Gravitational Wave Exercises @ Jena

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Exercises for the Gravitational Waves course. The exercise sheets can be found on the webpage.

1 Quadrupole approximation

The procedure to find the quadrupole approximation for the GW emitted by a Newtonian system is in the form:

1. determine the density $\rho(\vec{x}, t)$;
2. calculate the trace-free inertia tensor $Q^{ij}(t)$;
3. calculate the gravitational wave strain h_{ij}.

Two point particles

In order to model the point-like nature of the particles we can write an expression for the density as a sum of two delta-functions, whose positions oscillate on the x axis with an amplitude R:

$$\rho(\vec{x}, t) = m\delta(\vec{x} - \vec{x}_1(t)) + m\delta(\vec{x} - \vec{x}_2(t)) \,, \tag{1.1}$$

where the positions of the particles, $\vec{x}_1(t)$ and $\vec{x}_2(t)$, will be the solutions to the differential equation $\vec{F}_{12} = -k(\vec{x}_1 - \vec{x}_2)$. If they start out along the x axis they will remain along it, so the equation will read $F_{12} = -k(x_1 - x_2)$, so $\ddot{x}_1 = -\omega_0^2(x_1 - x_2)$ and $\ddot{x}_2 = +\omega_0^2(x_1 - x_2)$, where $\omega^2 = k/m$.

This can be rewritten, by taking the difference of the two, as $\ddot{\Delta x} = -2\omega_0^2 \Delta x$, with $\Delta x = x_1 - x_2$. Therefore, the pulsation of the system is $\omega = \sqrt{2}\omega_0$ — this could also have been calculated as $\sqrt{k/m_r}$, where $m_r = m^2/(m+m) = m/2$ is the reduced mass.

So, the position vectors will look like

$$\vec{x}_1(t) = R \begin{bmatrix} \cos(\omega t) \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \vec{x}_2(t) = -\vec{x}_1(t) \tag{1.2}$$

up to a phase.

So, the inertia tensor will be given by

$$I^{ij}(t) = \int \rho(\vec{x}, t) x^i x^j \, \mathrm{d}^3 x \tag{1.3}$$

$$= m \sum_{k=1,2} x_k^i x_k^j \tag{1.4}$$

$$= -mR^2 \begin{bmatrix} \cos^2(\omega t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{1.5}$$

Its trace is equal to its only nonzero component, I^{xx}, therefore its traceless version will look like

$$Q^{ij}(t) = mR^2 \begin{bmatrix} -(2/3)\cos^2 \omega t & 0 & 0 \\ 0 & (1/3)\cos^2 \omega t & 0 \\ 0 & 0 & (1/3)\cos^2 \omega t \end{bmatrix} . \tag{1.6}$$

The second derivative of this tensor will be given by

$$\ddot{Q}^{ij}(t) = \frac{2mR^2}{3}\omega^2 \begin{bmatrix} 2\cos(2\omega t) & 0 & 0 \\ 0 & -\cos(2\omega t) & 0 \\ 0 & 0 & -\cos(2\omega t) \end{bmatrix} . \tag{1.7}$$

Now, in order to compute the gravitational wave strain we need the projection tensor $\Lambda_{ij,kl}$. If the propagation direction we are interested in is $\vec{k}$, then the tensor

$$P_{ij} = \delta_{ij} - \frac{k_i k_j}{|k|^2} = \delta_{ij} - n_i n_j \tag{1.8}$$

will project a vector onto the subspace orthogonal to $\vec{k}$; and the tensor

$$\Lambda_{ij,kl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \tag{1.9}$$

$$= \delta_{ik}\delta_{jl} - n_i n_k \delta_{jl} - \delta_{ik} n_j n_l + \frac{1}{2} n_i n_k n_j n_l - \frac{1}{2}\delta_{ij}\delta_{kl} + \frac{1}{2}\delta_{ij} n_k n_l + \frac{1}{2} n_i n_j \delta_{kl} \, , \tag{1.10}$$

will project a rank-2 tensor onto the corresponding subspace.

Then, the gravitational wave emission will look like:

$$h_{ij}(t) = \frac{2}{r}\frac{G}{c^4}\Lambda_{ij,kl}\ddot{Q}_{kl}(t - r/c) \tag{1.11}$$

$$= \frac{2}{r}\frac{G}{c^4}\frac{2mR^2\omega^2}{3}\cos\big(2\omega(t - r/c)\big) F_{ij}(\theta, \varphi) \, , \tag{1.12}$$

where $F_{ij}(\theta, \varphi)$ is a tensor depending on the two angles which define the observation-direction unit vector:

$$\vec{n} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} . \tag{1.13}$$

The explicit shape of F_{ij} depends on the shape of $\ddot{Q}_{ij}$, the one for this specific case can be calculated analytically with a computer algebra system — see here.

However, that specific analytic form is not really interesting — we want to extract the two actual degrees of freedom of the gravitational wave. If we define two vectors $\vec{m}$ and $\vec{l}$ which are orthogonal to $\vec{n}$ and to each other, we will be able to extract the degrees of freedom as $h_+(t) = h_{ij}(t) m^i m^i$ and $h_\times(t) = h_{ij} m^i l^j$. We can select, for example, the $\hat{\theta}$ and $\hat{\varphi}$ unit vectors in a spherical coordinate system in which $\vec{n} = \hat{r}$ is the radial vector for this purpose.

The computation yields:

$$h_+(t) = \frac{2}{r}\frac{G}{c^4} m R^2 \omega^2 \left(\cos^2\theta \cos^2\varphi + \cos^2\varphi - 1\right) \cos(2\omega t) \tag{1.14}$$

$$h_\times(t) = -\frac{2}{r}\frac{G}{c^4} m R^2 \omega^2 \cos\theta \sin\left(2\varphi\right) \cos(2\omega t)\,. \tag{1.15}$$

2 Estimate of GW magnitude

We want a way to estimate the GW strain, $h \sim \delta L / L$, by simple considerations about the parameters of the system which is generating the waves. In the quadrupole approximation we have $h \sim (G/c^4 r)\ddot{I}$, where $\ddot{I}$ is the typical magnitude of the second derivative of the inertia tensor, while r is the distance from the system to Earth.

We know that $I \sim MR^2$, where M is the mass of the system while R is its characteristic radius (for example, a solid sphere has $I = (2/5)MR^2$ — the $2/5$ factor is relevant but we will neglect it in this order-of-magnitude estimate). If T is the characteristic timescale in which the components of the inertia tensor vary, we can estimate $\ddot{I}$ as $\ddot{I} \sim MR^2/T^2 = Mv^2$, where v is the typical velocity of the various parts of the system.

With this estimate, we have:

$$h \sim \frac{G}{c^4 r} M v^2 = \frac{GM}{c^2 r}\frac{v^2}{c^2} \sim \frac{R_s}{r}\frac{v^2}{c^2}\,. \tag{2.1}$$

Let us compute this for a few simple examples:

1. a car crash, with $M \sim 10^3\,\mathrm{kg}$, $v \sim 100\,\mathrm{km/h}$, $r \sim 10\,\mathrm{m}$;
2. a supernova explosion, with $M \sim M_\odot$, $v \sim 0.2c$, $r \sim 2\,\mathrm{kpc}$;
3. a binary black hole system, with $M \sim 50 M_\odot$, $v \sim 0.1c$, $r \sim 400\,\mathrm{Mpc}$.

We shall use the `units` system[1] provided by the python library `astropy` in order to aid us in this computation: we start out with the imports

```python
import astropy.units as u
from astropy.constants import codata2018 as ac
```

Now we may define a function yielding the desired estimate:

[1] https://docs.astropy.org/en/stable/units/index.html

```python
@u.quantity_input(M='mass', D='length', v='speed')
def estimate_h(M, D, v) -> u.dimensionless_unscaled:
    h = ac.G * M / D / ac.c**2 * (v / ac.c)**2
    # the units system takes care of all the unit conversion for us
    # since we specified that we want the result to be a pure number
    return(h)

print(f'Car crash: {estimate_h(1e3*u.kg, 10*u.m, 100* u.km/u.hr):.0e}')
print(f'Supernova: {estimate_h(1*u.Msun, 2*u.kpc, .2 * ac.c):.0e}')
print(f'Binary BH: {estimate_h(50*u.Msun, 400*u.Mpc, .1 * ac.c):.0e}')
```

which yields as output

```
Car crash: 6e-40
Supernova: 1e-18
Binary BH: 6e-23
```

The `u.quantity_input` decorator is not strictly necessary, but this way the function will raise a very clear `UnitsError` if we try to give it something with the wrong dimensionality as an input.

3 GW gauge

Detector frame Newtonian waves

As we will discuss later, the expression for the geodesic deviation is

$$\frac{\mathrm{d}^2 \xi^i}{\mathrm{d}\tau^2} = R^i{}_{\mu\nu j} u^\mu u^\nu \xi^j \,, \tag{3.1}$$

therefore we need to find out what the four-velocity is. To zeroth order it is $u^\mu = (1, \vec{0})$, and the first order contributions would result in a second-order term since the Riemann tensor itself is only nonzero at first order. Also, to zeroth order $\tau = t$. Also, the Christoffel symbols themselves are first order, so the $\Gamma\Gamma$ terms are of second order. With all of these considerations, the expression reduces to

$$\frac{\mathrm{d}^2 \xi^i}{\mathrm{d}t^2} = R^i{}_{ttj} \xi^j = \left(\partial_t \Gamma^i_{tj} - \partial_j \Gamma^i_{tt} \right) \xi^j \,. \tag{3.2}$$

The relevant Christoffel symbols for a perturbed metric $g = \eta + h$, where the only nonzero components of h are the spatial ones (which holds as long as we are in the TT gauge), are to first order

$$\Gamma^i_{tj} = \frac{1}{2} \eta^{ik} \left(2 g_{k(j,t)} - g_{tj,k} \right) \tag{3.3}$$

$$= \frac{1}{2} \eta^{ik} h_{kj,t} \tag{3.4}$$

$$\Gamma^i_{tt} = \frac{1}{2} \eta^{ik} \left(2 g_{k(t,t)} - g_{tt,k} \right) = 0 \,, \tag{3.5}$$

so we get

$$R^i{}_{ttj} = \partial_t \Gamma^i_{tj} - \underbrace{\partial_j \Gamma^i_{tt}}_{=0} \tag{3.6}$$

$$= \frac{1}{2}\eta^{ik} h_{kj,tt} \,. \tag{3.7}$$

Therefore, the equation for geodesic deviation in the proper detector frame to first order reads

$$\frac{\mathrm{d}^2 \xi^i}{\mathrm{d}t^2} = \left(\frac{1}{2} \frac{\partial^2 h^i_j}{\partial t^2} \right) \xi^j \,. \tag{3.8}$$

This can be interpreted as a Newtonian force, which acts on a particle with mass m as

$$F^i_{(h)} = m \left(\frac{1}{2} \frac{\partial^2 h^i_j}{\partial t^2} \right) \xi^j \,. \tag{3.9}$$

Fictional waves

We want to show that the perturbation

$$h_{\mu\nu} = -2\partial_{(\mu} \zeta_{\nu)} \tag{3.10}$$

$$\zeta_\mu = c_\mu \exp(-ik_\alpha x^\alpha) \tag{3.11}$$

is "fictional", that is, it is indistinguishable flat spacetime.

One way to do so is to explicitly show the transformation that maps this to flat spacetime: the perturbation transforms under diffeomorphisms as

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) + 2\partial_{(\mu}\chi_{\nu)}(x) \tag{3.12}$$

for a generic vector χ_ν, therefore if we set $\chi_\nu = \zeta_\nu$ we directly have $h'_{\mu\nu}(x') = 0$ identically.

Alternatively, we can compute the Riemann tensor: to linear order

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\Big(h_{\mu(\alpha,\beta)} - h_{\alpha\beta,\mu} \Big) \tag{3.13}$$

$$R_{\mu\nu\rho\sigma} = -2\Gamma_{\mu\nu[\rho,\sigma]} \,, \tag{3.14}$$

so we get

$$h_{\mu\nu} = 2ic_{(\mu}k_{\nu)} e^{-ik\cdot x} \tag{3.15}$$

$$h_{\mu\nu,\alpha} = -2c_{(\mu}k_{\nu)} k_\alpha e^{-ik\cdot x} \tag{3.16}$$

$$h_{\mu\nu,\alpha\beta} = +2ic_{(\mu}k_{\nu)} k_\alpha k_\beta e^{-ik\cdot x} \tag{3.17}$$

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\Big(h_{\mu\alpha,\beta} + h_{\mu\beta,\alpha} - h_{\alpha\beta,\mu} \Big) \tag{3.18}$$

$$= \frac{1}{2}(-2)\Big(c_{(\mu}k_{\alpha)}k_\beta + c_{(\mu}k_{\beta)}k_\alpha - c_{(\alpha}k_{\beta)}k_\mu\Big)e^{-ik\cdot x} \tag{3.19}$$

$$R_{\mu\nu\rho\sigma} = 2i\Big(c_{(\mu}k_{\nu)}k_{[\rho}k_{\sigma]} + c_{(\mu}k_{\rho)}k_{[\nu}k_{\sigma]} - c_{(\nu}k_{\rho)}k_{[\mu}k_{\sigma]}\Big)e^{-ik\cdot x}\,, \tag{3.20}$$

so all the terms in the Riemann tensor contain a term in the form $k_{[\mu}k_{\nu]} = 0$, since it is the antisymmetrization of two equal vectors.

4 NS surface tidal forces

We want to estimate the tidal forces on a $h = 1.6\,\mathrm{m}$ tall person standing on the surface of a nonrotating neutron star with $M = 1.4M_\odot$ and $R = 12\,\mathrm{km}$.[2]

We shall do so by comparing three different methods: Newtonian mechanics, weak-field GR and then full GR (still with the spherical, stationary NR assumption). The weak-field GR formulation will make the various approximations of the Newtonian approach more explicit.

Newtonian mechanics

Here, the gravitational acceleration $a(r)$ looks like

$$a(r) = -\frac{GM}{r^2}\,, \tag{4.1}$$

so the differential acceleration between the head and feet of our test person will be

$$\Delta a = -\frac{GM}{(r+h)^2} + \frac{GM}{r^2} \approx 2GM\frac{h}{r^3} + \mathcal{O}\Big((h/r)^2\Big)\,, \tag{4.2}$$

where the scale of the term we are neglecting is $(h/r)^2 \sim (10^{-4})^2 = 10^{-8}$ — definitely acceptable for this rough calculation.

With this expression we get $\Delta a \approx 3.4 \times 10^8\,\mathrm{m/s^2}$ (both in the exact case and with the approximation).

Weak-field GR

In general, the expression for the geodesic deviation d^μ corresponding to an observer described by a space-like vector ξ^μ moving along a geodesic with four-velocity u^μ is [Car19, eq. 3.208]

$$d^\mu = R^\mu{}_{\nu\rho\sigma}u^\nu u^\rho \xi^\sigma\,. \tag{4.3}$$

Our stationary observer on the surface is not moving along a geodesic, but as a working hypothesis we can assume that the acceleration due to being stationary in Schwarzschild

[2] A first sanity check: the corresponding Schwarzschild radius is around 4 km, so the compactness ratio Rc^2/GM is almost 6, well above the causal constraint of ~ 2.83 [LP07, eq. 5, fig. 2].

coordinates is shared by head and feet — it is the zeroth-order contribution, so to speak. The geodesic deviation d^μ goes directly to the first order.

In the weak-field approximation, we directly consider the component d^r to estimate $|\Delta a|$, since we assume (wrongly, in this case!) that the metric is not far off from the Minkowski one. The space-like displacement vector describing the person is $\xi^\mu = h\hat{e}^r$, while their four-velocity in this weak-field context is $u^\mu = \hat{e}^t$.

Therefore, the formula simplifies to

$$|\Delta a| \approx |d^r| = |hR^r{}_{ttr}| \,, \tag{4.4}$$

and we can approximate this component of the Riemann tensor as such:

$$-R^r_{ttr} = +R^r_{trt} = \partial_r \Gamma^r_{tt} - \underbrace{\partial_t \Gamma^r_{rt}}_{\text{zero by stationarity}} \underbrace{\Gamma^r_{r\mu}\Gamma^\mu_{tt} - \Gamma^r_{t\mu}\Gamma^\mu_{rt}}_{\text{second order}} \tag{4.5}$$

$$\approx \partial_r \Gamma^r_{tt} = \frac{1}{2}\partial_r \left[g^{r\mu} (\underbrace{2\partial_t g_{(\mu t)}}_{\text{zero by stationarity}} - \partial_\mu g_{00}) \right] \tag{4.6}$$

$$\approx -\frac{1}{2}\partial_r \partial_r g_{00} \approx \partial_r \partial_r \Phi \tag{4.7}$$

$$\approx \partial_r \partial_r \left(-\frac{GM}{r} \right) = \partial_r \frac{GM}{r^2} = -\frac{2GM}{r^3} \,, \tag{4.8}$$

so our result is exactly the same as the Newtonian one: $\Delta a = 2GMh/r^3 = 3.4\,\mathrm{m/s^2}$.

Full GR

Now that we have established the weak-field calculation, let us approach it accounting for the strong-field regime.

The geodesic deviation equation is still valid, but we must adapt the expressions for ξ^μ and u^μ: we want their moduli to be h and -1 respectively, and their directions to be $\hat{e}^r$ and $\hat{e}^t$. In order for this to be true we need to rescale the, so that their nonzero components will read $\xi^r = h/\sqrt{g_{rr}}$ and $u^t = 1/\sqrt{g_{tt}}$.

Also, we cannot only compute d^r: we must consider the full vector d^μ and compute its modulus $\sqrt{d^\mu d_\mu}$. Notice the lowered index: the component d^r appears in the modulus of the acceleration as $d^r\sqrt{g_{rr}}$, and $\sqrt{g_{rr}} \approx 1.23$ is noticeably different from 1 in this situation.

This way, we have all the components to plug into a computer algebra system like Cadabra:[3] the acceleration comes out to have d^r as its only nonvanishing component,

$$d^r = \frac{1}{r}\frac{h}{r}\frac{2GM}{r}\left(1 - \frac{2GM}{rc^2}\right)^{-1/2} \,, \tag{4.9}$$

whose modulus is then

$$|d| = d^r\sqrt{g_{rr}} = \frac{2GMh}{r^3} \,, \tag{4.10}$$

[3] See the code here!

which, incidentally, is proportional to the square root of the Kretschmann scalar $K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \propto G^2M^2c^{-4}r^{-6}$.

This is the same result we found in the weak field regime! The factors from the metric cancelled out.

Full GR, revised

There is a problem in the computation we did before: we applied the formula for geodesic deviation despite the fact that our observer is not moving along a geodesic.

An alternative way to calculate the tidal deformation would be to simply compute the acceleration at the feet and head and subtract them. Let us see how this approach goes.

Both the head and feet are stationary in Schwarzschild coordinates; their four-velocities are therefore time-directed, and normalized like before as $u^\mu = \hat{e}^t / \sqrt{g_{tt}}$.

The four-acceleration of an observer is the derivative of the four-velocity along itself (with respect to proper time):

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = u^\nu \nabla_\nu u^\mu \,, \tag{4.11}$$

whose only nonvanishing component is

$$a^r = \Gamma^r_{tt}\left(u^t\right)^2 = \frac{GM}{r^2} \,, \tag{4.12}$$

but, like before, we need to account for the metric component if we want to compute its modulus:

$$|a| = a^r \sqrt{g_{rr}} = \frac{GM}{r^2}\frac{1}{\sqrt{1 - 2GM/r}} \,. \tag{4.13}$$

Now, all that is left is to compute the difference between this quantity at r and $r + h$:

$$\Delta a = \frac{GM}{(r+h)^2}\frac{1}{\sqrt{1 - \frac{2GM}{c^2(r+h)}}} - \frac{GM}{r^2}\frac{1}{\sqrt{1 - \frac{2GM}{c^2 r}}} \,, \tag{4.14}$$

which could be approximated to linear order like before, but we can directly compute the difference: it comes out to be $\Delta a \approx 4.8 \times 10^8\,\mathrm{m/s^2}$.

This shows that, accounting for relativistic effects, an observer staying still on the surface of the star experiences $\sim 40\,\%$ more tidal acceleration.

References

[Car19] Sean M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. New York: Cambridge University Press, 2019. 1 p. ISBN: 978-1-108-77038-5 (cit. on p. 6).

[LP07] James M. Lattimer and Maddapa Prakash. “Neutron Star Observations: Prognosis for Equation of State Constraints”. In: *Physics Reports* 442.1-6 (Apr. 2007), pp. 109–165. ISSN: 03701573. DOI: 10.1016/j.physrep.2007.02.003. arXiv: astro-ph/0612440. URL: http://arxiv.org/abs/astro-ph/0612440 (visited on 2021-04-23) (cit. on p. 6).