

# Notes on Complements of Analysis

Jacopo Tissino

November 16, 2016

# Chapter 1

## Set theory

### 1.1 The ZFC axioms

#### Extensionality

$$\forall x : \forall y : \forall a : x = y \iff (a \in x \iff a \in y) \quad (1.1.1)$$

#### Existence of the null set

$$\exists x : \forall y : y \notin x \quad (1.1.2)$$

**Foundation** Every nonempty set contains an  $\in$ -minimal element:

$$\forall A : \exists x \in A : \forall y \in A : y \notin x \quad (1.1.3)$$

This means that there cannot be an infinite  $\in$  chain like  $A_1 \ni A_2 \ni A_3 \ni \dots$ .

We can also say that  $\forall A : \exists x \in A : x \cap A = \emptyset$ .

This also excludes the existence of the set of all sets:  $\nexists x : \forall y : y \in x$ .

**Separation** Given a well-defined property  $P(x)$ , there exists a set such that

$$\forall y : \forall x : x \in y \wedge P(x) \quad (1.1.4)$$

This implies the existence of the empty set, and excludes Russel's paradox.

#### Pair sets

$$\forall a : \forall b : \exists x : \forall y : y \in x \iff (y = a \vee y = b) \quad (1.1.5)$$

This implies the existence of singlets, and of ordered pairs, defined as:  $(a, b) := \{\{a\}, \{a, b\}\}$  ( $a = \cap(a, b)$ ,  $b = \cup(a, b) \setminus \cap(a, b)$ ).

Of course,

$$(a, b) = (c, d) \iff (a = c) \wedge (b = d) \quad (1.1.6)$$

### Union set axiom

$$\forall x : \exists u : \forall z : \exists y : z \in u \iff (z \in y \wedge y \in x) \quad (1.1.7)$$

The usual notation is  $u = \cup x$ , or  $A \cup B$ . This also enables us to define intersections:

$$A \cap B = \{x \in \{A \cup B\} : x \in A \wedge x \in B\} \quad (1.1.8)$$

### Power set axiom

$$\forall x : \exists p : \forall y : y \in p \iff y \subseteq x \quad (1.1.9)$$

The usual notation is:  $p = \mathcal{P}(x)$ .

### Infinity

$$\exists x : \forall y : \emptyset \in x \wedge (y \in x \implies y \cup \{y\} \in x) \quad (1.1.10)$$

**Replacement** Given the set  $A$ , we can construct the set  $\{x \in A : R(x)\}$ . This allows us to construct infinite unions: given the sets  $A_i$ ,  $i \in \mathbb{N}$ ,

$$W = \mathbb{N} \rightarrow \{A_0, A_1, A_2 \dots\} = I \quad (1.1.11)$$

then  $\exists \cup I = \cup_{n \in \mathbb{N}} A_n$ .

**Choice** Given a set  $A$  of nonempty sets, such that any two are disjoint, we can always find a set  $B$  containing exactly one element for any element of  $A$ .

(Reformulate as:  $A \times B = \emptyset$  iff  $A = \emptyset$  or  $B = \emptyset$ .)

#### 1.1.1 Goedel

In 1938 Goedel proved that ZFC is coherent. In 1931 he proved that any coherent axiom set contains undecidable propositions. One example for ZFC is the continuum hypothesis.

## 1.2 The Von Neumann Integers

We define  $0_{VN} = \emptyset$ , and  $S(n_{VN}) = n \cup \{n_{VN}\}$ . So  $1_{VN} = \{\emptyset\}$ ,  $2_{VN} = \{\emptyset, \{\emptyset\}\}$ ,  $3_{VN} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\dots$

The  $\leq$  relation is thus replaced by  $\subseteq$ , and  $<$  by  $\in$ .

The axiom of infinity seems to define the VN integers, but many sets could have those properties. So, we define the property  $P(x) = \emptyset \in x \wedge (y \in x \implies y \cup \{y\} \in x)$

We'd like to intersect all the sets satisfying  $P(x)$  (HOW?)

$$\omega := \{k \in x : k \in Y \iff \forall Y \in \mathcal{P}(x) : P(y)\} \quad (1.2.1)$$

to get the actual set  $\mathbb{N}$ .

## 1.3 Cardinality

Given the sets  $A$  and  $B$ , we say they have the same cardinality if there exists a bijective  $f : A \rightarrow B$ .

Having the same cardinality is an equivalence relation, but the set of all the sets with the same cardinality is not a set.

We say that  $|a| \leq |B|$  if  $\exists f : A \rightarrow B$  injective. This is an order relation: it is

- reflexive:  $|A| \leq |A|$ ;
- transitive:  $|A| \leq |B| \wedge |B| \leq |C| \implies |A| \leq |C|$ ;
- antisymmetric:  $|A| \leq |B| \wedge |B| \leq |A| \implies |A| = |B|$ ;
- connected if there is a good ordering:  $\forall A, B : |A| \leq |B| \vee |B| \leq |A|$ .

Saying that there exists a surjective  $g : B \rightarrow A$  is equivalent to saying there exists an injective  $f : A \rightarrow B$ .

*Proof.* If there exists an injective  $f : A \rightarrow B$ , we define  $g(y) := f^{-1}(y)$  if  $y \in f(A)$ , and any  $a_0$  otherwise.  $g$  is surjective.

If there exists a surjective  $g : B \rightarrow A$ , we define  $f(x)$  as any element in  $g^{-1}\{x\}$ . We need the axiom of choice for this.  $\square$

*Proof of point 1.3, by Cantor-Bernstein-Schroeder.* We have the two bijective functions  $f : A \rightarrow B$  and  $g : B \rightarrow A$ . Then  $\square$

# Chapter 2

## Inequalities

### 2.1 Means

We shall treat sequences ( $n$ -uples) in the form  $a = (a_1, a_2, \dots, a_n)$  where  $\forall i : a_i \geq 0$ .

**Definition 2.1.1.** The  $r$ -mean of  $a$ , denoted  $M_r(a)$  or just  $M_r$ , is:

$$M_r(a) := \left( \frac{1}{n} \sum_{i=1}^n a_i^r \right)^{\frac{1}{r}} \quad (2.1.1)$$

If  $r < 0$  and  $\exists i : a_i = 0$ , we take  $M_r = 0$ .

Some notable means are:

- the arithmetic mean  $A := M_1$ ; any mean can be written as  $M_r = A(a_i^r)^{1/r}$
- the harmonic mean  $H := M_{-1}$
- the geometric mean  $G(a) := \sqrt[n]{\prod_i a_i} = \exp[A(\log a_i)]$

**Definition 2.1.2.** Given the set of weights  $p = (p_1, p_2, \dots, p_n)$ , where  $\forall i : p_i > 0$ , the weighted sum  $M_r$  is:

$$M_r = M(a, p) := \left( \frac{\sum_i p_i a_i^r}{\sum_i p_i} \right)^{1/r} \quad (2.1.2)$$

The weighted geometric mean, from the definition, is

$$G := \left( \prod_i a_i^{p_i} \right)^{1/\sum_i p_i} \quad (2.1.3)$$

We notice that means are 1-homogeneous, that is,  $\forall \lambda \in \mathbb{R}^+ : M_r(\lambda a) = \lambda M_r(a)$ .

We can always suppose that the weights all add up to 1. If this is true, we call then  $q_i$ .

We notice that  $\min a_i \leq M_r(a) \leq \max a_i$ , with equality iff all the  $a_i$  are equal, or if  $r < 0$  and  $\exists a_i = 0$ . The same is true for the geometric mean.

*Theorem 2.1.1.*

$$\lim_{r \rightarrow 0} M_r(a) = G(a) \quad (2.1.4)$$

*Proof.* Suppose that  $\forall i : a_i > 0$ . We look at the log of  $M_r$ :

$$\lim_{r \rightarrow 0} \log M_r = \lim_{r \rightarrow 0} \frac{1}{r} \log \sum_i q_i a_i^r \quad (2.1.5)$$

Now,  $\sum_i q_i a_i^r$  goes to 1 under our hypotheses.

So, adding and subtracting 1 to the sum (since the  $q_i$  add up to 1), and multiplying and dividing, we get:

$$\lim_{r \rightarrow 0} \frac{1 + \log \left( \sum_i q_i (a_i^r - 1) \right) \sum_i q_i (a_i^r - 1)}{\sum_i q_i (a_i^r - 1) r} \quad (2.1.6)$$

but by the limit  $\lim_{x \rightarrow 0} \log(1+x)/x = 1$  the first fraction goes to 1, so we get:

$$\lim_{r \rightarrow 0} \frac{\sum_i q_i (a_i^r)}{r} \quad (2.1.7)$$

which by the linearity of the limit and the limit  $\lim_{x \rightarrow 0} (a^x - 1)/x = \log a$  equals:

$$\sum_i q_i \log a_i = \log \left( \prod_i a_i^{q_i} \right) = \log G(a) \quad (2.1.8)$$

In the case where  $\exists a_i = 0$ , we take the sets  $b = \{a_i \neq 0\}$ ,  $s = \{\text{the corresponding } q_i\}$ . Now, in the limit

$$\lim_{r \rightarrow 0^+} M_r(a, q) = \lim_{r \rightarrow 0^+} \left( \sum_i q_i a_i^r \right)^{1/r} \quad (2.1.9)$$

we would like to swap the  $a$ s for the  $b$ s and the  $q$ s for the  $s$ s, but we need to account for the fact that  $\sum_i s_i < 1$ . So

$$\lim_{r \rightarrow 0^+} \left( \sum_i s_i \right)^{1/r} M_r(b, s) = 0 = G(a) = \lim_{r \rightarrow 0^-} M_r(a, q) \quad (2.1.10)$$

by definition. So we define  $M_0 := G$ . □

*Theorem 2.1.2.*

$$\lim_{r \rightarrow +\infty} M_r(a_i) = \max(a_i) \quad (2.1.11)$$

$$\lim_{r \rightarrow -\infty} M_r(a_i) = \min(a_i) \quad (2.1.12)$$

*Proof.* We take  $a_k$  to be the maximum  $a_i$ . Then, since  $(q_i a_i^r)^{1/r} \leq (\sum_i q_i a_i^r)^{1/r}$ , we can write

$$q_k^{1/r} a_k \leq M_r(a_i) \leq \max(a_i) \quad (2.1.13)$$

which, by the squeeze theorem, implies the thesis.

For  $r \rightarrow -\infty$ , we just need to notice that

$$M_{-r} a_i = \frac{1}{M_r \left( \frac{1}{a_i} \right)} \quad (2.1.14)$$

and that the maximum of the  $1/a_i$  corresponds to the minimum of the  $a_i$ .  $\square$

### Cauchy's Inequality

*Theorem 2.1.3.* Given two sequences of numbers  $a_i$  and  $b_i$  with the usual properties:

$$\left( \sum_i a_i b_i \right)^2 \leq \left( \sum_i a_i^2 \right) \left( \sum_i b_i^2 \right) \quad (2.1.15)$$

The equality holds iff the vectors  $a$  and  $b$  are linearly dependent.

*Proof.* We can rearrange the inequality like:

$$\left( \sum_i a_i^2 \right) \left( \sum_j b_j^2 \right) - \left( \sum_k a_k b_k \right)^2 \geq 0 \quad (2.1.16)$$

$$\left( \sum_i a_i^2 \right) \left( \sum_j b_j^2 \right) - \left( \sum_i a_i b_i \right) \left( \sum_j a_j b_j \right) \geq 0 \quad (2.1.17)$$

$$\sum_{i,j} a_i^2 b_j^2 - \sum_{i,j} a_i b_i a_j b_j \geq 0 \quad (2.1.18)$$

$$\frac{1}{2} \sum_{i,j} 2 \left( a_i^2 b_j^2 - a_i b_i a_j b_j \right) \geq 0 \quad (2.1.19)$$

$$\frac{1}{2} \sum_{i,j} (a_i b_j - b_i a_j)^2 \geq 0 \quad (2.1.20)$$

Where, in the last passage, we have swapped some indices which would have been summed over in another iteration anyway. Now, this is clearly true.

$a$  and  $b$  are proportional iff  $\forall i, j : a_i b_j - b_i a_j = 0$ , that is, the matrix they span has rank 1.  $\square$

This implies that  $\forall r > 0 : M_r \leq M_{2r}$ , with equality iff all the  $a_i$  are equal. This can be easily proven by setting  $a_i := \sqrt{p_i}$  and  $b_i := \sqrt{p_i} a_i^r$  and applying the theorem.

*Theorem 2.1.4.*  $G \leq A$ .

*Proof.*  $A = M_1 \geq M_{1/2} \geq M_{1/4} \geq M_{1/8} \geq \dots \geq \lim_{r \rightarrow 0} M_r = G$   $\square$

*Theorem 2.1.5* (Young's Inequality).

$$\forall a, b \geq 0 : \forall p > 1 : ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'} \quad (2.1.21)$$

where  $p^{-1} + p'^{-1} = 1$ . Equality holds iff  $b = a^{p-1}$ .

*Proof.* We just need to use  $G \leq A$  with the set  $(a^p, b^{p'})$  and as weights  $(p^{-1}, p'^{-1})$ .  $\square$

*Integral version of the proof.* Suppose WLOG that  $b \leq a^{p-1}$ . Now, graph the function  $y = x^{p-1}$ . Now, in  $[0; +\infty] \times [0; +\infty]$ , consider the area between  $x = a$  and  $y = b$ , it is  $ab$  and surely less than the sum of these integrals:

$$ab \leq \int_0^a x^{p-1} dx + \int_0^b y^{\frac{1}{p-1}} dy = \frac{a^p}{p} + \frac{b^{\frac{1}{p-1}+1}}{\frac{1}{p-1}+1} = \frac{a^p}{p} + \frac{b^{p'}}{p'} \quad (2.1.22)$$

$\square$

## Hoelder

*Theorem 2.1.6.* Given some  $n$ -uples,  $(a_{ji})$  ( $j$  is the index of the tuple number, and  $i$  is the element in the tuple) and some weights  $\alpha_i$  such that  $\sum_i \alpha_i = 1$ , the following holds:

$$\sum_i \left( \prod_j a_{ji}^{\alpha_j} \right) \leq \prod_j \left( \sum_i a_{ji} \right)^{\alpha_j} \quad (2.1.23)$$

with equality iff all the  $n$ -uples are proportional.

*Proof.* If one of the tuples is 0 in every position, then the theorem is automatically proven.

Otherwise, we can divide the left side of the inequality by the right to get:

$$\frac{\sum_i \left( \prod_j a_{ji}^{\alpha_j} \right)}{\prod_j \left( \sum_i a_{ji} \right)^{\alpha_j}} = \sum_i \left( \prod_j \left( \frac{a_{ji}}{\sum_k a_{jk}} \right)^{\alpha_j} \right) \leq 1 \quad (2.1.24)$$



but

$$\sum_i \left( \prod_j \left( \frac{a_{ji}}{\sum_k a_{jk}} \right)^{\alpha_j} \right) \leq \sum_i \left( \sum_j \alpha_j \left( \frac{a_{ji}}{\sum_k a_{jk}} \right) \right) = \sum_j \alpha_j \frac{\sum_i a_{ji}}{\sum_k a_{jk}} = 1 \quad (2.1.25)$$

by  $G \leq A$ , and since  $\sum_k \alpha_k = 1$ . □

# Chapter 3

## Convex functions

### 3.1 Definition

**Definition 3.1.1.** Given an interval  $I \subseteq \mathbb{R}$ ,  $f : I \rightarrow \mathbb{R}$  is convex if,  $\forall x_1, x_2 \in I$  and  $\forall \lambda \in [0, 1]$ :

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \quad (3.1.1)$$

# Chapter 4

## Miscellaneous

### 4.1 Stirling's formula

For the proof we will need another result first:

*Theorem 4.1.1* (Wallis).

$$\lim_{n \rightarrow +\infty} \frac{\prod_{i=1}^n (2i)^2}{(2n+1) \prod_{i=1}^{n-1} (2i+1)^2} = \frac{\pi}{2} \quad (4.1.1)$$

*Proof.* We define the succession  $I_n$  as:

$$I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n dx \quad (4.1.2)$$

Clearly a first property is  $I_{n+1} \leq I_n$ . Then, if  $n \geq 2$ , we can calculate:

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^2(x) \sin^{n-2} x dx \\ &= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \sin^{n-2} x dx \\ &= I_{n-2} - \int_0^{\frac{\pi}{2}} (\cos^2 x) \left( \frac{\sin^{n-1} x}{n-1} \right)' dx \\ &= I_{n-2} - \frac{1}{n-1} \left( \cos x \sin^{n-1} x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \sin^n x dx \right) \\ &= I_{n-2} - \frac{1}{n-1} I_n \end{aligned}$$

therefore  $I_n = I_{n-2}((n-1)/n)$ .

Now, consider:

$$\frac{I_{2n+1}}{I_{2n}} = \frac{2n}{2n+1} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1 \quad (4.1.3)$$

this, by the squeeze theorem, implies

$$\lim_{n \rightarrow +\infty} \frac{I_{2n+1}}{I_{2n}} = 1 \quad (4.1.4)$$

But we can expand the  $I_n$  into products of the even or odd  $I_n$  preceding them, so we get:

$$\lim_{n \rightarrow +\infty} \frac{I_{2n+1}}{I_{2n}} = \frac{I_1 \prod_{i=1}^n \frac{2i}{2i+1}}{I_0 \prod_{i=1}^n \frac{2i-1}{2i}} = \frac{\prod_{i=1}^n (2i)^2}{(2n+1) \prod_{i=1}^{n-1} (2i+1)^2} \frac{I_1}{I_0} \quad (4.1.5)$$

and  $I_0 = \pi/2$ ,  $I_1 = 1$ . □

*Theorem 4.1.2.* For large enough numbers, the factorial can be approximated as  $n! \sim n^n e^{-n} \sqrt{2\pi n}$ ; that is:

$$\lim_{n \rightarrow +\infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1 \quad (4.1.6)$$

*Proof.* We define the succession  $a_n$  as:

$$a_n := \log \left( \frac{n!}{n^n e^{-n} \sqrt{n}} \right) = \log n! - \log \left( \left( n + \frac{1}{2} \right) \log n - n \right) \quad (4.1.7)$$

So we just need to prove that  $\lim_{n \rightarrow +\infty} a_n = \log(\sqrt{2\pi})$ . First, we will show that it is strictly decreasing:

$$\begin{aligned} a_n - a_{n+1} &= \left( n + \frac{1}{2} \right) \log \left( \frac{n+1}{n} \right) - 1 \\ &= (2n+1) \left( \frac{1}{2} \log \left( \frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}} \right) - \frac{1}{2n+1} \right) \end{aligned} \quad (4.1.8)$$

We will now use the fact that,  $\forall t \in [0; 1]$ :

$$\frac{1}{2} \log \left( \frac{1+t}{1-t} \right) - t < \frac{t^3}{3(1-t^2)} \quad (4.1.9)$$

this can be proven with derivatives,<sup>1</sup> but a deeper reason is the fact that  $\forall t \in [-1; 1]$  we can write the following MacLaurin series:

$$\log(1+t) = \sum_{i=1}^{+\infty} \frac{(-1)^{i+1} t^i}{i} \quad (4.1.11)$$

$$\log(1-t) = - \sum_{i=1}^{+\infty} \frac{t^i}{i} \quad (4.1.12)$$

Then we can expand:

$$\log\left(\frac{1+t}{1-t}\right) = \log(1+t) - \log(1-t) = 2 \sum_{i=0}^{+\infty} \frac{t^{2i+1}}{2i+1} \quad (4.1.13)$$

so

$$\frac{1}{2} \log\left(\frac{1+t}{1-t}\right) < t + \frac{t^3}{3} \left( \sum_{i=1}^{+\infty} t^{2i} \right) = t + \frac{t^3}{3(1-t^2)} \quad (4.1.14)$$

Now we can take  $t = (2n+1)^{-1}$  in equation (4.1.8), to get the following formula for  $a_n - a_{n+1}$ :

$$f(t) = \frac{1}{t} \left( \frac{1}{2} \log\left(\frac{1+t}{1-t}\right) - t \right) < \frac{t^2}{3(1-t^2)} \quad (4.1.15)$$

$$= \frac{(2n+1)^{-2}}{3(1-(2n+1)^{-2})} \quad (4.1.16)$$

$$= \frac{1}{12n^2 + 12n} \quad (4.1.17)$$

$$= 12 \left( \frac{1}{n} - \frac{1}{n+1} \right) \quad (4.1.18)$$

So, we have proven that the function  $a_n - (12n)^{-1}$  is strictly increasing; that is, its limit is either real or  $+\infty$ . Now, we shall prove that  $a_n$  is strictly decreasing, that is,  $0 < a_n - a_{n+1}$ .

It is enough for  $f(t)$  to satisfy this condition:

$$f'(t) = \frac{1}{2} \frac{1}{1+t} + \frac{1}{2} \frac{1}{1-t} > 0 \quad \forall t \in [0, 1] \quad (4.1.19)$$

---

1

$$\frac{d}{dt} \left( \frac{1}{2} \left( \log\left(\frac{1+t}{1-t}\right) - t \right) - \frac{t^3}{3(1-t^2)} \right) = \frac{-2t^4 - 3t^2 + 3}{3(t^2-1)^2} < 0 \quad (4.1.10)$$

and we have equality for  $t = 0$  in (4.1.9).

and  $f(0) = 0$ , so  $\forall t \in [0, 1] : f(t) > 0$ . So  $a_n$  is strictly decreasing, thus its limit is either real or  $-\infty$ . But the limits of  $a_n$  and  $a_n - (12n)^{-1}$  must be the same since their difference has limit 0: so  $\lim_{n \rightarrow +\infty} a_n = c \in \mathbb{R}$ .

□

# Contents

<b>1</b>	<b>Set theory</b>	<b>1</b>
1.1	The ZFC axioms . . . . .	1
	Extensionality . . . . .	1
	Existence of the null set . . . . .	1
	Foundation . . . . .	1
	Separation . . . . .	1
	Pair sets . . . . .	1
	Union set axiom . . . . .	2
	Power set axiom . . . . .	2
	Infinity . . . . .	2
	Replacement . . . . .	2
	Choice . . . . .	2
	1.1.1 Goedel . . . . .	2
1.2	The Von Neumann Integers . . . . .	2
1.3	Cardinality . . . . .	3
<b>2</b>	<b>Inequalities</b>	<b>4</b>
2.1	Means . . . . .	4
	Cauchy's Inequality . . . . .	6
	Hoelder . . . . .	7
<b>3</b>	<b>Convex functions</b>	<b>9</b>
3.1	Definition . . . . .	9
<b>4</b>	<b>Miscellaneous</b>	<b>10</b>
4.1	Stirling's formula . . . . .	10