Gravitational Waves @ Jena University

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Introduction

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The syllabus can be found here.

Interesting things on the Indico server of Jena university.

In this first lecture, a basic introduction to the theory of gravitational waves: Einstein's first papers, the sticky bead argument by Bondi & Feynman, the quadrupole formula:

$$\overline{h}_{ij}(t,r) = \frac{2G}{c^4 r}\ddot{I}_{ij}(t-r)\,. \tag{0.1}$$

The idea behind the multipole expansion is that we are solving the Poisson equation $\nabla^2\phi = \rho$, so

$$\phi(\vec{r}) = \int \frac{\rho(\vec{x})\,\mathrm{d}^3x}{|\vec{r}-\vec{x}|}\,, \tag{0.2}$$

so as long as we are far away from the source we will see

$$\phi(\vec{r}) = -\frac{q}{r} - \frac{p_i n^i}{r^2} - \frac{Q_{ij}n^i n^j}{r^3} + \dots\,. \tag{0.3}$$

Quiz: which of these are GW sources?

1. spherical star: no, its quadrupole is vanishing;
2. rotating star: no, its quadrupole is constant;
3. star with a mountain: yes, its quadrupole evolves (potential source of continuous GW);
4. supernova explosion: yes, if there is asymmetry (potential source of burst GW);
5. binary system: yes, already detected!

Claim 0.1. *Order of magnitude expression:*

$$h \lesssim \frac{GM}{c^2D}\frac{v^2}{c^2} = \frac{R}{D}\frac{GM}{c^2R}\left(\frac{v}{c}\right)^2, \tag{0.4}$$

where D is the distance to the object, R is the characteristic scale of the object (so that GM/c^2R is the compactness), while v is the characteristic velocity. The quantity we calculate is $h \sim \delta L/L$, the strain.

Proof. To do. □

The Hulse-Taylor pulsar. The two-body problem in GR is difficult.

The typical waveform in the PN region looks like:

$$h_+(t) \approx \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\text{gw}}(t)}{c}\right)^{2/3}\cos\Big(2\pi f_{\text{gw}}(t)t\Big)\,, \tag{0.5}$$

then we need numerical relativity to simulate the plunge and merger, and finally the ring-down is simulated using BH perturbation methods. The mass scale is

$$h(t) \sim \nu\frac{1}{r/M}(Mf_{\text{gw}})^{2/3}\,, \tag{0.6}$$

while

$$\phi_{\text{gw}}(t) \sim 2\phi_{\text{orb}}(t) = 2M_c^{-5/8}t^{5/8} = 2\nu^{-3/8}\left(\frac{t}{M}\right)^{5/8}\,, \tag{0.7}$$

where $\nu = \mu/M$, and $\mu = 1/(1/M_1 + 1/M_2)$.

Multiple detectors are crucial for sky localization, as well as for the measurement of polarization.

At leading order, the two-body problem in GR is scale-invariant: the length of the signal can be estimated simply from the mass of the stars involved.

R-process nucleosynthesis might have something to do with BNS mergers, if the stars are torn apart by the collision.

1 Weak-field GR

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This is the limit of GR for weak gravitational fields: the metric is assumed to be in the form of the Minkowski one plus a perturbation. We are seeking the equations of motion under this assumption.

How do we quantify the term "small"? We assume that there is a **global inertial coordinate system** such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{1.1}$$

where, like in the rest of the course, we will use the letters α, β, γ or μ, ν for the coordinates x^μ; while letters like a, b represent the abstract notation.

The term "small", then, means that each component of $h_{\mu\nu}$ has an absolute value which is much smaller than 1. We are using the metric signature $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.

What does this approximation describe?

1. Newtonian gravity;

2. gravito-electric / magnetic effects (this will be discussed in more detail later, an example is the Lense-Thirring effect);

3. propagation of gravitational waves.

In the case of the gravitational field around the Sun, in terms of orders of magnitude we have[1]

$$\left|h_{\mu\nu}\right| \sim \frac{\phi}{c^2} \sim \frac{GM_\odot}{c^2 R_\odot} \sim 10^{-6}\,. \tag{1.2}$$

From a field-theoretic point of view:

1. η is a background metric;
2. h is the "main" field;
3. the metric does *not* backreact on the matter ($T_{\mu\nu}$).

The metric perturbation h transform like a tensor on flat spacetime under Lorentz transformations: if $\Lambda^\top \eta \Lambda = \eta$, then the coordinates change like $x = \Lambda x'$, then the full metric transforms like

$$\begin{aligned} g_{\mu'\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} && (1.3)\\ &= \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \left(\eta_{\mu\nu} + h_{\mu\nu}\right) && (1.4)\\ &= \eta_{\mu'\nu'} + \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} h_{\mu\nu}\,, && (1.5) \end{aligned}$$

therefore the transformation for h is

$$h_{\mu\nu} \to h_{\mu'\nu'} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} h_{\mu\nu}\,. \tag{1.6}$$

Mind the notation: the meaning of $h_{\mu'\nu'}$ is $h_{\mu\nu}(x')$.

Symmetry of linearized GR

Full GR is diffeomorphism invariant, while linearized GR is *infinitesimal* diffeomorphism invariant. The relevant transformations are

$$x^\mu \to x^{\mu'} = x^\mu + \xi^\mu(x^\alpha)\,, \tag{1.7}$$

where the vector field ξ is selected so that $\left|\partial_\mu \xi^\alpha\right| \sim \left|h_{\mu\nu}\right| \ll 1$.

The Jacobian of this transformation is

$$\frac{\partial x^{\mu'}}{\partial x^\mu} = \delta^{\mu'}{}_\mu + \partial_\mu \xi^{\mu'}\,, \tag{1.8}$$

while the inverse Jacobian is

$$\frac{\partial x^\mu}{\partial x^{\mu'}} + \delta^\mu{}_{\mu'} - \partial_{\mu'} \xi^\mu + \mathcal{O}\left(|\partial \xi|^2\right), \tag{1.9}$$

[1] We make the c explicit here for clarity, but we will use geometric units $c = G = 1$ for the rest of the course.

since $(\mathbb{1}+\delta)(\mathbb{1}-\delta) = \mathbb{1} + \mathcal{O}\left(\delta^2\right)$.

Under this change of coordinates, we have

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} \tag{1.10}$$

$$\sim (\delta\delta - \partial\delta - \partial\delta + \partial\partial)(\eta + h) = \delta\delta\eta + h - \partial\delta - \partial\delta + \mathcal{O}\left(\delta^2\right) \tag{1.11}$$

$$= \delta^\mu_{\mu'}\delta^\nu_{\nu'}\eta_{\mu\nu} - \partial_{\mu'}\xi^\mu\delta^\nu_{\nu'}\eta_{\mu\nu} - \partial_{\nu'}\xi^\nu\delta^\mu_{\mu'} + \delta^\mu_{\mu'}\delta^\nu_{\nu'}h_{\mu\nu} \tag{1.12}$$

$$= \eta_{\mu'\nu'} + h_{\mu'\nu'} - 2\partial_{(\mu'}\xi_{\nu')}\,, \tag{1.13}$$

therefore we have our transformation law:

$$h_{\mu'\nu'} = h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}\,. \tag{1.14}$$

This can also be written in terms of the Lie derivative as

$$h_{\mu\nu} \to h_{\mu\nu} + \mathscr{L}_\xi \eta_{\mu\nu}\,. \tag{1.15}$$

This is the analogous of a gauge transformation in electromagnetism: $A_\alpha \to A_\alpha + \partial_\alpha\chi$, where A is the vector potential.

Equations of motion

The equations of motion will come through plugging $g = \eta + h$ into the EFE $G_{ab} = 8\pi T_{ab}$ and keeping only the linear order in h.

We will need the following quantities:

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} + \mathcal{O}\left(h^2\right) \tag{1.16}$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\eta^{\mu\lambda}\left(\partial_\alpha h_{\lambda\beta} + \partial_\beta h_{\lambda\alpha} - \partial_\lambda h_{\alpha\beta}\right) + \mathcal{O}\left(h^2\right) \tag{1.17}$$

$$R_{\mu\nu} = \partial\Gamma - \partial\Gamma + \mathcal{O}\left(h^2\right), \tag{1.18}$$

where we already simplified the expressions by removing the higher-order terms. The result is

$$R_{\mu\nu} = \partial^\alpha\partial_{(\mu}h_{\nu)\alpha} - \frac{1}{2}\partial_\lambda\partial^\lambda h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h + \mathcal{O}\left(h^2\right), \tag{1.19}$$

where $h = h^\alpha_\alpha = \eta^{\alpha\beta}h_{\alpha\beta}$. Note that we are allowed to use η instead of g to lower indices. The Einstein tensor reads

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \tag{1.20}$$

$$= \partial^\alpha\partial_{(\mu}h_{\nu)} - \frac{1}{2}\partial_\lambda\partial^\lambda h_{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \frac{1}{2}\eta_{\mu\nu}\partial_\lambda\partial^\lambda h\,, \tag{1.21}$$

which can be simplified if we consider the trace-reversed metric

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,, \tag{1.22}$$

so that $\overline{h} = \eta^{\mu\nu} h_{\mu\nu} - \eta^{\mu\nu}\eta_{\mu\nu} h/2 = -h$. See equation 2.13 in the notes for a full explanation, but the idea is to insert $\overline{h}_{\mu\nu}$ and $\overline{h}$ into $G_{\alpha\beta}$ and to make some simplifications. We get

$$G_{\mu\nu} = -\frac{1}{2}\eta_{\alpha\beta}\partial^\alpha\partial^\beta\overline{h}_{\mu\nu} + \partial^\alpha\partial_{(\mu}\overline{h}_{\nu)\alpha} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta\overline{h}_{\alpha\beta}\,, \tag{1.23}$$

which is in the form $\Box_\eta \overline{h}_{\mu\nu} + \dots \partial^\alpha \overline{h}_{\alpha\beta}$. We still have gauge freedom, so we can simplify the equation a great deal by setting $\partial^\alpha \overline{h}_{\alpha\beta} = 0$ — the Hilbert, or Lorentz gauge.

With this choice, we have

$$\boxed{\Box_\eta \overline{h}_{\mu\nu} = -\frac{16G}{c^4} T_{\mu\nu}\,,} \tag{1.24}$$

a relatively simple tensor wave equation.

Is is always possible to impose the Hilbert gauge? Yes: we can make an infinitesimal coordinate transformation to send a generic $h_{\mu\nu}$ to $h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$, so that $h \to h + 2\eta^{\alpha\beta}\partial_{(\alpha}\xi_{\beta)}$. Therefore,

$$\overline{h}_{\mu\nu} \to \overline{h}_{\mu\nu} + 2\partial_{(\mu}h_{\nu)} - \eta_{\mu\nu}\partial_\alpha\xi^\alpha\,, \tag{1.25}$$

and we can send

check indices here

$$\partial^\alpha\overline{h}_{\mu\alpha} \to \partial^\alpha\overline{h}_{\mu\alpha} + \Box\xi_\mu + \partial^\mu\partial_\nu\xi_\mu + \partial_\nu\partial^\lambda\xi_\lambda\,, \tag{1.26}$$

so if we set $\Box\xi_\mu = -\partial^\alpha\overline{h}_{\mu\alpha} = v_\mu$ we can reduce ourselves to the Hilbert gauge from any starting point. All we need to do is solve the wave equation $\Box\xi_\mu = v_\mu$.

Now, to linear order $T_{\mu\nu}$ does not depend on h. So, we can find formal solutions using Green's functions, like in electromagnetism.

The Bianchi identities are now given by $\partial_\nu G^{\mu\nu} = 0$, so $\partial_\nu T^{\mu\nu} = 0$, which gives us the EOM for matter — note that this is a partial, not a covariant derivative! This means that there is no backreaction on the metric.

The linear EFE correspond to the equations of motion of a massless spin-2 field.

Weak-field solutions

Let us consider a *static source*: suppose that $T_{\mu\nu} = \rho t_\mu t_\nu$, where $t^\mu = (\partial_t)^\mu$ is the time vector along the time direction of the global inertial coordinate system while ρ is an energy density.

If $t^\mu = (1,0,0,0)$ then $T_{00} = \rho$ while $T_{0i} = 0 = T_{ij}$.

In this case, then, the stress-energy tensor is time-independent: therefore also on the other side we will have $\partial_t\overline{h}_{\mu\nu} = 0$.

Therefore, the left-hand side of the equation will read $\nabla^2\overline{h}_{\mu\nu} = -16\pi\rho$ for $\mu = \nu = 0$ and $\nabla^2 h_{\mu\nu} = 0$ for all the other components.

These Poisson equations can be solved as boundary-value problems if we assume that $h_{\mu\nu} \to 0$ for $r \gg R$.

This looks very similar to the Newton equation $\nabla^2\phi = 4\pi\rho$; therefore $\bar{h}_{00} = -4\phi$, while $\bar{h}_{\mu\nu} = 0$ for all other components.

We can reconstruct the metric using the fact that $\bar{h} = 4\phi$, so

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} = -4\phi t_\mu t_\nu - \frac{1}{2}\eta_{\mu\nu}4\phi\,, \tag{1.27}$$

so the metric reads

$$g_{\mu\nu} = \eta_{\mu\nu}(1-2\phi) - 4\phi t_\mu t_\nu\,, \tag{1.28}$$

therefore

$$g = -(1+2\phi)\,\mathrm{d}t^2 + (1-2\phi)\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,. \tag{1.29}$$

We know that far away from the source, the Newtonian field decays like $\phi \approx -M/r + \mathcal{O}\left(r^{-2}\right)$.

Therefore, this metric approximation already includes special relativity as well: we have $g \to \eta$ for large r, but also $g = \eta$ for $M = 0$.

The geodesic equation for this weak field metric reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\partial^i\phi\,. \tag{1.30}$$

However, these Newtonian equations of motion are *not* consistent with $\partial_\mu T^{\mu\nu} = 0$. These describe the motion of the source which generates gravity, whereas the Newtonian EOM describe the motion of test particles in the weak-field metric.

The dual meaning of the full EFE — matter deforms the spacetime, the spacetime shapes the trajectories of matter — *cannot* be realized at linear order.