Notes on Complements of Analysis

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Chapter 1

Set theory

1.1 The ZFC axioms

Extensionality

$$\forall x : \forall y : \forall a : x = y \iff (a \in x \iff a \in y) \tag{1.1.1}$$

Existence of the null set

$$\exists x : \forall y : y \notin x \tag{1.1.2}$$

Foundation Every nonempty set contains an $\in$-minimal element:

$$\forall A : \exists x \in A : \forall y \in A : y \notin x \tag{1.1.3}$$

This means that there cannot be an infinite $\in$ chain like $A_1 \ni A_2 \ni A_3 \ni \ldots$.
We can also say that $\forall A : \exists x \in A : x \cap A = \emptyset$.
This also exclude the existence of the set of all sets: $\nexists x : \forall y : y \in x$.

Separation Given a well-defined property $P(x)$, there exists a set such that

$$\forall y : \forall x : x \in y \wedge P(x) \tag{1.1.4}$$

This implies the existence of the empty set, and excludes Russel's paradox.

Pair sets

$$\forall a : \forall b : \exists x : \forall y : y \in x \iff (y = a \vee y = b) \tag{1.1.5}$$

This implies the existence of singlets, and of ordered pairs, defined as: $(a, b) := \{\{a\}, \{a, b\}\}$ ($a = \cap(a, b)$, $b = \cup(a, b) \setminus \cap(a, b)$).
Of course,

$$(a, b) = (c, d) \iff (a = c) \wedge (b = d) \tag{1.1.6}$$

Union set axiom

$$\forall x : \exists u : \forall z : \exists y : z \in u \iff (z \in y \land y \in x) \tag{1.1.7}$$

The usual notation is $u = \cup x$, or $A \cup B$. This also enables us to define intersections:

$$A \cap B = \big\{x \in \{A \cup B\} : x \in A \land x \in B\big\} \tag{1.1.8}$$

Power set axiom

$$\forall x : \exists p : \forall y : y \in p \iff y \subseteq x \tag{1.1.9}$$

The usual notation is: $p = \mathcal{P}(x)$.

Infinity

$$\exists x : \forall y : \emptyset \in x \land (y \in x \implies y \cup \{y\} \in x) \tag{1.1.10}$$

Replacement

1.2 The Von Neumann Integers

We define $0_{VN} = \emptyset$, and $S(n_{VN}) = n \cup \{n_{VN}\}$. So $1_{VN} = \{\emptyset\}$, $2_{VN} = \{\emptyset, \{\emptyset\}\}$, $3_{VN} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$...

The $\leq$ relation is thus replaced by $\subseteq$, and $<$ by $\in$.

The axiom of infinity seems to define the VN integers, but many sets could have those properties. So, we define the property $P(x) = \emptyset \in x \land (y \in x \implies y \cup \{y\} \in x)$

We'd like to intersect all the sets satisfying $P(x)$ (HOW?)

$$\omega := \{k \in x : k \in Y \iff \forall Y \in \mathcal{P}(x) : P(y) \tag{1.2.1}$$

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