General Relativity handbook

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Length contraction and time dilation

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{\gamma} \qquad \text{and} \qquad \mathrm{d}s = \gamma \, \mathrm{d}x \,. \tag{1}$$

Energy momentum tensor

$$\Delta p^\alpha = T^{\alpha\beta} n_\beta \Delta V \,, \tag{2}$$

for non interacting dust

$$T^{\alpha\beta} = n_* m u^\alpha u^\beta \,. \tag{3}$$

Variational principle The proper time is: $\mathrm{d}\tau^2 = -\,\mathrm{d}s^2 = -\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, so the variational principle is

$$\delta\left(\int_A^B \mathrm{d}\tau\right) = 0 \iff \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = 0 \,. \tag{4}$$

In general

$$u^\mu \nabla_\mu u^\nu = \frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta \iff \delta\left(\int_A^B \mathrm{d}\tau\right) = 0 \,, \tag{5}$$

where $\mathrm{d}\tau = -g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$.

Lagrangian view:

$$\mathscr{L} = \frac{\mathrm{d}\tau}{\mathrm{d}\sigma} = \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}} \,. \tag{6}$$

Light The 4-velocity is *defined* as

$$k^\mu = (\omega, \vec{k})\,, \tag{7}$$

with $\left|\vec{k}\right| = \omega$. Doppler effect: the observed frequency is $-u^\mu k_\mu$, so observers moving with different velocities see different wavevectors.

Observers The measured energy of a particle with p^μ by an obs with velocity u^μ is $E = -u^\mu p_\mu$.

Gravitational time dilation

$$\frac{\Delta\tau_A}{1+\Phi_A} = \frac{\Delta\tau_B}{1+\Phi_B}\,, \tag{8}$$

and in the inertial frame $\Delta t = \Delta\tau$ for both.

Christoffel

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha}\left(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha}\right)\,. \tag{9}$$

Defined by assuming $\nabla_\mu A_\nu$ is a tensor, and the tensor differentiation law:

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho \qquad \text{and} \qquad \nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\rho} A^\rho\,. \tag{10}$$

Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{T_{\mu\nu}}{M_P^2}\,, \tag{11}$$

where

$$R^\mu_{\nu\rho\sigma} = -2\left(\Gamma^\mu_{\nu[\rho,\sigma]} + \Gamma^\beta_{\nu[\rho}\Gamma^\mu_{\sigma]\beta}\right)\,, \tag{12}$$

$M_P = 1/\sqrt{8\pi G}$ is the reduced Planck mass.

$$V_{\text{eff}} = -\frac{GM}{r} + \frac{l^2}{2r^2}\left(1 - \frac{2GM}{r}\right)\,, \tag{13}$$

$$V_{\text{eff}} = -\frac{GM}{r} + \frac{l^2 - a^2(e^2-1)}{2r^2} + \frac{GM(l-ae)^2}{r^3}\,, \tag{14}$$