

Notes on Calculus I

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Chapter 1

Naïve set theory

1.1 Basic sets

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}, \quad 0 \notin \mathbb{N} \quad (1.1.1)$$

$$\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\} \quad (1.1.2)$$

Remarks

- “ \in ” is for elements belonging to sets, “ \subseteq ” is for subsets
- $\{x\} \neq x$: the first is a set with x as its only element, and is called a “singlet”
- \subsetneq means “is a subset of, but not equal to”
- the elements of $\mathcal{P}(A)$ are precisely all the subsets of A
- $\#A$ is the cardinality of A
- $\#\mathcal{P}(A) = 2^{\#A}$

The naïve definitions of $A \cup B$, $A \cap B$, $A \setminus B$ are given.

Properties

- $A = (A \cap B) \cup (A \setminus B)$
- $(A \cap B) \cap (A \setminus B) = \emptyset$
- $C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$
- $C \cup (A \cap B) = (C \cup A) \cap (C \cup B)$

Complement

Definition 1.1.1. With respect to a “universe” set U , we define the complement of A as $U \setminus A$, denoted A^C .

The following hold:

- $(A \cup B)^C = A^C \cap B^C$
- $(A \cap B)^C = A^C \cup B^C$

Cartesian product

Definition 1.1.2. An *ordered pair* is a set of the form $\{\{x\}, \{x, y\}\}$, denoted (x, y) (where order matters).

Definition 1.1.3. We define the *cartesian product* $A \times B$ of two sets A and B as:

$$A \times B := \{(a, b) : a \in A, b \in B\} \quad (1.1.3)$$

1.2 Propositional logic

Implication

Definition 1.2.1.

$$p \implies q \iff (\neg p) \vee q \quad (1.2.1)$$

Double implication

Definition 1.2.2.

$$(p \iff q) \iff (p \implies q \wedge q \implies p) \quad (1.2.2)$$

Quantifiers $P(x)$ is a *predicate*. We say that $\forall x : P(x)$ if $P(x)$ is true independently of x , and that

$$\exists x : P(x) \iff \neg(\forall x : \neg P(x)) \quad (1.2.3)$$

Chapter 2

Number sets

2.1 The set \mathbb{N}

The Peano axioms

1. $1 \in \mathbb{N}$
2. $\forall n \in \mathbb{N} : \exists S(n) : \mathbb{N} \rightarrow \mathbb{N}$
3. $\forall n \in \mathbb{N} : S(n) \neq 1$
4. $\forall m, n \in \mathbb{N} : m \neq n \implies S(n) \neq S(m) \text{ or } S(n) = S(m) \implies n = m$
5. $(A \subseteq \mathbb{N}) \wedge (1 \in A) \wedge (n \in A \implies S(n) \in A) \implies A = \mathbb{N}$

Any set that verifies these axioms is isomorphic to \mathbb{N} . \mathbb{R}^+ , for example only satisfies the first 4.

2.2 Induction

If $P(n)$ is a proposition, $P(1)$ and $P(n) \implies P(n+1)$ ¹ (the *inductive hypothesis*); then $\forall n \in \mathbb{N}, P(n)$.

Proof. Define $A := \{n \in \mathbb{N} : P(n)\}$. By axiom 5, $A = \mathbb{N}$. \square

If a property $a' : P(k)$ holds for some $k \in \mathbb{N}$ and $b' : P(n) \implies P(n+1)$, then $\forall n \geq k : P(n)$.

Examples Example: we can show by induction that

$$\sum_{i=1}^n = \frac{n(n+1)}{2} \tag{2.2.1}$$

Example 2: we can show by induction that $P(\#A) : \#\mathcal{P}(A) = 2^{\#A}$.

¹We introduce the notation $n+1$ to signify $S(n)$.

Proof. $P(1)$ is true. We see that for any A we can take an element such that $A = \{a\} \cup B$. Then for any subset I , either $a \in I$ or $a \notin I$. If $a \in I$, $I = \{a\} \cup J$, but there are 2^n possible J s. If $a \notin I$, we have 2^n I 's. So there are 2^{n+1} possible subsets. \square

Example 3: show that $n! > 2^n$, which is true for $n > 3$.

Proof. We will use the second form of the induction principle. $P(4)$ is true. If $n > 4$ and $n! > 2^n$, we need to show that $(n+1)n! > 2 \cdot 2^n$. But $n+1 > 2$ by hypothesis, so the inequality always holds. \square

Observation: the notation $1 + 2 + 3 + 4 + \cdots + n$ is unclear, we should use $\sum_{i=1}^n i$.

Recursive formulas: we know the first term, and an algorithm to derive any term from the one before it, such as the definition of the factorial:

$$\begin{cases} 0! = 1 \\ (n+1)! = (n+1)n! \end{cases} \quad (2.2.2)$$

Another example is the sequence:

$$\begin{cases} S_1 = 2 \\ S_{n+1} = S_n + (2n+1) \end{cases} \quad (2.2.3)$$

Proof that $S_n = n^2 + 1$. Assume that $S_n = n^2 + 1$. Then $S_{n+1} = n^2 + 1 + 2n + 1 = (n+1)^2 + 1$. \square

Formal definition of summation:

$$\sum_{i=0}^n a_i = a_0 + a_1 + a_2 + a_3 + \cdots + a_n \quad (2.2.4)$$

by recursion: for an increment in n , we just add the $n+1$ -th term. So it comes down to the formal definition of induction.

Example: show by induction that

$$\forall a \in \mathbb{N} \vee a = 0 \quad (a \neq 1) : \quad \sum_{k=0}^n a^k = \frac{1 - a^{n+1}}{1 - a} \quad (2.2.5)$$

See the property for $n = 0$. Suppose that the property holds for n , show it for $n+1$:

$$\sum_{i=0}^{n+1} a_k = \sum_{k=0}^n a^k + a^{n+1} = \frac{1 - a^{n+1}}{1 - a} + a^{n+1} = \frac{1 - a^{n+2}}{1 - a} \quad (2.2.6)$$

To do: given two real numbers, $\forall n \in \mathbb{N}$, show that

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} \quad (2.2.7)$$

2.3 Groups

We have a set A with an operation $*$: $a, b \in A \rightarrow a * b \in A$. Es: A strings, $*$ concatenation. $(A, *)$ is a group if the following are satisfied:

1. Associativity: $\forall a, b, c \in A : \quad a * (b * c) = (a * b) * c$
2. Identity: $\exists e \in A : \forall a \in A : \quad e * a = a * e = a$
3. Inverse: $\forall a \in A : \exists a^{-1} \in A : \quad a * a^{-1} = a^{-1} * a = e$

$(A, *)$ is also a commutative group if $\forall a, b \in A : \quad a * b = b * a$

Examples: $(\mathbb{N}, +)$ is not a group: there is no identity, but even if we add 0 there is no inverse. $(\mathbb{Z}, +)$ is a commutative group. (\mathbb{R}, \times) is not a group because 0 has no inverse. (\mathbb{R}_0, \times) is hower a group, as is $(\mathbb{R}_0, +)$. (\mathbb{Z}_0, \times) is not a group because there is no inverse: (\mathbb{Q}_0, \times) is a commutative group.

So we introduce \mathbb{Q} :

$$\mathbb{Q} := \left\{ \frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{N} \right\} / \sim \quad (2.3.1)$$

but $ad \sim bc$ and \mathbb{Q} is isomorphic to \mathbb{Z}

Definition 2.3.1. Order relation:

$$\frac{a}{b} \leq \frac{c}{d} \iff ad \leq bc \quad (2.3.2)$$

Definition 2.3.2. Sum and product, subtraction and division are analogous:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad (2.3.3)$$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \quad (2.3.4)$$

$(\mathbb{Q}, +, \times)$ is then a field:

1. $(\mathbb{Q}, +)$ is a commutative group;
2. (\mathbb{Q}_0, \times) is a commutative group
3. $p(q + r) = pq + pr$

Roots The square root of a number $a \geq 0$ is a $b \geq 0$ such that $b^2 = a$.

Show that in \mathbb{Q} there is no $\sqrt{2}$:

Proof. By contradiction:² if there were $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that

²In the form: to show that $p \implies q$, we show that $\neg q \implies \neg p$, since

$$(p \implies q) \iff (\neg q \implies \neg p) \quad (2.3.5)$$

$$\left(\frac{a}{b}\right)^2 = 2 \quad (2.3.6)$$

Suppose that a, b have $\gcd(a, b) = 1$. Then $a^2 = 2b^2$. So a is even, and $a = 2k$. Then $4k^2 = 2b^2 \implies b = 2n$: so $\gcd(a, b) \neq 1$ \square

With the same reasoning we can show that numbers such that $\sqrt{3}$ and $\sqrt[3]{2}$ are irrational.

2.4 The set \mathbb{R}

The set is given. \mathbb{R} is a completely ordered field:

1. $(\mathbb{R}, +)$ is a commutative group
2. (\mathbb{R}_0, \times) is a commutative group
3. $a(b + c) = ab + ac$

There also exists a relation called \leq , such that $\forall a, b, c \in \mathbb{R}$:

1. $a \leq a$
2. $a \leq b \wedge b \leq a \implies a = b$
3. $a \leq b \wedge b \leq c \implies a \leq c$
4. $a \leq b \vee b \leq a$ (*completely* ordered)
5. $a \leq b \implies a + c \leq b + c$
6. $a \geq 0 \wedge b \geq 0 \implies ab \geq 0$ (of course, $a \geq 0$ means that $0 \leq a$)

$(\mathbb{Q}, +, \times)$ is also a completely ordered field?

Take the set of all the real numbers whose squares are greater or equal to 2: it has a minimum.

In \mathbb{Q} , it has no minimum.

Other statements: show that $a \leq 0 \wedge b \geq 0 \implies ab \leq 0$

Proof. Is $a \geq 0 \wedge -b \geq 0$? We first need to show that $a \geq 0 \iff -a \leq 0$: it suffices to add $-a$ to both sides. We also need to show that $a(-b) = -ab$: by an inverse application of the distributive property. \square

Useful inequalities $\forall x \in \mathbb{R} : x^2 \geq 0$. Also, $\forall a, b \in \mathbb{R} : ab \geq (a^2 + b^2)/2$ (one of the inequalities between the means. From this, we can also show that between the rectangles of perimeter p , the square is the one with the largest area.

TO DO: show this for parallelograms and trapezes.

Integer part of a number Given an $x \in \mathbb{R}$, we define its integer part $\lfloor x \rfloor = n \in \mathbb{Z}$ as the largest integer such that $n \leq x$.³

The fractionary part $\{x\}$ is defined as

$$\{x\} = x - \lfloor x \rfloor \quad (2.4.1)$$

It is clear that $x - 1 < \lfloor x \rfloor \leq x$ and that $0 \leq \{x\} < 1$, and that $\lfloor x \rfloor = x \iff x \in \mathbb{Z}$

2.5 Set notation

The notation $[a; b]$ means $\{x \in \mathbb{R} : a \leq x \leq b\}$, and $(a; b) =]a; b[$ means $\{x \in \mathbb{R} : a < x < b\}$. These are *closed* and *open* sets. We can combine the two types of brackets as we wish.

In the notation $(a; +\infty)$, the symbol ∞ is not a number.

Definition 2.5.1. We define the maximum of a set $E \subseteq \mathbb{R}$, denoted $\max E$ (and analogously $\min E$), as a number $M \in \mathbb{R}$ with the following properties:

1. $\forall x \in E : M \geq x$ (M is an upper bound of E)
2. $M \in E$

A set is limited from above if it has an upper bound, and from below if it has a lower bound. Open sets can be limited, but they do not have maximums and minimums: we can show this by taking the average between the first real number outside of the set and a number we suppose to be this maximum, getting to a contradiction.

Definition 2.5.2. We define the *least upper bound* of a set E as the minimum of the set of the upper bounds, and analogously for lower bounds. We can do this $\forall E \subseteq \mathbb{R} : E \neq \emptyset$, and we denote them as $\inf E$ and $\sup E$.

If E does not have an upper limit, we write $\sup E = +\infty$, but this is just notation. The same goes for $\inf E = -\infty$. It is also common to write $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$, but defining them this way makes the fact $\inf E \leq \sup E$ not true.

E has a maximum iff $\sup E \in E$. If $E \neq \emptyset \neq F$ and $E \subseteq F$, then $\sup E \leq \sup F$ and $\inf F \leq \inf E$.

Theorem 2.5.1 (Archimedes' axiom). Given $a, b \in \mathbb{R}$, where $a > 0$ and $b > 0$, $\exists n \in \mathbb{N} : na > b$.

Proof. We choose $n = \lfloor b/a \rfloor + 1$. □

³This is also the case for negative numbers: so the integer part of a negative real number may have greater absolute value than the number itself.

Axiom of continuity or completeness Given $E \subseteq \mathbb{R}$, $E \neq \emptyset$, with at least an upper bound, there exists $\sup E \in \mathbb{R}$. The inverse is easily proven from this.

\mathbb{R} (and sets which can be bijectively mapped to it, via a map that preserves addition and multiplication) is the only totally ordered set which verifies the axiom of continuity.

Theorem 2.5.2. $\sup E = M \iff \forall x \in M : x \leq M \text{ and } \forall \epsilon > 0 : \exists z \in E : z > M - \epsilon.$

Proof. We will prove the leftward implication by contradiction. Suppose there exists an $M' < M$. Then $\forall \epsilon \in E : z \leq M'$. But define $\epsilon := (M - M')/2$: then ???
Prove the rightward implication \square

Halved intervals

Theorem 2.5.3. Take $a_k, b_k \in \mathbb{R} : \forall k \in \mathbb{N} : a_k < b_k$ and $\forall k \in \mathbb{N} : [a_{k+1}; b_{k+1}]$ is one of the halves of $[a_k; b_k]$:

$$\forall k : [a_{k+1}, b_{k+1}] = \left[a_k, \frac{a_k + b_k}{2} \right] \vee \left[\frac{a_k + b_k}{2}, b_k \right] \quad (2.5.1)$$

Then $\exists! \lambda \in \mathbb{R} : \forall k \in \mathbb{N} : \lambda \in [a_k; b_k]$. We can write this as

$$\bigcap_{k=1}^{+\infty} [a_k; b_k] = \{\lambda\} \quad (2.5.2)$$

λ is clearly the $\sup\{a_k : k \in \mathbb{N}\} = \inf\{b_k : k \in \mathbb{N}\}$.

Proof. First, we will show that $\forall k, j \in \mathbb{N} : a_k < b_j$.

- $k \geq j$: $a_k < b_k \leq b_j$;
- $k < j$: $a_k \leq a_j < b_j$.

so, $\forall k \in \mathbb{N}$, a_k is a lower bound for $\{b_j : j \in \mathbb{N}\}$: $\forall k : a_k \leq \inf\{b_j\}$. So, $\inf\{b_j\} \leq \sup\{a_k\}$ (these must exist since the sets are bounded).

We can see this as:

$$\forall n \in \mathbb{N} : 0 \leq \inf\{b_j\} - \sup\{a_k\} \leq b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \quad (2.5.3)$$

It is clear that if $x \geq 0$ and $\forall \varepsilon > 0 : x \leq \varepsilon$, then $x = 0$.

Also, $\forall \varepsilon > 0 : \exists n \in \mathbb{N} : n\varepsilon > b_1 - a_1$ by Archimedes' property. But $\forall n \in \mathbb{N} : n \leq 2^{n-1}$. So

$$\forall \varepsilon > 0 : \exists n \in \mathbb{N} : 0 \leq \inf\{b_j\} - \sup\{a_k\} \leq b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} < \varepsilon \quad (2.5.4)$$

which implies $\inf\{b_j\} = \sup\{a_k\} := \lambda \in \mathbb{R}$. \square

This doesn't just work for halved intervals: any set of intervals for which $[a_{k+1}, b_{k+1}] \subseteq [a_k, b_k]$ and $\lim_{k \rightarrow +\infty} b_k - a_k = 0$:

Proof. We need to show the two inclusions in

$$\bigcap_{k=1}^{+\infty} [a_k, b_k] = \lambda \tag{2.5.5}$$

$\lambda = \sup\{a_k\} = \inf\{b_k\}$, so $\forall k : \lambda \in [a_k, b_k]$.

We need to show that if $x \in \bigcap_{k=1}^{+\infty} [a_k, b_k]$, then $x = \lambda$.

If $\forall k : a_k \leq x \leq b_k$, then x is an upper bound of $\{a_k\}$ ($x \geq \sup\{a_k\}$) and a lower bound of $\{b_k\}$ ($x \leq \inf\{b_k\}$). So, $x = \lambda$. \square

Chapter 3

Topology

We will focus on the topology of \mathbb{R} , sometimes generalizing to \mathbb{R}^n .

A *metric space* is couple (X, d) , where $d : X \times X \rightarrow \mathbb{R}$ is a *distance* function, satisfying $\forall x, y, z \in X$:

1. $d(x, y) \geq 0$; $d(x, y) = 0 \iff x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq d(x, z) + d(y, z)$ (the triangular inequality).

On \mathbb{R} , the distance function is usually $d(x, y) = |x - y|$.¹

This extends to \mathbb{R}^n :

$$d(x, y) = \sqrt{\sum_{i=0}^n (x_i - y_i)^2} \quad (3.0.1)$$

This clearly satisfies conditions 1 and 2, but we have to prove condition 3:

Proof. This follows from the subadditivity of the norm of vectors: \mathbb{R}^n is a vector space, and we can interpret $d(x, y)$ as $\|\mathbf{x} - \mathbf{y}\|$, so in the equation $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ we substitute $\mathbf{x} = u - z$ and $\mathbf{y} = z - v$. \square

3.0.1 Balls and openness

Given the metric space (X, d) , with $x_0 \in X$ and $r > 0, r \in \mathbb{R}$, we denote as $B(x_0, r)$ (or sometimes $B_r(x_0)$) the “ball”:

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\} \quad (3.0.2)$$

For example, in \mathbb{R} this looks like $(x_0 - r, x_0 + r)$.

Definition 3.0.1. Given the metric space (X, d) , and $E \subseteq X$, $x \in X$, we say that x is internal to E if $\exists r \in \mathbb{R}, r > 0 : B(x, r) \subseteq E$ (E intorno di X).

¹This is not our only option: something like $d(x, y) = \sqrt{|x - y|}$ would work as well.

Definition 3.0.2. Given the metric space (X, d) , and $E \subseteq X$, $x \in X$, we say that x is external to E if $\exists r \in \mathbb{R}, r > 0 : B(x, r) \cap E = \emptyset$. x is thus internal to the complement of E .

Definition 3.0.3. x is a frontier (or boundary) point if it is neither internal nor external.

We denote the set of internal points of E as $\overset{\circ}{E}$ or $\text{int}E \subseteq E$, the boundary as ∂E , and the external points as E^e .

For example, in \mathbb{R}^n , if $E = B(x_0, r)$, $\|x - x_0\| < r \iff x \in \overset{\circ}{E}$ because of the triangular inequality: $B(x, (r - \|x - x_0\|)) \subseteq B(x_0, r)$. We can see that for any y $d(y, x_0) \leq d(y, x) + d(x_0, x)$.

Also, if $\|x - x_0\| > r$, $x \in E^e$, since we can construct $B(x, \|x - x_0\| - r) \in E^C$.

If $\|x - x_0\| = r$, $x \in \partial E$ since we can construct $B(x, p)$ for some $p > 0$, and supposing WLOG that $p < r$ we can show that:

$$B(x, p) \setminus E \neq \emptyset \iff (x \in E^C \implies \forall t > 0 x \in B(x, t)) \quad (3.0.3)$$

$$B(x, p) \cap E \neq \emptyset \quad (3.0.4)$$

We can prove the last statement by defining

$$y := x + \frac{p}{2} \left(\frac{x - x_0}{r} \right) \quad (3.0.5)$$

and showing that $d(y, x_0) < r$.

It is easily proven that $\partial \mathbb{Q} = \mathbb{R}$, and that $\partial E = \partial(E^C)$.

Open sets

Theorem 3.0.1. $E \subseteq X$ is open, that is, $\overset{\circ}{E} = E$, iff $E \cap \partial E = \emptyset$.

This clearly implies that $\forall E \subseteq X : \overset{\circ}{E}$ is open. E^e is also open. We conventionally say that \emptyset is open.

Proof. Rightward: if $x \in E$, and E is open, then every point is internal, so it is not a frontier point.

Leftward: if $x \in E$, $x \notin \partial E$, and it is not external, so $x \in \overset{\circ}{E}$. □

Theorem 3.0.2. A union (finite or infinite) of open sets is open.

Proof. If $x \in \cup E_i$, $\exists k : x \in E_k$, so $\exists r > 0 : B(x, r) \subseteq E_k \subseteq \cup E_i$. □

Theorem 3.0.3. If A and B are open, $A \cap B$ is open.

Proof. Take $x \in A \cap B$. Then, $\exists r_1, r_2 > 0 : B(x, r_1) \subseteq A$ and $B(x, r_2) \subseteq B$. If we take $r := \min\{r_1, r_2\}$ we have $B(x, r) \subseteq A \cap B$. □

By induction we can see that, if $\forall i : A_i$ is open, then $\cap_{i=0}^{\infty} A_i$ is open. This, however, does not generalize to infinite intersections.

Closed sets

In a metric space X , $D \subseteq X$ is closed if D^C is open. Intervals like $[a, b] \subseteq \mathbb{R}$ are closed. $\{a\} \subseteq \mathbb{R}$ is also closed.

Theorem 3.0.4. D is closed iff $\partial D \subseteq D$, that is, $D = \overset{\circ}{D} \cup \partial D$.

D is closed iff $D^C \cap \partial(D^C) = \emptyset$, that is, $\partial D \subseteq (D^C)^C = D$.

Theorem 3.0.5. A finite union of closed sets is closed.

Proof.

$$\left(\bigcup_{i=0}^n D_i \right)^C = \bigcap_{i=0}^n (D_i)^C \quad (3.0.6)$$

□

Theorem 3.0.6. A (finite or infinite) intersection of closed sets is closed.

Proof.

$$\left(\bigcap_{i=0}^n D_i \right)^C = \bigcup_{i=0}^n (D_i)^C \quad (3.0.7)$$

□

Closure

Definition 3.0.4. The *closure* of a set E , denoted \bar{E} , is $\bar{E} := \overset{\circ}{E} + \partial E$.

E is closed iff $\bar{E} = E$. \bar{E} is always closed since $(\bar{E})^C$ is open. $\overset{\circ}{E} \subseteq E \subseteq \bar{E}$.

Theorem 3.0.7. \bar{E} is the smallest closed set containing E : if D is closed and $E \subseteq D$, then $\bar{E} \subseteq D$.

Proof. $D^e \subseteq E^e = \bar{E}^e$. If we take an $x \in \bar{E}$, x cannot belong to \bar{E}^e . This means it also does not belong to D^e , so $x \in \overset{\circ}{D} \vee x \in \partial D \implies x \in D$. □

3.1 Limit points

Definition 3.1.1. Take the set $E \subseteq \mathbb{R}^n$: $x_0 \in \mathbb{R}^n$ is a limit point for E if

$$\forall r > 0 : \exists x \in E, x \neq x_0 : x \in B(x_0, r) \quad (3.1.1)$$

Equivalently, $B(x_0, r)$ contains infinite points of E .

The set of the limit points of E is denoted $\mathcal{D}E$: it is called “derived set”.

If $x \in E^e$, then $x \notin \mathcal{D}E$; also if $x \in \overset{\circ}{E}$ then $x \in \mathcal{D}E$. This holds in \mathbb{R}^n , but it might not in pathological metric spaces.

So we have $\overset{\circ}{E} \subseteq \mathcal{D}E \subseteq \bar{E}$.

Some examples: $\mathcal{D}\mathbb{Z} = \emptyset$, $\mathcal{D}(a, b) = [a, b]$.

Theorem 3.1.1. D is closed iff it contains all of its limit points.

Proof. Rightward implication: $\mathcal{D}D \subseteq \bar{D} = D$.

Leftward implication: by contradiction. Suppose that $\partial D \not\subseteq D$. Then $\exists x \in \partial D \setminus D$, so $\forall r > 0 : \exists y \in B(x, r) \cap D$. But $y \in D$, and $x \notin D$, so $y \neq x$: x is a limit point, which implies $x \in D$. \square

Theorem 3.1.2. Given $a, b \in \mathbb{R}$, $\exists r \in \mathbb{Q} : a < r < b$. (\mathbb{Q} is dense in \mathbb{R}).

Proof. By Archimedes' property, the multiples of k^{-1} ($k \in \mathbb{N}$) will always exceed a . We can choose a k such that $k(b - a) > 1$. Then,

$$ka < \lfloor ka \rfloor + 1 \leq ka + 1 < kb \quad (3.1.2)$$

$$a < \frac{\lfloor ka \rfloor + 1}{k} < b \quad (3.1.3)$$

\square

So, $\bar{\mathbb{Q}} = \mathcal{D}\mathbb{Q} = \mathbb{R}$. We can also easily prove that $\mathcal{D}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$, and $\overset{\circ}{\mathbb{Q}} = \emptyset$.
If $E \subseteq \mathbb{R}$ is finite, $\mathcal{D}E = \emptyset$.

Theorem 3.1.3. If $E \subseteq \mathbb{R}$ is limited and infinite, then $\mathcal{D}E \neq \emptyset$.

Proof. We can split E in two parts, taking the average of $\sup E$ and $\inf E$. Then, in one of the sets into which we split E there must still be infinitely many points. We can apply the same splitting process to it, and so on: by the Halved Intervals theorem in the end the intersection of the sets we selected on each iteration will contain just one element $\lambda \in E$, and by construction $\lambda \in \mathcal{D}E$. \square

This also holds in \mathbb{R}^n .

Chapter 4

Functions

4.1 Basics

Given two nonempty sets A and B , a function $f : A \rightarrow B$ is a subset of $A \times B$ such that each element of A appears in exactly one of the ordered pairs $(a, b) : a \in A, b \in B$.

A is called *domain*, B is called *range*. The *image* of a set $X \subseteq A$ is $f(X) = \{y \in B : \exists x \in X : f(x) = y\}$. A function is *surjective* if $f(A) = B$. Similarly, the *preimage* of a set $Y \subseteq B$ is $f^{-1}(Y) = \{x \in A : \exists y \in Y : f(x) = y\}$.

Composition Given $f : A \rightarrow B$ and $g : B \rightarrow C$, $\exists h := g \circ f : A \rightarrow C$ such that $g \circ f(x) = g(f(x))$.

For convenience, given an expression in the variable x , we automatically assume it represents a function $f : A \rightarrow \mathbb{R}$, which associates every x with the expression evaluated at that point. A is the largest subset of \mathbb{R} for which the function is defined.

Injectivity $f : A \rightarrow B$ is injective if $\forall x_1, x_2 \in A : x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

Given an injective function, we can define an inverse: $f^{-1} : f(A) \rightarrow A$, where $f^{-1}(y) = x \in A : f(x) = y$.

$\forall x \in A : f^{-1} \circ f(x) = x$ and $\forall y \in f(A) : f \circ f^{-1}(y) = y$.

An inverse function is always bijective.

Definition 4.1.1. Given the subset $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$ is said to be increasing if $\forall x_1, x_2 \in A : x_2 > x_1 \implies f(x_2) \geq f(x_1)$, decreasing if $f(x_2) \leq f(x_1)$. It is *strictly* increasing or decreasing if the inequalities are strict.

Definition 4.1.2. f is even if $f(-x) = f(x)$, and odd if $f(-x) = -f(x)$.

4.2 Basic functions

Powers $f(x) = x^\alpha$ is easily defined $\forall x \in \mathbb{R}$ if $\alpha \in \mathbb{Q}$ and its denominator is odd, and only for positive numbers if it is even.

In general, though, if $\alpha \notin \mathbb{Q}$ and $\alpha > 0$:

$$x^\alpha = \begin{cases} \sup \{x^p : p \in \mathbb{Q}, p < \alpha\} & x \geq 1 \\ \inf \{x^p : p \in \mathbb{Q}, p < \alpha\} & 0 \leq x < 1 \end{cases} \quad (4.2.1)$$

If $\alpha < 0$, $x^\alpha := (x^{-\alpha})^{-1}$.

Trig functions As usual.

Logs

Definition 4.2.1. Given $a > 0$, $a \neq 1$, $x > 0$, we define: $\log_a x = p \iff a^p = x$.

Some properties are: $\log_{a^{-1}} x = -\log_a x$, $\log_b x = \log_b a \cdot \log_a x$.

Misc $\operatorname{sgn} x$ is defined as usual, and $\operatorname{sgn} 0 = 0$.

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