High Energy Theoretical Astroparticle Physics

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Pasquale Blasi gives this part of the course. He might ask us to do exercises at the blackboard. Monday 2021-11-8, compiled 2021-11-18

Introduction

This is a theory course, but the professor wants to have lots of connections to the physical interpretation.

In general, in order to produce high-energy particles we need *acceleration* and *transport* mechanisms.

The particles we observe are *non-thermal*: the spectra from these phenomena seem to be powerlaws, not the exponentially suppressed tail of a Maxwellian.

Therefore, the system must be, to a fist, approximation, collisionless: if things happen *faster* than the collisions between the particles, they do not have enough time to thermalize.

We typically see charged particles making up cosmic rays, and the (interstellar, intergalactic) medium they move in is mostly ionized. Thus, they will move under the action of electromagnetic fields. Each of these charged particles also produces EM fields of its own: they all "feel what the others are doing".

There is a deep connection between the micro and the macro physics. David N. Schramm called this an "inner world — outer world" connection.

In order to understand the behaviour of cosmic rays, we need both particle physics and plasma physics!

Electroscope discharge: a brief history of the discovery of cosmic rays

What follows is a brief history lesson, roughly based on the work of de Angelis [dAng14].

In the late 1700s Coulomb discovered that an electroscope, left on its own, will discharge. This phenomenon seemed to decrease in speed as the pressure decreased: the air was being ionized, but by what?

In the late 1800s, after the discovery of the emission of charged particles by radioisotopes (which definitely managed to discharge electroscopes), ambient radioactivity was blamed for the phenomenon as a whole.

Domenico Pacini explored the variation of the rate of ionization as one moved underwater: it was decreasing. So, it seemed likely that the origin was extraterrestrial!

Later explorations, especially on balloons, showed a slight decrease followed by a sharp increase as altitude went above a couple kilometers. It took a while for the scientific community to accept this, but the origin of the discharge of electroscopes was not Earth-bound: it was cosmic.

Millikan though they might be the "birth cry" of elements: γ rays with energies corresponding to the mass defects of certain nuclides. He used Dirac's theory of Compton scattering. He was wrong, but he was also the first to use the term "cosmic rays".

Bruno Rossi put lead blocks between Geiger counters: these are not γ rays, since they would be absorbed.

They have to be charged: their flux is influenced by the Earth's magnetic field, so the one from the East and the West is different.

In the 1930s there started a boom of discoveries. At the end of the 30s Auger measured 100 TeV cosmic rays.

At the end of the 60s the CMB was discovered: therefore, in the cosmic ray spectrum there should be a cutoff around 10^{20} eV: the **GZK feature** [Alo+18, sec. 5.1]. This is because the cosmic ray "sees" CMB photons as γ rays, so it can undergo production of pions, losing a lot of energy, in a process like:

$$\mathrm{N} + \gamma \to \mathrm{N} + \pi^0 \,. \tag{0.1}$$

Specifically, the average CMB photon has an energy of $E \sim 2.7\,\mathrm{K} \approx 245\,\mu\mathrm{eV}$; in order for it to be blueshifted up to the mass of a pion, $m_{\pi^+} \approx 140\,\mathrm{MeV}$ it needs a γ factor of about $\gamma \sim 6 \times 10^{11}$; this is achieved when a proton has an energy of $E_p \approx 6 \times 10^{11}\,\mathrm{GeV}$.[1]

The threshold for pair production, $\sim 1\,\mathrm{MeV}$, is lower: "only" $E_p \approx 4 \times 10^9\,\mathrm{GeV}$. These are not hard thresholds: they are computed from the average temperature of CMB photons, but these are distributed according to a thermal distribution, so there is a tail at high energies, albeit exponentially suppressed.

There are peaks in the spectral power of the discovery of fossils: a 62 Myr period corresponds to the period of the oscillation of the Solar system with respect to the galactic disk.

There's lots of hydrogen and helium in the interstellar medium, almost no Beryllium, Boron and Lithium, while elements heavier than Carbon, which are formed in stars, are found in decent amounts.

On the other hand, in cosmic rays there is a much higher amount of Be, B, Li. This is due to *spallation*, or x-process nucleosynthesis: a process by which a cosmic ray hits a nucleus, thereby splitting it into lighter components. The cross-section for spallation is $\sigma \sim 45 A^{2/3}$mb, where A is the mass number of the nuclide.

The motion needs to be "diffusive". If this is happening, the timescale is quadratic in the distance, and matches with our observations; if the motion was "ballistic" (straight on, basically) the timescale would be linear in the distance travelled.

[1] A proper calculation shows that the expression also depends on the angle at which the particles interact, and specifically the threshold for the formation of a particle x by the interaction of a high energy cosmic ray with Lorentz factor γ with a CMB photon with energy ϵ is $\gamma \geq m_x / 2\epsilon(1 - \cos\theta)$, where θ is the interaction angle.

Such "ballistic" trajectories would contradict observations, since they would not be able to produce enough spallation product.

But, it cannot be collisions, otherwise the distribution would be thermal! So, what is it? Magnetic fields.

There is Ballmer emission in the shockwave from SNe.

The Larmor radius is in general given by

$$r_{\text{Larmor}} \approx \frac{\gamma m v_\perp}{|q||B|}, \tag{0.2}$$

where the magnitude of the galactic B-field is of the order of $100\,\text{pT}$.

Is this a typical value, an underestimate maybe?

This means that even at knee-level, $E \sim 3 \times 10^{15}\,\text{eV}$, the gyroradius is on the order of $3\,\text{pc}$: extremely *small*, in terms of the scale of the galaxy, $10\,\text{kpc}$ or more.

The loss length for radiation decreases with energy: first we can do pair production when a proton interacts with a CMB photon, and then pion production can start.

The *goal of the course* is to understand the behaviour of charged particles in a magnetic field they generate themselves.

Diffusive transport takes into account charged particles, as well as ordered and turbulent B fields, but plasma instabilities are what happen when charged particles interact with a turbulent magnetic field.

There are many experiments measuring this kind of stuff!

Course outline

The plan is to discuss:

1. basics of plasma physics;
2. basics of MHD;
3. basics of transport of charged particles;
4. basic aspects of the supernova paradigm for cosmic rays;
5. test particle theory of diffusive shock acceleration;
6. cosmic ray transport in the galaxy;
7. some non-linear aspects of particle transport;
8. a taste of advanced topics: recent findings, positrons, end of galactic CR, UHECR...

Regarding books, the course is a synthesis of several topics, so many books cover them but they also include lots of other material. Wednesday 2021-11-10, compiled 2021-11-18

For plasma physics, "Plasma physics for astrophysics" by Russel M. Kulsrud.

For transport, "Astrophysics of cosmic rays".

1 Basics of plasma physics

From the microphysical point of view, cosmic rays are just electric charges moving in a plasma, which they in turn affect.

Loosely, a plasma is ionized gas; however the interstellar medium is also at temperatures of $10^4\,\mathrm{K}$ to $10^6\,\mathrm{K}$, this is also true for the intergalactic medium, which has a much lower density and similar temperatures,

This also applies to the medium in clusters, which has a higher temperature, of the order of $10^8\,\mathrm{K}$ but lower densities, $10^{-3}\,\mathrm{cm}^{-3}$.

Magnetic fields are sourced by currents, which cosmic rays affect. Electric fields, on the other hand, are "short-circuited" since the conductivity is very large.

If it is difficult to have an electric field, how can particles be accelerated? We will simply assume that cosmic rays, which are non-thermal, exist.

First, though, we will try to understand how the plasma works, then we will look at the non-thermal particles, and finally we will put them together.

For simplicity, let us consider a plasma made of protons (with density n_i) and electrons (with density n_e). Maxwell' equations will read

$$\vec{\nabla} \cdot \vec{E} = 4\pi\zeta \tag{1.1a}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.1b}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{1.1c}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\,, \tag{1.1d}$$

where $\zeta = n_i e - n_e e$, while the current reads $\vec{J} = n_i e \vec{v}_i - n_e e \vec{v}_e$.

The interactions between these will be Coulomb ones. At thermal equilibrium, we will have charge neutrality, so $n_i = n_e = n_0$.

Screening Suppose we add a positive charge in a neutral medium, at $\vec{r} = 0$. Then, the divergence of $\vec{E}$ will be

$$\vec{\nabla} \cdot \vec{E} = 4\pi n_i e - 4\pi n_e e + 4\pi e \delta(\vec{r})\,, \tag{1.2}$$

where the tilde means that the densities are perturbed.

We will assume that the perturbation induced by this is small. The potential energy drop of a particle at a distance d from another is e^2/d, so if we assume $e^2/d \ll k_B T$ the thermal background remains fixed, and is not "broken" by Coulomb interactions. The typical distance between particles will be $d \sim n_0^{-1/3}$, so the weak perturbation condition will read $e^2 n_0^{1/3} \ll k_B T$.

What we want to do now is to solve this equation (1.2) for the electric field.

In order to do so, we need to discuss the dependence of the phase space distribution on the energy of each particle, which we call ϵ. At zeroth order, we will have

$$\widetilde{n}_i = n_0 \exp\left(-\frac{\epsilon}{k_B T}\right)\,, \tag{1.3}$$

which we assume, on mesoscopic scales, to average out to $\left\langle e^{-\epsilon/k_B T}\right\rangle = 1$. In the following discussion we apply this mesoscopic approximation by setting $\epsilon \equiv k_B T$, but we do leave in the external perturbation to the potential.

The external potential will perturb the energy ϵ by $e\varphi$ where φ is the electric potential, so

$$\widetilde{n}_i = n_0 \exp\left(\frac{-\epsilon + e\varphi}{k_B T}\right) \tag{1.4a}$$

$$\widetilde{n}_e = n_0 \exp\left(\frac{-\epsilon - e\varphi}{k_B T}\right), \tag{1.4b}$$

and we can use $\vec{E} = -\vec{\nabla}\varphi$: then, for $r \neq 0$ we will have

$$\vec{\nabla}^2 \varphi = -4\pi e n_0 \exp\left(-\frac{\epsilon}{k_B T}\right) \exp\left(-\frac{e\varphi}{k_B T}\right) + 4\pi e n_0 \exp\left(-\frac{\epsilon}{k_B T}\right) \exp\left(\frac{e\varphi}{k_B T}\right), \tag{1.5}$$

which we can expand up to linear order:

$$\begin{aligned}
\vec{\nabla}^2 \varphi &= -4\pi e n_0 \exp\left(-\frac{\epsilon}{k_B T}\right)\left(\exp\left(-\frac{e\varphi}{k_B T}\right) - \exp\left(\frac{e\varphi}{k_B T}\right)\right) &\text{(1.6a)}\\
&\approx -4\pi e n_0 \exp\left(-\frac{\epsilon}{k_B T}\right)\left(1 - \frac{e\varphi}{k_B T} - 1 - \frac{e\varphi}{k_B T}\right) &\text{(1.6b)}\\
&= 8\pi e \underbrace{n_0 \exp\left(-\frac{\epsilon}{k_B T}\right)}_{=n_0} \frac{e}{k_B T}\varphi\,. &\text{(1.6c)}
\end{aligned}$$

The prefactor had the dimensions of an inverse square length; further, we include the exponential $e^{-\epsilon/k_B T}$ into the unperturbed density n_0.

We are being a bit cavalier in the distinction between the phase space density $f(\vec{x}, \vec{p})$ and the spatial number density $n(\vec{x})$.

Roughly speaking, the first is defined so that its integral over all of $\mathrm{d}^3x\,\mathrm{d}^3p$ yields the total number of particles, while for the second the integral need only be done over d^3x. We can recover the number density by integrating the phase space density: specifically, the correct normalization in natural units reads

$$n(\vec{x}) = \frac{g}{(2\pi)^3}\int f(\vec{x}, \vec{p})\,\mathrm{d}^3p\ , \tag{1.7}$$

where g is the number of helicity states of the particle at hand.

The isotropic Maxwellian phase space distribution (which approximates both the Bose-Einstein and Fermi ones) reads $f(\vec{x}, \vec{p}) = \exp\left(-\epsilon(\vec{p})/k_B T\right)$, where $\epsilon(\vec{p}) = \sqrt{m^2 + \vec{p}^2}$ is the energy corresponding to the momentum $\vec{p}$.

The way the specific integral is computed for this density does not really matter (and is actually rather complicated in general), but we call the result n_0 — it is independent of $\vec{x}$, since the phase space density is as well.

Now, we insert an external potential energy term to the *phase space* density, mapping $\epsilon \to \epsilon + U(\vec{x})$ (which for us will be $U(\vec{x}) = -e\varphi(\vec{x})$). The integral then reads

$$n(\vec{x}) = \frac{g}{(2\pi)^3} \int \exp\left(-\frac{\epsilon}{k_B T}\right) \exp\left(-\frac{U(\vec{x})}{k_B T}\right) \mathrm{d}^3 p = n_0 \exp\left(-\frac{U(\vec{x})}{k_B T}\right). \tag{1.8}$$

We define the **Debye length**

$$\lambda_D = \left(\frac{k_B T}{8\pi n_0 e^2}\right)^{1/2}, \tag{1.9}$$

in terms of which (with the assumption of spherical symmetry for our problem) the equation reads

$$\nabla^2 \varphi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r}\right) = \frac{1}{\lambda_D^2} \varphi, \tag{1.10}$$

which is solved by looking at $f = r\varphi$: then,

$$\frac{\mathrm{d}^2 f}{\mathrm{d} r^2} = \frac{f}{\lambda_D^2}, \tag{1.11}$$

which means $f = A \exp(-r/\lambda_D)$ (we discard the unphysical, exponentially diverging solution). Inserting back our particle boundary condition to fix A, we get

$$\varphi = \frac{e}{r} \exp\left(-\frac{r}{\lambda_D}\right). \tag{1.12}$$

This is physically meaningful: there is a **screening effect** on charges, on length scales of λ_D.

For this to happen, however, we need to have enough charges to screen the inserted one: the number of particles in the Debye volume, $\sim n_0 \lambda_D^3$, must be much larger than 1. In a way, this is also a mesoscopic statistical requirement.

This can be written as

$$n_0 \frac{(k_B T)^{3/2}}{(8\pi e^2)^{3/2} n_0^{3/2}} = \frac{(k_B T)^{3/2}}{(8\pi e^2)^{3/2} n_0^{1/2}} \gg 1, \tag{1.13}$$

which shows that, counter-intuitively, this condition is easier to fulfill for under-dense plasmas.

The path length for Coulomb scattering, λ_C, should be much larger than both $\Delta r \approx n^{-1/3}$ (the separation between particles) and λ_D.

It can be estimated by

$$\lambda_C = \frac{1}{n_0 \sigma_C} = \frac{(k_B T)^2}{n_0 e^4} \gg n^{-1/3}, \tag{1.14}$$

where $e^2/b = k_B T$ gives us a limit for the Coulomb interaction length, therefore $\sigma_C \approx b^2 \approx (e^2/k_B T)^2$.

It can be shown that $\lambda_C \gg \lambda_D$ is equivalent to the condition that many particles should be contained in a single Debye length: the condition reads

$$\lambda_C = \frac{(k_B T)^2}{n_0 e^4} \gg \sqrt{\frac{k_B T}{8\pi n_0 e^2}} = \lambda_D \tag{1.15}$$

$$(8\pi)^2 n_0 \left(\frac{k_B T}{8\pi n_0 e^2}\right)^2 \gg \left(\frac{k_B T}{8\pi n_0 e^2}\right)^{1/2} \tag{1.16}$$

$$\lambda_D^3 \gg \frac{1}{(8\pi)^2 n_0} \,. \tag{1.17}$$

Collective effects dominate the dynamics of a plasma, while the effect of a single charge quickly becomes negligible.

Propagation modes in a plasma We will now do an exercise in perturbation theory: which perturbations are allowed, beyond electromagnetic waves? Certain modes are "allowed", in that they do not die out.

A lot of interesting physics come from the fact that each particle interacts with the collection of all the others.

Perturbations which are unstable are interesting, but they break our perturbative approach.

The fields in Maxwell's equations (1.1) are assumed to start out at zero in the unperturbed configuration.

The current J is given by the Generalized Ohm's law:

$$J_r = \sigma_{rs} E_s \,, \tag{1.18}$$

where the proportionality constant σ_{rs} is the *conductivity tensor* (whose components are, roughly, 1 over resistance).

It is also convenient to define the displacement current $\vec{D}$ by:

$$\frac{4\pi \vec{J}}{c} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \,, \tag{1.19}$$

which means that the current can be recovered by

$$\vec{J} = \frac{1}{4\pi}\left(\frac{\partial \vec{D}}{\partial t} + \frac{\partial \vec{E}}{\partial t}\right) . \tag{1.20}$$

The idea is to decompose the perturbation in Fourier modes:

$$\vec{E} \to \widetilde{E}_k(\vec{k}, \omega) \exp\left(-i\omega t + i\vec{k} \cdot \vec{r}\right) . \tag{1.21}$$

The full expression for the electric field is a superposition of these modes, but we can look at them just one at a time. This simplifies things: time derivatives become $i\omega$, curls become $\vec{k}\times$ and so on.

The current reads, with this as well as Ohm's law,

$$\widetilde{J}_r = \frac{-i\omega}{4\pi}\left(\widetilde{D}_r - \widetilde{E}_r\right) = \sigma_{rs}E_s\,. \tag{1.22}$$

The displacement field is therefore

$$\widetilde{D}_r = \frac{4\pi}{i\omega}\left(\frac{i\omega}{4\pi}\widetilde{E}_r - \sigma_{rs}E_s\right) = \widetilde{E}_r + i\frac{4\pi}{\omega}\sigma_{rs}\widetilde{E}_s = \mathbb{K}_{rs}\widetilde{E}_s\,, \tag{1.23}$$

where $\mathbb{K}_{rs} = \delta_{rs} + (4\pi i/\omega)\sigma_{rs}$ is called the **Dielectric tensor**.

What we are trying to do is to write a *dispersion relation*, $F(\vec{k},\omega)E = 0$.

Now we move to Lenz's law: we take another curl, to get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right) \tag{1.24a}$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\vec{D}}{\partial t}\right) \tag{1.24b}$$

$$= -\frac{1}{c^2}\frac{\partial^2\vec{D}}{\partial t^2} \tag{1.24c}$$

$$[-\vec{k}\times(\vec{k}\times\widetilde{E})]_r = +\frac{\omega^2}{c^2}\widetilde{D}_r = \frac{\omega^2}{c^2}\mathbb{K}_{rs}\widetilde{E}_s\,. \tag{1.24d}$$

We are almost done: the only issue here is that $\mathbb{K}_{rs}$ contains the conductivity tensor σ_{rs}.

Supposing for simplicity that our plasma is non-relativistic, we have the EoM

$$m_e\frac{\mathrm{d}v_e}{\mathrm{d}t} = -eE \implies -i\omega m_e\widetilde{v}_e = -e\widetilde{E}\,, \tag{1.25}$$

but the current $\vec{J}$ is $\vec{J} = -en_e\vec{v}_e$, so $\widetilde{J}_r = -(ne^2/\omega m_e)\widetilde{E}_r$, therefore the conductivity reads

$$\sigma_{rs} = \frac{ine^2}{\omega m_e}\delta_{rs}\,, \tag{1.26}$$

so we can write out the dielectric tensor explicitly:

$$\mathbb{K}_{rs} = \delta_{rs}\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right] \qquad \text{where} \qquad \omega_p = \sqrt{\frac{4\pi n_e e^2}{m_e}} \tag{1.27}$$

is called the plasma frequency.

We have found our dispersion relation:

$$\vec{k}\times\left(\vec{k}\times\vec{E}\right) + \left(\frac{\omega^2 - \omega_p^2}{c^2}\right)\widetilde{E} = 0\,. \tag{1.28}$$

Let us separate out *longitudinal perturbations*, which have $\vec{k} \propto \vec{E}$, from *transverse* ones, since we are always able to write $\vec{E} = \vec{E}_{\parallel} + \vec{E}_{\perp}$.

In the **longitudinal case**,

$$(\omega^2 - \omega_p^2)E_\parallel = 0\,, \tag{1.29}$$

which means that the propagation must happen exactly at the plasma frequency: these are called **Langmir waves**, or plasma waves. In the transverse case, we get

$$\left[-k^2c^2 + (\omega^2 - \omega_p^2)\right]E_\perp = 0\,. \tag{1.30}$$

The pulsation must be

$$\omega^2 = \omega_p^2 + c^2k^2 \qquad \text{or} \qquad k^2 = \frac{\omega^2 - \omega_p^2}{c^2}\,. \tag{1.31}$$

If $\omega > \omega_p$, then $k^2 > 0$: these are allowed perturbations, which indeed exhibit oscillatory behavior.

Actually, we can compute their group velocity

$$v_g = \frac{\partial \omega}{\partial k} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}, \tag{1.32}$$

which shows that if $\omega \gg \omega_p$ the speed is close to c.

If $\omega < \omega_p$, on the other hand, k is imaginary. The solution corresponding to exponential growth is unphysical (for one, there is no mechanism providing the energy for the magnitude of the oscillation to exponentially grow), so we only look at the exponentially damped solutions.

They are exponentially damped over a scale $\left|\vec{k}\right|^{-1} = c/\omega_p$. This is the **skin depth** of the plasma, the largest distance a perturbation can penetrate in the plasma if it oscillates too slowly.

This characteristic wavelength's ratio to the Debye length reads

$$\frac{\lambda_{\text{skin depth}}}{\lambda_{\text{Debye}}} = \sqrt{\frac{m_e c^2}{4\pi n_e e^2}\frac{8\pi n_e e^2}{k_B T}} = \sqrt{\frac{2m_e c^2}{k_B T}} \gg 1\,, \tag{1.33}$$

since the plasmas we are considering are typically non-relativistic.

An example: fast radio bursts Fast radio bursts are like γ-ray bursts in the radio band. These are relatively high-frequency, around 1.4 GHz. We do not really know what their sources are; some were false positives due to microwave ovens making lunch, but others were certified to be true detections.

We know of one which came from a galaxy $L \sim 1\,\text{Gpc}$ away, and which had a dispersion of about $\Delta\omega \sim 300\,\text{MHz}$.

Photons of different frequencies arrived at different times, with a spread of about 300 ms. We are in the second case ($\vec{k} \perp \vec{E}$) since they are EM waves. The group velocity is, again

$$v_g = c\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]^{1/2}, \tag{1.34}$$

and we can assume (we will check *a posteriori*) that $\omega \gg \omega_p$.

The time difference will be

$$\Delta t = \frac{L}{v_g(\omega)} - \frac{L}{v_g(\omega + \Delta\omega)} \approx \frac{L}{v_g(\omega)} \frac{1}{v_g(\omega)} \frac{\partial v_g}{\partial \omega} \Delta\omega \tag{1.35a}$$

$$\approx \frac{L}{c^2} \underbrace{c \frac{\omega_p^2}{\omega^3}}_{\approx \partial v_g / \partial \omega} \Delta\omega \tag{1.35b}$$

$$\omega_p \approx \sqrt{\frac{c\omega^3 \Delta t}{L \Delta\omega}}\,, \tag{1.35c}$$

but we know that

$$\omega_p^2 = \frac{4\pi n_e e^2}{m_e}\,. \tag{1.36}$$

This allows us to measure $\omega_p \sim 5.2\,\text{Hz}$, and therefore also the density, which comes out to be $n_e \sim 8.2 \times 10^{-9}\,\text{cm}^{-3}$.

How well does this match the baryon density computed from the CMB? The computation goes

$$\underbrace{\underbrace{\frac{\omega_p^2 m_e}{4\pi e^2}}_{n_e} m_p}_{\rho_{\text{plasma}}} \times \frac{1}{\Omega_{0b}\rho_c} \approx 0.03\,. \tag{1.37}$$

Therefore, from this observation we can estimate that about 3 % of baryonic matter is in the ISM.

This can be generalized a bit: maybe, not all the path L from the source to here had this much plasma in it. Suppose, for simplicity's sake, that a fraction α of the path was constituted by uniform-density plasma.

Then, our estimate for ω_p will be multiplied by a factor $\alpha^{-1/2}$, while our estimate for n_e will be multiplied by α^{-1}. On the other hand, in the estimate for ρ_{plasma}, averaged over the whole universe, will need to be shifted by some factor.

If the path we were looking at is a fair sample for the population (not a given, but let's approximate as such), then about a fraction α of the universe is filled with this plasma — this precisely cancels the correction to our estimate, so it all works out.

This is done in a better way through the measurement of several pulsars for which we have more accurate distance measurements.

Check: the frequency given was ν, not ω!

1.1 Statistical descriptions of plasmas

Thursday 2021-11-18

If the length scales we consider are larger than the Debye length, we can safely neglect the effect of the Coulomb potential of each individual particle. This does not mean that there are no electric fields, but the electric fields are mesoscopic or larger.

This is "desirable", in that we'd like to use statistical descriptions of the plasma. Such a statistical description will necessarily work in phase space.

Let us start out in the nonrelativistic approximation: if we only have one particle, we can write its phase space distribution function as

$$N(\vec{x}, \vec{v}, t) = \delta(\vec{x} - \vec{X}(t))\delta(\vec{v} - \dot{\vec{X}}(t)) . \tag{1.38}$$

If we have several particles, this can be readily generalized:

$$N(\vec{x}, \vec{v}, t) = \sum_i \delta(\vec{x} - \vec{X}_i(t))\delta(\vec{v} - \dot{\vec{X}}_i(t)) , \tag{1.39}$$

but this only describes a single particle species: we know that at the very least we will have electrons and protons, in order to preserve charge neutrality. So, let us call the quantity defined above N_s, where s is an index spanning $\{e, i\}$ for electrons and ions respectively.

We want to describe the evolution of this quantity: its **total** time derivative can be computed through the chain rule, and if we set it to zero we find that $\mathrm{d}N_s / \mathrm{d}t = 0$ is equivalent to:

$$\frac{\partial N_s}{\partial t} = -\sum_i \dot{\vec{X}} \cdot \nabla_x \delta(\vec{x} - \vec{X}_i(t))\delta(\vec{v} - \vec{V}_i(t)) - \sum_i \delta(\vec{x} - \vec{X}_i(t))\dot{\vec{V}}_i(t) \cdot \nabla_v \delta(\vec{v} - \vec{V}_i(t)) . \tag{1.40}$$

Now, these charges will evolve under the actions of the electric and magnetic fields: but we also need to describe what is the source of these fields.

The action of these fields on a particle with velocity $\vec{V}_i$ will be described by the Lorentz force,

$$m_s \dot{\vec{V}}_i(t) = q_s \vec{E}^{\text{microscopic}}(\vec{x}) + \frac{q_s}{c} \vec{V}_i \times \vec{B}^{\text{microscopic}}(\vec{x}, t) . \tag{1.41}$$

This acceleration term will then be put into the aforementioned evolution equation. We can already understand why this problem will be hard: the fields acting on a particle will be sourced by all the others.

The microscopic EM fields will satisfy Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E}^{\text{micro}}(\vec{x}, t) = 4\pi \zeta^{\text{micro}} \tag{1.42}$$

$$\vec{\nabla} \cdot \vec{B}^{\text{micro}} = 0 \tag{1.43}$$

$$\vec{\nabla} \times E^{\text{micro}} = -\frac{1}{c}\frac{\partial \vec{B}^{\text{micro}}}{\partial t} \tag{1.44}$$

$$\vec{\nabla} \times \vec{B}^{\text{micro}} = \frac{4\pi}{c}\vec{J}^{\text{micro}} + \frac{1}{c}\frac{\partial \vec{E}^{\text{micro}}}{\partial t} , \tag{1.45}$$

where the density and current density read

$$\zeta^{\mathrm{micro}}(\vec{x},t) = \sum_{s=i,e} q_s \int \mathrm{d}^3\vec{v}\, N_s(\vec{x},\vec{v},t) \tag{1.46}$$

$$\vec{J}^{\mathrm{micro}} = \sum_{s=i,e} q_s \int \mathrm{d}^3\vec{v}\, \vec{v} N_s(\vec{x},\vec{v},t)\,. \tag{1.47}$$

The Boltzmann equation plus the Lorentz one can be more compactly written as

$$\frac{\partial N_s}{\partial t} = -\vec{v}\cdot\vec{\nabla}_x N_s - \sum_{i=1}^{N} \frac{q_s}{m_s}\left[\vec{E}^{\mathrm{micro}} + \frac{1}{c}\vec{v}\times\vec{B}^{\mathrm{micro}}\right]\cdot\vec{\nabla}_v\left[\delta(\vec{v}-\vec{\dot{X}}_i(t))\right]\delta(\vec{x}-\vec{X}_i(t))\,. \tag{1.48}$$

The equation then becomes

$$\frac{\partial N_s}{\partial t} + \vec{v}\cdot\vec{\nabla}_x N_s = -\frac{q_s}{m_s}\left[\vec{E}^{\mathrm{micro}} + \frac{\vec{v}\times\vec{B}^{\mathrm{micro}}}{c}\right]\cdot\nabla_v N_s\,. \tag{1.49}$$

This equation is called the Klimontovich-Dupree equation.[2]

This equation, however, is basically useless, unless we do a mean-field approximation.

Let us introduce the quantity $f_s(\vec{x},\vec{v},t)$, which we want to compute through average on mesoscopic scales:

$$f_s(\vec{x},\vec{v},t) = \left\langle N_s(\vec{x},\vec{v},t)\right\rangle_{\Delta V}\,. \tag{1.50}$$

The true phase space density will then be

$$N_s(\vec{x},\vec{v},t) = f_s(\vec{x},\vec{v},t) + \delta f_s\,, \tag{1.51}$$

where the fluctuations are assumed to average to zero. We can write a similar expression for the electric and magnetic fields:

$$\vec{E}^{\mathrm{micro}} = \vec{E} + \delta\vec{E} \tag{1.52}$$

$$\vec{B}^{\mathrm{micro}} = \vec{B} + \delta\vec{B}\,. \tag{1.53}$$

The averaged KD equation then becomes:

$$\frac{\mathrm{d}f_s}{\mathrm{d}t} + \vec{v}\cdot\nabla_x f_s + \frac{q_s}{m_s}\left[\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right]\vec{\nabla}_v f_s = -\frac{q_s}{m_s}\left\langle\left(\delta\vec{E} + \frac{1}{c}\vec{v}\times\delta\vec{B}\right)\cdot\vec{\nabla}_v\delta f_s\right\rangle\,. \tag{1.54}$$

The interaction and collision terms are *quadratic* in the fluctuations. If we neglect this term (which is typically called a "correlation" term), we get the **Vlasov** equation:

$$\frac{\partial f_s}{\partial t} + \vec{v}\cdot\vec{\nabla}_x f_s + \frac{q_s}{m_s}\left[\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right]\vec{\nabla}_v f_s = 0\,. \tag{1.55}$$

[2] As many things done during the Cold War, it was developed by military personnel independently in the two blocks.

We can do the same thing to the Maxwell equations, using an averaged version of the charge and current densities.

We will need to generalize to the relativistic case: we move to $(\vec{x}, \vec{p})$ phase space. For relativistic particles, the Lorentz force reads

$$\vec{p} = q_s \left[\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right] . \tag{1.56}$$

The source terms in the Maxwell equations will be integrated in $\mathrm{d}^3 p$, but for the charge current we will have an integral $\int \mathrm{d}^3 p \, \vec{v} f_s$.

The relativistic Vlasov equation is then readily derived with minor modifications, and reads

$$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_s + q_s \left[\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right] \vec{\nabla}_p f_s = 0 . \tag{1.57}$$

Let us now move to the nonrelativistic plasma again, and assume we are at zero temperature. We will then make a small perturbation: the ions will be stationary in first approximation. Will the Vlasov equation contain the plasma waves we derived earlier?

We only write it for electrons, so

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_s - \frac{e}{m_e} \left[\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right]_\alpha \frac{\partial f}{\partial v_\alpha} = 0 , \tag{1.58}$$

where in the second term we are adopting the Einstein convention, summing over α.

We will assume that there is no magnetic field perturbation: this is the same thing we did in the plasma waves, specifically in assuming that $\mathbb{K}$ is diagonal.

is this correct?

In the stationary configuration there is $\vec{E} = 0$. To linear order, the perturbed equation will read

$$\frac{\partial \delta f}{\partial t} + \vec{v} \cdot \nabla \delta f + \frac{e}{m_e} \nabla \varphi_\alpha \frac{\partial f}{\partial v_\alpha} = 0 . \tag{1.59}$$

But isn't $f = \text{const}$ at zeroth order?

We want to assume that the plasma is cold!

The electric potential φ will satisfy

$$-\nabla^2 \varphi = -4\pi e \int \mathrm{d}^3 v \, \delta f . \tag{1.60}$$

We then move to Fourier space:

$$-i\omega \widetilde{\delta f} + i\vec{k} \cdot \vec{v} \widetilde{\delta f} + \frac{e}{m_e} i k_\alpha \widetilde{\varphi} \frac{\partial f}{\partial v_\alpha} = 0 , \tag{1.61}$$

but the electric potential will satisfy

$$k^2 \widetilde{\varphi} = -4\pi e \int \mathrm{d}^3 v \, \widetilde{\delta f} \,, \tag{1.62}$$

so we get

$$\widetilde{\delta f} \left[i\vec{k} \cdot \vec{v} - i\omega \right] = -\frac{e}{m_e} \widetilde{\varphi} k_\alpha \frac{\partial f}{\partial v_\alpha} \tag{1.63}$$

$$\widetilde{\delta f} = -\frac{e}{m_e} \widetilde{\varphi} \frac{k_\alpha \, \partial f / \partial v_\alpha}{\vec{k} \cdot \vec{v} - \omega} \,, \tag{1.64}$$

which we substitute into the integral for $\widetilde{\varphi}$:

$$\widetilde{\varphi} = \frac{4\pi e^2}{k^2 m_e} \widetilde{\varphi} k_\alpha \int \mathrm{d}^3 v \, \frac{\partial f}{\partial v_\alpha} \frac{1}{\vec{k} \cdot \vec{v} - \omega} \,, \tag{1.65}$$

so the allowed perturbations are those which make this equation true (for arbitrary φ). For now we have not assumed that the electrons are cold: this will enter in how we write the unperturbed f.

The assumptions of the electrons being cold can be modelled as $f = n_0 \delta(\vec{v})$: therefore, we need to integrate by parts.

Suppose that the z axis is along k: then, the integrand reads

$$\int \mathrm{d}v_x \, \mathrm{d}v_y \, \mathrm{d}v_z \, \frac{\partial f}{\partial v_\alpha} \frac{1}{k v_z - \omega} = \int \mathrm{d}v_x \, \mathrm{d}v_y \, \mathrm{d}v_z \, f \frac{1}{(k v_z - \omega)^2} \tag{1.66}$$

$$= \int \mathrm{d}v_x \, \mathrm{d}v_y \, \delta(v_x) \delta(v_y) n_0 \frac{k}{\omega^2} = \frac{n_0 k}{\omega^2} \,, \tag{1.67}$$

so the equation just reads

$$1 - \frac{4\pi e^2 n_0}{k^2 m_e} \frac{k^2}{\omega^2} \implies \omega^2 = \frac{4\pi e^2 n_0}{m_e} = \omega_p^2 \,. \tag{1.68}$$

At the very least, this more complicated approach allows us to recover the results we expected.

2 Alfvén waves

Next time, we will add one complication: a global, ordered magnetic field.

We know that this happens, for example, in spiral galaxies like our own: we observe large-scale magnetic fields.

Much of the physics of the transport of non-thermal particles will be affected by these magnetic fields.

The perturbation of the two coupled Vlasov equations under the effect of this external $\vec{B}$ field will yield what are called **Alfvén waves**.

We will also find a simplification of the

References

[Alo+18] Roberto Aloisio et al. “Selected Topics in Cosmic Ray Physics”. In: *Multiple Messengers and Challenges in Astroparticle Physics*. Ed. by Roberto Aloisio, Eugenio Coccia, and Francesco Vissani. Cham: Springer International Publishing, 2018, pp. 1–95. ISBN: 978-3-319-65423-2 978-3-319-65425-6. DOI: 10.1007/978-3-319-65425-6_1. URL: http://link.springer.com/10.1007/978-3-319-65425-6_1 (visited on 2021-11-08).

[dAng14] Alessandro de Angelis. “Atmospheric Ionization and Cosmic Rays: Studies and Measurements before 1912”. In: *Astroparticle Physics* 53 (Jan. 2014), pp. 19–26. ISSN: 09276505. DOI: 10.1016/j.astropartphys.2013.05.010. URL: https://linkinghub.elsevier.com/retrieve/pii/S0927650513000856 (visited on 2021-11-08).