# General Relativity handbook

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## 1 Theory

**Length contraction and time dilation**

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{\gamma} \qquad \text{and} \qquad \mathrm{d}s = \gamma \, \mathrm{d}x \ . \tag{1}$$

**Energy momentum tensor**

$$\Delta p^\alpha = T^{\alpha\beta} n_\beta \Delta V \,, \tag{2}$$

for non interacting dust

$$T^{\alpha\beta} = n_* m u^\alpha u^\beta \,. \tag{3}$$

**Variational principle** The proper time is: $\mathrm{d}\tau^2 = -\mathrm{d}s^2 = -\eta_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu$, so the variational principle is

$$\delta\left( \int_A^B \mathrm{d}\tau \right) = 0 \iff \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = 0 \,. \tag{4}$$

In general

$$u^\mu \nabla_\mu u^\nu = \frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta \iff \delta\left( \int_A^B \mathrm{d}\tau \right) = 0 \,, \tag{5}$$

where $\mathrm{d}\tau = -g_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu$.

Lagrangian view:

$$\mathscr{L} = \frac{\mathrm{d}\tau}{\mathrm{d}\sigma} = \sqrt{-g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}} \,. \tag{6}$$

**Light** The 4-velocity is *defined* as

$$k^\mu = (\omega, \vec{k})\,, \tag{7}$$

with $\left|\vec{k}\right| = \omega$. Doppler effect: the observed frequency is $-u^\mu k_\mu$, so observers moving with different velocities see different wavevectors.

**Observers** The measured energy of a particle with $p^\mu$ by an obs with velocity $u^\mu$ is $E = -u^\mu p_\mu$.

**Gravitational time dilation**

$$\frac{\Delta\tau_A}{1+\Phi_A} = \frac{\Delta\tau_B}{1+\Phi_B}\,, \tag{8}$$

and in the inertial frame $\Delta t = \Delta\tau$ for both.

**Christoffel**

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha}\left(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha}\right). \tag{9}$$

Defined by assuming $\nabla_\mu A_\nu$ is a tensor, and the tensor differentiation law:

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho \qquad \text{and} \qquad \nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\rho} A^\rho\,. \tag{10}$$

**Riemann tensor**

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha_{\beta\mu\nu} V^\beta\,, \tag{11}$$

where

$$R^\mu_{\nu\rho\sigma} = -2\left(\Gamma^\mu_{\nu[\rho,\sigma]} + \Gamma^\beta_{\nu[\rho}\Gamma^\mu_{\sigma]\beta}\right), \tag{12}$$

**Parallel transport**

$$u^\mu \nabla_\mu V^\nu = 0\,, \tag{13}$$

**Einstein Field Equations**

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{T_{\mu\nu}}{M_P^2}\,, \tag{14}$$

where $M_P = 1/\sqrt{8\pi G}$ is the reduced Planck mass.

**Schwarzschild** $R_{00} = 0$ and $R_{11} = 0$ imply

$$(AB)' = 0\,, \tag{15}$$

while $R_{22} = 0$ implies

$$1 = B + r\frac{B'}{B}\,. \tag{16}$$

$R_{33} = 0$ is not linearly independent.
Limits to use:

1. $g_{00} \to -1$ and $g_{11} \to 1$ as $r \to \infty$;
2. $g_{00} \sim -(1 + 2\Phi)$ as $M \to 0$.

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2\left(1 - \frac{2GM}{r}\right) + \frac{\mathrm{d}r^2}{1 - \frac{2GM}{r}} + r^2\,\mathrm{d}\Omega^2\,. \tag{17}$$

**Classical orbits** Pose $L = rv_\varphi$, which is conserved.

$$\Omega = \sqrt{\frac{GM}{r^3}} \tag{18}$$

follows from equating the forces. The effective potential is the Schwarzchild one without the $r^{-3}$ term.

**Schwarzschild orbits** $u^\mu u_\mu = -1$ plus the two Killing vectors:

$$\frac{1}{2}\dot{r}^2 + V_{\mathrm{eff}}(r) = \frac{e^2 - 1}{2}\,, \tag{19}$$

with

$$V_{\mathrm{eff}} = -\frac{GM}{r} + \frac{l^2}{2r^2}\left(1 - \frac{2GM}{r}\right), \tag{20}$$

then: change variable to $u = 1/r$, and

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{l}{r^2}\frac{\mathrm{d}}{\mathrm{d}\varphi}\,. \tag{21}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{GM}{l^2} + 3GMu^2\,, \tag{22}$$

where $u = 1/r$. Upon perturbation, we get

$$\delta\varphi = 6\pi\left(\frac{GM}{l}\right)^2, \tag{23}$$

**Radial infall** $e = 1$, $l = 0$. The horizon is regular. We recover the escape velocity: An observer sees

$$E = -u^{\mu}_{\text{obs}} p_{\mu} = \frac{m}{\sqrt{1 - \frac{2GM}{r}}} \stackrel{!}{=} m\gamma\,, \tag{24}$$

which means $v = \sqrt{2GM/r}$.

**Impact parameter** $u^2 = 0$ (light), impact parameter $b = l/e$: to prove it draw, recall $l = \mathrm{d}\varphi/\mathrm{d}\lambda$ and $e = \mathrm{d}t/\mathrm{d}\lambda$. The light equation is

$$\frac{1}{l^2}\dot{r}^2 + W_{\text{eff}}(r) = \frac{1}{b^2}\,, \tag{25}$$

where

$$W_{\text{eff}} = \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right). \tag{26}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = 3GMu^2\,, \tag{27}$$

perturb $u_0 = b^{-1}\sin(\varphi)$ with $w/b$. Ansatz $w = A + B\sin^2\varphi$. Find $u(\varphi)$, pose $u = 0$, discard $\sin^2$ and linearize sin.

**Geodetic precession** $s \cdot u = 0$, $u^2 = -1$ and $u$ is equatorial circular orbit

$$u^{\nu}\nabla_{\nu}s^{\mu} = 0\,, \tag{28}$$

gives

$$\overline{\Omega} = \Omega\sqrt{1 - \frac{3GM}{r}}\,, \tag{29}$$

so, compute $\cos(\Delta\varphi)$ as ratio of radial components and pose $\Delta\varphi = -\overline{\Omega}t$.

**Slowly rotating metric — Lense-Thirring precession** Work in orders of negative powers of $c$, so no $M$ and Minkowski plus $g_{03}$ terms, $g_{03} = -2GJ\sin^2\theta/r$.

Turn $\mathrm{d}\varphi$ into $\mathrm{d}x$ and $\mathrm{d}y$, the sine of $\theta$ cancels.

Show that $s^t$ and $s^z$ are constant. For $s^z$: $\vec{J}\cdot\vec{s} = 0$ is constant. For $s^t$ compute.

Write $u^{\nu}\nabla_{\nu}s^{\mu}$. In the end

$$\Omega = \frac{2GJ}{z^3}\,. \tag{30}$$

**Kerr** Horizon $g_{rr}$ diverges, ergoregion $g_{00} > 0$. Hozizon is a null surface, null vector is $(1, 0, 0, \Omega_H)$, where $\Omega_H = a/2GMr$.

The effective potential is:

$$V_{\text{eff}} = -\frac{GM}{r} + \frac{l^2 - a^2(e^2 - 1)}{2r^2} + \frac{GM(l - ae)^2}{r^3} \,. \tag{31}$$

The ISCO goes from $9GM$ for extreme counterrotating to $GM$ for extreme corotating; it is found by setting

$$V_{\text{eff}} = \frac{e^2 - 1}{2} \qquad \text{and} \qquad \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}r^2} = 0 \,. \tag{32}$$

**Penrose process** Project $p^\mu_{\text{in}} = p^\mu_{\text{out}} + p^\mu_{\text{BH}}$ onto $-\xi^\mu = (-1, \vec{0})$: $e_{BH} < 0$. Also, $l_{\text{BH}} < 0$ (project onto rotating obs $u^\mu$) so angular momentum is removed from the BH.

**Cosmology** EFE applied to $-\,\mathrm{d}t^2 + a^2\,\mathrm{d}\vec{x}^2$ with $T^{\mu\nu} = \rho u^\mu u^\nu + p h^{\mu\nu}$ give

$$\Gamma^0_{ij} = \delta_{ij} a \dot{a} \qquad \text{and} \qquad \Gamma^i_{0j} = \delta_{ij} \frac{\dot{a}}{a} \,. \tag{33}$$

$R_{0i} = 0$, which means $T^{0i} = 0$. The 00 and $ij$ equations are

$$3H^2 = \frac{\rho}{M_P^2} \qquad \text{and} \qquad -2\frac{\ddot{a}}{a} - H^2 = \frac{P}{M_P^2} \,, \tag{34}$$

where $H = \dot{a}/a$, and the conservation of the SEMT (0th component, the others vanish) is

$$\frac{\dot{\rho}}{\rho} + 3(1 + w)\frac{\dot{a}}{a} = 0 \,. \tag{35}$$

Assume $P = w\rho$ and solve EFE 00 with the conservation of the SEMT. We get either $a \sim t^{2/(3+3w)}$ or $a \sim \exp(t)$ when $w = -1$. That is vacuum energy, add $\Lambda g_{\mu\nu}$, evaluate the 00 and $ij$ components and move it to the RHS into the $T_{\mu\nu}$.

**Cosmological redshift** Start from

$$\int_A^B \mathrm{d}r = \int_{t_A}^{t_B} \frac{\mathrm{d}t}{a} = \int_{t_A}^{t_B} \frac{\mathrm{d}t}{a} \,, \tag{36}$$

get

$$a_A f_A = a_B f_B \,. \tag{37}$$

**Hubble law** Start from $d_L = a(t)\int \mathrm{d}r = t_0 - t_E$, then expand $a$ to first order to find

$$d_L = \frac{z}{H_0}\,. \tag{38}$$

**Gravitational waves** Linearized EFE for $g = \eta + h$ give $\Box h_{\mu\nu} = 0$ under the gauge choice $g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = 0$.

The gauge transformations are derived from shifts of the position vector, the transformation is

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\epsilon_{\nu)}\,, \tag{39}$$

$\epsilon_\mu$ being the infinitesimal generator of the transformation. We can fix the residual gauge with vectors such that $\Box\epsilon_\mu = 0$.

Moving to plane wave solutions, this is the TT gauge: transverse ($\partial_i h_{ij} = 0$) and traceless ($h^{\mu}_{\mu} = 0$).

**Transformations of the basis tensors** Slightly improper, but transform the metric, therefore the perturbation, therefore the basis tensors.

# 2 Exercises

5.1:

$$R_{\mu\nu\rho\sigma} = -2g_{[\mu|[\rho|,\nu]|\sigma]}\,, \tag{40}$$

5.3: the plane is $\mathrm{d}s^2 = y^{-2}\,\mathrm{d}s^2{}_E$, the equations of motion read

$$y\ddot{y} = \dot{y}^2 - \dot{x}^2 \qquad \text{and} \qquad y\ddot{x} - 2\dot{x}\dot{y}\,, \tag{41}$$

and the conserved quantity comes the $(1,0)$ KV ($\dot{x}/y^2$) plus $y = \|u\|$. Then, substitute for circles.

7.3: need the *derivative* of the normalization. Easier to start from

$$u = \frac{GM}{l^2} + 3GMu^2\,, \tag{42}$$

calculate $\mathrm{d}\varphi/\mathrm{d}\tau$ from $l$.

Use $\Omega$ to calculate $\mathrm{d}t/\mathrm{d}\tau$.

8.2: $a$ is just $a^r$.

9.1: just need to evaluate the potential at $R = GM$, it is there that it equals $(e^2 - 1)/2$.

9.2: recall $r^2 + a^2 = 2GMr$; form a square for the null vector. $l = (1, 0, 0, \Omega_{\mathrm{hor}})$.

10.1 final: $1 + R$ is always a factor.

10.2: watch out for $T_{ij} = ph_{ij} \neq p\delta_{ij}$.