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Relativistic non-ideal flows

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Abstract

After reviewing the basic concepts of general-relativistic fluid mechanics, I will focus on the treatment of non-ideal (viscous, thermo-conducting) flows. An application of non-ideal relativistic flows to spherical accretion onto black holes (generalized Bondi accretion) will be also discussed.

The table of contents will only include up to the subsection level in the final document, it is just convenient for drafts to be able to see the paragraph structure.

Contents

1 Introduction

In section 3 I will briefly recall the basics of special relativity, differential geometry and general relativity in order to introduce the Schwarzschild metric, which is the geometrical background on which the spherical accretion problem is studied.

Then, in section 4 I will state the equations of motion of a non-relativistic fluid and those of a relativistic fluid, thereby defining the stress-energy tensor.

I will use the formalism first introduced by Eckart [Eck40] for the decomposition the stress energy tensor, and following Taub [Tau78] distinguish in the spatial projection of the conservation equations the relativistic forces acting on the fluid due to viscosity and to heat transfer.

I will then give a proof of the relativistic Second Principle of thermodynamics, which will justify the definition of an ideal fluid.

I will then derive the equations which govern spherical accretion in the adiabatic case.

In section 5 I will introduce and apply the PSTF moments formalism by Thorne [Tho81] to the spherical accretion problem, still considering the fluid as ideal but introducing energy transfer terms due to radiation.

2 Notational preface

I will use Greek indices (μ, ν, $\rho\ldots$) to denote 4-dimensional indices ranging from 0 to 3, and Latin indices (i, j, $k\ldots$) to denote 3-dimensional indices ranging from 1 to 3.

I will use the "mostly plus" metric for flat Minkowski space-time, $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$: therefore four-velocities will have square norm $u^\mu u_\mu = -1$. I will use Einstein summation convention: if an index appears multiple times in the same monomial, it is meant to be summed over.

I will always use the abuse of notation in which a vector is denoted by its components, and a free index means we consider all of the possible values it can take, such as x^μ denoting a point in spacetime.

Take a diffeomorphism $x \to y$, with Jacobian matrix $\partial y^\mu / \partial x^\nu$. The indices of contravariant vectors, transforming as

$$V^\mu \to \left(\frac{\partial y^\mu}{\partial x^\nu} \right) V^\nu \tag{2.1}$$

will be denoted as upper indices, while the indices of covariant vectors, transforming as

$$V_\mu \to \left(\frac{\partial x^\nu}{\partial y^\mu} \right) V_\nu \tag{2.2}$$

will be denoted as lower indices; the same applies to higher rank tensors.

Unless otherwise specified, I will work in geometrized units, where $c = G = 1$.

In section 5 I will use the notation from Thorne [Tho81]: A_k will represent a sequence of k indices labelled as α_i, for i between 1 and k. The same will hold for $B_k \to \{\beta_i\}$ etc.

Take a tensor with many indices, $T_{A_k B_j}$. These indices can be symmetrized and antisymmetrized, and I will use the following conventions:

$$T_{(A_k)B_j} = \frac{1}{k!} \sum_{\sigma \in S_k} T_{\sigma(A_k)J} \tag{2.3}$$

$$T_{[A_k]B_j} = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign} \sigma T_{\sigma(A_k)B_j} \tag{2.4}$$

where $S_k \ni \sigma$ is the group of permutations of k elements, and $\operatorname{sign} \sigma$ is 1 if σ is an even permutation (it can be obtained in an even number of pair swaps) and -1 otherwise.

In the case a set of indices that are not nearby need to be (anti)symmetrized, I will use vertical bars: for example, $R_{(\mu|\nu|\rho\sigma)}$ means that we take the permutations of the indices μ, ρ and σ.

When reporting calculations I will show equations one under the other, and if one side of them stays the same (for example if it is always equal to 0) I will not report it after the first time: if an equals sign only has an expression on one side, it is implied that on the other is the last expression which appeared on that side above.

3 Relativity

3.1 Special relativity

Special Relativity is a theory which satisfies the following axioms [Lec14]:

1. space and time are homogeneous (i. e. shift-invariant), space is isotropic (i.e. rotation-invariant);
2. the speed of light is the same in every inertial reference frame;
3. all the laws of physics are written in the same way in every inertial reference frame.

In special relativity, instead of having vectors in 3D space and a time scalar coordinate, we denote events as points in 4D spacetime, which is an intrinsically flat semi-Riemannian manifold with metric signature $(-, +, +, +)$, with coordinates such as $x^\mu = (t, x, y, z)$. This is called Minkowski flat spacetime.

This difference is not just semantic: the spacetime formalism is needed because the axioms are equivalent to the conservation of the spacetime interval $\mathrm{d}s^2 = \eta_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu$, and the only transformations between inertial reference frames which leave it invariant regularly *mix* time and space: they are represented in 4D spacetime as $x^\mu \to \Lambda^\mu{}_\alpha x^\alpha + a^\mu$, with the $\Lambda_\mu{}^\alpha$ being $(1,1)$ tensors which satisfy $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$, and a^μ being a generic constant 4-vector.

The flat metric allows us to compute the lengths of vectors, and since it is indefinite there are nonzero vectors of positive, negative and zero spacetime length. These are respectively called spacelike, timelike and null-like.

We define the proper time τ by $\mathrm{d}\tau \stackrel{\text{def}}{=} \sqrt{-\mathrm{d}s^2}$. Unlike coordinate time $x^0 = t$ this has the advantage of being Lorentz-invariant.

We can define a tensorial velocity by differentiating the position with respect to proper time: $u^\mu \stackrel{\text{def}}{=} \mathrm{d}x^\mu / \mathrm{d}\tau$. Defined this way, the so-called 4-velocity trasforms like a tensor. If $\vec{v}$ is the regular three-velocity and v is its magnitude, we define $\gamma = 1/\sqrt{1-v^2}$ and then: $u^\mu = (\gamma, \gamma\vec{v})$.

Now, differentiating any function of position looks like $\mathrm{d}T^{A_k}(x^\mu) \big/ \mathrm{d}\tau = (\nabla_\mu T^{A_k})\, \mathrm{d}x^\mu / \mathrm{d}\tau = u^\mu \nabla_\mu T^{A_k}$.

Once we have this, we can define the 4-acceleration:

$$a^\nu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = u^\mu \nabla_\mu u^\nu \,. \tag{3.1}$$

The 4-velocity is a unit vector: $u^\mu u_\mu = -1$, and by differentiating this relation we get the often used identity $u^\mu a_\mu = 0$.

We also define the 4-momentum $p^\mu = m u^\mu$, where m is the rest mass of the body at hand.

The 0-th component of the 4-momentum vector is the energy of the body, while the i-th components define a new relativistic momentum $p^i = \gamma m v^i$: we then have $p^\mu p_\mu = m^2 = E^2 - |p|^2$.

3.2 Differential geometry and tensor calculus

3.2.1 Metric

The metric tensor $g_{\mu\nu}$ is a symmetric $(0,2)$ tensor which defines a scalar product at every point in our manifold: $x \cdot y = g_{\mu\nu} x^\mu y^\nu$. It is not intrinsic to the manifold. By integrating the velocity vector we can find the lengths of curves $x^\mu(\lambda)$:

$$L = \int \sqrt{g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\, \mathrm{d}\lambda \,. \tag{3.2}$$

For a flat spacetime we use the Minkowski metric $\eta_{\mu\nu}$. In general, in the presence of matter the manifold will be curved, so there will not be a coordinate transformation to cast $g_{\mu\nu}$ in the form $\eta_{\mu\nu}$. If we choose a certain point P, however, it is possible to find a transformation in order to impose the conditions $g_{\mu\nu}(P) = \eta_{\mu\nu}(P)$, $\partial_\rho g_{\mu\nu}(P) = 0$ [Car97, pages 49–50].

The metric defines an invariant called the *spacetime interval*:

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \,. \tag{3.3}$$

3.2.2 Tensor calculus

An object such as $\partial_\mu A^\nu$ for some vector A^ν does not in general transform as a tensor. The quantity we should actually compute is $\partial_\mu (A^\nu e_\nu)$ where e_ν are the basis vectors: we evaluate this with the product rule and are left with $(\partial_\mu A^\nu) e_\nu + A^\nu \partial_\mu e_\nu$. If we only had the first half the derivative would be covariant: in flat spacetime the second half is zero but in general it is not: confusing the derivatives of the components with the derivatives of the vector does not work here.

The covariant derivative keeps account of the shifting of the basis vectors:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\alpha\mu} A^\alpha \,. \tag{3.4}$$

The rank-3 objects Γ are called Christoffel symbols. They are not tensors: they depend on the choice of basis e_α, and they satisfy $\nabla_\mu e_\alpha = \Gamma^\nu_{\mu\alpha} e_\nu$. They are not intrinsic to the manifold.

The definition is posed in such a way that the objects $\nabla_\mu A^\nu$ are tensorial. This allows us to see that the covariant derivative is the same as the coordinate derivative for scalars.

We can define the covariant derivative of higher order tensor analogously; adding a Christoffel symbol for every new index. The symbols corresponding to lower indices have a minus sign: this can be seen by differentiating a scalar such as $\nabla_\nu (A_\mu B^\mu) \stackrel{!}{=} \partial_\nu (A_\mu B^\mu)$ and matching the Christoffel terms.

If we have the metric (and make reasonable assumptions of the connection being torsion-free), they can be calculated as:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho} \right) . \tag{3.5}$$

This also tells us that they are symmetric in the lower two indices: $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$.

The divergence of a vector field A^μ can be calculated as:

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} A^\mu \right) \tag{3.6}$$

where g is the determinant of the metric.

3.2.3 Stokes' theorem

Following [Ung16]. If we have a manifold of dimension n, and an n-dimensional region V of this manifold equipped with coordinates x^μ a metric $g_{\mu\nu}$, with a submanifold boundary ∂V equipped with coordinates y^μ and the induced metric $h_{\alpha\beta} = \left(\partial x^\mu / \partial y^\alpha \right) \left(\partial x^\nu \middle/ \partial y^\beta \right) g_{\mu\nu}$, and for which we have a properly oriented normal vector $n^\mu(y)$; then for any vector f^μ we have:

$$\int_V \nabla_\mu f^\mu \sqrt{|\det g|} \, \mathrm{d}^n x = \int_{\partial V} f^\mu n_\mu \sqrt{|\det h|} \, \mathrm{d}^{n-1} y \, . \tag{3.7}$$

3.2.4 Geodesics

A path $x^\mu(\lambda)$ is called a *geodesic* if it is stationary with respect to path length. To check whether a given path is a geodesic we can stationarize the action corresponding to the Lagrangian $\mathcal{L}(x, \dot{x}) = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ (where we use the notation $\dot{x} = \mathrm{d}x / \mathrm{d}\lambda$). The Lagrange equations then are:

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0 \tag{3.8}$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols, which can be calculated by differentiating the metric, as shown in (3.5). $\mathcal{L}$ is an integral of these Lagrange equations.

If the parameter λ is taken to be the proper time s, then the equation is

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}s} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0 \, . \tag{3.9}$$

Notice that this is equivalent to the covariant acceleration (3.1) being zero.

3.2.5 Fermi-Walker transport

Take a general vector field $V^\mu(s)$ defined along a curve, with the curve's tangent vector u^μ whose covariant acceleration is a^μ. Then we say that V^μ is transported according to Fermi-Walker iff it satisfies

$$\dot{V}^\mu = u^\nu \nabla_\nu V^\mu = 2 V_\rho u^{[\mu} a^{\rho]} \,. \tag{3.10}$$

This condition is always satisfied by $V^\mu = u^\mu$, since $a^\mu u_\mu = 0$, whether or not the curve is a geodesic. The tangent vector is *parallel* transported only for geodesics.

The justification of this definition is the fact that we want the transformations of our tetrad to be infinitesimal Lorentz boosts, which are generated by antisymmetric tensors, and we want to prohibit any rotations in the plane orthogonal to a^μ and u^μ.

3.2.6 Tetrads and projectors

We want to work in a reference in which the velocity u^μ is purely timelike. This can always be found by the equivalence principle. Such a reference can be completed into what is called a tetrad, for which the metric becomes the Minkowski metric in a neighbourhood of the point we consider.

We call the velocity $u^\mu = V^\mu_{(0)}$ and complement it with three other vectors $V^\mu_{(i)}$ such that

$$g_{\mu\nu} V^\mu_{(\alpha)} V^\nu_{(\beta)} = \eta_{(\alpha)(\beta)} \tag{3.11}$$

where the brackets around the indices denote the fact that they label four vectors, not the components of a tensor.

We can choose the vectors $V^\mu_{(i)}$ so that they are Fermi-Walker transported along the worldline defined by u^μ: this allows us to find the relativistic equivalent of a non-rotating frame of reference.

It is useful to project tensors onto the space-like and time-like subspaces defined by our tetrad (and we wish to do so in a coordinate-independent manner, so just taking the 0th and i-th components in the tetrad will not suffice). We therefore define the projectors:

$$h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu} \qquad\qquad \pi_{\mu\nu} = -u_\mu u_\nu \tag{3.12}$$

respectively onto the space- and time-like subspaces.

3.2.7 Killing vector fields

Following [MTW73, section 25.2, page 650]. Suppose there is a certain direction (which, for simplicity, we assume to be along one of our coordinate axes) along which the metric is preserved: an $\widetilde{\alpha}$ such that $\partial_{\widetilde{\alpha}} g_{\mu\nu} = 0$.

Then the metric properties of curves along the manifold are unchanged if we shift their coordinate representation by a constant along the $\widetilde{\alpha}$ coordinate axis.

Let us call the direction of this translation $\xi^\mu = \delta^\mu_{\widetilde{\alpha}}$ if we use this coordinate system. It can be shown by direct computation that

$$\nabla_\nu \xi_\mu = \frac{1}{2}\Big(\partial_{\widetilde{\alpha}} g_{\mu\nu} + \partial_\nu g_{\mu\widetilde{\alpha}} - \partial_\mu g_{\nu\widetilde{\alpha}}\Big) \tag{3.13}$$

but by hypothesis the first term on the RHS of (3.13) is zero, therefore we have shown that $\nabla_\nu \xi_\mu = \nabla_{[\nu} \xi_{\mu]}$ in this coordinate frame, but since this is a covariant equation it extends to every other one.

This can equivalently be stated by writing $\nabla_{(\nu}\xi_{\mu)} = 0$: this is called *Killing's equation*. This is useful since: given a geodesic $x^\mu(\lambda)$, for which we define $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$, it must be the case that $u^\nu \nabla_\nu u^\mu = 0$. Then, the component of u^μ along ξ^μ ($u^{\widetilde{\alpha}} = u^\mu \xi_\mu$) is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(u^\mu \xi_\mu\right) = u^\nu \nabla_\nu \left(u^\mu \xi_\mu\right) = \cancelto{0}{\xi^\mu u^\nu \nabla_\nu u_\mu} + \cancelto{0}{u^\nu u^\mu \nabla_\nu \xi_\mu} \equiv 0\,. \tag{3.14}$$

3.2.8 Surfaces in space-time and acceleration decomposition

Following [Tau78, section 4]. We consider 3D space-like slices in 4D space-time: if a fluid is moving with velocity u^μ, we denote the solutions of the differential equation associated with the vector field as $x^\mu(\xi^i, s)$, where ξ^i are the 3D coordinates of the starting position and s is the time at which we look at the solution. Then the "starting" hypersurface is $\Sigma = \left\{x^\mu(\xi^i, 0)\right\}$.

Suppose we have a curve $\xi^i(\tau)$ in Σ. Then we can define the two-dimensional surface defined by the evolution of $\xi^i(\tau)$: $x^\mu(\xi^i(\tau), s) \overset{\text{def}}{=} x^\mu(\tau, s)$. If we also define the "spatial" tangent vector $\lambda^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, it follows from Schwarz's theorem that:

$$\frac{\partial^2 x^\mu}{\partial\tau\partial s} = \frac{\partial^2 x^\mu}{\partial s\partial\tau} \implies \frac{\partial u^\mu}{\partial \tau} = \frac{\partial \lambda^\mu}{\partial s}\,. \tag{3.15}$$

Now let us take the spatial vectors of an orthonormal Fermi-Walker transported tetrad $V^\mu_{(a)}$ as described in 'Tetrads and projectors' on page 7, and express λ^μ in this frame: its covariant components will be

$$X_{(a)} = V_{(a)\mu}\lambda^\mu\,. \tag{3.16}$$

If we differentiate (3.16) with respect to s, and use (3.15) with the fact that $\frac{\mathrm{d}}{\mathrm{d}\tau} = \lambda^\mu \nabla_\mu$, we get:

$$\begin{aligned}
\frac{\mathrm{d}X_{(a)}}{\mathrm{d}s} &= \frac{\mathrm{d}V_{(a)\mu}}{\mathrm{d}s}\lambda^\mu + V_{(a)\mu}\frac{\mathrm{d}\lambda^\mu}{\mathrm{d}s} &\text{(3.17a)}\\
&= V^\rho_{(a)}\cancelto{0}{\lambda^\mu u_\mu} a_\rho - \cancelto{0}{V^\rho_{(a)} u_\rho}\lambda^\mu a_\mu + V^\nu_{(a)}\nabla_\mu u_\nu &\text{(3.17b)}\\
&= V^\rho_{(a)}\lambda^\mu \nabla_\mu u_\rho &\text{(3.17c)}\\
&= \left(\nabla_\mu u_\rho\right) V^\rho_{(a)} V^\mu_{(b)} X^{(b)} &\text{(3.17d)}
\end{aligned}$$

where in the last step we expressed everything with respect to the tetrad coordinate system. Therefore, in those coordinates, the evolution of the components $X^{(a)}$ is linear, and defined by the tetrad components of the two-form $\nabla_\mu u_\nu$. So, we want to decompose this tensor:

$$\nabla_\sigma u_\tau = \underbrace{\omega_{\sigma\tau}}_{\substack{\text{spatial}\\\text{rotation}}} + \underbrace{\sigma_{\sigma\tau}}_{\substack{\text{spatial}\\\text{shear}}} + \underbrace{\frac{1}{3}\theta h_{\sigma\tau}}_{\substack{\text{spatial}\\\text{compression}}} - \underbrace{a_\tau u_\sigma}_{\text{acceleration}} \tag{3.18}$$

1. $\theta = \nabla_\mu u^\mu$ is the bare trace of the tensor, corresponding to the expansion velocity;
2. $a_\mu = u^\nu \nabla_\nu u^\mu$ is the covariant acceleration;
3. $\sigma_{\sigma\tau} = \left(\nabla_{(\mu} u_{\nu)}\right) h^\mu_\sigma h^\nu_\tau - 1/3\theta h_{\sigma\tau} = \nabla_{(\sigma} u_{\tau)} + a_{(\sigma} u_{\tau)} - 1/3\theta h_{\sigma\tau}$ is the spatial symmetric trace-free part of the tensor, which corresponds to the shear stress;
4. $\omega_{\sigma\tau} = h^\nu_\sigma h^\mu_\tau \nabla_{[\nu} u_{\mu]} = \partial_{[\tau} u_{\sigma]} + a_{[\tau} u_{\sigma]}$ is the spatial (antisymmetric, trace-free) rotation tensor.

To verify the formulas given for $\sigma_{\sigma\tau}$ and $\omega_{\sigma\tau}$ it is enough to expand the definitions, simplifying the terms which contain products of the 4-acceleration and the 4-velocity; also, the terms such as $u^\mu u_\tau \nabla_\sigma u_\mu$ vanish since $0 = u_\tau \nabla_\sigma (u^\mu u_\mu) = 2u^\mu u_\tau \nabla_\sigma u_\mu$.

3.3 General Relativity

3.3.1 The Equivalence Principle

In the General theory of Relativity we make a stronger claim than that of the axioms of SR, which are only formulated for inertial reference frames.

The *Einstein Equivalence Principle* states [Car97, p. 100] that in small enough regions of spacetime the laws of physics are those of special relativity, and we cannot detect gravitational effect locally. The frame of reference in which we must write the laws for them to appear in their special-relativistic form is called the Locally Inertial Reference Frame or Local Rest Frame (LRF).

Unlike special relativity, the transformation laws between different reference frames are not linear, but can be in general be represented as diffeomorphisms.

We model spacetime as a manifold which has intrinsic (basis-independent) curvature; an object which is modelled in newtonian mechanics as being in free fall, accelerated by a gravitational force, is modelled in general relativity as following a geodesic in the manifold.

3.3.2 Curvature

The intrinsic curvature of spacetime is fully described by the Riemann curvature tensor, which is a fourth rank tensor: for any generic vector V^μ,

$$R^\mu_{\nu\rho\sigma} V^\nu \stackrel{\text{def}}{=} [\nabla_\rho, \nabla_\sigma] V^\mu . \tag{3.19}$$

It can be calculated using the Christoffel symbols, and while they are not tensors $R^\mu_{\nu\rho\sigma}$ is one. This result follows by expanding all the covariant derivatives in formula (3.19).

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu} \tag{3.20}$$

The Christoffel symbols can be nonzero if we choose certain coordinates even for flat spacetime, but the Riemann tensor is zero iff the spacetime is flat.

The Riemann tensor satisfies the following identities [MTW73, eqs. 8.45 and 8.76]:

$$\nabla_{[\lambda} R_{\mu\nu]\rho\sigma} = 0 \tag{3.21a}$$

$$R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]} = R_{[\rho\sigma][\mu\nu]} \tag{3.21b}$$

$$R_{[\mu\nu\rho\sigma]} = 0 = R_{\mu[\nu\rho\sigma]} . \tag{3.21c}$$

If we define the Ricci tensor $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$ and the curvature scalar $R = R_{\mu\nu} g^{\mu\nu}$, we can derive from (3.21a) the *contracted Bianchi identity* $\nabla_\mu R = 2\nabla_\nu R^\nu_\mu$, which means that $\nabla_\mu \left(R^{\mu\nu} - {}^1\!/_2 R g^{\mu\nu} \right) = 0$.

3.3.3 The Einstein Field Equations

They describe the way the presence of matter changes the geometry of spacetime. They involve the *stress-energy tensor* $T^{\mu\nu}$ which is defined in 'Stress-energy tensor' on page 12 and the Einstein tensor $G^{\mu\nu} = R^{\mu\nu} - {}^1\!/_2 R g^{\mu\nu}$, which is the only independent tensor satisfying the following properties: it can be constructed from only the Riemann tensor and the metric, it vanishes for flat spacetime and it identically satisfies the conservation laws $\nabla_\mu G^{\mu\nu} = 0$.

The equations are:

$$G^{\mu\nu} = 8\pi T^{\mu\nu} . \tag{3.22}$$

The constant comes by imposing continuity with the newtonian limit, in which we know the gravitational field Φ is determined by the matter density ρ_0 according to the Poisson equation $\partial_i \partial^i \Phi = 4\pi\rho_0$.

The matter density ρ_0 is substituted in the relativistic formulation by T_{00} while the gravitational field Φ is substituted by a small perturbation in the metric: $\Phi = -1/2 h_{00}$, with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (see [Car97, eq. 4.46]).

They can be written in a more general way by removing the condition that the LHS vanish for flat spacetime, and thus including there a *cosmological constant* term $\Lambda g^{\mu\nu}$ with constant Λ. It is unclear whether this term should appear, and what the value of Λ should be.

3.3.4 The Schwarzschild solution

Following [Car97, section 7].

The EFE are generally very difficult to solve, but they admit analytical solutions in certain special cases. One of the simplest is that of a central mass M described with spherical coordinates (t, r, θ, φ) and in the presence of spherical symmetry.

One imposes the condition that the stress energy tensor be identically zero for radii greater than a certain (arbitrarily small) radius, $r > r_c$.

One can write down the most general possible spherically symmetric metric, which turns out to be [Car97, eq. 7.13]:

$$\mathrm{d}s^2 = -e^{2\alpha(r,t)}\,\mathrm{d}t^2 + e^{2\beta(r,t)}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi\right) \tag{3.23}$$

and plug this into the equations $G_{\mu\nu} = 0 \implies R_{\mu\nu} = 0$ for all $r > r_c$. One gets the result that the metric possesses a timelike Killing vector field which is orthogonal to a family of hypersurfaces $t = \text{const}$: therefore it is *static*, unchanging with time. α and β only depend on r, and one gets the result that

$$e^{2\alpha(r)} = e^{-2\beta(r)} = \left(1 + \frac{C}{r}\right) \tag{3.24}$$

for some C. By continuity with the weak-field limit, for which one has the newtonian gravitational field $\Phi = -M/r$ and $g_{00} = -(1 + 2\Phi)$, one sets $C = -2M$. Keeping the notation $\Phi = -M/r$ we have:

$$\mathrm{d}s^2 = -(1 + 2\Phi)\,\mathrm{d}t^2 + \frac{1}{1 + 2\Phi}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi\right) \tag{3.25}$$

or, equivalently,

$$g_{\mu\nu} = \mathrm{diag}\left(-(1 + 2\Phi), \frac{1}{1 + 2\Phi}, r^2, r^2\sin^2\theta\right) . \tag{3.26}$$

We can see that it approaches the spherical-coordinates flat metric $\eta'_{\mu\nu} = \mathrm{diag}\left(-1, 1, r^2, r^2\sin^2\theta\right)$ both in the limit $M \to 0$ and the limit $r \to \infty$. Its determinant is $g = -r^4\sin^2\theta$, so $\sqrt{-g} = r^2\sin\theta$.

4 Fluid dynamics

4.1 Nonrelativistic fluid mechanics

Nonrelativistic (compressible) fluid mechanics are described by the equations:

$$\partial_t \rho_0 + \partial_i(\rho_0 v^i) = 0 \qquad \text{conservation of mass} \tag{4.1a}$$

$$\rho_0 \left(\partial_t v^i + v^j \partial_j v^i\right) = \partial_j \sigma^{ij} \qquad \text{conservation of momentum} \tag{4.1b}$$

$$\rho_0 \partial_t E + v^i \partial_i E = \partial_i \left(\sigma^{ij} v_j + \kappa \partial^i T\right) \qquad \text{conservation of energy} \tag{4.1c}$$

where ρ_0 is the density of the fluid, v^i are the components of the velocity vector field, σ^{ij} is the classical stress tensor (or, equivalently, the *negative* of the space-like components of the stress-energy tensor T^{ij}), E is the energy density of the fluid, κ is the thermal conductivity, T is the temperature field of the fluid.

We use the compressible formulation in order for these to be closer to their relativistic counterpart, for which compressive effects cannot be ignored.

The nonrelativistic stress tensor can be written as:

$$\sigma_{ij} = -(p - \xi \partial_k v^k)\delta_{ij} + 2\eta \partial_{(i} v_{j)} \tag{4.2}$$

where p is the (isotropic) pressure, η the viscosity, ξ is the bulk viscosity.[1] We are assuming that the normal stresses are only those exerted by pressure, so the diagonal terms σ_{ii} (not summed) must just be $-p$. So, the term $-\xi \partial_k v^k$ must equal $\eta \partial_{(i} v_{i)} = 2\eta \partial_i v_i$ (not summed). Therefore, by isotropy, $\xi = -2\eta/3$.

Note that we are working in Euclidean 3D space, so the metric is the identity and upper and lower indices are equivalent.

The energy density is a sum of kinetic and specific energies:

$$E = v^i v_i / 2 + \varepsilon \tag{4.3}$$

where ε is the specific internal energy per unit mass.

4.2 The relativistic fluid

We want to develop a formalism to treat a fluid dynamical problem in the presence of relativistic speeds and strong gravitational fields, such as in spherical accretion onto a black hole. It will have to be fully relativistic: the conservation laws will have to be written as tensorial equations.

We treat the fluid as a continuous medium which will have a certain density of particles per unit volume n: that is, what we consider "infinitesimal" is not actually arbitrarily small but should be considered much smaller than the characteristic lengths of the problem, while still containing many particles.

The 4-current of particles is $N^\mu = n u^\mu$, where u^μ is the 4-velocity field of the fluid. If these particles have a certain rest mass m_0, we can then define the rest-mass-flow vector $\rho_0 u^\mu = m_0 N^\mu$, which is conserved: $\nabla_\mu(\rho_0 u^\mu) = 0$, since particles do not spontaneously appear or disappear nor change their rest mass. The presence of particles with different rest masses can be easily accounted for by adding the mass flow vectors.

[1] For consistency with the later sections, here we define ξ with the opposite sign to what appears in [Tau78, page 301].

Particles in a fluid can have three kinds of energy we concern ourselves with: mass, kinetic energy and other forms of energy (thermal, chemical, nuclear…). We can always perform a change of coordinates to bring us to a frame in which the kinetic energy is zero.

Do note that, since our volume element is not actually arbitrarily small, being in the LRF only means that, locally, the average velocity of the fluid is purely timelike: the temperature can be nonzero, so in the LRF the particles in the volume element will still have isotropically distributed nonzero velocities.

We write the sum of mass-energy and internal energy as $\rho = \rho_0(1+\epsilon)$, the *energy* density of the fluid in its Local Rest Frame, while ρ_0 is the *mass* density in the LRF. So, ϵ is the ratio of the internal non-mass energy to the mass.

4.2.1 Stress-energy tensor

The stress-energy tensor $T^{\mu\nu}$ is a symmetric $(2,0)$ tensor whose μ, ν component is defined as the flux of μ-th component of the four-momentum p^μ through a surface of constant coordinate x^ν.

Because of our choice of the metric signature, the spatial part of the tensor corresponds to the *negative* of the classical continuum-mechanics stress tensor: $T^{ij} = -\sigma^{ij}$, since that tensor describes the stresses *on* the "box" of fluid [Mor16].

To give an example: for a gas of non-interacting particles, the stress-energy tensor is very simple: the momentum density is ρu^μ, and then to obtain the flow through a surface of constant x^ν we just need to multiply by u^ν, so in the Local Rest Frame we have:

$$T^{\mu\nu} = \rho u^\mu u^\nu = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.4}$$

The fact that momentum is conserved (which follows from Noether's theorem applied to the translational invariance of spacetime) can be expressed as the statement that the stress-energy tensor is conserved: $\nabla_\mu T^{\mu\nu} = 0$.

In our example, this can be written as

$$\rho(a^\nu + u^\nu(\nabla_\mu u^\mu)) + u^\nu \frac{\mathrm{d}\rho}{\mathrm{d}\tau} = 0\,. \tag{4.5}$$

4.2.2 Relativistic non-ideal fluid dynamics

The evolution of the fluid is described by the conservation of the stress-energy tensor $\nabla_\mu T^{\mu\nu} = 0$ and the conservation of mass $\nabla_\mu(\rho_0 u^\mu) = 0$.

If we wanted to analyze our fluids without *any* approximation we would need to consider the stress-energy tensor of the fluid when solving the Einstein Field Equations: this can be done but it makes the geometry of the system significantly harder to work with, and since we wish to consider other effects such as heat transfer and viscosity throughout the fluid we assume the fluid is not self-gravitating, that is, we solve the conservation equations in a fixed Schwarzschild metric background. This assumption is reasonable in our case: the components of the stress-energy tensor of the infalling gas are much smaller than those of the black hole.

Any stress-energy tensor can be decomposed into its space and time-like parts in the local rest frame of the fluid:

$$T_{\mu\nu} = w u_\mu u_\nu + w_\mu u_\nu + u_\mu w_\nu + w_{\mu\nu} \tag{4.6}$$

where [Tau48, eqs. 8.2, 8.3, 8.5]:

$$
\begin{aligned}
w &= T_{\mu\nu}u^\mu u^\nu = \rho_0(1+\varepsilon) = \rho && \text{rest energy} && (4.7a)\\
w_\mu &= T_{\nu\sigma}h^\sigma_\mu u^\nu = -\kappa h^\sigma_\mu(\partial_\sigma T + Ta_\sigma) && \text{heat conduction} && (4.7b)\\
w_{\mu\nu} &= T_{\rho\sigma}h^\rho_\mu h^\sigma_\nu = (p-\xi\theta)h_{\mu\nu} - 2\eta\sigma_{\mu\nu} && \text{pressure and viscous stresses.} && (4.7c)
\end{aligned}
$$

Do note that by definition the tensors w^μ and $w^{\mu\nu}$ are purely spatial: $u^\mu w_\mu = u^\mu w_{\mu\nu} = 0$. For the definition of the acceleration, vorticity etc. see equation (3.18).

As in section Nonrelativistic fluid mechanics η is the viscosity, ξ is the bulk viscosity, κ is the thermal conductivity, T is the temperature field, p is the pressure field, ρ_0 is the rest mass density while $\rho = \rho_0(1+\varepsilon)$ is the rest energy density.

The fact that the decomposition can be written in this way is not proven in [Tau78], but it seems like a proof can be found in [Eck40], which is locked behind a paywall.

Equivalently, we can write

$$
\begin{aligned}
T_{\mu\nu} &= T^p_{\mu\nu} && - T^V_{\mu\nu} && + T^h_{\mu\nu} && (4.8a)\\
&= wu_\mu u_\nu + ph_{\mu\nu} && - \xi\theta h_{\mu\nu} - 2\eta\sigma_{\mu\nu} && + 2w_{(\mu}u_{\nu)} && (4.8b)\\
&\text{perfect fluid} && \text{viscous stresses} && \text{heat conduction.}
\end{aligned}
$$

In the nonrelativistic limit the velocity is approximately $u^\mu = (1, v^i)$, so the conservation of mass becomes (4.1a), while we can obtain both the conservation of energy (4.1c) and the conservation of momentum (4.1b) from the four equations of conservation of the stress-energy tensor.

4.2.3 Viscous and heat-flow relativistic forces

If we consider the spatial components of the conservation equations by applying $h^\sigma_\mu \nabla_\nu$ to the formulation of the stress-energy tensor given in (4.8) we get:

$$
\begin{aligned}
h^\sigma_\mu \nabla_\nu\left((p+\rho)u^\mu u^\nu + pg^{\mu\nu}\right) &= h^\sigma_\mu \nabla_\nu \left(T^{\mu\nu}_V - T^{\mu\nu}_h\right) && (4.9a)\\
(\rho+p)a^\sigma + h^\sigma_\nu \partial^\nu p &= h^\sigma_\mu \nabla_\nu\left(+\xi\theta h^{\mu\nu} + 2\eta\sigma^{\mu\nu} - 2w^{(\mu}u^{\nu)}\right) && (4.9b)\\
&= \underbrace{\nabla_\nu T^{\sigma\nu}_V - u^\sigma\left(\frac{\xi\theta^2}{3} + 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu}\right)}_{\mathscr{F}^\sigma_V} \underbrace{-\nabla_\nu(w^\sigma u^\nu) - w^\nu\nabla_\nu u^\sigma + a_\mu w^\mu u^\sigma}_{\mathscr{F}^\sigma_h}\,. && (4.9c)
\end{aligned}
$$

The vectors $\mathscr{F}^\sigma_{h,V}$ are relativistic forces on the fluid due respectively to heat flow and viscosity. In the perfect fluid case the RHS is zero we simply get Euler's equation from (4.9c).

4.2.4 The Second Principle in GR

Because of the conservation of the stress-energy tensor, we have:

$$
\nabla_\nu\left(u_\mu T^{\mu\nu}\right) = T^{\mu\nu}\nabla_\nu u_\mu\,. \tag{4.10}
$$

Let us also consider the *fundamental thermodynamic relation*, which follows from the first and second principles in classical thermodynamics, as a definition for the scalar entropy per unit rest mass S:

$$T\,\mathrm{d}S = \mathrm{d}\varepsilon + p\,\mathrm{d}\frac{1}{\rho_0}\,. \tag{4.11}$$

Claim 4.1. *We can mold equation (4.10) into a version of the second principle of thermodynamics*

$$T\nabla_\mu S^\mu = \xi\theta^2 + 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{w^\mu w_\mu}{\kappa T} \geq 0 \tag{4.12}$$

where we define $S^\mu = \rho_0 S u^\mu + w^\mu/T$.

Proof. We will need the decompositions of the derivative of velocity (3.18), of the stress-energy tensor (4.6), (4.7), the expression of differential entropy (4.11) and the conservation of mass $\nabla_\mu(\rho_0 u^\mu) = 0$.

First of all, the LHS of (4.10) can be greatly simplified by noticing that $u^\mu w_\mu = u^\mu w_{\mu\nu} = 0$, so it becomes

$$\begin{aligned}
\nabla_\nu\left(u_\mu T^{\mu\nu}\right) &= \nabla_\nu\left(w\,\underbrace{u_\mu u^\mu}_{-1}u^\nu + \underbrace{u_\mu w^\mu}_{0}u^\nu + \underbrace{u_\mu u^\mu}_{-1}w^\nu + \underbrace{u_\mu w^{\mu\nu}}_{0}\right) && (4.13a)\\
&= \nabla_\nu\left(-u^\nu\rho_0(1+\varepsilon) - w^\nu\right) && (4.13b)\\
&= -\rho_0 u^\nu\partial_\nu\varepsilon - \nabla_\nu w^\nu\,. && (4.13c)
\end{aligned}$$

In the RHS of (4.10) many terms are cancelled as well because they contain contractions of space and timelike indices: we get

$$\begin{aligned}
\left(\nabla_\nu u_\mu\right)T^{\mu\nu} &= \left(\omega_{\nu\mu} + \sigma_{\nu\mu} + \frac{1}{3}\theta h_{\nu\mu} - a_\mu u_\nu\right)\left(wu^\mu u^\nu + w^\mu u^\nu + u^\mu w^\nu + w^{\mu\nu}\right) && (4.14a)\\
&= w^{\mu\nu}\left(\omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{\theta h_{\mu\nu}}{3}\right) + a_\mu w^\mu && (4.14b)\\
&= \left((p-\xi\theta)h^{\mu\nu} - 2\eta\sigma^{\mu\nu}\right)\left(\omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{\theta h_{\mu\nu}}{3}\right) + a_\mu\left(-\kappa h^\mu_\sigma\left(\partial^\sigma T + Ta^\sigma\right)\right) && (4.14c)\\
&= (p-\xi\theta)\theta - 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} - \kappa a_\mu\partial^\mu T - \kappa T a_\mu a^\mu\,. && (4.14d)
\end{aligned}$$

So far, we have:

$$-\rho_0 u^\nu\partial_\nu\varepsilon - \nabla_\nu w^\nu = (p-\xi\theta)\theta - 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} - \kappa a_\mu\partial^\mu T - \kappa T a_\mu a^\mu\,. \tag{4.15}$$

Let us rearrange (4.15) in a convenient way:

$$+\rho_0 u^\nu\partial_\nu\varepsilon + p\theta = -\nabla_\nu w^\nu + \xi\theta^2 + 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \kappa a_\mu\partial^\mu T + \kappa T a_\mu a^\mu\,. \tag{4.16}$$

Now let us consider a quantity we wish to obtain from these manipulations: $T\nabla_\mu S^\mu$. It can be expanded using the continuity equation into:

$$T\nabla_\mu S^\mu = T\nabla_\mu\left(\rho_0 S u^\mu + \frac{1}{T}w^\mu\right) = T\rho_0 u^\mu\partial_\mu S + \nabla_\mu w^\mu - w^\mu\frac{\nabla_\mu T}{T}\,. \tag{4.17}$$

We can turn the differentials in (4.11) into proper-time derivatives: $\mathrm{d} \to \mathrm{d}/\mathrm{d}\tau = u^\mu\partial_\mu$. Also, we can use the continuity equation to see that $u^\mu\partial_\mu\rho_0 = -\rho_0\theta$. Then (4.11) becomes:

$$T\frac{\mathrm{d}S}{\mathrm{d}\tau} = \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} - \frac{p}{\rho_0^2}\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} = \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} + \frac{p\theta}{\rho_0}\,. \tag{4.18}$$

So we can write the LHS of (4.16), using the identities in equation (4.17) and (4.18):

$$\rho_0 \left(u^\nu \partial_\nu \varepsilon + \frac{p\theta}{\rho_0} \right) = \rho_0 T u^\nu \partial_\nu S = T \nabla_\mu S^\mu - \nabla_\mu w^\mu + \frac{1}{T} w^\mu \nabla_\mu T \,. \tag{4.19}$$

Let us substitute (4.19) into (4.16), and then subtract the desired result (4.12) from the equation: this way, if we get an identity the proof will be complete (this may seem circular, but it is done just for convenience in the algebraic manipulations: to get a more rigorous argument one may just reverse the steps, using the identity (4.20b) in equation (4.20a) to get equation (4.12)).

$$T \nabla_\mu S^\mu - \nabla_\mu w^\mu + \frac{1}{T} w^\mu \nabla_\mu T = -\nabla_\nu w^\nu + \xi \theta^2 + 2\eta \sigma_{\mu\nu} \sigma^{\mu\nu} + \kappa a_\mu \partial^\mu T + \kappa T a_\mu a^\mu \tag{4.20a}$$

$$-\nabla_\mu w^\mu + \frac{1}{T} w^\mu \nabla_\mu T = -\nabla_\nu w^\nu + \kappa a_\mu \partial^\mu T + \kappa T a_\mu a^\mu - \frac{w^\mu w_\mu}{\kappa T} \tag{4.20b}$$

$$+\frac{1}{T} w^\mu \nabla_\mu T = +\kappa a_\mu \partial^\mu T + \kappa T a_\mu a^\mu - \frac{w^\mu w_\mu}{\kappa T} \,. \tag{4.20c}$$

The last term in (4.20c) looks like:

$$\frac{w^\mu w_\mu}{\kappa T} = \frac{1}{\kappa T} \kappa^2 h^\sigma_\mu h^{\mu\nu} (\partial_\sigma T + T a_\sigma)(\partial_\nu T + T a_\nu) = \kappa \left(\frac{h^{\mu\nu}}{T} \partial_\mu T \partial_\nu T + 2 a^\mu \partial_\mu T + T a_\mu a^\mu \right) . \tag{4.21}$$

Inserting the identity in (4.21) and making the last w^μ explicit in (4.20c) we get:

$$-\frac{1}{T} \kappa h^\mu_\sigma (\partial^\sigma T + T a^\sigma) \partial_\mu T = +\kappa a_\mu \partial^\mu T + \kappa T a_\mu a^\mu - \kappa \left(\frac{h^{\mu\nu}}{T} \partial_\mu T \partial_\nu T + 2 a^\mu \partial_\mu T + T a_\mu a^\mu \right) \tag{4.22a}$$

$$+\frac{1}{T} h^{\mu\nu} \partial_\nu T \partial_\mu T + h^{\mu\nu} a_\nu \partial_\mu T = -a_\mu \partial^\mu T - T a_\mu a^\mu + \left(\frac{h^{\mu\nu}}{T} \partial_\mu T \partial_\nu T + 2 a^\mu \partial_\mu T + T a_\mu a^\mu \right) \tag{4.22b}$$

$$0 = -a_\mu \partial^\mu T - T a_\mu a^\mu + \left(a^\mu \partial_\mu T + T a_\mu a^\mu \right) \tag{4.22c}$$

Thus we have proved the equation in (4.12), the inequality follows directly from the fact that we are considering square moduli of spacelike vectors, and the coefficients such as ξ are assumed to be positive. □

If we assume that the fluid is in equilibrium ($\nabla_\mu S^\mu = 0$) then we must have $\theta = 0$ (no compression), $\sigma_{\mu\nu} = 0$ (no shear stresses), $w_\mu = 0$: the interpretation of this last equation is slightly harder, but it is equivalent to the statement that, in the LRF, the log-temperature gradient is purely spatial and it defines the acceleration, by $a_\sigma = -\partial_\sigma \log T$.

4.2.5 Ideal fluids

They are fluids with $\eta = \xi = \kappa = 0$, that is, without viscosity (neither compressive nor shear) nor heat transmission. They are described by the following stress-energy tensor:

$$T^{\mu\nu} = \rho u^\mu u^\nu + p h^{\mu\nu} = \rho_0 h u^\mu u^\nu + p g^{\mu\nu} \tag{4.23}$$

where $h = (p + \rho)/\rho_0$ is the specific enthalpy. If our fluid is ideal then the RHS of (4.12) is zero and so is w^μ, therefore $T \nabla_\mu S^\mu = T \nabla_\mu (\rho_0 S u^\mu) = T \rho_0 u^\mu \partial_\mu S$ by the continuity equation. So, S is conserved along the world-lines of the fluid.

Also, the RHS of (4.9c) is zero, therefore we get the Euler equation:

$$(p + \rho) a^\mu + h^{\mu\nu} \partial_\nu p = 0 \,. \tag{4.24}$$

4.2.6 Speed of sound

Following [Yos11].

The definition of the adiabatic speed of sound is $v_s^2 = (\partial p / \partial \rho)_s$: here we give a justification for it.

We work in Minkowski spacetime, where $g_{\mu\nu} = \eta_{\mu\nu}$, and with an ideal fluid, for which $T_{\mu\nu} = (p+\rho)u_\mu u_\nu + p\eta_{\mu\nu}$. Then, the equations of conservation of mass and momentum read:

$$\rho_0 \partial_\mu u^\mu + u^\mu \partial_\mu \rho_0 = 0 \qquad \text{mass} \tag{4.25a}$$

$$u^\mu \left(\partial_\mu \rho - h \partial_\mu \rho_0 \right) = 0 \qquad \text{momentum along } u^\mu \tag{4.25b}$$

$$(p+\rho) a^\mu + h^{\mu\rho} \partial_\rho p = 0 \qquad \text{momentum in the span of } h_{\mu\nu} \tag{4.25c}$$

where h is the specific enthalpy. If we assume small perturbations $p \to p + \delta p$, $\rho_0 \to \rho_0 + \delta \rho_0$, $\rho \to \rho + \partial \rho$, $u^\mu \to (1, \delta u^x, 0, 0)$ (the normalization condition is satisfied to first order in δu^x) we get our three equations, up to first order in the perturbations:

$$\partial_x (\delta u^x) = -\frac{\partial_t (\delta \rho_0)}{\rho_0} \tag{4.26a}$$

$$\partial_t (\delta \rho) - h \partial_t (\delta \rho_0) = 0 \tag{4.26b}$$

$$-(p+\rho) \partial_t (\delta u^x) = \partial_x p \,. \tag{4.26c}$$

We manipulate these by differentiating and substituting, in order to eliminate the dependence on δu^x and ρ_0, and get:

$$\partial_t \delta \rho + (p+\rho) \partial_x (\partial u^x) = 0 \tag{4.27a}$$

$$\partial_x \delta p + (p+\rho) \partial_t (\partial u^x) = 0 \tag{4.27b}$$

which simplifies to $\partial_{tt}^2 \delta \rho - \partial_{xx}^2 \delta p = 0$.

We can write this as the wave equation $\left(v_s^{-2} \partial_{tt}^2 - \partial_{xx}^2 \right) \delta p = 0$, where we to define $v_s^2 = \partial p / \partial \rho$, the square of the characteristic velocity of propagation v_s.

4.3 Bondi accretion: the adiabatic case

We now apply the formalism developed in this section to the problem of spherical accretion into a black hole the *adiabatic* case, where we model the gas as an ideal fluid.

We assume the geometry of the spacetime is described by the Schwarzschild metric (3.25) and make the following assumptions:

1. spherical symmetry: $\partial_\theta = \partial_\varphi = 0$;

2. stationarity: $\partial_t = 0$.

4.3.1 Fiducial congruence

The comoving tetrad defined by the fluid's motion is called the *fiducial congruence reference*.

Claim 4.2. *When written with respect to the usual spherical coordinates (t, r, θ, φ) the fiducial congruence looks like:*

$$\hat{t} = e_t = u^\mu = \left(\gamma^2/y, -yv, 0, 0\right) \tag{4.28a}$$

$$\hat{r} = e_r = a^\mu/\sqrt{a^\rho a_\rho} = \left(-v\gamma^2/y, y, 0, 0\right) \tag{4.28b}$$

$$\hat{\theta} = e_\theta = (0, 0, 1/r, 0) \tag{4.28c}$$

$$\hat{\varphi} = e_\varphi = \left(0, 0, 0, 1/(r\sin(\theta))\right) \tag{4.28d}$$

where γ is the Lorentz factor ($\gamma = 1/\sqrt{1-v^2}$), and $y = \gamma\sqrt{1+2\Phi} = \gamma\sqrt{1-2M/r}$ is the "energy-at-infinity per unit rest mass" (see [TFŻ81, equation 3]).

Proof. First of all, we prove that we can form a comoving tetrad by rescaling the vectors $(u^\mu, a^\mu, e_\theta, e_\varphi)$. They are clearly orthogonal, let us show that they are Fermi-Walker transported (see equation (3.10)); notice that the condition of being FW-transported is 1-homogeneous, so proving it before or after normalization makes no difference. For the velocity $u^\mu \propto \hat{t}$ we have

$$u^\nu \nabla_\nu u^\mu = a^\mu = u_\rho(u^\mu a^\rho - u^\rho a^\mu) = -(u_\rho u^\rho)a^\mu\,. \tag{4.29}$$

For the acceleration $a^\mu \propto \hat{r}$ we need the following identity: $0 = \mathrm{d}(u^\mu a_\mu)\big/\mathrm{d}\tau = a^\mu a_\mu + u^\mu\,\mathrm{d}a_\mu\big/\mathrm{d}\tau$. We multiply this by u^μ to get: $u^\mu(a^\rho a_\rho) = \mathrm{d}a^\mu/\mathrm{d}\tau$.

Then, we can prove:

$$u^\nu \nabla_\nu a^\mu = a_\rho(u^\mu a^\rho) = a_\rho(u^\mu a^\rho - u^\rho a^\mu)\,. \tag{4.30}$$

In the proof for the $\hat{\theta}$ and $\hat{\varphi}$ vectors, the RHS vanishes immediately since the time-radial surface to which the velocity and acceleration belong is orthogonal to the sphere's surface; the LHS instead vanishes since it is a derivative with respect to proper time, therefore along the fluid's flow lines, which all lie in the time-radial surface.

Now, we need to show that the velocity and the acceleration actually have the form shown in (4.28): for the velocity, it is enough to impose the normalization $u^\mu u_\mu = -1$, and to consider an observer with velocity k^μ who is stationary with respect to the spherical coordinates: by normalization their 4-velocity will be $k^\mu = (1/\sqrt{g_{00}}, 0, 0, 0)$, and the local transformation between the frames will be a Lorentz transformation with factor γ defined by the fluid's velocity, therefore it must hold that $\gamma = -k^\mu u_\mu$.

Since $g_{00} = -y^2/\gamma^2$ we have:

$$\gamma = -k^\mu u^\nu g_{\mu\nu} = -u^0 \frac{g_{00}}{\sqrt{-g_{00}}} = u^0\sqrt{-g_{00}} = u^0\frac{y}{\gamma} \tag{4.31}$$

which gives us $u^0 = \gamma^2/y$, while for u^1 we must impose (already knowing that $u^2 = u^3 = 0$):

$$-1 = (u^0)^2 g_{00} + (u^1)^2 g_{11} \tag{4.32a}$$

$$= -\left(\frac{\gamma^2}{y}\right)^2\left(\frac{y}{\gamma}\right)^2 + (u^1)^2\frac{\gamma^2}{y^2} \tag{4.32b}$$

$$-1+\gamma^2 = (u^1)^2\frac{\gamma^2}{y^2} \tag{4.32c}$$

$$y^2\left(-\frac{1}{\gamma^2}+1\right) = (u^1)^2 \tag{4.32d}$$

$$y^2v^2 = (u^1)^2\,. \tag{4.32e}$$

As for a^μ, we need not compute the covariant derivative since the two components of the normalized vector we want are determined by $u^\mu a_\mu = 0 = u^\mu \hat{r}^\nu g_{\mu\nu}$ and the normalization condition $\hat{r} \cdot \hat{r} = 1$. The first one translates to $\hat{r}^0 y^2 + \hat{r}^1 v \gamma^2 = 0$ and the second one to $(\hat{r}^0)^2(-y^2/\gamma^2) + (\hat{r}^1)^2(\gamma^2/y^2) = 1$, since we know the other two components of the acceleration are zero. Solving this system for the two unknown components yields precisely the desired expression.

The $\hat{\theta}$ and $\hat{\varphi}$ vectors just need to be rescaled, and they will need to become $1/\sqrt{g_{22}}$ and $1/\sqrt{g_{33}}$ respectively. □

The first equation we consider is the conservation of mass: if ρ_0 is the rest mass density of the fluid, we must have $\nabla_\mu(\rho_0 u^\mu) = 0$. This, using the formula for covariant divergence (3.6), yields:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho_0 y v r^2\right) = 0\,. \tag{4.33}$$

In the newtonian limit both γ and y approach 1; also, the infalling mass rate $\dot{M}$ at a certain radius is $\rho_0(r)v(r)4\pi r^2$. Then, by continuity to the newtonian limit, the quantity which is constant with respect to the radius must be $\dot{M}/(4\pi)$: therefore

$$\dot{M} = 4\pi\rho_0 y v r^2\,. \tag{4.34}$$

The second equation we consider is the Euler equation (4.24), which follows from the spatial projection of the conservation of the stress-energy tensor: because of spherical symmetry, the only nontrivial component of this is the radial one, so we need to calculate $a^1 = u^\mu \nabla_\mu u^1 = \mathrm{d}u^1 \big/ \mathrm{d}\tau + \Gamma^1_{\mu\nu} u^\mu u^\nu$. To do this we will need the radial Schwarzschild Christoffel coefficients:

$$\Gamma^1_{\mu\nu} = \begin{bmatrix} \frac{M(-2M+r)}{r^3} & 0 & 0 & 0 \\ 0 & \frac{M}{r(2M-r)} & 0 & 0 \\ 0 & 0 & 2M-r & 0 \\ 0 & 0 & 0 & (2M-r)\sin^2(\theta) \end{bmatrix} \tag{4.35}$$

while the proper-time derivative is $\mathrm{d}/\mathrm{d}\tau = u^\mu \partial_\mu = y v \partial_1$. Plugging in the expression for the only relevant component of $h^{\mu\nu}$, $h^{11} = g^{11} + u^1 u^1 = (1+2\Phi)(1+v^2\gamma^2) = y^2$ we get, after a lengthy computation,

$$a^1 = y^2\left(\gamma^2 v \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2}\right). \tag{4.36}$$

Substituting this into the (radial component of the) Euler equation (4.24) we get

$$(p+\rho)y^2\left(\gamma^2 v \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2}\right) = -h^{11}\partial_1 p = -y^2 \partial_1 p \tag{4.37a}$$

$$\gamma^2 v \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2} + \frac{1}{p+\rho}\frac{\mathrm{d}p}{\mathrm{d}r} = 0\,. \tag{4.37b}$$

The third equation we consider is the projection of the conservation of the ideal fluid stress energy tensor onto the 4-velocity, $-u_\mu \nabla_\nu T^{\mu\nu} = 0$, which can be written as $\mathrm{d}\rho/\mathrm{d}\tau + (p+\rho)\theta = 0$; using the conservation of mass, it can be cast into $\mathrm{d}\rho_0/\mathrm{d}\tau + \rho_0\theta = 0$ or $\theta = -\rho_0^{-1}\,\mathrm{d}\rho_0/\mathrm{d}\tau$ therefore we get an equation for the variation of the total internal energy, which holds for ideal fluids at constant entropy:

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \frac{p+\rho}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} \qquad \text{or} \qquad \left(\frac{\partial\rho}{\partial\rho_0}\right)_s = \frac{p+\rho}{\rho_0} \stackrel{\text{def}}{=} h \tag{4.38}$$

where h is the specific enthalpy.

4.3.2 The Bernoulli equation

From these we can show that

Claim 4.3. *The quantity $\gamma h\sqrt{1+2\Phi} = yh$, is a constant of motion.*

Proof. First of all, by direct computation it can be shown from the definition of y that

$$\gamma^2 v \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{M}{(1+2\Phi)r^2} = \frac{\mathrm{d}\log y}{\mathrm{d}r}\,. \tag{4.39}$$

Then, following Gourgoulhon [Gou06, section 6.3] we find that $\mathrm{d}p = \rho_0\,\mathrm{d}h$ in the isentropic case, therefore

$$\frac{1}{\rho+p}\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\mathrm{d}\log h}{\mathrm{d}r} \tag{4.40}$$

so we can substitute the results in (4.39) and (4.40) into (4.37b):

$$\frac{\mathrm{d}\log h}{\mathrm{d}r} + \frac{\mathrm{d}\log y}{\mathrm{d}r} = \frac{\mathrm{d}\log(hy)}{\mathrm{d}r} = 0\,. \tag{4.41}$$

□

In the nonrelativistic, weak-field limit this becomes the classical conservation of density of energy:

$$\gamma h\sqrt{1+2\Phi} \approx \frac{p}{\rho_0} + \frac{v^2}{2} - \frac{M}{r} + \epsilon = \text{const}\,. \tag{4.42}$$

4.3.3 The full adiabatic equations of motion

We want to write the equations of motion with the formalism of logarithmic derivatives: we express all the derivatives which were with respect to r, $\mathrm{d}/\mathrm{d}r$, with derivatives with respect to $\log r$, which properly speaking would be ill-defined but we understand to mean $\mathrm{d}/\mathrm{d}\log r \overset{\text{def}}{=} r\,\mathrm{d}/\mathrm{d}r$; a more formal approach to this definition would be to use an adimensional radial coordinate $r/(2M)$, but for consistency with [Nob00] I will not use that notation.

We can recast the mass conservation equation (4.33) using logarithmic derivatives, since when a quantity has zero derivative its logarithm also does:

$$\frac{\mathrm{d}\log\rho_0}{\mathrm{d}\log r} + \frac{\mathrm{d}\log yv}{\mathrm{d}\log r} + 2 = 0\,. \tag{4.43}$$

With the same approach we can reframe all the equations we found as the following system, in which we introduce the notation of primes denoting derivatives with respect to $\log r$:

$$\frac{y'}{y} + \frac{p'}{p+\rho} = 0 \qquad \text{Euler equation} \tag{4.44a}$$

$$\rho' - h\rho_0' = 0 \qquad \text{energy equation} \tag{4.44b}$$

$$\frac{(yv)'}{yv} + \frac{\rho_0'}{\rho_0} + 2 = 0 \qquad \text{mass conservation.} \tag{4.44c}$$

In order to reframe these we rewrite the gradients of ρ and P in terms of ρ_0 and T: the following equations are just a useful way to reframe the multivariate chain rule.

$$\frac{\rho'}{p+\rho} = A\frac{\rho_0'}{\rho_0} + B\frac{T'}{T} \tag{4.45a}$$

$$\frac{p'}{p+\rho} = a\frac{\rho_0'}{\rho_0} + b\frac{T'}{T} \tag{4.45b}$$

where the parameters A, B, a and b are defined by:

$$A = \frac{\rho_0}{p+\rho}\left(\frac{\partial\rho}{\partial\rho_0}\right)_T \qquad B = \frac{T}{p+\rho}\left(\frac{\partial\rho}{\partial T}\right)_{\rho_0} \tag{4.46a}$$

$$a = \frac{\rho_0}{p+\rho}\left(\frac{\partial p}{\partial\rho_0}\right)_T \qquad b = \frac{T}{p+\rho}\left(\frac{\partial p}{\partial T}\right)_{\rho_0}; \tag{4.46b}$$

the subscripts T and ρ_0 on the derivatives mean that the denoted quantity should be held constant when differentiating.

These are related by the reciprocity relation [Fla82, eq. B3]: $A + b = 1$.

When inserting these relations into equation (4.44b) we get:

$$\begin{aligned}
0 &= \left(A\frac{\rho_0'}{\rho_0} + B\frac{T'}{T} - \frac{\rho_0'}{\rho_0}\right)(p+\rho) && (4.47a)\\
&= \frac{T'}{T} + \frac{A-1}{B}\frac{\rho_0'}{\rho_0} && (4.47b)\\
&= \frac{T'}{T} - \frac{b}{B}\frac{\rho_0'}{\rho_0} && (4.47c)\\
&= \frac{T'}{T} - (\Gamma-1)\frac{\rho_0'}{\rho_0} && (4.47d)
\end{aligned}$$

where we define the *local adiabatic exponent* $\Gamma = 1 + b/B$.

When we insert the relations into equation (4.44a) we get:

$$\begin{aligned}
0 &= \frac{y'}{y} + a\frac{\rho_0'}{\rho_0} + b\frac{T'}{T} && (4.48a)\\
&= \frac{y'}{y} + a\frac{\rho_0'}{\rho_0} + \frac{b^2}{B}\frac{\rho_0'}{\rho_0} && (4.48b)\\
&= \frac{y'}{y} + v_s^2\left(-2 - \frac{(yv)'}{yv}\right) && (4.48c)\\
&= \left(\frac{(yv)'}{yv} - \frac{v'}{v}\right) + -v_s^2\left(2 + \frac{(yv)'}{yv}\right) && (4.48d)\\
&= (v^2 - v_s^2)\frac{(yv)'}{yv} - 2v_s^2 + \frac{M}{y^2 r} && (4.48e)
\end{aligned}$$

where: we used the fact that $v_s^2 = \left(\partial p/\partial\rho\right)_s = a + b^2/B = a + b(\Gamma-1)$ [Fla82, eq. B12], the energy equation (4.47d), the conservation of mass equation (4.43), and the identity we now derive, starting from the expression for the logarithmic derivative of y (4.39):

$$\begin{aligned}
\frac{y'}{y} &= \gamma^2 v^2\frac{v'}{v} + \frac{M\gamma^2}{y^2 r} && (4.49a)\\
&= (\gamma^2-1)\frac{v'}{v} + \frac{M\gamma^2}{y^2 r} && (4.49b)\\
&= \gamma^2\left(\frac{v'}{v} + \frac{M}{y^2 r}\right) - \frac{v'}{v} && (4.49c)
\end{aligned}$$

$$(1-v^2)\frac{(yv)'}{yv} = \frac{v'}{v} + \frac{M}{y^2 r} \tag{4.49d}$$

$$\frac{(yv)'}{yv} - \frac{v'}{v} = v^2\frac{(yv)'}{yv} + \frac{M}{y^2 r} \tag{4.49e}$$

$$\frac{y'}{y} = v^2\frac{(yv)'}{yv} + \frac{M}{y^2 r} \tag{4.49f}$$

The system which describes the motion is then given by equations (4.47d), (4.48e) and (4.44c), which remains unchanged. This is a system of three first-order differential equations in the varibles T, yv and ρ_0 (y and v can be recovered since we have an explicit definition in the form $y = y(v)$), which would ordinarily need three boundary conditions, but there is a mechanism by which we only need two.

When $v = v_s$, the logarithmic derivative of yv term in (4.48e) vanishes: then, either $r = 2v_s y^2/M$ or the logarithmic derivative diverges. Since the latter condition is unphysical, we must consider the former, which defines a radius r_s, and impose $v(r_s) = v_s$.

This constrains the accceptable solutions, and allows the accretive motion to be fully determined by just imposing two conditions, such as $\rho_0(r \to \infty) = \rho_\infty$ and $T(r \to \infty) = T_\infty$.

5 Radiative effects in spherical accretion

5.1 Thorne's PSTF moment formalism

The PSTF moment formalism was developed by Thorne [Tho81] in order to give a way to consider the radiation energy transfer problem to an arbitrarily high degree of accuracy: it is fully relativistic, and — in the presence of certain simmetries, such that it is possible to use its scalar-moments version, which will be developed in section 5.2.3 — it gives rise to an infinite number of ordinary linear first-order differential equations for the various scalar moments, which can be truncated at a certain order.

We start by introducing the mathematical operations which will need to be applied to the moment tensors.

Given any tensor $A^{\mu_1\cdots\mu_k} = A^{M_k}$ we can use the tensor $h^{\mu\nu}$ to project it into the space-like subspace defined by the velocity u^μ:

$$A^{M_k} \to \left(A^{M_k}\right)^P = \left(\prod_{i=1}^{k} h^{\mu_i}_{\nu_i}\right) A^{M_k} . \tag{5.1}$$

Then, we can take the symmetric part of any tensor as outlined in 'Notational preface' on page 3:

$$A^{M_k} \to \left(A^{M_k}\right)^S = A^{(M_k)} . \tag{5.2}$$

We can select the trace-free part of a projected, symmetric tensor by

$$A^{\mu_1\cdots\mu_k} \to \left(A^{\mu_1\cdots\mu_k}\right)^{TF} = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \frac{k!(2k-2i-1)!!}{(k-2i)!(2k-1)!!(2i)!!} h^{(\alpha_1\alpha_2} \ldots h^{\alpha_{2i-1}\alpha_{2i}} A^{\alpha_{2i+1}\ldots\alpha_k)\beta_1\ldots\beta_i}{}_{\beta_1\ldots\beta_i} . \tag{5.3}$$

To see what this is doing, let us consider its action on a rank-two projected tensor: it is just the subtraction of its trace,

$$A^{\mu\nu} \to A^{\mu\nu} - \frac{1}{3} h^{\mu\nu} A^\rho_\rho . \tag{5.4}$$

Now, let us consider all the unit vectors n^μ in the space normal to the velocity, which have $n_\mu u^\mu = 0$ and $n^\mu n_\mu = 1$. They span the sphere S^2.

If we have a function $F \colon S^2 \to \mathbb{R}$, we can decompose it into harmonics as:

$$F(n) = \sum_{k=0}^{\infty} \mathscr{F}_{\alpha_1 \dots \alpha_k} \prod_{i=0}^{k} n^{\alpha_i} \tag{5.5}$$

where the PSTF moments $\mathscr{F}_{\alpha_1 \dots \alpha_k}$ can be computed as

$$\mathscr{F}_{\alpha_1 \dots \alpha_k} = \frac{(2k+1)!!}{4\pi k!} \left(\int F \prod_{i=0}^{k} n^{\alpha_i} \, \mathrm{d}\Omega \right)^{TF} . \tag{5.6}$$

In particular, the function we will apply this to is the distribution of EM radiation around the BH. So, we consider a photon, whose trajectory in spacetime is parameterized as $\gamma(\xi)$, with a choice of ξ such that the photon's momentum is $p = \mathrm{d}/\mathrm{d}\xi$.

Now, our observer has a timelike velocity u^μ. We can find a spacelike vector n^μ corresponding to the space-like part of the movement of the photon, or

$$p^\mu = (-u^\nu p_\nu)(u^\mu + n^\mu) . \tag{5.7}$$

It must hold that $u^\mu u_\mu = -1$ while $n^\mu n_\mu = +1$ in order for p^μ to be null-like. Now, we define a parameter l which corresponds to the space distance the photon moved through in this frame (l is *not* covariant!)

$$l = \int (-u^\nu p_\nu) \, \mathrm{d}\xi \tag{5.8}$$

now, $\mathrm{d}/\mathrm{d}l$ is parallel to p but it has different length, in fact since $\mathrm{d}l/\mathrm{d}\xi = (-u^\nu p_\nu)$ it is $\mathrm{d}/\mathrm{d}l = (n^\mu + u^\mu)\partial_\mu$.

It holds ([Tho81, eq. 2.17], with the notation from (3.18)), that

$$\frac{\mathrm{d}\nu}{\mathrm{d}l} = (u^\mu + n^\mu)\nabla_\mu(-p^\nu u_\nu) = -\nu \left(n_\mu a^\mu + \frac{\theta}{3} + n_\mu n_\nu \sigma^{\mu\nu} \right) . \tag{5.9}$$

We want to quantify the number density of photons in relation to their momentum. We assume the radiation in unpolarized, therefore for each unit h^3 cell in phase space there can be 2 photons: so we denote the distribution function of the photons as $2N(x^\mu, p^\mu)$.

It is known that the volume element $\mathrm{d}V_p = \mathrm{d}^3 p \,/ p^0$ is Lorentz invariant (see [MTW73, box 22.5]). We can write this using the photons' frequency $\nu = -p^\mu u_\mu / h$ as $\mathrm{d}V_p = \nu \, \mathrm{d}\Omega \, \mathrm{d}\nu$.

Let us define the *specific radiative intensity* as

$$I_\nu = \frac{\delta E}{\delta A \delta t \delta \nu \delta \Omega} = \frac{h\nu \delta N}{\delta A \delta t \delta \nu \delta \Omega} \tag{5.10}$$

where δA denotes an infinitesimal area the photons are coming through, δt an infinitesimal time, $\delta \nu$ an infinitesimal photon frequency, $\delta \Omega$ an infinitesimal solid angle.

Then, the number density of photons in phase space is [MTW73, figure 22.2]

$$\frac{2N(x^\mu, p^\mu)}{h^3} = \frac{\delta N}{V_x V_p} = \frac{\delta N}{h^3 \nu^2 \delta A \delta t \delta \nu \delta \Omega} = \frac{1}{h^4 \nu^3} I_\nu \tag{5.11}$$

therefore $I_\nu = 2N\nu^3 h$.

Now, we want to describe the variation of the occupation number N with respect to the photons' trajectories' parameter l. We encapsulate all possible effects into a source term $\mathfrak{S}$:

$$\mathfrak{S} \overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}l} 2N(x^\mu, p^\mu) = 2 \left(\frac{\partial N}{\partial x^\mu} \frac{\mathrm{d}x^\mu}{\mathrm{d}l} + \frac{\partial N}{\partial p^\mu} \frac{\mathrm{d}p^\mu}{\mathrm{d}l} \right) \tag{5.12}$$

since the occupation number can be thought of as just a function of the spatial components of the momentum.

Since $\mathrm{d}/\mathrm{d}l = (n^\mu + u^\mu)\partial_\mu$ and the covariant derivative of p^γ is zero, we can compute

$$\frac{\mathrm{d}p^\gamma}{\mathrm{d}l} = (n^\mu + u^\mu)\nabla_\mu p^\gamma - \Gamma^\gamma_{\alpha\beta}p^\alpha(u^\beta + n^\beta) = -\Gamma^\gamma_{\alpha\beta}p^\alpha(u^\beta + n^\beta) \tag{5.13}$$

where the covariant derivative term vanishes since the photon's trajectory is a geodesic.

5.1.1 Moments' definitions

In this subsection I will follow Thorne [Tho81] in his usage of units where $c = h = 1$.

We define the (unprojected) k-th moments of radiative transfer:

$$M_\nu^{A_k} \stackrel{\text{def}}{=} \int 2N \frac{\delta(\nu - (-p^\nu u_\nu))}{\nu^{k-2}} \prod_i^k p^{\alpha_i}\,\mathrm{d}V_p \tag{5.14a}$$

$$= \int \left(2N\nu^3\right)\frac{1}{\nu}\delta(\nu + p^\nu u_\nu)\prod_i^k\left(\frac{p^{\alpha_i}}{\nu}\right)(\nu\,\mathrm{d}\Omega\,\mathrm{d}\nu) \tag{5.14b}$$

$$= \int I_\nu \prod_i^k (n^{\alpha_i} + u^{\alpha_i})\,\mathrm{d}\Omega\ . \tag{5.14c}$$

In general, we can compute the k-th moments of any function just as here we computed those of $2N = I_\nu/\nu^3$: if we apply this procedure to the the source term $\mathfrak{S}$ we get the following moments:

$$S_\nu^{A_k} = \nu^3 \int S\mathfrak{S}\prod_i^k(n^{\alpha_i} + u^{\alpha_i})\,\mathrm{d}\Omega\ . \tag{5.15}$$

5.1.2 Redshift-adapted version

Thorne [Tho81] also defines a redshift-adapted version of the moments' definition: if R is a universal redshift functions, such that $R(p^\nu u_\nu)$ is conserved along every photon geodesic $p^\mu\nabla_\mu p^\nu = 0$, that is, R allows us to calculate the redshift between any two points A, B which are connected by a geodesic as $\nu_A/\nu_B = R_B/R_A$.

Then, we define $M_f^{A_k} = M_\nu^{A_k}/R$.

5.1.3 Frequency-integrated version

We may not wish to consider the frequency dependence of the radiation, but instead to treat all radiative transfer "in bulk": to this end, we define the frequency-integrated moments:

$$M^{A_k} = \int M_\nu^{A_k}\,\mathrm{d}\nu \tag{5.16}$$

and the same is applied to the source moments $S_\nu^{A_k} \to S^{A_k}$.

Since this includes the radiation intensity from all frequencies, we have direct interpretations for the first moments:

$$M = \int I_\nu\,\mathrm{d}\Omega\,\mathrm{d}\nu \qquad \text{energy density of radiation} \tag{5.17a}$$

$$M^{\alpha} = \int I_{\nu}(n^{\alpha} + u^{\alpha})\, d\Omega\, d\nu \qquad (M^0, M^i) = (\text{energy density of radiation, energy flux}) \tag{5.17b}$$

$$M^{\alpha\beta} = \int I(n^{\alpha} + u^{\alpha})(n^{\beta} + u^{\beta})\, d\Omega\, d\nu \qquad \text{stress-energy tensor of radiation.} \tag{5.17c}$$

5.1.4 The moment equations

These can be derived from the transport equation, see [Tho81, eq. 3.14]. I present them only in the grey (frequency-integrated) case:

$$\nabla_{\beta} M^{A_k \beta} - (k-1) M^{A_k \beta\gamma} (\nabla_{\gamma} u_{\beta}) = S^{A_k}\,. \tag{5.18}$$

The moments (the frequency-integrated M^{A_k}, but also the full moments $M_{\nu}^{A_k}$ and the redshift-adapted ones $M_f^{A_k}$) satisfy the following:

$$M^{A_k\beta}{}_{\beta} = 0 \tag{5.19a}$$

$$u_{\beta} M^{A_k \beta} = -M^{A_k} \tag{5.19b}$$

$$h_{\beta\gamma} M^{A_k \beta\gamma} = M^{A_k}\,. \tag{5.19c}$$

So, the k-th moment contains all the information about the l-th moments with $l \leq k$; also, to get lower-order moments we take partial traces onto space- and time-like subspaces: therefore the unique information to the k-th moment, which is not redundantly expressed in lower-order moments, is in its PSTF part:

$$\mathscr{M}^{A_k} = \left(M^{A_k}\right)^{PSTF}\,. \tag{5.20}$$

The same can be applied to $M_{\nu}^{A_k}$ and $M_f^{A_k}$, to the moment equations (5.18) and to the source moments $S^{A_k} \to \mathscr{S}^{A_k}$. Since we are taking the projection onto the space-like subspaces, we can simplify the expression of the PSTF moments: all the terms which contain at least a four-velocity vanish, therefore:

$$\mathscr{M}^{A_k} = \left(\int I \prod_i n^{\alpha_i}\, d\Omega\right)^{TF} \tag{5.21}$$

where $I = \int I_{\nu}\, d\nu$. The first *PSTF* moments also have physical interpretations:

$$\mathscr{M} = \int I\, d\Omega \qquad \text{energy density of radiation} \tag{5.22a}$$

$$\mathscr{M}^{\alpha} = \int I n^{\alpha}\, d\Omega \qquad \text{energy flux of radiation} \tag{5.22b}$$

$$\mathscr{M}^{\alpha\beta} = \int I n^{\alpha} n^{\beta}\, d\Omega \qquad \text{shears in the stress-energy tensor of radiation.} \tag{5.22c}$$

We can write the stress-energy tensor $T^{\mu\nu} = M^{\mu\nu}$ with the PSTF moments (see [Tho81, eq. 4.9]):

$$T^{\mu\nu} = \mathscr{M} u^{\mu} u^{\nu} + 2\mathscr{M}^{(\mu} u^{\nu)} + \mathscr{M}^{\mu\nu} + \frac{1}{3}\mathscr{M} h^{\mu\nu} \tag{5.23}$$

and compare these to the expression of the components of the stress-energy tensor (4.7) to get the following identifications:

$$\mathscr{M} = w = \rho \tag{5.24a}$$
$$\mathscr{M}^{\mu} = w^{\mu} = -\kappa h^{\mu}_{\sigma}(\partial^{\sigma} T + T a^{\sigma}) \tag{5.24b}$$
$$\mathscr{M}^{\mu\nu} + \frac{1}{3}\mathscr{M} h^{\mu\nu} = (p - \zeta\theta) h^{\mu\nu} - 2\eta\sigma^{\mu\nu} \tag{5.24c}$$

and since the photons' paths are geodesics in this case $\theta = 0$, for the components proportional to $h^{\mu\nu}$ of equation (5.24c) we just get $\rho = {}^{1}/{}_{3}p$, which is what we expect for a photon gas. For the traceless part of the equation, we get $\mathscr{M}^{\mu\nu} = -2\eta\sigma^{\mu\nu}$.

5.1.5 The PSTF moment equations

We want to express the grey moment equations (5.18) in terms of the PSTF moments. This can be done as follows: an expression can be found for the full moments in terms of the PSTF moments in [Tho81, eq. 4.10c]:

$$M^{A_k} = \sum_{l=0}^{k} \sum_{j=0}^{\lfloor \frac{k-l}{2} \rfloor} \frac{1}{(2j)!!(k-l-2j)!} \frac{k!}{l!} \frac{(2l+1)!!}{(2l+1+2j)!!} \mathscr{M}^{(A_l} \prod_{i=l+1}^{l+2j-1} h^{\alpha_i \alpha_{i+1}} \prod_{x=l+2j+1}^{k} u^{\alpha_x)} \tag{5.25}$$

where all the indices of the $\mathscr{M}$, h and u are meant to be symmetrized.

We insert this into the moment equations and expand, making use of the decomposition of the covariant derivative of the 4-velocity (3.18).

Then, we should take the PSTF part of the equations. This yields a very complicated expression, so here I record only the implicit formula given in Thorne [Tho81, eq. 4.11c]:

$$\begin{aligned}\Bigg(&\nabla_{\beta}\mathscr{M}^{A_k\beta} + u^{\beta}\nabla_{\beta}\mathscr{M}^{A_k} + \frac{k}{2k+1}\nabla_{\alpha_k}\mathscr{M}^{A_{k-1}} - (k-1)\mathscr{M}^{A_k\beta\gamma}\sigma_{\beta\gamma} - (k-1)\mathscr{M}^{A_k\beta}a_{\beta} + \frac{4}{3}\mathscr{M}^{A_k}\theta \\ &+ \frac{5k}{2k+3}\mathscr{M}^{A_{k-1}\beta}\sigma^{\alpha_k}_{\beta} - k\mathscr{M}^{A_{k-1}\beta}\omega^{\alpha_k}_{\beta} + \frac{k(k+3)}{2k+1}\mathscr{M}^{A_{k-1}}a^{\alpha_k} + \frac{(k-1)k(k+2)}{(2k-1)(2k+1)}\mathscr{M}^{A_{k-2}}\sigma^{\alpha_{k-1}\alpha_k}\Bigg)^{PSTF} = \mathscr{S}^{A_k}\,.\end{aligned} \tag{5.26}$$

5.1.6 How to recover the intensity

Once one has solved the PSTF grey moment equations, one can compute the intensity from the moments by comparing (5.6) and (5.21):

$$I = \sum_{k=0}^{\infty} \frac{(2k+1)!!}{4\pi k!} \mathscr{M}^{A_k} \prod_{i=1}^{k} n_{\alpha_i}\,. \tag{5.27}$$

5.2 Generalized Bondi accretion

5.2.1 Simplifications under assumptions of symmetry

Instead of treating the general case as is done in [Tho81], we describe the specific choices made under the assumption of spherical symmetry, following Thorne, Flammang, and Żytkow [TFŻ81].

The fiducial frame defined in (4.28) can still be used here: we denote with a subscript "fid" the tensors expressed in that basis. We get the following expressions:

$$a^{\mu} = (0, \mathrm{d}y/\mathrm{d}r\ , 0, 0)_{\mathrm{fid}} \tag{5.28a}$$
$$\theta = -\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 v y\right) \tag{5.28b}$$
$$\sigma_{\mu\nu} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{vy}{r}\right)\frac{2r}{3}\begin{bmatrix} 0 & & & \\ & 1 & & \\ & & -1/2 & \\ & & & -1/2 \end{bmatrix}_{\mathrm{fid}} = \sigma \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & -1/2 & \\ & & & -1/2 \end{bmatrix}_{\mathrm{fid}} \tag{5.28c}$$
$$\Gamma_{\theta r \theta} = \Gamma_{\varphi r \varphi} = \frac{y}{r}\,. \tag{5.28d}$$

We can see that the shear has been heavily simplified. This is a specific case of a general statement about the PSTF moments: in the spherically symmetric case, the k-th PSTF moment only has one independent component. This is because it satisfies the following identities:

$$\mathscr{M}^{A_k} = 0 \text{ if } A_k \text{ contains an odd number of } \theta\text{s or } \varphi\text{s} \tag{5.29a}$$
$$\mathscr{M}^{A_k\theta\theta} = \mathscr{M}^{A_k\varphi\varphi} = -\frac{1}{2}\mathscr{M}^{A_k rr}\,. \tag{5.29b}$$

Equation (5.29a) comes from the fact that an odd number of θ or φ indices corresponds to and odd number of unit vectors which are integrated on the sphere (see the definition (5.21)): therefore the integrand is odd.

Equation (5.29b) comes from two observations: first of all, the moments corresponding to indices θ and φ respectively must be equal because of spherical symmetry; secondly the moments must be traceless, therefore the sum of the $\theta\theta$, $\varphi\varphi$ and rr moments must be zero (for any pair of indices).

So, with these every k-th moment is fully determined by the component $\mathscr{M}^{r\dots r}$ (k rs): therefore we give it a name: w_k. This fact is analogous to the statement that the only spherically symmetrical one of the spherical harmonics Y_{lm} is Y_{l0}, therefore as in that case we have only one independent component for every l.

5.2.2 Legendre polynomials complement

The l-th Legendre polynomial is:

$$P_l(x) = \frac{1}{2^l}\sum_{k=0}^{\lfloor l/2 \rfloor}\frac{(-1)^k(2l-2k)!}{k!(l-k)!(l-2k)!}x^{l-2k}\,. \tag{5.30}$$

We can see that the coefficient of x^l is $(2l)!/(2^l(l!)^2)$. We can rewrite this making use of the identities $(2n)! = (2n-1)!!(2n)!!$ and $(2n)!! = 2^n n!$, as:

$$\frac{(2l)!}{2^l(l!)^2} = \frac{1}{l!}\frac{(2l)!}{(2l)!!} = \frac{(2l-1)!!}{l!} = \frac{(2l+1)!!}{l!(2l+1)} \tag{5.31}$$

which is equation [Tho81, eq. 5.7d].

In Thorne [Tho81, eqs. 5.6] we find the statement that:

$$\int_{-1}^{1} I(\mu)P_k(\mu)\left(\frac{(2k-1)!!}{l!}\right)^{-1}\mathrm{d}\mu = \left(\int_{-1}^{1} I(\mu)\prod_{i=1}^{k} n^r\,\mathrm{d}\mu\right)^{TF} \tag{5.32}$$

where n^r denotes the radial component of a normal vector in spherical coordinates, P_k is the k-th Legendre polynomial and $\mu = \cos\theta$ where θ is the azimuthal coordinate of n. $I(\mu)$ is a generic function.

5.2.3 The scalar moments

It can be shown, using the identity (5.32) that the definition of w_k we gave is equivalent to

$$w_k = \int_{-1}^{1} I(\cos\theta) P_k(\cos\theta) \left(\frac{(2k-1)!!}{l!} \right)^{-1} 2\pi \,\mathrm{d}\cos\theta \tag{5.33}$$

where P_k is the k-th Legendre polynomial (5.30). Then the first moments are:

$$w_0 = \int I \,\mathrm{d}\Omega \qquad \text{radiation energy density} \tag{5.34a}$$

$$w_1 = \int I \cos\theta \,\mathrm{d}\Omega \qquad \text{radiation energy flux} \tag{5.34b}$$

$$w_2 = \int I \left(\cos^2\theta - \frac{1}{3} \right) \mathrm{d}\Omega \qquad \text{radiation shear stress.} \tag{5.34c}$$

We can explicitly write the stress-energy tensor in terms of the w_k using (5.23):

$$T^{\mu\nu} = \begin{bmatrix} w_0 & w_1 & 0 & 0 \\ w_1 & \frac{1}{3}w_0 + w_2 & 0 & 0 \\ 0 & 0 & \frac{1}{3}w_0 - \frac{1}{2}w_2 & \\ 0 & 0 & & \frac{1}{3}w_0 - \frac{1}{2}w_2 \end{bmatrix}_{\text{fid}} . \tag{5.35}$$

5.2.4 The source moments

We can get an explicit formula for the source moments $s_k = \mathscr{S}^{r\ldots r}$ (k rs) with the same procedure which was used in paragraph The scalar moments: we get

$$s_k = \int_{-1}^{1} \frac{\mathrm{d}I}{\mathrm{d}l}(\cos\theta) P_k(\cos\theta) \left(\frac{(2k-1)!!}{l!} \right)^{-1} 2\pi \,\mathrm{d}\cos\theta \tag{5.36}$$

where $\mathrm{d}I/\mathrm{d}l = \int \mathfrak{S}\nu^3 \,\mathrm{d}\nu$ is the frequency-integrated source term in the transfer equation.

5.2.5 The simplified moment equations

It is possible to write equations (5.26) explicitly in terms of the w_k and of derivatives wrt the fiducial basis: one gets [Tho81, eq. 5.10c]

$$\begin{aligned} &\frac{\partial w_{k+1}}{\partial \hat{r}} + [(2-k)a + (k+2)b] w_{k+1} + \frac{\partial w_k}{\partial \hat{t}} + \left[\frac{4}{3}\theta + \frac{5k(k+1)}{2(2k-1)(2k+3)}\sigma \right] w_k + \\ &\quad + \frac{k^2}{(2k-1)(2k+1)} \frac{\partial w_{k-1}}{\partial \hat{r}} + \frac{k^2[(k+3)a + (1-k)b]}{(2k-1)(2k+1)} w_{k-1} + \\ &\quad - \frac{3}{2}(k-1)\sigma w_{k+2} + \frac{3(k-1)^2 k^2 (k+2)}{2(2k-3)(2k-1)^2(2k+1)} \sigma w_{k-2} = s_k \end{aligned} \tag{5.37}$$

where $a = \mathrm{d}y/\mathrm{d}r = \sqrt{a^\mu a_\mu}$ is the magnitude of the 4-acceleration, $b = y/r$ is the extrinsic curvature, θ is the expansion velocity, σ is the scalar shear — the largest eigenvalue of the shear matrix. Explicit expressions for these are found in (5.28).

Nobili, Turolla, and Zampieri [NTZ91] only used the first two of the moment equations, so here is how the expression is simplified for $k = 0, 1$: for $k = 0$ we get:

$$\frac{\partial w_1}{\partial \hat{r}} + 2(a+b)w_1 + \frac{\partial w_0}{\partial \hat{t}} + \frac{4}{3}\theta w_0 + \frac{3}{2}\sigma w_2 = s_0 \,. \tag{5.38}$$

For $k = 1$ we get:

$$\frac{\partial w_2}{\partial \hat{r}} + (a+3b)w_2 + \frac{\partial w_1}{\partial \hat{t}} + \left[\frac{4}{3}\theta + \sigma\right] w_1 + \frac{1}{3}\frac{\partial w_0}{\partial \hat{r}} + \frac{4a}{3}w_0 = s_1 \,. \tag{5.39}$$

These have to be simplified further to be used: specifically, they can be expressed with respect to r, v, y.

5.2.6 Simplifying the moment equations further

Because of the hypothesis of stationarity, we can express the derivatives in (5.37) as:

$$\frac{\partial}{\partial \hat{t}} = \frac{\gamma^2}{y}\frac{\partial}{\partial t} - yv\frac{\partial}{\partial r} = -yv\frac{\partial}{\partial r} \tag{5.40a}$$
$$\frac{\partial}{\partial \hat{r}} = -v\frac{\gamma^2}{y}\frac{\partial}{\partial t} + y\frac{\partial}{\partial r} = y\frac{\partial}{\partial r} \,. \tag{5.40b}$$

Now, we can make the scalar PSTF moment equations (5.37) fully explicit: denoting derivation with respect to r with a prime, we have

$$yw_1' + 2\left(y' + \frac{y}{r}\right)w_1 - yvw_0' - \frac{4}{3}\frac{w_0}{r^2}\left(r^2 vy\right)' - w_2 r\left(\frac{vy}{r}\right)' = s_0 \tag{5.41a}$$
$$yw_2' + \left(y' + \frac{3y}{r}\right)w_2 - yvw_1' - \frac{4}{3r^2}\left(r^2 vy\right)' w_1 - \frac{2r}{3}\left(\frac{vy}{r}\right)' w_1 + \frac{y}{3}w_0' + \frac{4}{3}y' w_0 = s_1 \,. \tag{5.41b}$$

We can simplify these by expanding the derivatives of products $(vyr^2)'$ and $(vy/r)'$:

$$yw_1' + 2\left(y' + \frac{y}{r}\right)w_1 - yvw_0' - \frac{4}{3}w_0\left((vy)' + \frac{2vy}{r}\right) - w_2\left((vy)' - \frac{vy}{r}\right) = s_0 \tag{5.42a}$$
$$yw_2' + \left(y' + \frac{3y}{r}\right)w_2 - yvw_1' - 2w_1\left((vy)' + \frac{vy}{r}\right) + \frac{y}{3}w_0' + \frac{4}{3}y' w_0 = s_1 \,. \tag{5.42b}$$

Equations (5.42) are the ones which appear in Nobili, Turolla, and Zampieri [NTZ91, eq. 4] and which are reported in equation (5.43): the only difference is that their primes denote derivatives with respect to $\log(r/(2M))$.

$$w_1' - vw_0' - vw_2\left[\frac{(vy)'}{vy} - 1\right] + 2w_1\left(1 + \frac{y'}{y}\right) - \frac{4}{3}vw_0\left[\frac{(vy)'}{vy} + 2\right] = \frac{rs_0}{y} \tag{5.43a}$$
$$w_2' - vw_1' + \frac{1}{3}w_0' + w_2\left(3 + \frac{y'}{y}\right) - 2vw_1\left[\frac{(vy)'}{vy} + 1\right] + \frac{4}{3}\frac{y'}{y}w_0 = \frac{rs_1}{y} \,. \tag{5.43b}$$

5.2.7 Some properties of the accretion variables

From equation (4.34) we can find an expression [TFŻ81, eq. 18a] for y which only depends on r and constants:

$$y = \sqrt{y^2(1 - v^2 + v^2)} = \sqrt{\left(\frac{y^2}{\gamma^2}\right) + y^2v^2} = \sqrt{\left(1 - \frac{2M}{r}\right) + \left(\frac{\dot{M}}{4\pi r^2 \rho_0}\right)^2} \tag{5.44}$$

therefore v can also be expressed in terms of r and constants:

$$v = \frac{\dot{M}}{4\pi r^2 \rho_0 y(r)} \, . \tag{5.45}$$

5.2.8 The source term

The source term can be written [TFŻ81, eq. 15] as

$$s_k = \frac{l!(2l+1)}{(2l+1)!!} \int_{-1}^{1} \frac{\mathrm{d}I}{\mathrm{d}l} P_k(\mu) 2\pi \, \mathrm{d}\mu \, . \tag{5.46}$$

The general relation for the change in intensity is

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}l} = \rho_0(\varepsilon_\nu - \kappa_\nu I_\nu) \tag{5.47}$$

where ρ_0 is the rest mass density, while ε_ν and κ_ν are the specific coefficients of emission and absorption. If we integrate this relation over all frequencies and take its k-th moment, we get

$$s_k = \rho_0(\varepsilon_k - \kappa_k w_k) \tag{5.48}$$

where w_k are the PSTF scalar moments, ε_k is the k-th moment of emissivity and κ_k is the k-th moment of opacity of the gas.

Equation (5.48) can be taken to be a practical definition of ε_k and κ_k; we do the integral in (5.46) and get constant terms and terms proportional to w_k, which we split into the two terms on the RHS of (5.48).

We will only consider the $k = 0$ and $k = 1$ moments. If the emission is isotropic, then the emissivity moment ε_1 is 0 since its definition contains an integral of the product of an even and odd function.

Because of this, we just call the one moment we need $\varepsilon = \varepsilon_0$.

The source moments given in [NTZ91, eq. 6] are:

$$s_0 = \rho_0\left(\varepsilon - w_0\left(\kappa_0 - \kappa_{\mathrm{es}}\frac{4k_B}{m_e}(T - T_\gamma)\right)\right) \tag{5.49a}$$

$$s_1 = -\rho_0 w_1 \kappa_1 \tag{5.49b}$$

with

$$T_\gamma = \frac{1}{4k_B} \frac{\displaystyle\int_0^\infty h\nu w_0(r,\nu)\,\mathrm{d}\nu}{\displaystyle\int_0^\infty w_0(r,\nu)\,\mathrm{d}\nu} \, . \tag{5.50}$$

Further, we use the facts that $\varepsilon/\kappa_0 = aT^4$ if there is thermodynamic equilibrium and that we have the following expression for the emissivity ε in terms of the cooling function $\Lambda(T)$ (given in (5.57)):

$$\varepsilon = \frac{\rho_0 \Lambda(T)}{m_p^2} \tag{5.51}$$

to write the source term s_0 as

$$s_0 = \frac{\rho_0^2 \Lambda(T)}{m_p^2}\left(1 - \frac{w_0}{aT^4}\right) + \rho_0 \kappa_{\rm es} w_0 \frac{4k_B}{m_e}(T - T_\gamma)\,. \tag{5.52}$$

As for the s_1 term, we model κ_1 as

$$\kappa_1 = \kappa_{\rm es} + \langle \kappa_{\rm ff} \rangle = \kappa_{\rm es} + 6.4 \times 10^{22}\,\mathrm{cm\,g^{-2}}\rho_0 T^{-7/2} \tag{5.53}$$

where the second term is the conventional approximation of the *Rosseland mean opacity* computed taking into account only free-free transitions: the definition of the RMO is a harmonic mean of the opacities at every frequency, weighted by the derivatives with respect to temperature of the Planck function[1] at specific frequencies [RL04, eq. 1.110].

$$\frac{1}{\langle \kappa_{\rm ff} \rangle} = \frac{\displaystyle\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\frac{1}{\kappa_\nu^{\rm ff}}\,\mathrm{d}\nu}{\displaystyle\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\,\mathrm{d}\nu} \tag{5.54}$$

The expression we get for s_1 is:

$$s_1 = -\rho_0 w_1 \left(\kappa_{\rm es} + 6.4 \times 10^{22}\,\mathrm{cm\,g^{-2}}\rho_0 T^{-7/2}\right). \tag{5.55}$$

5.2.9 Cooling function

The cooling function $\Lambda(T)$ is defined by the following relation, which describes the variation in the energy density by radiative processes:

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = n_b^2\big(\Gamma(T) - \Lambda(T)\big) \tag{5.56}$$

where ρ is the energy density (measured in $\mathrm{erg\,cm^{-3}}$), n_b is the baryon density (measured in $\mathrm{cm^{-3}}$), while Γ and Λ are the heating and cooling functions, both measured in $\mathrm{erg\,cm^3\,s^{-1}}$, see [GH12, equation 1].

The cooling function of the infalling gas is plotted in figure 1 and given in equation (5.57):

$$\begin{aligned}\Lambda(T) = \Bigg(&\left(1.42 \times 10^{-27} T^{1/2}\left(1 + 4.4 \times 10^{-10} T\right) + 6.0 \times 10^{-22} T^{-1/2}\right)^{-1} \\ &+ 10^{25}\left(\frac{T}{1.5849 \times 10^4\,\mathrm{K}}\right)^{-12}\Bigg)^{-1} \mathrm{erg\,cm^3\,s^{-1}}\,.\end{aligned} \tag{5.57}$$

[1] The Planck function [RL04, eq. 1.51], here given reintroducing c explicitly,

$$B_\nu = \frac{2h\nu^3 c^{-2}}{\exp(h\nu/k_B T) - 1}$$

quantifies the spectral radiance emitted by a blackbody.

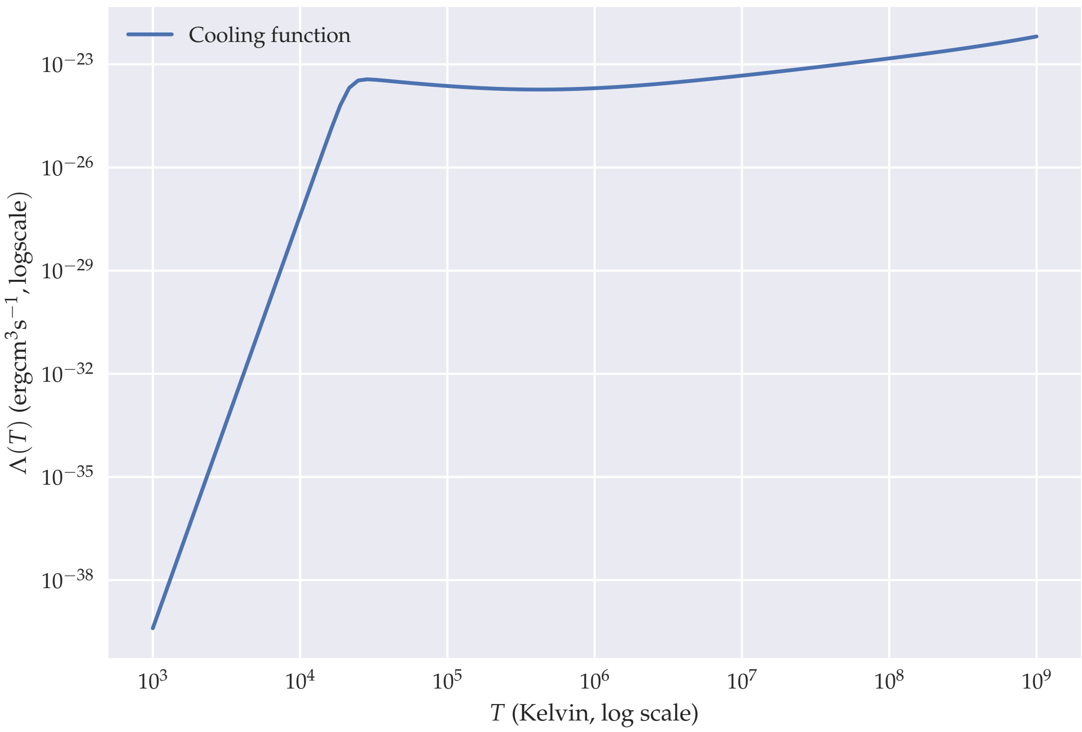

Figure 1: Cooling function graph.

5.2.10 How the conservation equations change using the moment equations

The way we use the grey moment equations (5.43) to study the accretion problem is by using them to simplify the conservation equations $\nabla_\mu T^{\mu\nu}$.

The full energy momentum tensor of the problem is given, in the fiducial reference frame, by combining an ideal-fluid stress-energy tensor (with pressure P and rest energy density ρ) with the one given in (5.35):

$$T^{\mu\nu} = T^{\mu\nu}_{\text{radiation}} + T^{\mu\nu}_{\text{matter}} = \begin{bmatrix} \rho + w_0 & w_1 & 0 & 0 \\ w_1 & P + \frac{w_0}{3} + w_2 & 0 & 0 \\ 0 & 0 & P + \frac{w_0}{3} - \frac{w_2}{2} & 0 \\ 0 & 0 & 0 & P + \frac{w_0}{3} - \frac{w_2}{2} \end{bmatrix}_{\text{fid}} . \tag{5.58}$$

The t, r by t, r components of the stress-energy tensor with contravariant indices in spherical coordinates are:

$$\begin{bmatrix} \frac{\gamma^4\left(\rho - v w_1 + \frac{v\left(v(3P + w_0 + 3w_2) - 3w_1\right)}{3} + w_0\right)}{y^2} & \gamma^2\left(-\frac{v(3P + w_0 + 3w_2)}{3} + v\left(-\rho + v w_1 - w_0\right) + w_1\right) \\ \gamma^2\left(-v\left(\rho + w_0\right) + \frac{v(-3P + 3v w_1 - w_0 - 3w_2)}{3} + w_1\right) & y^2\left(P - v w_1 + v\left(v\left(\rho + w_0\right) - w_1\right) + \frac{w_0}{3} + w_2\right) \end{bmatrix} . \tag{5.59}$$

5.2.11 The total luminosity $\dot{E}$

We can project the conservation of the stress energy tensor along the unit vector in the temporal direction in the spherical reference frame, $(e_t)_\nu$: we get $(e_t)_\nu \nabla_\mu T^{\nu\mu} = \nabla_\mu T^\mu_0 = 0$.

Then, applying (3.6) and our symmetry assumptions:

$$0 = \nabla_\mu T^\mu_t = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial r}\left(r^2 \sin\theta T^r_t \right) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 T^r_t \right) \implies r^2 T^r_t = \text{const} \, . \tag{5.60}$$

In [TFŻ81, before eq. 18c] this appears with an additional immaterial factor of 4π.

The quantity which is conserved is $r^2 g_{tt} T^{tr}$, where T^{tr} is the $(0,1)$ matrix element which appears in (5.59). We can simplify it by expressing it in terms of the luminosity L which is $L = 4\pi r^2 w_1$: we get

$$4\pi r^2 g_{tt} \left(\gamma^2 (1+v^2) \frac{L}{4\pi r^2} - v\gamma^2 \left(p + \rho + w_2 + \frac{4}{3} w_0 \right) \right) = \text{const} \, . \tag{5.61}$$

We can also substitute in the expression for $v = \dot{M} / \left(4\pi r^2 \rho_0 y \right)$, and recognize the expression for the specific enthalpy h and for $y^2 = -g_{tt}\gamma^2$:

$$- y^2 (1+v^2) L + y\dot{M} \left(h + \frac{4w_0/3 + w_2}{\rho_0} \right) \stackrel{\text{def}}{=} -\dot{E} = \text{const} \, . \tag{5.62}$$

This is the total luminosity of the accretion process.

5.2.12 The full equations of motion

The conservation of the stress-energy tensor can be projected onto $\hat{t}$ and $\hat{r}$ and cast into [NTZ91, eq. A7]:

$$(P+\rho)\frac{\mathrm{d}y}{\mathrm{d}r} + y\frac{\mathrm{d}P}{\mathrm{d}r} + \underbrace{\frac{4}{3} w_0 \frac{\mathrm{d}y}{\mathrm{d}r} + \frac{1}{3} y \frac{\mathrm{d}w_0}{\mathrm{d}r} + \frac{1}{yvr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(y^2 v^2 r^2 w_1 \right) + \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^3 y w_2 \right)}_{s_1} = 0 \tag{5.63a}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{P+\rho}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}r} + \underbrace{\frac{\mathrm{d}w_0}{\mathrm{d}r} + \frac{4}{3}\frac{w_0}{yvr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(yvr^2 \right) + \frac{1}{y^2 vr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(y^2 r^2 w_1 \right) + w_2 \frac{r}{yv}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{yv}{r} \right)}_{s_0/(yv)} = 0 \tag{5.63b}$$

where the identifications come from a reframing of the moment equations (5.43) (see [NTZ91, eq. A8]).

By adding the continuity equation (4.43) to the equations (5.63) — which are reworked slightly to get the expressions of logarithmic derivatives — we get the full system of the simplified conservation equations we will work with. Here, as in [NTZ91] and as in section 4.3.3, primes denote differentiation with respect to $\log r$.

$$(P+\rho)\frac{y'}{y} + P' + \frac{rs_1}{y} = 0 \tag{5.64a}$$

$$\rho' - (P+\rho)\frac{\rho_0'}{\rho_0} + \frac{rs_0}{vy} = 0 \tag{5.64b}$$

$$\frac{(vy)'}{vy} + \frac{\rho_0'}{\rho_0} + 2 = 0 \, . \tag{5.64c}$$

We can expresse these in terms of the variables $(yv)(r)$, $\rho_0(r)$ and $T(r)$. To this end, we apply the same manipulations used in section 4.3.3 and thus get [NTZ91, eqs. 15]:

$$(v^2 - v_s^2)\frac{(yv)'}{yv} - 2v_s^2 + \frac{M}{y^2 r} + \frac{r}{yv(p+\rho)}\left((\Gamma-1)s_0 + vs_1\right) = 0 \tag{5.65a}$$

$$\frac{T'}{T} - (\Gamma-1)\frac{\rho_0'}{\rho_0} - \frac{rs_0}{Bvy(p+\rho)} = 0 \tag{5.65b}$$

$$\frac{(yv)'}{yv} + \frac{\rho_0'}{\rho_0} + 2 = 0 \tag{5.65c}$$

where Γ, B, and v_s^2 are those defined in section 4.3.3.

The sign of the source term in the energy equation is a minus: why? it seems like it should be a plus. To get the first equation, however, we need the minus.

Their expressions will in general be unknown: they are however defined in terms of derivatives of ρ and p, so if we can write an equation for those variables in terms of T and ρ_0 we can compute all the desired thermodynamic variables. The desired equations of state, which need to account both for slow-moving and fully relativitic regimes, are [NTZ91, eqs. 16]:

$$p = \left(1 + \frac{F}{1+F}\right)\frac{\rho_0 k_B T}{m_p} \tag{5.66a}$$

$$\rho = \rho_0 + \left(\frac{3}{2}\frac{F}{(1+F)} + \left(\frac{\eta-1}{\theta} - 1\right)\right)\frac{\rho_0 k_B T}{m_p} + \left(1 - \frac{F}{1+F}\right)\frac{\rho_0 E_H}{m_p} \tag{5.66b}$$

where $F = 2(T/1\,\mathrm{K})\exp\left(-1.58 \times 10^5\,\mathrm{K}/T\right)$ and the quantity $F/(1+F)$ is the approximate degree of collisional ionization, $\theta = k_B T/m_e$ while η is defined with the use of the Bessel functions of order n, K_n: $\eta = K_3(\theta^{-1})/K_2(\theta^{-1})$; m_p and m_e are the masses of the proton and of the electron, k_B is the Boltzmann gas constant, while E_H is ???.

5.2.13 Eddington luminosity

It is the characteristic luminosity at which the radiation pressure from the photons moving outward equals the gravitational specific force on the infalling matter.

The gravitational force, in the newtonian limit, is

$$F_{\mathrm{grav}} = \frac{GMm}{r^2}\,. \tag{5.67}$$

The radiation pressure can be given in terms of the luminosity L (reinserting the units of c for this) as

$$P_{\mathrm{rad}} = \frac{L}{c4\pi r^2}\,, \tag{5.68}$$

then, the radiative force is given by $F_{\mathrm{rad}} = P_{\mathrm{rad}}\kappa m$, where m is the mass of the test object and κ is the opacity: the per-unit-mass cross-section. We usually assume $\kappa = \sigma_T/m_p$, that is, that the interaction between radiation and matter is all due to Thompson scattering and the matter is only composed of hydrogen atoms.

Equating the forces, we get our result:

$$\frac{L_{\mathrm{Edd}}}{M} = \frac{4\pi cG}{\kappa}\,. \tag{5.69}$$

In the $\kappa = \sigma_T / m_p \approx 0.04\,\mathrm{m}^2/\mathrm{kg}$ case, we get L_{Edd}/M to be around $6.32\,\mathrm{W\,kg}^{-1}$ (constants' values from [18]). If we express this in units of $L_\odot / M_\odot \approx 1.93 \times 10^{-4}\,\mathrm{W\,kg}^{-1}$ [Wil18] we get $L_{\mathrm{Edd}}/M \approx 3.27 \times 10^4 L_\odot / M_\odot$: the amount of radiation emitted by the Sun is much less than the Eddington limit.

It is, of course, important to note that this is a limit found with many approximations: non-relativistic gravity, spherical symmetry, only Thompson scattering, only hydrogen.

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