# Low Energy Theoretical Astroparticle Physics

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## 1 Neutrino physics

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Neutrinos are important because:

1. being neutral, they can trace sources;

   What does that mean?

2. they are connected with new physics;
3. there are open questions about their nature (Dirac or Majorana);
4. there could be a CP violation in the lepton sector;
5. they affect cosmology, with Baryon Acoustic Oscillations in the CMB, as well as the large scale structure.

nu.to.infn.it, "Neutrino Unbound"; Giunti et al, Giunti and Kim.
Outline of the course:

1. Dirac equation;
2. gauge theories;
3. standard EW model;
4. fermion masses and mixing;
5. neutrino oscillations in vacuo;
6. neutrino oscillations in matter;
7. current neutrino phenomenology;
8. extra on statistics and data analysis.

We know that neutrino masses are less than an eV; there are (at least) three flavours: $\nu_e$, $\nu_\mu$ and $\nu_\tau$, typically organized in doublets with the corresponding charged lepton.

These neutrinos can have charged interactions with a $W^\pm$ boson, as well as neutral interactions with a $Z$ boson (which is connected with photon production).

Looking at these decays tells us about the number of (interacting) neutrino families: $N_\nu = 3$.

There are also bounds from cosmology: both in the CMB spectrum and in primordial nucleosynthesis.

Neutrinos are chiral in nature: only left-handed neutrinos (with negative helicity) seem to interact.

We know that a left-handed particle has a right-handed component of order $\beta \sim m/E$.

At order $m^2/E$ there are neutrino flavor oscillations.

## The Dirac equation

We start from Lorentz transformations: linear transformations which preserve the Minkowski metric. They are rotations and boosts. They can be written in terms of infinitesimal transformations: for example,

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \approx \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} \right) \begin{bmatrix} y \\ z \end{bmatrix}, \tag{1.1}$$

therefore

$$\Lambda = \mathbb{1} + i\,\mathrm{d}\theta \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \mathcal{O}\left(\theta^2\right) = e^{i\theta J_1} + \mathcal{O}\left(\theta^2\right). \tag{1.2}$$

The same holds for the boosts:

$$\Lambda = \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix} = e^{iuk_1} \qquad \text{where} \qquad K_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{1.3}$$

With these generators we can make any transformation we want: a Lorentz transformation will be in general given by the composition of a rotation around $\vec{\omega} = \theta\hat{n}$ and a boost in $\vec{u} = u\hat{v}$. This will then read

$$\Lambda = \exp\left( i \left( \vec{\omega} \cdot \vec{J} + \vec{u} \cdot \vec{K} \right) \right). \tag{1.4}$$

In general these do not commute: $[J_i, J_k] \neq 0$, $[K_i, K_j] \neq 0$, $[J_i, K_j] \neq 0$. This, however, is a closed algebra: all these commutators are given in terms of other $J$, $K$ matrices.

This construction is not only useful if we need to make complicated transformations; it is also useful as a theoretical mean to parametrize a general transformation.

Pauli was trying to generalize the Schrödinger equation for a spin-1/2 particle. It can be found by mapping $\vec{p} \to -i\vec{\nabla}$ and $E \to i\partial_t$ in the eigenvalue equation for a classical kinematic Hamiltonian $H = p^2/2m + V$.

This is classical, so we forget boosts: let us try to at least have the 4D $J_i$ rotation algebra $[J_i, J_j] = i\epsilon_{ijk} J_k$ for two-component vectors: this is also obeyed by the 2D matrices $\sigma_i/2$, where

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.5}$$

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{1.6}$$

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{1.7}$$

So, we could think to have a spinor transform under

$$\xi' = \lambda \xi \qquad \text{where} \qquad \lambda = \exp\left( i \vec{\omega} \cdot \frac{\vec{\sigma}}{2} \right) . \tag{1.8}$$

In order for the momentum $\vec{p}$ to act on 2D objects we can use $\vec{p} \cdot \vec{\sigma}$.

The idea is to use minimal coupling: $\vec{p} \to \vec{p} - q\vec{A}$, and $\vec{V} \to \vec{V} + q\Phi$.

How can we generalize this to boosts? Not only is it possible to do, but there are two ways to do it.

The Lorentz group, to which the matrices $\Lambda$ in $x' = \Lambda x$ belong, satisfies

$$\Lambda = \exp\left( i \left( \vec{\omega} \cdot \vec{J} + \vec{u} \cdot \vec{K} \right) \right) \tag{1.9}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{1.10}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k \tag{1.11}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k . \tag{1.12}$$

Spinors will transform with

$$\xi' = \Lambda_\xi \xi \tag{1.13}$$

$$\Lambda_\xi = \exp\left( i \frac{\vec{\sigma}}{2} \cdot \vec{J} + ? \right) \tag{1.14}$$

$$\left[ \frac{\sigma_i}{2}, \frac{\sigma_j}{2} \right] = i\epsilon_{ijk} \frac{\sigma_k}{2} , \tag{1.15}$$

so what do we need to add? We can do either $\vec{k} = \pm i\vec{\sigma}/2$, so we get

$$\Lambda_\xi = \exp\left( i \frac{\vec{\sigma}}{2} \cdot \vec{\omega} \pm \vec{u} \cdot \frac{\vec{\sigma}}{2} \right) . \tag{1.16}$$

The plus sign is for right-handed spinors, the minus sign is for left-handed ones. If we have a spinor at rest, we cannot determine its helicity. If we boost it, we still cannot determine it!

But, it cannot be in a superposition of things which transform in different ways.

The idea is then to have a direct sum, an object which contains both the left and right-handed components:

$$\xi = \begin{bmatrix} \xi_L \\ \xi_R \end{bmatrix} . \tag{1.17}$$

These will be given by boosting the rest-frame spinor $\xi$ in two different ways:

$$\xi'_R = \exp\left(\vec{u} \cdot \frac{\vec{\sigma}}{2}\right)\xi \tag{1.18}$$

$$\xi'_L = \exp\left(-\vec{u} \cdot \frac{\vec{\sigma}}{2}\right)\xi . \tag{1.19}$$

The Dirac equation is a relation between the left- and right-handed components of a free spinor. In order to derive it, we need some relations:

$$\exp\left(\vec{u} \cdot \frac{\vec{\sigma}}{2}\right) = \cosh(u/2) + \hat{u} \cdot \vec{\sigma} \sinh(u/2) \tag{1.20}$$

$$(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot \left(\vec{a} \times \vec{b}\right) \tag{1.21}$$

$$\exp\left(i\vec{\omega} \cdot \frac{\vec{\sigma}}{2}\right) = \cos(\theta/2) + i\hat{u} \cdot \vec{\sigma} \sin(\theta/2) . \tag{1.22}$$

With these, we get

$$\exp\left(\pm\vec{u} \cdot \frac{\vec{\sigma}}{2}\right) = \frac{E + m \pm \vec{p} \cdot \vec{\sigma}}{\sqrt{2m(E+m)}} \tag{1.23}$$

$$\xi_{R,L} = \frac{E + m \pm \vec{p} \cdot \vec{\sigma}}{\sqrt{2m(E+m)}}\xi \tag{1.24}$$

$$\xi = \frac{E + m \mp \vec{p} \cdot \vec{\sigma}}{\sqrt{2m(E+m)}}\xi_{R,L} , \tag{1.25}$$

therefore we can put these equations together into

$$-m\xi_{R,L} + \left(E \pm \vec{p} \cdot \vec{\sigma}\right) \xi_{L,R} = 0 , \tag{1.26}$$

or

$$\begin{bmatrix} -m & E + \vec{p} \cdot \vec{\sigma} \\ E - \vec{p} \cdot \vec{\sigma} & -m \end{bmatrix} \begin{bmatrix} \xi_R \\ \xi_L \end{bmatrix} = 0 . \tag{1.27}$$

If $m = 0$, these decouple into two equations:

$$(p \pm \vec{p} \cdot \vec{\sigma})\xi_{L,R} = 0 . \tag{1.28}$$

Therefore, for a massless particle

$$\left(\hat{p} \cdot \sigma\right) = \pm\xi_{R,L} . \tag{1.29}$$

This is not the case when the particle is massive.

Helicity is the expectation value of $\hat{p} \cdot \vec{\sigma}$, chirality is "being $\xi_L$ or $\xi_R$".

This is all in momentum space, but it can also be written in position space by switching the momentum to a derivative: we introduce the Dirac matrices

$$\gamma^0 = \begin{bmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{bmatrix} \qquad \text{and} \qquad \vec{\gamma} = \begin{bmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}, \tag{1.30}$$

so that the Dirac equation becomes

$$(\gamma^\mu p_\mu - m)\psi = 0\,, \tag{1.31}$$

where $\psi = (\xi_L, \xi_R)^\top$.

This can then also be written in terms of derivatives as $(i\gamma^\mu \partial_\mu - m)\psi = 0$.

Left- and right-handed spinors transform in the same way under rotations; they differ under boosts.

It is useful to introduce

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\ : \tag{1.32}$$

if we introduce

$$P_{L,R} = \frac{1 \mp \gamma^5}{2} \tag{1.33}$$

this will select the left- or right-handed component of a spinor.

We have derived this result using the Weyl basis; Dirac used a different one.

If the spinor transforms with $\psi \to T\psi$ the Dirac matrices transform with $\gamma^\mu = T\gamma^\mu T^{-1}$.

In the Dirac basis, which is useful when one studies non-relatvistic particles, the matrices look like

$$\gamma^0 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{bmatrix} \qquad \text{and} \qquad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}, \tag{1.34}$$

while

$$\gamma^5 = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}. \tag{1.35}$$

In the nonrelativistic limit $\vec{p} \to 0$ the equation decouples into

$$\begin{bmatrix} E - m & 0 \\ 0 & E + m \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0\,, \tag{1.36}$$

so we have $E = \pm m$ solutions. Note that $\xi$ and $\eta$ are not left- and right-handed, this is a different basis. In Dirac's time this was seen as a tragedy.

The fact that these are actually particles and antiparticle can be properly explained in QFT; in the end, the four degrees of freedom correspond to the left- and right-handed components of particles and antiparticles.

We can write the particle and antiparticle solutions as

$$\psi_p = \begin{bmatrix} \xi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\xi \end{bmatrix} e^{-ip_\mu x^\mu} \qquad \text{and} \qquad \psi_A = \begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\eta \\ \eta \end{bmatrix} e^{+ip_\mu x^\mu} . \tag{1.37}$$

It is convenient to introduce a charge-conjugation operator:

$$\mathsf{C}(\psi) = \psi^c = i\gamma^2\psi^* . \tag{1.38}$$

This can be seen by proving that $C(\psi_P) = \psi_A$. Also, if we look at the minimally-coupled Dirac equation for a particle in an external EM field, and we apply the conjugation operator we find that the particle will obey the same equation, but with an opposite charge.

The last thing to introduce here is the adjoint spinor: we know how a 4-spinor $\psi$ behaves under a Lorentz transformation. What is the object which transforms with the inverse transformation? If we denote it as $\overline{\psi}$, we will be able to write *invariant* objects like $\overline{\psi}\psi$.

It takes some time to prove, but it comes out to be

$$\overline{\psi} = \psi^\dagger\gamma^0 . \tag{1.39}$$

We can make invariant objects like $\overline{\psi}\psi$, or objects like $\overline{\psi}\gamma^\mu\psi$: it can be proven that the latter is *divergenceless*, $\partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0$, so it can be used to describe currents.

## 2 The standard electroweak model

### Gauge theories

Tuesday
2021-12-7

Yesterday we wrote the equation for a free fermion:

$$\xi'_{R,L} = \exp\left(-i\vec{\omega}\cdot\frac{\vec{\sigma}}{2} \pm \vec{u}\cdot\frac{\vec{\sigma}}{2}\right)\xi_{R,L} . \tag{2.1}$$

The Dirac equation tells us that these are coupled:

$$\begin{bmatrix} -m & E + \vec{p}\cdot\vec{\sigma} \\ E - \vec{p}\cdot\vec{\sigma} & m \end{bmatrix} \begin{bmatrix} \xi_R \\ \xi_L \end{bmatrix} = 0 . \tag{2.2}$$

For nonzero mass these components are coupled, while in the zero-mass case they are decoupled and helicity is equal to chirality.

Helicity is the eigenvalue under $\hat{p}\cdot\vec{\sigma}$, chirality is the eigenvalue under $\gamma^5$.

What is the meaning of $\gamma^5$ in position space? Is it a parity transformation?

The coupling of a fermion to an external EM field can be represented with $\partial_\mu \to \partial_\mu + iqA_\mu$.

The SM Lagrangian can be written on a mug as

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\psi}\not{D}\psi + \text{h.c.} + i\psi_i y_{ij}\psi_j\phi + \text{h.c.} + \left|\not{D}\phi\right|^2 - V(\phi) . \tag{2.3}$$

We have scalars $\phi$, fermions $\psi$ and spin-1 gauge fields $A_\mu$.

From a Lagrangian $L(q, \dot{q})$ we get Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0\,. \tag{2.4}$$

If the Lagrangian is, say, $L = m\dot{q}^2/2 - V(q)$ we get Newton's law $m\ddot{q} = -\vec{\nabla} V = F$.

This is classical; in field theory we have Lagrangians in the form

$$\partial_\mu \frac{\partial \mathscr{L}(\phi, \partial\phi)}{\partial \partial_\mu \phi_i} - \frac{\partial \mathscr{L}}{\partial \phi_i} = 0\,. \tag{2.5}$$

We will write the Yukawa Lagrangian with terms like $\psi\phi$, and the QED Lagrangian with terms like $\psi A_\mu$.

The equation of motion for a scalar field is the Klein-Gordon equation: $(\partial_\mu \partial^\mu + m^2)\phi = 0$. The Lagrangian giving rise to this is

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m \phi^2\,. \tag{2.6}$$

We have written the EOM for a free fermion, the Dirac equation $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$; the Lagrangian giving it is the Dirac Lagrangian

$$\mathscr{L} = \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi\,. \tag{2.7}$$

Again, we have a mass-like term and a kinetic-like term.

The term $m\overline{\psi}\psi$ can be expanded into left and right components:

$$m\overline{\psi}\psi = m\left(\overline{\psi}_R \psi_L + \overline{\psi}_L \psi_R\right)\,. \tag{2.8}$$

What about the EM field? $A_\mu$ is gauge-dependent, but

$$F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} \tag{2.9}$$

is not. The Lagrangian giving Maxwell's equations in a vacuum (so, $\Box A^\mu = 0$ in the appropriate gauge) is

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\,. \tag{2.10}$$

The absence of a mass term means that the photon is massless.

What about interactions? We start from the Yukawa interaction between a fermion and a scalar. The Lagrangian, for starters, must contain the free terms for both:

$$\mathscr{L} = \underbrace{\overline{\psi}\left(i\gamma^\mu \partial_\mu - m_\psi\right)\psi + \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m_\phi \phi^2\right)}_{\text{free terms}} - \underbrace{g\overline{\psi}\psi\phi}_{\text{interaction}}\,. \tag{2.11}$$

The pictorial way to represent this is to draw quadratic terms for fermions like straight lines with an arrow, scalar fields as dashed lines, and cubic interaction terms like vertices.

What do fermion-photon interactions look like?

$$\mathscr{L}_{\text{QED}} = \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m_{\psi}\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - q\overline{\psi}\gamma^{\mu}\psi A_{\mu}\,. \tag{2.12}$$

The EOM for the electromagnetic field are Maxwell's equations with an external current $j^{\mu} = q\overline{\psi}\gamma^{\mu}\psi$.

This is the same Lagrangian we get if we do a minimal coupling substitution $\partial \to \partial + iqA$. It is invariant under gauge transformations.

These diagrams can be used to compute scattering amplitudes perturbatively.

The way to get a cross-section is to multiply the square modulus of the amplitude by the phase space term. The idea to get decay rates is similar.

The guiding principle to describe EW interactions is gauge invariance. The Yukawa Lagrangian has global phase invariance; what happens if we try to make a transformation $\psi \to e^{-iq\alpha(a)}\psi$? The term $q$ is only introduced here for later convenience.

The timezone analogy for local gauge invariance! If we have the freedom to choose a phase locally, we must have carriers of information moving at the maximum possible speed, otherwise processes would be disrupted.

We start with $U(1)$ gauge invariance, as written above, but we will also need $SU(2)$ invariance, written as

$$\exp\left(-ig\vec{\theta}\cdot\vec{T}\right), \tag{2.13}$$

where $\vec{T} = \vec{\sigma}/2$, but we write them differently to not confuse them with spacetime rotations.

Introducing gauge invariance for a term $\overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi$ will take us to the QED Lagrangian.

The mass term is already invariant, the problem is the kinetic one. The trick is to introduce a covariant derivative

$$\mathrm{D}_{\mu} = \partial_{\mu} + iqA_{\mu}\,. \tag{2.14}$$

If $\psi \to e^{-iq\alpha(x)}\psi$, also $\mathrm{D}_{\mu}\psi \to e^{-iq\alpha(x)}\mathrm{D}_{\mu}\psi$, as long as $A_{\mu}$ also transforms like $A_{\mu} \to A_{\mu} + \partial_{\mu}\alpha$.

Then, the Lagrangian

$$\mathscr{L} = \overline{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,. \tag{2.15}$$

A term like $m^2 A_{\mu}A^{\mu}$ would violate gauge invariance, so this principle also gives an explanation as to why the photon is massless if we accept the gauge invariance principle.

Protons and neutrons were thought to be part of a doublet under isospin SU(2) transformations.

But, we have interactions like protons and neutrons interacting with electrons and neutrinos. So, one might think that electrons + neutrinos have isospin-like SU(2) symmetry as well.

We want to make the doublet $\psi_1, \psi_2$ invariant under SU(2) transformations as written above: how do we do it? Again, we redefine the derivative:

$$\partial_\mu \to \mathrm{D}_\mu - ig\vec{T} \cdot \vec{A}_\mu \,, \tag{2.16}$$

so we need to introduce three fields $\vec{A}_\mu$. These are the gauge bosons related to SU(2) gauge invariance.

These now transform like

$$\vec{A}_\mu \to \vec{A}_\mu - \partial_\mu \vec{\theta}(x) + g\vec{\theta} \times \vec{A}_\mu \,. \tag{2.17}$$

The presence of this coupling means that the field is charged.

The field tensor here is

$$\vec{F}_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} + g\vec{A}_\mu \times \vec{A}_\nu \,. \tag{2.18}$$

We then get a Lagrangian like

$$\mathscr{L} = \overline{\psi} \left( i\gamma^\mu \mathrm{D}_\mu - m \right) \psi - \frac{1}{4} \vec{F}^{\mu\nu} \cdot \vec{F}_{\mu\nu} \,. \tag{2.19}$$

In the $\psi$ term we have $\psi\psi$ terms, as well as $\psi\psi A$ interactions. In this $F^2$ term we have quadratic, trilinear and quadrilinear terms in $A$!

The actual group for the EW theory is $SU(2)_L \otimes U(1)_Y$. The simplest EW world contains

1. 1 massive electron, with both right- and left-handed components;
2. 1 massless $\nu_e$, with only the left-handed component;
3. EM interactions (we want to have a diagram describing how the electron interacts with a massless photon);
4. weak chiral interactions like $\nu_L e_L W^\pm$, where $W^\pm$ is massive.

The Higgs mechanism accomplishes this, and it also gives mass to fermions. If neutrinos have Majorana masses, the Higgs mechanism cannot give them mass.

Let us define a doublet $L = (e_L, \nu_{e,L})$, and a singlet $R = e_R$. The theory must then be invariant if we redefine

$$L \to L' = \exp\Big( -ig\vec{\theta}(x) \cdot \vec{T} \Big) L \tag{2.20}$$

$$R \to R' = R \,. \tag{2.21}$$

The doublet $L$ corresponds to the $T_3 = \pm 1/2$ quantum numbers, while $T_3 = 0$ for the singlet.

Could the three bosons be some combinations of the photon and the two $W^\pm$ bosons? We know that the charge operator $Q$ has eigenvalues 0, $-1$ for the doublet, but $\operatorname{Tr} Q \neq 0$ while $\operatorname{Tr} T_i = 0$ for all $T_i$.

Why would $Q$ need to be able to be written as a function of the $T_i$? The proper answer lies in Nöther's theorem.

We therefore introduce a further hypercharge symmmetry:

$$L' = \exp\left(-ig'\beta(x)\frac{Y}{2}\right)L \tag{2.22}$$

$$R' = \exp\left(-ig'\beta(x)\frac{Y}{2}\right)R\,, \tag{2.23}$$

so that this generator commutes with the $T_i$, and indeed $[T_i, Y] = 0$. The only way to have this is $Y \propto Q - T_3$, and indeed $Y = 2(Q - T_3)$.

In the end, the Lagrangian must be chargeless.

The full transformation is therefore

$$L' = \exp\left(-ig\vec{\theta}(x)\cdot\vec{T} - ig'\beta(x)\frac{Y}{2}\right)L \tag{2.24}$$

$$R' = \exp\left(-ig'\beta(x)\frac{Y}{2}\right)R\,. \tag{2.25}$$

The procedure is then like before: we need to redefine the derivative, as

$$\mathrm{D}_\mu L = \left(\partial_\mu + ig\vec{T}\cdot\vec{A}_\mu - ig'\frac{Y}{2}B_\mu\right)L\,, \tag{2.26}$$

so we have four fields, the three $\vec{A}_\mu$ with their $\vec{F}_{\mu\nu}$ and $B_\mu$ with its $G_{\mu\nu}$.

The Lagrangian will then read

$$\mathscr{L} = \overline{L}i\gamma^\mu \mathrm{D}_\mu L + \overline{R}i\gamma^\mu \mathrm{D}_\mu R + \text{kinetic}\,, \tag{2.27}$$

but we cannot write mass terms like $m\overline{\psi}\psi$, which would be $\overline{L}R$ or $L\overline{R}$: the matrix dimensions don't match up!

So, everything's massless: the Higgs mechanism comes to the rescue. It gives masses to all the bosons except the photon, as well as giving mass to the leptons.

Suppose we have a $U(1)\otimes U(1)$ symmetry, spontaneously broken to $U(1)$. The thing we think of is a pencil about to fall on a table.

Thursday 2021-12-9

We have discussed the Dirac equation and SU(2) gauge invariance; now we will look at the Higgs mechanism and SSB, with the goal in mind to understand the Lagrangian written on the CERN mug.

We study this model in the context of an $L$ doublet $(\nu_{eL}, e_L)$ and an $R$ singlet $(e_R)$.

The Higgs mechanism yields neutral currents as a "bonus".

So far, we have written the gauge field Lagrangian:

$$\mathscr{L}_{\mathrm{GF}} = -\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}\,, \tag{2.28}$$

and the lepton kinetic Lagrangian:

$$\mathscr{L}_{\mathrm{F}} = \overline{L}(i\gamma^\mu \mathrm{D}_\mu)L + \overline{R}(i\gamma^\mu \mathrm{D}_\mu)R \tag{2.29}$$

$$\mathrm{D}_\mu L = \left(\partial_\mu - ig\vec{T}\cdot\vec{A}_\mu - ig'\frac{Y}{2}B_\mu\right)L \tag{2.30}$$

$$\mathrm{D}_\mu R = \left( \partial_\mu - i g' \frac{Y}{2} B_\mu \right) R \,. \tag{2.31}$$

We want to have a breaking like $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y \to \mathrm{U}(1)_Q$, where we want the ground state $Q$ to be chargeless (since it represents the photon).

In order to have a Dirac-like mass term, $m\overline{\psi}_L \psi_R$, we need to saturate a doublet with a singlet: it cannot be done! So, we need a new field $\phi$ which is at least a doublet, and then we can make a term $\overline{L}\phi R$.

A complex field $\phi \in \mathbb{C}^2$ works, and it must be chargeless for things to work. The Yukawa Lagrangian will then be

$$\mathscr{L}_{\text{YUK}} = -y_e \left( \overline{L}\phi R + \overline{R}\phi^\dagger L \right) \,. \tag{2.32}$$

A mass term for the field $\phi$ in the form $m^2 \phi^\dagger \phi$ would be stable, but we want to make it *un*stable at $\phi = 0$, o we need a $\phi^\dagger \phi$ term with a negative coefficient and a $(\phi^\dagger \phi)^2$ term with a positive coefficient:

$$V(\phi) = -\mu^2 (\phi \phi^\dagger) + \lambda (\phi \phi^\dagger)^2 \,. \tag{2.33}$$

The minimum of this potential is reached when $\phi\phi^\dagger = (1/2)\mu^2/\lambda = (1/2)v^2$.

Why $\phi\phi^\dagger$? That's 2x2...

So far, we have introduced the couplings $g$ and $g'$, the parameters $\mu$ and $v$ for the Higgs field potential, and the Yukawa coupling $y_e$.

The connection to experiment will be to write in terms of these parameters $e$, $G_F$, $M_Z$, $M_H$, $m_e$.

So a 5 parameter fit for 5 measurements?

The $\phi$ field near the minimum can be parametrized as

$$\phi = \begin{bmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{bmatrix} \,. \tag{2.34}$$

The upper, charged part is equal to zero. In the Yukawa Lagrangian we find

$$\begin{aligned}
\mathscr{L}_{\text{YUK}} &= -y_e \left[ \begin{bmatrix} \overline{\nu}_{eL} & e_L \end{bmatrix} \begin{bmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{bmatrix} e_R + \bar{e}_R \begin{bmatrix} 0 & \frac{v+H}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \nu_{eL} \\ e_L \end{bmatrix} \right] && (2.35) \\
&= -\frac{y_e}{\sqrt{2}} (v + H) \left( \bar{e}_L e_R + \bar{e}_R e_L \right) && (2.36) \\
&= -m_e \bar{e} e - \frac{y_e v}{\sqrt{2}} H \bar{e} e \,. && (2.37)
\end{aligned}$$

We have not obtained any mass for the neutrino, as we expected by not having a $\nu_R$ term.

The proportionality of the Yukawa coupling to the masses has been seen experimentally.

In general the degrees of freedom of the $\phi$ can be written as

$$\phi = \exp\left(i\vec{T}\cdot\frac{\vec{\sigma}}{2}\right)\left[\begin{array}{c} 0 \\ \frac{v+H}{\sqrt{2}} \end{array}\right], \tag{2.38}$$

so we can use SU(2) gauge invariance to select a ground state; the three remaining degrees of freedom become Goldstone bosons and are "eaten" by the three $W^{\pm}_{\mu}$, $Z^0_{\mu}$ bosons to give them a third, longitudinal polarization.

The Higgs sector of the Lagrangian becomes

$$\mathscr{L}_{\mathrm{H}} = \left(\mathrm{D}_{\mu}\phi^{\dagger}\right)\left(\mathrm{D}_{\mu}\phi\right) - V(\phi), \tag{2.39}$$

which contains the bosons. What we get if we diagonalize this Lagrangian is terms written in terms of the new fields

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}\left(A^1_{\mu} \pm A^2_{\mu}\right), \tag{2.40}$$

and

$$\left[\begin{array}{c} Z_{\mu} \\ A_{\mu} \end{array}\right] = \left[\begin{array}{cc} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{array}\right]\left[\begin{array}{c} A^3_{\mu} \\ B_{\mu} \end{array}\right], \tag{2.41}$$

where $\theta_W$ is the Weinberg angle, chosen such that $\tan\theta_W = g'/g$.

The masses of these bosons come out to be

$$M_W^2 = \frac{v^2}{4}g^2 \qquad \text{and} \qquad M_Z = \frac{v^2}{4}(g^2 + g'^2), \tag{2.42}$$

while $m_{\gamma} = 0$.

We get cubic couplings like $HZZ$ but also quartic ones like $HHZZ$ or $HHWW$.

The mass terms for the Higgs is $m_H = \sqrt{2}\mu$.

The photon does not couple directly to the Higgs, but we can have a $H \to \gamma\gamma$ process through loops.

When we rewrite the gauge field Lagrangian in terms of the physical fields $A_{\mu}$, $W_{\mu}$ and $Z_{\mu}$, we get quadratic, cubic and quartic terms in the gauge terms.

The terms we get are $\gamma WW$, $ZWW$, as well as the quartic $\gamma\gamma WW$, $WWWW$, $ZZWW$, $\gamma ZWW$.

What about the fermion field Lagrangian? We expect to get terms in the form $X^{\mu}J_{\mu}$, where $X^{\mu}$ is a gauge field while $J_{\mu}$ is a fermion current.

The result is

$$\mathscr{L}_{\mathrm{F}} = \mathscr{L}_{\text{electromagnetic}} + \mathscr{L}_{\text{charged current}} + \mathscr{L}_{\text{neutral current}}. \tag{2.43}$$

The electromagnetic term is

$$\mathscr{L}_{\text{electromagnetic}} = \underbrace{\frac{gg'}{\sqrt{g^2+g'^2}}}_{e} J^{\mu}_{EM}A_{\mu}, \tag{2.44}$$

where $e$ is the electric charge; the other terms read

$$\mathscr{L}_{\text{charged current}} = \frac{g}{\sqrt{2}} J^\mu_\pm W^\mp \tag{2.45}$$

$$\mathscr{L}_{\text{neutral current}} = \frac{g}{\cos\theta_W} J^\mu_{NC} Z_\mu \,. \tag{2.46}$$

The electromagnetic current will be $J^\mu_{EM} = \bar{e}\gamma^\mu Q e$ (by construction: we made the theory to reproduce Maxwell's equations).

The charged current is $J^\mu_+ = \bar{e}_L \gamma^\mu \nu_{eL}$ and $J^\mu_- = \bar{\nu}_{eL}\gamma^\mu e_L$.

These were the already-observed parts, while the new one is

$$J^\mu_{NC} = \bar{\nu}_{eL}\gamma^\mu T_3 \nu_{eL} + \bar{e}_L \gamma^\mu \left( T_3 - Q s^2_W \right) e_L + \bar{e}_R \gamma^\mu (-Q s^2_W) e_R \,. \tag{2.47}$$

This term is carrying a current $T_3 - Q s^2_W$, where $s^2_W = \sin^2\theta_W$. The generator $T_3$ is just half of the $\sigma_3$ Pauli matrix.

The phenomenology of this prediction is quite rich.

We have couplings in the form $\gamma^\mu P_{L,R}$, there is specific jargon to describe these interactions.

1. Vector ($V$) currents are those in the form $\overline{\psi}\gamma^\mu\psi$, which do not change sign under a parity transformation;

2. Axial ($A$) currents are those in the form $\overline{\psi}\gamma^\mu\gamma^5\psi$, which *do* change sign under a parity transformation.

EM interactions are of type $V$, charged currents are of type $V - A$ (or, $1 \pm \gamma^5$), while neutral currents are of the form $g_V V + g_A A$.

We can define left-handed couplings in the neutral current $g_L = T_3 - Q s^2_W$, as well as right-handed ones like $g_R = -Q s^2_W$ (since the action of $T_3$ on an $R$ singlet is zero).

The vector and axial couplings are then defined as

$$g_V = g_L + g_R = T_3 - 2Q s^2_W \tag{2.48}$$

$$g_A = g_L - g_R = T_3 \,. \tag{2.49}$$

When a short-range gauge boson mediates an interaction, its contribution will be approximately $1/M^2$ at low energies.

So, both processes which are mediated by $W^\pm$ and $Z$ must give the correct low-energy limit.

Specifically, we find

$$G_F \sim \frac{g^2}{M^2} \qquad \text{or} \qquad \frac{g^2}{8M_Z^2 \cos^2\theta_W} = \frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}} \,. \tag{2.50}$$

In the notes there are qualitative considerations about the amplitudes in $\beta$ decay, getting the energy spectrum of the electron.

A charged current process is $\mu \to \nu_\mu \overline{\nu}_e e$. The Fermi constant is best estimated through this process, since it is very well-known both theoretically and experimentally.

The decay $\pi \to \ell \overline{\nu}_e$ probes the $V - A$ nature of the interaction.

Another interesting process is $\nu_\mu e$ scattering, which allows us to estimate $s_W^2$.

We can have the recoil of an entire nucleus with a low-energy neutrino, which is mediated by a $Z$ boson, Coherent Elastic $\nu$ Nucleus Scattering, CE$\nu$NS.

This process turns out to be proportional to the number of neutrons squared, which goes up quickly — this is the only tabletop neutrino detector.

The idea is that the neutrino's wavelength is very long, so it cannot see the specific nucleons or quarks.

We separate out the parameters $g$, $g'$, $\mu$ and $v$ from the Yukawa coupling of the electron — the first four remain the same, while we will need to introduce more Yukawa coupling for the different families.

The data are $e$, $G_F$, $M_H$, $M_Z$, as well as the mass of the electron.

The relations are

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} \tag{2.51}$$

$$G_F = \frac{1}{\sqrt{2} v^2} \tag{2.52}$$

$$M_H = \sqrt{2} \mu \tag{2.53}$$

$$M_Z = \frac{v}{2} \sqrt{g^2 + g'^2} \,, \tag{2.54}$$

as well as

$$m_e = \frac{y_e v}{\sqrt{2}} \,. \tag{2.55}$$

These are all computed at tree-level; at more loops there can be corrections of the order of a few parts in a hundred. There are proposals to replace $e$ with $M_W$, so that all the terms refer to the same energy scale $\sim 100\,\text{GeV}$.

In natural units,

$$e = 0.3 \tag{2.56}$$

$$G_F = 1.17 \times 10^{-5}\,\text{GeV}^{-2} \tag{2.57}$$

$$M_H = 125\,\text{GeV} \tag{2.58}$$

$$M_Z = 91\,\text{GeV} \tag{2.59}$$

$$m_e = 0.51 \times 10^{-3}\,\text{GeV} \,, \tag{2.60}$$

$e = .3$ corresponds to $e^2/4\pi = 1/137 = \alpha$; why does this not match with the astropy code?

so

$$g = 0.64 \tag{2.61}$$

$$g' = 0.34 \tag{2.62}$$
$$\mu = 88\,\text{GeV} \tag{2.63}$$
$$v = 246\,\text{GeV} \tag{2.64}$$
$$y_e = 2.9 \times 10^{-6}\,, \tag{2.65}$$

so

$$\lambda = 0.13 \tag{2.66}$$
$$\sin^2\theta_W = 0.22 \tag{2.67}$$
$$M_W = 80\,\text{GeV}\,. \tag{2.68}$$

In terms of naturalness, it is weird that $y_e \ll 1$, which tells us that $m_e \ll 100\,\text{GeV}$.