Gravitational Wave Exercises @ Jena

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Exercises for the Gravitational Waves course. The exercise sheets can be found on the webpage.

1 Quadrupole approximation

The procedure to find the quadrupole approximation for the GW emitted by a Newtonian system is in the form:

1. determine the density $\rho(\vec{x}, t)$;
2. calculate the trace-free inertia tensor $Q^{ij}(t)$;
3. calculate the gravitational wave strain h_{ij}.

Two point particles

In order to model the point-like nature of the particles we can write an expression for the density as a sum of two delta-functions, whose positions oscillate on the x axis with an amplitude R:

$$\rho(\vec{x}, t) = m\delta(\vec{x} - \vec{x}_1(t)) + m\delta(\vec{x} - \vec{x}_2(t))\,, \tag{1.1}$$

where

$$\vec{x}_1(t) = R \begin{bmatrix} \cos(\omega t + \phi) \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \vec{x}_2(t) = -\vec{x}_1(t)\,. \tag{1.2}$$

So, the trace-free inertia tensor will be given by

$$\begin{aligned} Q^{ij}(t) &= \int \rho(\vec{x}, t) \left(x^i x^j - \frac{1}{3}\delta^{ij} r^2 \right) \mathrm{d}^3 x && (1.3) \\ &= m \sum_{k=1,2} \left(x_k^i x_k^j - \frac{1}{3}\delta^{ij} r^2 \right) && (1.4) \\ &= 2mR^2 \begin{bmatrix} \cos^2(\omega t + \phi) - 1/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}. && (1.5) \end{aligned}$$

The second derivative of this tensor will be given by

$$\ddot{Q}^{ij}(t) = 4mR^2\omega^2 \begin{bmatrix} \sin^2 - \cos^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{1.6}$$

Now, in order to compute the gravitational wave strain we need the projection tensor $\Lambda_{ij,kl}$. If the propagation direction we are interested in is $\vec{k}$, then the tensor

$$P_{ij} = \delta_{ij} - \frac{k_i k_j}{|k|^2} = \delta_{ij} - n_i n_j \tag{1.7}$$

will project a vector onto the subspace orthogonal to $\vec{k}$; and the tensor

$$\begin{aligned} \Lambda_{ij,kl} &= P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} && (1.8) \\ &= \delta_{ik}\delta_{jl} - n_i n_k \delta_{jl} - \delta_{ik} n_j n_l + \frac{1}{2} n_i n_k n_j n_l - \frac{1}{2}\delta_{ij}\delta_{kl} + \frac{1}{2}\delta_{ij} n_k n_l + \frac{1}{2} n_i n_j \delta_{kl} \,, && (1.9) \end{aligned}$$

will project a rank-2 tensor onto the corresponding subspace.

Then, the gravitational wave emission will look like:

$$\begin{aligned} h_{ij}(t) &= \frac{2}{r}\frac{G}{c^4}\Lambda_{ij,kl}\ddot{Q}_{kl}(t - r/c) && (1.10) \\ &= \frac{2}{r}\frac{G}{c^4}\Lambda_{ij,xx}\Big(\sin^2(\omega t - \omega r/c + \phi) - \cos^2(t - \omega r/c + \phi)\Big) && (1.11) \\ &= -\frac{2}{r}\frac{G}{c^4}\left(\delta_{ix}\delta_{jx} - n_i n_x \delta_{jx} - \delta_{ix} n_j n_x + \frac{1}{2} n_i n_x n_j n_x - \frac{1}{2}\delta_{ij}\delta_{xx} + \frac{1}{2}\delta_{ij} n_x n_x + \frac{1}{2} n_i n_j \delta_{xx}\right) \\ &\quad \times \cos\left(2\omega\left(r - \frac{r}{c}\right) + \phi\right) \,, && (1.12) \end{aligned}$$

which we can express explicitly by writing out

$$\vec{n} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} . \tag{1.13}$$

A free-falling particle

Ellipsoid rotating along its axis

Two particles in circular Newtonian orbit

$$\rho(\vec{x}, t) = m_1\delta(\vec{x} - \vec{x}_1(t)) + m_2\delta(\vec{x} - \vec{x}_2(t)) \,, \tag{1.14}$$

where

$$in\vec{x}_1(t) = r \begin{bmatrix} \cos(\omega t + \phi) \\ -\sin(\omega t + \phi) \\ 0 \end{bmatrix} \qquad \text{and} \qquad \vec{x}_2(t) = -\vec{x}_1(t)\,. \tag{1.15}$$

For simplicity, we will also assume that the radius r is constant — this cannot be precisely the case, since there will be some amount of power lost because of the GW emission, but it is a reasonable approximation.

So, the trace-free inertia tensor will be given by

$$\begin{aligned} Q^{ij}(t) &= \int \rho(\vec{x}, t) \left(x^i x^j - \frac{1}{3} \delta^{ij} r^2 \right) \mathrm{d}^3 x && (1.16) \\ &= \sum_{k=1,2} m_k \left(x_k^i x_k^j - \frac{1}{3} \delta^{ij} r^2 \right) && (1.17) \\ &= 2mr^2 \begin{bmatrix} \cos^2 - 1/3 & -\cos\sin \\ -\cos\sin & \sin^2 - 1/3 \end{bmatrix}, && (1.18) \end{aligned}$$

where we write only the upper-left 2x2 submatrix in Q^{ij}, since the other entries are constant, and we omit the argument of the sines and cosines (which is always $\omega t + \phi$).

The second derivative of this tensor will be given by

$$\ddot{Q}^{ij}(t) = 4mr^2\omega^2 \begin{bmatrix} \sin^2 - \cos^2 & 2\sin\cos \\ 2\sin\cos & \cos^2 - \sin^2 \end{bmatrix}. \tag{1.19}$$

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