

Machine Learning Gravitational Waveforms for Binary Neutron Star mergers

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Contents

1	Gravitational wave theory	1
1.1	Linearized gravity	1
1.1.1	Transformations of the perturbation	3
1.1.2	Gauge fixing	4
1.1.3	Plane gravitational waves	5
1.1.4	Effect on test masses	6
1.1.5	The quadrupole formula	8
1.1.6	Energy loss through gravitational radiation	10
1.2	Compact binaries	12
1.2.1	Energy evolution	13
1.2.2	Parameters for a compact binary coalescence (CBC) waveform	14
1.3	Interferometers and data analysis	18
A	The Stationary Phase Approximation	18
B	Acronyms	21

1 Gravitational wave theory

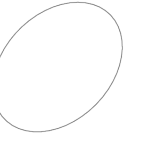
1.1 Linearized gravity

The simplest way to discuss gravitational radiation is to consider linearized gravity on a flat Minkowskian background.

We give a brief overview, roughly following the path taken by Maggiore [Mag07, chapter 1].

This means that we assume that our spacetime admits a reference frame for which the metric is in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \quad (1.1)$$



where the value of the components $h_{\mu\nu}$ is small enough that we can work to first order in them. Any equation in this section includes an implicit “ $+\mathcal{O}(h^2)$ ”. We work in this *global inertial frame*.

In order to study the evolution of the perturbation $h_{\mu\nu}$ we need to solve the Einstein Field Equations for it to linear order. In a vacuum, they can be written as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = 0 \quad (1.2)$$

$$R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu} \quad (1.3)$$

$$R = g^{\mu\nu}R_{\mu\nu}, \quad (1.4)$$

where $R_{\alpha\mu\beta\nu} \sim \partial\Gamma + \Gamma\Gamma$ is the Riemann tensor, which is written in terms of derivatives and squares of Christoffel symbols Γ :

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \quad (1.5)$$

$$= \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}). \quad (1.6)$$

We use the fact that $\partial\eta = 0$, and keep only linear order terms.

Since the Christoffel symbols are of first order in the perturbation, the term $\Gamma\Gamma$ in the Riemann tensor is of second order and can be neglected. Therefore, the relevant components are

$$R_{\mu\nu\rho\sigma} = 2\eta_{\mu\lambda}\Gamma_{\nu[\sigma,\rho]}^\lambda = h_{\mu[\sigma,\rho]\nu} - h_{\nu[\sigma,\rho]\mu}, \quad (1.7)$$

which gives us the following expression for the Ricci tensor:

$$R_{\mu\nu} = \frac{1}{2}(h^\sigma_{\mu,\sigma\nu} + h^\sigma_{\nu,\sigma\mu} - h_{,\mu\nu} - \square h_{\mu\nu}), \quad (1.8)$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$ is the trace of the perturbation (computed with respect to the flat metric), while $\square = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat space d’Alambertian.

This, in turn, allows us to write out the Einstein tensor:

$$G_{\mu\nu} = \frac{1}{2}(h^\sigma_{\mu,\sigma\nu} + h^\sigma_{\nu,\sigma\mu} - h_{,\mu\nu} - \square h_{\mu\nu} - \eta_{\mu\nu}h^{\rho\lambda}_{,\rho\lambda} + \eta_{\mu\nu}\square h). \quad (1.9)$$

This can be greatly simplified with two steps: first, we change variable from the perturbation $h_{\mu\nu}$ to the *trace-reversed* perturbation $\bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}h/2$ — the name comes from the fact that $\eta^{\mu\nu}\bar{h}_{\mu\nu} = -h$.

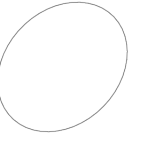
This substitution allows us to write the Einstein tensor as

$$G_{\mu\nu} = -\frac{1}{2}\square\bar{h}_{\mu\nu} + \bar{h}_{\alpha(\mu,\nu)}{}^\alpha - \frac{1}{2}\eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{\alpha\beta}. \quad (1.10)$$

We will shortly show that it is possible, as a *gauge choice*, to set the divergence of the trace-reversed perturbation to zero: $\partial^\mu\bar{h}_{\mu\nu} = 0$ (see section 1.1.2). The gauge imposed by this choice is called the *Hilbert Gauge*,¹ which in terms of the regular perturbation reads

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0. \quad (1.11)$$

¹ Despite the name, this choice was first suggested by De Sitter to Einstein [Ken07], who had been previously trying to impose the gauge $|g| = 1$ [Ein16, page 688].



With this choice the Einstein tensor becomes simply

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu}, \quad (1.12)$$

so the general form of the Einstein equations to linear order will be

$$\Box\bar{h}_{\mu\nu} = -16\pi G \left. T_{\mu\nu} \right|_{\text{linear}}, \quad (1.13)$$

where the stress-energy tensor is computed up to first order in the metric perturbation.

1.1.1 Transformations of the perturbation

The theory of General Relativity is constructed to be invariant under smooth changes of coordinates: under a map in the form $x \rightarrow x' = x'(x)$ (where $x'(x)$ is a diffeomorphism²).

Under such a coordinate transformation the metric transforms like any $(0,2)$ tensor:

$$g'_{\alpha\beta}(x') = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}(x). \quad (1.14)$$

A general transformation of this kind may break the condition that $g = \eta + h$ where h is small, so in order to preserve our framework we restrict ourselves to a small class of transformations.

One possibility is to consider infinitesimal transformations in the form

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x), \quad (1.15)$$

where ξ^μ is a vector field such that $|\partial_\mu \xi_\nu|$ is small — specifically, the condition to impose is that the first order in $\partial_\mu \xi_\nu$ should match the first order in $h_{\mu\nu}$.

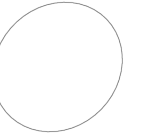
This condition is all we need in order to write the transformation law for the perturbation: the full equation reads

$$\eta'_{\mu\nu} + h'_{\mu\nu} \approx \left(\delta_\mu^\alpha - \partial^\alpha \xi_\mu \right) \left(\delta_\nu^\beta - \partial^\beta \xi_\nu \right) \left(\eta_{\alpha\beta} + h_{\mu\nu} \right), \quad (1.16) \quad \text{Used } 1/(1+x) = 1-x + \mathcal{O}(x^2).$$

so the zeroth order contribution is $\eta'_{\mu\nu} = \eta_{\mu\nu}$, while the first order one is

$$h'_{\mu\nu} = h_{\mu\nu} - 2\partial_{(\mu} \xi_{\nu)}, \quad (1.17)$$

² In the physics parlance this property is known as “diffeomorphism invariance”, while a mathematician would call the kinds of transformations considered “isometries”, since we ask that they preserve the metric structure of the manifold.



which is our transformation law for the metric perturbation.

We will also need a transformation law for the trace-reversed perturbation $\bar{h}_{\mu\nu}$: the trace transforms as $h' \rightarrow h - 2\partial_\mu \xi^\mu$, therefore the required law is

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - 2\partial_{(\mu} \xi_{\nu)} + \eta_{\mu\nu} \partial_\alpha \xi^\alpha, \quad (1.18)$$

A second class of transformations is a subset of Lorentz boosts and rotations: substituting $\partial x^\mu / \partial x'^\nu = \Lambda_\nu^\mu$ into the transformation law (1.14) we find that the flat metric is unchanged, while

$$h'_{\mu\nu}(x') = \Lambda_\mu^\alpha \Lambda_\nu^\beta h_{\alpha\beta}, \quad (1.19)$$

which may remain in the class of small metric perturbations: this is not guaranteed, but it is true for a certain subset of boosts and for all rotations [Mag07].

Finally, the perturbation is invariant under shifts in the form $x'^\mu = x^\mu + a^\mu$.

1.1.2 Gauge fixing

Now that we know how the perturbation $h_{\mu\nu}$ transforms under an infinitesimal transformation, we can use this to impose the condition we want — specifically, the Hilbert gauge (1.11).

The way to show that this is possible is to write out the way $\partial^\mu \bar{h}_{\mu\nu}$ transforms for an arbitrary choice of ξ , and to see that with an appropriate choice of ξ we can always map it to zero. The transformation reads

$$\partial^\mu \bar{h}'_{\mu\nu} = \partial^\mu \left(h_{\mu\nu} - 2\partial_{(\mu} \xi_{\nu)} \right) - \frac{1}{2} \partial_\nu \left(\eta^{\alpha\beta} \left(h_{\alpha\beta} - 2\partial_{(\alpha} \xi_{\beta)} \right) \right) \quad (1.20)$$

$$= \partial^\mu \bar{h}_{\mu\nu} - \square \xi_\nu - \partial_\nu \left(\partial^\mu \xi_\mu \right) + \partial_\nu \left(\partial^\mu \xi_\mu \right) \quad (1.21)$$

$$= \partial^\mu \bar{h}_{\mu\nu} - \square \xi_\nu. \quad (1.22)$$

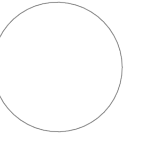
Therefore, from any starting gauge we must only find a ξ_ν such that $\partial^\mu \bar{h}_{\mu\nu} = \square \xi_\nu$, and we will be in the correct gauge. This can always be done, since the D'Alembert equation $\square f = g$ can always be solved for f — if we needed to compute ξ_ν explicitly (which we typically do not) we could use the Green's function $G(z)$ for the operator, defined by $\square G(z) = \delta^{(4)}(z)$.

While the equation is solvable, the solution is not unique: if we were to define an “inverse” of the D'Alembertian it would not be a function but a one-to-many relation. Specifically, while keeping fixed the value of $\square \xi_\nu$ we can add any function ζ_ν to ξ_ν as long as $\square \zeta_\nu = 0$. A trivial example is $\zeta_\nu = \text{const}$, but other wave-like choices are of more interest.

These still induce a transformation on $h_{\mu\nu}$ according to the usual law (1.17), and they can be used to further specify the form of the gravitational radiation.

In terms of **degrees of freedom**, the full perturbation $h_{\mu\nu}$ starts with 10 as any symmetric 4D, rank-2 tensor; the four Hilbert gauge conditions (1.11) reduce them to 6, while the four residual gauge conditions will allow us to reduce them to 2.

Doing so in full generality is not useful for us, let us instead fix the residual gauge in the specific context of a plane wave solution to the wave equation $\square \bar{h}_{\mu\nu} = 0$.



1.1.3 Plane gravitational waves

From an analogy to electromagnetic theory it seems reasonable to work with an ansatz in the form

$$\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\alpha x^\alpha}, \quad (1.23)$$

where $A_{\mu\nu}$ is a constant symmetric tensor.

Imposing the wave equation sets $k_\alpha k^\alpha = 0$, and the Hilbert gauge (1.11) condition can be written as $A_{\mu\nu} k^\mu = 0$, where $A = \eta^{\mu\nu} A_{\mu\nu}$.

In this framework we can impose the residual gauge condition explicitly: a function which satisfies $\square \zeta_\mu = 0$ is $\zeta_\mu = B_\mu \exp(id_\alpha x^\alpha)$, where d_α is a null vector ($d_\alpha d^\alpha = 0$) while B_μ is a generic constant vector.

In these terms, the transformation equation (1.18) reads

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} + \left(-2iB_{(\mu} d_{\nu)} + i\eta_{\mu\nu} B_\beta d^\beta \right) e^{id_\alpha x^\alpha} \quad (1.24)$$

$$A_{\mu\nu} e^{ik_\alpha x^\alpha} \rightarrow A_{\mu\nu} e^{ik_\alpha x^\alpha} + \left(-2iB_{(\mu} d_{\nu)} + i\eta_{\mu\nu} B_\beta d^\beta \right) e^{id_\alpha x^\alpha}. \quad (1.25)$$

This tells us that if we set the vector d_α to be equal to k_α the amplitude $A_{\mu\nu}$ will transform according to the algebraic system

$$A_{\mu\nu} \rightarrow A_{\mu\nu} - 2iB_{(\mu} k_{\nu)}, \quad (1.26)$$

which allows us to impose four conditions on $A_{\mu\nu}$, one for each of the free components of B_μ . It is customary to choose $A = 0 = A_{0i}$: these are known together as the transverse-traceless (TT) gauge.

The condition $A = 0$ also means that $h = 0 = \bar{h}$: thus, from this point onward we can stop distinguishing between $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$, and for simplicity's sake we write the former.

If we orient our axes such that $\vec{k} = k^i$ is along the \hat{z} direction (which means $k^\mu = (k, 0, 0, k)^\top$) the conditions can be written as

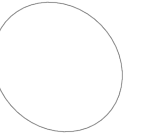
1. Hilbert gauge + traceless: $A_{\mu 0} + A_{\mu 3} = 0$;
2. traceless: $A = -A_{00} + A_{11} + A_{22} + A_{33} = 0$;
3. transverse: $A_{0i} = 0$.

The Hilbert gauge combined with the transverse conditions show that $A_{00} = 0$ as well, followed by $A_{13} = 0$, $A_{23} = 0$ and $A_{33} = 0$.

Finally, the traceless condition imposes $A_{11} = -A_{22}$. These conditions tell us that the plane gravitational wave must have the form

$$h_{\mu\nu}(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \exp(ik_\alpha x^\alpha), \quad (1.27)$$

where $h_+ = A_{11}$ and $h_\times = A_{12}$.



1.1.4 Effect on test masses

The **TT** gauge we defined has the rather peculiar characteristic of “moving with the wave”, so that the position of any observer initially at rest (so, such that $u^\mu(\tau = 0) = (1, \vec{0})^\top$) is unchanged: the geodesic equation evaluated at $\tau = 0$ reads

$$\left. \frac{d^2 x^\mu}{d\tau^2} \right|_{\tau=0} + \Gamma_{\nu\rho}^\mu u^\nu u^\rho \Big|_{\tau=0} = 0 \quad (1.28)$$

$$\left. \frac{d^2 x^\mu}{d\tau^2} \right|_{\tau=0} = -\Gamma_{00}^\mu = 0, \quad (1.29)$$

since the Christoffel symbols computed Γ_{00}^μ with the **TT** gauge perturbation (1.27) all vanish: $\Gamma_{00}^\mu = \eta^{\mu\nu}(2g_{\nu 0,0} - g_{00,\nu})/2 = 0$.

Does this mean that gravitational waves are merely an artefact, and have no effect on particles? No, since while in the **TT** gauge the *positions* of the points do not change, the *distance* among them does.

In order to understand this effect we can make use of the geodesic deviation equation [Car19, section 3.10], which states that if we take two geodesics whose four-velocities are both approximately u^μ , separated by a short vector ξ^μ ,³ they might diverge or converge, and the evolution of ξ^μ will be described by

$$\ddot{\xi}^\mu = R_{\nu\rho\sigma}^\mu u^\nu u^\rho \xi^\sigma. \quad (1.30)$$

Let us consider the same geodesics as before, whose four-velocity is uniformly $u^\mu \equiv (1, \vec{0})^\top$: the “acceleration” will then be given by the matrix product $R_{00\sigma}^\mu \xi^\sigma$.

These components of the Riemann tensor read [Car19, eq. 7.106]:

$$R_{00\sigma}^\mu = \frac{1}{2} \ddot{h}_\sigma^\mu. \quad (1.31)$$

Therefore, the temporal component of the acceleration is $\ddot{\xi}^0 \propto \ddot{h}_\sigma^0 = 0$, while the spatial components read

$$\ddot{\xi}^i = \frac{1}{2} \ddot{h}_j^i \xi^j. \quad (1.32)$$

These equations can then be explicitly solved. It is important to remark that $\ddot{\xi}^i$ is *not* coordinate acceleration: the points are *stationary* in the **TT** gauge coordinates, but the distance among them changes. Intuitively, we can interpret this as due to our carefully constructed gauge choice, which was determined through residual gauge, so it satisfied a wave equation $\square \zeta_\mu = 0$: it precisely oscillates with the wave, allowing the position of any point to not change (to linear order). Fortunately **GWs** are physical, so they must still manifest regardless of our gauge, and they do so through the distortion of distances as described here.

³ There is a technical note to be made here: in GR the notion of a vector between two points in the manifold is meaningless, however we can get around this problem by considering a one-parameter family of geodesics, and identifying the separation vector between them to be the tangent vector associated to the parameter.

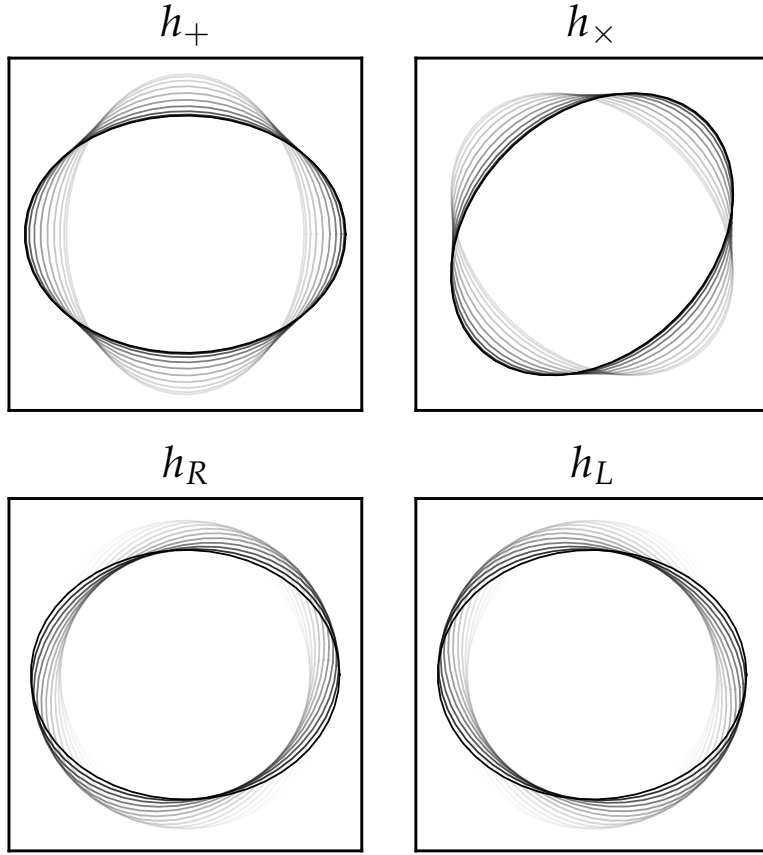


Figure 1: Polarizations of gravitational waves. Time evolution is represented through transparency: darker ellipses correspond to later points in time. The four cases are described in the text. We stress that in the **TT** gauge the ellipses do not represent changes in coordinate position but only in distance among points. In all cases the starting configuration is circular. The code generating this figure is available [here](#).

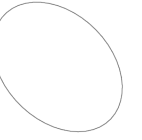
The positions of points at a fixed position (x, y, z) change (up to a phase) according to the expression

$$x(t) = x(t=0) \left(1 + h_+ e^{i\omega t} \right) + y(t=0) h_\times e^{i\omega t} \quad (1.33)$$

$$y(t) = y(t=0) \left(1 - h_+ e^{i\omega t} \right) + x(t=0) h_\times e^{i\omega t}, \quad (1.34)$$

which is shown graphically in figure 1 for four configurations: only h_+ or h_\times being nonzero, or only $h_R \propto h_+ + ih_\times$ or $h_L \propto h_+ - ih_\times$ being nonzero. The last two are called *circular polarizations*.

This description is not the one an experimentalist might use: if we need to consider the noise due to other sources of gravitation, it is convenient to move from the **TT** gauge to the *proper detector frame*, in which the metric is expanded around a fiducial point of our detector



— for instance, around the beam-splitter in an interferometer. In such a frame, it can be shown that the effect of **GWs** can be described as a Newtonian force on points a certain distance away from it — for instance, the mirrors of the interferometer. This comes in at second order in the distance, and it is given by [Mag07, eq. 1.96]

$$F_i = \frac{m}{2} \ddot{h}_{ij}^{TT} \zeta^j, \quad (1.35)$$

which unsurprisingly closely mirrors the acTT gauge variation of the distance: the physically measurable effect must be independent of gauge choice. The reason the **TT** gauge perturbation appears here as well is that the Riemann tensor, which is used in these computations, is *invariant* (a stronger condition than covariant! the values of the components are actually unchanged) under gauge transformations.

1.1.5 The quadrupole formula

The lowest order contribution to the generation of gravitational waves can be calculated starting from the linearized Einstein equation (1.13). The Green's function method for the inversion of the D'Alambertian is a common technique: if we can find a function $G(z)$ such that $\square G(z) = \delta^{(4)}(z)$, then we can do the following manipulation (denoting the gravitational constant as G_N for clarity):

$$\square_x \bar{h}_{\mu\nu}(x) = -16\pi G_N T_{\mu\nu}(x) \quad (1.36)$$

$$= -16\pi G_N \int d^4y T_{\mu\nu}(y) \delta^{(4)}(x - y) \quad (1.37)$$

$$= -16\pi G_N \int d^4y T_{\mu\nu}(y) \square_x G(x - y) \quad (1.38)$$

$$= \square_x \left(-16\pi G_N \int d^4y T_{\mu\nu}(y) G(x - y) \right) \quad (1.39)$$

$$\bar{h}_{\mu\nu} = -16\pi G_N \int d^4y T_{\mu\nu}(y) G(x - y) \quad (1.40)$$

$$= 4G_N \int d^3y T_{\mu\nu}(x^0 - |\vec{x} - \vec{y}|, \vec{y}) \frac{1}{|\vec{x} - \vec{y}|}, \quad (1.41)$$

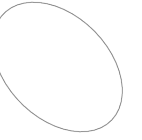
Removing the D'Alambertian is valid inasmuch as it will give us *a* solution for the wave equation, which will not be unique: $\square f = \square g$ does not imply $f = g$, but the reverse is true.

where in the last step we introduced the explicit expression for the Green's function of the D'Alambertian: $G(z) = -[z^0 > 0]/4\pi|\vec{z}|^4$ [Mag07, eq. 3.6].

We then have a formula for the trace-reversed perturbation, and it can be shown [Mag07, page 10] that far from the source we can recover the **TT** gauge perturbation by projecting this equation thanks to a tensor $\Lambda_{ij,kl}(\hat{n})$:

$$h_{ij}^{TT}(t, \vec{x}) = 4G_N \Lambda_{ij,kl}(\hat{n}) \int \frac{d^3y}{|\vec{x} - \vec{y}|} T_{kl} \left(t - |\vec{x} - \vec{y}|, \vec{y} \right), \quad (1.42)$$

⁴ The square bracket here is the Iverson bracket [Knu92], which maps a boolean expression to 1 (if true) or 0 (if false); in this context it is thus equivalent to the Heaviside Theta: $[z^0 > 0] = \Theta(z^0)$.



where the tensor Λ is constructed to be a projector which sends 2-tensors into the subspace of traceless tensors, which are transverse to the direction defined by $\hat{n} = \vec{x}/|\vec{x}|$:

$$\Lambda_{ij,kl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \quad \text{where} \quad P_{ij} = \delta_{ij} - n_i n_j. \quad (1.43)$$

This expression can be further simplified by making use of the assumption that the source is far away: the integral over d^3y ranges over a region which has a maximum size of R , the scale of the source, while the wave is observed at a distance $r = |\vec{x}| \gg R$. This allows us to expand and neglect terms of the order R^2/r^2 and over,⁵ the result we get is

$$h_{ij}^{TT} = \frac{4G_N}{r} \Lambda_{ij,kl}(\hat{n}) \int d^3y T_{kl}(t - r + \vec{y} \cdot \hat{n}, \vec{y}). \quad (1.45)$$

This equation uses the stress-energy tensor as computed without considering the higher-order effects of gravity on matter (this is the meaning of “linear” in the linear wave equation (1.13)); thus it is only a good approximation in the case of objects whose typical scale R is much larger, than, say, their Schwarzschild radius $2GM$. The problem with this assumption is that it fails to hold precisely for the sources which we are most interested in since, as we shall see, they give out some of the most easily detectable gravitational radiation: binary compact objects near coalescence. Let us forget about this issue for now.

For a gravitationally bound source like a binary, with total mass M and reduced mass μ , moving with speed v , the virial theorem dictates that

$$\frac{1}{2}\mu v^2 = \frac{1}{2}G \frac{\mu M}{r} \implies v^2 = \frac{GM}{r} \sim \frac{R_s}{r}. \quad (1.46)$$

This means that the source not being very compact ($R_s \ll r$) is equivalent to it moving slowly ($v \ll 1$). Neither of these assumptions will hold in the end, but since we are using one we might as well use both.

In Fourier space, this amounts to saying that the typical frequencies ω for which the amplitude of the Fourier transform of the stress-energy tensor is large will satisfy $\omega \vec{y} \cdot \hat{n} \lesssim \omega R \ll 1$. This means that we can expand the exponential in the Fourier transform; in the time domain this amounts to expanding in time around the retarded time $t - r$:

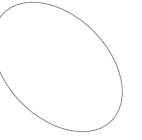
$$T_{kl}(t - r + \vec{y} \cdot \hat{n}, \vec{y}) \approx T_{kl}(t - r, \vec{y}) + \vec{y} \cdot \hat{n} \partial_t T_{kl} + \frac{1}{2}(\vec{y} \cdot \hat{n})^2 \partial_{tt}^2 T_{kl} + \mathcal{O}\left((\vec{y} \cdot \hat{n})^3\right). \quad (1.47)$$

This allows us to write the resulting wave as

$$h_{ij}^{TT}(t, \vec{x}) = \frac{4G_N}{r} \Lambda_{ij,kl} \left(S_{kl} + \sum_{L=1}^{\infty} \frac{1}{L!} \partial_t^L S_{kl|i_1 \dots i_L} n_{i_1} \dots n_{i_L} \right), \quad (1.48) \quad \text{To be computed at retarded time.}$$

⁵ Specifically, we expand

$$|\vec{x} - \vec{y}| = \sqrt{x^2 + y^2 + 2\vec{x} \cdot \vec{y}} = r \sqrt{1 - 2\frac{\hat{n} \cdot \vec{y}}{r} + \frac{y^2}{r^2}} \approx r \left(1 - 2\frac{\hat{n} \cdot \vec{y}}{r} + \mathcal{O}(R^2/r^2) \right). \quad (1.44)$$



where the moments of the stress tensor are defined as

$$S_{kl|i_1\dots i_L} = \int d^3y T_{kl} y_{i_1} \dots y_{i_L}. \quad (1.49)$$

We can also analogously calculate moments for the energy density T_{00} , which we denote $M_{i_1\dots i_L}$, and for the momentum density T_{0i} , which we denote $P_{i|i_1\dots i_L}$.

With integration by parts combined with the conservation of the (flat space!) stress-energy tensor $\partial_\mu T^{\mu\nu} = 0$ we can relate the M , P and S with equations such as [Mag07, eqs. 3.45–51]

$$\dot{M} = 0 \quad (1.50)$$

$$\dot{P}_i = 0 \quad (1.51)$$

$$\dot{M}_{ij} = 2P_{(i|j)} \quad (1.52)$$

$$S_{ij} = P_{i|j}. \quad (1.53)$$

These all tell us something interesting: $\dot{M} = 0$ and $\dot{P}_i = 0$ are energy and momentum conservation, which seem to tell us that there is no energy nor momentum loss from gravitational wave (GW) emission. This is an artifact, due to the assumptions of linear theory which neglect back-action on the source; fortunately we will still be able to compute the energy loss of the system.

The second useful fact is $S_{ij} = \ddot{M}_{ij}/2$: this allows us to write the lowest-order approximation of the expression of the wave from the source in terms of moments (1.48) as

$$h_{ij}^{TT}(t, \vec{x}) = \frac{2G_N}{r} \Lambda_{ij,kl} \ddot{M}_{kl}, \quad (1.54)$$

where

$$M_{kl}(t-r) = \int d^3y T_{00}(t-r, \vec{y}) y_k y_l. \quad (1.55)$$

since this expression only depends on the trace-free part of the moment M we can write it as a function of the traceless **quadrupole moment**

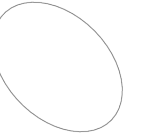
$$Q_{kl} = M_{kl} - \delta_{kl} M_{nn}/3 = \int d^3y \rho(t, \vec{y}) \left(y^i y^j - \frac{1}{2} \delta^{ij} y^2 \right). \quad (1.56)$$

This gives rise to the quadrupole formula

$$h_{ij}^{TT}(t, \vec{x}) = \frac{2G_N}{r} \Lambda_{ij,kl} \ddot{Q}_{kl}(t-r). \quad (1.57)$$

1.1.6 Energy loss through gravitational radiation

The problem of quantifying the energy carried by gravitational radiation is thorny: first of all, there is no universally valid way to split the perturbation from the background in the



general case; also, at each point we can always apply the equivalence principle to recover flat spacetime up to first order. It is impossible to construct a *local*, gauge invariant stress-energy tensor $T_{\mu\nu}^{\text{GW}}$ for GW radiation: the limit of the energy density contained in any volume will always vanish as that volume goes to zero.

However, we *can* define a tensor through an averaging procedure over many wavelengths and periods of the wave. There are different ways to do so, but a common one is the Landau-Lifshitz pseudotensor:

$$t_{\mu\nu} = -\frac{1}{8\pi G} \left\langle R_{\mu\nu}^{(2)} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \right\rangle_{\text{mesoscopic}} \quad (1.58)$$

$$= \frac{1}{32\pi G} \left\langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \right\rangle, \quad (1.59)$$

where $R_{\mu\nu}^{(2)}$ and $R^{(2)}$ are the components of the Ricci tensor and scalar which are quadratic in the perturbation $h_{\mu\nu}$; $\bar{g}_{\mu\nu}$ is the background metric, and the averaging procedure is done on scales (wavelengths/periods) which are (much) larger than the typical wavelengths of the gravitational radiation considered, but (much) smaller than the typical wavelengths of the background [Mag07, sections 1.4.2, 1.4.3].

This pseudotensor can be used to describe the way in which, on large enough scales, the presence of GWs does indeed curve spacetime. Also, we can compute the energy flux passing through a surface a large distance from the source: if we use the quadrupole formula (1.57) for the gravitational perturbation, we can give an expression for the emitted power in terms of the third derivatives of the quadrupole as [Mag07, eq. 3.3.2]

$$\frac{dE}{dt} = \frac{G_N}{5} \left\langle \ddot{Q}_{ij} \ddot{Q}_{ij} \right\rangle. \quad (1.60)$$

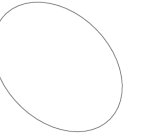
Dimensionally, $Q \sim \int d^3y \rho r^2$ has units of kgm^2 ; so \ddot{Q} has units of $\text{kgm}^2/\text{s}^3 = \text{W}$.

This means that the prefactor, which we wrote using $c = 1$, must really be $G_N c^n / 5$ with units of inverse power, which implies $n = -5$. Numerically, it is the inverse of $5c^5/G_N \approx 2 \times 10^{53} \text{ W}$. This means that in order for GW emission to be efficient we must have a large value for \ddot{Q} : let us estimate it in terms of the typical size of the system, R , of its typical velocity $v = \Omega R$ and of its mass M . Each time derivative will roughly correspond to multiplication by a factor Ω , so $\ddot{Q} \sim \Omega^3 M R^2 = v^3 M / R$.

The power can then be estimated as

$$\frac{dE}{dt} \sim \frac{G_N}{5c^5} v^6 \frac{M^2}{R^2} = \underbrace{\frac{1}{5} \frac{c^5}{G_N}}_{\sim 10^{52} \text{ W}} \left(\frac{v}{c} \right)^6 \left(\frac{G_N M}{c^2 R} \right)^2, \quad (1.61)$$

which tells us that the most significant sources of GW will be relativistic and compact. This prediction has been validated in 2015 with the first detection of GW from a binary black hole (BBH) system [LIG+16], and again in 2017 with the first detection of GW from a binary neutron star (BNS) system [Abb+17].



1.2 Compact binaries

We focus our attention towards a pair of inspiraling (compact) objects, which we will initially model as point masses. The first thing we need to do is to write an expression the amplitudes in the two polarizations $h_{+,\times}$ of the waves generated by a generic mass distribution with a mass moment M_{ij} [Mag07, eqs. 3.67–68]:⁶

$$h_{+}(t, \hat{n}) = \frac{G_N}{r} \left(\ddot{M}'_{11} - \ddot{M}'_{22} \right) \quad (1.62)$$

$$h_{\times}(t, \hat{n}) = \frac{G_N}{r} \left(2\ddot{M}'_{12} \right), \quad (1.63)$$

where M'_{ij} are the components of the mass moment tensor in a frame whose z' axis is aligned to the observation direction.

This frame is not the natural one with which to describe a binary system: typically, we would want to align the z axis of the coordinates with the rotation axis. The rotation matrix between two systems depends on two angles, however one of these is more important than the other: aligning the z axes can be accomplished by a rotation of angle ι , while another rotation of angle ϕ is needed in order to align the x and y axes as well. However, since (as we will discuss in a short while) the system is rotating in quasi-circular orbits, the second rotation only amounts to a time shift, or equivalently a phase in the oscillatory functions. For simplicity, we will neglect this phase freedom here and only include a variable phase at the end of the computations.

If the position of the bodies in the center-of-mass frame is

$$\vec{x}(t) = R \begin{bmatrix} \cos(\Omega t) \\ \sin(\Omega t) \\ 0 \end{bmatrix}, \quad (1.64)$$

then the mass moments read $M_{ij} = \mu x_i(t) x_j(t)$ (where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the system), and going through the computation yields [Mag07, eq. 3.332]

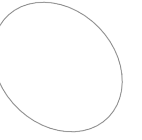
$$h_{+}(t) = \underbrace{\frac{4G_N \mu \Omega^2 R^2}{r}}_A \left(\frac{1 + \cos^2 \iota}{2} \right) \cos(2\Omega t) \quad (1.65)$$

$$h_{\times}(t) = \underbrace{\frac{4G_N \mu \Omega^2 R^2}{r}}_A \cos \iota \sin(2\Omega t) \quad (1.66)$$

$$A = \frac{4}{r} \left(\frac{G \mathcal{M}_c}{c^2} \right)^{5/3} \left(\frac{\pi f_{\text{gw}}}{c} \right)^{2/3}. \quad (1.67)$$

Approximating the motion of the bodies as circular during an orbit is called the *adiabatic approximation*, since it assumes that no energy is lost in each orbit (as that would deform

⁶ We could also write these in terms of the traceless quadrupole moment Q_{ij} ; we follow the convention set forward by Maggiore [Mag07].



the circle). The variation of the orbital frequency Ω , which we shall compute shortly, must satisfy $\dot{\Omega} \ll \Omega^2$. It is a rather good approximation for the early stages of the inspiral, as we shall see.

The last expression we wrote for the amplitude of the emission reintroduces factors of c , is expressed in terms of the emission frequency $f_{\text{gw}} = \omega_{\text{gw}}/2\pi = \Omega/\pi$ and of the *chirp mass*

$$\mathcal{M}_c = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} = \nu^{3/5} M \quad \text{where} \quad \nu = \frac{\mu}{M}. \quad (1.68)$$

1.2.1 Energy evolution

Writing the emission in this way allows us to give an expression for the emitted power

$$P = \frac{32}{5} \frac{c^5}{G} \left(\frac{G \mathcal{M}_c \omega_{\text{gw}}}{2c^3} \right)^{10/3}. \quad (1.69)$$

This allows us to describe the evolution of the orbit in this quasi-adiabatic context: the total energy of the binary system, by the virial theorem combined with Kepler's third law $\Omega^2 R^3 = GM$, reads

$$E = -\frac{Gm_1 m_2}{2R} = -\frac{Gm_1 m_2}{2} \frac{\Omega^{2/3}}{G^{1/3} M^{1/3}} = -\left(\frac{G^2 \mathcal{M}_c^5 \omega_{\text{gw}}^2}{32} \right)^{1/3}. \quad (1.70)$$

With these two expressions at hand, we can impose $P = -\dot{E}$, which we can write in the form $\dot{\omega}_{\text{gw}} \propto \omega^n$ for some n : since $\dot{E} \propto \omega_{\text{gw}}^{-1/3} \dot{\omega}_{\text{gw}}$, n will be $11/3$, and the specific expression will look like

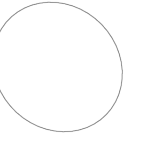
$$\dot{\omega}_{\text{gw}} = \frac{12}{5} \sqrt[3]{2} \left(\frac{G \mathcal{M}_c}{c^3} \right)^{5/3} \omega_{\text{gw}}^{11/3}. \quad (1.71)$$

This equation is in the form $\omega^{-11/3} d\omega = K dt$; integrating from point 0 to point 1 we find $3K(t_1 - t_0)/8 = \omega_0^{-8/3} - \omega_1^{-8/3}$; if we treat this as an initial value problem we can see that as t_1 increases there is only a finite “budget” on the right-hand side: when $3K(t_1 - t_0)/8 = \omega_0^{-8/3}$, $\omega_1^{-8/3}$ must go to 0 so ω_1 must diverge. The divergence itself is unphysical since, besides corresponding to a breakdown of some of our approximations, it comes about when considering point masses, while compact objects have a finite size and at some point collide.

Nevertheless, the moment of divergence happens close enough to the actual merger of the objects, therefore it is useful to call it t_c and to define a time coordinate as $\tau = t_c - t$. In terms of this, the solution to the equation reads

$$f_{\text{gw}}(t) = \frac{\omega_{\text{gw}}(t)}{2\pi} = \frac{1}{2\pi} \left(\frac{3K\tau}{8} \right)^{-3/8} = \frac{1}{\pi} \left(\frac{5}{256\tau} \right)^{3/8} \left(\frac{G \mathcal{M}_c}{c^3} \right)^{-5/8}. \quad (1.72)$$

At this point we can notice an interesting scaling property: if we map $f_{\text{gw}} \rightarrow \alpha f_{\text{gw}}$, $\tau \rightarrow \tau/\alpha$ and $\mathcal{M}_c \rightarrow \mathcal{M}_c/\alpha$ (or, specifically, we divide each mass by α) the equation still



holds! This result turns out to hold in general, not only at the linear, quadrupole level. [FIND CITATION FOR THIS STATEMENT]

This expression for f_{gw} can be substituted into the gravitational waveforms (1.65); we must also update the phase term, since $2\Omega t$ is the phase for a uniformly circular orbit: we will want to use integrate the angular velocity, to find

$$\Phi(t) = \int^t \omega_{\text{gw}}(\tilde{t}) d\tilde{t} = 2 \int^t \Omega(\tilde{t}) d\tilde{t} \quad (1.73)$$

$$= -2 \left(\frac{5G\mathcal{M}_c}{c^3} \right)^{-5/8} \tau^{-5/8} + \text{const.} \quad (1.74)$$

In terms of this, the amplitude in the two polarizations will read [Mag07, eqs. 4.31–32]

$$h_+(t) = \frac{1}{r} \left(\frac{G\mathcal{M}_c}{c^2} \right)^{5/4} \left(\frac{5}{c\tau} \right)^{1/4} \left(\frac{1 + \cos^2 \iota}{2} \right) \cos \Phi(t) \quad (1.75)$$

$$h_\times(t) = \frac{1}{r} \left(\frac{G\mathcal{M}_c}{c^2} \right)^{5/4} \left(\frac{5}{c\tau} \right)^{1/4} \cos \iota \sin \Phi(t). \quad (1.76)$$

As we will discuss, it is convenient to have expressions for the Fourier transforms of these. An analytic computation of the integrals is intractable, but we can make use of a technique known as stationary phase approximation (SPA), which is discussed in detail in appendix A. The final expressions for the Fourier-domain waveforms are equations (A.18).

1.2.2 Parameters for a CBC waveform

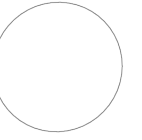
The waveform we wrote explicitly depends on:

1. the chirp mass \mathcal{M}_c , defined in (1.68);
2. the distance r , which in a more general cosmological context should be replaced by the *luminosity distance* D_L ;
3. the inclination ι , which is the angle between the observation direction and the total angular momentum of the system.

These are only some of the parameters which must be considered when discussing a CBC. We will now discuss the full set of parameters which can be used to fully describe the binary system generating the waveform [BGB21, eq. 21].

Arrival time and initial phase The time at which any given waveform arrives at Earth is arbitrary, as is the global phase of the waveform. In practice, one will typically analyze the output from a detector in batches, and in each of these the analysis will be performed in Fourier space. Therefore, it is relevant to see how the Fourier-domain waveform responds to a time and phase shift: what is the transform of $f(t - t_0)e^{i\varphi_0}$? It comes out to be

$$\int_{\mathbb{R}} f(t - t_0) e^{i\varphi_0} e^{i\omega t} dt = e^{i\omega t_0 + i\varphi_0} \underbrace{\int_{\mathbb{R}} f(t - t_0) e^{i\omega(t - t_0)} d(t - t_0)}_{\tilde{f}(\omega)}, \quad (1.77)$$



so a time shift corresponds to the addition of a linear term to the phase, while the phase can be directly moved from the time to the frequency domain.

Sky position and polarization The wave will be coming from a sky position, which we can describe with respect to a given coordinate system through two angles; it is convenient to use right ascension α and declination δ . Also, the polarization of the gravitational waves can have an arbitrary angle with respect to the observation direction: we denote this angle as ψ .

In more technical terms, a general waveform arriving our detector will be a superposition of [GWs](#) in the TT gauge (1.27) with varying α , δ , ψ as well as different frequencies; we can write it as [Mag07, eq. 1.58]

$$h_{ab}(t, \vec{x}) = \sum_{\text{pol}=+, \times} \int_{-\infty}^{\infty} df \int d^2\hat{n} \tilde{h}_{\text{pol}}(f, \hat{n}) e_{ab}^{\text{pol}}(\hat{n}, \psi) e^{-2\pi i f(t - \hat{n} \cdot \vec{x}/c)}, \quad (1.78)$$

where $\hat{n} = \hat{n}(\alpha, \delta)$ is the vector describing the propagation direction of a specific component — in the context of a [CBC](#) the distribution of the $e_{ab}^{\text{pol}}(\hat{n}, \psi)$ will include a Dirac delta centered on the position of the source in the sky, this more general expression can be useful, for example, in the context of stochastic gravitational wave background ([SGWB](#)).

The tensors e_{ab} are basis tensors: in a frame where \hat{n} and ψ are chosen such that the + polarization stretches the x and y axes, they read

$$e_{ij}^+(\hat{n} = \hat{z}, \psi = 0) = u_i u_j - v_i v_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1.79)$$

$$e_{ij}^\times(\hat{n} = \hat{z}, \psi = 0) = u_i v_j + v_i u_j = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (1.80)$$

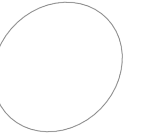
where u and v are two unit vectors defining the polarization direction; they are always chosen to be orthogonal, so fixing the angle ψ is enough to determine them both once we give \hat{n} .

The aforementioned expansion glosses over a technical issue: the three-dimensional Fourier transform of the [TT](#) gauge waveform is in the form $h_{ij}(x) \sim \int d^3k A_{ij}(k) e^{ikx}$, so in order to write it like we did we need to define

$$\frac{f^2}{c^3} A_{ij}(f, \hat{n}) = \sum_{\text{pol}=+, \times} \tilde{h}_{\text{pol}}(f, \hat{n}) e_{ij}^{\text{pol}}(\hat{n}), \quad (1.81)$$

where we reparametrized the wavevector \vec{k} through frequency f and direction \hat{n} , and exploited the decomposition $d^3k = k^2 dk d^2\hat{n}$.

Masses and mass ratio The waveform to the lowest order we described only depends on the chirp mass \mathcal{M}_c , but as we will see later at higher order there appears a dependence



on the ratio of the two masses: $q = m_1/m_2$, where m_i are the masses of the two compact objects. It is customary to fix one of the two masses to always be the largest, therefore constraining $q \gtrless 1$; we choose the convention $q \geq 1$.

A possible parametrization for the two masses is (\mathcal{M}_c, q) , another is (M, q) : these are equivalent, but in the following work we will typically use the latter because of the aforementioned scaling property of the waveforms: we conventionally work with respect to the dimensionless frequency Mf and the dimensionless time t/M in order to have one less parameter.

Spin Compact objects can spin, and the spin of each of the two is a vector with three independent components. These vectors \vec{S}_i have units of angular momentum, but they are typically rescaled as

$$\vec{\chi}_i = \frac{c\vec{S}_i}{Gm_i} \in [-1, 1] \text{ in magnitude.} \quad (1.82)$$

The constraint of $|\chi_i| < 1$ can be approached by realistic models of BHs, while realistic models of NSs typically are constrained to spin much less, at the most $|\chi_i| \lesssim 0.7$ [LL11]. This constraint is hard to experimentally verify since one is faced with the degeneracy between the components $\chi_{z,i}$ aligned with the direction of the angular momentum and the mass ratio q ; in the analysis of the merger GW170807 two sets of prior distributions for the spins were used, $|\chi| < 0.05$ and $|\chi| < 0.89$, for this reason [Abb+17]. We have theoretical reasons to believe that the low-spins priors might correspond to a more physically meaningful scenario, but the degeneracy means that the high-spin case is not experimentally excluded, so we must still consider it a possibility.

Tidal polarizability of neutron stars If our compact objects are not black holes, they might be able to deform. The astrophysically motivated scenario we think of is that of neutron stars, but the following characterization can also apply to any extended compact objects. For concreteness' sake, we will refer to NSs.

This is a complicated process, but we can try to capture its most significant part by limiting ourselves to the quadrupole order of the deformation of the star. The discussion of this effect is clearest in the Newtonian context, but it can be generalized to general relativity (GR) [Mag18, section 14.4.1].

Tidal effects are described by the tidal tensor $\mathcal{E}_{ij} = -U_{,ij}$, the traceless⁷ Hessian of the Newtonian potential U . In the weak-field relativistic case, this corresponds to part of the Riemann tensor: $\mathcal{E}_{ij} = c^2 R_{0i0j}$.

We can describe the effects on the deformed star of such a tidal stress by looking at its quadrupole moment Q_{ij} (1.56). To linear order and neglecting any time dependence, we will have the relation

$$\mathcal{E}_{ij} = -\lambda Q_{ij}. \quad (1.83)$$

⁷ This tidal tensor describes the effect of one star's gravitational field on the other, so its source is not where we compute it, which is a convoluted way of saying that it should be taken to be a solution of $\nabla^2 U = 4\pi\rho$ in vacuum: therefore, $\nabla^2 U = -\text{Tr}[\mathcal{E}_{ij}] = 0$.

find better explanation for minus sign here and in def of tidal tensor

This parameter λ describes the deformability of the star; if it is smaller the star deforms less in response to tidal perturbations. In practice the parameter used is not λ but two ways to rescale it: the $l = 2$ (quadrupole) **Love number**

$$k_2 = \frac{3}{2} \frac{G\lambda}{R^5}, \quad (1.84)$$

where R is the radius of the deformed neutron star (NS), or the **tidal deformability**

$$\Lambda = \frac{2}{3} k_2 \left(\frac{Rc^2}{Gm} \right)^5, \quad (1.85)$$

where m is the mass of the deformed NS. The value of these parameters depends on the specific equation of state used in the model; k_2 is typically of the order of 10^{-1} , and in the expression for Λ we have a fifth power of the compactness $\sigma = Rc^2/Gm$ of the NS — this is expected to be a small number, but larger than 3,⁸ so its fifth power will be of order $10^{3 \div 4}$. This means that the deformability Λ will typically be of the order $10^{2 \div 3}$.

The Love number is useful since it is an expansion parameter for the gravitational field of the deformed NS. This field has contributions from both its quadrupole moment, which is due to the tidal deformation from the other star, and from the gravitational field of the other star which we can expand in terms of the tidal field around the center of the deformed star: $U_{\text{ext}} \approx -\mathcal{E}_{ij}x_i x_j / 2$.

The gravitational field of the deformed star also changes under the effect of the deformation: including the first term in its expansion, which depends on the quadrupole moment, the potential reads

$$U_{\text{int}} \approx \frac{Gm}{r} + \frac{3G}{2r^3} \frac{x_i}{r} \frac{x_j}{r} Q_{ij}, \quad (1.86)$$

where $r = |x|$ is the radius from the center of the deformed star.

We can then see that under the assumption of linear dependence of the quadrupole on the tidal tensor the two potentials can be added (since we are still in the realm of linear theory):⁹

$$U \approx \frac{Gm}{r} - \frac{1}{2} \mathcal{E}_{ij} x_i x_j \left[1 + 2k_2 (R/r)^5 \right]. \quad (1.87)$$

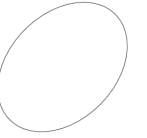
The meaning of Λ is less immediate — the first correction to the phasing of the waveform which appears is¹⁰ [Mag18, eq. 14.231; FH08]

$$\Delta \Psi_{\text{5PN}}^{\text{tidal}} = -\frac{117}{256} \frac{m^2}{m_1 m_2} \tilde{\Lambda} \left(\frac{v}{c} \right)^5 \quad \text{where} \quad \tilde{\Lambda} = \frac{16}{13} \frac{(m_1 + 12m_2)m_1^4 \Lambda_1 + (m_2 + 12m_1)m_2^4 \Lambda_2}{(m_1 + m_2)^5}. \quad (1.88)$$

⁸ A limit on the maximum redshift of radiation from the surface of a NS was calculated by Lindblom [Lin84] under the assumptions of causality and stability for the nuclear matter, which is equivalent to a compactness limit of $\sigma \geq 2.83$ [LP07].

⁹ A technical fact to remember here is that the expansions do not naturally match: the expansion of the internal potential is in orders of $1/r$, while the expansion of the external one is in orders of r .

¹⁰ The meaning of the post-Newtonian (PN) expansion and of the expansion in orders of v/c will be discussed in [SEC].



The parameter $\tilde{\Lambda}$, which depends on the tidal deformabilities of the two stars $\Lambda_{1,2}$, is called the *reduced tidal parameter*.

1.3 Interferometers and data analysis

Having seen how a gravitational waveform from a [CBC](#) might look to lowest order, we move to a discussion of the detection of these waveforms with interferometric techniques.

The response of any detector to a [GW](#) is a¹¹ scalar output, which will be in the form

$$h(t) = D_{ij}h_{ij}, \quad (1.89)$$

where D_{ij} is known as the *detector tensor*. We want to apply this expression to the generic expression for a gravitational wave ([1.78](#)); however

A The Stationary Phase Approximation

We can apply [SPA](#) to approximate the Fourier transform of an oscillatory function $f(t) = A(t)e^{-i\Phi(t)}$ as long as

1. $\Phi(t)$ is monotonic;
2. the evolution of the phase, $\dot{\Phi}$, is slowly changing: $\ddot{\Phi} \ll \dot{\Phi}^2$;
3. the amplitude $A(t)$ is slowly changing.

These seem like the right conditions intuitively, but are they correct?

If these conditions hold, then in the computation of the transform

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt A(t)e^{-i\Phi(t)+i\omega t} \quad (A.1)$$

the exponential term will be quickly oscillating, and therefore destructively interfering, almost everywhere; while near the time in which $\omega = \dot{\Phi}$ it will have a positive contribution.

With this in mind, we can approximate the integral: specifically, looking at the local integral contribution to the integral in [figure 2](#) we can see that it approaches a Gaussian, which suggests expanding the argument of the exponential to second order around its stationary point,

$$-i\Phi(t) + i\omega t \approx \left(-i\Phi(t) + i\omega t\right)\Big|_{t_s} - \frac{i}{2} \ddot{\Phi}\Big|_{t_s} (t - t_s)^2, \quad (A.2)$$

¹¹ Current interferometric detectors have a single scalar response because of their Michelson-Morley design with arms at 90°; planned detector such as the Einstein Telescope will exhibit multiple scalar inputs [[Tea11](#), section 5.3.2]. The following analysis still applies, each scalar input can be treated analogously; having multiple (from one multi-output detector or from a network) is incredibly beneficial for the accuracy of measurements.

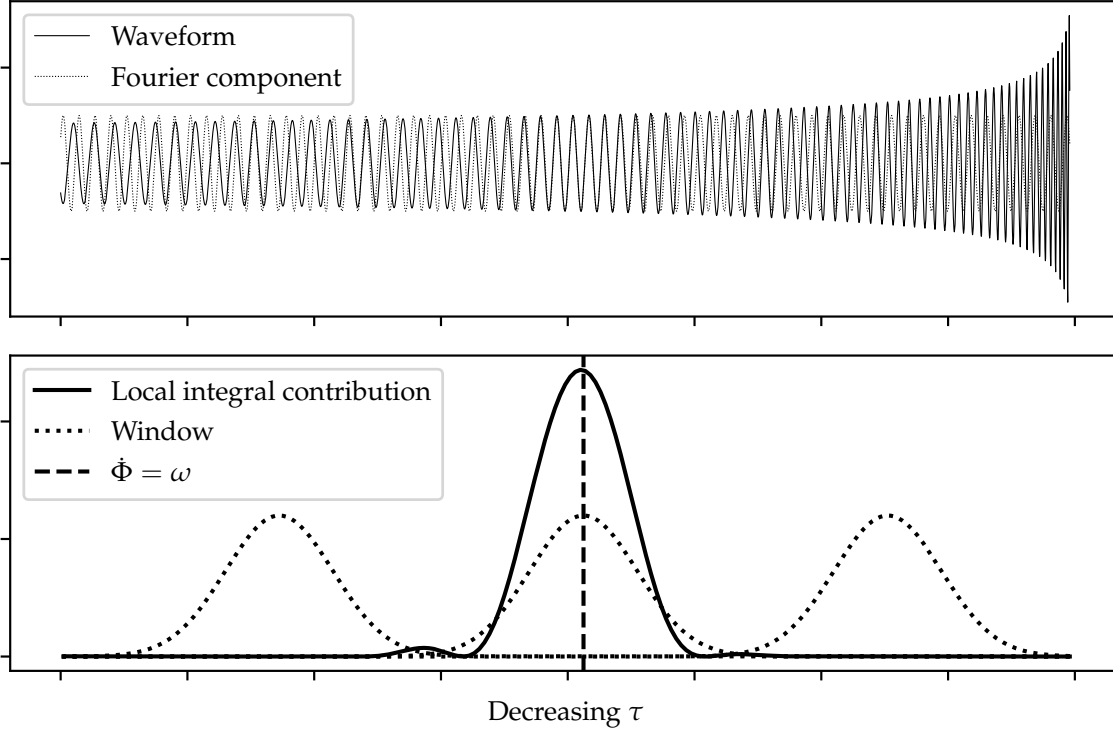


Figure 2: Visualization of the idea behind SPA. In the upper plot we show a typical waveform for a CBC together with a sinusoid corresponding to the ones used in the computation of the Fourier transform. The lower plot shows a windowed Fourier-like integral, in the form $\int h(t) \cos(\omega t) e^{-(t-t_0)^2/\sigma^2} dt$ for some window width σ , as well as the window. It can be clearly seen that in the regions where there is a frequency mismatch the contribution vanishes.

where the stationary time t_s is precisely the one such that $\dot{\Phi}(t_s) = \omega$ (which means that the linear term of the expansion vanishes).

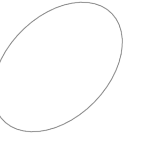
The Fourier integral then looks like

$$\tilde{f}(\omega) \approx \int_{-\infty}^{\infty} dt A(t) \exp(-i\Phi(t_s) + i\omega t_s) \exp\left(-\frac{i}{2}\ddot{\Phi}(t - t_s)^2\right), \quad (\text{A.3})$$

so the t -dependent part of the integral looks like the product of the amplitude and a “complex Gaussian”. There is an analytic formula for a complex Gaussian integral

$$\int e^{i\alpha x^2} dx = \sqrt{\frac{\pi i}{\alpha}}, \quad (\text{A.4})$$

however, in order to apply it, we need to approximate the amplitude as constant: $A(t) \approx A(t_s)$. This will be valid as long as the variation of the amplitude is small compared to the width of the Gaussian we are integrating it against, which is $\sigma = 1/\sqrt{\ddot{\Phi}}$, so a condition we might write is $\dot{A}/A \lesssim 1/\sqrt{\ddot{\Phi}}$.



If this is the case, we can write the Fourier integral as

$$\tilde{f}(\omega) \approx A(t_s) e^{-i\Phi(t_s) + i\omega t_s} \sqrt{\frac{-2\pi i}{\ddot{\Phi}}} \quad (\text{A.5})$$

$$|\tilde{f}(\omega)| \approx A(t_s(\omega)) \sqrt{\frac{2\pi}{\ddot{\Phi}(t_s(\omega))}} \quad (\text{A.6})$$

$$\angle \tilde{f}(\omega) \approx -\Phi(t_s(\omega)) + \omega t_s(\omega) - \frac{\pi}{4}, \quad (\text{A.7})$$

where the $-\pi/4$ comes from the $\sqrt{-i}$.

We want to apply this to the specific case of A and Φ being given by polynomial expressions like the formulas describing quadrupole emission: (1.75) and (1.73); for simplicity, let us write these as

$$\Phi(t) = K_\Phi (-t)^\alpha \quad \text{and} \quad A(t) = K_A (-t)^\beta, \quad (\text{A.8})$$

where the reason for writing $(-t)$ is that the aforementioned equations are written in terms of the time until merger $\tau - t = -\tau$ is increasing and equal to 0 at the merger. We do not specify the polarization, the argument will apply for both. The true values for the exponents in the quadrupole approximation are $\alpha = 5/8$ and $\beta = -1/4$.

The first thing to do is to find $t_s(\omega)$: it is given by

$$\omega = \dot{\Phi} = \alpha K_\Phi (-t_s)^{\alpha-1} \quad (\text{A.9})$$

$$t_s = -\left(\frac{\omega}{\alpha K_\Phi}\right)^{1/(\alpha-1)}. \quad (\text{A.10})$$

The other quantities we need to compute are:

$$A(t_s) = K_A \left(\frac{\omega}{\alpha K_\Phi}\right)^{\beta/(\alpha-1)} \quad (\text{A.11})$$

$$\Phi(t_s) = K_\Phi \left(\frac{\omega}{\alpha K_\Phi}\right)^{\alpha/(\alpha-1)} \quad (\text{A.12})$$

$$\ddot{\Phi}(t_s) = K_\Phi \alpha(\alpha-1) (-t_s)^{\alpha-2} = -K_\Phi \alpha(\alpha-1) \left(\frac{\omega}{\alpha K_\Phi}\right)^{(\alpha-2)/(\alpha-1)}, \quad (\text{A.13})$$

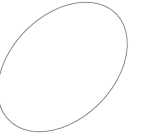
therefore the phase reads

$$\angle \tilde{f}(\omega) = -K_\Phi \left(\frac{\omega}{\alpha K_\Phi}\right)^{\alpha/(\alpha-1)} - \omega \left(\frac{\omega}{\alpha K_\Phi}\right)^{1/\alpha} - \frac{\pi}{4} \quad (\text{A.14})$$

$$= -\left(\frac{\omega^\alpha}{K_\Phi} (\alpha^{-\alpha} + \alpha^{-1})\right)^{1/(\alpha-1)} - \frac{\pi}{4}, \quad (\text{A.15})$$

while the amplitude reads

$$|\tilde{f}(\omega)| = K_A \left(\frac{\omega}{\alpha K_\Phi}\right)^{\beta/(\alpha-1)} \underbrace{\sqrt{2\pi} \left(K_\Phi \alpha(\alpha-1) \left(\frac{\omega}{\alpha K_\Phi}\right)^{(\alpha-2)/(\alpha-1)}\right)^{-1/2}}_{1/\sqrt{\ddot{\Phi}}} \quad (\text{A.16})$$



$$= K_A (\alpha K_\Phi)^{\frac{\beta-1/2}{\alpha-1}} \omega^{\frac{\beta-(\alpha-2)/2}{\alpha-1}} \sqrt{\frac{2\pi}{\alpha-1}}. \quad (\text{A.17})$$

The last complication is the fact that the phase appears in the expression for the waveform (1.75) not with a complex exponential, but instead with a cosine or a sine: this is not a problem, we just need to use the relations $\cos x = (e^{ix} + e^{-ix})/2$ and $\sin x = (e^{ix} - e^{-ix})/2i$. The factor of i in the sine will correspond to a $\pi/2$ phase difference between the two polarizations, while the average of the two different exponentials will correspond to an amplitude reduction of a factor 2 if we restrict ourselves to positive frequencies.¹²

Substituting the values for α , β , K_Φ and K_A we get the following expressions [Mag07, eqs. 4.34–37], written in terms of $f = \omega/2\pi$:

$$\tilde{h}_+(f) = \frac{1}{\pi^{2/3}} \sqrt{\frac{5}{24}} \frac{c}{r} \left(\frac{G\mathcal{M}_c}{c^3} \right)^{5/6} f^{-7/6} \exp \left(\frac{3i}{4} \left(\frac{G\mathcal{M}_c}{c^3} 8\pi f \right)^{-5/3} - i\frac{\pi}{4} \right) \left(\frac{1 + \cos^2 \iota}{2} \right) \quad (\text{A.18})$$

$$\tilde{h}_-(f) = \frac{1}{\pi^{2/3}} \sqrt{\frac{5}{24}} \frac{c}{r} \left(\frac{G\mathcal{M}_c}{c^3} \right)^{5/6} f^{-7/6} \exp \left(\frac{3i}{4} \left(\frac{G\mathcal{M}_c}{c^3} 8\pi f \right)^{-5/3} + i\frac{\pi}{4} \right) \cos \iota. \quad (\text{A.19})$$

The frequency dependence of the amplitude comes out to be $|\tilde{f}(\omega)| \propto \omega^{-7/6} \propto f^{-7/6}$: it might be surprising to see that this is decreasing while the chirping waveform rises in amplitude! The reason for this fact is that each portion of the Fourier integral is weighted not only by the amplitude of the envelope $A(t)$ but also by the time the waveform “spends” in that frequency region, which decreases with time. These two effects compete, but $A \sim \tau^{-1/4}$ while the time spent in the frequency region is measured by $1/\sqrt{\ddot{\Phi}} \sim 1/\sqrt{\tau^{-11/8}} = \tau^{+11/16}$, so the latter wins: $|\tilde{h}(f)| \sim \tau^{7/16} \propto \omega^{-7/6}$.

B Acronyms

GR general relativity

GW gravitational wave

EOB effective one-body

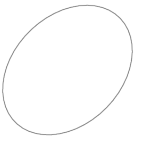
NR numerical relativity

PN post-Newtonian

PCA principal component analysis

BNS binary neutron star

¹² Since the Fourier transform is linear the two exponentials can be treated separately. The transform of a real signal is fully encoded by its positive-frequency components; in order for the integral not to vanish in the Gaussian approximation there needs to be a stationary point; the phase of the **GW** signal is strictly monotonic. These three assumptions lead to the fact that only one of the two exponentials will give a nonzero contribution while the other will have no stationary point, so only the division by 2 will matter in the average.



BBH binary black hole

NS neutron star

BH black hole

SPA stationary phase approximation

NN neural network

MOTPE multi-objective tree-structured Parzen estimator

TPE tree-structured Parzen estimator

mlgw_bns machine learning gravitational waves from binary neutron stars

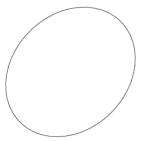
TT transverse-traceless

CBC compact binary coalescence

SGWB stochastic gravitational wave background

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