High Energy Theoretical Astroparticle Physics

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Pasquale Blasi gives this part of the course. He might ask us to do exercises at the blackboard. Monday 2021-11-8, compiled 2022-01-18

Introduction

This is a theory course, but the professor wants to have lots of connections to the physical interpretation.

In general, in order to produce high-energy particles we need *acceleration* and *transport* mechanisms.

The particles we observe are *non-thermal*: the spectra from these phenomena seem to be powerlaws, not the exponentially suppressed tail of a Maxwellian.

Therefore, the system must be, to a fist, approximation, collisionless: if things happen *faster* than the collisions between the particles, they do not have enough time to thermalize.

We typically see charged particles making up cosmic rays, and the (interstellar, intergalactic) medium they move in is mostly ionized. Thus, they will move under the action of electromagnetic fields. Each of these charged particles also produces EM fields of its own: they all "feel what the others are doing".

There is a deep connection between the micro and the macro physics. David N. Schramm called this an "inner world — outer world" connection.

In order to understand the behaviour of cosmic rays, we need both particle physics and plasma physics!

Electroscope discharge: a brief history of the discovery of cosmic rays

What follows is a brief history lesson, roughly based on the work of de Angelis [dAng14].

In the late 1700s Coulomb discovered that an electroscope, left on its own, will discharge. This phenomenon seemed to decrease in speed as the pressure decreased: the air was being ionized, but by what?

In the late 1800s, after the discovery of the emission of charged particles by radioisotopes (which definitely managed to discharge electroscopes), ambient radioactivity was blamed for the phenomenon as a whole.

Domenico Pacini explored the variation of the rate of ionization as one moved underwater: it was decreasing. So, it seemed likely that the origin was extraterrestrial!

Later explorations, especially on balloons, showed a slight decrease followed by a sharp increase as altitude went above a couple kilometers. It took a while for the scientific community to accept this, but the origin of the discharge of electroscopes was not Earth-bound: it was cosmic.

Millikan though they might be the "birth cry" of elements: γ rays with energies corresponding to the mass defects of certain nuclides. He used Dirac's theory of Compton scattering. He was wrong, but he was also the first to use the term "cosmic rays".

Bruno Rossi put lead blocks between Geiger counters: these are not γ rays, since they would be absorbed.

They have to be charged: their flux is influenced by the Earth's magnetic field, so the one from the East and the West is different.

In the 1930s there started a boom of discoveries. At the end of the 30s Auger measured 100 TeV cosmic rays.

At the end of the 60s the CMB was discovered: therefore, in the cosmic ray spectrum there should be a cutoff around 10^{20} eV: the **GZK feature** [Alo+18, sec. 5.1]. This is because the cosmic ray "sees" CMB photons as γ rays, so it can undergo production of pions, losing a lot of energy, in a process like:

$$\mathrm{N} + \gamma \to \mathrm{N} + \pi^0 . \tag{0.1}$$

Specifically, the average CMB photon has an energy of $E \sim 2.7\,\mathrm{K} \approx 245\,\mu\mathrm{eV}$; in order for it to be blueshifted up to the mass of a pion, $m_{\pi^+} \approx 140\,\mathrm{MeV}$ it needs a γ factor of about $\gamma \sim 6 \times 10^{11}$; this is achieved when a proton has an energy of $E_p \approx 6 \times 10^{11}\,\mathrm{GeV}$.[1]

The threshold for pair production, $\sim 1\,\mathrm{MeV}$, is lower: "only" $E_p \approx 4 \times 10^9\,\mathrm{GeV}$. These are not hard thresholds: they are computed from the average temperature of CMB photons, but these are distributed according to a thermal distribution, so there is a tail at high energies, albeit exponentially suppressed.

There are peaks in the spectral power of the discovery of fossils: a 62 Myr period corresponds to the period of the oscillation of the Solar system with respect to the galactic disk.

There's lots of hydrogen and helium in the interstellar medium, almost no Beryllium, Boron and Lithium, while elements heavier than Carbon, which are formed in stars, are found in decent amounts.

On the other hand, in cosmic rays there is a much higher amount of Be, B, Li. This is due to *spallation*, or x-process nucleosynthesis: a process by which a cosmic ray hits a nucleus, thereby splitting it into lighter components. The cross-section for spallation is $\sigma \sim 45 A^{2/3}$mb, where A is the mass number of the nuclide.

The motion needs to be "diffusive". If this is happening, the timescale is quadratic in the distance, and matches with our observations; if the motion was "ballistic" (straight on, basically) the timescale would be linear in the distance travelled.

[1] A proper calculation shows that the expression also depends on the angle at which the particles interact, and specifically the threshold for the formation of a particle x by the interaction of a high energy cosmic ray with Lorentz factor γ with a CMB photon with energy ϵ is $\gamma \geq m_x / 2\epsilon(1 - \cos\theta)$, where θ is the interaction angle.

Such "ballistic" trajectories would contradict observations, since they would not be able to produce enough spallation product.

But, it cannot be collisions, otherwise the distribution would be thermal! So, what is it? Magnetic fields.

There is Ballmer emission in the shockwave from SNe.

The Larmor radius is in general given by

$$r_{\text{Larmor}} \approx \frac{\gamma m v_\perp}{|q||B|}, \tag{0.2}$$

where the magnitude of the galactic B-field is of the order of $100\,\text{pT}$.

Is this a typical value, an underestimate maybe?

This means that even at knee-level, $E \sim 3 \times 10^{15}\,\text{eV}$, the gyroradius is on the order of $3\,\text{pc}$: extremely *small*, in terms of the scale of the galaxy, $10\,\text{kpc}$ or more.

The loss length for radiation decreases with energy: first we can do pair production when a proton interacts with a CMB photon, and then pion production can start.

The *goal of the course* is to understand the behaviour of charged particles in a magnetic field they generate themselves.

Diffusive transport takes into account charged particles, as well as ordered and turbulent B fields, but plasma instabilities are what happen when charged particles interact with a turbulent magnetic field.

There are many experiments measuring this kind of stuff!

Course outline

The plan is to discuss:

1. basics of plasma physics;
2. basics of MHD;
3. basics of transport of charged particles;
4. basic aspects of the supernova paradigm for cosmic rays;
5. test particle theory of diffusive shock acceleration;
6. cosmic ray transport in the galaxy;
7. some non-linear aspects of particle transport;
8. a taste of advanced topics: recent findings, positrons, end of galactic CR, UHECR...

Wednesday 2021-11-10, compiled 2022-01-18

Regarding books, the course is a synthesis of several topics, so many books cover them but they also include lots of other material.

For plasma physics, "Plasma physics for astrophysics" by Russel M. Kulsrud.

For transport, "Astrophysics of cosmic rays".

1 Basics of plasma physics

From the microphysical point of view, cosmic rays are just electric charges moving in a plasma, which they in turn affect.

Loosely, a plasma is ionized gas; however the interstellar medium is also at temperatures of 10^4 K to 10^6 K, this is also true for the intergalactic medium, which has a much lower density and similar temperatures,

This also applies to the medium in clusters, which has a higher temperature, of the order of 10^8 K but lower densities, $10^{-3}\,\mathrm{cm}^{-3}$.

Magnetic fields are sourced by currents, which cosmic rays affect. Electric fields, on the other hand, are "short-circuited" since the conductivity is very large.

If it is difficult to have an electric field, how can particles be accelerated? We will simply assume that cosmic rays, which are non-thermal, exist.

First, though, we will try to understand how the plasma works, then we will look at the non-thermal particles, and finally we will put them together.

For simplicity, let us consider a plasma made of protons (with density n_i) and electrons (with density n_e). Maxwell' equations will read

$$\vec{\nabla} \cdot \vec{E} = 4\pi\zeta \tag{1.1a}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.1b}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{1.1c}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\,, \tag{1.1d}$$

where $\zeta = n_i e - n_e e$, while the current reads $\vec{J} = n_i e \vec{v}_i - n_e e \vec{v}_e$.

The interactions between these will be Coulomb ones. At thermal equilibrium, we will have charge neutrality, so $n_i = n_e = n_0$.

Screening Suppose we add a positive charge in a neutral medium, at $\vec{r} = 0$. Then, the divergence of $\vec{E}$ will be

$$\vec{\nabla} \cdot \vec{E} = 4\pi n_i e - 4\pi n_e e + 4\pi e \delta(\vec{r})\,, \tag{1.2}$$

where the tilde means that the densities are perturbed.

We will assume that the perturbation induced by this is small. The potential energy drop of a particle at a distance d from another is e^2/d, so if we assume $e^2/d \ll k_B T$ the thermal background remains fixed, and is not "broken" by Coulomb interactions. The typical distance between particles will be $d \sim n_0^{-1/3}$, so the weak perturbation condition will read $e^2 n_0^{1/3} \ll k_B T$.

What we want to do now is to solve this equation (1.2) for the electric field.

In order to do so, we need to discuss the dependence of the phase space distribution on the energy of each particle, which we call ϵ. At zeroth order, we will have

$$\widetilde{n}_i = n_0 \exp\left(-\frac{\epsilon}{k_B T}\right)\,, \tag{1.3}$$

which we assume, on mesoscopic scales, to average out to $\left\langle e^{-\epsilon/k_B T}\right\rangle = 1$. In the following discussion we apply this mesoscopic approximation by setting $\epsilon \equiv k_B T$, but we do leave in the external perturbation to the potential.

The external potential will perturb the energy ϵ by $e\varphi$ where φ is the electric potential, so

$$\widetilde{n}_i = n_0 \exp\left(\frac{-\epsilon + e\varphi}{k_B T}\right) \tag{1.4a}$$

$$\widetilde{n}_e = n_0 \exp\left(\frac{-\epsilon - e\varphi}{k_B T}\right), \tag{1.4b}$$

and we can use $\vec{E} = -\vec{\nabla}\varphi$: then, for $r \neq 0$ we will have

$$\vec{\nabla}^2 \varphi = -4\pi e n_0 \exp\left(-\frac{\epsilon}{k_B T}\right) \exp\left(-\frac{e\varphi}{k_B T}\right) + 4\pi e n_0 \exp\left(-\frac{\epsilon}{k_B T}\right) \exp\left(\frac{e\varphi}{k_B T}\right), \tag{1.5}$$

which we can expand up to linear order:

$$\vec{\nabla}^2 \varphi = -4\pi e n_0 \exp\left(-\frac{\epsilon}{k_B T}\right) \left(\exp\left(-\frac{e\varphi}{k_B T}\right) - \exp\left(\frac{e\varphi}{k_B T}\right)\right) \tag{1.6a}$$

$$\approx -4\pi e n_0 \exp\left(-\frac{\epsilon}{k_B T}\right) \left(1 - \frac{e\varphi}{k_B T} - 1 - \frac{e\varphi}{k_B T}\right) \tag{1.6b}$$

$$= 8\pi e \underbrace{n_0 \exp\left(-\frac{\epsilon}{k_B T}\right)}_{=n_0} \frac{e}{k_B T}\varphi . \tag{1.6c}$$

The prefactor had the dimensions of an inverse square length; further, we include the exponential $e^{-\epsilon/k_B T}$ into the unperturbed density n_0.

We are being a bit cavalier in the distinction between the phase space density $f(\vec{x}, \vec{p})$ and the spatial number density $n(\vec{x})$.
Roughly speaking, the first is defined so that its integral over all of $\mathrm{d}^3 x \, \mathrm{d}^3 p$ yields the total number of particles, while for the second the integral need only be done over $\mathrm{d}^3 x$. We can recover the number density by integrating the phase space density: specifically, the correct normalization in natural units reads

$$n(\vec{x}) = \frac{g}{(2\pi)^3} \int f(\vec{x}, \vec{p}) \, \mathrm{d}^3 p \ , \tag{1.7}$$

where g is the number of helicity states of the particle at hand.
The isotropic Maxwellian phase space distribution (which approximates both the Bose-Einstein and Fermi ones) reads $f(\vec{x}, \vec{p}) = \exp\big(-\epsilon(\vec{p})/k_B T\big)$, where $\epsilon(\vec{p}) = \sqrt{m^2 + \vec{p}^2}$ is the energy corresponding to the momentum $\vec{p}$.
The way the specific integral is computed for this density does not really matter (and is actually rather complicated in general), but we call the result n_0 — it is independent of $\vec{x}$, since the phase space density is as well.

Now, we insert an external potential energy term to the *phase space* density, mapping $\epsilon \to \epsilon + U(\vec{x})$ (which for us will be $U(\vec{x}) = -e\varphi(\vec{x})$). The integral then reads

$$n(\vec{x}) = \frac{g}{(2\pi)^3} \int \exp\left(-\frac{\epsilon}{k_B T}\right) \exp\left(-\frac{U(\vec{x})}{k_B T}\right) \mathrm{d}^3 p = n_0 \exp\left(-\frac{U(\vec{x})}{k_B T}\right). \tag{1.8}$$

We define the **Debye length**

$$\lambda_D = \left(\frac{k_B T}{8\pi n_0 e^2}\right)^{1/2}, \tag{1.9}$$

in terms of which (with the assumption of spherical symmetry for our problem) the equation reads

$$\nabla^2 \varphi = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) = \frac{1}{\lambda_D^2} \varphi\,, \tag{1.10}$$

which is solved by looking at $f = r\varphi$: then,

$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} = \frac{f}{\lambda_D^2}\,, \tag{1.11}$$

which means $f = A\exp(-r/\lambda_D)$ (we discard the unphysical, exponentially diverging solution). Inserting back our particle boundary condition to fix A, we get

$$\varphi = \frac{e}{r} \exp\left(-\frac{r}{\lambda_D}\right). \tag{1.12}$$

This is physically meaningful: there is a **screening effect** on charges, on length scales of λ_D.

For this to happen, however, we need to have enough charges to screen the inserted one: the number of particles in the Debye volume, $\sim n_0 \lambda_D^3$, must be much larger than 1. In a way, this is also a mesoscopic statistical requirement.

This can be written as

$$n_0 \frac{(k_B T)^{3/2}}{(8\pi e^2)^{3/2} n_0^{3/2}} = \frac{(k_B T)^{3/2}}{(8\pi e^2)^{3/2} n_0^{1/2}} \gg 1\,, \tag{1.13}$$

which shows that, counter-intuitively, this condition is easier to fulfill for under-dense plasmas.

The path length for Coulomb scattering, λ_C, should be much larger than both $\Delta r \approx n^{-1/3}$ (the separation between particles) and λ_D.

It can be estimated by

$$\lambda_C = \frac{1}{n_0 \sigma_C} = \frac{(k_B T)^2}{n_0 e^4} \gg n^{-1/3}\,, \tag{1.14}$$

where $e^2/b = k_B T$ gives us a limit for the Coulomb interaction length, therefore $\sigma_C \approx b^2 \approx (e^2/k_B T)^2$.

It can be shown that $\lambda_C \gg \lambda_D$ is equivalent to the condition that many particles should be contained in a single Debye length: the condition reads

$$\lambda_C = \frac{(k_B T)^2}{n_0 e^4} \gg \sqrt{\frac{k_B T}{8\pi n_0 e^2}} = \lambda_D \tag{1.15}$$

$$(8\pi)^2 n_0 \left(\frac{k_B T}{8\pi n_0 e^2}\right)^2 \gg \left(\frac{k_B T}{8\pi n_0 e^2}\right)^{1/2} \tag{1.16}$$

$$\lambda_D^3 \gg \frac{1}{(8\pi)^2 n_0}\,. \tag{1.17}$$

Collective effects dominate the dynamics of a plasma, while the effect of a single charge quickly becomes negligible.

Propagation modes in a plasma We will now do an exercise in perturbation theory: which perturbations are allowed, beyond electromagnetic waves? Certain modes are "allowed", in that they do not die out.

A lot of interesting physics come from the fact that each particle interacts with the collection of all the others.

Perturbations which are unstable are interesting, but they break our perturbative approach.

The fields in Maxwell's equations (1.1) are assumed to start out at zero in the unperturbed configuration.

The current J is given by the Generalized Ohm's law:

$$J_r = \sigma_{rs} E_s\,, \tag{1.18}$$

where the proportionality constant σ_{rs} is the *conductivity tensor* (whose components are, roughly, 1 over resistance).

It is also convenient to define the displacement current $\vec{D}$ by:

$$\frac{4\pi\vec{J}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\,, \tag{1.19}$$

which means that the current can be recovered by

$$\vec{J} = \frac{1}{4\pi}\left(\frac{\partial\vec{D}}{\partial t} + \frac{\partial\vec{E}}{\partial t}\right). \tag{1.20}$$

The idea is to decompose the perturbation in Fourier modes:

$$\vec{E} \to \widetilde{E}_k(\vec{k}, \omega) \exp\left(-i\omega t + i\vec{k}\cdot\vec{r}\right). \tag{1.21}$$

The full expression for the electric field is a superposition of these modes, but we can look at them just one at a time. This simplifies things: time derivatives become $i\omega$, curls become $\vec{k}\times$ and so on.

The current reads, with this as well as Ohm's law,

$$\widetilde{J}_r = \frac{-i\omega}{4\pi}\left(\widetilde{D}_r - \widetilde{E}_r\right) = \sigma_{rs} E_s \,. \tag{1.22}$$

The displacement field is therefore

$$\widetilde{D}_r = \frac{4\pi}{i\omega}\left(\frac{i\omega}{4\pi}\widetilde{E}_r - \sigma_{rs} E_s\right) = \widetilde{E}_r + i\frac{4\pi}{\omega}\sigma_{rs}\widetilde{E}_s = \mathbb{K}_{rs}\widetilde{E}_s \,, \tag{1.23}$$

where $\mathbb{K}_{rs} = \delta_{rs} + (4\pi i/\omega)\sigma_{rs}$ is called the **Dielectric tensor**.

What we are trying to do is to write a *dispersion relation*, $F(\vec{k}, \omega)E = 0$.

Now we move to Lenz's law: we take another curl, to get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right) \tag{1.24a}$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \vec{D}}{\partial t}\right) \tag{1.24b}$$

$$= -\frac{1}{c^2}\frac{\partial^2 \vec{D}}{\partial t^2} \tag{1.24c}$$

$$[-\vec{k} \times (\vec{k} \times \widetilde{E})]_r = +\frac{\omega^2}{c^2}\widetilde{D}_r = \frac{\omega^2}{c^2}\mathbb{K}_{rs}\widetilde{E}_s \,. \tag{1.24d}$$

We are almost done: the only issue here is that $\mathbb{K}_{rs}$ contains the conductivity tensor σ_{rs}.

Supposing for simplicity that our plasma is non-relativistic, we have the EoM

$$m_e \frac{\mathrm{d}v_e}{\mathrm{d}t} = -eE \implies -i\omega m_e \widetilde{v}_e = -e\widetilde{E} \,, \tag{1.25}$$

but the current $\vec{J}$ is $\vec{J} = -en_e\vec{v}_e$, so $\widetilde{J}_r = -(ne^2/\omega m_e)\widetilde{E}_r$, therefore the conductivity reads

$$\sigma_{rs} = \frac{ine^2}{\omega m_e}\delta_{rs} \,, \tag{1.26}$$

so we can write out the dielectric tensor explicitly:

$$\mathbb{K}_{rs} = \delta_{rs}\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right] \qquad \text{where} \qquad \omega_p = \sqrt{\frac{4\pi n_e e^2}{m_e}} \tag{1.27}$$

is called the plasma frequency.

We have found our dispersion relation:

$$\vec{k} \times \left(\vec{k} \times \vec{E}\right) + \left(\frac{\omega^2 - \omega_p^2}{c^2}\right)\widetilde{E} = 0 \,. \tag{1.28}$$

Let us separate out *longitudinal perturbations*, which have $\vec{k} \propto \vec{E}$, from *transverse* ones, since we are always able to write $\vec{E} = \vec{E}_\parallel + \vec{E}_\perp$.

In the **longitudinal case**,

$$(\omega^2 - \omega_p^2)E_\parallel = 0\,, \tag{1.29}$$

which means that the propagation must happen exactly at the plasma frequency: these are called **Langmir waves**, or plasma waves. In the transverse case, we get

$$\left[-k^2c^2 + (\omega^2 - \omega_p^2)\right]E_\perp = 0\,. \tag{1.30}$$

The pulsation must be

$$\omega^2 = \omega_p^2 + c^2k^2 \qquad \text{or} \qquad k^2 = \frac{\omega^2 - \omega_p^2}{c^2}\,. \tag{1.31}$$

If $\omega > \omega_p$, then $k^2 > 0$: these are allowed perturbations, which indeed exhibit oscillatory behavior.

Actually, we can compute their group velocity

$$v_g = \frac{\partial \omega}{\partial k} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}\,, \tag{1.32}$$

which shows that if $\omega \gg \omega_p$ the speed is close to c.

If $\omega < \omega_p$, on the other hand, k is imaginary. The solution corresponding to exponential growth is unphysical (for one, there is no mechanism providing the energy for the magnitude of the oscillation to exponentially grow), so we only look at the exponentially damped solutions.

They are exponentially damped over a scale $\left|\vec{k}\right|^{-1} = c/\omega_p$. This is the **skin depth** of the plasma, the largest distance a perturbation can penetrate in the plasma if it oscillates too slowly.

This characteristic wavelength's ratio to the Debye length reads

$$\frac{\lambda_{\text{skin depth}}}{\lambda_{\text{Debye}}} = \sqrt{\frac{m_e c^2}{4\pi n_e e^2}\frac{8\pi n_e e^2}{k_B T}} = \sqrt{\frac{2m_e c^2}{k_B T}} \gg 1\,, \tag{1.33}$$

since the plasmas we are considering are typically non-relativistic.

An example: fast radio bursts Fast radio bursts are like γ-ray bursts in the radio band. These are relatively high-frequency, around 1.4 GHz. We do not really know what their sources are; some were false positives due to microwave ovens making lunch, but others were certified to be true detections.

We know of one which came from a galaxy $L \sim 1\,\text{Gpc}$ away, and which had a dispersion of about $\Delta\omega \sim 300\,\text{MHz}$.

Photons of different frequencies arrived at different times, with a spread of about 300 ms.

We are in the second case ($\vec{k} \perp \vec{E}$) since they are EM waves. The group velocity is, again

$$v_g = c\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]^{1/2}\,, \tag{1.34}$$

and we can assume (we will check *a posteriori*) that $\omega \gg \omega_p$.

The time difference will be

$$\Delta t = \frac{L}{v_g(\omega)} - \frac{L}{v_g(\omega + \Delta\omega)} \approx \frac{L}{v_g(\omega)} \frac{1}{v_g(\omega)} \frac{\partial v_g}{\partial \omega} \Delta\omega \tag{1.35a}$$

$$\approx \frac{L}{c^2} \underbrace{c \frac{\omega_p^2}{\omega^3}}_{\approx \partial v_g / \partial \omega} \Delta\omega \tag{1.35b}$$

$$\omega_p \approx \sqrt{\frac{c\omega^3 \Delta t}{L \Delta\omega}}, \tag{1.35c}$$

but we know that

$$\omega_p^2 = \frac{4\pi n_e e^2}{m_e}. \tag{1.36}$$

This allows us to measure $\omega_p \sim 5.2\,\text{Hz}$, and therefore also the density, which comes out to be $n_e \sim 8.2 \times 10^{-9}\,\text{cm}^{-3}$.

How well does this match the baryon density computed from the CMB? The computation goes

$$\underbrace{\underbrace{\frac{\omega_p^2 m_e}{4\pi e^2}}_{n_e} m_p}_{\rho_{\text{plasma}}} \times \frac{1}{\Omega_{0b}\rho_c} \approx 0.03. \tag{1.37}$$

Therefore, from this observation we can estimate that about 3 % of baryonic matter is in the ISM.

This can be generalized a bit: maybe, not all the path L from the source to here had this much plasma in it. Suppose, for simplicity's sake, that a fraction α of the path was constituted by uniform-density plasma.

Then, our estimate for ω_p will be multiplied by a factor $\alpha^{-1/2}$, while our estimate for n_e will be multiplied by α^{-1}. On the other hand, in the estimate for ρ_{plasma}, averaged over the whole universe, will need to be shifted by some factor.

If the path we were looking at is a fair sample for the population (not a given, but let's approximate as such), then about a fraction α of the universe is filled with this plasma — this precisely cancels the correction to our estimate, so it all works out.

This is done in a better way through the measurement of several pulsars for which we have more accurate distance measurements.

Check: the frequency given was ν, not ω!

1.1 Statistical descriptions of plasmas

Thursday 2021-11-18

If the length scales we consider are larger than the Debye length, we can safely neglect the effect of the Coulomb potential of each individual particle. This does not mean that there are no electric fields, but the electric fields are mesoscopic or larger.

This is "desirable", in that we'd like to use statistical descriptions of the plasma. Such a statistical description will necessarily work in phase space.

Let us start out in the nonrelativistic approximation: if we only have one particle, we can write its phase space distribution function as

$$N(\vec{x}, \vec{v}, t) = \delta(\vec{x} - \vec{X}(t))\delta(\vec{v} - \dot{\vec{X}}(t)) \,. \tag{1.38}$$

If we have several particles, this can be readily generalized:

$$N(\vec{x}, \vec{v}, t) = \sum_i \delta(\vec{x} - \vec{X}_i(t))\delta(\vec{v} - \dot{\vec{X}}_i(t)) \,, \tag{1.39}$$

but this only describes a single particle species: we know that at the very least we will have electrons and protons, in order to preserve charge neutrality. So, let us call the quantity defined above N_s, where s is an index spanning $\{e, i\}$ for electrons and ions respectively.

We want to describe the evolution of this quantity: its **total** time derivative can be computed through the chain rule, and if we set it to zero we find that $\mathrm{d}N_s / \mathrm{d}t = 0$ is equivalent to:

$$\frac{\partial N_s}{\partial t} = -\sum_i \dot{\vec{X}} \cdot \nabla_x \delta(\vec{x} - \vec{X}_i(t))\delta(\vec{v} - \vec{V}_i(t)) - \sum_i \delta(\vec{x} - \vec{X}_i(t))\dot{\vec{V}}_i(t) \cdot \nabla_v \delta(\vec{v} - \vec{V}_i(t)) \,. \tag{1.40}$$

Now, these charges will evolve under the actions of the electric and magnetic fields: but we also need to describe what is the source of these fields.

The action of these fields on a particle with velocity $\vec{V}_i$ will be described by the Lorentz force,

$$m_s \dot{\vec{V}}_i(t) = q_s \vec{E}^{\text{microscopic}}(\vec{x}) + \frac{q_s}{c} \vec{V}_i \times \vec{B}^{\text{microscopic}}(\vec{x}, t) \,. \tag{1.41}$$

This acceleration term will then be put into the aforementioned evolution equation. We can already understand why this problem will be hard: the fields acting on a particle will be sourced by all the others.

The microscopic EM fields will satisfy Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E}^{\text{micro}}(\vec{x}, t) = 4\pi \zeta^{\text{micro}} \tag{1.42}$$

$$\vec{\nabla} \cdot \vec{B}^{\text{micro}} = 0 \tag{1.43}$$

$$\vec{\nabla} \times E^{\text{micro}} = -\frac{1}{c} \frac{\partial \vec{B}^{\text{micro}}}{\partial t} \tag{1.44}$$

$$\vec{\nabla} \times \vec{B}^{\text{micro}} = \frac{4\pi}{c} \vec{J}^{\text{micro}} + \frac{1}{c} \frac{\partial \vec{E}^{\text{micro}}}{\partial t} \,, \tag{1.45}$$

where the density and current density read

$$\zeta^{\text{micro}}(\vec{x},t) = \sum_{s=i,e} q_s \int \mathrm{d}^3\vec{v}\, N_s(\vec{x},\vec{v},t) \tag{1.46}$$

$$\vec{J}^{\text{micro}} = \sum_{s=i,e} q_s \int \mathrm{d}^3\vec{v}\, \vec{v} N_s(\vec{x},\vec{v},t)\,. \tag{1.47}$$

The Boltzmann equation plus the Lorentz one can be more compactly written as

$$\frac{\partial N_s}{\partial t} = -\vec{v}\cdot\vec{\nabla}_x N_s - \sum_{i=1}^{N} \frac{q_s}{m_s}\left[\vec{E}^{\text{micro}} + \frac{1}{c}\vec{v}\times\vec{B}^{\text{micro}}\right]\cdot\vec{\nabla}_v\left[\delta(\vec{v}-\dot{\vec{X}}_i(t))\right]\delta(\vec{x}-\vec{X}_i(t))\,. \tag{1.48}$$

The EM fields are *a priori* computed at the position and velocity of the i-th particle, $(\vec{X}_i,\vec{V}_i)$, but because of the δ-functions at each point we can substitute this position for the generic one $(\vec{x},\vec{v})$.

The equation then becomes

$$\frac{\partial N_s}{\partial t} + \vec{v}\cdot\vec{\nabla}_x N_s = -\frac{q_s}{m_s}\left[\vec{E}^{\text{micro}} + \frac{\vec{v}\times\vec{B}^{\text{micro}}}{c}\right]\cdot\nabla_v N_s\,. \tag{1.49}$$

This equation is called the Klimontovich-Dupree equation.[2]

This equation, however, is basically useless, unless we do a mean-field approximation.

Let us introduce the quantity $f_s(\vec{x},\vec{v},t)$, which we want to compute through average on mesoscopic scales:

$$f_s(\vec{x},\vec{v},t) = \left\langle N_s(\vec{x},\vec{v},t)\right\rangle_{\Delta V}\,. \tag{1.50}$$

The true phase space density will then be

$$N_s(\vec{x},\vec{v},t) = f_s(\vec{x},\vec{v},t) + \delta f_s\,, \tag{1.51}$$

where the fluctuations are assumed to average to zero. We can write a similar expression for the electric and magnetic fields:

$$\vec{E}^{\text{micro}} = \vec{E} + \delta\vec{E} \tag{1.52}$$

$$\vec{B}^{\text{micro}} = \vec{B} + \delta\vec{B}\,. \tag{1.53}$$

The averaged KD equation then becomes:

$$\frac{\mathrm{d}f_s}{\mathrm{d}t} + \vec{v}\cdot\nabla_x f_s + \frac{q_s}{m_s}\left[\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right]\vec{\nabla}_v f_s = -\frac{q_s}{m_s}\left\langle\left(\delta\vec{E} + \frac{1}{c}\vec{v}\times\delta\vec{B}\right)\cdot\vec{\nabla}_v\delta f_s\right\rangle\,. \tag{1.54}$$

The interaction and collision terms are *quadratic* in the fluctuations. If we neglect this term (which is typically called a "correlation" term), we get the **Vlasov** equation:

$$\frac{\partial f_s}{\partial t} + \vec{v}\cdot\vec{\nabla}_x f_s + \frac{q_s}{m_s}\cdot\left[\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right]\vec{\nabla}_v f_s = 0\,. \tag{1.55}$$

[2] As many things done during the Cold War, it was developed by military personnel independently in the two blocks.

We can do the same thing to the Maxwell equations, using an averaged version of the charge and current densities.

We will need to generalize to the relativistic case: we move to $(\vec{x}, \vec{p})$ phase space. For relativistic particles, the Lorentz force reads

$$\vec{p} = q_s \left[\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right] . \tag{1.56}$$

The source terms in the Maxwell equations will be integrated in $\mathrm{d}^3 p$, but for the charge current we will have an integral $\int \mathrm{d}^3 p \, \vec{v} f_s$.

The relativistic Vlasov equation is then readily derived with minor modifications, and reads

$$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_s + q_s \left[\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right] \vec{\nabla}_p f_s = 0 . \tag{1.57}$$

Let us now move to the nonrelativistic plasma again, and assume we are at zero temperature. We will then make a small perturbation: the ions will be stationary in first approximation. Will the Vlasov equation contain the plasma waves we derived earlier?

We only write it for electrons, so

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_s - \frac{e}{m_e} \left[\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right]_\alpha \frac{\partial f}{\partial v_\alpha} = 0 , \tag{1.58}$$

where in the second term we are adopting the Einstein convention, summing over α.

We will assume that there is no magnetic field perturbation: this is the same thing we did in the plasma waves, specifically in assuming that $\mathbb{K}$ is diagonal.

is this correct?

In the stationary configuration there is $\vec{E} = 0$. To linear order, the perturbed equation will read

$$\frac{\partial \delta f}{\partial t} + \vec{v} \cdot \nabla \delta f + \frac{e}{m_e} \nabla \varphi_\alpha \frac{\partial f}{\partial v_\alpha} = 0 . \tag{1.59}$$

But isn't $f = \text{const}$ at zeroth order?

We want to assume that the plasma is cold!

The electric potential φ will satisfy

$$-\nabla^2 \varphi = -4\pi e \int \mathrm{d}^3 v \, \delta f . \tag{1.60}$$

We then move to Fourier space:

$$-i\omega \widetilde{\delta f} + i\vec{k} \cdot \vec{v} \widetilde{\delta f} + \frac{e}{m_e} i k_\alpha \widetilde{\varphi} \frac{\partial f}{\partial v_\alpha} = 0 , \tag{1.61}$$

but the electric potential will satisfy

$$k^2 \widetilde{\varphi} = -4\pi e \int \mathrm{d}^3 v \, \widetilde{\delta f} \,, \tag{1.62}$$

so we get

$$\widetilde{\delta f} \left[i\vec{k} \cdot \vec{v} - i\omega \right] = -\frac{e}{m_e} \widetilde{\varphi} k_\alpha \frac{\partial f}{\partial v_\alpha} \tag{1.63}$$

$$\widetilde{\delta f} = -\frac{e}{m_e} \widetilde{\varphi} \frac{k_\alpha \, \partial f / \partial v_\alpha}{\vec{k} \cdot \vec{v} - \omega} \,, \tag{1.64}$$

which we substitute into the integral for $\widetilde{\varphi}$:

$$\widetilde{\varphi} = \frac{4\pi e^2}{k^2 m_e} \widetilde{\varphi} k_\alpha \int \mathrm{d}^3 v \, \frac{\partial f}{\partial v_\alpha} \frac{1}{\vec{k} \cdot \vec{v} - \omega} \,, \tag{1.65}$$

so the allowed perturbations are those which make this equation true (for arbitrary φ). For now we have not assumed that the electrons are cold: this will enter in how we write the unperturbed f.

The assumptions of the electrons being cold can be modelled as $f = n_0 \delta(\vec{v})$: therefore, we need to integrate by parts.

Suppose that the z axis is along k: then, the integrand reads

$$\int \mathrm{d}v_x \, \mathrm{d}v_y \, \mathrm{d}v_z \, \frac{\partial f}{\partial v_\alpha} \frac{1}{k v_z - \omega} = \int \mathrm{d}v_x \, \mathrm{d}v_y \, \mathrm{d}v_z \, f \frac{1}{(k v_z - \omega)^2} \tag{1.66}$$

$$= \int \mathrm{d}v_x \, \mathrm{d}v_y \, \delta(v_x) \delta(v_y) n_0 \frac{k}{\omega^2} = \frac{n_0 k}{\omega^2} \,, \tag{1.67}$$

so the equation just reads

$$1 - \frac{4\pi e^2 n_0}{k^2 m_e} \frac{k^2}{\omega^2} \implies \omega^2 = \frac{4\pi e^2 n_0}{m_e} = \omega_p^2 \,. \tag{1.68}$$

At the very least, this more complicated approach allows us to recover the results we expected.

2 Alfvén waves

Next time, we will add one complication: a global, ordered magnetic field.

We know that this happens, for example, in spiral galaxies like our own: we observe large-scale magnetic fields.

Much of the physics of the transport of non-thermal particles will be affected by these magnetic fields.

The perturbation of the two coupled Vlasov equations under the effect of this external $\vec{B}$ field will yield what are called **Alfvén waves**. Wednesday 2021-11-24

Each of the particle species in our plasma satisfy a Vlasov equation,

$$\frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \nabla_x f_\alpha + q_\alpha \left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right]_\beta \frac{\partial f_\alpha}{\partial p_\beta} = 0\,, \tag{2.1}$$

where α runs over electrons, ions, cosmic rays and so on: any collisionless particles; while the index β is a spatial index, to be interpreted in the Einstein summation convention.

The $\vec{E}$ and $\vec{B}$ fields do *not* have an index α: they are produced by all the components of the plasma.

An astrophysically significant situation is one in which there is a "baseline" magnetic field, which adds to the one generated within the plasma.

We align our axes so that $\vec{B}_0 = B_0 \hat{z}$. The baseline electric field is $\vec{E}_0 = 0$.

1. We ask that $\vec{B}_0$ is ordered: its variations should be on much larger scales than the scales of the plasma.

2. The equilibrium phase space density, $f_{0,\alpha}$, is assumed to be known: it could, for example, be a Maxwellian, or a delta-function on $\vec{v} = 0$.

Again, we will perturb this system in order to find a dispersion relation $F(k, \omega)\delta = 0$, where δ is our perturbation.

Perturbing the Vlasov equation The perturbed components in our fields will read

$$f_\alpha = f_{0,\alpha} + \delta f_\alpha \tag{2.2}$$

$$\vec{B} = \vec{B}_0 + \delta \vec{B} \tag{2.3}$$

$$\vec{E} = \delta \vec{E}\,. \tag{2.4}$$

The perturbed equation, to first order, reads

$$\frac{\partial \delta f_\alpha}{\partial t} + \vec{v} \cdot \vec{\nabla} \delta f_\alpha + q_\alpha \delta \vec{E}_\beta \frac{\partial f_{0\alpha}}{\partial p_\beta} + q_\alpha \left(\frac{\vec{v}}{c} \times \vec{B}_0 \right)_\beta \frac{\partial \delta f_\alpha}{\partial p_\beta} + q_\alpha \left(\frac{\vec{v}}{c} \times \delta \vec{B} \right) \frac{\partial f_{0\alpha}}{\partial t} = 0\,, \tag{2.5}$$

and, as usual, we will move to Fourier space, without tildes for simplicity:

$$-i\omega \delta f_\alpha + i\vec{k} \cdot \vec{v} \delta f_\alpha + q_\alpha \delta \vec{E}_\beta \frac{\partial f_{0\alpha}}{\partial p_\beta} + q_\alpha \left(\frac{\vec{v}}{c} \times \vec{B}_0 \right)_\beta \frac{\partial \delta f_\alpha}{\partial p_\beta} + q_\alpha \left(\frac{\vec{v}}{c} \times \delta \vec{B} \right) \frac{\partial f_{0\alpha}}{\partial p_\beta} = 0\,. \tag{2.6}$$

Perturbed Maxwell equations The divergence of the electric field perturbation reads

$$\vec{\nabla} \cdot \delta \vec{E} = 4\pi \zeta = 4\pi \sum_\alpha q_\alpha \int \mathrm{d}^3 p \, \delta f_\alpha\,, \tag{2.7}$$

where ζ denotes the charge density (which is purely a perturbation, since its unperturbed value is zero due to charge neutrality).

The divergence of $\vec{B}$ vanishes, but so does the divergence of the constant $\vec{B}_0$, so we have $\vec{\nabla} \cdot \delta \vec{B} = 0$ as well.

The other two equations read

$$\vec{\nabla} \times \delta\vec{E} = -\frac{1}{c}\frac{\partial \delta\vec{B}}{\partial t} \tag{2.8}$$

$$\vec{\nabla} \times \delta\vec{B} = \frac{4\pi\vec{J}}{c} + \frac{1}{c}\frac{\partial \delta\vec{E}}{\partial t}\,. \tag{2.9}$$

In Fourier space, these read

$$i\vec{k}\cdot\delta\vec{E} = 4\pi q_\alpha \int \mathrm{d}^3p\,\delta f_\alpha \tag{2.10}$$

$$i\vec{k}\cdot\delta\vec{B} = 0 \tag{2.11}$$

$$i\vec{k}\times\delta\vec{E} = \frac{i\omega}{c}\delta\vec{B} \tag{2.12}$$

$$i\vec{k}\times\delta\vec{B} = \frac{4\pi}{c}\sum_\alpha q_\alpha \int \mathrm{d}^3p\,\vec{v}\delta f_\alpha - \frac{i\omega}{c}\delta\vec{E}\,. \tag{2.13}$$

We are now in a position to make a further simplifying assumption.

3. We restrict ourselves to perturbations moving parallel to the magnetic field, $\vec{k} \parallel \vec{B}_0$.

Boron, Beryllium and Lithium are absent in the ISM but present in cosmic rays: there must be spallation, but for them to have significant interaction chances they must move very slowly. Something must be deflecting them.

Since $\vec{k}\cdot\delta\vec{B} = 0$, the magnetic field perturbation must be in the form $\delta\vec{B} = (\delta B_x, \delta B_y, 0)$. From Faraday's law, we find $\omega\delta\vec{B} = c\vec{k}\times\delta\vec{E}$:

$$\frac{\omega}{c}\begin{bmatrix}\delta B_x \\ \delta B_y \\ 0\end{bmatrix} = \det\begin{bmatrix}\hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & k \\ \delta E_x & \delta E_y & \delta E_z\end{bmatrix} = \begin{bmatrix}-k\delta E_y \\ k\delta E_x \\ 0\end{bmatrix}. \tag{2.14}$$

Therefore, $\delta B_x = -(kc/\omega)\delta E_y$ and $\delta B_y = (kc/\omega)\delta E_x$.

Manipulating the Lorentz terms in Vlasov Let us now try to simplify the last three terms in the perturbed, Fourier space Vlasov equation (2.6): they read

$$q_\alpha \delta\vec{E}_\beta \frac{\partial f_{0\alpha}}{\partial p_\beta} + q_\alpha\left(\frac{\vec{v}}{c}\times\vec{B}_0\right)_\beta \frac{\partial \delta f_\alpha}{\partial p_\beta} + q_\alpha\left(\frac{\vec{v}}{c}\times\delta\vec{B}\right)\frac{\partial f_{0\alpha}}{\partial p_\beta}\,, \tag{2.15}$$

so, explicitly:

$$\begin{aligned}&\frac{q_\alpha}{c}\det\begin{bmatrix}\hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ \delta B_x & \delta B_y & 0\end{bmatrix}_\beta \frac{\partial f_{0\alpha}}{\partial p_\beta} \\ &+ \frac{q_\alpha}{c}\det\begin{bmatrix}\hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B_0\end{bmatrix}_\beta \frac{\partial \delta f_\alpha}{\partial p_\beta} + q_\alpha\left(\delta E_x \frac{\partial f_{0\alpha}}{\partial p_x} + \delta E_y \frac{\partial f_{0\alpha}}{\partial p_y}\right) =\end{aligned} \tag{2.16}$$

$$
\begin{aligned}
&= \frac{q_\alpha}{c}\left(-v_z\delta B_y\frac{\partial f_{0\alpha}}{\partial p_x} + v_z\delta B_x\frac{\partial f_{0\alpha}}{\partial p_y} + \left(v_x\delta B_y - v_y\delta B_x\right)\frac{\partial f_{0\alpha}}{\partial p_z}\right) \\
&+ \frac{q_\alpha}{c}\left(v_y B_0\frac{\partial \delta f_\alpha}{\partial p_x} - v_x B_0\frac{\partial \delta f_\alpha}{\partial p_y}\right) + q_\alpha\left(\delta E_x\frac{\partial f_{0\alpha}}{\partial p_x} + \delta E_y\frac{\partial f_{0\alpha}}{\partial p_y}\right).
\end{aligned}
\tag{2.17}
$$

Moving to a circular polarization basis We can always do the calculations for a perturbation in terms of circularly polarized waves, and then add them together. We will write this as $\delta E_y = \pm i\delta E_x$. The idea is then to rewrite everything in terms of δE_x, which we can then just denote as δE. Thanks to this, we get:

$$\delta E_x = \delta E \qquad \delta E_y = \pm i\delta E \tag{2.18}$$

$$\delta B_x = \mp\frac{kc}{\omega}\delta E \qquad \delta B_y = \frac{kc}{\omega}\delta E\,. \tag{2.19}$$

With these substitutions, the three terms read

$$
\begin{aligned}
&\frac{q_\alpha}{c}\left(-v_z\frac{kc}{\omega}\delta E\frac{\partial f_{0\alpha}}{\partial p_x} - v_z i\frac{kc}{\omega}\delta E\frac{\partial f_{0\alpha}}{\partial p_y} + \left[v_x\frac{ck}{\omega}\delta E \pm iv_y\frac{ck}{\omega}\delta E\right]\frac{\partial f_{0\alpha}}{\partial p_z}\right) + \\
&+ q_\alpha\delta E\frac{\partial f_{0\alpha}}{\partial p_x} \pm q_\alpha i\delta E\frac{\partial f_{0\alpha}}{\partial p_y} + \frac{q_\alpha}{c}B_0\left[v_y\frac{\partial f_{0\alpha}}{\partial p_x} - v_x\frac{\partial \delta f_\alpha}{\partial p_y}\right].
\end{aligned}
\tag{2.20}
$$

Moving to cylindrical coordinates The unperturbed distribution function $f_{0\alpha}$ cannot depend on the angle around the direction of the magnetic field $\vec{B}_0$ because of the cylindrical symmetry: it will be written like $f_{0\alpha} = f_{0\alpha}(p_\parallel, p_\perp)$, therefore it is useful to move to cylindrical coordinates.

From p_x, p_y, p_z we move to $p_\parallel$, $p_\perp$, φ, where

$$p_x = p_\perp\cos\varphi \tag{2.21}$$

$$p_y = p_\perp\sin\varphi \tag{2.22}$$

$$p_z = p_\parallel\,. \tag{2.23}$$

One can invert the Jacobian in order to get

$$\mathrm{d}p_\perp = \cos\varphi\,\mathrm{d}p_x + \sin\varphi\,\mathrm{d}p_z \tag{2.24}$$

$$\mathrm{d}\varphi = -\frac{1}{p_\perp}\sin\varphi\,\mathrm{d}p_x + \frac{1}{p_z}\cos\varphi\,\mathrm{d}p_z \tag{2.25}$$

$$\mathrm{d}p_\parallel = \mathrm{d}p_z\,. \tag{2.26}$$

Therefore,

$$\frac{\partial f_{0\alpha}}{\partial p_x} = \frac{\partial f_{0\alpha}}{\partial p_\perp}\cos\varphi \qquad \text{and} \qquad \frac{\partial f_{0\alpha}}{\partial p_y} = \frac{\partial f_{0\alpha}}{\partial p_\perp}\sin\varphi\,, \tag{2.27}$$

since the term containing a derivative with respect to φ vanishes.

With this, the term reads

$$\begin{aligned}
&\frac{q_\alpha}{c} v_z \frac{ck}{\omega} \delta E \left(\frac{\partial f_{0\alpha}}{\partial p_\perp} \cos\varphi \pm i \frac{\partial f_{0\alpha}}{\partial p_\perp} \sin\varphi \right) + \frac{q_\alpha}{c} \frac{ck}{\omega} \delta E \left(v_\perp \cos\varphi \pm i v_\perp \sin\varphi \right) \frac{\partial f_{0\alpha}}{\partial p_\parallel} \\
&\quad + q_\alpha \delta E \frac{\partial f_{0\alpha}}{\partial p_\perp} \left(\cos\varphi \pm i \sin\varphi \right) - \frac{q_\alpha}{c} B_0 \frac{v_\perp}{p_\perp} \frac{\partial \delta f_\alpha}{\partial p}
\end{aligned} \tag{2.28}$$

$$= \frac{q_\alpha}{c} v_z \frac{ck}{\omega} \delta E \frac{\partial f_{0\alpha}}{\partial p_\perp} e^{\pm i\varphi} + \frac{q_\alpha}{c} \frac{ck}{\omega} \delta E v_\perp e^{\pm i\varphi} \frac{\partial f_{0\alpha}}{\partial p_\parallel} + q_\alpha \delta E \frac{\partial f_{0\alpha}}{\partial p_\perp} e^{\pm i\varphi} - \frac{q_\alpha}{c} B_0 \frac{v_\perp}{p_\perp} \frac{\partial \delta f_\alpha}{\partial \varphi} \,. \tag{2.29}$$

Recombining the Vlasov equation Now we just need to add in the two remaining terms of the Vlasov equation:

$$\begin{aligned}
&- i\omega \delta f_\alpha + i\vec{k} \cdot \vec{v} \delta f_\alpha + \frac{q_\alpha}{c} v_z \frac{ck}{\omega} \delta E \frac{\partial f_{0\alpha}}{\partial p_\perp} e^{\pm i\varphi} + \frac{q_\alpha}{c} \frac{ck}{\omega} \delta E v_\perp e^{\pm i\varphi} \frac{\partial f_{0\alpha}}{\partial p_\parallel} + q_\alpha \delta E \frac{\partial f_{0\alpha}}{\partial p_\perp} e^{\pm i\varphi} \\
&\quad - \frac{q_\alpha}{c} B_0 \frac{v_\perp}{p_\perp} \frac{\partial \delta f_\alpha}{\partial \varphi} = 0 \,.
\end{aligned} \tag{2.30}$$

The last term includes a factor in which we can recognize the cyclotron frequency corresponding to particle species α moving in the magnetic field $\vec{B}_0$:

$$\frac{q_\alpha}{c} B_0 \frac{v_\perp}{p_\perp} = \frac{q_\alpha B_0}{c} \frac{v_\perp}{m_\alpha v_\perp \gamma} = \Omega_{\text{gyr}, \alpha} \,. \tag{2.31}$$

Making a rotating ansatz If δf_α was in the form $\delta g_\alpha e^{\pm i\varphi}$, then the equation would automatically be satisfied, if g_α does not depend on the phase, since in that case the last derivative would read $\mp i \delta g_\alpha e^{\pm i\varphi}$.

With this substitution all the phase factors would cancel, and we would find

$$\delta g_\alpha = \frac{\frac{q_\alpha}{c} \delta E \left[v_z \frac{ck}{\omega} \frac{\partial f_0}{\partial p_\perp} - v_\perp \frac{ck}{\omega} \frac{\partial f_0}{\partial p_\perp} - \frac{\partial f_0}{\partial p_\perp} \right]}{-i\omega + ikv_z \mp i\Omega_{\text{gyr}}} \,. \tag{2.32}$$

Closing the system with the Ampére-Maxwell law We have one step left in order to close the system: we need a relation between δf_α and δE; we know that

$$i\vec{k} \times \delta\vec{B} = \frac{4\pi}{c} \sum_\alpha q_\alpha \int \mathrm{d}^3 p \, \vec{v} \delta f_\alpha - i \frac{\omega}{c} \delta\vec{E} \,. \tag{2.33}$$

However, we already know how to connect $\delta\vec{B}$ and $\delta\vec{E}$:

$$-ik \frac{ck}{\omega} \delta E = \frac{4\pi}{c} \sum_\alpha q_\alpha \int \mathrm{d}p_\perp \, p_\perp \int \mathrm{d}p_\parallel \int \mathrm{d}\varphi \, v_\perp \cos\varphi - i \frac{\omega}{c} \delta E \,, \tag{2.34}$$

where we are considering the x component only, which was what we called δE.

If we decompose $\delta f_\alpha = \delta g_\alpha e^{\pm i\varphi}$, the integral in φ reads

$$\int_0^{2\pi} \mathrm{d}\varphi \cos\varphi e^{\pm i\varphi} = \pi\,, \tag{2.35}$$

so we get, substituting in what we know for δg_α:

$$\frac{-ik^2c}{\omega}\delta E = \frac{4\pi}{c}\sum_\alpha q_\alpha \int \mathrm{d}p_\perp\, p_\perp v_\perp \int \mathrm{d}p_\parallel\, \frac{\frac{q_\alpha}{c}\delta E\left[v_z\frac{ck}{\omega}\frac{\partial f_0}{\partial p_\perp} - v_\perp\frac{ck}{\omega}\frac{\partial f_0}{\partial p_\perp} - \frac{\partial f_0}{\partial p_\perp}\right]}{-i\omega + ikv_z \mp i\Omega_{\text{gyr, }\alpha}} - i\frac{\omega}{c}\delta E\,, \tag{2.36}$$

which looks like a dispersion relation: we just need to collect the generic δE, to get

$$\frac{k^2c^2}{\omega^2} = 1 - \sum_\alpha \frac{4\pi^2 q_\alpha^2}{\omega}\int \mathrm{d}p_\parallel \int \mathrm{d}p_\perp \left(p_\perp v_\perp\left(\frac{v_\parallel k}{\omega} - 1\right)\frac{\partial p_{0\alpha}}{\partial p_\perp} - \frac{kv_\perp}{\omega}\frac{\partial f_{0\alpha}}{\partial p_\parallel}\right)\left[\omega - kv_\parallel \pm \Omega_\alpha\right]^{-1}. \tag{2.37}$$

We can write an alternative version of this using $v_\parallel = v\mu$ and $v_\perp(1-\mu^2)^{1/2}v$, where $\mu = \cos\theta$:

$$\frac{k^2c^2}{\omega^2} = 1 - \sum_\alpha \frac{4\pi^2 q_\alpha^2}{\omega}\int \mathrm{d}p \int \mathrm{d}\mu\, \frac{p^2 v(1-\mu^2)}{\omega - kv\mu \pm \Omega_\alpha}\left[-\frac{\partial f_{0\alpha}}{\partial p} + \frac{1}{p}\frac{\partial f_{0\alpha}}{\partial \mu}\left(\mu - \frac{kv}{\omega}\right)\right]. \tag{2.38}$$

Application: a cold electron-proton plasma This is very general, and we can specify it to the case in which we only have electrons and protons, at $T = 0$.

This means that $f_{0\alpha} = \delta(p)n_\alpha/4\pi p^2$ for some constant. This way, we have

$$\int_0^\infty \mathrm{d}p\, 4\pi p^2 \frac{n_\alpha}{4\pi p^2}\delta(p) = n_\alpha\,. \tag{2.39}$$

This will be true for both protons and electrons, and their densities will be equal to preserve neutrality: $n_e = n_p$. There is no dependence on μ, so we are left with

$$\begin{aligned}\frac{k^2c^2}{\omega^2} &= 1 - \sum_\alpha \frac{4\pi^2 q_\alpha^2}{\omega}\int \mathrm{d}p \int \mathrm{d}\mu\, \frac{p^2 v(1-\mu^2)}{\omega - kv\mu \pm \Omega_\alpha}\left(-\frac{\partial}{\partial p}\frac{n_\alpha \delta(p)}{4\pi p^2}\right) &(2.40)\\ &= 1 - \frac{4\pi e^2 n}{\omega}\int \mathrm{d}p\, \mathrm{d}\mu\, \frac{\delta(p)}{4\pi p^2}\frac{\partial}{\partial p}\left[\frac{p^2 v(1-\mu^2)}{\omega - kv\mu \pm \Omega_\alpha}\right]. &(2.41)\end{aligned}$$

Let us look at the integral with respect to momentum, using $p = m_\alpha v$:

$$\begin{aligned}&\int \mathrm{d}p\, \mathrm{d}\mu\, \frac{\delta(p)}{4\pi p^2}\frac{\partial}{\partial p}\left[\frac{p^2 v(1-\mu^2)}{\omega - kv\mu \pm \Omega_\alpha}\right] = \\ &= \int \mathrm{d}p\, \frac{\delta(p)}{4\pi p^2}\left(\frac{3p^2}{m_\alpha}(\omega - kv\mu \pm \Omega_\alpha) + \frac{k_\mu}{m_\alpha}\frac{p^3}{m_\alpha}\right)\left(\omega - kv\mu \pm \Omega_\alpha^2\right)^{-1} &(2.42)\\ &= \frac{3}{4\pi m_\alpha}\frac{1}{\omega \pm \Omega_\alpha}\,. &(2.43)\end{aligned}$$

The integral in μ is just $\int_{-1}^{1} \mathrm{d}\mu\,(1-\mu^2) = 4/3$. This finally yields

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{4\pi e^2 n}{\omega m_p}\frac{1}{\omega \pm eB_0/m_p c} - \frac{4\pi e^2 n}{\omega m_e}\frac{1}{\omega \mp eB_0/m_e c} \tag{2.44}$$

$$= 1 \mp \frac{4\pi e^2 n}{\omega m_p}\frac{m_p c}{eB_0}\frac{1}{1 \pm \omega/\Omega_p} \pm \frac{4\pi e^2 n}{\omega m_e}\frac{m_e c}{eB_0}\frac{1}{1 \mp \omega/|\Omega_e|}\,, \tag{2.45}$$

We can replace m_e/m_e by m_p/m_p.

so if we consider the case in which $\omega \ll \Omega_p \ll |\Omega_e|$, we get

$$\frac{k^2c^2}{\omega^2} = 1 \mp \frac{4\pi e^2 n}{\omega m_p}\frac{m_p c}{eB_0}\left(1 \mp \frac{\omega}{\Omega_p}\right) \pm \frac{4\pi e^2 n}{\omega m_p}\frac{1}{\Omega_p} \tag{2.46}$$

$$= 1 + \frac{4\pi e^2 n}{\omega m_p}\frac{\omega}{\Omega_p^2} \tag{2.47}$$

$$= 1 + \frac{4\pi e^2 n}{m_p}\frac{m_p^2 c^2}{e^2 B_0^2} \tag{2.48}$$

$$= 1 + 4\pi \underbrace{n m_p}_{\rho} \frac{c^2}{B_0^2} \tag{2.49}$$

$$= 1 + 4\pi\rho\frac{c^2}{B_0^2} = 1 + \left(\frac{c}{v_A}\right)^2, \tag{2.50}$$

where we introduced

$$v_A = \frac{B_0}{\sqrt{4\pi\rho}}\,, \tag{2.51}$$

so in the galaxy, where $B_0 \sim \mu\mathrm{G}$ and $\rho \sim m_p/\mathrm{cm}^3$, we have $v_A \sim 2\,\mathrm{km/s}$.

This means that

$$\frac{k^2c^2}{\omega^2} = \frac{c^2}{v_A^2} \implies \pm k v_A = \omega\,. \tag{2.52}$$

These perturbations travel very slowly! They are called Alfvén waves. They move with $\vec{k}$ along $\vec{B}_0$ (well, we assumed so), and they have oscillating electric and magnetic fields, both perpendicular to the driving magnetic fields.

Without these, cosmic rays would be basically travelling at the speed of light all the time. They also create a small, induction-generated electric field, which is interesting!

3 Magneto-HydroDynamics

Thursday 2021-11-25

We are going to briefly treat a topic which is often discussed in whole books.

The Vlasov equation has a large amount of information in it, way more than what we can actually measure for astrophysical plasmas.

So, we would like to treat our plasma like a magnetized fluid. Further, we know that astrophysical plasmas are extremely conductive. Therefore, the only significant electric field we can have is the one associated with induction.

Still, we must retain a kinetic description (with the full phase space distribution) for anything which is not the background plasma, like the nonthermal particles. We only want to simplify the treatment of the background.

We always start with the Vlasov equation, which we can write in index notation:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{q}{m} \left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right] \cdot \vec{\nabla}_v f = 0 \tag{3.1}$$

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial x_r} + \frac{q}{m} \left[E_r + \frac{1}{c} \epsilon_{rst} v_s B_t \right] \frac{\partial f}{\partial v_r} = 0 \,. \tag{3.2}$$

Now, we need to define local mean values with respect to the phase space distribution of any function:

$$\left\langle \varphi \right\rangle = \frac{\int \mathrm{d}^3 v \, \varphi f}{\int \mathrm{d}^3 v \, f} = \frac{1}{n} \int \mathrm{d}^3 v \, \varphi f \,. \tag{3.3}$$

Suppose the function we are averaging is only dependent on the velocity, which we denote as $\psi(\vec{v})$, we can multiply it by the Vlasov equation and integrate over $\mathrm{d}^3 v$: the various terms become

$$\int \frac{\partial}{\partial t} \psi(\vec{v}) f \, \mathrm{d}^3 v = \frac{\partial}{\partial t} \int \mathrm{d}^3 v \, \psi f = \frac{\partial}{\partial t} \Big[n \left\langle \psi \right\rangle \Big] \tag{3.4}$$

$$\int \mathrm{d}^3 v \, \psi(\vec{v}) v_r \frac{\partial f}{\partial x_r} = \frac{\partial}{\partial x_r} \mathrm{d}^3 v \, \psi v_r f = \frac{\partial}{\partial x_r} \Big[n \left\langle \psi v_r \right\rangle \Big] \tag{3.5}$$

$$\frac{q}{m} \int \mathrm{d}^3 v \, E_r \psi(\vec{v}) \frac{\partial f}{\partial v_r} = - \frac{q}{m} \int \mathrm{d}^3 v \, f \frac{\mathrm{d}\psi}{\mathrm{d}v_r} E_r = - \frac{q}{m} E_r n \left\langle \frac{\mathrm{d}\psi}{\mathrm{d}v_r} \right\rangle \tag{3.6}$$

We integrate by parts, and at the boundary $|v| \to \infty$ all quantities vanish.

$$\frac{q}{mc} \int \mathrm{d}^3 v \, \psi \epsilon_{rst} v_s B_t \frac{\partial f}{\partial v_r} = - \frac{q}{mc} \int \mathrm{d}^3 v \, f \epsilon_{rst} B_t \frac{\partial}{\partial v_r} \big(v_s \psi \big) \tag{3.7}$$

$$= - \frac{q}{mc} \epsilon_{rst} B_t n \left\langle v_s \frac{\partial \psi}{\partial v_r} \right\rangle \,. \tag{3.8}$$

We brought v_s outside the derivative since the term it yields is symmetric but contracted with the Levi-Civita symbol.

If we take $\psi \equiv 1$, we get

$$\frac{\mathrm{d}n}{\mathrm{d}t} + \frac{\partial}{\partial x_r} \left(n \left\langle v_r \right\rangle \right) = 0 \,, \tag{3.9}$$

since all the derivatives of ψ vanish. We found the continuity equation! Keep in mind that for now this only holds for one component of the fluid.

Let us take $\psi = v_r$. We then get

$$\frac{\partial}{\partial t} [n \left\langle v_r \right\rangle] + \frac{\partial}{\partial x_s} [n \left\langle v_r v_s \right\rangle] - \frac{q}{m} E_r n - \frac{q}{mc} n \epsilon_{rst} B_t \left\langle v_s \right\rangle = 0 \,. \tag{3.10}$$

If the fluid is *at rest* in our frame its mean velocity is zero. Let us be more general than this: suppose there is a bulk motion $u_r = \langle v_r \rangle$. Then, the equation reads

$$\frac{\partial}{\partial t}[n u_r] + \frac{\partial}{\partial x_s}[n \langle v_r v_s \rangle] - \frac{q}{m} E_r n - \frac{q}{mc} n \epsilon_{rst} B_t u_s = 0 \,. \tag{3.11}$$

We have one of these equations for the electrons and one for the protons. We have a few component masses in the denominator, but if we multiply through my m the densities in the first two terms just become mass densities.

Let us define the total density $\rho = n_p m_p + n_e m_e$. The density of charges will read $\zeta = (n_p - n_e)e$,

For now, the bulk motions of electrons and protons may differ, so we will have $u_{p,r}$ and $u_{e,r}$. The current density will then read

$$J_r = \Big(n_p u_{p,r} - n_e u_{e,r} \Big) e \,. \tag{3.12}$$

We can define a global, mass-averaged bulk velocity:

$$U_r = \frac{n_p m_p u_{p,r} + n_e m_e u_{e,r}}{\rho} \,. \tag{3.13}$$

We still need to describe internal energy through fluctuations over this mean: the peculiar velocity will read

$$w_{p,r} = v_{p,r} - U_r \tag{3.14}$$

$$w_{e,r} = v_{e,r} - U_r \,. \tag{3.15}$$

Let us now multiply the continuity equation for protons by the proton mass, and similarly for the electrons:

$$\frac{\partial}{\partial t} + \frac{\partial}{\partial x_r} \Big(n_p m_p u_{p,r} \Big) = 0 \tag{3.16}$$

$$\frac{\partial}{\partial t} + \frac{\partial}{\partial x_r} (n_e m_e u_{e,r}) = 0 \,, \tag{3.17}$$

which we can add together: we get a global mass conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r} (\rho U_r) = 0 \,. \tag{3.18}$$

Applying a similar procedure to the $\psi = v_r$ equation we find

$$\frac{\partial}{\partial t} \Big(n_p m_p u_r \Big) + \frac{\partial}{\partial x_s} \Big(n_p m_p \langle v_r v_s \rangle \Big) - e E_r n_p - \frac{e}{c} n_p \epsilon_{rst} B_t n_s = 0 \,, \tag{3.19}$$

which we can add to the corresponding equation for the electrons. First, though, let us look at the $\langle v_r v_s \rangle$ term:

$$\langle v_r v_s \rangle = \Big\langle (w_r + U_r)(w_s + U_s) \Big\rangle \tag{3.20}$$

$$= \langle w_r w_s \rangle + \langle w_r \rangle U_s + \langle w_s \rangle U_r + U_r U_s \tag{3.21}$$

$$= \langle w_r w_s \rangle + (u_r - U_r) U_s + (u_s - U_s) U_r + U_r U_s \tag{3.22}$$

$$= \langle w_r w_s \rangle + u_r U_s + u_s U_r - U_r U_s \,. \tag{3.23}$$

Let us now introduce a stress (pressure) tensor:

$$P_{rs,p} = m_p \int \mathrm{d}^3 v \, f w_{p,r} w_{p,s} = n m_p \left\langle w_{p,r} w_{p_s} \right\rangle \,, \tag{3.24}$$

which yields

$$\langle v_r v_s \rangle = \frac{P_{rs}}{mn} + u_r U_s + u_s U_r - U_r U_s \,. \tag{3.25}$$

We can now sum the two components:

$$n_p m_p \left[\frac{P_{rs,p}}{m_p n_p} + u_{r,p} U_s + u_{s,p} U_r - U_r U_s \right] + n_e m_e \left[\frac{P_{rs,e}}{m_e n_e} + u_{r,e} U_s + u_{s,e} U_r - U_r U_s \right] =$$
$$= P_{rs} + \rho U_r U_s + \rho U_r U_s - \rho U_r U_s = P_{rs} + \rho U_r U_s \,. \tag{3.26}$$

The full equation now looks like:

$$\frac{\mathrm{d}}{\mathrm{d}t} (\rho U_r) + \frac{\partial}{\partial x_s} \left[P_{rs} + \rho U_r U_s \right] - \zeta E_r - \frac{1}{c} \epsilon_{rst} J_s B_t = 0 \,. \tag{3.27}$$

We can expand the derivatives to get something which looks more like an equation of motion, and simplify through the continuity equation:

$$\underbrace{\frac{\partial \rho}{\partial t} U_r}_{U_r \times \text{continuity}} + \rho \frac{\mathrm{d} U_r}{\mathrm{d}t} + \frac{\partial P_{rs}}{\partial x_s} + \underbrace{\frac{\partial}{\partial x_s} (\rho U_s) U_r}_{U_r \times \text{continuity}} + \frac{\partial}{\partial x_s} (\rho U_r) U_s + \dots \,, \tag{3.28}$$

so the equation reads

$$\underbrace{\rho \frac{\partial U_r}{\partial t} + \rho U_s \frac{\partial U_r}{\partial x_s}}_{\rho D_t U} = - \frac{\partial P_{rs}}{\partial x_s} + \zeta E_r + \frac{1}{c} \epsilon_{rst} J_s B_t \,. \tag{3.29}$$

Now this system does not know about protons and electrons anymore: it is just a fluid. In vector terms,

$$\rho \left[\frac{\partial \vec{U}}{\partial t} + \vec{U} \cdot \vec{\nabla} U \right] = - \vec{\nabla} P + \zeta \vec{E} + \frac{1}{c} \vec{J} \times \vec{B} \,. \tag{3.30}$$

We haven't done the energy conservation equation since it will not be useful for our purposes, while being very lengthy. Still, in principle we should use it.

We will now make the **ideal MHD approximation**, in which the conductivity is assumed to be very high. Still, there are situations in which the resistivity becomes high locally. This can happen through a phenomenon called *reconnection*, for example.

Suppose we have an electric field in the lab frame: in the frame comoving with the plasma, it will look like $\vec{E} \to \vec{E} + \vec{U} \times \vec{B}/c$.

This can be described by Ohm's law,

$$\vec{E} + \frac{1}{c}\vec{U} \times \vec{B} = \eta \vec{J}\,. \tag{3.31}$$

Using the Ampére-Maxwell equation we get that the current reads

$$\vec{J} = \frac{c}{4\pi}\left[\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right]. \tag{3.32}$$

We assume that η is very small. In the $\eta = 0$ case, in the lab reference frame we get

$$\vec{E} = -\frac{1}{c}\vec{U} \times \vec{B}\,. \tag{3.33}$$

What is the order of magnitude of $\vec{J} \times \vec{B}/c$?

$$\frac{1}{4\pi}\left[\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right] \times \vec{B} \sim \frac{B^2}{UT} + \frac{U}{c^2}\frac{B^2}{T}\,, \tag{3.34}$$

The first term has scale B^2/L, the second has scale $(U/c^2)(B^2/T)$.

We can then see that under this low-resistivity assumption the second term can be neglected.

Then, the equation just reads

$$\rho\left[\frac{\partial \vec{U}}{\partial t} + \vec{U} \cdot \vec{\nabla} U\right] = -\vec{\nabla} P + \frac{1}{4\pi}(\vec{\nabla} \times \vec{B}) \times \vec{B}\,. \tag{3.35}$$

There is an interesting implication to this equation known as **flux freezing**.

The assumption of ideal MHD is that there is no dissipation: the field lines cannot cross; we can "squeeze" them, but the flux will be preserved.

Let us take a surface S_0, and compute the magnetic field flux across it at $t = t_0$:

$$\Phi_0 = \int \mathrm{d}\vec{S}_0\, \vec{B}(\vec{x}, t_0)\,, \tag{3.36}$$

which we want to compare to the flux across the same surface, transported along the fluid motion:

$$\begin{aligned}\Phi_1 &= \int \mathrm{d}S_1\, \vec{B}(\vec{x} + \vec{v}\,\mathrm{d}t\,, t_0 + \mathrm{d}t) & (3.37)\\ &= \int \mathrm{d}S_1 \left[\vec{B}(\vec{x} + \vec{v}\,\mathrm{d}t\,, t_0) + \frac{\partial \vec{B}}{\partial t}\,\mathrm{d}t\right]. & (3.38)\end{aligned}$$

If we include S_0, S_1 as well as the boundary S_2 we will get a closed surface. The integral over $S = -S_0 + S_1 + S_2$ will be

$$\oint_S \vec{B} \cdot \mathrm{d}S = 0\,, \tag{3.39}$$

where we write $-S_0$ since the normal to that surface must be oriented outward.

From this equation we get that

$$\Phi_0 = \int \mathrm{d}S_1\, B + \int \mathrm{d}S_2\, B\,. \tag{3.40}$$

What is $\mathrm{d}S_2$? It can be written as $\mathrm{d}\ell \wedge \vec{v}\,\mathrm{d}t$, since we are asking that we move along the flux lines of the fluid.

We know that

$$\begin{aligned}
\Phi_1 &= \Phi_0 - \int \mathrm{d}\vec{S}_1 \cdot \vec{B} + \mathrm{d}t \int \mathrm{d}\vec{S}_1 \frac{\partial \vec{B}}{\partial t} &(3.41)\\
&= \Phi_0 - \mathrm{d}t \int (\mathrm{d}\vec{\ell} \times \vec{c}) \cdot \vec{B} + \mathrm{d}t \int \mathrm{d}\vec{S}_1 \frac{\partial \vec{B}}{\partial t} &(3.42)\\
&= \Phi_0 - \int \mathrm{d}t\, (\vec{v} \times \vec{B}) \cdot \mathrm{d}\ell + \mathrm{d}t \int \frac{\partial \vec{B}}{\partial t}\, \mathrm{d}S &(3.43)\\
&= \Phi_0 - \mathrm{d}t \int \mathrm{d}S \left[\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) + \frac{\partial \vec{B}}{\partial t} \right], &(3.44)
\end{aligned}$$

but since

$$\vec{E} = -\frac{1}{c} \vec{v} \times \vec{B} \qquad \text{and} \qquad \frac{\partial \vec{B}}{\partial t} = -c \vec{\nabla} \times \vec{E}\,, \tag{3.45}$$

we get that the integrand vanishes.

Check this derivation, something seems fishy. We should compute everything at the same time!

Tuesday 2021-11-30

We want to introduce some thermodynamic quantities for the plasma.

Let us start from the MHD equations:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 &(3.46)\\
&\rho \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \rho \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} P + \frac{1}{4\pi} \left(\vec{\nabla} \times \vec{B} \right) \times \vec{B}\,. &(3.47)
\end{aligned}$$

We introduce the additional assumption that the plasma is isentropic:

$$\left[\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla} \right] P \rho^{-\gamma} = 0\,, \tag{3.48}$$

as well as the magnetic field equations:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3.49}$$

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \left(\vec{u} \times \vec{B} \right) = 0 \,. \tag{3.50}$$

We can always put ourselves in a frame which is at equilibrium: $\vec{u} = 0$, which we then perturb with $\delta\vec{u}$.

Then, we move to Fourier space and do first-order perturbation theory: we find

$$-i\omega\delta\rho + i\vec{k} \cdot \delta\vec{u}\rho = 0 \,. \tag{3.51}$$

The entropy equation becomes

$$\frac{\partial P}{\partial t}\rho^{-\gamma} - \gamma\rho^{-\gamma-1}P\frac{\partial \rho}{\partial t} = 0 \tag{3.52}$$

$$-i\omega\delta P + \gamma\rho^{-\gamma-1}Pi\omega\delta\rho = 0 \tag{3.53}$$

$$-i\omega\delta P + \gamma\frac{P}{\rho}i\omega\delta\rho = 0 \tag{3.54}$$

$$\frac{\partial \delta P}{\partial \delta\rho} = \gamma\frac{P}{\rho} = c_s^2 \,, \tag{3.55}$$

so we have found acoustic perturbations by allowing the plasma to be compressible.

The no-monopoles equation becomes $i\vec{k} \cdot \delta\vec{B} = 0$; while the last one can only be perturbed like

$$-i\omega\delta\vec{B} = i\vec{k} \times \left(\delta\vec{u} \times \vec{B} \right) \tag{3.56}$$

$$\delta\vec{B} = -\frac{\vec{k}}{\omega} \times \left(\delta\vec{u} \times \vec{B} \right) . \tag{3.57}$$

The momentum conservation law becomes

$$-i\omega\rho\delta\vec{u} = -i\vec{k}\delta P + \frac{i\vec{k}}{4\pi} \times \delta\vec{B} \times \vec{B}_0 \tag{3.58}$$

$$= -i\vec{k}c_s^2\delta\rho - \frac{i}{4\pi\omega}\vec{k} \times \left(\vec{k} \times \left(\delta\vec{u} \times \vec{B}_0 \right) \right) \times \vec{B}_0 \tag{3.59}$$

$$-i\omega\rho\delta\vec{u} = -i\vec{k}c_s^2\frac{\rho}{\omega}\vec{k} \cdot \delta\vec{u} - \frac{i}{4\pi\omega}\vec{k} \times \left(\vec{k} \times \left(\delta\vec{u} \times \vec{B}_0 \right) \right) \times \vec{B}_0 \,. \tag{3.60}$$

Used $\delta\rho = (\rho/\omega)\vec{k} \cdot \delta\vec{u}$.

This is now all written in terms of $\delta\vec{u}$, and it contains all MHD perturbations.

Let us look at some specific cases: first, assume $\vec{B}_0 = B_0\hat{z}$, and that $\vec{k}$ is parallel to it, so $\vec{k} = k\hat{z}$.

The cross product term reads:

$$\delta\vec{u} \times \vec{B}_0 = \begin{bmatrix} \delta u_y B_0 \\ -\delta u_x B_0 \\ 0 \end{bmatrix} \tag{3.61}$$

$$\vec{k} \times (\delta\vec{u} \times \vec{B}_0) = \begin{bmatrix} k\delta u_x B_0 \\ k\delta u_y B_0 \\ 0 \end{bmatrix} \tag{3.62}$$

$$\vec{k} \times \left(\vec{k} \times (\delta\vec{u} \times \vec{B}_0)\right) = \begin{bmatrix} -k^2 \delta u_y B_0 \\ k^2 \delta u_x B_0 \\ 0 \end{bmatrix} \tag{3.63}$$

$$\left[\vec{k} \times \left(\vec{k} \times (\delta\vec{u} \times \vec{B}_0)\right)\right] \times \vec{B}_0 = \begin{bmatrix} k^2 \delta u_x B_0^2 \\ k^2 \delta u_y B_0^2 s \\ 0 \end{bmatrix} . \tag{3.64}$$

The equation is therefore

$$-i\omega\rho \begin{bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{bmatrix} = -i c_s^2 \frac{\rho}{\omega} \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} k \delta u_z - \frac{i}{4\pi\omega} \begin{bmatrix} k^2 \delta u_x B_0^2 \\ k^2 \delta u_y B_0^2 s \\ 0 \end{bmatrix} , \tag{3.65}$$

where we can see that the perpendicular and parallel modes are decoupled: the z mode reads

$$\omega\rho\delta u_z = c_s^2 \frac{\rho}{\omega} k^2 \delta u_z \implies \omega^2 = k^2 c_s^2 , \tag{3.66}$$

therefore the *longitudinal* modes are *sound waves*! For those modes the magnetic field is irrelevant since $\delta\vec{u} \times \vec{B}_0 = 0$.

What about the other directions? We find

$$\omega\rho = \frac{k^2 B_0^2}{4\pi\omega} \implies \omega^2 = k^2 \frac{B_0^2}{4\pi\rho} = k^2 v_A^2 , \tag{3.67}$$

so these are Alfén waves. This tells us that these are *transverse* waves, but still propagating *along* the magnetic field.

What is the magnetic field perturbation?

$$\delta\vec{B} = -\frac{1}{\omega} \vec{k} \times \left(\delta\vec{u} \times \vec{B}_0\right) \tag{3.68}$$

$$= -\frac{1}{\omega} \begin{bmatrix} k B_0 \delta u_x \\ k B_0 \delta u_y \\ 0 \end{bmatrix} , \tag{3.69}$$

which means that the magnetic field is also only oscillating along the x and y directions.

We can also compute the induced electric field:

$$\delta\vec{E} = -\frac{1}{c} \delta\vec{u} \times \vec{B}_0 = \begin{bmatrix} \delta u_y B_0 / c \\ \delta u_x B_0 / c \\ 0 \end{bmatrix} . \tag{3.70}$$

The perturbations in the magnetic field, $\sim B_0 \delta u / v_A$ are *larger* than the ones in the electric field, $\sim B_0 \delta u / c$! This can be also checked when looking at the Lorentz force expression.

Let us then look at the perpendicular modes: $\vec{k} \perp B_0$, specifically $\vec{k} = k\hat{x}$.

$$\delta\vec{u} \times \vec{B}_0 = \begin{bmatrix} \delta u_y B_0 \\ -\delta u_x B_0 \\ 0 \end{bmatrix} \tag{3.71}$$

$$\vec{k} \times (\delta\vec{u} \times \vec{B}_0) = \begin{bmatrix} 0 \\ 0 \\ -k\delta u_x B_0 \end{bmatrix} \tag{3.72}$$

$$\vec{k} \times \left(\vec{k} \times (\delta\vec{u} \times \vec{B}_0)\right) = \begin{bmatrix} 0 \\ k^2\delta u_x B_0 \\ 0 \end{bmatrix} \tag{3.73}$$

$$\left[\vec{k} \times \left(\vec{k} \times (\delta\vec{u} \times \vec{B}_0)\right)\right] \times \vec{B}_0 = \begin{bmatrix} k^2\delta u_x B_0^2 \\ 0 \\ 0 \end{bmatrix} . \tag{3.74}$$

Plugging this into the equation yields

$$-i\omega\rho \begin{bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{bmatrix} = -i \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} c_s^2 \frac{\rho}{\omega} k\delta u_x - \frac{i}{4\pi\omega} \begin{bmatrix} k^2 B_0^2 \delta u_x \\ 0 \\ 0 \end{bmatrix} , \tag{3.75}$$

which means that the only nonzero component, along x, reads

$$\omega^2 = \left(c_s^2 + \frac{B_0^2}{4\pi\rho}\right)k^2 = \left(c_s^2 + v_A^2\right)k^2 , \tag{3.76}$$

which is commonly called a "fast" **magnetosonic mode**.

For the rest of the course we will focus on the Alfvén modes.

4 High energy particles

We start by assuming we already have a non-thermal, high energy particle.

The momentum of the particle is $\vec{p} = m\vec{v}\gamma$, while the field is $\vec{B}_0 = B_0\hat{z}$.

The energy of this particle will not change under the action of the magnetic field; the Lorentz force's action means we will have

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q\frac{\vec{v}}{c}\vec{B}_0 , \tag{4.1}$$

therefore

$$m\gamma\frac{\mathrm{d}v_x}{\mathrm{d}t} = q\frac{v_y}{c}B_0 \tag{4.2}$$

γ is a constant: the energy cannot change.

$$m\gamma\frac{\mathrm{d}v_y}{\mathrm{d}t} = -q\frac{v_x}{c}B_0 \tag{4.3}$$

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = 0\,. \tag{4.4}$$

Putting these together we get

$$\left(\frac{m\gamma}{qB_0}\right)^2 \frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} = -v_x \tag{4.5}$$

$$\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} = -\Omega^2 v_x\,. \tag{4.6}$$

If $\gamma = 1$, this is the cyclotron frequency; otherwise, it is just $\Omega = qB_0/(mc\gamma)$.

If μ is the cosine of the angle between the initial velocity and the magnetic field, the solution will read

$$v_x(t) = v_0(1-\mu^2)^{1/2}\cos(\Omega t) \tag{4.7}$$

$$v_y(t) = v_0(1-\mu^2)^{1/2}\sin(\Omega t) \tag{4.8}$$

$$v_z(t) = v_0\mu\,. \tag{4.9}$$

What happens if this trajectory is perturbed, say, by an Alfvén wave? The field will now be $\vec{B} = \vec{B}_0 + \delta\vec{B}$, and we will work to first order in $\delta\vec{B}$.

Let us assume one of these two things:

1. we sit in a reference frame moving with the waves, so that the electric field perturbation vanishes;

2. we decide to neglect the electric field since we know its effect to be small.

There is a subtle difference between the two.

We assume that the polarization of the electric field perturbation is circular: $\delta E_y = \pm i\delta E_x$, so that $\delta B_x = \mp\delta B_y$.

We take

$$\delta B_y = \exp\Big(i\big(kz - \omega t + \varphi\big)\Big)\,, \tag{4.10}$$

where we insert an arbitrary initial phase we will later average over.

If we take the real part of the perturbation, we get

$$\delta B_x = \pm\sin\big(kz - \omega t + \varphi\big) \tag{4.11}$$

$$\delta B_y = \pm\cos\big(kz - \omega t + \varphi\big)\,. \tag{4.12}$$

This is an Alfvén wave, so $\omega = kv_A$.

If we assume that the electric field is negligible, the equations of motion for the particle will be modified as

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q\frac{\vec{v}}{c}\times\left(\vec{B}_0 + \delta\vec{B}\right) = \frac{q}{c}\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ \delta B_x & \delta B_y & B_0 \end{bmatrix} \tag{4.13}$$

$$m\gamma \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_y B_0 - v_z \delta B_y \\ -v_x B_0 + \delta B_x v_z \\ v_x \delta B_y - v_y \delta B_x \end{bmatrix} . \tag{4.14}$$

Looking at the z component, we see that the perturbation changes v_z, but v_0 is constant, so the pitch angle μ must be changeing.

Why is this relevant? We know that spallation must occur, so the cosmic rays must remain in the galaxy for a while. This can be a mechanism to explain that.

We focus on the z equation:

$$m\gamma \frac{\mathrm{d}v_z}{\mathrm{d}t} = \frac{q}{c} \delta B v_0 (1-\mu^2)^{1/2} \Big(\cos(\Omega t) \cos\big(kz - \omega t + \varphi\big) \pm \sin(\Omega t) \sin\big(kz - \omega t + \varphi\big) \Big) \tag{4.15}$$

$$m\gamma v_0 \frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{q\delta B}{mc\gamma} v_0 (1-\mu^2)^{1/2} \Big(\cos(\Omega t) \cos\big(kz - \omega t + \varphi\big) \pm \sin(\Omega t) \sin\big(kz - \omega t + \varphi\big) \Big) \tag{4.16}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{q\delta B}{mc\gamma} v_0 (1-\mu^2)^{1/2} \cos\big(\Omega t \mp kz \pm \omega t \mp \varphi\big) . \tag{4.17}$$

Used some ugly prostapheresis formulae.

Now, however, $kz \approx k v_0 \mu t$; however, $\omega t \approx k v_A t$, so unless $\mu \lesssim v_A / c$ the latter is negligible.

In normal conditions, $v_A \ll v_0 \sim c$, and this is a requirement of the order of 10^{-6}.

The falsifying example is a particle moving basically moving in a circle without any z component to its velocity, but this is very unlikely.

The wave, for the relativistic particle, is basically stationary.

So,

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{q\delta B}{mc\gamma} (1-\mu)^{1/2} \cos\big(\Omega t \mp k v_0 \mu t \mp \varphi\big) . \tag{4.18}$$

To find out what the effect of this is over a long period of time we need to integrate.

The mean value of the cosine vanishes, however, we can have a diffusion-like process. On average, $\langle \Delta\mu \rangle = 0$, but we might want to compute $\langle \Delta\mu\Delta\mu \rangle$: this will be

$$\langle \Delta\mu\Delta\mu \rangle = \left(\frac{q\delta B}{mc\gamma} \right)^2 (1-\mu^2) \int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' \cos\big((\Omega \mp v_0 k \mu) t \mp \varphi \big) \cos\big((\Omega \mp v_0 k \mu) t' \mp \varphi \big) , \tag{4.19}$$

where the first thing we want to do is to also take an average over φ:

$$\begin{aligned} \langle \Delta\mu\Delta\mu \rangle &= \left(\frac{q\delta B}{mc\gamma} \right)^2 (1-\mu^2) \frac{1}{2} \int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' \cos\big((\Omega \mp v_0 k\mu)(t - t') \big) && (4.20) \\ &= \left(\frac{q\delta B}{mc\gamma} \right) (1-\mu^2) \frac{2\pi}{2} T \delta(\Omega \mp v_0 k \mu) && (4.21) \\ &= \left(\frac{q\delta B}{mc\gamma} \right) (1-\mu^2) \pi T \delta(k \mp \frac{\Omega}{v_0 \mu}) , && (4.22) \end{aligned}$$

so we do indeed get a diffusion-like motion with variance $\sim T$.

However, this is only the case if

$$k = \pm \frac{\Omega}{v_0 \mu} = \pm \frac{1}{r_L \mu} \,. \tag{4.23}$$

So, only if the wave is *resonant* with the Larmor radius then its pitch angle *diffuses*. This remains quite close to being true even if the assumption of small δB is relaxed. More specifically,

$$k = \pm \frac{\operatorname{sign} q}{|r_L| \mu} \,. \tag{4.24}$$

It is customary to introduce a *diffusion coefficient*:

$$\begin{aligned}
D_{\mu\mu} &= \frac{1}{2} \left\langle \frac{\Delta\mu \Delta\mu}{\Delta t} \right\rangle = \frac{1}{2} \left(\frac{q \delta B}{m c \gamma} \right)^2 (1 - \mu^2) \frac{\pi}{v_0 \mu} \delta(k \mp \frac{\Omega}{v_0 \mu}) && (4.25)\\
&= \left(\frac{q B_0}{m c \gamma} \right)^2 \frac{\delta B^2}{B_0^2} (1 - \mu^2) \frac{\pi}{2 v_0 \mu} \delta(k \mp \frac{\Omega}{\mu v_0}) \,. && (4.26)
\end{aligned}$$

This is useful since in general δB will depend on the scale: there are no monochromatic Alfvén waves in nature. Therefore,

$$\begin{aligned}
D_{\mu\mu} &= \Omega^2 (1 - \mu^2) \frac{\pi}{2 v_0 \mu} \int \mathrm{d}k \, \frac{\delta B^2(k)}{B_0^2} \delta(k \mp \frac{\Omega}{v_0 \mu}) && (4.27)\\
&= \Omega^2 (1 - \mu^2) \frac{\pi}{2 v_0 \mu} \frac{\delta B^2(k_{\text{res}})}{B_0^2} && (4.28)\\
&= \frac{\pi}{2} \Omega (1 - \mu^2) k_{\text{res}} F(k_{\text{res}}) \,, && (4.29)
\end{aligned}$$

where

$$F(k_{\text{res}}) = \frac{1}{B_0^2} \delta B^2(k_{\text{res}}) \,. \tag{4.30}$$

We can also compute $D_{\theta\theta}$:

$$D_{\theta\theta} = \frac{\pi}{2} \Omega k_{\text{res}} F(k_{\text{res}}) \,. \tag{4.31}$$

How long does it take to have a diffusion by, say, 1 radian? it will be

$$\tau \sim \frac{1}{D_{\theta\theta}} \sim \frac{1}{\Omega k_{\text{res}} F(k_{\text{res}})} \,. \tag{4.32}$$

There seems to be a high tail at high k!

Tuesday 2021-12-14

The equations of motion we found read

$$m \gamma \frac{\mathrm{d} v_x}{\mathrm{d} t} = \frac{q}{c} \left(v_y B_0 - v_z \delta B_y \right) \tag{4.33}$$

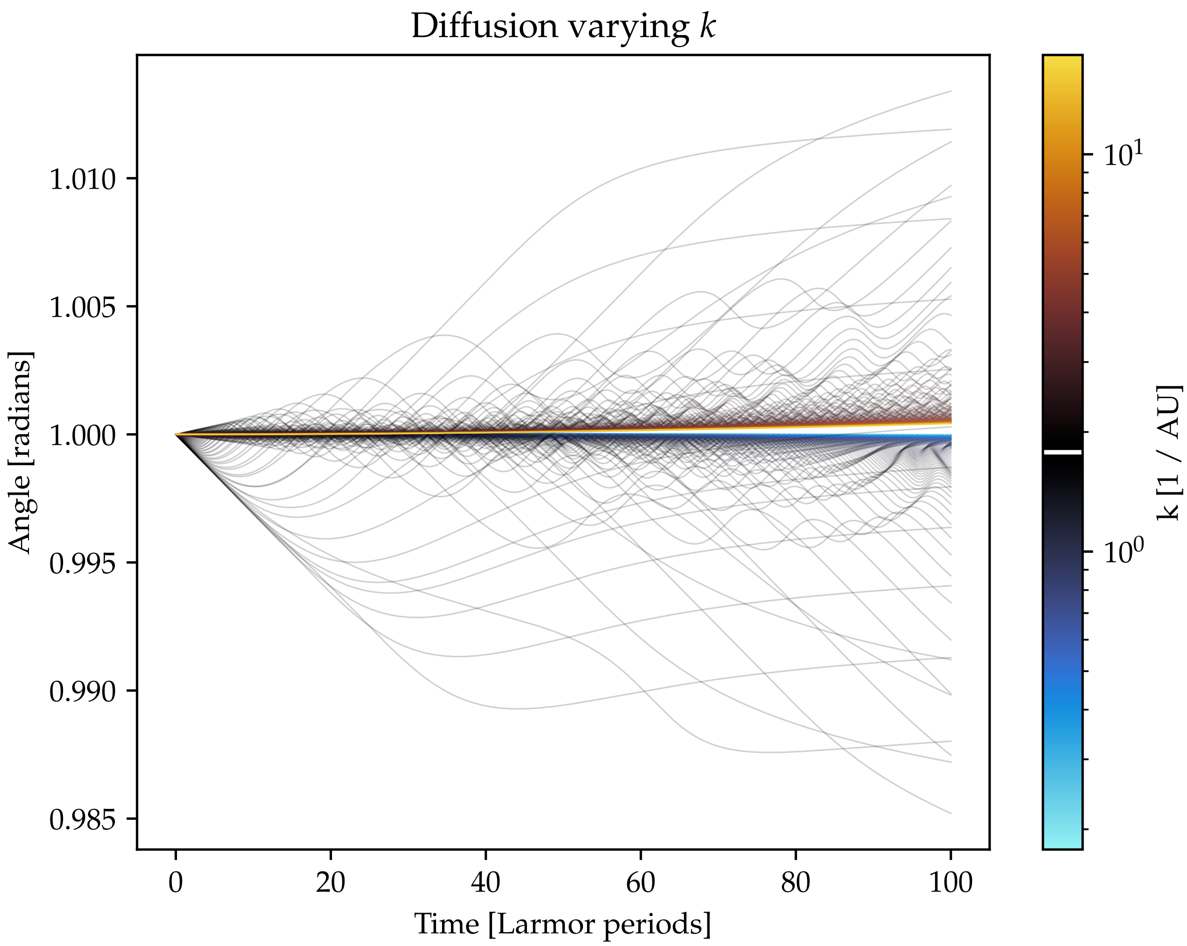

Figure 1: Diffusion over time. The evolution is shown for a relativistic proton with $\gamma = 10$, moving across a field $B_0 = 2\,\mu\mathrm{G}$ with a perturbation $\delta B = 10^{-4} B_0$, with the $+$ circular polarization (i. e. all the $\pm$ become $+$, all the $\mp$ become $-$). A white mark on the wavenumber scale marks $k = \Omega v_0 \cos\theta$, at which the "resonance" happens.

$$m\gamma \frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{q}{c}\left(v_x B_0 - v_z \delta B_y\right) \tag{4.34}$$

$$m\gamma \frac{\mathrm{d}v_z}{\mathrm{d}t} = \frac{q}{c}\left(v_x \delta B_y - v_y \delta B_x\right)\,, \tag{4.35}$$

and we found

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} \sim \cos\left(\Omega t \mp kz \pm \omega t \mp \varphi\right)\,. \tag{4.36}$$

We then neglected $\omega t \sim k v_A t$ compared to $kz = kv\mu t$.

We found $\langle \Delta\mu\Delta\mu \rangle = \Omega^2 (\delta B / B_0)^2 \Delta t (\pi / v\mu) \delta(k \mp \Omega / v\mu)$. The $\Delta\mu^2 \sim \Delta t$ dependence tells us that it is *diffusive* motion, as opposed to *ballistic* motion, which would have $\Delta\mu \sim \Delta t$.

The diffusive motion is then described as

$$D_{\mu\mu} = \frac{1}{2}\left\langle \frac{\Delta\mu\Delta\mu}{t} \right\rangle = \frac{\pi}{2}\Omega(1-\mu^2) k_{\mathrm{res}} F(k_{\mathrm{res}})\,, \tag{4.37}$$

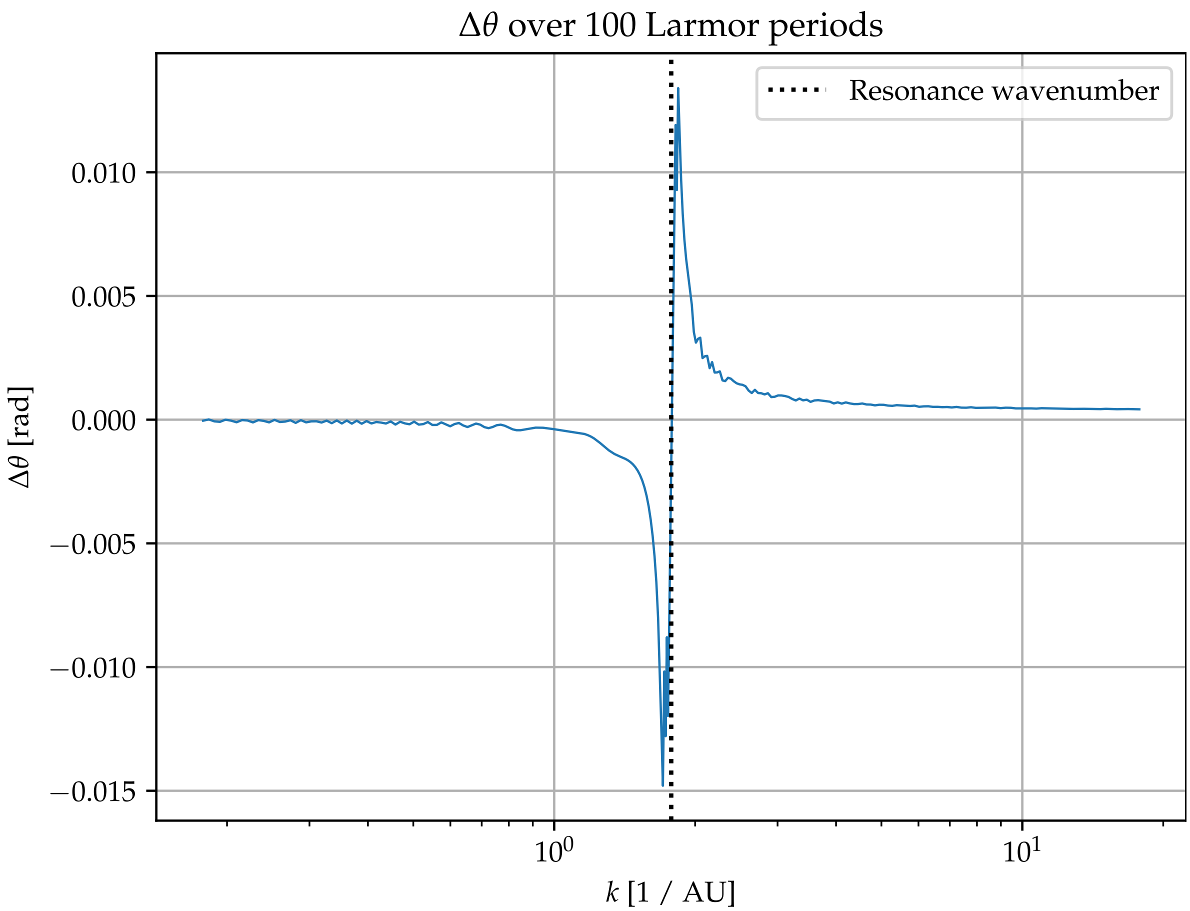

Figure 2: $\Delta\theta$ corresponding to the end of the integrated curves in figure 1.

where $F(k_{\rm res}) = \delta B^2(k_{\rm res})/B_0^2$.

Diffusion in pitch angle means diffusion in space! Once the angle changes enough, the particle changes direction completely. The timescale for diffusion by $\pi/2$ is

$$\tau \sim \frac{1}{\Omega k_{\rm res} F(k_{\rm res})} \,. \tag{4.38}$$

We will have a diffusion length scale of the order of $\lambda_d \approx v\tau$, and the diffusion coefficient in space will be

$$D_{zz} = \frac{1}{3} v \lambda_d = \frac{1}{3} v^2 \tau \,, \tag{4.39}$$

so we get

$$D = \frac{1}{3} v^2 \frac{1}{\Omega k_{\rm res} F(k_{\rm res})} \tag{4.40}$$

$$= \frac{1}{2} r_L(p) \frac{v}{k_{\rm res} F(k_{\rm res})} \,. \tag{4.41}$$

This is *Bohm diffusion*! This theoretical result allows us to compute a timescale for diffusion, which with some reasonable numbers is

$$\tau = \frac{H^2}{D} \approx 25 \times 10^{20} \times \text{resonant power} \stackrel{!}{=} \text{million years}\,, \tag{4.42}$$

so adding a very small $k_{\text{res}} F(k_{\text{res}}) \sim 10^{-6}$ perturbation allows for strong diffusive motion.

Do calculation with units

It's easy to understand how a particle gets to $\mu = 0$, but crossing that threshold is complicated. Slightly nonlinear theories of this problem allow for the crossing of this threshold more easily.

If we include the electric field, the energy can change a bit! Proceeding in this way allows us to compute a corrected diffusion coefficient for the momentum:

$$\left\langle \frac{\Delta p \Delta p}{\Delta t} \right\rangle \approx (m v_A \gamma)^2 \left\langle \frac{\Delta \mu \Delta \mu}{\Delta t} \right\rangle \tag{4.43}$$

$$\approx (mc\gamma)^2 \Omega k_{\text{res}} F(k_{\text{res}}) \tag{4.44}$$

$$\approx (mc\gamma)^2 \frac{v_A^2}{c^2} \frac{1}{\tau} = p^2 \left(\frac{v_A}{c} \right)^2 \frac{1}{\tau}\,, \tag{4.45}$$

therefore

$$\tau_{\text{acceleration}} \approx \left(\frac{c}{v_A} \right)^2 \tau\,, \tag{4.46}$$

it takes $\sim 10^{10}$ times longer to accelerate/decelerate a particle than to make it diffuse over time.

There can be situations in which v_A is not so small compared to c! For example, in a GRB the plasma is relativistic, and even diffusive processes will be fast.

The effect of diffusion is to isotropize the distribution of the particles! The net speed of these relativistic particles will roughly become v_A!

We will now try to understand this quantitatively. Once we get the result of this equation, the rest of the course will be applications.

There are two ways to get this result: we will use the Fokker-Planck approach, which is less heavy in terms of mathematics. The alternative would be to look at the zeroth-order term in the Vlasov equation which would describe collisions, a big average on the right. We do have a way to express it: we write the δf for cosmic rays, the δB for Alfvén waves, and compute away.

Here is the alternative approach: we need to introduce a distribution function in phase space for the cosmic rays.

We will need to introduce the probability that a certain cosmic ray momentum $\vec{p}$ changes to $\vec{p} + \Delta\vec{p}$: $\Psi(\vec{p}, \Delta\vec{p})$, which will satisfy

$$\int \Psi(\vec{p}, \Delta\vec{p})\, \mathrm{d}\Delta\vec{p} = 1\,. \tag{4.47}$$

We will then have the following:

$$f(\vec{x}+\vec{v}\Delta t,\vec{p},t+\Delta\vec{t}) = \int \mathrm{d}\Delta\vec{p}\, f(\vec{x},\vec{p}-\Delta\vec{p},t)\Psi(\vec{p}-\Delta\vec{p},\Delta\vec{p})\,, \tag{4.48}$$

which we can expand:

$$f(\vec{x}+\vec{v}\Delta t,\vec{p},t+\Delta t)[] = f + \vec{v}\cdot\frac{\partial}{\partial\vec{x}}f\Delta t + \frac{\partial}{\partial t}f\Delta t \tag{4.49}$$

$$f(\vec{x},\vec{p}-\Delta\vec{p},t) = f - \frac{\partial f}{\partial\vec{p}}\Delta\vec{p} + \frac{1}{2}\frac{\partial}{\partial p}\frac{\partial}{\partial p}f\Delta\vec{p}\Delta\vec{p} \tag{4.50}$$

$$\Psi(\vec{p}-\Delta\vec{p},\Delta\vec{p}) = \Psi - \Delta\vec{p}\frac{\partial\Psi}{\partial\vec{p}} + \frac{1}{2}\frac{\partial}{\partial\vec{p}}\frac{\partial}{\partial\vec{p}}\Psi\Delta\vec{p}\Delta\vec{p}\,. \tag{4.51}$$

Inserting these terms in the integral equation yields

$$f + \frac{\partial f}{\partial t}\Delta t + \vec{v}\cdot\frac{\partial f}{\partial\vec{x}}\Delta t = \int \mathrm{d}\Delta\vec{p}\left(f - \frac{\partial f}{\partial\vec{p}}\Delta\vec{p} + \frac{1}{2}\frac{\partial^2 f}{\partial p^2}\Delta p^2\right)\left(\Psi - \frac{\partial\Psi}{\partial p}\Delta p + \frac{1}{2}\Delta p^2\frac{\partial^2\Psi}{\partial p^2}\right) \tag{4.52}$$

$$= \int \mathrm{d}\Delta p\left(f\Psi - f\frac{\partial\Psi}{\partial p}\Delta p + \frac{1}{2}f\frac{\partial^2\Psi}{\partial p^2} - \Psi\frac{\partial f}{\partial p}\Delta p + \frac{\partial f}{\partial p}\frac{\partial\Psi}{\partial p}\Delta p^2 + \frac{1}{2}\frac{\partial^2 f}{\partial p^2}\Delta p^2\right) \tag{4.53}$$

$$= f - \int \mathrm{d}\Delta\vec{p}\,\Delta\vec{p}\frac{\partial}{\partial p} + \frac{1}{2}\frac{\partial}{\partial p}\frac{\partial}{\partial p}\left(\mathrm{d}\Delta p\,\Delta\vec{p}\Delta\vec{p}f\Psi\right)\,, \tag{4.54}$$

where we need the result

$$\frac{\partial^2(f\Psi)}{\partial p^2} = \frac{\partial^2 f}{\partial p^2}\Psi + \frac{\partial^2\Psi}{\partial p^2}f + 2\frac{\partial f}{\partial p}\frac{\partial\Psi}{\partial p}\,. \tag{4.55}$$

We also need to define

$$\left\langle\frac{\Delta\vec{p}}{\Delta t}\right\rangle = \frac{1}{\Delta t}\int\Psi(\Delta\vec{p})\Delta\vec{p}\,\mathrm{d}\Delta\vec{p} \tag{4.56}$$

$$\left\langle\frac{\Delta\vec{p}\Delta\vec{p}}{\Delta t}\right\rangle = \frac{1}{\Delta t}\int\Psi(\Delta\vec{p})\Delta\vec{p}\Delta p\,\mathrm{d}\Delta\vec{p}\,. \tag{4.57}$$

Then, we get

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial\vec{x}} = -\frac{\partial}{\partial\vec{p}}\left(f\left\langle\frac{\Delta p}{\Delta t}\right\rangle\right) + \frac{1}{2}\frac{\partial}{\partial p}\frac{\partial}{\partial p}\left(f\left\langle\frac{\Delta p\Delta p}{\Delta t}\right\rangle\right)\,. \tag{4.58}$$

We also need to introduce an assumption: a detailed balance principle,

$$\Psi(\vec{p},-\Delta\vec{p}) = \Psi(\vec{p}-\Delta\vec{p},\Delta p) \tag{4.59}$$

$$\Psi(\vec{p},-\Delta\vec{p}) = \Psi(\vec{p},\Delta\vec{p}) - \Delta\vec{p}\frac{\partial\Psi}{\partial\vec{p}} + \frac{1}{2}\frac{\partial^2\Psi}{\partial p^2}\Delta p\Delta p \tag{4.60}$$

$$\frac{\partial}{\partial p}\left\langle\frac{\Delta\vec{p}}{\Delta t}\right\rangle = \frac{1}{2}\frac{\partial}{\partial p}\frac{\partial}{\partial p}\left\langle\frac{\Delta\vec{p}\Delta\vec{p}}{\Delta t}\right\rangle\,, \tag{4.61}$$

meaning that the quantity

$$\left\langle \frac{\Delta\vec{p}}{\Delta t} \right\rangle - \frac{1}{2}\frac{\partial}{\partial p}\left\langle \frac{\Delta\vec{p}\Delta\vec{p}}{\Delta t} \right\rangle \tag{4.62}$$

is a constant, which we set to zero. We will verify *a posteriori* that this is correct, but it kind of makes sense. So,

$$\underbrace{\left\langle \frac{\Delta\vec{p}}{\Delta t} \right\rangle}_{R} = \underbrace{\frac{1}{2}\frac{\partial}{\partial p}\left\langle \frac{\Delta\vec{p}\Delta\vec{p}}{\Delta t} \right\rangle}_{\partial/\partial p} . \tag{4.63}$$

The equation then becomes

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{x}} = -\frac{\partial f}{\partial p}R - f\frac{\partial R}{\partial p} + \frac{\partial}{\partial p}\frac{\partial}{\partial p} = \frac{\partial}{\partial p}\left(D_{pp}\frac{\partial f}{\partial p} \right) . \tag{4.64}$$

In one dimension, and neglecting interactions, this simply becomes

$$\frac{\partial f}{\partial t} + v\mu\frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f}{\partial \mu} \right) , \tag{4.65}$$

à la Fokker-Planck. The beautiful thing is that this is valid without assuming slow velocities or isotropy!

This exactly describes the diffusion of the beam we were considering. It's a mess to solve this.

Let us try to integrate this between -1 and 1: we apply the operator $I = (1/2)\int_{-1}^{1} \mathrm{d}\mu$. We define the distribution function $M(z,t) = I(f)$.

The equation then becomes

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial z} I(v\mu f) = 0 , \tag{4.66}$$

since $D_{\mu\mu}$ vanishes at ± 1.

The term $J = I(v\mu f)$ looks like a current: we then have the conservation equation

$$\frac{\partial M}{\partial t} + \frac{\partial J}{\partial z} = 0 . \tag{4.67}$$

We can rewrite J using $\mu = -(1/2)\partial_\mu(1-\mu^2)$:

$$J = -\frac{1}{2}v\int \mathrm{d}\mu\, \frac{\partial}{\partial \mu} f \tag{4.68}$$

$$= +\frac{v}{4}\int_{-1}^{1} \mathrm{d}\mu\, (1-\mu^2)\frac{\partial f}{\partial \mu} . \tag{4.69}$$

If we integrate up to an arbitrary μ instead, we get

$$\frac{\partial}{\partial t}\int_{-1}^{\mu} \mathrm{d}\mu'\, f + v\int_{-1}^{\mu} \mathrm{d}\mu'\, \mu'\frac{\partial f}{\partial z} = D_{\mu\mu}\frac{\partial f}{\partial \mu} \tag{4.70}$$

$$(1-\mu^2)\frac{\partial f}{\partial \mu} = \frac{(1-\mu^2)}{D_{\mu\mu}}\frac{\partial}{\partial t}\int_{-1}^{\mu} \mathrm{d}\mu' \, f + \frac{v(1-\mu^2)}{D_{\mu\mu}}\int_{-1}^{\mu} \mathrm{d}\mu' \, \mu' \frac{\partial f}{\partial z}, \tag{4.71}$$

but we are close to isotropy, so

$$(1-\mu^2)\frac{\partial f}{\partial \mu} = \frac{(1-\mu^2)}{D_{\mu\mu}}(\mu+1)\frac{\partial M}{\partial t} + \frac{v(1-\mu^2)}{D_{\mu\mu}}\frac{1}{2}(M^2-1)\frac{\partial M}{\partial z}, \tag{4.72}$$

so we can multiply everything by $v/4$ and integrate in μ between -1 and 1:

$$J = \int_{-1}^{1} \mathrm{d}\mu \left(\frac{(1-\mu^2)}{D_{\mu\mu}}(1+\mu)\frac{\partial M}{\partial t} - \frac{v^2(1-\mu^2)^2}{8D_{\mu\mu}}\frac{\partial M}{\partial z} \right), \tag{4.73}$$

so we can define

$$k_{tt} = \frac{v}{4}\int_{-1}^{1} \mathrm{d}\mu \, \frac{(1-\mu^2)(1+\mu)}{D_{\mu\mu}} \tag{4.74}$$

$$k_{zz} = \frac{v^2}{8}\int_{-1}^{1} \mathrm{d}\mu \, \frac{(1-\mu^2)^2}{D_{\mu\mu}}, \tag{4.75}$$

therefore

$$J = k_{tt}\frac{\partial M}{\partial t} - k_{zz}\frac{\partial M}{\partial z}. \tag{4.76}$$

Together with the continuity equation we find

$$\frac{\partial M}{\partial t} = -\frac{\partial}{\partial z}\left(k_{tt}\frac{\partial M}{\partial t} - k_{zz}\frac{\partial M}{\partial z} \right), \tag{4.77}$$

but f will be close to isotropy — the simplest thing is a dipole, $f_0(1+\delta\mu)$, where δ is a number $\ll 1$:

$$J = \frac{1}{2}\int_{-1}^{1} \mathrm{d}\mu \, f_0 v \mu (\delta\mu) = \frac{f_0 v \delta}{3}. \tag{4.78}$$

If δ is small, then the k_{tt} part will be small: therefore, we find

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial z}\left(k_{zz}\frac{\partial M}{\partial z} \right). \tag{4.79}$$

Clarify this. The first term is $\sim vM$, the second is $\sim v^2 M/z$.

Wednesday 2021-12-22

Last time we derived the transport equation in pitch angle:

$$\frac{\partial f}{\partial t} + v\mu\frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f}{\partial \mu} \right), \tag{4.80}$$

where $D_{\mu\mu} = (1/2)\left\langle \Delta\mu\Delta\mu/\Delta t \right\rangle$.

This already contains a lot of information about what scattering does to cosmic rays.

If we have the classic system with a "gun" of particle emitting them into a plasma with Alfvén waves, those will be isotropized.

Because of this, after some time we expect to see the bulk velocity of the Alfvén waves and the cosmic rays to be the same.

This is not realistic in every situation: if the plasma is relativistic, the assumption of the particle velocity being much higher than that of the waves breaks down. This can be treated, but we will not discuss it in the course; the professor can provide the necessary references upon request.

After integrating the aforementioned equation between -1 and 1, as well as between -1 and μ, we found

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial z}\left(k_{zz}\frac{\partial M}{\partial z}\right) \tag{4.81}$$

$$k_{zz} = \frac{v^2}{8}\int \mathrm{d}\mu \, \frac{(1-\mu)^2}{D_{\mu\mu}}, \tag{4.82}$$

where the first equation tells us that M changes in time due to a process of *spatial* diffusion, which is caused (as the second equation tells us) by diffusion in pitch angle.

The second-order nature of this equation is a trademark of a diffusive process. With a heuristic line of reasoning we got a typical time for diffusion by 90°, called

$$\tau_{90} = \frac{1}{D_{\mu\mu}}, \tag{4.83}$$

which allows us to define $\lambda = v\tau_{90}$, while the correct expression is the one for k_{zz} above.

One thing we are still missing is the effect of the plasma! That is currently only described through the fact that the particles are diffusing in pitch angle due to Alfvén waves.

Can we not ignore the bulk velocity if we move to the correct frame? Yes! But, we cannot do the same if there are velocity *gradients*.

Suppose we have a nonrelativistic plasma velocity $u(z)$. The change in the particle velocity term $v\mu$ can be approximated as $v\mu \to u + v\mu$.

Risky! $u + v\mu$ could be larger than c! Wouldn't a better approximation be $v\mu$?

What about the momentum? We transform it as

$$\begin{bmatrix} E' \\ p'_z c \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} E \\ p_z c \end{bmatrix}. \tag{4.84}$$

So, the z-axis momentum p_z changes to $p'_z c = -\beta\gamma E + \gamma p_z c$, where $\beta = u(z)/c$.

The derivative of the phase space distribution changes to

$$\frac{\partial f}{\partial z} \to \frac{\partial f'}{\partial z'} = \frac{\partial f'}{\partial p'_z}\frac{\partial p'_z}{\partial z'} \tag{4.85}$$

$$= \frac{\partial f'}{\partial p'_z}\left(-\frac{\mathrm{d}u(z)}{\mathrm{d}c}\right)\frac{E'}{c} \tag{4.86}$$

$$v\mu \frac{\partial f}{\partial z} \to -(u + v\mu) \left(\frac{\partial f'}{\partial p'} \frac{E'}{c} \frac{\mathrm{d}u(z)}{\mathrm{d}c} \right) \tag{4.87}$$

$$= (u + v\mu) \frac{\partial f'}{\partial z'} - (u + v\mu) \frac{\partial f'}{\partial p'} \frac{E'}{c^2} \frac{\mathrm{d}u}{\mathrm{d}z'} \tag{4.88}$$

$$\approx (u + v\mu) \frac{\partial f'}{\partial z'} - v\mu \frac{\partial f'}{\partial p'} \frac{E'}{c^2} \frac{\mathrm{d}u}{\mathrm{d}z'} \,. \tag{4.89}$$

$u \ll v\mu$

We neglect the derivative of γ since $\gamma = 1 + \mathcal{O}\left(\beta^2\right)$, and we are working to first order around 0 in the β corresponding to the plasma.

The change of coordinates formulas read

$$\mathrm{d}\mu = \frac{1 - \mu^2}{p} \mathrm{d}p_z - \mu (1 - \mu^2)^{1/2} \mathrm{d}p_\perp \tag{4.90}$$

$$\mathrm{d}p = \mu \, \mathrm{d}p_z + (1 - \mu^2)^{1/2} \mathrm{d}p_\perp \,, \tag{4.91}$$

which means we get

$$\frac{\partial f}{\partial p_z} = \frac{\partial f}{\partial p} \mu + \frac{\partial f}{\partial \mu} \frac{1 - \mu^2}{p} \,, \tag{4.92}$$

therefore

$$v\mu \frac{\partial f}{\partial z} \approx (u + v\mu) \frac{\partial f}{\partial z} - v\mu \frac{E'}{c^2} \frac{\mathrm{d}u}{\mathrm{d}z} \left(\frac{\partial f}{\partial \mu} p + \frac{\partial f}{\partial p} \frac{1 - \mu^2}{p} \right) \,. \tag{4.93}$$

Now we can apply the exact same approach as before! We just added the term $u \, \partial f / \partial z$.

The term $\partial f / \partial \mu$ can be neglected, because f is assumed to be close to isotropy. If we do not neglect it, we only get higher order corrections.

This yields

$$\frac{vE'}{c^2} \frac{\partial \mu}{\partial p} \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu \, \mu^2 = \frac{1}{3} \frac{vE'}{c^2} \frac{\partial \mu}{\partial p} \,. \tag{4.94}$$

The object in front can be written in terms of the Lorentz factor of the particle Γ:

$$\frac{vE'}{c^2} = \frac{v m_p c^2 \Gamma}{c^2} = p \,, \tag{4.95}$$

so the final result is

$$\frac{\partial M}{\partial t} + u \frac{\partial M}{\partial z} - \frac{1}{3} \frac{\mathrm{d}u}{\mathrm{d}z} p \frac{\partial M}{\partial p} = \frac{\partial}{\partial z} \left(k_{zz} \frac{\partial M}{\partial z} \right) \,. \tag{4.96}$$

This is the transport equation for any non-thermal particles. The assumption of those being non-thermal is implicit in the assumption that $v \gg u$. This is conceptually important, since we need to answer the mechanism by which these particles emerge out of the thermal distribution.

The *injection problem* is about how we go from a thermal distribution to a thermal distribution with non-thermal particles.

The effect of the particles on the plasma is also something which we should consider. This would be a dependence of u and k_{zz} on the distribution of non-thermal particles.

The term $p\,\partial M/\partial p$ is sensitive to the *spectrum* of the particles! This can change the distribution of particles in p.

Exercise: suppose we have a system with particles injected and removed from surfaces orthogonal to k. Further, suppose that k_{zz} is very small: this means that there is a lot of diffusion, since $D_{\mu\mu}$ is very large.
This basically means that the particles are "glued" to the plasma. Therefore, we also get $\partial M/\partial t = 0$.
What happens to the distribution function in this case? Suppose there is a linear gradient in the plasma velocity,

$$u = u_0 + \frac{\Delta u}{L} z\,. \tag{4.97}$$

Hint: using the method of characteristics.
The expected result is to get adiabatic compression. If the velocity decreases we get energy increasing, and vice versa. This is the same as cosmological redshift.

The way this works is that since there is a velocity gradient, in some reference frame we need to transform the magnetic field, therefore we also get an electric field which does the work.

4.1 Particle transport

In nature there are basically two acceleration mechanisms:

1. when $\left\langle \vec{E} \right\rangle \neq 0$ — *regular processes*;
2. when $\left\langle \vec{E} \right\rangle = 0$ but $\left\langle E^2 \right\rangle = 0$ — *stochastic processes*.

Unipolar inductors are when we have a magnet spinning very fast generating an electric field. This might happen, for example, near a pulsar! In the ideal MHD approximation E is orthogonal to B, but near a pulsar this can be broken! Then, we get a parallel component. Another way this can happen is the accretion of a black hole.

Near a neutron star the magnetic field is roughly dipolar; if $E \parallel B$ we get an electric field which is about 8 orders of magnitude larger than the gravitational field. So, particles feeling this field are stripped off.

When this happens, the particles still feel the very strong magnetic field. So, electrons undergoing this process with a tilted dipolar magnetic field emit curvature radiation (bremsstrahlung).

We need to give a quantum description of such a magnetic field; the virtual photons of this field can do pair production.

This process can repeat for 10^4 to 10^6 times: electrons stripped from the star are multiplied, and we get a lot of positrons as well. These particles form a plasma of their own!

This will not be really discussed in this course, but there will be a short course about it.

Ideal MHD is locally violated, giving rise to regular processes, in plasma reconnection as well.

Magnetic flux is not conserved anymore, and we get heating of particles.

The rest of this course will mainly be about **stochastic** processes. We will discuss *second order Fermi acceleration*, the first investigation of particle acceleration historically.

Fermi, after getting the Nobel in '38, moved to Chicago and then worked on the Manhattan Project.

After the war, he started to work on many new problems. The professor has met the last student of Fermi, Simpson.

At uni Chicago there is a quadrangle, which in 1948 was used to hold important dinners. Enrico Fermi was not there a few minutes from midnight on New Year's, at which point he came running and sweaty, having understood how cosmic rays were accelerated. The idea came to Fermi after Alfvén gave a talk about waves.

The model by Fermi does not really work, but it was important historically.

The momentum of the particles in an Alfvén wave will change by $\Delta p / p \sim v_A / c$, but this will happen stochastically with

$$D_{pp} = \left\langle \frac{\Delta p \Delta p}{\Delta t} \right\rangle = \frac{p^2}{T^2} \left(\frac{v_A}{c} \right)^2 , \tag{4.98}$$

so the acceleration timescale is $\tau_{\text{acc}} = p^2 / D_{pp} = (c/v_A)^2 T$. This is too long to actually be the origin of cosmic rays, but in 1948 this was not clear.

This is called second-order because of two reasons: the square in (c/v_A), and the fact that the diffusive motion can both increase and decrease the momentum.

Suppose we have regions of plasma moving in different directions. Also suppose we have an electric field E in the reference frame of the laboratory. A particle will sometimes "interact" with one of these magnetized regions, in a purely elastic way.

We introduce β and γ for the plasma clouds, with $\beta \ll 1$.

After the scattering with the cloud, the energy of the particle in the reference frame of the cloud will become

$$E' = \gamma E + \beta \gamma p \mu . \tag{4.99}$$

Suppose that the scattering is purely elastic: μ is simply reversed. The cloud is acting like a mirror. This is not really necessary, it is just a simplifying assumption.

The final energy in the lab frame will be

$$E'' = \gamma E' + \beta \gamma p'_z \mu \tag{4.100}$$

$$= \gamma^2 E \left(1 + p^2 + 2\beta\mu \frac{p}{E} \right) , \tag{4.101}$$

The momentum is reversed, hence the plus sign.

where p/E is the velocity of the particle.

The quantity we are interested in is

$$\frac{E'' - E}{E} = \gamma^2 \left(1 + 2pv\mu + \beta^2\right) - 1 \tag{4.102}$$

$$\approx 2\beta^2 + 2\beta^2 + 2\beta v\mu \,. \tag{4.103}$$

In the relativistic approximation, $\beta \ll 1$ but $v \sim 1$.

What is the mean value of this? It can attain positive and negative values, depending on the scattering angle μ. This is the reason this process does not work!

If there is a situation for which μ is bound to only allow acceleration, that can work.

We need to average this, weighed by the probability to have an interaction with a given μ. The probability will be proportional to the relative velocity,

Why? If the particle is chasing the cloud it will catch it!

$$\mathbb{P}(\mu) = Av_{\text{rel}} = A\frac{\beta\mu + v}{1 + v\beta\mu} \approx A(1 + \beta\mu) \,, \tag{4.104}$$

and A must be $1/2$ for normalization. We can then compute

$$\left\langle \frac{\Delta E}{E} \right\rangle = \int_{-1}^{-1} \mathrm{d}\mu \, \frac{1}{2} \left(2\beta^2 + 2\beta^2 + 2\beta v\mu\right) (1 + \beta\mu) = \frac{8}{3}\beta^2 \,, \tag{4.105}$$

so we have a mechanism for acceleration, but it is $\propto \beta^2$. The typical velocity, as we know now, is of a few tens of km/s, which means we have something of the order 10^{-9}.

If the plasma is relativistic, even second order processes can be important, and actually for long gamma ray bursts they are believed to be the main mechanism.

The problem of injection, as mentioned by Fermi, is about the fact that we have nuclei as well as protons in cosmic rays. How do we deal with ionization losses? There is no solution to this problem, the mechanism does not really work.

This was understood about 30 years later. The thing we need to discuss are explosions and shock waves.

Monday 2022-1-10

We have seen how Enrico Fermi thought that there could be second-order acceleration in a plasma.

The problems yet to be resolved are

1. injection: how does one plasma particle become nonthermal?
2. the timescale for energy gain in second-order processes seems close to the timescale for energy loss through ionization, which can hamper the process as a whole.

Fermi recognized the “problem of injection” in the abstract of his first paper, meaning the second of these.

These problems are only now being solved with particle-in-a-cell simulations.

The people who studied these problems in the early days were the ones involved with the Manhattan Project: they were interested in the physics of bombs.

When such an explosion occurs, shock fronts form! These are “event horizons”, preventing the transmission of information in one direction.

At the beginning of the course, from the Vlasov equation we derived mass and momentum conservation, while neglecting energy conservation since we were thinking of adiabatic processes.

The continuity equation reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{u}\right) = 0\,, \tag{4.106}$$

which in a stationary situation translates to $\nabla \cdot \left(\rho \vec{u}\right) = 0$. In one spatial dimension, this simply means $\rho u = \text{const}$.

The momentum conservation equation reads

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} P\,, \tag{4.107}$$

which in 1D and without time-dependence reads

$$\rho u \frac{\partial u}{\partial x} = -\frac{\partial P}{\partial x}\,, \tag{4.108}$$

but because ρu is constant we get

$$\frac{\partial}{\partial x}\left(\rho u^2 + P\right) = 0\,, \tag{4.109}$$

the conservation of momentum, or Bernoulli's law.

The conservation of entropy for our isentropic system reads

$$u \frac{\partial}{\partial x}\left(P \rho^{-\gamma}\right) = 0 \tag{4.110}$$

$$\frac{\partial P}{\partial x} = \gamma \frac{P}{\rho} \frac{\partial \rho}{\partial x}\,, \tag{4.111}$$

but the derivative of P can be expressed in terms of ρu^2:

$$\frac{\partial}{\partial x}\left(\rho u^2\right) = -\gamma \frac{P}{\rho} \frac{\partial \rho}{\partial x}\,, \tag{4.112}$$

which we can manipulate and integrate by parts:

$$u \frac{\partial}{\partial x}\left(\rho u^2\right) = +\gamma \frac{P}{\rho} \rho \frac{\partial u}{\partial x} = \gamma P \frac{\partial u}{\partial x}\,. \tag{4.113}$$

Manipulating this further, we find

$$u \frac{\partial}{\partial x}\left(\rho u^2\right) = \gamma \frac{\partial}{\partial x} - \gamma u \frac{\partial u^2}{\partial x} \tag{4.114}$$

$$u \frac{\partial}{\partial x}(1-\gamma) = \gamma \frac{\partial}{\partial x} \tag{4.115}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{2}\rho u^3 + \frac{\gamma}{\gamma - 1} P u\right) = 0\,. \tag{4.116}$$

This is analogous to energy conservation ("the flux of energy is constant").

Through these equations we can connect two points in a solution; a trivial solution will be $u_1 = u_2$ and so on, but there may be another nontrivial solution.

We can define the sound speed as $c_s^2 = \gamma P/\rho$, the Mach number is $M = u/c_s$; it can be proven that if $M < 1$ only the trivial solution exists, while if $M > 1$ at least on one side of the boundary we are looking at there is a solution

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} \tag{4.117}$$

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\,. \tag{4.118}$$

This is a *shock front* solution of the plasma equations. If the shock is *strong*, meaning that $M_1 \gg 1$, the compression ratio ρ_2/ρ_1 approaches $(\gamma+1)/(\gamma-1)$, which for the $\gamma = 5/3$ of a monoatomic gas means we have a compression factor $r = \rho_2/\rho_1 = 4$.

What happens to the pressure? That tends to infinity! The plasma is heated in such a shock front.

What is happening is we are converting bulk motion ρu^2 into compression and heat P, in the Bernoulli equation!

The plasma is slowed down and heated. Since the pressure is $P \propto N k_B T$, the ratio of the pressures is also a ratio of temperatures!

The new temperature will be

$$T_2 = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}\frac{u_1^2}{\gamma}\frac{\rho_1}{P_1}T_1\,, \tag{4.119}$$

but $P_1 \propto T_1/m_p$, so we get

$$T_2 \sim \underbrace{\frac{2(\gamma-1)}{(\gamma+1)^2}}_{\text{order-1 normalization}} m_p u_1^2\,, \tag{4.120}$$

so the kinetic energy is being converted to thermal energy! The shock isotropizes the motion of the particles.

This is quite different in the atmosphere and in astrophysics. In the atmosphere the collision path length is quite short, while in astrophysics the cross-sections are way too small; still, we can observe these shocks, and they are quite thin!

Mainly, these are *collisionless* shocks. Instead of particle-particle interaction, we have interactions mediated by local electromagnetic fields.

The averaged approach we are using allows us to neglect the microphysics, but the price is that we don't know what the microphysics is!

Let us look at a shock at $x = 0$, in the reference frame in which this shock is stationary.

The $x < 0$ region is called *upstream*: a fast and cold plasma, whose Mach number is large (while the Mach number below the shock is forced to be < 1).

The $x > 0$ region is called *downstream*: slow and warm plasma, with velocity u_2.

The relative velocity between the two sides is $u_1 - u_2$.

If you are downstream, the shock is moving towards you; but so if you are upstream!

The idea is that photons coming from the other side will *always* be blueshifted! Maybe we got rid of the situations in which a particle could lose energy?

We are neglecting the magnetic field in that $B^2/8\pi$ is small compared to P or to ρu^2, however magnetic fields can be there! It is customary to define an Alfvén Mach number $M_{A,1} = u_1/v_{A,1}$. Typically, the sonic Mach number is of order 100, the Alfvén Mach number is of the order of 1000.

Notice that $\rho v_A^2 = B^2/4\pi$: the Alfvén velocity is determined by the energy density of the magnetic field, while $\rho c_s^2 = \gamma P$. The assumption that the magnetic field energy is small is not very good, really.

Particles diffuse slowing to $\sim v_A$ in the upstream reference, but that reference is approaching the shock with a speed larger than v_A, so particles will always come back to the shock eventually.

This holds true even for particles initially moving away from it!

Suppose we have a particle coming towards the shock with a pitch angle $0 \leq \mu < 1$; its energy in the downstream frame will be $E_d = \gamma(1 + \beta\mu)$, where $\beta = (u_1 - u_2)/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

The particle, in the downstream, has a chance of diffusing away to infinity, but also a chance to come back to the shock.

The energy the upstream sees is $E_u = \gamma^2 E(1 + \beta\mu)(1 - \beta\mu')$, where $-1 \leq \mu' < 0$.

Therefore, $E_u > E$ necessarily. In each crossing, *every single particle* which goes upstream-downstream-upstream gains energy!

But, what is the probability that a particle will have a certain pitch angle μ? The flux will read

$$\Phi = \int_{\mu=0}^{\mu=1} \mathrm{d}\Omega \, \frac{N}{4\pi} v\mu = \frac{Nv}{4} \,, \tag{4.121}$$

therefore

$$P(\mu) = \frac{4ANv\mu}{Nv} = 2\mu \,, \tag{4.122}$$

and $P(\mu')$ will be exactly the same.

Therefore, we can compute

$$\left\langle \frac{E_u - E}{E} \right\rangle = \int_0^1 \mathrm{d}\mu \int_{-1}^0 \mathrm{d}\mu' \, 2\mu 2\mu' \left(\gamma^2 \left(1 + \beta\mu\right) \left(1 - \beta\mu'\right) - 1 \right) = \frac{4}{3}\beta = \frac{4}{3} \frac{u_1 - u_2}{c} \,. \tag{4.123}$$

This is very nice! We have something to first order in the shock velocity, which is large, larger than the Alfvén speed by a couple orders of magnitude.

This mechanism is called DSA, Diffusive Shock Acceleration, or "first order Fermi" even though Fermi didn't develop this.

We assumed there is diffusion, but we don't care at all about the mechanism! The problem with diffusion is that it must be fast, fast enough that a particle can do several passes through.

The probability of returning from upstream is 1 or close to it, the thing that can happen for it to leave is that its Larmor radius gets so large it grows out of the system.

The incoming flux to the downstream will be

$$\varphi_{\text{in}} = \int_{-u_2/c}^{1} \mathrm{d}\mu \, f_0(u_2 + c\mu) = f_0 \frac{1}{2} \left(1 + \frac{u_2}{c}\right)^2 , \tag{4.124}$$

where now we must be careful to only consider particles which are actually moving downstream.

The particles which are reapproching the shock to get out will be

$$\varphi_{\text{out}} = \int_{-1}^{-u_2/c} \mathrm{d}\mu \, f_0(u_2 + c\mu) = f_0 \frac{1}{2} \left(1 + \frac{u_2}{c}\right)^2 = f_0 \frac{1}{2} \left(1 - \frac{u_2}{c}\right)^2 . \tag{4.125}$$

Therefore, the probability is

$$P_{\text{return}} = \frac{\varphi_{\text{out}}}{\varphi_{\text{in}}} = \frac{(1 - u_2/c)^2}{(1 + u_2/c)^2} \approx 1 - 4\frac{u_2}{c} . \tag{4.126}$$

This is quite sizeable, although it is not 1.

Counterintuitive scaling with u_2!

Suppose we start with N_0 particles with energy E_0: after one cycle their energy will be $E_1 = E_0(1 + \Delta E/E)$, which comes out to

$$E_1 = E_0 \left(1 + \frac{4}{3}\beta\right) , \tag{4.127}$$

and $N_1 = N_0 P_{\text{ret}}$ particles will have at least this energy.

$N_2 = N_0 P_{\text{ret}}^2$ particles will have at least $E_2 = E_0(1 + 4\beta/3)^2$ and so on.

We have

$$\log \frac{E_k}{E_0} = k \log \left(1 + \frac{4}{3}\beta\right) , \tag{4.128}$$

so

$$k = \frac{\log E_k/E_0}{\log(1 + 4\beta/3)} = \frac{\log N_k/N_0}{(1 - 4u_2/c)} . \tag{4.129}$$

Taylor expanding, we find

$$\log \frac{N_k}{N_0} = -\alpha \log \frac{E_k}{E_0} , \tag{4.130}$$

where $\alpha = 3/(r-1)$, with $r = u_1/u_2$.

The powerlaw only depends on r! For a strong shock, we get $r = 4$, so $\alpha = 1$.

This is the *integral spectrum*, while the differential spectrum $n(E)\,\mathrm{d}E$ will scale like $n(E) \propto N_k/E \propto E^{-(r+2)/(r-1)}$, which means E^{-2} with $r = 4$.

This only holds for relativistic particles! For non-relativistic ones, this does not hold as we shall see.

The fact that this produces a powerlaw is great! We observe many powerlaws for non-thermal particles.

The energy of these high-energy particle scales like $\mathcal{E} \sim \int_{E_0}^{E_{\max}} \mathrm{d}E \, A(E/E_0)^{-2} E \sim \log E_{\max}/E_0$.

If $E_{\max}$ is large enough, this quantity can become comparable with ρu^2, therefore the non-thermal particles will contribute a significant amount to the overall physics of the shock, invalidating the theory we used. This is the crux of the issue behind the *nonlinear physics* of shock acceleration.

What we need to do now is to get a connection with microphysics, which will give us a hint about where to go to deal with nonlinear theory.

Properly speaking, u_1 and u_2 should be replaced with $u_1 \pm v_{A,1}$ and so on. This is relevant because the compression factor inside the spectrum is really

$$r = \frac{u_1 \pm v_{A,1}}{u_2 \pm v_{A,2}}, \tag{4.131}$$

making the spectrum a bit harder or steeper. Tuesday 2022-1-11

Anywhere we have a supersonic plasma, we will form a shock front. The conditions at the shock are called Rankine-Hugoniot equations.

What the kinetic energy is dissipated into can vary: accelerated particles are a possibility, but we can also have heat, magnetic fields and so on.

The transport equation for the distribution function in one dimension reads

$$\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left(D \frac{\partial f}{\partial x} \right) + \frac{1}{3} \frac{\mathrm{d}u}{\mathrm{d}x} p \frac{\partial f}{\partial p} + Q(x, p), \tag{4.132}$$

where u is the velocity of the plasma, D the diffusion coefficient, while $Q(x, p)$ is the injection term: without it the equation is linear, so unless we put particles in there will not be any.

This distribution in phase space will be roughly isotropic, so that $f(p) \, \mathrm{d}p = 4\pi p^2 f(p) \, \mathrm{d}p$.

Conceptually, there is a neat separation between the plasma with its velocity u and the nonthermal particles with their distribution f.

The price to pay by inserting Q is to split the thermal and non-thermal particles.

The form of $u(x)$ near the shock looks like a step function. What happens to the non-thermal particles?

We can define these as those for which the shock seems to be infinitesimally small.

The thickness of the shock will be roughly of the order of the Larmor radius of the thermal particles, since the shock forms through EM interactions (while collisions are negligible).

A practical example is a SN explosion: it leads to a plasma moving at about 10 000 km/s.

The magnetic field in the field is roughly $B \sim 1\,\mu\mathrm{G}$; the Larmor radius is something like 10^9 to 10^{10} cm.

This is tiny compared to the scale of the remnant. The nonthermal particles are then chosen to be the ones such that their Larmor radius is much larger than the width of the shock.

This implies that f, the phase space distribution of the *nonthermal* particles, must be continuous across the shock: they move so fast that their properties don't have time to change discontinuously across the shock.

The distribution function of the thermal particles downstream will be thermal, where the mean particle will have a Larmor radius on the order of magnitude of the shock thickness.

The injected particles are those at the upper tail of the distribution, whose Larmor radii are roughly larger than the shock width, and which are then amplified.

The fraction of the particles which are accelerated is tiny!

The velocity gradient is modelled as

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (u_2 - u_1)\delta(x) . \tag{4.133}$$

The injection term can also be modelled as

$$Q(x, p) = Q_0 \delta(x) \delta(p - p_{\text{in}}) \frac{1}{4\pi p_{\text{in}}^2} , \tag{4.134}$$

since the part of the tail which is high enough to inject but still has particles in it is quite low.

We will assume stationarity, thereby neglecting the $\partial_t f$ term (and also taking Q not to be function of time as well).

The approach is to solve the equation upstream and downstream, using the shock as a boundary condition.

Upstream, we have

$$u_1 \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left(D \frac{\partial f}{\partial x} \right) \tag{4.135}$$

$$\frac{\partial}{\partial x} \left(\underbrace{u_1 f}_{\text{advective}} - D \underbrace{\frac{\partial f}{\partial x}}_{\text{current}} \right) = 0 , \tag{4.136}$$

where $D\partial f$ is the current term. This equation is basically the conservation of the total flux of the particles!

The source is downstream: upstream infinity is fast and cold, so there, in the interstellar medium we have $f = \partial f / \partial x = 0$: therefore, we can say that that term vanishes, so

$$u_1 f = D \frac{\partial f}{\partial x} . \tag{4.137}$$

We take

$$f(x, p) = A e^{\alpha x} = f_0(p) \exp(\alpha x) \tag{4.138}$$

$$\frac{\partial f}{\partial x} = \alpha f_0(p) \exp(\alpha x) , \tag{4.139}$$

therefore we have $u_1 f_0 = D \alpha f_0$, so $\alpha = u_1 / D$.

Upstream for us is $x < 0$; the distribution is therefore a decreasing exponential. The diffusion length is $1/\alpha = D(p)/u_1$.

What this is telling us is that particles cannot leave! They could leave if their Larmor radius became larger than the system, but this would lead to a peaked spectrum!

Downstream the distribution is weird: if there were gradients, it could not be stationary since we also have convection. So, we have $\partial f / \partial x = 0$ downstream.

Let us integrate the function in $[-\epsilon, +\epsilon]$ around 0: the term $u \, \partial f / \partial x$ must vanish, since the distribution is continuous.

We get

$$D \left. \frac{\partial f}{\partial x} \right|_2 - D \left. \frac{\partial f}{\partial x} \right|_1 + \frac{1}{3}(u_2 - u_1) p \frac{\partial f_0}{\partial p} + k\delta(p - p_{\text{in}}) = 0 \,. \tag{4.140}$$

We know that $D \, \partial f / \partial x$ downstream is zero, so we get

$$-D \left. \frac{\partial f}{\partial x} \right|_1 = -D \frac{u_1}{D} f_0 \,, \tag{4.141}$$

therefore

$$-u_1 f_0 + \frac{1}{3}(u_2 - u_1) p \frac{\partial f_0}{\partial p} = 0 \tag{4.142}$$

everywhere but at p_{in} (that will only serve towards normalization):

$$\frac{\mathrm{d} f_0}{f_0} = \frac{3u_1}{u_2 - u_1} \frac{\mathrm{d} p}{p} \,, \tag{4.143}$$

or

$$f_0 = k p^{-3u_1/(u_2 - u_1)} \,. \tag{4.144}$$

We find the same powerlaw as yesterday! Also, the index is precisely $3r/(r-1)$. Here, however, the result is in momentum, and it also holds for non-relativistic particles. For a strong shock we find $f \sim p^{-4}$; the spectrum we found yesterday was

$$n(E) \, \mathrm{d}E = 4\pi p^2 f(p) \, \mathrm{d}p \,, \tag{4.145}$$

meaning that in the relativistic regime $E \sim p$ we get

$$n(E) \propto E^2 E^{-4} = E^{-2} \,. \tag{4.146}$$

In the non-relativistic case, we have $E \sim p^2/2m$, so

$$n(E) \propto p^2 p^{-4} \frac{\mathrm{d}p}{\mathrm{d}E} \propto E E^{-2} \frac{1}{p} \propto E^{-3/2} \,. \tag{4.147}$$

In momentum, the spectrum is always p^{-4}; in energy we must distinguish between the two cases.

Here, we do not have any suppression at high p: the powerlaw keeps going.

There are problems: the energy is infinite if we take a strong shock with $r = 4$. Even if we put a E_{max} by hand, we are completely neglecting the loss of energy by the plasma. The available energy is at most ρu^2!

The proper way to deal with this is a *nonlinear extension*; we will just try to give a rough idea.

The mass conservation equation reads $\rho_0 u_0 = \rho_1 u_1$; now we need to add a new term for the nonthermal particles to the momentum conservation equation, which now reads

$$\rho_0 u_0^2 + P_0 = \rho_1 u_1^2 + P_1 + P_{\text{CR}} \,. \tag{4.148}$$

We do not add this to the mass conservation since their mass is tiny. Further, we assume that the shock is strong, therefore the gas pressure is negligible:

$$\rho_0 u_0^2 = \rho_1 u_1^2 + P_{\text{CR}} \tag{4.149}$$

$$1 = \frac{u_1}{u_0} + \frac{P_{\text{CR}}}{\rho_0 u_0^2} \,. \tag{4.150}$$

The term $P_{\text{CR}}/\rho_0 u_0^2$ is the efficiency of CR production, ξ_{CR}: the velocity at the shock is damped when we add cosmic rays. The plasma slows down when it accelerates cosmic rays.

The velocity upstream near the shock will be smaller; the compression factor "seen" by the particles which remain close to the shock (since they have smaller Larmor radii) will be smaller, meaning that the spectrum is harder at smaller radii.

The backreaction makes the problem of energy divergence even harder! With the stationarity assumption the system is doomed to break, if we don't use a nonlinear theory.

This thing works relatively well if we have an efficiency $\lesssim 10\,\%$.

Is this process able to accelerate particles up to the very high energies we detect cosmic rays at?

The problems in raising the energy is in the lifetime of the source, or its scale. The conditions are $\tau_{\text{acc}} \lesssim \tau_{\text{age}}$ and $D/u_1 \lesssim R_{\text{source}}$. These two roughly give the same bound.

A rough estimate of D in the galaxy can be given through the abundance of spallation products; it comes out to be

$$D \approx 3 \times 10^{28}\,\text{cm}^2/\text{s} \left(\frac{E}{\text{GeV}}\right)^{1/2} \,. \tag{4.151}$$

What is u_1? It is the velocity of the shock (which is also the velocity of the ISM in the shock frame). This is typically $v \sim 10^9\,\text{cm/s}$ (or smaller).

The scale of the source is typically the velocity of the shock times its timescale: $\tau_{\text{source}} \approx 3 \times 10^{10}\,\text{s}$ (1 thousand years).

From here we get

$$\left(\frac{E}{1\,\text{GeV}}\right) < \left(\frac{\tau}{1000\,\text{yr}}\right)^2 \,. \tag{4.152}$$

If this is the case, we need a thousand years to get to a GeV! We need to get to a million GeV. This mechanism seems to be way too slow!

Once we expressed the diffusion coefficient as

$$D = \frac{1}{3} r_L v \frac{1}{\mathscr{F}(k)}\,, \tag{4.153}$$

so we can say that this quantity should be

$$\frac{1}{3} r_L v \frac{1}{\mathscr{F}(k)} \leq u_1^2 \tau_{\text{age}}\,, \tag{4.154}$$

meaning that the power at the resonant frequency should be

$$\mathscr{F}(k) > \frac{1}{3} \frac{r_L}{R_{\text{source}}} \frac{v}{u_1}\,, \tag{4.155}$$

so for the Larmor radius of a 1 PeV particle we get $r_L = pc/eB \approx 10^{19}\,\text{cm}$, while the source's scale is a few parsecs, so we get

$$\mathscr{F} > \frac{1}{3} \frac{10^{19}\,\text{cm}}{3 \times 10^{19}\,\text{cm}} \frac{3 \times 10^{10}\,\text{cm/s}}{10^{9}\,\text{cm/s}} \gtrsim 1\,, \tag{4.156}$$

meaning that there must be a *lot* of resonant power!

This will also mean that the perturbative approach to particle diffusion will be broken.

Something very different must be happening directly upstream! The accelerating particles are the only thing which can be doing something upstream of the shocks: are their own magnetic fields strengthening the diffusion? Friday 2022-1-14

4.1.1 The effect of a finite source size

We take our usual picture with the upstream and downstream in one dimension; we take a certain x_0 in the upstream (so $x_0 < 0$) such that when a particle reaches it it completely leaves the system.

Why does he draw the exponential as concave?

We know that the solution is exponential, and it satisfies

$$D \left.\frac{\partial f}{\partial x}\right|_1 = u_1 f_0\,. \tag{4.157}$$

We suppose that particles are leaving the surface at x_0 with a flux $f_* c/2 \approx u_1 f_0$, therefore $f_* \approx 2(u_1/c) f_0 \ll f_0$.

The free escape boundary condition is $f(x_0, p) = 0$.

The equation we want to solve, in a stationary situation, is always

$$u \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(D \frac{\partial f}{\partial x}\right) + \frac{1}{3} \frac{\mathrm{d}u}{\mathrm{d}x} p \frac{\mathrm{d}f}{\mathrm{d}p} + Q\,, \tag{4.158}$$

and we have already seen that

$$\left.\frac{\partial f}{\partial x}\right|_2 = 0\,, \tag{4.159}$$

since in the downstream the solution must be constant.

On the other hand, in the upstream

$$u\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right) \implies uf - D\frac{\partial f}{\partial x} = \text{constant}\,. \tag{4.160}$$

We now write the flux as

$$f = A\exp(\alpha x) + B\,, \tag{4.161}$$

imposing our conditions: they tell us that

$$u_1 A\exp(\alpha x) + u_1 B - DA\alpha\exp(\alpha x) = k\,, \tag{4.162}$$

therefore $\alpha = u_1/D$ and $B = k/u_1$.

The solution then reads

$$f = A\exp\left(\frac{u_1 x}{D}\right) + \frac{k}{u_1}\,, \tag{4.163}$$

but we also have

$$f(x=0) = A + k/u_1 \tag{4.164}$$

$$f(x=x_0) = A\exp\left(\frac{u_1 x_0}{D}\right) + \frac{k}{u_1} = 0\,. \tag{4.165}$$

Subtracting these, we get

$$A(1-\exp\left(\frac{u_1 x_0}{D}\right)) = f_0\,, \tag{4.166}$$

while

$$\frac{k}{u_1} = -\frac{\exp(u_1 x_0/D)}{1-\exp(u_1 x_0/D)}\,. \tag{4.167}$$

The full solution then reads

$$f(x,p) = \frac{f_0}{1-\exp(u_1 x_0/D)}\left(\exp\left(\frac{u_1 x}{D}\right) - \exp\left(\frac{u_1 x_0}{D}\right)\right) - \,. \tag{4.168}$$

The strategy is always to join these solutions at the boundary: integrating the solution there yields

$$D\left.\frac{\partial f}{\partial x}\right|_2 - D\left.\frac{\partial f}{\partial x}\right|_1 + \frac{1}{3}(u_2-u_1)p\frac{\partial f}{\partial p} = 0\,, \tag{4.169}$$

but the derivative of the distribution function evaluated at 0^- is

$$D \left.\frac{\partial f}{\partial x}\right|_1 = \frac{f_0}{1 - \exp(u_1 x_0 / D)} u_1 \,, \tag{4.170}$$

and we can approximate the exponential as

$$\exp\left(\frac{u_1 x_0}{D}\right) \approx 1 + \frac{u_1 x_0}{D} \,, \tag{4.171}$$

as long as the particles are leaving the system at a distance which is small compared to the diffusion length scale. The opposite limit, $|x_0| \gg u_1/D$, is the one in which we retrieve the situation we discussed previously.

Note that both of these could be included in the same system, since D depends on the particle momentum!

We can reframe the equation as

$$\frac{\mathrm{d}f_0}{f_0} = \frac{\mathrm{d}p}{p} \frac{3u_1}{u_2 - u_1} \frac{1}{1 - \exp(u_1 x_0 / D)} \,. \tag{4.172}$$

The solution is therefore

$$f_0(p) \propto \exp\left(\int_{p_{\min}}^{p} \frac{\mathrm{d}p'}{p'} \frac{3u_1}{u_2 - u_1} \frac{1}{1 - \exp(u_1 x_0 / D)} \right) . \tag{4.173}$$

There is a momentum p_* such that $|x_0| = D(p_*)/u_1$.

If $p \ll p_*$, we get $D/u_1 \ll |x_0|$, so the exponential is of a large negative quantity, meaning we can neglect it and find

$$f_0(p) \propto \left(\frac{p}{p_{\min}}\right)^{-3\frac{u_1}{u_1 - u_2}} . \tag{4.174}$$

This is the usual solution.

On the other hand, if $p \gg p_*$ we get the exponential of a quantity close to zero, which we can expand:

$$f_0(p) \sim \left(\frac{p_*}{p_{\min}}\right)^{-3r/(r-1)} \exp\left(- \int_{p_*}^{p} \frac{\mathrm{d}p'}{p'} \frac{3u_1}{u_1 - u_2} \frac{D}{u_1 |x_0|} \right) . \tag{4.175}$$

At a momentum close to p_*, we get an exponential suppression with respect to the powerlaw! The integrand in the big negative exponential is large (as the exponential $\exp(u_1 x_0 / D)$ is close to 1).

When particles have enough momentum to reach the "leaving surface", they simply leave!

Exercise: take the derivative of the distribution function at $x = x_0$. Then, at p very small compared to p_*, the escaping flux goes to zero. On the other hand, at very large p the flux also goes to zero!

This means that only particles with momentum very close to p_* leave! We get a peaked flux! Since f is zero there, we will only have the $D\,\partial f/\partial x$ term.

We also want to know how broad the distribution is.

In real systems, u_1 is not constant: it slows down in time, selecting lower and lower energies, and the peak at p_* is moving left. In the end, even though at a given time the spectrum is peaked, if we integrate over the size of the system a spectrum is still found.

We are approaching the problem that the accelerated particles are producing their own magnetic field, but far enough from the shock there will be few particles and the ones which reach this length will be able to leave.

A second exercise: what if we impose the boundary condition that $f(x_0, p)$ is not zero, but instead it equals the (small) background cosmic ray density in the galaxy? The PDE system to solve is

$$u_1 \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(D \frac{\partial f}{\partial x} \right) \tag{4.176}$$

$$D\left.\frac{\partial f}{\partial x}\right|_{x=0^-} = \frac{1}{3}(u_2 - u_1) p \left.\frac{\partial f}{\partial p}\right|_{x=0^-} \tag{4.177}$$

$$f(x = x_0) = g(p) = A\left(\frac{p}{p_{\text{min}}}\right)^{-\alpha} . \tag{4.178}$$

Let us assume the solution looks like

$$f(x, p) \propto \exp\left(\frac{x u_1}{D}\right)\left(\frac{p}{p_{\text{min}}}\right)^{-\beta(x)} , \tag{4.179}$$

and for convenience let us define the characteristic diffusion wavenumber $k = u_1/D$. Then, the first equation upstream tells us that

$$(\beta')^2 = \beta'' + k\beta' , \tag{4.180}$$

while the condition at the shock tells us that

$$(k - \beta') = \frac{1}{3}\frac{r-1}{r} k\beta . \tag{4.181}$$

Finally, the condition at the x_0 border tells us that

$$\beta(x = x_0) = \alpha_{\text{CR}} , \tag{4.182}$$

a fixed number.

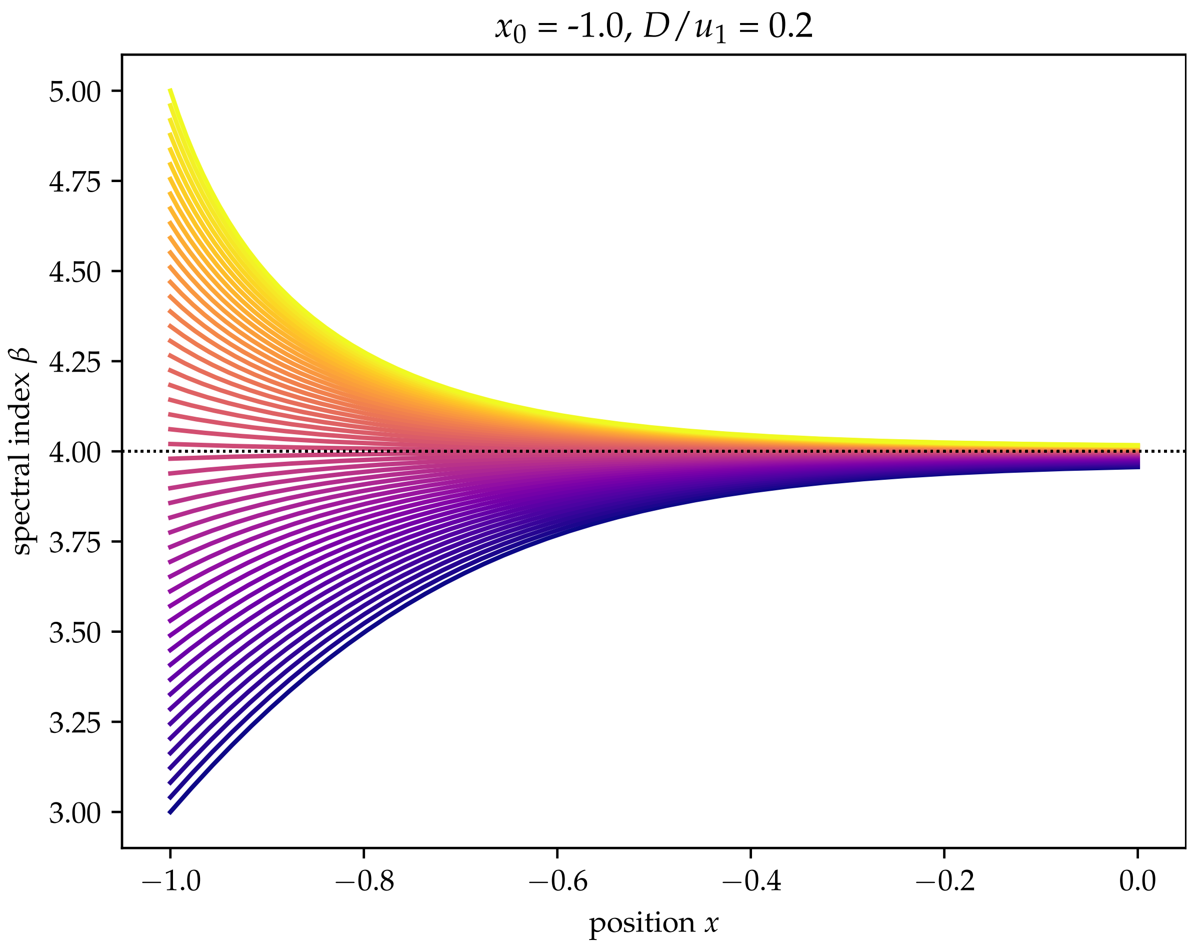

Figure 3: Reacceleration with $D/u_1 \gg |x_0|$.

We will get an interesting phenomenon: *reacceleration*. Particles from the galaxy are pushed up in energy.

We suppose to have a powerlaw spectrum:

$$f(x = x_0, p) = A(p/p_{\min})^{-\alpha}, \tag{4.183}$$

with the two cases of α being smaller or larger than $3r/(r-1)$.

We now want to give an estimate of the *acceleration time*: this is useful since we typically have information about the age of the systems which may produce cosmic rays.

The proper way would be to start from the transport equation, and move to a Laplace transform.

We then still join solutions at the boundary as usual; we then take the limit as $s \to 0$, which corresponds to infinite time.

This is very instructive: we find a decreasing exponential in time, in the form of $\exp(-t/\tau)$, where τ is the acceleration time. This will also yield a formal answer as to why the derivative of the distribution downstream must vanish.

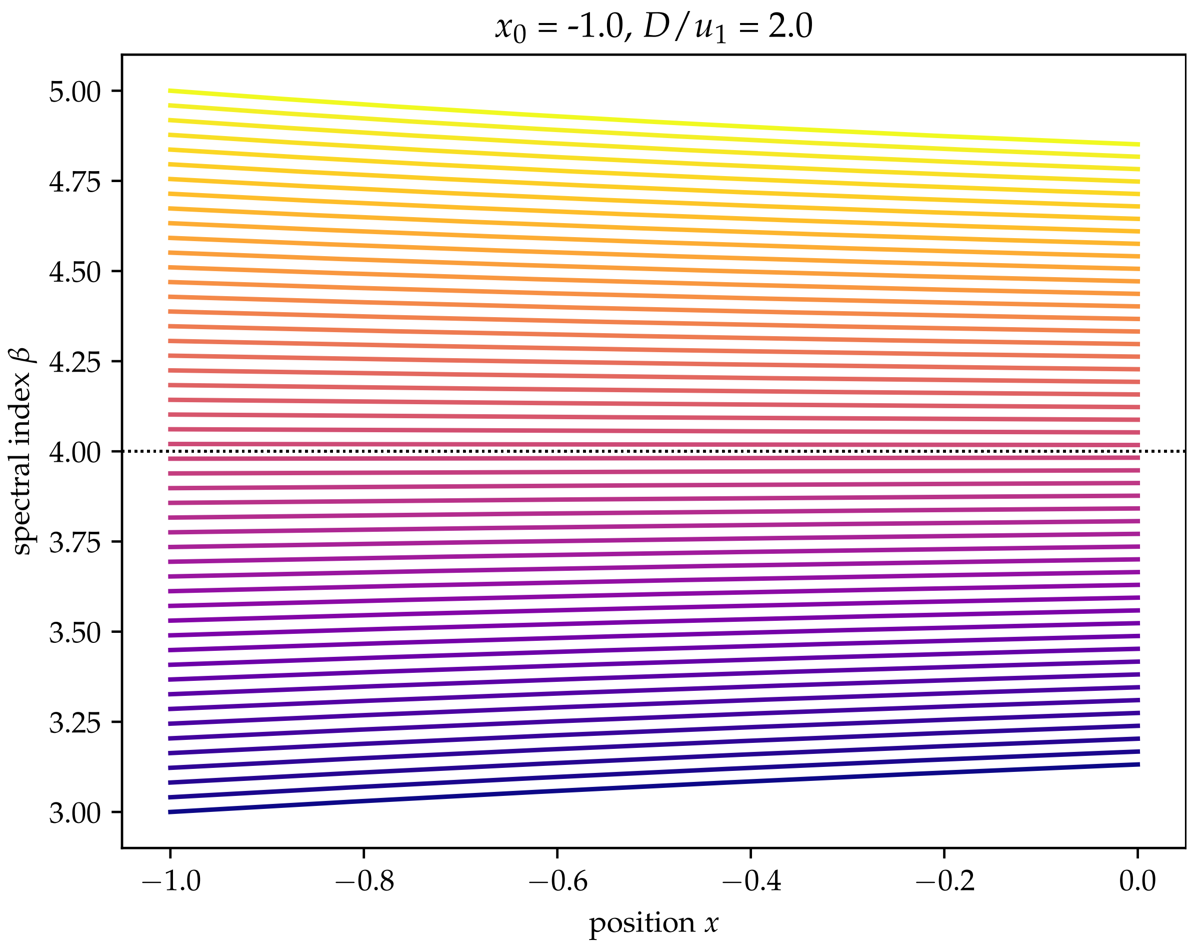

Figure 4: Reacceleration with $D/u_1 \lesssim |x_0|$.

We are welcome to try to do this exercise.

We will derive the acceleration time in a simpler way.

If the cosmic rays have a density n_{CR}, and supposing the flux per unit area is Σ, ??? will be

what? what is this quantity?

$$
\frac{n_{\text{CR}} v}{4} \Sigma \tau_1 = \frac{n_{\text{CR}} \Sigma D}{u_1} . \tag{4.184}
$$

Therefore,

$$
\tau_1 = 4 \frac{D}{u_1 v} , \tag{4.185}
$$

and we can do the same thing downstream, which yields $\tau_2 = 4D_2/vu_2$.

The total timescale is therefore $\tau = \tau_1 + \tau_2$.

We also know that

$$\frac{\Delta E}{E} = \frac{4}{3} \frac{u_1 - u_2}{v} , \tag{4.186}$$

so

$$\tau_{\text{acc}} = \frac{E}{\frac{\mathrm{d}E}{\mathrm{d}t}} = \frac{E}{\Delta E} \Delta t = \frac{3}{4} \frac{v}{u_1 - u_2} \frac{4}{v} \left(\frac{D}{u_1} + \frac{D}{u_2} \right) , \tag{4.187}$$

therefore

$$\tau = \frac{3}{u_1 - u_2} \left(\frac{D}{u_1} + \frac{D}{u_2} \right) , \tag{4.188}$$

while the Laplace transform approach yields

$$\tau_{\text{acc}} = \frac{3}{u_1 - u_2} \int_{p_{\min}}^{p} \frac{\mathrm{d}p'}{p'} \left(\frac{D}{u_1} + \frac{D}{u_2} \right) , \tag{4.189}$$

so, as an order of magnitude, $\tau \sim D/u_1^2$.

This expression tells us something important: even if D_2 is small, since downstream there is a mess, we still need D_1 to be small as well in order for the acceleration timescale to allow for decent particle acceleration.

We were always assuming that the particles were protons.

Synchrotron emission scales like E^2, so we cannot neglect it for electrons. Since electrons emit energy so fast, it is easier to see the effect of their acceleration.

We need to modify the transport equation to account for energy losses, and we need a way to inject them.

The temperature of the protons will be larger than the temperature of the electrons. The electrons will have much smaller Larmor radii, meaning that their injection will be harder.

How to account for energy losses?

$$u \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left(D \frac{\partial f}{\partial x} \right) + \frac{1}{3} \frac{\mathrm{d}u}{\mathrm{d}x} p \frac{\partial f}{\partial p} + Q - \frac{f}{\tau_{\text{losses}}(p)} , \tag{4.190}$$

this term, known as "catastrophic energy loss", is not really accurate; the proper way to do it would be

$$\frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 \dot{p} f \right) , \tag{4.191}$$

but if we write it that way there is no analytic solution. If the spectrum is a powerlaw, these two formulation are close to each other.

Let us integrate around the shock: the new term is continuous, so it integrates to zero and we get

$$D \frac{\partial f}{\partial x}\bigg|_2 - D \frac{\partial f}{\partial x}\bigg|_1 + \frac{u_2 - u_1}{3} p \frac{\partial f_0}{\partial p} . \tag{4.192}$$

Let us start by assuming that f is constant downstream; upstream on the other hand we have

$$\frac{\partial}{\partial x}\left(u_1 f - D\frac{\partial f}{\partial x}\right) = -\frac{\delta f}{\delta \tau_{\text{loss}}(p)}\,. \tag{4.193}$$

We can try to solve this with an exponential ansatz:

$$f = A\exp(x/\lambda) \tag{4.194}$$

$$\frac{\partial f}{\partial x} = \frac{A}{\lambda}\exp\left(\frac{x}{\lambda}\right), \tag{4.195}$$

which yields

$$A\frac{\partial}{\partial x}\left(\exp\left(\frac{x}{\lambda}\right) - \frac{D}{\lambda}\exp\left(\frac{x}{\lambda}\right)\right) = A\exp\left(\frac{x}{\lambda}\right) \tag{4.196}$$

$$\frac{1}{\lambda}\left(u - \frac{D}{\lambda}\right) = -\frac{1}{\tau_{\text{loss}}} \tag{4.197}$$

$$\lambda u\tau - D\tau + \lambda^2 = 0\,. \tag{4.198}$$

This means we have

$$\lambda = \frac{-u_1\tau + \sqrt{(u_1\tau)^2 + 4D\tau}}{2}\,, \tag{4.199}$$

where we must choose the plus solution since λ must be positive, so that we get a decreasing exponential.

The boundary term reads

$$-D_1\left.\frac{\partial f}{\partial x}\right|_1 = -D_1\frac{f_0}{\lambda}\,, \tag{4.200}$$

so we have

$$-D_1\frac{f_0}{\lambda} + \frac{u_2 - u_1}{3}p\frac{\partial f_0}{\partial p} = 0\,. \tag{4.201}$$

The terms in the square root we are comparing are: $(u\tau)^2$, the square of the distance the particles move over a loss time, and $D\tau$, the square of the distance they diffuse over a loss time.

Comparing these is equivalent to comparing

$$\frac{4D}{u_1^2} = \tau_{\text{acc}} \lesseqgtr \tau_{\text{loss}}\,. \tag{4.202}$$

Let us suppose that the acceleration timescale is indeed much smaller than the energy loss scale: then, we can approximate λ as

$$\lambda \approx u\frac{\tau}{2}\left(-1 + \sqrt{1 + \frac{4D\tau}{(u\tau)^2}}\right) \tag{4.203}$$

$$\approx \frac{u\tau}{2}\frac{2D\tau}{u^2\tau} = \frac{2D\tau}{u}\,, \tag{4.204}$$

so the result is the same as the case without losses.

What if, instead, the acceleration is slow compared to energy losses? The scale λ will be

$$\lambda \approx \sqrt{D\tau}\,. \tag{4.205}$$

This means that the particles are only getting as far from the shock as $\sqrt{D\tau}$, not very far.

In this case we get

$$f_0 \propto \exp\left(\int_{p_{\text{min}}}^{p} \frac{\mathrm{d}p'}{p'} \frac{3}{u_1 - u_2} \frac{D_1}{\lambda(p)}\right)\,. \tag{4.206}$$

Now, we know that

$$\tau_{\text{loss}} \sim \frac{E}{\Delta E/\Delta t} \sim \frac{E}{E^2} \sim \frac{1}{p}\,, \tag{4.207}$$

while D/u^2 is a growing function of p! We get two regimes! At low energies acceleration dominates, while at high energies losses start to dominate, and we get an exponential suppression of the spectrum as well.

For Bond diffusion, we get a hard exponential since $D \sim p$ while $\lambda \sim 1/p$: we get a suppression like $\exp\left(-(p/p_*)^2\right)$.

We assumed $\partial f/\partial x = 0$ downstream, but instead we can solve the equation like upstream now.

Next time we will briefly look at the supernova remnant paradigm, but that is not the only possibility.

We could do the same thing for protons, but we would get a much larger size than any concrete source.

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