General Relativity handbook

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Length contraction and time dilation

$$d\tau = \frac{dt}{\gamma} \qquad \text{and} \qquad ds = \gamma \, dx \,. \tag{1}$$

Energy momentum tensor

$$\Delta p^\alpha = T^{\alpha\beta} n_\beta \Delta V \,, \tag{2}$$

for non interacting dust

$$T^{\alpha\beta} = n_* m u^\alpha u^\beta \,. \tag{3}$$

Variational principle The proper time is: $d\tau^2 = -ds^2 = -\eta_{\mu\nu} \, dx^\mu \, dx^\nu$, so the variational principle is

$$\delta \left(\int_A^B d\tau \right) = 0 \iff \frac{du^\mu}{d\tau} = 0 \,. \tag{4}$$

In general

$$u^\mu \nabla_\mu u^\nu = \frac{d^2 x^\nu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta \iff \delta \left(\int_A^B d\tau \right) = 0 \,, \tag{5}$$

where $d\tau = -g_{\mu\nu} \, dx^\mu \, dx^\nu$.

Lagrangian view:

$$\mathscr{L} = \frac{d\tau}{d\sigma} = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} \,. \tag{6}$$

Light The 4-velocity is *defined* as

$$k^\mu = (\omega, \vec{k}) \,, \tag{7}$$

with $\left|\vec{k}\right| = \omega$. Doppler effect: the observed frequency is $-u^\mu k_\mu$, so observers moving with different velocities see different wavevectors.

Observers The measured energy of a particle with p^μ by an obs with velocity u^μ is $E = -u^\mu p_\mu$.

Gravitational time dilation

$$\frac{\Delta\tau_A}{1+\Phi_A} = \frac{\Delta\tau_B}{1+\Phi_B} \,, \tag{8}$$

and in the inertial frame $\Delta t = \Delta\tau$ for both.

Christoffel

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha} \right) . \tag{9}$$

Defined by assuming $\nabla_\mu A_\nu$ is a tensor, and the tensor differentiation law:

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho \qquad \text{and} \qquad \nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\rho} A^\rho \,. \tag{10}$$

Riemann tensor

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha_{\beta\mu\nu} V^\beta \,, \tag{11}$$

where

$$R^\mu_{\nu\rho\sigma} = -2 \left(\Gamma^\mu_{\nu[\rho,\sigma]} + \Gamma^\beta_{\nu[\rho} \Gamma^\mu_{\sigma]\beta} \right) , \tag{12}$$

Parallel transport

$$u^\mu \nabla_\mu V^\nu = 0 \,, \tag{13}$$

Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{T_{\mu\nu}}{M_P^2} \,, \tag{14}$$

where $M_P = 1/\sqrt{8\pi G}$ is the reduced Planck mass.

Schwarzschild $R_{00} = 0$ and $R_{11} = 0$ imply

$$(AB)' = 0\,, \tag{15}$$

while $R_{22} = 0$ implies

$$1 = B + r\frac{B'}{B}\,. \tag{16}$$

$R_{33} = 0$ is not linearly independent.
Limits to use:

1. $g_{00} \to -1$ and $g_{11} \to 1$ as $r \to \infty$;
2. $g_{00} \sim -(1 + 2\Phi)$ as $M \to 0$.

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2\left(1 - \frac{2GM}{r}\right) + \frac{\mathrm{d}r^2}{1 - \frac{2GM}{r}} + r^2\,\mathrm{d}\Omega^2\,. \tag{17}$$

Classical orbits Pose $L = rv_\varphi$, which is conserved.

$$\Omega = \sqrt{\frac{GM}{r^3}} \tag{18}$$

follows from equating the forces. The effective potential is the Schwarzchild one without the r^{-3} term.

Schwarzschild orbits $u^\mu u_\mu = -1$ plus the two Killing vectors:

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{e^2 - 1}{2}\,, \tag{19}$$

with

$$V_{\text{eff}} = -\frac{GM}{r} + \frac{l^2}{2r^2}\left(1 - \frac{2GM}{r}\right), \tag{20}$$

then: change variable to $u = 1/r$, and

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{l}{r^2}\frac{\mathrm{d}}{\mathrm{d}\varphi}\,. \tag{21}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = \frac{GM}{l^2} + 3GMu^2\,, \tag{22}$$

where $u = 1/r$. Upon perturbation, we get

$$\delta\varphi = 6\pi\left(\frac{GM}{l}\right)^2, \tag{23}$$

Radial infall $e = 1$, $l = 0$. The horizon is regular.

Impact parameter $u^2 = 0$ (light), impact parameter $b = l/e$: to prove it draw, recall $l = \mathrm{d}\varphi/\mathrm{d}\lambda$ and $e = \mathrm{d}t/\mathrm{d}\lambda$. The light equation is

$$\frac{1}{l^2}\dot{r}^2 + W_{\text{eff}}(r) = \frac{1}{b^2}\,, \tag{24}$$

where

$$W_{\text{eff}} = \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)\,. \tag{25}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = 3GMu^2\,, \tag{26}$$

perturb $u_0 = b^{-1}\sin(\varphi)$ with w/b. Ansatz $w = A + B\sin^2\varphi$. Find $u(\varphi)$, pose $u = 0$, discard $\sin^2$ and linearize sin.

Geodetic precession $s \cdot u = 0$, $u^2 = -1$ and u is equatorial circular orbit

$$u^\nu \nabla_\nu s^\mu = 0\,, \tag{27}$$

gives

$$\overline{\Omega} = \Omega\sqrt{1 - \frac{3GM}{r}}\,, \tag{28}$$

so, compute $\cos(\Delta\varphi)$ as ratio of radial components and pose $\Delta\varphi = -\overline{\Omega}t$.

Slowly rotating metric Work in orders of negative powers of c, so no M and Minkowski plus g_{03} terms, $g_{03} = -2GJ\sin^2\theta/r$.

Turn $\mathrm{d}\varphi$ into $\mathrm{d}x$ and $\mathrm{d}y$, the sine of θ cancels.

Show that s^t and s^z are constant. For s^z: $\vec{J} \cdot \vec{s} = 0$ is constant. For s^t compute.

Write $u^\nu \nabla_\nu s^\mu$. In the end

$$\Omega = \frac{2GJ}{z^3}\,. \tag{29}$$

Kerr Horizon g_{rr} diverges, ergoregion $g_{00} > 0$. Hozizon is a null surface, null vector is $(1, 0, 0, \Omega_H)$, where $\Omega_H = a/2GMr$.

The effective potential is:

$$V_{\text{eff}} = -\frac{GM}{r} + \frac{l^2 - a^2(e^2 - 1)}{2r^2} + \frac{GM(l - ae)^2}{r^3}\,, \tag{30}$$