Notes on Geometry and Linear Algebra

Jacopo Tissino

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Chapter 1

Vectors

1.1 Euclidean space

We will use the Euclidean n-dimentional space as a model for the physical space. Space is a set, whose elements are points, and whose subsets are lines, planes and hyperplanes, which we will treat algebraically. We will need notions of parallelism, measure and orthogonality.

1.2 Segments

An oriented segment (or "applied vector") is a subset of a space S_n characterized by an ordered pair of points. It is denoted as such: P_1P_2 is the segment (P_1, P_2), where $\forall k \in \mathbb{N} : P_k \in S_n$.

There exist "trivial" segments P_1P_1.

We define an equivalence relation on $S_n \times S_n$: $\sim$, where $P_1P_2 \sim Q_1Q_2 \iff P_1P_2Q_2Q_1$ is a parallelogram.[1] Notably, we do not need a notion of distance for this.

We denote the representative as such: $[(P_1, P_2)] := \overrightarrow{P_1P_2} = \mathbf{v}$. These are "free vectors", or "geometrical vectors".

We have the following operations between them:

- addition: $\mathbf{a} + \mathbf{b}$
- multiplication by a scalar: $\alpha\mathbf{a}, \quad \alpha \in \mathbb{R}$

Addition If we want to add together two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$, first we must represent both as starting from the same point: so we change $\overrightarrow{CD}$ to $\overrightarrow{AE} = \overrightarrow{CD}$. Then, there exists a K such that $\overrightarrow{EK} = \overrightarrow{AB}$, and

$$\overrightarrow{AK} := \overrightarrow{AB} + \overrightarrow{CD} \tag{1.2.1}$$

[1] It is easy enough to check the three conditions.

Multiplication by a scalar We use a real number as a scalar because of the continuum hypothesis.[2]

For any $\alpha \in \mathbb{R}$ and any vector , we define $\alpha\overrightarrow{AB}$ as a vector $\overrightarrow{AC}$ such that $\left|\overrightarrow{AC}\right| = |\alpha|\left|\overrightarrow{AB}\right|$ and C is on the same side of A as B if $\alpha > 0$ and on the opposite side if $\alpha < 0$.

Defining the zero vector $\mathbf{0} = \overrightarrow{AA}$ we immediately see that:

- $0\mathbf{v} = \mathbf{0}$
- $\alpha\mathbf{0} = \mathbf{0}$

Properties of vector operations The following hold:

1. addition is commutative: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. addition is associative: $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$, since they define a parallelepiped
3. there exists a zero vector: $\exists\mathbf{0} : \mathbf{a} + \mathbf{0} = \mathbf{a}$
4. there exists an opposite for every vector: $\exists(-\mathbf{a}) : \mathbf{a} + (-\mathbf{a}) = \mathbf{0}$, also, we notice that $(-\mathbf{a}) = -1\mathbf{a}$
5. scalar multiplication is distributive: $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$
6. scalar addition distributes as vector addition over scalar multiplication: $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{b}$
7. scalar multiplication is associative: $(\alpha\beta)\mathbf{a} = \alpha(\beta\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$
9. $0\mathbf{a} = \mathbf{0}$

1.3 Reference frames

For an n-dimensional reference frame we need n vectors (the “basis vectors”) which are neither parallel to one another nor lying in the same (hyper)plane, so that we can express any vector we want through a linear combination of them; they, however, need not be orthogonal. We will call our three-dimensional basis vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$. Any vector $\mathbf{p}$ can then be seen as:

$$\mathbf{p} = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{1.3.1}$$

[2] CH states that there is no set whose cardinality is between that of the integers and that of the reals.

We can verify through the identities that, as long as we work in a consistent reference system, we can express vector addition and scalar mutiplication through the coordinates as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}, \qquad c \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ cx_3 \end{pmatrix} \tag{1.3.2}$$

1.4 Linear dependence

Parallelism To write the fact that $\mathbf{a} \parallel \mathbf{b}$, we could say that $\exists \lambda \in \mathbb{R} : \mathbf{a} = \lambda \mathbf{b}$, but this fails to account for one of the vectors being $\mathbf{0}$. A better formula is this one:

$$\exists \alpha, \beta \in \mathbb{R} : \neg(\alpha = \beta = 0) : \alpha \mathbf{a} + \beta \mathbf{b} = \mathbf{0} \tag{1.4.1}$$

Co-hyper-planar vectors With a similar argument, we say that n vectors $\mathbf{x}_i$, where $0 < i \leq n$ are in the same $n-1$-plane if

$$\forall i \in \mathbb{N} : 0 < i \leq n : \exists \alpha_i \in \mathbb{R} : \exists \alpha_i \neq 0 : \sum_{i=1}^{n} \alpha_i \mathbf{x}_i = \mathbf{0} \tag{1.4.2}$$

and we call this a *linear combination* of the vectors $\mathbf{x}_i$.

If there is no such linear combination (with at least one nonzero index) yielding $\mathbf{0}$, the vectors are said to be *linearly independent*; otherwise they are *linearly dependent*. If one of the vectors is the zero vector, all of them are automatically linearly dependent.

Definition 1.4.1. A *subspace* is a subset S of the geometrical vector space such that:

1. $\mathbf{0} \in S$;
2. $\forall \mathbf{a}, \mathbf{b} \in S : \mathbf{a} + \mathbf{b} \in S$;
3. $\forall \mathbf{a} \in S : \forall \lambda \in \mathbb{R} : \lambda \mathbf{a} \in S$;

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