Theoretical Gravitation and Cosmology

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1 General Relativity basics

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Course given by Andrea Maselli. There's lots of stuff to say, and the course cannot possibly be comprehensive.

The exam can be two things: there will be a list of "old" papers, literature by now, to study in more detail. We can prepare a presentation on this, to be shared among each other.

The alternative is to study more "book-like" topics: say, the TOV solution.

Three standard reference books are

1. Bernard Schultz, "A first course in GR";
2. Clifford and Will, "Theory and experiments of gravitational physics", which is more experimentally oriented;
3. Ferrari, Gualtieri and Pani, "GR and its applications", which is very complete.

The equivalence principles

There is a plural in the title. Principia impose constraints which modify the theory, not the other way around.

The thing to be careful with is whether we are testing *principia* or a specific theory. For example, the Pound-Rebka redshift experiment was testing the equivalence principle.

Alternative theories, such as scalar-tensor theories which were pioneered by Brams and Dicke, could also be written satisfying the same equivalence principle.

We can then classify alternative theories of gravity according to the principles they satisfy.

Newton Equivalence Principle This is also dubbed "NEP".

"The quantity that I mean hereafter by the name of mass. . . "

He distinguishes "mass" and "weight", and his formulation is not a principle yet.

> In the Newtonian limit, the inertial and gravitational masses are all proportional.

Newton's theory says that a density $\rho(x(t), t)$ will source a gravitational potential

$$\nabla^2 \phi(x(t), t) = 4\pi G \rho \,, \tag{1.1}$$

and particles will move according to the law

$$m\ddot{x}(t) = -m\nabla\phi(x(t), t) \,. \tag{1.2}$$

The density ρ and the mass m are conceptually different: $\rho = \rho_{\text{act}}$ is the *active* gravitational density, the mass $m = m_{\text{in}}$ multiplying $\ddot{x}$ is the inertial mass, while the $m = m_{\text{pass}}$ multiplying $\nabla\phi$ is the passive gravitational mass.

We can show that the active and passive gravitational masses are proportional by exploiting the third of Newton's laws: consider two objects 0 and 1, the attraction forces exerted by 0 on 1 and respectively by 1 on 0 will be

$$f_1 = m_1^{\text{in}} a_1 = \frac{G m_0^{\text{act}} m_1^{\text{pass}}}{r^2} \tag{1.3a}$$

$$f_0 = m_0^{\text{in}} a_0 = \frac{G m_1^{\text{act}} m_0^{\text{pass}}}{r^2} \,, \tag{1.3b}$$

but the third law imposes $f_0 = f_1$, which means

$$\frac{m_0^{\text{pass}}}{m_0^{\text{act}}} = \frac{m_1^{\text{pass}}}{m_1^{\text{act}}} \,. \tag{1.4}$$

Therefore, these are proportional. Setting the proportionality constant to 1 is just a matter of the units with which we measure them. Therefore, we can say $m_{\text{act}} = m_{\text{pass}} = m_{\text{grav}}$.

If we have one more body, 2, the force exerted by 0 will be

$$f_2 = \frac{G m_0^{\text{act}} m_2^{\text{pass}}}{r^2} \,, \tag{1.5}$$

so the accelerations of bodies 1 and 2 will be $a_1 = f_1 / m_1^{\text{in}}$ and $a_2 = f_2 / m_2^{\text{in}}$.

If these are equal, then we get $m_1^{\text{pass}} / m_1^{\text{in}} = m_2^{\text{pass}} / m_2^{\text{in}}$.

A way to quantify discrepancies from this principle is

$$\eta = \frac{\frac{m_1^{\text{grav}}}{m_1^{\text{in}}} - \frac{m_2^{\text{grav}}}{m_2^{\text{in}}}}{\frac{m_1^{\text{grav}}}{m_1^{\text{in}}} + \frac{m_2^{\text{grav}}}{m_2^{\text{in}}}} \,. \tag{1.6}$$

We know from experiments that $|\eta| < 10^{-13}$. In the Newtonian language, this is purely a coincidence.

Consider a nonrelativistic object moving in an external gravitational field, so that

$$m_{\text{in}} \ddot{x} = m_{\text{grav}} g + \underbrace{\sum_k f_k}_{F} \,, \tag{1.7}$$

where f_k are all other forces affecting this body. We are assuming that g is a constant.

We can make a transformation $y(t) = x(t) - gt^2/2$, $t' = t$. The equation of motion will be

$$m_{\text{in}}\ddot{y} = (m_{\text{grav}} - m_{\text{in}})g + F\,. \tag{1.8}$$

If the equivalence principle holds, the equation of motion becomes $m_{\text{in}}\ddot{y} = F$. In this reference system, gravity disappears completely.

We can see hints of the locality of this procedure: we are neglecting any derivatives of g, which will never be truly constant. This is the very meaning of a "local inertial observer".

How does this work in GR? (we are actually doing this in the simplest possible, SR terms)

Weak Equivalence Principle

> "The trajectory of an uncharged test particle in a point of spacetime is independent of that particle's structure and composition."

What "test particle" means is that it has negligible self-gravity, and that it is small enough that it does not couple to the inhomogeneities of the gravitational field.

The first statement can be quantified by the compactness parameter: GM/c^2R. For the Sun, this is of the order of 10^{-6}.

The second statement can be quantified by saying that the size of the body should be small compared to the length scale of the curvature of spacetime.

These two constraints are logically independent. In GR, the term "mass" is quite difficult to deal with. Therefore, instead of discussing them we only talk of trajectories.

"Test experiment" means that its effects are "weak".

Einstein Equivalence Principle There are two alternative formulations.

> "The outcome of any non-gravitational test experiment is not affected locally, at any point in spacetime, by the presence of the gravitational field."

> "The outcome of any non-gravitational test experiment is independent of the position of the lab in space and of the velocity of the free-falling apparatus."

The second formulation more explicitly requires invariance under the Poincaré group.

In the WEP we are only talking about mechanical laws, while here we extend to all the laws of physics.

This means that we can always find a reference system which cancels gravity.

This is establishing the connection between local frames in a gravitational field, and frames in the absence of gravity.

This sounds a bit circular without a careful definition of a "gravitational experiment"...

Strong equivalence principle

"The outcome of any experiment is not affected at any point in spacetime by the presence of the gravitational field."

There is no proof that SEP implies GR, but GR surely satisfies it.

The EEP is what tells us that the theory must be a metric one, since it includes SR, which has the Minkowski tensor.

There must be (at least one) metric tensor, so that it can be made equal to the Minkowski tensor at each point in spacetime, up to a conformal transformation:

$$[g_1, g_2 \dots] \to [\phi_1(P)\eta, \phi_2(p)\eta, \dots] \,. \tag{1.9}$$

If this is the case, we will be able to rescale $\phi_i = C_i \phi(P)$, but then we can rescale our units so that $C_i = 1$, and the metric as $\overline{g} = \phi^{-1} g$, which finally yields the transformation $g \to \eta$.

In the end, then, there must be a reference frame such that

$$g_{\mu\nu}(P) = \eta_{\mu\nu} + \sum_\alpha \mathcal{O}\left(\left| x^\alpha - x^\alpha(P) \right|^2 \right) . \tag{1.10}$$

In a Local Lorentz Frame there is a family of preferred curves for $g_{\mu\nu}$, geodesics, which are straight lines: free-falling trajectories are straight lines for free-falling observers.

There are purely metric theories, with a single metric tensor, but also ones in which other fields mediate the interaction.

The presence of the LLF means that there is a frame such that, around a point, $\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}y^\mu\, \mathrm{d}y^\nu$.

What are the EoM for a free-falling particle? We know that they read $\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = 0$ in the LLF.

Suppose we then move to a different frame, $x^\alpha = x^\alpha(y)$: then the interval will transform as

$$\mathrm{d}s^2 = \underbrace{\eta_{\mu\nu} \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta}}_{g_{\alpha\beta}} \mathrm{d}x^\alpha\, \mathrm{d}x^\beta \ . \tag{1.11}$$

The EoM then reads

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial y^\alpha}{\partial x^\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \right) = \frac{\mathrm{d}^2 x^\beta}{\mathrm{d}\tau^2} \frac{\partial y^\beta}{\partial x^\beta} + \frac{\partial^2 y^\alpha}{\partial x^\beta \partial x^\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} \tag{1.12a}$$

$$= \frac{\mathrm{d}^2 x^\beta}{\mathrm{d}\tau^2} \underbrace{\frac{\partial y^\beta}{\partial x^\beta} \frac{\partial x^\rho}{\partial x^\beta}}_{\delta_\beta{}^\rho} + \underbrace{\left(\frac{\partial x^\rho}{\partial y^\alpha} \frac{\partial^2 y^\alpha}{\partial x^\beta \partial x^\gamma} \right)}_{\Gamma^\rho_{\beta\gamma}} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} \tag{1.12b}$$

$$= \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\tau^2} + \Gamma^\rho_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} \,. \tag{1.12c}$$

This is the **Geodesic equation**. The crucial thing is that Γ vanishes when the spacetime is flat.

The WEP implies the NEP. On the other hand, the NEP does not imply the WEP. A theory in which the equations of motion contain weird stuff and not just m_I / m_G satisfies the NEP and not the WEP.

Differential geometry

Monday 2021-11-15

A point in $\mathbb{R}^n$ is described by the n-tuple $(x_1, \dots, x_n)$. These are a dense set (and also a complete one).

Open sets in $\mathbb{R}^n$ are defined as follows: $S \subseteq \mathbb{R}^n$ is open if for all $x \in S$ we can make a ball $B_r(x)$ with $r > 0$ such that $B_x \subseteq S$, where a ball is a set

$$B_r(x) = \left\{ y \in \mathbb{R}^n \middle| \left| x - y \right| < r \right\}. \tag{1.13}$$

A topological space is a collection of open sets which is closed under arbitrary intersections and finite unions.

A map ρ between two sets M and N takes a point $x \in M$ to a point $y = f(x) \in N$.

If we consider a set $S \subseteq M$ we can look at the set of the images of S under f, denoted as $T = \rho(S) \subseteq N$. We can also define the inverse image, $S = \rho^{-1}(T)$.

Composition of maps between different sets can also be defined: suppose we have three spaces M, N, Z, with some maps $f \colon M \to N$ and $g \colon N \to Z$; their composition will be $g \circ f \colon M \to Z$.

A map of M is **into** N if all points of M are mapped to N; a map of M is **onto** N if all points of N have inverse maps to M.

A **continuous** map f between topological spaces M and N is one for which, given any point $x \in M$ such that $y = f(x) \in N$, there is an open set of N containing $f(x)$, which is the image of an open set of M.

This is less stringent than the ϵ-δ definition of continuity because there is no size requirement...??

A differentiable map is defined as follows: a function $f(x)$, where $x \in S$ and $S \subseteq \mathbb{R}^n$ is open, is of order $\mathcal{C}^k$ if all partial derivatives of order up to k exist and are continuous.

A **manifold** M is a collection of points such that each of them has an open neighborhood which has a continuous one-to-one map with an open set of $\mathbb{R}^n$. The number n is called the **dimension** of the manifold.

He talks about the locality of this definition as being about the fact that M is a subset of a larger space...

A coordinate system or *chart* is a pair (M, ρ), where M is an open set while ρ is a map.

Suppose we have two coordinate systems (U, f) and (V, g), where U and V overlap at least at a point (so, in a region as well).

Consider the region $f(U \cap V)$: in order to move from it to $g(U \cap V)$ we need to apply the map $g \circ f^{-1}$. This is, therefore, a **change of coordinates**, which we will be able to write as $\vec{y} = \vec{y}(\vec{x})$.

A $\mathcal{C}^k$ differentiable **manifold** is one for which

1. each point of M belongs at least to an open set (and a corresponding chart);

2. each chart is $\mathcal{C}^k$ related to other charts it overlaps with.

These changes of coordinates can be written as $y^i = F^i(x^j)$, for which we can define a Jacobian determinant:

$$J = \left| \frac{\partial F^i}{\partial x^j} \right| . \tag{1.14}$$

If $J \neq 0$ at a point P, there is an onto, one-to-one map in some neighborhood of P.

A **curve** is a path in M such that we associate a number, called the parameter, to each point in it, and this provides a map between it and a path in $\mathbb{R}^n$, called the image of the curve.

This is typically denoted as $\gamma\colon s \in [a, b] \to [x^1(s), \dots, x^\mu(s)]$.

The quantities

$$\frac{\mathrm{d}x^i}{\mathrm{d}s} = \begin{bmatrix} \frac{\mathrm{d}x^1}{\mathrm{d}s} & \dots & \frac{\mathrm{d}x^n}{\mathrm{d}s} \end{bmatrix} \tag{1.15}$$

are the *components of the tangent vector*.

Suppose we want to make a coordinate change, $x'(x)$: how does our tangent vector change? By the chain rule, it will read

$$\frac{\mathrm{d}x'^i}{\mathrm{d}s} = \frac{\partial x'^i}{\partial x^j} \frac{\mathrm{d}x^j}{\mathrm{d}s} . \tag{1.16}$$

This kind of vector is called a **contravariant vector**, since it transforms with the Jacobian matrix.

We want to show that directional derivatives form a vector space at a point P. Consider a curve with its parameter λ and a differentiable function $\phi(x_1, \dots, x_m)$. The derivative

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \frac{\partial \phi}{\partial x^i} \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} . \tag{1.17}$$

With this in mind, we can define the **directional derivative operator**

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\partial}{\partial x^i} . \tag{1.18}$$

However, we already know that the $\mathrm{d}x^i \Big/ \mathrm{d}\lambda$ are the *components of the tangent vector*.

If we have two curves, $x^i(\lambda)$ and $x^i(s)$, we can define a directional derivative for each of them. We can then sum these, by defining the sum as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} + \frac{\mathrm{d}}{\mathrm{d}s} = \left(\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} + \frac{\mathrm{d}x^i}{\mathrm{d}s} \right) \frac{\partial}{\partial x^i} . \tag{1.19}$$

This vector will still be tangent to the curve, and we will be able to find a third parameter μ such that

$$\frac{\mathrm{d}}{\mathrm{d}\mu} = \frac{\mathrm{d}x^i}{\mathrm{d}\mu} \frac{\partial}{\partial x^i} . \tag{1.20}$$

We can also scale the directional derivative operator:

$$a\frac{\mathrm{d}}{\mathrm{d}\lambda} = \underbrace{\left(a\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\right)}_{\mathrm{d}x^i/\mathrm{d}\sigma}\frac{\partial}{\partial x^i}\,. \tag{1.21}$$

This proves (?) that the space of directional derivatives is a vector space. This space is denoted as T_P, where $\mathrm{d}/\mathrm{d}\lambda$ is a vector.

The crucial idea here is that vectors at different points belong to different vector spaces, and cannot be directly compared.

Coordinate lines are the ones for which all but one of the coordinates we are using remain constant. What this means is that directional derivatives along these read

$$\frac{\mathrm{d}}{\mathrm{d}x^i} = \frac{\mathrm{d}x^j}{\mathrm{d}x^i}\frac{\partial}{\partial x^j} = \delta^j_i\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^i}\,. \tag{1.22}$$

This also means that any directional derivatives can be expressed (uniquely) as linear combinations of these ∂_i, which are therefore a basis for the tangent space at each point.

We have a one-to-one connection between the tangents of curves at a point P and the derivatives along a curve at P. Because of this, we associate the directional derivative $\mathrm{d}/\mathrm{d}\lambda$ to a tangent vector to a curve $x^i(\lambda)$.

We denote the i-th basis vector as $\vec{e}_{(i)} = \frac{\partial}{\partial x^i}$. The number between round brackets enumerates the vectors in the basis, it is not a spatial index.

We can then express any vector in a more conventional notation as $\vec{A} = A^i\vec{e}_{(i)}$. If we change coordinates, the basis will shift like

$$\vec{e}_{k'} = \widetilde{\Lambda}^j_{k'}\vec{e}_j\,, \tag{1.23}$$

where $\widetilde{\Lambda}$ is the inverse of the Jacobian.

One-forms A one-form $\widetilde{q}$ is a real-valued, linear function of vectors:

$$\widetilde{q}(a\vec{V} + \vec{W}) = a\widetilde{q}(\vec{V}) + \widetilde{q}(\vec{W})\,. \tag{1.24}$$

We can sum one-forms by summing their action, and multiply them by scalars by multiplying the result of their application by a scalar. These are then a vector space, which is called the *dual*: T^*_P.

The reason for the name is the dual symmetry: taking $\widetilde{q}(\vec{V}) \simeq \vec{V}(\widetilde{q})$.

The basis for the dual is called the conjugate basis $\widetilde{\omega}$, and it is convenient to select it so that $\widetilde{\omega}^{(i)}(\vec{V}) = V^i$. Alternatively, we can write this as $\widetilde{\omega}^{(i)}(\vec{e}_{(j)}) = \delta^i_j$.

Any 1-form can also be written in terms of the basis covectors, $\widetilde{q} = q_i\widetilde{\omega}^{(i)}$.

We can make the manipulation

$$\widetilde{q}(\vec{V}) = \widetilde{q}(V^i\vec{e}_{(i)}) = V^i\widetilde{q}(\vec{e}_{(i)}) = \widetilde{\omega}^{(i)}(\vec{V})\widetilde{q}(\vec{e}_{(i)}) = q^iV_i\,. \tag{1.25}$$

The conjugate basis is often denoted as $\widetilde{\omega}^{(J)} = \mathrm{d}\widetilde{x}^{(J)}$. Wednesday 2021-11-17

The application of a one-form on a vector, $\widetilde{q}(\vec{V}) = q_J V^J$, can be computed as a *contraction*. A one-form transforms with $\partial x^{\mu} / \partial x'^{\alpha}$, a vector transforms with $\partial x'^{\mu} / \partial x^{\alpha}$.

Since a basis vector transforms like

$$\vec{e}_{(\alpha')} = \Lambda^{\mu}{}_{\alpha'} \vec{e}_{(\mu)} \tag{1.26}$$

we have

$$q_J = \widetilde{q}(\vec{e}_{(J)}) = \widetilde{q}(\vec{e}_{(k')} \Lambda^{k'}{}_J) = \Lambda^{k'}{}_J \widetilde{q}(\vec{e}_{(k')}) = \Lambda^{k'}_J q_{k'} . \tag{1.27}$$

This shows that the one-form transforms with the inverse of the Jacobian.

An example of a one-form is the gradient of a scalar function ϕ:

$$\frac{\partial \phi}{\partial x^{(J')}} = \frac{\partial \phi}{\partial x^k} \frac{\partial x^k}{\partial x'^{J'}} = \frac{\partial \phi}{\partial x^k} \Lambda^k{}_{J'} . \tag{1.28}$$

Tensors A tensor of type (N, R) is a function which takes N one-forms, R vectors, and yields a number. We require that it is multilinear (linear in all its arguments),

One-forms are $(0, 1)$ tensors, vectors are $(1, 0)$ tensors.

The components of the tensor are defined by its application to the basis vectors / covectors of the vector space. For a $(0, 2)$ tensor, we have

$$F_{\alpha\beta} = F(\vec{e}_{(\alpha)}, \vec{e}_{(\beta)}) , \tag{1.29}$$

from which we can recover by linearity any application of the tensor: $F(\vec{A}, \vec{B}) = A^{\alpha} B^{\beta} F_{\alpha\beta}$.

We can then try to construct a basis for the tensor space such that

$$F = F_{\alpha\beta} \widetilde{\omega}^{(\alpha,\beta)} . \tag{1.30}$$

We do this by writing

$$F(\vec{A}, \vec{B}) = \left(F_{\alpha\beta} \widetilde{\omega}^{(\alpha,\beta)} \right) (\vec{A}, \vec{B}) = F_{\alpha\beta} \widetilde{\omega}^{\alpha} (\vec{A}) \widetilde{\omega}^{(\beta)} (\vec{B}) , \tag{1.31}$$

therefore we can write the $(0, 2)$ basis as an outer product:

$$\widetilde{\omega}^{(\alpha,\beta)} = \widetilde{\omega}^{(\alpha)} \otimes \widetilde{\omega}^{(\beta)} . \tag{1.32}$$

We can iterate this procedure to get a basis for any tensor space. How do tensors transform?

$$F = F_{\alpha'\beta'} \Lambda^{\alpha'}_{\gamma} \widetilde{\omega}^{(\gamma)} \otimes \widetilde{\omega}^{(\rho)} \Lambda^{\beta'}_{\gamma} = F_{\gamma\rho} \widetilde{\omega}^{(\gamma)} \otimes \widetilde{\omega}^{(\rho)} . \tag{1.33}$$

this looks wrong...

The transformation law is therefore

$$F_{\alpha'\beta'} \Lambda^{\alpha'}{}_{\gamma} \Lambda^{\beta'}{}_{\delta} = F_{\gamma\delta} , \tag{1.34}$$

or equivalently

$$F_{\alpha'\beta'} = \Lambda^{\rho}{}_{\alpha'}\Lambda^{\sigma}{}_{\beta'}F_{\rho\sigma}\,. \tag{1.35}$$

Tensors of the same kind can be added, and tensors of any order can be multiplied and contracted.

Tensors can be symmetric: in an algebraic sense, F is symmetric if $F(\vec{A},\vec{B}) = F(\vec{B},\vec{A})$. In terms of its indices, this means $F_{\alpha\beta} = F_{\beta\alpha}$.

We can symmetrize a tensor by $F_{(\alpha\beta)} = (F_{\alpha\beta} + F_{\beta\alpha})/2$.

Symmetric n-dimensional tensors have $n(n+1)/2$ independent components.

Tensors can be antisymmetric: in an algebraic sense, F is antisymmetric if $F(\vec{A},\vec{B}) = -F(\vec{B},\vec{A})$. In terms of its indices, this means $F_{\alpha\beta} = -F_{\beta\alpha}$.

We can antisymmetrize a tensor by $F_{[\alpha\beta]} = (F_{\alpha\beta} - F_{\beta\alpha})/2$.

Antisymmetric n-dimensional tensors have $n(n-1)/2$ independent components.

The **metric tensor**! We call its application to two vectors their scalar product, $g(\vec{A},\vec{B}) = \vec{A}\cdot\vec{B}$. It is symmetric, therefore it has 10 free components.

The distance between two points separated by an infinitesimal $\mathrm{d}s$ is

$$\mathrm{d}s^2 = \mathrm{d}s\cdot\mathrm{d}s = g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}\,. \tag{1.36}$$

A curve $\gamma\colon [a,b] \to \mathcal{M}$ can be measured: its length will be

$$s = \int_a^b \mathrm{d}s = \int_a^b \mathrm{d}\lambda\,\underbrace{\sqrt{g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda}}}_{\mathrm{d}s/\mathrm{d}\lambda}\,. \tag{1.37}$$

This gives the length of the **path**, which is invariant with respect to the parametrization.

Thanks to the metric tensor we can lower or raise indices: the map $\omega_V\colon U \to g(U,V)$ is a one-form by the properties we require of the metric, so we can say it is the "dual" form of V, typically just denoted as $V_{\mu} = g_{\mu\nu}V^{\nu}$.

In order to do the inverse, we need the inverse of the metric tensor, $g^{\mu\nu}$, which satisfies $g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho}$.

Consider the metric tensor

$$\eta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\,. \tag{1.38}$$

Let us rotate the coordinate system:

$$x^0 = x^{0'} \tag{1.39}$$
$$x^1 = r\cos\theta \tag{1.40}$$
$$x^2 = r\sin\theta\,, \tag{1.41}$$

where r and θ are polar coordinates for the unprimed system.

How does the metric transform?

$$g_{0'0'} = \Lambda^{\mu}_{0'}\Lambda^{\nu}_{0'}\eta_{\mu\nu} = \eta_{00} = -1 \tag{1.42}$$
$$g_{0'i'} = \Lambda^{\mu}_{0'}\Lambda^{\nu}_{i'}\eta_{\mu\nu} = 0 \tag{1.43}$$
$$g_{1'1'} = \Lambda^{\mu}_{1'}\Lambda^{\nu}_{1'}\eta_{\mu\nu} = \cos^2\theta + \sin^2\theta = 1 \tag{1.44}$$
$$g_{2'2'} = \Lambda^{\mu}_{2'}\Lambda^{\nu}_{2'}\eta_{\mu\nu} = r^2(\cos^2\theta + \sin^2\theta) = r^2 \,. \tag{1.45}$$

So, we find the line element in polar coordinates

$$g_{\alpha'\beta'} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{bmatrix} . \tag{1.46}$$

Covariant differentiation

Monday 2021-11-22

Consider a vector $\vec{V} = V^{\alpha}\vec{e}_{\alpha}$.

How do we compute $\partial\vec{V} \big/ \partial x^{\mu}$? It will read

$$\frac{\partial\vec{V}}{\partial x^{\beta}} = \frac{\partial V^{\alpha}}{\partial x^{\beta}}\vec{e}_{(\alpha)} + V^{\alpha}\frac{\partial\vec{e}_{(\alpha)}}{\partial x^{\beta}} \,. \tag{1.47}$$

What we need to argue now is that the second term is a vector just like the first one: we need to use the equivalence principle. We can move to a reference in which $g_{\mu\nu} = \eta_{\mu\nu}$, such that $\vec{e}_{(\alpha)}$ also become constant.

The new and old will be related through transformation matrices:

$$\vec{e}_{(\alpha')} = \Lambda^{\mu}_{\alpha'}\vec{e}_{\mu} \,, \tag{1.48}$$

so we will get

$$\frac{\partial}{\partial x^{\beta}}\vec{e}_{(\alpha)} = \frac{\partial}{\partial x^{\beta}}\left[\Lambda^{\mu'}_{\alpha}\right]\vec{e}_{(\mu')} \,. \tag{1.49}$$

This object can be expressed as a linear combination of the basis vectors in the new basis, and is therefore a vector. We call it

$$\frac{\partial\vec{e}_{(\alpha)}}{\partial x^{\beta}} = \Gamma^{\rho}_{\alpha\beta}\vec{e}_{(\rho)} \,. \tag{1.50}$$

This is an *affine connection*. With this expression, we write

$$\frac{\partial\vec{V}}{\partial x^{\beta}} = \frac{\partial V^{\alpha}}{\partial x^{\beta}} + V^{\alpha}\Gamma^{\rho}_{\alpha\beta}\vec{e}_{(\rho)} \,. \tag{1.51}$$

The formalism is as follows:

$$V^{\alpha}{}_{,\mu} = \partial_{\mu}V^{\alpha} = \frac{\partial V^{\alpha}}{\partial x^{\mu}} \tag{1.52}$$

$$V^{\alpha}{}_{;\mu} = \nabla_{\mu} V^{\alpha} = \frac{\partial V^{\alpha}}{\partial x^{\mu}} + V^{\beta} \Gamma^{\alpha}_{\mu\beta} \,. \tag{1.53}$$

The covariant derivative can also be written as

$$\nabla \vec{V} = \left[V^{\alpha}{}_{;\mu} \vec{e}_{(\alpha)} \right] \otimes \widetilde{\omega}^{(\mu)} \,. \tag{1.54}$$

Therefore, $\nabla \vec{V}$ is a $(1,1)$ tensor.

In a Local Inertial Frame, covariant and component derivatives are equal, since $\Gamma^{\mu}_{\nu\rho} = 0$. Also, the covariant derivative of a scalar is equal to its partial derivative.

We can also differentiate a one-form: this will be a $(0,2)$ tensor. Its components will read

$$q_{\alpha;\beta} = \partial_{\beta} q_{\alpha} - \Gamma^{\rho}_{\alpha\beta} q_{\rho} \,. \tag{1.55}$$

When taking the covariant derivative of a tensor with any amount of indices we need to include a Christoffel term for all indices.

Let us write out a mixed tensor's derivative:

$$\nabla_{\beta} T^{\mu}_{\nu} = T^{\mu}_{\nu,\beta} + \Gamma^{\mu}_{\rho\beta} T^{\rho}_{\nu} - \Gamma^{\rho}_{\nu\beta} T^{\mu}_{\rho} \,. \tag{1.56}$$

In GR, we always have $g_{\mu\nu;\rho} = 0$. Why is this? If we move to a LIF, it equals $\eta_{\mu\nu,\rho} = 0$.

The Christoffel symbols are also symmetric: $\Gamma^{\rho}_{\alpha\beta} = \Gamma^{\rho}_{(\alpha\beta)}$. There is an explicit expression for them in terms of the metric:

$$\Gamma^{\rho}_{\alpha\beta} = \frac{1}{2} g^{\rho\sigma} \left(g_{\sigma\alpha,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma} \right) . \tag{1.57}$$

Are the Christoffel symbols the components of a tensor? No. They vanish in the LIF, so if they were a tensor they would always be zero. If one tries to compute their transformation law, it comes out to depend on the mixed second derivatives of the coordinate change.

The fact that in the LIF $g_{\mu\nu,\alpha} = 0$ and $\Gamma^{\rho}_{\alpha\mu} = 0$ is the mathematical representation of the equivalence principle.

Parallel transport It helps us understand the intrinsic geometry of the manifold. We will do it for a vector along a path. We ask that, for each infinitesimal displacement, the displaced vector remains parallel to itself and does not change magnitude.

This means that we cannot, in general, define a *global* constant vector field, or "comb a hairy ball".

Suppose we have a curve parametrized by λ, whose tangent vector is

$$u^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda} \,. \tag{1.58}$$

We want to transport a vector $\vec{V}$ along it. How do we do it? We can move to a LIF, with coordinates ξ^{α}. If we move $\vec{V}$ by $\mathrm{d}\lambda$ along the curve, we will change it by

$$\frac{\mathrm{d}V^{\mu}}{\mathrm{d}\lambda} = \frac{\mathrm{d}V^{\mu}}{\mathrm{d}\xi^{\alpha}} \frac{\mathrm{d}\xi^{\alpha}}{\mathrm{d}\lambda} = u^{\alpha} \partial_{\alpha} V^{\mu} \,. \tag{1.59}$$

However, $\partial = \nabla$ in a LIF, so this non-tensorial expression is equal in this reference frame to the tensorial expression $u^\alpha \nabla_\alpha V^\mu$, which we set to zero in order to impose that the vector does not change.

This is often also written like $\nabla_{\vec{u}} \vec{V} = 0$, or

$$\nabla_{\vec{u}} V^\alpha = \frac{\mathrm{d}V^\alpha}{\mathrm{d}\lambda} + V^\mu u^\beta \Gamma^\alpha_{\mu\beta} = 0 \,. \tag{1.60}$$

In the presence of a curved spacetime, the components of the vector do change, depending on the curve!

Geodesics are curves which parallel-transport their own tangent vector: $u^\mu \nabla_\mu u^\nu = 0$. This means that

$$u^\beta u^\alpha_{;\beta} = \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \left[\frac{\partial x^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\rho} u^\rho \right] = \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha_{\beta\rho} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} \,. \tag{1.61}$$

Any affine transformation $\lambda \to a\lambda + b$ leaves this unchanged.

Can't we do any monotone differentiable transformation?

The curvature tensor

It is the only four-index tensor which can be built only from derivatives of the metric.

A way to do it is to try and build an object which transforms like a tensor from the (derivatives of the) Christoffel symbols.

The Riemann tensor is written as

$$R^\lambda_{\mu\nu\kappa} = -\left(\partial_\alpha \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\lambda_{\kappa\eta} \Gamma^\eta_{\mu\nu} - \Gamma^\lambda_{\nu\eta} \Gamma^\eta_{\mu\kappa} \right) . \tag{1.62}$$

It can be schematically represented as $R \sim \partial\Gamma + \Gamma\Gamma$. In the LIF, $\Gamma\Gamma = 0$ but the first part does not vanish; this allows it to be tensorial.

In that case, it is also written as

$$R^\alpha_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu} \right) . \tag{1.63}$$

Let us write out its symmetries in the $(0,4)$ version:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta} \,, \tag{1.64}$$

and also

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0 \,. \tag{1.65}$$

These reduce the number of independent components: it would have $4^4 = 256$, but the symmetries reduce that number to 20.

If we parallel-transport a vector along a loop of coordinate directions 1 and 2 it changes by

$$\delta V^\alpha = R^\alpha_{\beta 12} V^\beta \,. \tag{1.66}$$

So, we can see that the spacetime is globally flat iff $R^{\alpha}_{\beta\mu\nu} = 0$.

The Riemann tensor can also be defined as a measure of the non-commutativity of the covariant derivative:

$$\left[\nabla_{\mu}, \nabla_{\nu}\right] V^{\alpha} = R^{\alpha}_{\beta\mu\nu} V^{\beta} \,. \tag{1.67}$$

We will also need the Bianchi identities:

$$R_{\alpha\beta[\mu\nu;\lambda]} = 0 \,. \tag{1.68}$$

The stress-energy tensor

Wednesday 2021-11-24

A massive particle is characterized by a four-momentum $pp^{\mu} = mu^{\mu}$, where $u^{\mu} = \mathrm{d}\xi^{\mu}/\mathrm{d}\tau$, and ξ^{μ} are the flat spacetime coordinates in a LIF parametrizing the particle's trajectory.

This can be written as

$$p^{\mu} = mc\frac{\mathrm{d}\xi^{0}}{\mathrm{d}\tau}\left[1, \frac{\mathrm{d}\xi^{i}}{\mathrm{d}\xi^{0}}\right] = \left[mc\gamma, m\gamma v^{i}\right] . \tag{1.69}$$

Here, we are using $\gamma = (1 - v^2/c^2)^{-1/2}$, so that $E = \gamma mc^2$.

The stress-energy tensor Suppose we have a collection of n particles, with their locations $\xi^{i}_{n}(t)$.

We define the 00 component of the stress-energy tensor, or *energy density*, as

$$T^{00} = \sum_{n} c p^{0}_{n} \delta^{3}(\vec{\xi} - \vec{\xi}_{n}(t)) \,, \tag{1.70}$$

the *momentum density* T^{0i}/c through

$$T^{0i} = \sum_{n} c p^{i}_{n} \delta^{3}(\vec{\xi} - \vec{\xi}_{n}(t)) \,, \tag{1.71}$$

and the *momentum flux* as

$$T^{ij} = \sum_{n} p^{i}_{n} \frac{\mathrm{d}\xi^{j}}{\mathrm{d}\tau} \delta^{3}(\vec{\xi} - \vec{\xi}_{n}(t)) \,. \tag{1.72}$$

In more generality, we can define

$$T^{\alpha\beta} = \sum_{n} p^{\alpha}_{n} \frac{\mathrm{d}\xi^{\beta}_{n}}{\mathrm{d}\tau} \delta^{3}(\vec{\xi} - \vec{\xi}_{n}(t)) \,. \tag{1.73}$$

Since

$$p^{\alpha}_{n} = \frac{E_{n}}{c^{2}} \frac{\mathrm{d}\xi^{\alpha}_{n}}{\mathrm{d}t} \,, \tag{1.74}$$

we can write this as

$$T^{\alpha\beta} = c^2 \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(\vec{\xi} - \vec{\xi}_n(t))\,, \tag{1.75}$$

which tells us that this is a manifestly symmetric object.

In integral form, it reads

$$T^{\alpha\beta} = c \sum_n \int p_n^\alpha \frac{\mathrm{d}\xi^\beta}{\mathrm{d}\tau} \delta^4(\xi^\mu - \xi_n^\mu(\tau))\, \mathrm{d}\tau_n\ . \tag{1.76}$$

The 3D delta function is defined through

$$\int \mathrm{d}^3\xi\, \rho(\vec{\xi}) \delta^3(\vec{\xi} - \vec{\xi}_n) = \rho(\vec{\xi}_n)\,, \tag{1.77}$$

which tells us that the dimensions of a $[\delta^3]$ are $[\text{length}^{-3}]$; therefore the dimensions of T^{00} are indeed those of an energy density.

In the non-relativistic limit, $v^i \ll c$, the time component of each four-momentum is approximately $p_n^0 \approx m_n c$; so in this first approximation we find

$$T^{00} \approx \sum_n m_n c^2 \delta^3(\vec{\xi} - \vec{\xi}_n(t)) = \underbrace{\sum_n m_n c^2 \delta^3(\vec{\xi} - \vec{\xi}_n(t))\, c^2}_{\rho}\,. \tag{1.78}$$

The spatial components of the three-momentum, p^i, have dimensions of [energy/velocity].

Therefore, the components $[cT^{0i}] = [E/tS]$ have dimensions of the energy per unit time per unit area. This is therefore referred to as *energy flow* across a surface orthogonal to x^i.

Similarly, the T^{ij} have the dimensions of the flux of the i-th component of the momentum flowing across the unit surface orthogonal to axis ξ^j.

One can show that this object does indeed transform like a tensor, and its most general expression, in a non-flat frame x^μ reads

$$T^{\alpha\beta} = c \sum_n \int \frac{\mathrm{d}\tau_n}{\sqrt{-g}} p_n^\alpha \frac{\mathrm{d}x_n^\beta}{\mathrm{d}\tau_n} \delta^4(x^\mu x_n^\mu(\tau))\,. \tag{1.79}$$

In the flat case, $\sqrt{-g} = 1$. The invariant volume element is written as $\sqrt{-g}\,\mathrm{d}^4x$.

The δ-function has its own transformation rule:

$$\frac{\delta^4(x^\mu - x_n^\mu)}{\sqrt{-g}} = \frac{\delta^4(x^{\mu\prime} - x_n^{\mu\prime})}{\sqrt{-g'}}\,. \tag{1.80}$$

If we have a scalar field with a Lagrangian $\mathscr{L} = -\partial^\mu\phi\partial_\mu\phi/2 + V(\phi)$, we can associate a stress-energy tensor to it by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathscr{L}\,. \tag{1.81}$$

If we have a perfect fluid with energy density ϵ and pressure P, it will have a diagonal stress-energy tensor in its own rest frame:

$$T_{\mu\nu} = (\epsilon + P)u_\mu u_\nu + P g_{\mu\nu}\,. \tag{1.82}$$

For the electromagnetic field we get

$$T_{\mu\nu} = C\left(g^{\alpha\beta}F_{\alpha(\mu}F_{\nu)\beta} - \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\right), \tag{1.83}$$

where the electromagnetic tensor is written in terms of the four-potential A^μ:

$$F_{\alpha\beta} = 2\nabla_{[\alpha}A_{\beta]}\,. \tag{1.84}$$

We now seek a conservation law of some sort. Moving back to flat spacetime, we can compute its three-divergence:

$$\frac{\partial T^{\alpha i}}{\partial \xi^i} = \sum_n p_n^\alpha(t)\frac{\mathrm{d}\xi_n^i(t)}{\mathrm{d}t}\frac{\partial}{\partial \xi^i}\delta^3(\vec{\xi} - \vec{\xi}_n(t)) \tag{1.85}$$

$$= -\sum_n p_n^\alpha(t)\frac{\mathrm{d}\xi_n^i(t)}{\mathrm{d}t}\frac{\partial}{\partial \xi_n^i}\delta^3(\vec{\xi} - \vec{\xi}_n(t)) \tag{1.86}$$

$$= -\sum_n p_n^\alpha(t)\frac{\mathrm{d}}{\mathrm{d}t}\delta^3(\vec{\xi} - \vec{\xi}_n(t))\,. \tag{1.87}$$

On the other hand, by taking the derivative of the zeroth component we find

$$\frac{\partial T^{\alpha 0}}{\partial \xi^0} = \sum_n\left[\frac{\mathrm{d}p_n^\alpha}{\mathrm{d}t}\delta^3(\vec{\xi} - \vec{\xi}_n(t)) + p_n^\alpha(t)\frac{\mathrm{d}}{\mathrm{d}t}\delta^3(\vec{\xi} - \vec{\xi}_n(t))\right], \tag{1.88}$$

but the derivative of the four-momentum can be referred back to the quadri-force, which vanishes for an isolated system:

$$\frac{\mathrm{d}p_n^\alpha}{\mathrm{d}t} = \frac{\mathrm{d}p_n^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = F_n^\alpha\frac{\mathrm{d}\tau}{\mathrm{d}t}, \tag{1.89}$$

therefore we get

$$\frac{\partial T^{\alpha i}}{\partial \xi^i} = -\frac{\partial T^{\alpha 0}}{\partial \xi^0}, \tag{1.90}$$

which means that $T^{\alpha\mu}{}_{,\mu} = 0$: in flat spacetime, the divergence of the stress-energy tensor vanishes.

This is associated with a conservation law: for example, we can integrate $0 = T^{0\mu}{}_{,\mu}$ on a volume at fixed time (a $\xi^0 = \text{const}$ hypersurface),

$$\frac{\partial}{\partial \xi^0}\int_V T^{00}\,\mathrm{d}^3x = -\int_V \frac{\partial T^{0i}}{\partial \xi^i}\,\mathrm{d}^3x = -\int_{\partial V} T^{0i}n_i\,\mathrm{d}S\,. \tag{1.91}$$

If the system is isolated, the term on the boundary ∂V vanishes, which means that the total energy $E = \int T^{00}\,\mathrm{d}^3x$ is conserved. Similarly, we can find conservation laws for $P^i = \int T^{0i}\,\mathrm{d}^3x$.

We have so far only found this result for flat spacetime, or in a LIF, but we can now introduce the **principle of general covariance**:

"A physical law is true if:

1. with no gravity, it reduces to the laws of special relativity;
2. it is generally covariant, meaning that it can be expressed in a tensorial form."

What we can then do is generalize $T^{\alpha\beta}{}_{,\beta} = 0$ to $T^{\alpha\beta}{}_{;\beta} = 0$.

Be careful about this: it is *not a conservation law*! If one does all the calculations, it comes out to

$$\frac{\partial}{\partial x^{\mu}} \left(\sqrt{-g} T^{\mu\nu} \right) = -\sqrt{-g} \Gamma^{\nu}_{\lambda\mu} T^{\lambda\mu} . \tag{1.92}$$

The reasons this is not a conservation law are the gravitational field and gravitational waves.

If we include the Landau-Lifschitz pseudo-tensor $\tau_{\mu\nu}$, then *that* is conserved.

The Einstein Equations

Newtonian gravity is described by Poisson's equation: $\nabla^2 \phi = 4\pi G \rho$. We want to write an equation which reproduces this in the non-relativistic limit.

Let us try to take some of these limits: the weak, and stationary field limits. The low-velocity limit means that $\mathrm{d}x^i \Big/ \mathrm{d}\tau \ll c$, which also means that

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \ll c \frac{\mathrm{d}t}{\mathrm{d}\tau} . \tag{1.93}$$

Further, we take the *weak field limit*: we assume that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where the components of the perturbation $h_{\mu\nu}$ all have magnitudes much smaller than 1. We will then work to first order in this as well.

We can do this in the geodesic equation

$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2} + \Gamma^{\mu}_{\alpha\beta} \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\beta}}{\mathrm{d}\tau} = 0 . \tag{1.94}$$

Since the largest component of the four-velocity is the zeroth one (1.93) the most relevant contribution will be given by the $\Gamma^{\mu}_{00} u^0 u^0$ term.

Those Christoffel symbols read

$$\Gamma^{\mu}_{00} = \frac{1}{2} \eta^{\mu\nu} (2 h_{\nu 0,0} - h_{00,\nu}) . \tag{1.95}$$

By the stationary approximation, the time derivatives vanish; this yields $\Gamma^{\mu}_{00} = -h_{00}{}^{,\mu}$.

The geodesic equation is therefore

$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2} = \frac{1}{2} h_{00}{}^{,\mu} \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \right)^2 , \tag{1.96}$$

but the $\mu = 0$ equation just reads $\mathrm{d}^2 x^0 \Big/ \mathrm{d}\tau^2 = 0$, so we can write $x^0 = ct = \tau$ with an affine transformation.

The spatial equations then read:

$$\frac{1}{c^2} \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2} \partial^i h_{00} \,. \tag{1.97}$$

This looks very similar to Newton's law

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\partial^i \phi \,, \tag{1.98}$$

so long as we identify $\phi = -c^2 h_{00}/2$, or $h_{00} = -2\phi/c^2$ (up to an additive constant).

In the Newtonian case, the field reads $\phi = -GM/r$: this yields

$$g_{00} = \eta_{00} + h_{00} = -1 + \frac{2GM}{c^2 r} \,, \tag{1.99}$$

again up to an additive constant, which we can however neglect if we ask that the field vanishes at infinity.

We might be therefore tempted to write something like

$$\nabla^2 g_{00} = \frac{8\pi G}{c^4} T_{00} \,, \tag{1.100}$$

however this is not covariant (not even under Lorentz transformations).

The way to make this covariant is to write

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{1.101}$$

where the Einstein tensor $G_{\mu\nu}$ is made up of differential operators acting on the metric.

The guidelines to arrive at this equation are:

1. the expression must be tensorial;
2. the dimensions of $G_{\mu\nu}$ must be those of inverse length, so we can only have terms like $\partial g \partial g$, or $\partial \partial g$, unless we want to include a new characteristic length scale — therefore, we want to write $G_{\mu\nu}$ in terms of the Riemann tensor;
3. $G_{\mu\nu}$ must be symmetric;
4. $G_{\mu\nu}$ must satisfy $G^{\mu\nu}{}_{;\nu} = 0$, since the stress-energy tensor does;
5. the expression reduces to equation (1.100).

Only two quantities can be constructed in this way from the Riemann tensor: the Ricci tensor $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ and the term $R g_{\mu\nu}$, where $R = g^{\mu\nu} R_{\mu\nu}$.

With this, we can say that $G_{\mu\nu} = c_1 R_{\mu\nu} + c_2 g_{\mu\nu} R$.

Monday
2021-11-29

The Bianchi identities read $R_{\mu\nu[\rho\sigma;\alpha]} = 0$; if we contract them with $g^{\mu\lambda}g^{\nu\rho}$ we find

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\nu} = 0\,. \tag{1.102}$$

This tells us that in order for the Einstein tensor to satisfy $G^{\mu\nu}{}_{;\nu} = 0$ we should set $c_2/c_1 = -1/2$.

How do we find the absolute scaling? In the weak field limit, we know that $\left|T_{ij}\right| \ll |T_{00}|$, since we can write

$$T^{\mu\nu} = c^2 \sum_n \frac{p_n^\mu p_n^\nu}{E_n}\delta^3(\vec{\xi} - \vec{\xi}_n(t)) \tag{1.103}$$

$$= c^2 \sum_n m_n \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\delta^3(\vec{\xi} - \vec{\xi}_n(t)) \tag{1.104}$$

$$= \rho c^2 \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\,, \tag{1.105}$$

where $\rho = \sum_n m_n \delta^3(\vec{\xi} - \vec{\xi}_n(t))$ is the density. Then we can use the fact that

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{v^i}{c}\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \ll \frac{\mathrm{d}x^0}{\mathrm{d}\tau}\,. \tag{1.106}$$

This means that, because of the equation we want to write, we must also have $\left|G_{ij}\right| = \left|c_1(R_{ij} - (1/2)g_{ij}R)\right| \ll |G_{00}|$.

If we approximate $g_{ij} \sim \eta_{ij}$, we must also have $R_{ij} \sim (1/2)\eta_{ij}R$, so R_{ij} must also be approximately diagonal, with $R_{kk} \sim (1/2)R$.

This means that

$$R = g^{\mu\nu}R_{\mu\nu} = -R_{00} + \sum_k R_{kk} = -R_{00} + \frac{3}{2}R\,, \tag{1.107}$$

so in this limit $R \approx 2R_{00}$. Then, the 00 component of the weak field Einstein equation will read

$$G_{00} = c_1\left(R_{00} - \frac{1}{2}g_{00}R\right) \tag{1.108}$$

$$\approx 2c_1 R_{00}\,. \tag{1.109}$$

We can compute the R_{00} component explicitly:

$$R_{00} = -\frac{1}{2}\nabla^2 g_{00}\,, \tag{1.110}$$

where $\nabla^2 = \eta^{ij}\partial_i\partial_j$. This means that $G_{00} = -c_1\nabla^2 g_{00}$.

Together with the Newtonian limit expression we found, $G_{00} = -\nabla^2 g_{00}$, this means that $c_1 = 1$.

Finally, then, we can write the Einstein equation as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \,. \tag{1.111}$$

It can also be written as

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) , \tag{1.112}$$

since if we contract the Einstein equation with a $g^{\mu\nu}$ we get $-R = (8\pi G/c^4)T$.

In a vacuum, $T_{\mu\nu} = 0$, therefore $R_{\mu\nu} = 0$, but in general $R_{\mu\nu\rho\sigma} \neq 0$.

One could also add a term like

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,. \tag{1.113}$$

This cosmological constant term satisfies all the conditions, except for the Newtonian limit. In order for this to be true, we need $\Lambda \lesssim 1.11 \times 10^{-52}\,\mathrm{m}^{-2}$. This is a new characteristic length scale!

Degrees of freedom There are 10 independent Einstein equations for the 10 degrees of freedom in $g_{\mu\nu}$.

However, we know that ${T^{\mu\nu}}_{;\nu} = 0$ implies ${G^{\mu\nu}}_{;\nu} = 0$: this means that only 6 of the components of the Einstein tensor are independent.

It looks like the system is underdetermined, but in fact the equations are uniquely determined *up to a gauge transformation*.

So, in fact, there are 6 equations. If $g_{\mu\nu}$ solves the Einstein Equations, and we transform it with a coordinate transformation to $g'_{\mu\nu}$, we will still find something which satisfies these equations.

This is similar to the EM case: the potential A_μ satisfies the equation

$$\Box A_\mu - \frac{\partial^2 A_\alpha}{\partial x^\mu \partial x^\alpha} = -\frac{4\pi}{c} J_\mu \,, \tag{1.114}$$

but this equation does not uniquely determine A, since J satisfies ${J^\mu}_{,\mu} = 0$. This is solved by the gauge invariance of the electromagnetic field, which is unchanged if we map $A_\mu \to A_\mu - \partial_\mu \phi$. (similarly, Maxwell's equation is unchanged in that case).

A common choice in EM theory is the Lorenz gauge, $A^\mu_{;\mu} = 0$.

In gravitational theory, on the other hand, a gauge which is often chosen is the *harmonic gauge*: $\Gamma^\lambda = g^{\alpha\beta} \Gamma^\lambda_{\alpha\beta} = 0$. More explicitly, this reads

$$\frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\lambda} \right) = 0 \,. \tag{1.115}$$

In curved coordinates, the Dalambertian reads

$$\Box \phi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = g^{\alpha\beta} \partial_\alpha \partial_\beta \phi - \Gamma^\alpha \partial_\alpha \phi \,. \tag{1.116}$$

In the harmonic gauge, this simplifies to $g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = g^{\alpha\beta} \partial_\alpha \partial_\beta \phi$.

2 Gravitational waves

They are a prediction made by any relativistic theory of gravity: something needs to propagate in order to mediate the interaction.

The connection of the "gravitational potential" to the metric means that these are *metric waves*, the proper distance between points itself changes when a GW passes by.

We assume that the metric is written like $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$. To first order, the inverse metric will read $g^{\mu\nu} = g^{\mu\nu(0)} - h^{\mu\nu}$, where $h^{\mu\nu}$ is computed raising the indices with $g^{\mu\nu(0)}$.

The first order contribution to the Christoffel symbols reads

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\left[g^{\lambda\rho(0)} - g^{\lambda\rho}\right]\left(2g^{0}_{\rho(\mu,\nu)} + 2h_{\rho(\mu,\nu)} - g^{(0)}_{\mu\nu,\rho} - h_{\mu\nu,\rho}\right) \tag{2.1}$$

$$= \underbrace{\frac{1}{2}g^{\lambda\rho(0)}\left(2g^{(0)}_{\rho(\mu,\nu)} - g^{(0)}_{\mu\nu,\rho}\right)}_{\mathcal{O}\left(h^{0}\right)} + \underbrace{\frac{1}{2}g^{\lambda\rho(0)}\left(2h_{\rho(\mu,\nu)} - h_{\mu\nu,\rho}\right) + \frac{1}{2}h^{\lambda\rho}\left(2g^{(0)}_{\rho(\mu,\nu)} - g^{(0)}_{\mu\nu,\rho}\right)}_{\mathcal{O}(h)} + \mathcal{O}\left(h^{2}\right). \tag{2.2}$$

The general expression for the Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\sigma\alpha}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\alpha}\,. \tag{2.3}$$

In the $\Gamma\Gamma$ terms we will have expressions like $(\Gamma^{(0)} + \Gamma^{(1)})(\Gamma^{(0)} + \Gamma^{(1)})$, so we can also isolate 0th-order, 1st-order and 2nd-order terms; similarly in the $\partial\Gamma$ term we will have 0th and 1st-order terms.

The zeroth and first order EFE will read

$$R^{(0)}_{\mu\nu} = \kappa\left(T^{(0)}_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}T^{(0)}\right) \tag{2.4}$$

$$R^{(1)}_{\mu\nu} = \kappa\left(T^{(1)}_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}T^{(1)} - \frac{1}{2}h^{\mu\nu}T^{(0)}\right), \tag{2.5}$$

where $\kappa = 8\pi G/c^4$.

We now take Minkowski: $\eta_{\mu\nu} = g^{(0)}_{\mu\nu}$. With this, and substituting in h to the Ricci tensor, we get

$$R_{\mu\nu} = \frac{1}{2}\left[-\Box h_{\mu\nu} + \frac{\partial^2 h^{\lambda}_{\mu}}{\partial x^{\lambda}\partial x^{\mu}} + \frac{\partial^2 h^{\lambda}_{\nu}}{\partial x^{\lambda}\partial x^{\nu}} - \frac{\partial^2 x^{\lambda}_{\lambda}}{\partial x^{\mu}\partial x^{\nu}}\right], \tag{2.6}$$

where $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$.

With these, the equations already look quite similar to a wave equation. We'd like to make a gauge transformation in order to cancel the second derivatives, but we must be careful to preserve the perturbative character of the metric — we can do so by mapping $x^\mu \to x^\mu + \epsilon^\mu$, where $\epsilon_{\mu,\nu} = \mathcal{O}\left(h_{\mu\nu}\right)$.

The transformed metric perturbation will read

$$h'_{\alpha\beta} = h_{\alpha\beta} - 2\epsilon_{(\alpha,\beta)}\,. \tag{2.7}$$

We then try to impose the harmonic gauge: $0 = g^{\alpha\beta}\Gamma^{\lambda}_{\alpha\beta}$. This yields

$$0 = h^{\mu}{}_{\kappa,\mu} - \frac{1}{2}h_{,\kappa} \,. \tag{2.8}$$

This precisely means that the second derivatives in the Ricci tensor vanish.
So, we get $R_{\mu\nu} = -(1/2)\Box h_{\mu\nu}$, and the Einstein equations become

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right) . \tag{2.9}$$

If we introduce $\overline{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}h/2$, the *trace-reversed perturbation*, we find

$$\Box\overline{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \,, \tag{2.10}$$

to be computed under $\overline{h}^{\mu}{}_{\nu,\mu} = 0$.

In a vacuum, we just have $\Box h_{\mu\nu} = 0$. The solution to this equation will be given in terms of retarded quantities

$$\overline{h}_{\mu\nu}(t,\vec{x}) = \frac{4G}{c^4}\int_V \frac{T_{\mu\nu}(t - |\vec{x} - \vec{x}'|/c, \vec{x}')}{|\vec{x} - \vec{x}'|}\, \mathrm{d}^3x' \ . \tag{2.11}$$

We have yet to show that choosing the harmonic gauge is indeed always possible with an infinitesimal coordinate transformation. If we make the coordinate transformation for the Christoffel system, we get

$$\Gamma^{\lambda'} = \eta^{\lambda k}\left[h^{\mu}{}_{\kappa,\mu} - \frac{1}{2}h_{,\kappa}\right] + \Box\epsilon^{\lambda} \,. \tag{2.12}$$

Since solving the Dalambertian equation is always possible, we can always set this to zero.

A monochromatic plane wave in a vacuum reads

$$\overline{h}_{\mu\nu} = \mathrm{Re}\, A_{\mu\nu}e^{ik_\sigma x^\sigma} \,. \tag{2.13}$$

The Einstein equation means that k^2 must vanish. The gauge condition tells us that $k^\mu A_{\mu\nu} = 0$.

Exercise: Einstein's carousel (2.2).
The transformation law to and from flat spacetime is according to the matrix

$$L^{\mu}_{\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \tag{2.14}$$

so that $\xi^\mu = L^\mu_\nu x^\nu$, and $x^\mu = \widetilde{L}^\mu_\nu \xi^\nu$, where $\widetilde{L}$ is the inverse of L, which can be obtained

by mapping $t \to -t$ in it.
Now, we need to compute the derivatives of this transformation in the expression for the Christoffel symbols:

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial^2 \xi^\rho}{\partial x^\alpha \partial x^\beta} \frac{\partial x^\mu}{\partial \xi^\rho} \,. \tag{2.15}$$

The transformation matrix $\widetilde{L}^\mu_\nu$ depends on ξ only through its component $\xi^0 = t$:

$$\begin{aligned} \frac{\partial x^\mu}{\partial \xi^\rho} &= \frac{\partial}{\partial \xi^\rho}\left(\widetilde{L}^\mu_\nu \xi^\nu\right) && (2.16)\\ &= \widetilde{L}^\mu_\nu \delta^\nu_\rho + \frac{\partial \widetilde{L}^\mu_\nu}{\partial \xi^\rho} \xi^\nu && (2.17)\\ &= \widetilde{L}^\mu_\rho + \dot{\widetilde{L}}^\mu_\nu \xi^\nu \delta^t_\rho \,, && (2.18) \end{aligned}$$

and the reciprocal transformation is quite similar:

$$\begin{aligned} \frac{\partial \xi^\rho}{\partial x^\beta} &= \frac{\partial}{\partial x^\beta}\left(L^\rho_\sigma x^\sigma\right) && (2.19)\\ &= L^\rho_\beta + \dot{L}^\rho_\sigma \xi^\sigma \delta^t_\beta \,. && (2.20) \end{aligned}$$

Now, however, we need to take a second derivative of this term:

$$\begin{aligned} \frac{\partial^2 \xi^\rho}{\partial x^\alpha \partial x^\beta} &= \frac{\partial}{\partial x^\alpha}\left(L^\rho_\beta + \dot{L}^\rho_\sigma \xi^\sigma \delta^t_\beta\right) && (2.21)\\ &= \dot{L}^\rho_\beta \delta^t_\alpha + \dot{L}^\rho_\sigma \delta^\sigma_\alpha \delta^t_\beta + \ddot{L}^\rho_\sigma \delta^t_\alpha \xi^\sigma \delta^t_\beta && (2.22)\\ &= 2\dot{L}^\rho_{(\alpha} \delta^t_{\beta)} + \ddot{L}^\rho_\sigma \xi^\sigma \delta^t_\alpha \delta^t_\beta \,, && (2.23) \end{aligned}$$

which we can combine into the full Christoffel symbol expression, which is (thankfully) manifestly symmetric in $\alpha\beta$:

$$\begin{aligned} \Gamma^\mu_{\alpha\beta} &= \left(\dot{L}^\rho_\beta \delta^t_\alpha + \dot{L}^\rho_\alpha \delta^t_\beta + \ddot{L}^\rho_\sigma \xi^\sigma \delta^t_\alpha \delta^t_\beta\right)\left(\widetilde{L}^\mu_\rho + \dot{\widetilde{L}}^\mu_\nu \xi^\nu \delta^t_\rho\right) && (2.24)\\ &= \dot{L}^\rho_\beta \delta^t_\alpha \widetilde{L}^\mu_\rho + \dot{L}^\rho_\alpha \delta^t_\beta \widetilde{L}^\mu_\rho + \ddot{L}^\rho_\sigma \widetilde{L}^\mu_\rho \xi^\sigma \delta^t_\alpha \delta^t_\beta \,, && (2.25) \end{aligned}$$

Terms like $\dot{L}^\rho \delta^t_\rho$ vanish!

which is already starting to look like Coriolis + centrifugal terms.
We can write

$$\ddot{L}^\rho_\sigma = -\omega^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = -\omega^2\left(L^\rho_\sigma - \delta^\rho_t \delta^t_\sigma - \delta^\rho_z \delta^z_\sigma\right) , \tag{2.26}$$

using which we can rewrite the centrifugal term like

$$\ddot{L}^\rho_\sigma \widetilde{L}^\mu_\rho \xi^\sigma \delta^t_\alpha \delta^t_\beta = -\omega^2\left(\delta^\mu_\sigma \xi^\sigma \delta^t_\alpha \delta^t_\beta - \delta^\rho_t \delta^t_\sigma \widetilde{L}^\mu_\rho \xi^\sigma \delta^t_\alpha \delta^t_\beta - \delta^\rho_z \delta^z_\sigma \widetilde{L}^\mu_\rho \xi^\sigma \delta^t_\alpha \delta^t_\beta\right) \tag{2.27}$$

$$= \omega^2 \xi^\mu \delta^t_\alpha \delta^t_\beta = \omega^2 L^\mu_\nu x^\nu \delta^t_\alpha \delta^t_\beta \,. \tag{2.28}$$

Let us now move to the Coriolis term: we will get identical results if we swap α and β, and we know that one of them must be t, so let us set $\alpha = t$: we get

$$\left(\Gamma^\mu_{t\beta}\right)_{\text{Coriolis}} = \dot{L}^\rho_\beta L^\mu_\rho \tag{2.29}$$

$$= \omega \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin(\omega t) & -\cos(\omega t) & 0 \\ 0 & -\cos(\omega t) & \sin(\omega t) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^\rho_\beta \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\omega t) & \sin(\omega t) & 0 \\ 0 & -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^\mu_\rho \tag{2.30}$$

$$= \omega \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^\mu_\beta \tag{2.31}$$

$$= \omega \epsilon^{\mu z}_{t\beta} \,, \tag{2.32}$$

where the ϵ is the four-dimensional Levi-Civita symbol. This also shows that the symmetric $\beta = t$ term equals $\omega \epsilon^{\mu z}_{\alpha t}$.
We can thus compactly write

$$\Gamma^\mu_{\alpha\beta} = \omega \left(\epsilon^{\mu z}_{\alpha t} \delta^t_\beta + \epsilon^{\mu z}_{t\beta} \delta^t_\alpha \right) + \omega^2 L^\mu_\nu x^\nu \delta^t_\alpha \delta^t_\beta \,. \tag{2.33}$$

Now, let us write the geodesic equation:

$$\ddot{x}^\mu = -\Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \tag{2.34}$$

$$= -\left[\omega \left(\epsilon^{\mu z}_{\alpha t} \delta^t_\beta + \epsilon^{\mu z}_{t\beta} \delta^t_\alpha \right) + \omega^2 L^\mu_\nu x^\nu \delta^t_\alpha \delta^t_\beta \right] \dot{x}^\alpha \dot{x}^\beta \,. \tag{2.35}$$

The sum includes several terms: for example, the $\alpha = \beta = t$ one reads

$$-\omega^2 L^\mu_\nu x^\nu (\dot{x}^t)^2 \,. \tag{2.36}$$

Let us look at this more concretely: the four-velocity of a timelike trajectory is $u^\mu = \mathrm{d}x^\mu / \mathrm{d}\tau$; which in terms of the three-velocity $\vec{v}$ reads $u^\mu = [\gamma, \gamma \vec{v}]$, where $\gamma = 1/\sqrt{1 - |\vec{v}|^2}$.
In our case, $\dot{x}^\mu = u^\mu$.
Let us then look at the $\mu = i$, spatial components of this equation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\gamma v^i \right) = -\omega \left[\epsilon^{iz}_{jt} u^j + \epsilon^{iz}_{tj} u^j \right] u^t - \omega^2 \xi^i (u^t)^2 \tag{2.37}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} (\gamma v^x) = -2\omega u^y u^t - \omega^2 \xi^x (u^t)^2 \tag{2.38}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\gamma v^y) = +2\omega u^x u^t - \omega^2 \xi^y (u^t)^2 \,. \tag{2.39}$$

There's a wrong sign in here somewhere.

In the nonrelativistic limit $\gamma \to 1$, $u^t \to 1$, $\tau = t$, so we just get

$$\ddot{x} = -2\omega\dot{y} - \omega^2 x \tag{2.40}$$

$$\ddot{y} = +2\omega\dot{x} - \omega^2 y \,. \tag{2.41}$$

Exercise: variation of a vector (1.5).

We want to show that, when parallel-transporting a vector along a loop, it varies by

$$\delta a^\lambda = -\frac{1}{2} a^\rho R^\lambda_{\rho\alpha\beta} \,\mathrm{d}x^\alpha \,\mathrm{d}x^\beta \,. \tag{2.42}$$

We denote the closed curve along which we transport the vector as γ. Its tangent vector will be $\dot{\gamma}$, and its parameter will be t (not the time coordinate). The parallel transport equation reads

$$0 = \dot{\gamma}^\mu \nabla_\mu a^\lambda = \dot{\gamma}^\mu \left(\partial_\mu a^\lambda + \Gamma^\lambda_{\mu\alpha} a^\alpha \right) . \tag{2.43}$$

The quantity we want to compute is

$$\oint_\gamma \mathrm{d}a^\lambda = \oint_\gamma \frac{\mathrm{d}a^\lambda}{\mathrm{d}t} \,\mathrm{d}t \tag{2.44}$$

$$= \oint_\gamma \dot{\gamma}^\mu \partial_\mu a^\lambda \,\mathrm{d}t \tag{2.45}$$

$$= -\oint_\gamma \dot{\gamma}^\mu \Gamma^\lambda_{\mu\alpha} a^\alpha \,\mathrm{d}t \tag{2.46}$$

$$= -\oint_{\partial\Sigma} \Gamma^\lambda_{\mu\alpha} a^\alpha \,\mathrm{d}x^\mu \tag{2.47}$$

We identify γ with the border of a region Σ.

$$= -\int_\Sigma \partial_\nu \left(\Gamma^\lambda_{\mu\alpha} a^\alpha \right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{2.48}$$

$$= -\int_\Sigma \partial_\nu \Gamma^\lambda_{\mu\alpha} - \Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\nu\alpha} a^\alpha \,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{2.49}$$

$$= \int \left(\partial_{[\mu} \Gamma^\lambda_{\nu]\alpha} + \Gamma^\lambda_{\rho[\mu} \Gamma^\rho_{\nu]\alpha} \right) a^\alpha \,\mathrm{d}x^\mu \,\mathrm{d}x^\nu \tag{2.50}$$

$$= -\int_\Sigma \frac{1}{2} R^\lambda_{\alpha\mu\nu} a^\alpha \,\mathrm{d}x^\mu \,\mathrm{d}x^\nu \,. \tag{2.51}$$

The result then follows by taking the limit of a small integration region, so that the Riemann tensor inside the expression is roughly constant.