Notes on Calculus I

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Chapter 1

Naïve set theory

1.1 Basic sets

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}, \qquad 0 \notin \mathbb{N} \tag{1.1.1}$$

$$\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\} \tag{1.1.2}$$

Remarks

- "$\in$" is for elements belonging to sets, "$\subseteq$" is for subsets
- $\{x\} \neq x$: the first is a set with x as its only element, and is called a "singlet"
- $\subsetneq$ means "is a subset of, but not equal to"
- the elements of $\mathcal{P}(A)$ are precisely all the subsets of A
- $\sharp A$ is the cardinality of A
- $\sharp \mathcal{P}(A) = 2^{\sharp A}$

The naïve definitions of $A \cup B$, $A \cap B$, $A \smallsetminus B$ are given.

Properties

- $A = (A \cap B) \cup (A \smallsetminus B)$
- $(A \cap B) \cap (A \smallsetminus B) = \emptyset$
- $C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$
- $C \cup (A \cap B) = (C \cup A) \cap (C \cup B)$

Complement

Definition 1.1.1. With respect to a "universe" set U, we define the complement of A as $U \setminus A$, denoted A^C.

The following hold:

- $(A \cup B)^C = A^C \cap B^C$
- $(A \cap B)^C = A^C \cup B^C$

Cartesian product

Definition 1.1.2. An *ordered pair* is a set of the form $\{\{x\}, \{x, y\}\}$, denoted (x, y) (where order matters).

Definition 1.1.3. We define the *cartesian product* $A \times B$ of two sets A and B as:

$$A \times B := \big\{(a, b) : a \in A, b \in B\big\} \tag{1.1.3}$$

1.2 Propositional logic

Implication

Definition 1.2.1.

$$p \implies q \iff (\neg p) \vee q \tag{1.2.1}$$

Double implication

Definition 1.2.2.

$$(p \iff q) \iff (p \implies q \wedge q \implies p) \tag{1.2.2}$$

Quantifiers $P(x)$ is a *predicate.* We say that $\forall x : P(x)$ if $P(x)$ is true independently of x, and that

$$\exists x : P(x) \iff \neg(\forall x : \neg P(x)) \tag{1.2.3}$$

Chapter 2

Number sets

2.1 The set $\mathbb{N}$

The Peano axioms

1. $1 \in \mathbb{N}$
2. $\forall n \in \mathbb{N} : \exists S(n) : \mathbb{N} \to \mathbb{N}$
3. $\forall n \in \mathbb{N} : S(n) \neq 1$
4. $\forall m, n \in \mathbb{N} : m \neq n \implies S(n) \neq S(m)$ or $S(n) = S(m) \implies n = m$
5. $(A \subseteq \mathbb{N}) \wedge (1 \in A) \wedge (n \in A \implies S(n) \in A) \implies A = \mathbb{N}$

Any set that verifies these axioms is isomorphic to $\mathbb{N}$. $\mathbb{R}^+$, for example only satisfies the first 4.

2.2 Induction

If $P(n)$ is a proposition, $P(1)$ and $P(n) \implies P(n+1)$[1] (the *inductive hypothesis*); then $\forall n \in \mathbb{N}, P(n)$.

Proof. Define $A := \{n \in \mathbb{N} : P(n)\}$. By axiom 5, $A = \mathbb{N}$. $\square$

If a property $a' : P(k)$ holds for some $k \in \mathbb{N}$ and $b' : P(n) \implies P(n+1)$, then $\forall n \geq k : P(n)$.

Examples Example: we can show by induction that

$$\sum_{i=1}^{n} = \frac{n(n+1)}{2} \tag{2.2.1}$$

Example 2: we can show by induction that $P(\sharp A) : \sharp\mathcal{P}(A) = 2^{\sharp A}$.

[1] We introduce the notation $n+1$ to signify $S(n)$.

Proof. $P(1)$ is true. We see that for any A we can take an element such that $A = \{a\} \cup B$. Then for any subset I, either $a \in I$ or $a \notin I$. If $a \in I$, $I = \{a\} \cup J$, but there are 2^n possible Js. If $a \notin I$, we have 2^n I's. So there are 2^{n+1} possible subsets. $\square$

Example 3: show that $n! > 2^n$, which is true for $n > 3$.

Proof. We will use the second form of the induction principle. $P(4)$ is true. If $n > 4$ and $n! > 2^n$, we need to show that $(n+1)n! > 2 \cdot 2^n$. But $n + 1 > 2$ by hypothesis, so the inequality always holds. $\square$

Observation: the notation $1 + 2 + 3 + 4 + \cdots + n$ is unclear, we should use $\sum_{i=1}^{n} i$.

Recursive formulas: we know the first term, and an algorithm to derive any term from the one before it, such as the definition of the factorial:

$$\begin{cases} 0! = 1 \\ (n+1)! = (n+1)n! \end{cases} \tag{2.2.2}$$

Another example is the sequence:

$$\begin{cases} S_1 = 2 \\ S_{n+1} = S_n + (2n+1) \end{cases} \tag{2.2.3}$$

Proof that $S_n = n^2 + 1$. Assume that $S_n = n^2+1$. Then $S_{n+1} = n^2+1+2n+1 = (n+1)^2 + 1$. $\square$

Formal definition of summation:

$$\sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + a_3 + \cdots + a_n \tag{2.2.4}$$

by recursion: for an increment in n, we just add the $n+1$-th term. So it comes down to the formal definition of induction.

Example: show by induction that

$$\forall a \in \mathbb{N} \vee a = 0 \quad (a \neq 1): \quad \sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} \tag{2.2.5}$$

See the property for $n = 0$. Suppose that the property holds for n, show it for $n + 1$:

$$\sum_{i=0}^{n+1} a_k = \sum_{k=0}^{n} a^k + a^{n+1} = \frac{1 - a^{n+1}}{1 - a} + a^{n+1} = \frac{1 - a^{n+2}}{1 - a} \tag{2.2.6}$$

To do: given two real numbers, $\forall n \in \mathbb{N}$, show that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \tag{2.2.7}$$

2.3 Groups

We have a set A with an operation $*$: $a, b \in A \to a * b \in A$. Es: A strings, $*$ concatenation. $(A, *)$ is a group if the following are satisfied:

1. Associativity: $\forall a, b, c \in A : \quad a * (b * c) = (a * b) * c$
2. Identity: $\exists e \in A : \forall a \in A : \quad e * a = a * e = a$
3. Inverse: $\forall a \in A : \exists a^{-1} \in A : \quad a * a^{-1} = a^{-1} * a = e$

$(A, *)$ is also a commutative group if $\forall a, b \in A : \quad a * b = b * a$

Examples: $(\mathbb{N}, +)$ is not a group: there is no identity, but even if we add 0 there is no inverse. $(\mathbb{Z}, +)$ is a commutative group. $(\mathbb{R}, \times)$ is not a group because 0 has no inverse. $(\mathbb{R}_0, \times)$ is hower a group, as is $(\mathbb{R}_0, +)$. $(\mathbb{Z}_0, \times)$ is not a group because there is no inverse: $(\mathbb{Q}_0, \times)$ is a commutative group.

So we introduce $\mathbb{Q}$:

$$\mathbb{Q} := \left\{ \frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{N} \right\} / \sim \tag{2.3.1}$$

but $ad \sim bc$ and $\mathbb{Q}$ is isomorphic to $\mathbb{Z}$

Definition 2.3.1. Order relation:

$$\frac{a}{b} \leq \frac{c}{d} \iff ad \leq bc \tag{2.3.2}$$

Definition 2.3.2. Sum and product, subtraction and division are analogous:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \tag{2.3.3}$$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \tag{2.3.4}$$

$(\mathbb{Q}, +, \times)$ is then a field:

1. $(\mathbb{Q}, +)$ is a commutative group;
2. $(\mathbb{Q}_0, \times)$ is a commutative group
3. $p(q + r) = pq + pr$

Roots The square root of a number $a \geq 0$ is a $b \geq 0$ such that $b^2 = a$.

Show that in $\mathbb{Q}$ there is no $\sqrt{2}$:

Proof. By contradiction:[2] if there were $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that

[2] In the form: to show that $p \implies q$, we show that $\neg q \implies \neg p$, since

$$(p \implies q) \iff (\neg q \implies \neg p) \tag{2.3.5}$$

$$\left(\frac{a}{b}\right)^2 = 2 \tag{2.3.6}$$

Suppose that a, b have $\gcd(a, b) = 1$. Then $a^2 = 2b^2$. So a is even, and $a = 2k$. Then $4k^2 = 2b^2 \implies b = 2n$: so $\gcd(a, b) \neq 1$ □

With the same reasoning we can show that numbers such that $\sqrt{3}$ and $\sqrt[3]{2}$ are irrational.

2.4 The set $\mathbb{R}$

The set is given. $\mathbb{R}$ is a completely ordered field:

1. $(\mathbb{R}, +)$ is a commutative group
2. $(\mathbb{R}_0, \times)$ is a commutative group
3. $a(b + c) = ab + ac$

There also exists a relation called $\leq$, such that $\forall a, b, c \in \mathbb{R}$:

1. $a \leq a$
2. $a \leq b \wedge b \leq a \implies a = b$
3. $a \leq b \wedge b \leq c \implies a \leq c$
4. $a \leq b \vee b \leq a$ (*completely* ordered)
5. $a \leq b \implies a + c \leq b + c$
6. $a \geq 0 \wedge b \geq 0 \implies ab \geq 0$ (of course, $a \geq 0$ means that $0 \leq a$)

$(\mathbb{Q}, +, \times)$ is also a completely ordered field?

Take the set of all the real numbers whose squares are greater or equal to 2: it has a minimum.

In $\mathbb{Q}$, it has no minimum.

Other statements: show that $a \leq 0 \wedge b \geq 0 \implies ab \leq 0$

Proof. Is $a \geq 0 \wedge -b \geq 0$? We first need to show that $a \geq 0 \iff -a \leq 0$: it suffices to add $-a$ to both sides. We also need to show that $a(-b) = -ab$: by an inverse application of the distributive property. □

Useful inequalities $\forall x \in \mathbb{R} : x^2 \geq 0$. Also, $\forall a, b \in \mathbb{R} : ab \geq (a^2 + b^2)/2$ (one of the inequalities between the means. From this, we can also show that between the rectangles of perimeter p, the square is the one with the largest area.

TO DO: show this for parallelograms and trapezes.

Integer part of a number Given an $x \in \mathbb{R}$, we define its integer part $\lfloor x \rfloor = n \in \mathbb{Z}$ as the largest integer such that $n \leq x$.[3]

The fractionary part $\{x\}$ is defined as

$$\{x\} = x - \lfloor x \rfloor \tag{2.4.1}$$

It is clear that $x - 1 < \lfloor x \rfloor \leq x$ and that $0 \leq \{x\} < 1$, and that $\lfloor x \rfloor = x \iff x \in \mathbb{Z}$

2.5 Set notation

The notation $[a; b]$ means $\{x \in \mathbb{R} : a \leq x \leq b\}$, and $(a; b) =]a; b[$ means $\{x \in \mathbb{R} : a < x < b\}$. These are *closed* and *open* sets. We can combine the two types of brackets as we wish.

In the notation $(a; +\infty)$, the symbol ∞ is not a number.

Definition 2.5.1. We define the maximum of a set $E \subseteq \mathbb{R}$, denoted $\max E$ (and analogously $\min E$), as a number $M \in \mathbb{R}$ with the following properties:

1. $\forall x \in E : M \geq a$ (M is an upper bound of E)
2. $M \in E$

A set is limited from above if it has an upper bound, and from below if it has a lower bound. Open sets can be limited, but they do not have maximums and minimums: we can show this by taking the average between the first real number outside of the set and a number we suppose to be this maximum, getting to a contradiction.

Definition 2.5.2. We define the *least upper bound* of a set E as the minimum of the set of the upper bounds, and analogously for lower bounds. We can do this $\forall E \subseteq \mathbb{R} : E \neq \emptyset$, and we denote them as $\inf E$ and $\sup E$.

If E does not have an upper limit, we write $\sup E = +\infty$, but this is just notation. The same goes for $\inf E = -\infty$. It is also common to write $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$, but defining them this way makes the fact $\inf E \leq \sup E$ not true.

E has a maximum iff $\sup E \in E$. If $E \neq \emptyset \neq F$ and $E \subseteq F$, then $\sup E \leq \sup F$ and $\inf F \leq \inf E$.

Theorem 2.5.1 (Archimedes' axiom). Given $a, b \in \mathbb{R}$, where $a > 0$ and $b > 0$, $\exists n \in \mathbb{N} : na > b$.

Proof. We choose $n = \lfloor b/a \rfloor + 1$. □

[3] This is also the case for negative numbers: so the integer part of a negative real number may have greater absolute value than the number itself.

Axiom of continuity or completeness Given $E \subseteq \mathbb{R}$, $E \neq \emptyset$, with at least an upper bound, there exists $\sup E \in \mathbb{R}$. The inverse is easily proven from this.

$\mathbb{R}$ (and sets which can be bijectively mapped to it, via a map that preserves addition and multiplication) is the only totally ordered set which verifies the axiom of continuity.

Theorem 2.5.2. $\sup E = M \iff \forall x \in M : x \leq M$ and $\forall \epsilon > 0 : \exists z \in E : z > M - \epsilon$.

Proof. We will prove the leftward implication by contradiction. Suppose there exists an $M' < M$. Then $\forall \epsilon \in E : z \leq M'$. But define $\epsilon := (M - M')/2$:then ???
Prove the rightward implication □

Halved intervals

Theorem 2.5.3. Take $a_k, b_k \in \mathbb{R} : \forall k \in \mathbb{N} : a_k < b_k$ and $\forall k \in \mathbb{N} : [a_{k+1}; b_{k+1}]$ is one of the halves of $[a_k; b_k]$:

$$\forall k : [a_{k+1}, b_{k+1}] = \left[a_k, \frac{a_k + b_k}{2}\right] \vee \left[\frac{a_k + b_k}{2}, b_k\right] \tag{2.5.1}$$

Then $\exists! \lambda \in \mathbb{R} : \forall k \in \mathbb{N} : \lambda \in [a_k; b_k]$. We can write this as

$$\bigcap_{k=1}^{+\infty} [a_k; b_k] = \{\lambda\} \tag{2.5.2}$$

λ is clearly the $\sup\{a_k : k \in \mathbb{N}\} = \inf\{b_k : k \in \mathbb{N}\}$.

Proof. First, we will show that $\forall k, j \in \mathbb{N} : a_k < b_k$.

- $k \geq j$: $a_k < b_k \leq b_j$;
- $k < j$: $a_k \leq a_j < b_j$.

so, $\forall k \in \mathbb{N}$, a_k is a lower bound for $\{b_j : j \in \mathbb{N}\}$: $\forall k : a_k \leq \inf\{b_j\}$. So, $\inf\{b_j\} \leq \sup\{a_k\}$ (these must exist since the sets are bounded).

We can see this as:

$$\forall n \in \mathbb{N} : 0 \leq \inf\{b_j\} - \sup\{a_k\} \leq b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \tag{2.5.3}$$

It is clear that if $x \geq 0$ and $\forall \varepsilon > 0 : x \leq \varepsilon$, then $x = 0$.

Also, $\forall \varepsilon > 0 : \exists n \in \mathbb{N} : n\varepsilon > b_1 - a_1$ by Archimedes' property. But $\forall n \in \mathbb{N} : n \leq 2^{n-1}$. So

$$\forall \varepsilon > 0 : \exists n \in \mathbb{N} : 0 \leq \inf\{b_j\} - \sup\{a_k\} \leq b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} < \varepsilon \tag{2.5.4}$$

which implies $\inf\{b_j\} = \sup\{a_k\} := \lambda \in \mathbb{R}$. □

This doesn't just work for halved intervals: any set of intervals for which $[a_{k+1}, b_{k+1}] \leq [a_k, b_k]$ and $\lim_{k \to +\infty} b_k - a_k = 0$:

Proof. We need to show the two inclusions in

$$\cap_{k=1}^{+\infty} [a_k, b_k] = \lambda \tag{2.5.5}$$

$\lambda = \sup\{a_k\} = \inf\{b_j\}$, so $\forall k : \lambda \in [a_k, b_k]$.

We need to show that if $x \in \cap_{k=1}^{+\infty}[a_k, b_k]$, then $x = \lambda$.

If $\forall k : a_k \leq x \leq b_k$, then x is an upper bound of $\{a_k\}$ ($x \geq \sup\{a_k\}$) and a lower bound of $\{b_k\}$ ($x \leq \inf\{b_k\}$). So, $x = \lambda$. □

Chapter 3

Topology

We will focus on the topology of $\mathbb{R}$, sometimes generalizing to $\mathbb{R}^n$.

A *metric space* is couple (X, d), where $\mathrm{d} : X \times X \to \mathbb{R}$ is a *distance* function, satisfying $\forall x, y, z \in X$:

1. $\mathrm{d}(x, y) \geq 0$; $\mathrm{d}(x, y) = 0 \iff x = y$;
2. $\mathrm{d}(x, y) = \mathrm{d}(y, x)$;
3. $\mathrm{d}(x, y) \leq \mathrm{d}(x, z) + \mathrm{d}(y, z)$ (the triangular inequality).

On $\mathbb{R}$, the distance function is usually $\mathrm{d}(x, y) = |x - y|$.[1]

This extends to $\mathbb{R}^n$:

$$\mathrm{d}(x, y) = \sqrt{\sum_{i=0}^{n} (x_i - y_i)^2} \tag{3.0.1}$$

This clearly satisfies conditions 1 and 2, but we have to prove condition 3:

Proof. This follows from the subadditivity of the norm of vectors: $\mathbb{R}^n$ is a vector space, and we can interpret $\mathrm{d}(x, y)$ as $\|\mathbf{x} - \mathbf{y}\|$, so in the equation $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ we substitute $\mathbf{x} = u - z$ and $\mathbf{y} = z - v$. $\square$

3.1 Balls and openness

Given the metric space (X, d), with $x_0 \in X$ and $r > 0, r \in \mathbb{R}$, we denote as $B(x_0, r)$ (or sometimes $B_r(x_0)$) the "ball":

$$B(x_0, r) = \{x \in X : \mathrm{d}(x, x_0) < r\} \tag{3.1.1}$$

For example, in $\mathbb{R}$ this looks like $(x_0 - r, x_0 + r)$.

Definition 3.1.1. Given the metric space (X, d), and $E \subseteq X$, $x \in X$, we say that x is internal to E if $\exists r \in \mathbb{R}, r > 0 : B(x, r) \subseteq E$ (E intorno di X).

[1] This is not our only option: something like $\mathrm{d}(x, y) = \sqrt{|x - y|}$ would work as well.

Definition 3.1.2. Given the metric space (X, d), and $E \subseteq X$, $x \in X$, we say that x is external to E if $\exists r \in \mathbb{R}, r > 0 : B(x,r) \cap E = \emptyset$. x is thus internal to the complement of E.

Definition 3.1.3. x is a frontier (or boundary) point if it is neither internal nor external.

We denote the set of internal points of E as $\overset{\circ}{E}$ or $\text{int}E \subseteq E$, the boundary as ∂E, and the external points as E^e.

For example, in $\mathbb{R}^n$, if $E = B(x_0, r)$, $\|x - x_0\| < r \iff x \in \overset{\circ}{E}$ because of the triangular inequality: $B(x, (r - \|x - x_0\|)) \subseteq B(x_0, r)$. We can see that for any y $\mathrm{d}(y, x_0) \leq \mathrm{d}(y, x) + \mathrm{d}(x_0, x)$.

Also, if $\|x - x_0\| > r$, $x \in E^e$, since we can construct $B(x, \|x - x_0\| - r) \in E^C$.

If $\|x - x_0\| = r$, $x \in \partial E$ since we can construct $B(x, p)$ for some $p > 0$, and supposing WLOG that $p < r$ we can show that:

$$B(x,p) \setminus E \neq \emptyset \Leftarrow (x \in E^C \implies \forall t > 0 x \in B(x,t)) \tag{3.1.2}$$
$$B(x,p) \cap E \neq \emptyset \tag{3.1.3}$$

We can prove the last statement by defining

$$y := x + \frac{p}{2}\left(\frac{x - x_0}{r}\right) \tag{3.1.4}$$

and showing that $\mathrm{d}(y, x_0) < r$.

It is easily proven that $\partial\mathbb{Q} = \mathbb{R}$, and that $\partial E = \partial(E^C)$.

Open sets

Theorem 3.1.1. $E \subseteq X$ is open, that is, $\overset{\circ}{E} = E$, iff $E \cap \partial E = \emptyset$.

This clearly implies that $\forall E \subseteq X : \overset{\circ}{E}$ is open. E^e is also open. We conventionally say that $\emptyset$ is open.

Proof. Rightward: if $x \in E$, and E is open, then every point is internal, so it is not a frontier point.

Leftward: if $x \in E$, $x \notin \partial E$, and it is not external, so $x \in \overset{\circ}{E}$. □

Theorem 3.1.2. A union (finite or infinite) of open sets is open.

Proof. If $x \in \cup E_i$, $\exists k : x \in E_k$, so $\exists r > 0 : B(x, r) \subseteq E_k \subseteq \cup E_i$. □

Theorem 3.1.3. If A and B are open, $A \cap B$ is open.

Proof. Take $x \in A \cap B$. Then, $\exists r_1, r_2 > 0 : B(x, r_1) \subseteq A$ and $B(x, r_2) \subseteq B$. If we take $r := \min\{r_1, r_2\}$ we have $B(x, r) \subseteq A \cap B$. □

By induction we can see that, if $\forall i : A_i$ is open, then $\cap_{i=0}^{\infty} A_i$ is open. This, however, does not generalize to infinite intersections.

Closed sets

In a metric space X, $D \subseteq X$ is closed if D^C is open. Intervals like $[a, b] \subseteq \mathbb{R}$ are closed. $\{a\} \subseteq \mathbb{R}$ is also closed.

Theorem 3.1.4. D is closed iff $\partial D \subseteq D$, that is, $D = \overset{\circ}{D} \cup \partial D$.

D is closed iff $D^C \cap \partial(D^C) = \emptyset$, that is, $\partial D \subseteq (D^C)^C = D$.

Theorem 3.1.5. A finite union of closed sets is closed.

Proof.

$$\left(\bigcup_{i=0}^{n} D_i \right)^C = \bigcap_{i=0}^{n} (D_i)^C \tag{3.1.5}$$

□

Theorem 3.1.6. A (finite or infinite) intersection of closed sets is closed.

Proof.

$$\left(\bigcap_{i=0}^{n} D_i \right)^C = \bigcup_{i=0}^{n} (D_i)^C \tag{3.1.6}$$

□

Closure

Definition 3.1.4. The *closure* of a set E, denoted $\bar{E}$, is $\bar{E} := \overset{\circ}{E} + \partial E$.

E is closed iff $\bar{E} = E$. $\bar{E}$ is always closed since $(\bar{E})^C$ is open. $\overset{\circ}{E} \subseteq E \subseteq \bar{E}$.

Theorem 3.1.7. $\bar{E}$ is the smallest closed set containing E: if D is closed and $E \subseteq D$, then $\bar{E} \subseteq D$.

Proof. $D^e \subseteq E^e = \bar{E}^e$. If we take an $x \in \bar{E}$, x cannot belong to $\bar{E}^e$. This means it also does not belong to D^e, so $x \in \overset{\circ}{D} \vee x \in \partial D \implies x \in D$. □

3.2 Limit points

Definition 3.2.1. Take the set $E \subseteq \mathbb{R}^n$: $x_0 \in \mathbb{R}^n$ is a limit point for E if

$$\forall r > 0 : \exists x \in E, x \neq x_0 : x \in B(x_0, r) \tag{3.2.1}$$

Equivalently, $B(x_0, r)$ contains infinite points of E.

The set of the limit points of E is denoted $\mathcal{D}E$: it is called "derived set".

If $x \in E^e$, then $x \notin \mathcal{D}E$; also if $x \in \overset{\circ}{E}$ then $x \in \mathcal{D}E$. This holds in $\mathbb{R}^n$, but it might not in pathological metric spaces.

So we have $\overset{\circ}{E} \subseteq \mathcal{D}E \subseteq \bar{E}$.

Some examples: $\mathcal{D}\mathbb{Z} = \emptyset$, $\mathcal{D}(a, b) = [a, b]$.

Theorem 3.2.1. D is closed iff it contains all of its limit points.

Proof. Rightward implication: $\mathcal{D}D \subseteq \bar{D} = D$.

Leftward implication: by contradiction. Suppose that $\partial D \nsubseteq D$. Then $\exists x \in \partial D \setminus D$, so $\forall r > 0 : \exists y \in B(x, r) \cap D$. But $y \in D$, and $x \notin D$, so $y \neq x$: x is a limit point, which implies $x \in D$. $\square$

Theorem 3.2.2. Given $a, b \in \mathbb{R}$, $\exists r \in \mathbb{Q} : a < r < b$. ($\mathbb{Q}$ is dense in $\mathbb{R}$).

Proof. By Archimedes' property, the multiples of k^{-1} ($k \in \mathbb{N}$) will always exceed a. We can choose a k such that $k(b - a) > 1$. Then,

$$ka < \lfloor ka \rfloor + 1 \leq ka + 1 < kb \tag{3.2.2}$$

$$a < \frac{\lfloor ka \rfloor + 1}{k} < b \tag{3.2.3}$$

$\square$

So, $\bar{\mathbb{Q}} = \mathcal{D}\mathbb{Q} = \mathbb{R}$. We can also easily prove that $\mathcal{D}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$, and $\mathring{\mathbb{Q}} = \emptyset$.

If $E \subseteq \mathbb{R}$ is finite, $\mathcal{D}E = \emptyset$.

Theorem 3.2.3. If $E \subseteq \mathbb{R}$ is limited and infinite, then $\mathcal{D}E \neq \emptyset$.

Proof. We can split E in two parts, taking the average of $\sup E$ and $\inf E$. Then, in one of the sets into which we split E there must still be infinitely many points. We can apply the same splitting process to it, and so on: by the Halved Intervals theorem in the end the intersection of the sets we selected on each iteration will contain just one element $\lambda \in E$, and by construction $\lambda \in \mathcal{D}E$. $\square$

This also holds in $\mathbb{R}^n$.

Chapter 4

Functions

4.1 Basics

Given two nonempty sets A and B, a function $f : A \to B$ is a subset of $A \times B$ such that each elements of A appears in exactly one of the ordered pairs $(a, b) : a \in A, b \in B$.

A is called *domain*, B is called *range*. The *image* of a set $X \subseteq A$ is $f(X) = \{y \in B : \exists x \in X : f(x) = y\}$. A function is *surjective* if $f(A) = B$. Similarly, the *preimage* of a set $Y \subseteq B$ is $f^{-1}(Y) = \{x \in A : \exists y \in Y : f(x) = y\}$.

Composition Given $f : A \to B$ and $g : B \to C$, $\exists h := g \circ f : A \to C$ such that $g \circ f)(x) = g(f(x))$.

For convenience, given an expression in the variable x, we automatically assume it represents a function $f : A \to \mathbb{R}$, which associates every x with the expression evaluated at that point. A is the largest subset of $\mathbb{R}$ for which the function is defined.

Injectivity $f : A \to B$ is injective if $\forall x_1, x_2 \in A : x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

Given an injective function, we can define an inverse: $f^{-1} : f(A) \to A$, where $f^{-1}(y) = x \in A : f(x) = y$.

$\forall x \in A : f^{-1} \circ f(x) = x$ and $\forall y \in f(A) : f \circ f^{-1}(y) = y$.

An inverse function is always bijective.

Definition 4.1.1. Given the subset $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ is said to be increasing if $\forall x_1, x_2 \in A : x_2 > x_1 \implies f(x_2) \geq f(x_1)$, decreasing if $f(x_2) \leq f(x_1)$. It is *strictly* increasing or decreasing if the inequalities are strict.

Definition 4.1.2. f is even if $f(-x) = f(x)$, and odd if $f(-x) = -f(x)$.

4.2 Basic functions

Powers $f(x) = x^\alpha$ is easily defined $\forall x \in \mathbb{R}$ if $\alpha \in \mathbb{Q}$ and its denominator is odd, and only for positive numbers if is even.

In general, though, if $\alpha \notin \mathbb{Q}$ and $\alpha > 0$:

$$x^\alpha = \begin{cases} \sup\{x^p : p \in \mathbb{Q}, p < \alpha\} & x \geq 1 \\ \inf\{x^p : p \in \mathbb{Q}, p < \alpha\} & 0 \leq x < 1 \end{cases} \tag{4.2.1}$$

If $\alpha < 0$, $x^\alpha := (x^{-\alpha})^{-1}$.

Trig functions As usual.

Logs

Definition 4.2.1. Given $a > 0$, $a \neq 1$, $x > 0$, we define: $\log_a x = p \iff a^p = x$.

Some properties are: $\log_{a^{-1}} x = -\log_a x$, $\log_b x = \log_b a \cdot \log_a x$.

Misc $\operatorname{sgn} x$ is defined as usual, and $\operatorname{sgn} 0 = 0$.

Definition 4.2.2. Given a function $f : A \to B$, and $D \subset A$, we can restrict f to D: the function $f_{|D} : D \to B$ is defined by $\forall x \in D : f_{|D}(x) = f(x)$.

Inverses Since sin, cos and tan are not injective, to invert them we have to restrict their domain: arcsin is the inverse of $\sin_{|[-\pi/2,\pi/2]} x$, $\arccos x$ is the inverse of $\cos_{|[0,\pi]} x$ and arctan is the inverse of $\tan_{|]\pi/2,\pi/2[} x$.

4.3 Some other properties

Definition 4.3.1. A function $f : A \to \mathbb{R}$ is bounded by above if $\exists M \in \mathbb{R} : \forall x \in A : f(x) \leq M$; that is, $\sup\{f(A)\} \in \mathbb{R}$. The definition is analogous for functions that have a lower bound.

A function is bounded if it has a lower and upper bound; that is, $\left|f(x)\right|$ has an upper bound.

Definition 4.3.2. The upper bound of f in A is the $\sup\{f(A)\}$, it is often denoted $\sup_A f$ or $\sup_{x \in A} f$.

Definition 4.3.3. $x_0 \in A$ is a maximum for f in A if $\forall x \in A : f(x) \leq f(x_0)$. It is often denoted $\max_A f$ or $\max_{x \in A} f$.

The following hold:

$$\forall x \in A : f(x) \leq \sup_A f \tag{4.3.1}$$

$$\forall \varepsilon > 0 : \exists z \in A : f(z) > \sup_A f - \varepsilon \tag{4.3.2}$$

Chapter 5

Limits

5.1 Definitions

Definition 5.1.1. Given $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and a limit point $x_0 \in D$, we say that $\lim_{x \to x_0} f(x) = L$ iff

$$\forall \varepsilon > 0 : \exists \delta > 0 : f(D \cap B(x_0, \delta) \setminus \{x_0\}) \subseteq B(L, \varepsilon) \tag{5.1.1}$$

Since x_0 is a limit point, $\forall \delta > 0 :]x_0 - \delta, x_0 + \delta[\setminus \{x_0\} \neq \emptyset$. The value of $f(x_0)$ is irrelevant. The value of δ we choose depends both on ε and x_0; if it works for ε_1 it must also $\forall \varepsilon > \varepsilon_1$.

5.2 Properties

Theorem 5.2.1. Given $D \subseteq \mathbb{R}$, $f, g : D \to \mathbb{R}$, and a limit point $x_0 \in D$. If

$$\lim_{x \to x_0} f(x) = L \in \mathbb{R} \wedge \lim_{x \to x_0} g(x) = M \in \mathbb{R} \tag{5.2.1}$$

then

$$\lim_{x \to x_0} f(x) + g(x) = L + M \tag{5.2.2}$$

$$\lim_{x \to x_0} f(x)g(x) = LM \tag{5.2.3}$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{as long as } M \neq 0 \tag{5.2.4}$$

Additivity.

$$\forall \varepsilon > 0 : \exists \delta_1 > 0 : \forall x \in D : 0 < |x - x_0| < \delta_1 \big| f(x) - L \big| < \varepsilon \tag{5.2.5}$$

$$\forall \varepsilon > 0 : \exists \delta_2 > 0 : \forall x \in D : 0 < |x - x_0| < \delta_2 \big| g(x) - M \big| < \varepsilon \tag{5.2.6}$$

We can take $\delta := \min\{\delta_1, \delta_2\}$. Then

$$\forall \eta = 2\varepsilon > 0 : \exists \delta > 0 : \forall x \in D :$$
$$0 < |x - x_0| < \delta \big| f(x) + g(x) - (L + M) \big| \leq \big| f(x) - L \big| + \big| g(x) - M \big| < \eta \tag{5.2.7}$$

□

Multiplicativity. We will use f and g as shorthand for $f(x)$ and $g(x)$. Then, setting the same δ as before, we have that $\forall x \in D : 0 < |x - x_0| < \delta$:

$$|fg - LM| \leq |fg - Lg| + |Lg - LM| \tag{5.2.8}$$
$$= |g||f - L| + |L||g - M| < \big(|g| + |L|\big)\, \varepsilon \tag{5.2.9}$$
$$< \big(1 + |M| + |L|\big)\, \varepsilon \tag{5.2.10}$$

The last step is justified since, if $\varepsilon < 1$, $g \in [M - 1, M + 1]$. □

Divisibility. We can say that $\forall \varepsilon > 0$ there $\exists \delta > 0$ such that:

$$\left| \frac{f}{g} - \frac{L}{M} \right| \leq \frac{|M||f - L| + |L||g - M|}{|g||M|} \tag{5.2.11}$$
$$< 2 \left(\frac{|M| + |L|}{M^2} \right) \varepsilon \tag{5.2.12}$$

□

Theorem 5.2.2. If the limit exists, it is unique.

Proof. By contradiction: suppose both L and M were limits. Then by linearity

$$L - M = \lim_{x \to x_0} f(x) - f(x) = \lim_{x \to x_0} 0 = 0 \tag{5.2.13}$$

□

Change of variable

Theorem 5.2.3. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{y \to L} g(y) = M$, and g is continuous in L, then

$$\lim_{x \to x_0} g(f(x)) = M \tag{5.2.14}$$

Squeeze theorem Given $D \subseteq \mathbb{R}$, a limit point $x_0 \in D$, and $f, g, h : D \to \mathbb{R}$ such that $\exists r : \forall x \in B(x_0, r) : f \leq g \leq h$:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L \implies \lim_{x \to x_0} g(x) = L \tag{5.2.15}$$

Proof. From the definition of limit we have that, $\forall \varepsilon > 0 : \exists \delta_1, \delta_2 :$

$$\begin{aligned} \forall x \in B(x_0, r) :& 0 < |x - x_0| < \delta_1 : \quad f(x) > L - \varepsilon \\ \forall x \in B(x_0, r) :& 0 < |x - x_0| < \delta_2 : \quad g(x) < M + \varepsilon \end{aligned}$$

We can then take $\delta := \min\{r, \delta_1, \delta_2\}$. Then $\forall x \in D : 0 < |x - x_0| < \delta$ we have:

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon \tag{5.2.16}$$

□

This allows us to prove that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{5.2.17}$$

5.3 Infinite limits

We can write that $\lim_{x \to x_0} f(x) = +\infty$ if

$$\forall M > 0 : \exists \delta > 0 : \forall x \in B(x_0, \delta) \setminus \{x_0\} : f(x) > M \tag{5.3.1}$$

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