & 0 & 0 \\ 0 & \frac{M}{r(2M-r)} & 0 & 0 \\ 0 & 0 & 2M-r & 0 \\ 0 & 0 & 0 & (2M-r)\sin^2(\theta) \end{bmatrix} \tag{3.30}$$

while the proper-time derivative is $\mathrm{d}/\mathrm{d}\tau = u^\mu\partial_\mu = yv\partial_1$. Plugging in the expression for the only relevant component of $h^{\mu\nu}$, $h^{11} = g^{11} + u^1u^1 = (1+2\Phi)(1+v^2\gamma^2) = y^2$ we get, after lengthy computation,

$$a^1 = y^2\left(\gamma^2 v\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2}\right) \tag{3.31}$$

Substituting this into the (radial component of the) Euler equation (3.25) we get

$$(p+\rho)y^2\left(\gamma^2 v\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2}\right) = -h^{11}\partial_1 p = -y^2\partial_1 p \tag{3.32a}$$

$$\gamma^2 v\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2} + \frac{1}{p+\rho}\frac{\mathrm{d}p}{\mathrm{d}r} = 0 \tag{3.32b}$$

In equations 3.12.7, 8 in [Nob00] there is most definitely a sign error: the term proportional to M/r^2 should be positive.

We also have the equation for the variation of the total internal energy, which holds for ideal fluids at constant entropy:

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \frac{p+\rho}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} \quad \text{or} \quad \frac{\partial\rho}{\partial\rho_0} = \frac{p+\rho}{\rho_0} \overset{\text{def}}{=} h \tag{3.33}$$

(where we have defined h, the specific enthalpy).

The Bernoulli equation From these we can show that

Claim 3.2. *The quantity $\gamma h\sqrt{1+2\Phi} = yh$, is a constant of motion.*

Proof. First of all, by direct computation it can be shown that

$$\gamma^2 v \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2} = \frac{\mathrm{d}\log y}{\mathrm{d}r} \tag{3.34}$$

Then, following Gourgoulhon [Gou06, section 6.3] we find that $\mathrm{d}p = \rho_0 \,\mathrm{d}h$ in the isentropic case, therefore

$$\frac{1}{\rho+p}\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\mathrm{d}\log h}{\mathrm{d}r} \tag{3.35}$$

we can substitute the results in (3.34) and (3.35) into (3.32b):

$$\frac{\mathrm{d}\log h}{\mathrm{d}r} + \frac{\mathrm{d}\log y}{\mathrm{d}r} = \frac{\mathrm{d}\log(hy)}{\mathrm{d}r} = 0 \tag{3.36}$$

□

In the nonrelativistic, weak-field limit this becomes the classical conservation of density of energy:

$$\gamma h\sqrt{1+2\Phi} \approx \frac{p}{\rho_0} + \frac{v^2}{2} - \frac{M}{r} + \epsilon = \mathrm{const} \tag{3.37}$$

Also, we can rewrite the last term of the Euler equation (3.32b) using (3.33) as:

$$\frac{\partial_1 p}{p+\rho} = \frac{\rho_0}{\rho_0}\frac{\partial_1 p}{p+\rho} = \frac{1}{\rho_0}\frac{\partial p}{\partial \rho}\frac{\partial \rho_0}{\partial \rho}\frac{\partial \rho}{\partial p}\partial_1 p = \frac{v_s^2}{\rho_0}\partial_1 \rho_0 \tag{3.38}$$

where we define the speed of sound $v_s^2 = (\partial p/\partial \rho)_s$ (the index s means the derivative is to be taken at constant entropy, and for an adiabatic process).

This can be rewritten as

$$\mathrm{d}p = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial \rho_0}\,\mathrm{d}\rho_0 = v_s^2 h\,\mathrm{d}\rho_0 \tag{3.39}$$

Somehow in [Nob00, page 175] this becomes:

$$\frac{\mathrm{d}\log p}{\mathrm{d}\log r} = \rho_0 v_s^2 \frac{\mathrm{d}\log \rho_0}{\mathrm{d}\log r} \tag{3.40}$$

but it seems to me it should be

$$\frac{\mathrm{d}\log p}{\mathrm{d}\log r} = \frac{p+\rho}{p} v_s^2 \frac{\mathrm{d}\log \rho_0}{\mathrm{d}\log r} \tag{3.41}$$

Coming back to (3.25), we get:

$$\gamma^2 v \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2} + \frac{v_s^2}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}r} = 0 \tag{3.42}$$

We can replace every occcurence of $(\partial x)/x$ with $\partial \log x$:

$$\gamma^2 v^2 \frac{\mathrm{d}\log v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2} + v_s^2 \frac{\mathrm{d}\log \rho_0}{\mathrm{d}r} = 0 \tag{3.43a}$$

$$\frac{\gamma^2-1}{r}\frac{\mathrm{d}\log v}{\mathrm{d}\log r} + \frac{M}{(y^2/\gamma^2)r^2} + \frac{v_s^2}{r}\frac{\mathrm{d}\log \rho_0}{\mathrm{d}\log r} = 0 \tag{3.43b}$$

$$v^2 \frac{\mathrm{d}\log v}{\mathrm{d}\log r} + \frac{M}{y^2 r} + (1-v^2)v_s^2 \frac{\mathrm{d}\log \rho_0}{\mathrm{d}\log r} = 0 \tag{3.43c}$$

Are the references to equations at page 174 in [Nob00] wrong? Equations 2.7.3, 5 seem not to be related at all...

Somehow this equation comes up:

$$\frac{\mathrm{d}\log\rho_0}{\mathrm{d}r} + \gamma^2 v^2 \frac{\mathrm{d}\log v}{\mathrm{d}r} + \frac{2v^2}{r} + \frac{M}{(1+2\Phi)r^2} = 0 \tag{3.44}$$

and we wind up with the system

$$(v^2 - v_s^2)\frac{\mathrm{d}\log\rho_0}{\mathrm{d}\log r} = -2v^2 + \frac{M}{y^2 r} \tag{3.45a}$$

$$(v^2 - v_s^2)\frac{\mathrm{d}\log yv}{\mathrm{d}\log r} = 2v_s^2 - \frac{M}{y^2 r} \tag{3.45b}$$

$$\frac{\mathrm{d}\log p}{\mathrm{d}\log r} = \rho_0 v_s^2 \frac{\mathrm{d}\log\rho_0}{\mathrm{d}\log r} \tag{3.45c}$$

but I do not see at all how to get to it, it does not seem to work! If we add the first two equations we get back (3.28) which is good but that is the only part which seems to make sense.

Typo in [Nob00, page 175]: shouldn't it be "il movimento avviene nella direzione opposta ($v > 0, \dot{M} > 0$))"?

To add: commentary on the equation system, its boundary conditions ρ_∞ and T_∞, and the fact that to make the system regular at $v = v_s$ the problem becomes uniquely determined.

4 Radiative effects in spherical accretion

4.1 Thorne's PSTF moment formalism

Following [Tho81].

Given any tensor $A^{\mu_1\cdots\mu_k}$ we can use the tensor $h^{\mu\nu}$ to project it into the space-like subspace defined by the velocity u^μ:

$$A^{\mu_1\cdots\mu_k} \to \left(A^{\mu_1\cdots\mu_k}\right)^P = \left(\prod_i h^{\mu_i}_{\nu_i}\right) A^{\nu_1\cdots\nu_k} \tag{4.1}$$

Then, we can take the symmetric part of any (?) tensor as outlined in 'Notational preface' on page 3:

$$A^{\mu_1\cdots\mu_k} \to \left(A^{\mu_1\cdots\mu_k}\right)^S = A^{(\mu_1\cdots\mu_k)} \tag{4.2}$$

We can select the trace-free part of a projected, symmetric tensor by

$$A^{\mu_1\cdots\mu_k} \to \left(A^{\mu_1\cdots\mu_k}\right)^{TF} = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \frac{k!(2k-2i-1)!!}{(k-2i)!(2k-1)!!(2i)!!} h^{(\alpha_1\alpha_2} \dots h^{\alpha_{2i-1}\alpha_{2i}} A^{\alpha_{2i+1}\dots\alpha_k)\beta_1\dots\beta_i}{}_{\beta_1\dots\beta_i} \tag{4.3}$$

To see what this is doing, let us consider its action on a rank-two projected tensor: it is just the subtraction of its trace,

$$A^{\mu\nu} \to A^{\mu\nu} - \frac{1}{3} h^{\mu\nu} A^\rho_\rho \tag{4.4}$$

Now, let us consider all the unit vectors n^μ in the space normal to the velocity, which have $n_\mu u^\mu = 0$ and $n^\mu n_\mu = 1$. They span a three-dimensional sphere.

If we have a function $F\colon S^2 \to \mathbb{R}$, we can decompose it into harmonics as such:

$$F(n) = \sum_{k=0}^{\infty} \mathscr{F}_{\alpha_1 \dots \alpha_k} \prod_{i=0}^{k} n^{\alpha_i} \tag{4.5}$$

Where the PSTF moments $\mathscr{F}_{\alpha_1 \dots \alpha_k}$ can be computed as

$$\mathscr{F}_{\alpha_1 \dots \alpha_k} = \frac{(2k+1)!!}{4\pi k!} \left(\int F \prod_{i=0}^{k} n^{\alpha_i} \, \mathrm{d}\Omega \right)^{TF} \tag{4.6}$$

In particular, the function we will apply this to is the distribution of EM radiation around the BH. So, let us consider a photon, whose trajectory in spacetime is parameterized as $\gamma(\xi)$, with a choice of ξ such that the photon's momentum is

$$p = \frac{\mathrm{d}}{\mathrm{d}\xi} \tag{4.7}$$

Now, our observer has a timelike velocity u^μ. We can find a spacelike vector n^μ corresponding to the space-like part of the movement of the photon, or

$$p^\mu = (-u^\nu p_\nu)(u^\mu + n^\mu) \tag{4.8}$$

It must hold that $u^\mu u_\mu = -1$ while $n^\mu n_\mu = +1$ in order for p^μ to be null-like. Now, we define a parameter l which corresponds to the space distance the photon moved through in this frame (this is *not* covariant!)

$$l = \int (-u^\nu p_\nu) \, \mathrm{d}\xi \tag{4.9}$$

now, $\mathrm{d}/\mathrm{d}l$ is parallel to p but it has different length, in fact since $\mathrm{d}l/\mathrm{d}\xi = (-u^\nu p_\nu)$ it is $\mathrm{d}/\mathrm{d}l = u + n$.

It holds [Tho81, eq. 2.17], (with the notation from (2.18)), that

$$\frac{\mathrm{d}\nu}{\mathrm{d}l} = (u^\mu + n^\mu)\nabla_\mu(-p^\nu u_\nu) = -\nu \left(n_\mu a^\mu + \frac{\theta}{3} + n_\mu n_\nu \sigma^{\mu\nu} \right) \tag{4.10}$$

We want to quantify the number density of photons in relation to their momentum. We assume the radiation in unpolarized, therefore for each unit h^3 cell in phase space there can be 2 photons: so we denote the distribution function of the photons as $2N(x^\mu, p^\mu)$.

It is known that the volume element $\mathrm{d}V_p = \mathrm{d}^3 p \,/\, p^0$ is Lorentz invariant (see [MTW73, box 22.5]). We can write this using the photons' frequency $\nu = -p^\mu u_\mu / h$ as $\mathrm{d}V_p = \nu \, \mathrm{d}\Omega \, \mathrm{d}\nu$.

Let us define the *specific radiative intensity* as

$$I_\nu = \frac{\delta E}{\delta A \delta t \delta \nu \delta \Omega} = \frac{h\nu \delta N}{\delta A \delta t \delta \nu \delta \Omega} \tag{4.11}$$

where δA denotes an infinitesimal area the photons are coming through, δt an infinitesimal time, $\delta\nu$ an infinitesimal photon frequency, $\delta\Omega$ an infinitesimal solid angle.

Then, [MTW73, figure 22.2] the number density of photons in phase space is

$$2N(x^\mu, p^\mu) = \frac{\delta N}{V_x V_p} = \frac{\delta N}{h^3 \nu^2 \delta A \delta t \delta \nu \delta \Omega} = \frac{1}{h^4 \nu^3} I_\nu \tag{4.12}$$

therefore $I_\nu = 2N\nu^3 h^4$.

This is from [MTW73, figure 22.2], but it seems to conflict with [Tho81, equation 6.2]!

Now, we want to describe the variation of the occupation number N with respect to the photons' trajectories' parameter l. We encapsulate all possible effects into a source term $\mathfrak{S}$:

$$\mathfrak{S} \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}l} 2N(x^\mu, p^\mu) = 2\left(\frac{\partial N}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}l} + \frac{\partial N}{\partial p^i}\frac{\mathrm{d}p^i}{\mathrm{d}l} \right) \tag{4.13}$$

since the occupation number can be thought of as just a function of the spatial components of the momentum.

Why so? Surely $N = N(\nu) = N(p^0)$ also!

Since $\mathrm{d}/\mathrm{d}l = (n^\mu + u^\mu)\partial_\mu$ and the covariant derivative of p^j is zero, we can compute

$$\frac{\mathrm{d}p^j}{\mathrm{d}l} = (n^\mu + u^\mu)\nabla_\mu p^j - \Gamma^j_{\alpha\beta} p^\alpha (u^\beta + n^\beta) = -\Gamma^j_{\alpha\beta} p^\alpha (u^\beta + n^\beta) \tag{4.14}$$

where the covariant derivative term vanishes since the photon's trajectory is a geodesic.

Moments' definitions In units where $c = h = 1$,

$$M_\nu^{A_k} \stackrel{\text{def}}{=} \int 2N \frac{\delta(\nu - (-p^\nu u_\nu))}{\nu^{k-2}} \prod_i^k p^{\alpha_i} \,\mathrm{d}V_p \tag{4.15a}$$

$$= \int \left(2N\nu^3\right) \frac{1}{\nu} \delta(\nu + p^\nu u_\nu) \prod_i^k \left(\frac{p^{\alpha_i}}{\nu}\right) (\nu \,\mathrm{d}\Omega \,\mathrm{d}\nu) \tag{4.15b}$$

$$= \int I_\nu \prod_i^k (n^{\alpha_i} + u^{\alpha_i}) \,\mathrm{d}\Omega \tag{4.15c}$$

This is a general procedure we can use to associate a function f (in this case we started with $2N$) with ν^3 times the integral (4.15c) (where one might substitute I_ν with the function f). We need it for the source moments:

$$S_\nu^{A_k} = \nu^3 \int S\mathfrak{S} \prod_i^k (n^{\alpha_i} + u^{\alpha_i}) \,\mathrm{d}\Omega \tag{4.16}$$

Redshift-adapted version Thorne [Tho81] also defines a redshift-adapted version of the moments' definition: if R is a universal redshift functions, such that $R(p^\nu u_\nu)$ is conserved along every photon geodesic $p^\mu \nabla_\mu p^\nu = 0$, that is, R allows us to calculate the redshift between any two points A, B which are connected by a geodesic as $\nu_A/\nu_B = R_B/R_A$.

Then, we define $M_f^{A_k} = M_\nu^{A_k}/R$

Frequency-integrated version The definition is:

$$M^{A_k} = \int M_\nu^{A_k} \,\mathrm{d}\nu \tag{4.17}$$

and the same is applied to the source moments $S_\nu^{A_k} \to S^{A_k}$.

Since this includes the radiation intensity from all frequencies, we have direct interpretations for the first moments:

$$M = \int I_\nu \,\mathrm{d}\Omega \,\mathrm{d}\nu \qquad \text{energy density of radiation} \tag{4.18a}$$

$$M^{\alpha} = \int I_{\nu}(n^{\alpha} + u^{\alpha})\, \mathrm{d}\Omega\, \mathrm{d}\nu \qquad (M^0, M^i) = \text{(energy density of radiation, energy flux)} \tag{4.18b}$$
$$M^{\alpha\beta} = \int I(n^{\alpha} + u^{\alpha})(n^{\beta} + u^{\beta})\, \mathrm{d}\Omega\, \mathrm{d}\nu \qquad \text{stress-energy tensor of radiation} \tag{4.18c}$$

The moment equations These can be derived from the transport equation, see [Tho81, p. 3.14]. I present them only in the grey (frequency-integrated) case:

$$\nabla_{\beta} M^{A_k \beta} - (k-1) M^{A_k \beta\gamma}(\nabla_{\gamma} u_{\beta}) = S^{A_k} \tag{4.19}$$

Also, the moments (M^{A_k}, but also $M_{\nu}^{A_k}$ and $M_f^{A_k}$) satisfy the following:

$$M^{A_k \beta}{}_{\beta} = 0 \tag{4.20a}$$
$$u_{\beta} M^{A_k \beta} = -M^{A_k} \tag{4.20b}$$
$$h_{\beta\gamma} M^{A_k \beta\gamma} = M^{A_k} \tag{4.20c}$$

So, the k-th moment contains all the information about the l-th moments with $l \leq k$; also, to get lower-order moments we take partial traces onto space- and time-like subspaces: therefore the unique information to the k-th moment, which is not redundantly expressed in lower-order moments, is in its PSTF part:

$$\mathscr{M}^{A_k} = \left(M^{A_k} \right)^{PSTF} \tag{4.21}$$

The same can be applied to $M_{\nu}^{A_k}$ and $M_f^{A_k}$ and to the moment equations (4.19). Since we are taking the projection onto the space-like subspaces, we can simplify the expression of the PSTF moments: all the terms which contain at least a four-velocity vanish, therefore:

$$\mathscr{M}^{A_k} = \left(\int I \prod_i n^{\alpha_i}\, \mathrm{d}\Omega \right)^{TF} \tag{4.22}$$

where $I = \int I_{\nu}\, \mathrm{d}\nu$. The first *PSTF* moments also have physical interpretations:

$$\mathscr{M} = \int I\, \mathrm{d}\Omega \qquad \text{energy density of radiation} \tag{4.23a}$$
$$\mathscr{M}^{\alpha} = \int I n^{\alpha}\, \mathrm{d}\Omega \qquad \text{energy flux of radiation} \tag{4.23b}$$
$$\mathscr{M}^{\alpha\beta} = \int I n^{\alpha} n^{\beta}\, \mathrm{d}\Omega \qquad \text{shears in the stress-energy tensor of radiation} \tag{4.23c}$$

We can write the stress-energy tensor $T^{\mu\nu} = M^{\mu\nu}$ with the PSTF moments (see [Tho81, eq. 4.9]):

$$T^{\mu\nu} = \mathscr{M} u^{\mu} u^{\nu} + 2\mathscr{M}^{(\mu} u^{\nu)} + \mathscr{M}^{\mu\nu} + \frac{1}{3}\mathscr{M} h^{\mu\nu} \tag{4.24}$$

we can compare these to (3.6) to get the following identifications:

$$\mathscr{M} = w = \rho \tag{4.25a}$$
$$\mathscr{M}^{\mu} = w^{\mu} = -\kappa h^{\mu}_{\sigma}(\partial^{\sigma} T + T a^{\sigma}) \tag{4.25b}$$
$$\mathscr{M}^{\mu\nu} + \frac{1}{3}\mathscr{M} h^{\mu\nu} = (p - \xi\theta) h^{\mu\nu} - 2\eta\sigma^{\mu\nu} \tag{4.25c}$$

but since the photons' paths are geodesics in this case $\theta = 0$, so for the components proportional to $h^{\mu\nu}$ of equation (4.25c) we just get $\rho = {}^{1}\!/_{3} p$, which is what we expect for the photon gas. For the traceless part of the equation, we get $\mathscr{M}^{\mu\nu} = -2\eta\sigma^{\mu\nu}$.

The PSTF moment equations We want to express the grey moment equations (4.19) in terms of the PSTF moments. This can be done as follows: an expression can be found for the full moments in terms of the PSTF moments in [Tho81, eq. 4.10c]:

$$M^{A_k} = \sum_{l=0}^{k} \sum_{j=0}^{\lfloor \frac{k-l}{2} \rfloor} \frac{1}{(2j)!!(k-l-2j)!} \frac{k!}{l!} \frac{(2l+1)!!}{(2l+1+2j)!!} \mathscr{M}^{(A_l} \prod_{i=l+1}^{l+2j-1} h^{\alpha_i \alpha_{i+1}} \prod_{x=l+2j+1}^{k} u^{\alpha_x)} \tag{4.26}$$

where all the indices of the $\mathscr{M}$, h and u are meant to be symmetrized.

We insert this into the moment equations and expand, making use of the decomposition of the covariant derivative of the 4-velocity (2.18).

Then, we take the PSTF part of the equations. This yields a very complicated expression, so here I record only the implicit formula [Tho81, eq. 4.11c]:

$$\begin{aligned}&\left(\nabla_\beta \mathscr{M}^{A_k \beta} + u^\beta \nabla_\beta \mathscr{M}^{A_k} + \frac{k}{2k+1} \nabla_{\alpha_k} \mathscr{M}^{A_{k-1}} - (k-1)\mathscr{M}^{A_k \beta\gamma} \sigma_{\beta\gamma} - (k-1)\mathscr{M}^{A_k \beta} a_\beta + \frac{4}{3} \mathscr{M}^{A_k} \theta \right.\\ &\left. + \frac{5k}{2k+3} \mathscr{M}^{A_{k-1}\beta} \sigma^{\alpha_k}_\beta - k \mathscr{M}^{A_{k-1}\beta} \omega^{\alpha_k}_\beta + \frac{k(k+3)}{2k+1} \mathscr{M}^{A_{k-1}} a^{\alpha_k} + \frac{(k-1)k(k+2)}{(2k-1)(2k+1)} \mathscr{M}^{A_{k-2}} \sigma^{\alpha_{k-1}\alpha_k} \right)^{PSTF} = \mathscr{S}^{A_k}\end{aligned} \tag{4.27}$$

How to recover the intensity Once one has solved the PSTF grey moment equations, one can compute the intensity from the moments by comparing (4.6) and (4.22):

$$I = \sum_{k=0}^{\infty} \frac{(2k+1)!!}{4\pi k!} \mathscr{M}^{A_k} \prod_{i=1}^{k} n_{\alpha_i} \tag{4.28}$$

Simplifications under assumptions of symmetry Instead of treating the general case as is done in [Tho81], we describe the specific choices made under the assumption of spherical symmetry, following [TFŻ81].

The fiducial congruence reference, written with respect to the regular spherical coordinates (t, r, θ, φ), is:

$$\hat{t} = e_t = u^\mu = \left(\gamma^2/y, yv, 0, 0\right) \tag{4.29a}$$

$$\hat{r} = e_r = \left(-v\gamma^2/y, y, 0, 0\right) \tag{4.29b}$$

$$\hat{\theta} = e_\theta = (0, 0, 1/r, 0) \tag{4.29c}$$

$$\hat{\varphi} = e_\varphi = \left(0, 0, 0, 1/(r \sin(\theta))\right) \tag{4.29d}$$

In this frame (denoted with a subscript "fid"), we have the following expressions:

$$a^\mu = (0, \mathrm{d}y/\mathrm{d}r\,, 0, 0)_{\text{fid}} \tag{4.30a}$$

$$\theta = -\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 v y\right) \tag{4.30b}$$

$$\sigma_{\mu\nu} = -\frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{vy}{r}\right) \frac{2r}{3} \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & -1/2 & \\ & & & -1/2 \end{bmatrix}_{\text{fid}} = \sigma \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & -1/2 & \\ & & & -1/2 \end{bmatrix}_{\text{fid}} \tag{4.30c}$$

$$\Gamma_{\theta r \theta} = \Gamma_{\varphi r \varphi} = \frac{y}{r} \tag{4.30d}$$

Now, we can see that the shear has been heavily simplified. This is a specific case of a general statement about the PSTF moments: in the spherically symmetric case, the k-th PSTF moment only has one independent component. This is because it satisfies the following identities:

$$\mathscr{M}^{A_k} = 0 \text{ if } A_k \text{ contains an odd number of } \theta\text{s or } \varphi\text{s} \tag{4.31a}$$

$$\mathscr{M}^{A_k\theta\theta} = \mathscr{M}^{A_k\varphi\varphi} = -\frac{1}{2}\mathscr{M}^{A_k rr} \tag{4.31b}$$

Equation (4.31a) comes from the fact that an odd number of θ or φ indices corresponds to and odd number of unit vectors which are integrated on the sphere (see the definition (4.22)): therefore the integrand is odd.

Equation (4.31b) comes from two observations: first of all, the moments corresponding to indices θ and φ respectively must be equal because of spherical symmetry; secondly the moments must be traceless, therefore the sum of the $\theta\theta$, $\varphi\varphi$ and rr moments must be zero (for any pair of indices).

So, with these every k-th moment is fully determined by the component $\mathscr{M}^{r\dots r}$ (k rs): therefore we give it a name: w_k. This fact is analogous to the statement that the only spherically symmetrical one of the spherical harmonics Y_{lm} is Y_{l0}, therefore as in that case we have only one independent component for every l.

Legendre polynomials complement The l-th Legendre polynomial is:

$$P_l(x) = \frac{1}{2^l}\sum_{k=0}^{\lfloor l/2 \rfloor} \frac{(-1)^k(2l-2k)!}{k!(l-k)!(l-2k)!}x^{l-2k} \tag{4.32}$$

We can see that the coefficient of x^l is $(2l)!/(2^l(l!)^2)$. We can rewrite this making use of the identities $(2n)! = (2n-1)!!(2n)!!$ and $(2n)!! = 2^n n!$, as:

$$\frac{(2l)!}{2^l(l!)^2} = \frac{1}{l!}\frac{(2l)!}{(2l)!!} = \frac{(2l-1)!!}{l!} = \frac{(2l+1)!!}{l!(2l+1)} \tag{4.33}$$

which is equation [Tho81, eq. 5.7d].

Thorne [Tho81, eqs. 5.6] claims that

$$\int_{-1}^{1} I(\mu)P_k(\mu)\left(\frac{(2k-1)!!}{l!}\right)^{-1} \mathrm{d}\mu = \left(\int_{-1}^{1} I(\mu)\prod_{i=1}^{k} n^r \,\mathrm{d}\mu\right)^{TF} \tag{4.34}$$

where n^r denotes the radial component of a normal vector in spherical coordinates, P_k is the k-th Legendre polynomial and $\mu = \cos\theta$ where θ is the azimuthal coordinate of n. $I(\mu)$ is a generic function.

The scalar moments It can be shown, using the identity (4.34) that the definition of w_k we gave is equivalent to

$$w_k = \int_{-1}^{1} I(\cos\theta)P_k(\cos\theta)\left(\frac{(2k-1)!!}{l!}\right)^{-1} 2\pi \,\mathrm{d}\cos\theta \tag{4.35}$$

where P_k is the k-th Legendre polynomial (4.32). Then the first moments are:

$$w_0 = \int I \,\mathrm{d}\Omega \qquad \text{radiation energy density} \tag{4.36a}$$

$$w_1 = \int I \cos\theta \,\mathrm{d}\Omega \qquad \text{radiation energy flux} \tag{4.36b}$$

$$w_2 = \int I\left(\cos^2\theta - \frac{1}{3}\right) \mathrm{d}\Omega \qquad \text{radiation shear stress} \tag{4.36c}$$

We can explicitly write the stress-energy tensor in terms of the w_k using (4.24):

$$T^{\mu\nu} = \begin{bmatrix} w_0 & w_1 & 0 & 0 \\ w_1 & \frac{1}{3}w_0 + w_2 & 0 & 0 \\ 0 & 0 & \frac{1}{3}w_0 - \frac{1}{2}w_2 & \\ 0 & 0 & & \frac{1}{3}w_0 - \frac{1}{2}w_2 \end{bmatrix} \tag{4.37}$$

The simplified moment equations It is possible to write equations (4.27) explicitly in terms of the w_k and of derivatives wrt the fiducial basis: one gets [Tho81, eq. 5.10c]

$$\begin{aligned} &\frac{\partial w_{k+1}}{\partial \hat{r}} + [(2-k)a + (k+2)b]w_{k+1} + \frac{\partial w_k}{\partial \hat{t}} + \left[\frac{4}{3}\theta + \frac{5k(k+1)}{2(2k-1)(2k+3)}\sigma\right] w_k + \\ &+ \frac{k^2}{(2k-1)(2k+1)}\frac{\partial w_{k-1}}{\partial \hat{r}} + \frac{k^2[(k+3)a + (1-k)b]}{(2k-1)(2k+1)} w_{k-1} + \\ &- \frac{3}{2}(k-1)\sigma w_{k+2} + \frac{3(k-1)^2 k^2 (k+2)}{2(2k-3)(2k-1)^2(2k+1)}\sigma w_{k-2} = s_k \end{aligned} \tag{4.38}$$

where $a = \mathrm{d}y/\mathrm{d}r = \sqrt{a^\mu a_\mu}$ is the magnitude of the 4-acceleration, $b = y/r$ is the extrinsic curvature, θ is the expansion velocity, σ is the scalar shear — the largest eigenvalue of the shear matrix. Explicit expressions for these are found in (4.30).

Nobili, Turolla, and Zampieri [NTZ91] only use the first two of the moment equations, so here is how the expression is simplified for $k = 0, 1$: for $k = 0$ we get:

$$\frac{\partial w_1}{\partial \hat{r}} + 2(a+b)w_1 + \frac{\partial w_0}{\partial \hat{t}} + \frac{4}{3}\theta w_0 + \frac{3}{2}\sigma w_2 = s_0 \tag{4.39}$$

For $k = 1$ we get:

$$\frac{\partial w_2}{\partial \hat{r}} + (a+3b)w_2 + \frac{\partial w_1}{\partial \hat{t}} + \left[\frac{4}{3}\theta + \sigma\right] w_1 + \frac{1}{3}\frac{\partial w_0}{\partial \hat{r}} + \frac{4a}{3}w_0 = s_1 \tag{4.40}$$

These have to be simplified further to be used: specifically, they can be expressed with respect to r, v, y.

How do we get rid of the time derivatives?

Some properties of the accretion variables Following Thorne, Flammang, and Żytkow [TFŻ81].

From equation (3.29) we can find an expression [TFŻ81, eq. 18a] for y which only depends on r and constants:

$$y = \sqrt{y^2(1 - v^2 + v^2)} = \sqrt{\left(\frac{y^2}{\gamma^2}\right) + y^2 v^2} = \sqrt{\left(1 - \frac{2M}{r}\right) + \left(\frac{\dot{M}}{4\pi r^2 \rho_0}\right)^2} \tag{4.41}$$

therefore v can also be expressed in terms of r and constants:

$$v = \frac{\dot{M}}{4\pi r^2 \rho_0 y(r)} \tag{4.42}$$

The transfer equations (? to rename) The source term can be written [TFŻ81, eq. 15] as

$$s_k = \frac{l!(2l+1)}{(2l+1)!!}\int_{-1}^{1} \frac{\mathrm{d}I}{\mathrm{d}\tau}_{\text{interaction}} P_k(\mu) 2\pi \,\mathrm{d}\mu \tag{4.43}$$

Why is this a proper-time derivative of the intensity instead of a derivative wrt to the photon spatial distance parameter l as in [Tho81]?

5 Extra sections, possibly to remove

5.1 Tensor calculus

We can define the Dalambertian operator $\Box = \nabla_\mu \nabla^\mu = \nabla_\mu \partial^\mu$, which can only act on scalars, and it does so like:

$$\Box A = \nabla_\mu (\partial^\mu A) = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu A \right) \tag{5.1}$$

If we differentiate and antisymmetrize (so, take the rotor of) an antisymmetric tensor $F_{[\mu\nu]}$, the Christoffel symbols cancel:

$$\nabla_{[\mu} F_{\nu\rho]} = \partial_{[\mu} F_{\nu\rho]} \tag{5.2}$$

Lie derivative Following [Tau78, section 6]. The Lie derivative of a generic tensor $T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m}$ along a vector ξ^μ is defined as:

$$\underset{\xi}{\mathscr{L}} T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \overset{\text{def}}{=} \xi^\rho \nabla_\rho T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} + \sum_{i=1}^{m} T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_{i-1} \rho \nu_{i+1} \dots \nu_m} \nabla_{\nu_i} \xi^\rho - \sum_{i=1}^{n} T^{\mu_1 \dots \nu_{i-1} \rho \nu_{i+1} \dots \mu_n}_{\nu_1 \dots \nu_m} \nabla_\rho \xi^{\mu_i} \tag{5.3}$$

Some special cases are: a scalar $\mathscr{L}_\xi f = \xi^\rho \partial_\rho f$, a vector $\mathscr{L}_\xi u^\mu = \xi^\rho \nabla_\rho u^\mu - u^\rho \nabla_\rho \xi^\mu$, and an antisymmetric covariant two-tensor $\omega_{\mu\nu} = \omega_{[\mu\nu]}$ which represents a closed form $\nabla_{[\mu} \omega_{\nu\rho]} = 0$ we have $\mathscr{L}_\xi \omega_{\mu\nu} = 2\nabla_{[\nu} \left(\omega_{\mu]\rho} \xi^\rho \right)$

5.2 Fluid dynamics

Wave velocity Following [Tau78, section 5].

An equation in the form $\varphi(x^\mu) = 0$ defines a 3D surface Σ; its constant-x^0 slices are 2D surfaces. We can decompose $\partial_\mu \varphi$ in a component proportional to the velocity, $a u_\mu$, and one which is orthogonal to it, $W_\mu = h^\sigma_\mu \partial_\sigma \varphi$. Then $W_\mu W^\mu = h^\sigma_\mu h^\mu_\beta \partial_\sigma \varphi \partial^\beta \varphi = h^{\sigma\beta} \partial_\sigma \varphi \partial_\beta \varphi$ by the idempotency of the projector h. Thus we can see that $W^2 \geq 0$, therefore it is a spacelike vector. We can define its corresponding unit vector: $W^\mu = t^\mu \sqrt{W^\nu W_\nu}$.

We can choose a velocity v such that $k^\mu = u^\mu - v t^\mu$ is parallel to Σ, or $k^\mu \partial_\mu \varphi = 0$. If this condition is satisfied, then v is the wave velocity of Σ as measured by an observer with velocity u^μ.

Now, we can multiply the equation $k^\mu = u^\mu - v t^\mu$ by $\partial_\mu \varphi$: we get $v t^\mu \partial_\mu \varphi = u^\mu \partial_\mu \varphi$. What multiplies v in the LHS of this equation is the length of the spatial component of $\partial_\mu \varphi$, or $\sqrt{h^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi}$. Using the definition of $h^{\mu\nu}$, we can arrive at

$$\gamma^{-2} = 1 - v^2 = \frac{g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi}{h^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi} \tag{5.4}$$

The denominator in (5.4) is positive, so we can see that $1 - v^2$ is positive iff $\partial_\mu \varphi$ is spacelike, and $v^2 = 1$ iff it is null.

If we are dealing with a timelike surface Σ, or equivalently $\partial_\mu \varphi$ is spacelike, then we define $n^\mu \propto \partial_\mu \varphi$ such that $n^\mu n_\mu = 1$ and we find:

$$v = \frac{u^\mu \partial_\mu \varphi}{\sqrt{h^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi}} = \frac{u^\mu n_\mu}{\sqrt{h^{\mu\nu} n_\nu n_\mu}} = \frac{u^\mu n_\mu}{\sqrt{1 + (u^\mu n_\mu)^2}} \tag{5.5}$$

From this it can be shown that $1 - v^2 = (1 + (u^\mu n_\mu)^2)^{-1}$, and then $v\gamma = u^\mu n_\mu$.

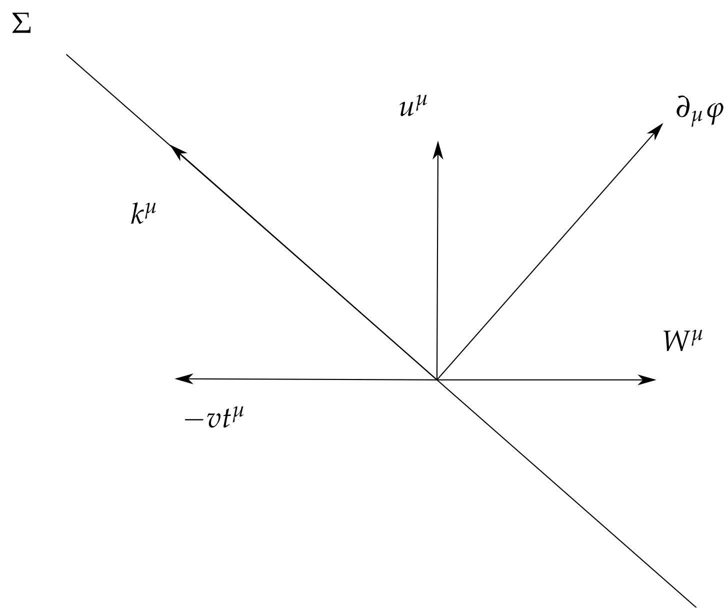

Figure 1: Wave velocity diagram

Stationarity Will skip most of this section. A spacetime is stationary if it admits a timelike Killing vector field? Ehlers [Ehl71] proved that from a hypothesis of stationarity we get $\nabla_\mu S^\mu = 0$.

Speed of sound Following [Yos11]. We want to justify the expression $v_s^2 = \partial p / \partial \rho$. We work in Minkowski spacetime, where $g_{\mu\nu} = \eta_{\mu\nu}$, and with an ideal fluid, for which $T_{\mu\nu} = (p+\rho) u_\mu u_\nu + p \eta_{\mu\nu}$. Then, the equations of conservation of mass and momentum read:

$$\rho_0 \partial_\mu u^\mu + u^\mu \partial_\mu \rho_0 = 0 \qquad \text{mass} \tag{5.6a}$$
$$u^\mu \left(\partial_\mu \rho - h \partial_\mu \rho_0 \right) = 0 \qquad \text{momentum along } u^\mu \tag{5.6b}$$
$$(p+\rho) a^\mu + h^{\mu\rho} \partial_\rho p = 0 \qquad \text{momentum in the span of } h_{\mu\nu} \tag{5.6c}$$

where h is the specific enthalpy. If we assume small perturbations $p \to p + \delta p$, $\rho_0 \to \rho_0 + \delta \rho_0$, $\rho \to \rho + \partial \rho$, $u^\mu \to (1, \delta u^x, 0, 0)$ (it is *almost* normalized) we get our three equations, up to first order in the perturbations:

$$\partial_x (\delta u^x) = -\frac{\partial_t (\delta \rho_0)}{\rho_0} \tag{5.7a}$$
$$\partial_t (\delta \rho) - h \partial_t (\delta \rho_0) = 0 \tag{5.7b}$$
$$-(p+\rho) \partial_t (\delta u^x) = \partial_x p \tag{5.7c}$$

We manipulate these by differentiating and substituting, in order to eliminate the dependence on δu^x and ρ_0, and get:

$$\partial_t \delta \rho + (p+\rho) \partial_x (\partial u^x) = 0 \tag{5.8a}$$
$$\partial_x \delta p + (p+\rho) \partial_t (\partial u^x) = 0 \tag{5.8b}$$

which simplifies to $\partial^2_{tt} \delta \rho - \partial^2_{xx} \delta p = 0$: in order to get the wave equation in the form $\left(v_s^{-2} \partial^2_{tt} - \partial^2_{xx} \right) \delta p = 0$ we *need* to define $v_s^2 = \partial p / \partial \rho$.

Barotropic flows The definition of a barotropic flow is: a flow for which the density is just a function of temperature, $\rho = \rho(p)$.

What is d in [Tau78, equation 11.10]? Is it a typo? I do not get that equation. Also, I quote: "Equations 11.3 and 11.5 do not hold. However, it is a consequence of equations 11.3 and...".

Insert here: definition of the variable s.

The quantity s obeys the conservation law $\nabla_\mu(su^\mu) = 0$. We define a quantity $Q = \log\big((\rho + p)/s\big)$, which for $s = \rho_0$ is the log-enthalpy. Then, the Euler equation (3.25) can be substituted by

$$a^\sigma = -h^{\sigma\mu}\partial_\mu Q \tag{5.9}$$

or, more explicitly:

$$(\rho + p)a^\mu = -sh^{\mu\nu}\partial_\nu\left(\frac{\rho + p}{s}\right) \tag{5.10}$$

Unproven in [Tau78]!? where does it come from?

We now define a "current" of $\exp(Q)$: $V^\mu = \exp(Q)u_\mu$.

Then we define $\Omega_{\mu\nu} = 2\nabla_{(\mu}V_{\nu)}$: this satisfies $\Omega_{\mu\nu}u^\nu = -T\partial_\mu S$ in general, and $\Omega_{\mu\nu}u^\nu = 0$ in the isentropic case.

Right? Underneath equation [Tau78, p. 11.14] what is meant is "in the isentropic" case, I think, since then $\partial_\mu S = 0$...

Also, what does the distinction between the two definitions in 11.14 and 11.13 mean? Is 11.13 not a subcase of 11.14?

Then, we define

$$v^\mu = \frac{1}{2}\eta^{\mu\nu\rho\sigma}u_\nu\Omega_{\rho\sigma} \tag{5.11}$$

which satisfies

$$3u_{[\alpha}\Omega_{\beta\gamma]} = v^\mu\eta_{\mu\alpha\beta\gamma} \tag{5.12}$$

The normalization does not seem right: $\eta^{\mu\nu\rho\sigma}\eta_{\mu\alpha\beta\gamma} = -\delta^{\nu\rho\sigma}_{\alpha\beta\gamma}$, so when substituting in the result seems off by $-3!$...

We use this result to get an explicit formula for $\Omega_{\alpha\beta}$ in terms of u^μ, v^μ. This comes out by multiplying by u^γ, and is:

$$\Omega_{\alpha\beta} = -v^\mu u^\gamma\eta_{\mu\gamma\alpha\beta} + u_\alpha\Omega_{\beta\gamma}u^\gamma - u_\beta\Omega_{\alpha\gamma}u^\gamma = -v^\mu u^\gamma\eta_{\mu\gamma\alpha\beta} + 2T(u_{[\beta}\partial_{\alpha]}S) \tag{5.13}$$

We can also rewrite the perfect-fluid stress-energy tensor as $T^{\mu\nu} = sV^\mu u^\nu + pg^{\mu\nu}$: then, since the metric has zero covariant derivative, if we look at the projection of $T^{\mu\nu}$ along a Killing vector field $\nabla_{(\mu}\xi_{\nu)} = 0$ we get

$$\nabla_\nu\Big(\xi_\mu T^{\mu\nu}\Big) = \nabla_\nu\Big(\xi_\mu sV^\mu u^\nu\Big) = 0 \tag{5.14}$$

but in the (barotropic?) case $s = \rho_0$ we can use the conservation of mass to get $\rho_0 u^\nu\nabla_\nu(\xi_\mu V^\mu) = 0$, so the quantity $H = \xi_\mu V^\mu$ is conserved along the worldlines of the fluid.

If we are in the coordinate system which would be the LRF for an observer with velocity ξ^μ, then the conserved quantity is $H = V^0$.

Shocks and conservation laws through boundaries Following [Tau78, section 13]. The differential formulations of the conservation of mass and momentum will not hold at points of non-differentiability, such as through shock waves. There, they must be replaced by an integral law. We can reframe our conservation laws as the statements that, for any scalar f and vector λ_μ we will have

$$\nabla_\mu \left(f \rho_0 u^\mu \right) = \rho_0 u^\mu \partial_\mu f \tag{5.15a}$$
$$\nabla_\mu (\lambda_\nu T^{\mu\nu}) = T^{\mu\nu} \nabla_\mu \lambda_\nu \tag{5.15b}$$

We can integrate these on a generic volume V using Stokes' theorem (2.7), choosing coordinates on the boundary such that the determinant of the induced metric on the submanifold is uniformly 1.

$$\int_{\partial V} n_\mu \left(f \rho_0 u^\mu \right) \mathrm{d}^3 y = \int_V \rho_0 u^\mu \partial_\mu f \sqrt{-g}\, \mathrm{d}^4 x \tag{5.16a}$$
$$\int_{\partial V} n_\mu (\lambda_\nu T^{\mu\nu}) \mathrm{d}^3 y = \int_V T^{\mu\nu} \nabla_\mu \lambda_\nu \sqrt{-g}\, \mathrm{d}^4 x \tag{5.16b}$$

Now, to deal with the shock we do the following: take the hypersurface at which the shock happens, enclose it in an infinitesimally thin 4D volume: then, as the volume decreases the RHSs of equations (5.16) go to 0, therefore the LHSs also must: we can write them as the difference of the integrands at the boundary. Therefore, introducing the notation $[F] = F_+ - F_-$, selecting one of the two outward vectors and denoting only it as n_μ and using the arbitrariness of f, λ_ν we have:

$$\left[n_\mu \rho_0 u^\mu \right] = n_\mu \left[\rho_0 u^\mu \right] = 0 \qquad \text{and} \qquad \left[n_\mu T^{\mu\nu} \right] = n_\mu [T^{\mu\nu}] = 0 \tag{5.17}$$

We denote $m = \rho_0 u^\mu n_\mu$ (at the boundary), and then the first equation is the fact that m is the same on either side:

$$\rho_{0+} u^\mu_+ n_\mu = \rho_{0-} u^\mu_- n_\mu \overset{\text{def}}{=} m \tag{5.18}$$

The other, for an ideal flud, is

$$m \left(V^\mu_+ - V^\mu_- \right) + n^\mu \left(p_+ - p_- \right) = 0 \tag{5.19}$$

with $V^\mu = h u^\mu$. Now, consider a set of vectors Y^μ such that $Y^\mu n_\mu = 0$ and $Y^\mu Y_\mu = 1$. These form a two-parameter family ($\sim S^2$), and by (5.19) must satisfy

$$m \left(V^\mu_+ - V^\mu_- \right) Y_\mu = 0 \tag{5.20}$$

then

- either $m = 0$: this is called a *slip-stream* discontinuity, because the normal component of the velocity is zero, so the fluid is flowing *along* the boundary, no matter is crossing it;
- or $V^\mu_+ Y_\mu = V^\mu_- Y_\mu$ for *all* the Y^μ: this is called a *shock wave*.

There are *three* independent Y^μ, right?

In the shock wave case, we must write two (?) more equations to fully recover the (5.19). To do so, we define $\tau = h/\rho_0$ and use the fact that $V^2 = -h^2$. Then, we multiply (5.19) by V^+_μ and V^-_μ, and use the fact that $n_\mu V^\mu_\pm = m\tau_\pm$. We get

$$h_+^2 - h_-^2 - (p_+ - p_-)(\tau_+ - \tau_-) = 0 \tag{5.21}$$

then, by multiplying (5.19) and using the fact that n^μ is timelike we get:

$$m^2 = \frac{p_+ - p_-}{\tau_+ - \tau_-} \tag{5.22}$$

equations (5.18), (5.21) and (5.22) are the Rankine-Hugoniot equations.

TODO: show that the nonrelativistic limit of these is

$$\rho_0 u^i n_i = \text{const} \tag{5.23a}$$
$$\rho_0 u^i u_i + p = \text{const} \tag{5.23b}$$
$$h + u^2/2 = \text{const} \tag{5.23c}$$

5.3 Radiative processes

Question posed in [TFŻ81, Introduction]: will a black hole inside a star eat it whole in a free-fall time-scale (around 1 year) or on an Eddington-limited time-scale (around 10^8 years)? Is this known now?

Eddington luminosity It is the characteristic luminosity at which the radiation pressure from the photons moving outward equals the gravitational specific force on the infalling matter.

The gravitational force, in the newtonian limit, is

$$F_{\text{grav}} = \frac{GMm}{r^2} \tag{5.24}$$

The radiation pressure can be given in terms of the luminosity L (reinserting the units of c for this) as

$$P_{\text{rad}} = \frac{L}{c 4\pi r^2} \tag{5.25}$$

then, the radiative force is given by $F_{\text{rad}} = P_{\text{rad}} \kappa m$, where m is the mass of the test object and κ is the opacity: the per-unit-mass cross-section. We usually assume $\kappa = \sigma_T / m_p$, that is, that the interaction between radiation and matter is all due to Thompson scattering and the matter is only composed of hydrogen atoms.

Equating the forces, we get our result:

$$\frac{L_{\text{Edd}}}{M} = \frac{4\pi c G}{\kappa} \tag{5.26}$$

In the $\kappa = \sigma_T / m_p \approx 0.04\,\text{m}^2/\text{kg}$ case, we get L_{Edd}/M to be around $6.32\,\text{W}\,\text{kg}^{-1}$ (constants' values from [18]). If we express this in units of $L_\odot / M_\odot \approx 1.93 \times 10^{-4}\,\text{W}\,\text{kg}^{-1}$ [Wil18] we get $L_{\text{Edd}}/M \approx 3.27 \times 10^4 L_\odot / M_\odot$: the amount of radiation emitted by the Sun is much less than the Eddington limit.

It is, of course, important to note that this is a limit found with many approximations: nonrelativistic gravity, spherical symmetry, only Thompson scattering, only hydrogen.

Cooling function From [NTZ91].

The cooling function $\Lambda(T)$ is defined by the following relation, which describes the variation in the energy density by radiative processes:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = n_b^2 \big(\Gamma(T) - \Lambda(T)\big) \tag{5.27}$$

where U is the energy density (measured in $\text{erg}\,\text{cm}^{-3}$), n_b is the baryon density (measured in cm^{-3}), while Γ and Λ are the heating and cooling functions, both measured in $\text{erg}\,\text{cm}^3\,\text{s}^{-1}$, see [GH12, equation 1].

The cooling function of the infalling gas is

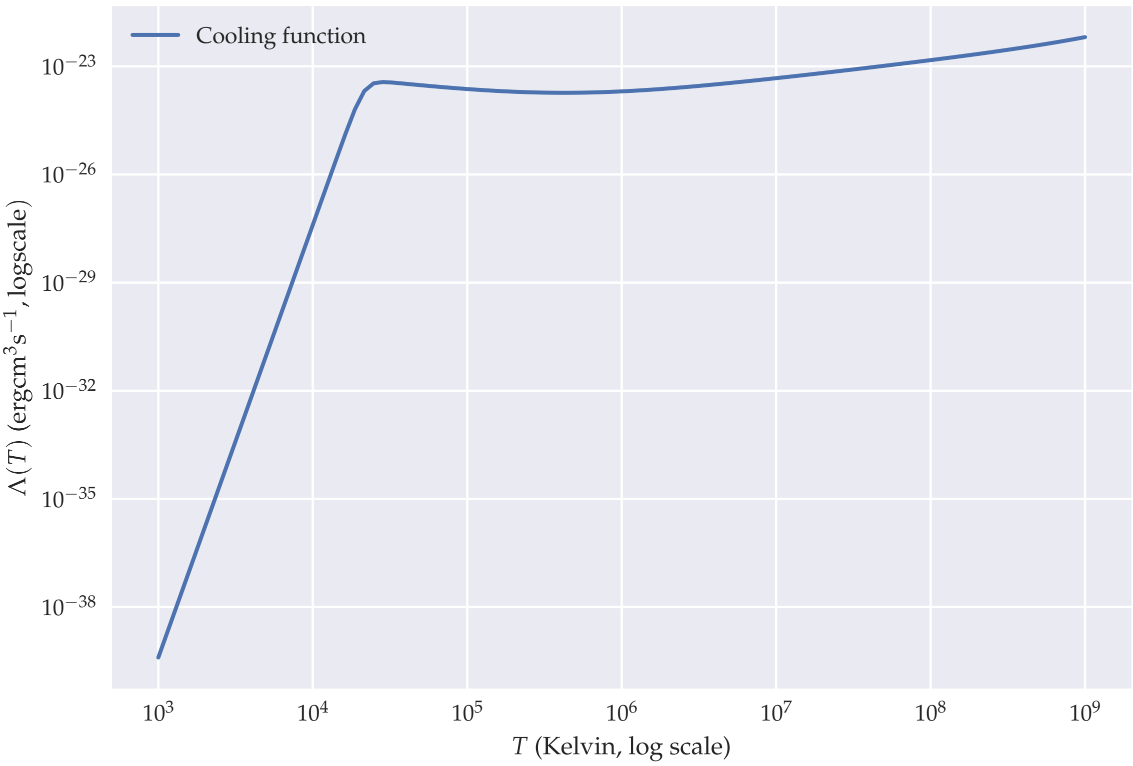

Figure 2: Cooling function graph.

$$\begin{aligned}\Lambda(T) = \Bigg(\bigg(1.42 \times 10^{-27} T^{1/2}\Big(1 + 4.4 \times 10^{-10} T\Big) + 6.0 \times 10^{-22} T^{-1/2}\bigg)^{-1} \\ + 10^{25}\left(\frac{T}{1.5849 \times 10^4\,\mathrm{K}}\right)^{-12}\Bigg)^{-1} \mathrm{erg\,cm^3\,s^{-1}}\end{aligned} \tag{5.28}$$

The version of this equation in Stellingwerf and Buff [SB82, equation 10] is similar: the first constant is 2.4×10^{-27} instead of 1.42×10^{-27}, and the factor $\Big(1 + 4.4 \times 10^{-10} T\Big)$ is just 1.

Bibliography

[Eck40] C. Eckart. "The Thermodynamics of Irreversible Processes. III. Relativistic Theory of the Simple Fluid". In: *Phys. Rev.* 58 (10 Nov. 1940), pp. 919–924. DOI: 10.1103/PhysRev.58.919. URL: https://link.aps.org/doi/10.1103/PhysRev.58.919.

[Tau48] A. H. Taub. "Relativistic Rankine-Hugoniot Equations". In: *Phys. Rev.* 74 (3 Aug. 1948), pp. 328–334. DOI: 10.1103/PhysRev.74.328. URL: https://link.aps.org/doi/10.1103/PhysRev.74.328.

[Ehl71] J. Ehlers. "General relativity and kinetic theory." In: *General Relativity and Cosmology*. Ed. by R. K. Sachs. 1971, pp. 1–70.

[MTW73] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. Ed. by Misner, C. W., Thorne, K. S., & Wheeler, J. A. 1973. ISBN: 0-7167-0344-0.

[Tau78] A. H. Taub. "Relativistic Fluid Mechanics". In: *Annual Review of Fluid Mechanics* 10.1 (1978), pp. 301–332. URL: https://doi.org/10.1146/annurev.fl.10.010178.001505.

[Tho81] K. S. Thorne. "Relativistic radiative transfer: moment formalisms". In: *Monthly Notices of the Royal Astronomical Society* 194.2 (Feb. 1981), pp. 439–473. ISSN: 0035-8711. DOI: 10.1093/mnras/194.2.439. URL: https://doi.org/10.1093/mnras/194.2.439.

[TFŻ81] K. S. Thorne, R. A. Flammang, and A. N. Żytkow. "Stationary spherical accretion into black holes – I. Equations of structure". In: *Monthly Notices of the Royal Astronomical Society* 194.2 (Feb. 1981), pp. 475–484. ISSN: 0035-8711. DOI: 10.1093/mnras/194.2.475. URL: https://doi.org/10.1093/mnras/194.2.475.

[SB82] R. F. Stellingwerf and J. Buff. "X-radiation limited accretion flow. I - Steady flow regime". In: *ApJ* 260 (Sept. 1982), pp. 755–767. DOI: 10.1086/160295.

[NTZ91] L. Nobili, R. Turolla, and L. Zampieri. "Spherical accretion onto black holes - A complete analysis of stationary solutions". In: *ApJ* 383 (Dec. 1991), pp. 250–262. DOI: 10.1086/170781.

[Car97] S. M. Carroll. *Lecture notes on general relativity*. 1997. arXiv: gr-qc/9712019 [gr-qc].

[Nob00] L. Nobili. *Astrofisica relativistica — fondamenti di relatività generale e fluidodinamica relativistica*. 1st ed. Via G. Prati, 19 — Padova: CLEUP, Oct. 2000. ISBN: 88-7178-583-5.

[Gou06] E. Gourgoulhon. "An Introduction to relativistic hydrodynamics". In: *EAS Publ. Ser.* 21 (2006), pp. 43–79. DOI: 10.1051/eas:2006106. arXiv: gr-qc/0603009 [gr-qc].

[Yos11] S. Yoshida. "Sound speed in relativistic fluid". Jan. 2011. URL: http://ea.c.u-tokyo.ac.jp/astro/Members/yoshida/notes/sound_speed.pdf.

[GH12] N. Y. Gnedin and N. Hollon. "Cooling and heating functions of photoionized gas". In: *The Astrophysical Journal Supplement Series* 202.2 (Sept. 2012), p. 13. DOI: 10.1088/0067-0049/202/2/13. URL: https://doi.org/10.1088%2F0067-0049%2F202%2F2%2F13.

[Lec14] K. Lechner. *Elettrodinamica classica*. 2014. ISBN: 978-88-470-5210-9. DOI: 10.1007/978-88-470-5211-6.

[Mor16] V. Moretti. *Stress-energy tensor: Its relation to force?* Physics Stack Exchange. 2016. eprint: https://physics.stackexchange.com/q/301254. URL: https://physics.stackexchange.com/q/301254.

[Ung16] R. Unger. *Stokes theorem in Lorentzian manifolds*. Physics Stack Exchange. 2016. eprint: https://physics.stackexchange.com/q/240747. URL: https://physics.stackexchange.com/q/240747.

[18] *CODATA Internationally recommended 2018 values of the Fundamental Physical Constants*. CODATA. 2018. URL: https://pml.nist.gov/cuu/Constants/.

[Wil18] David R. Williams. *Sun Fact Sheet*. NASA. 2018. URL: https://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html.