Numerical Relativity @ Jena

Jacopo Tissino

2021-06-22

Introduction

Tuesday 2021-4-13, compiled 2021-06-22

The syllabus can be found here. There will also be exercise sessions; the exercises will be posted on the same webpage.

The hydrodynamic part of the program is more pertinent to the computational hydrodynamics course.

There will also be a final project. The most recommended books are those by Alcubierre and Baumgarte-Shapiro. A good standard GR reference is Wald. The notes by Ghourghoulon are good and complete. The numerical methods reference by Choptuik is quite good.

There are **four pillars**:

1. we need to **formulate GR as a set of PDEs** (a Cauchy problem, really), including relativistic hydrodynamics (also, at this point we should classify these equations — are they hyperbolic, elliptic?);

2. some issues we encounter are the problems with **coordinates and singularities**: after all, we are interested in black holes and other extreme objects, so we must be able to work around horizons — the gauge is arbitrary, but some choices are better than others, we can gauge away coordinate singularities, but we must also work around the physical singularities;

3. we should use **numerical methods for the solution of PDEs on adaptive grids**;

4. the calculations are quite expensive, so we need **high performance computing** and easily parallelizable code.

We will mostly discuss the first two pillars in this course.

A landmark paper is one by Pretorius in 2006; his code already had several very useful characteristics.

We need **excision** to ignore the interior of the BHs, but we need to track them since they move.

In the 1970s there were already precursors to NR, such as throwing test masses at black holes.

Do the horizons merge in a continuous fashion? Yes, but it's hard to write shock-free gauge equations. We shall explore this later.

The definition of an event horizon depends on global properties of the spacetime, so it cannot be done "at runtime": we must store the data for all the simulation and then do raytracing.

On the other hand, an apparent horizon is local and gauge dependent, and we can compute these at runtime. It is useful to find it since, if it exists, it is always inside or coincident with the event horizon.

The result of a BNS merger is not necessarily a BH immediately.

We also have gravitational collapse of a scalar field with a variable energy. Letting it evolve according to GR leads to BH formation. There is a phase transition, the parameter is p and we get $M_{BH} \sim |p - p_*|^\beta$.

Do we have self-similarity for all p or only for $p = p_*$?

NR can also be used to study stringy higher-dimensional BHs.

1 Setting up the geometry

The idea is to introduce a notion of time by foliating the manifold. We must restrict ourselves to globally hyperbolic spacetimes. There must be no ugly things like CTCs.

The lapse function controls the foliation: it defines the proper time of Eulerian observers.

How is the foliation Σ "bent" into the 4D manifold? How does $\hat{n}$ change as we transport it along Σ?

The "velocity" is defined by a curvature $K_{\mu\nu}$, which is defined as

$$K_{\mu\nu} = -\gamma^\alpha{}_\mu \nabla_\alpha n_\nu \,, \tag{1.1}$$

where γ is the metric restricted to the Σ_t foliation.

The fundamental variables are then: the three-metric γ_{ij}, the "velocity" K_{ij}, the lapse α and β^i, as well as j_i and S_{ij}.

The equations we will write for these by manipulating the Einstein equations can be decomposed into *spacetime dynamical equations*:

$$\left(\partial_t - \mathcal{L}_{\vec{\beta}}\right)\gamma_{ik} = -2\alpha K_{ik} \tag{1.2}$$

$$\left(\partial_t - \mathcal{L}_{\vec{\beta}}\right)K_{ik} = -D_i D_k \alpha + \alpha\left({}^{(3)}R_{ik} - 2K_{ik}K^j{}_k + K_{ik}\right) - 8\pi\alpha\left(S_{ik} - \frac{1}{2}\gamma_{ik}(S - E)\right), \tag{1.3}$$

as well as two *constraints*:

$$ {}^{(3)}R + K^2 - K_{ik}K^{ik} = 16\pi E \tag{1.4}$$

$$D_k(K\gamma^k{}_i) = 8\pi j_i \,, \tag{1.5}$$

and the latter do not involve any Lie derivatives: they are specific to a single Σ_t. Also, we will need matter dynamical equations: from the stress-energy tensor we define

$$S_{ik} = \gamma^\mu{}_i \gamma^\nu{}_k T_{\mu\nu} \tag{1.6}$$

$$
\begin{aligned}
S &= S^i{}_i & (1.7)\\
j_i &= -\gamma^\mu{}_i n^\nu T_{\mu\nu} & (1.8)\\
E &= T^{\mu\nu} n_\mu n_\nu\,, & (1.9)
\end{aligned}
$$

and from $\nabla_\mu T^{\mu\nu} = 0$ we can write

$$
\partial_t q_\mu + \partial_i F^i_\mu(q) = s_\mu\,. \tag{1.10}
$$

We will need to make sure that the problem is well-posed, so that solutions exist and depend continuously on the parameters.

Slicing and coordinate choices

The gauge is defined by α and β^i: we can freely choose them. We want to have smoothness, to avoid singularities, to minimize grid distortion and for the problem to be well-posed.

The time is defined by the lapse: if we take $\alpha \equiv 1$ everywhere we have $\vec{a} = \nabla_n \vec{n} = D \log \alpha = 0$; this is called **geodesic slicing**, and we know that Eulerian observers follow geodesics. However, geodesics are "looking for" singularities, in that they easily fall inside them.

We could ask our hypersurfaces to bend as little as possible: we could minimize the trace of the extrinsic curvature, or even set $K = \nabla_a n^a = 0$. This allows our gauge not to create black holes. This translates to an elliptic equation for the lapse, to be solved together with the others.

This condition can be found to be equivalent to the maximization of the contained volume.

If we use geodesic slicing the simulation fails at time $\tau = \pi$ as the first gridpoint reaches the singularity. If we do excision by removing gridpoints as they fall in things are better but the simulation still fails.

If, instead, we impose $\partial_t \alpha = -\alpha K$, the foliation "freezes" as it passes the horizon, since the lapse function will prevent it. However, this means we will have large gradients in space as well as in time: as the grid points inside freeze the ones outside keep evolving. However, we can use a clever technique to minimize the distortion.

2 Introduction

What is the problem in NR? We have the Einstein equation $G_{ab} = 8\pi T_{ab}$, in geometric units $c = G = 1$, and we want to discuss its solutions which cannot be expressed analytically:

1. gravitational collapse;
2. BH or NS collisions;
3. dynamical stability of some stationary solutions — for example, Kerr black holes.

The commonalities among these are: strong gravity, the absence of symmetries (no Killing vectors), and the fact that these are dynamical.

We need to:

1. formulate the EFE as a PDE system;
2. check that it is well-posed;
3. simulate this using certain numerical algorithms;
4. extract information: we should use gauge-invariant quantities, such as gravitational waves or to-be-specified "energies".

Regarding the third point, we need to be able to solve nonlinear PDES: these can be elliptic or hyperbolic, deal with space-time decomposition, deal with singularities, and use high-performance computing effectively.

The PDE system

Let us start by treating the equations in vacuo: they reduce to $R_{\mu\nu} = 0$. The Ricci tensor can be expressed explicitly as

$$0 = R_{\mu\nu} = \underbrace{-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} - g_{\alpha(\mu}\partial_{\nu)}H^\alpha}_{\text{principal part}} + Q_{\mu\nu}[g, \partial g] \tag{2.1}$$

$$H^\alpha = \partial_\mu g^{\alpha\mu} + \frac{1}{2}g^{\alpha\beta}g^{\rho\sigma}\partial_\beta g_{\rho\sigma}\,, \tag{2.2}$$

where the part denoted as the principal contains the highest derivatives of the metric, while the part denoted as Q is less important.

It seems like we have 10 equations for $g_{\mu\nu}$, so we are done! We actually do not, because of the Bianchi identities: $\nabla_a G^{ab} = 0$. These are four more equations, which we need to consider.

Our questions are:

1. what type of PDEs are these?
2. how do we formulate an initial/boundary value problem?
3. are these PDE problems well-posed?

Definition 2.1 (Well-posed PDE problem)**.** *It is a problem in which a unique solution exists, it is continuous, and it depends continuously on the boundary data.*

Maxwell equations in flat spacetime

Let us start with these as a reference. They are written in terms of the antisymmetric Maxwell-Faraday tensor $F_{\alpha\beta}$, and they read

$$0 = \partial^\alpha F_{\alpha\beta} = \partial^\alpha \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha \right) , \tag{2.3}$$

where A^α is the vector potential. We could naively interpret these as four wave-like equations for the vector potential because of the $\Box A$ operator — however, this is not true. Let us show it by looking at the $\beta = 0$ equation:

$$\begin{aligned} 0 &= \partial^\alpha \partial_\alpha A_0 - \partial^\alpha \partial_0 A_\alpha = \Box A_0 - \partial_0 \partial^\alpha A_\alpha & (2.4) \\ &= -\partial_0^2 A_0 + \partial_i \partial^i A_0 + \partial_0^2 A_0 - \partial_0 \partial^i A_i & (2.5) \\ &= \partial^i (\partial_i A_0 - \partial_i A_0) = \partial^i F_{0i} = \partial^i E_i = C \, , & (2.6) \end{aligned}$$

where $E_\alpha = F_{\alpha 0} = F_{\alpha\beta} n^\beta$, where n^β is a unit four-vector in the time direction.

Importantly, this equation ($C = 0$) does not contain second time derivatives (they cancelled out). Can we get a wave equation from $\partial^0 C = 0$, maybe? No: the computation goes like

$$\begin{aligned} \partial^0 C &= \partial^0 (\partial^\alpha \partial_\alpha A_0 - \partial_0 A_\alpha) & (2.7) \\ &= \partial^i \left[\partial^\alpha (\partial_\alpha A_i - \partial_i A_\alpha) \right] , & (2.8) \end{aligned}$$

where we used the fact that $\partial^\alpha \partial^\beta F_{\alpha\beta} = 0$. The thing we are taking a derivative of is the left-hand side of the Maxwell equations for $\beta = i$! If A is a solution, the equation $\partial^0 C = 0$ is an *identity*, it does not constrain the solution in any way.

In summary, the Maxwell equations for A^α are 3 evolution equations (which contain second time derivatives) and the one $C = 0$ equation, which is a *constraint.*

What this means is that the Maxwell equations are **undetermined**! Three evolution equations, four unknowns. They are not well posed according to our mathematical definition.

However, from a physical point of view this is not a problem: we know that there is gauge freedom in electromagnetism, so the couple E_α, B_α are calculated from A_α up to a gauge transformation. A_α and $A_\alpha + \partial_\alpha \phi$ for any scalar field ϕ represent the same electric and magnetic fields.

What we can then do is to fix the gauge: we can choose, for example the Lorentz gauge $\partial_\alpha A^\alpha = 0$.

After doing so, the Maxwell equations reduce to $0 = \partial_\alpha \partial^\alpha A_\beta = 0$. These are 4 dynamical equations (containing $\partial_0 \partial_0$) for 4 unknowns A_α. This, then is a well-posed Cauchy problem.

Does the Lorentz Gauge hold for all time? Yes, because

$$0 = \partial^\beta \left(\Box A_\beta \right) = \Box (\partial^\beta A_\beta) \, , \tag{2.9}$$

so the quantity in parentheses is zero initially and its derivative is always zero: then, it remains zero.

What happened to the $C = 0$ constraint? If $C = 0$ initially then the condition $C = 0$ remains:

$$C = \Box A_\alpha - \partial^\alpha \partial_0 A_\alpha = \Box A_\alpha - \partial_0 \partial^\alpha A_\alpha = 0\,, \tag{2.10}$$

which is usually stated as: "**the constraints are transported along the dynamics**".

Finally then we can say that the Maxwell equations, in an appropriate gauge, are well-posed.

The EFE

In the EFE case, the Bianchi identities have the same role as our $C = 0$ constraint.

If n^b is a timelike vector field, then the projection of the Einstein equations

$$0 = G_{ab} n^a = C_b[\partial_i^2 g, \partial g, g]\,, \tag{2.11}$$

are *constraint* equations. On the other hand, the Bianchi identities guarantee that these are transported along the dynamics.

If we pick a gauge such that $0 = C^\mu = G^{0\mu}$, then the Bianchi identities read

$$\nabla_\alpha G^{\alpha\mu} = 0 \tag{2.12}$$

$$\partial_0 G^{0\mu} = -\partial_k G^{k\mu} - \Gamma^\mu_{\alpha\beta} G^{\alpha\beta} - \Gamma^\alpha_{\alpha\rho} G^{\mu\rho}\,, \tag{2.13}$$

and they contain at most $\partial_0^2 g$.

Can we write the Einstein equations in a way that makes them explicitly well-posed? Let us ignore the term $Q_{\mu\nu}$ — the important part is the principal one. The first term is already $g^{\alpha\beta}\partial_\alpha\partial_\beta$, the Dalambertian. The part we do not like is the derivative of H^α.

Can we say that $H^\alpha = 0$? Yes! This is the Hilbert / Lorentz / Harmonic gauge. Then the EFE read

$$\Box_g g_{\mu\nu} \simeq 0\,, \tag{2.14}$$

where the $\simeq$ sign denotes the fact that this is only considering the principal part.

Causality and globally hyperbolic spacetime

What we have seen so far is still not enough to show that the PDE system is, as a whole, hyperbolic. For that, we need to consider the eingenvalues of the problem.

We have initially assumed a "global motion of time".

Note that we do not mean a specific global time: in SR we had such a notion, because of the light-cone structure. An equation like $\Box_f \phi = 0$ is always "clear", its character is always determined by the lightcone structure — the condition at a point is affected by the past light cone, and it affects the future lightcone.

In GR this is not the case in general! An example are closed timelike curves.

In order to not have this problem, we need to restrict ourselves to a smaller class of spacetimes.

Definition 2.2. *An achronal set $S \subset M$ if made of events which are not connected by timelike curves.*

Definition 2.3. *The future domain of dependence $D_+(S)$ is the set of events such that every causal curve starting from a point in $D_+(S)$ intersects S in the past.*

Definition 2.4. *The future Cauchy horizon $H_+(S)$ is the boundary of $D_+(S)$.*

We can make an analogous definition by substituting the past for the future, and the plus for a minus.

Definition 2.5. *The domain of dependence of S is $D(S) = D_+(S) \cup D_-(S)$.*

Definition 2.6. *A **Cauchy surface** is a hypersurface $\Sigma \subset M$ of spatial character such that $D(\Sigma) = M$, or $H(\Sigma) = 0$.*

A property of Cauchy surface is that every causal curve intersects Σ exactly once.

Then we can say that

Definition 2.7. *A manifold (M, g) is **globally hyperbolic** iff there exists a Cauchy surface $\Sigma \subset M$.*

This is a key hypothesis in numerical relativity. We restrict our class of solutions to globally hyperbolic spacetimes.

There is a theorem ensuring that the initial value problem $\Box_G \phi = 0$ is well-posed in globally hyperbolic spacetimes.

3 3 + 1 geometry

Tuesday 2021-4-27, compiled 2021-06-22

We take our 4D spacetime (M, g, ∇) and equip it with a scalar field $t \colon M \to \mathbb{R}$ such that the vector

$$(\mathrm{d}t)^a = g^{ab} (\mathrm{d}t)_b \tag{3.1}$$

is timelike.

This defines 3D spatial hypersurfaces, whose normal vector is $n^a = -\alpha(\mathrm{d}t^a)$.

These surfaces are denoted as Σ_t, and their tangent vectors $v^a \in T_P(\Sigma_t)$ are all spacelike.

Embedding is a bijective map ϕ from an $n - 1$ dimensional manifold $\hat{\Sigma}$ to a subset Σ of an n dimensional manifold M.

We can move tensor fields from $\hat{\Sigma}$ to M and back: these are known as the pullback and pushforward operations. Not all fields can be moved in this way.

The idea is to identify Σ with the manifold $\hat{\Sigma}$ — however, they are distinct conceptually.

A metric γ on Σ is given by the pullback of g on M: this is the **induced** metric $\gamma = \phi^* g$. The components of this metric are calculated as

$$\gamma_{ij} = \frac{\partial x^\alpha}{\partial x^i} \frac{\partial x^\beta}{\partial x^j} g_{\alpha\beta} \,. \tag{3.2}$$

Also, we can define a connection D and a Riemann tensor $\mathcal{R}$ on the submanifold; its components are $\mathcal{R}_{ijkl}$. This is the "internal", or "intrinsic curvature" of the submanifold.

There is another curvature we could define: how does Σ deform into M? This is the **extrinsic curvature**.

For example, a 1D curve has zero intrinsic curvature, but it can bend. This is determined by the map

$$K\colon T_P(\Sigma) \times T_P(\Sigma) \to \mathbb{R}\,, \tag{3.3}$$

which acts on two vectors $u, v \in T_P(\Sigma)$ as $K(u,v) = K_{ab} u^a v^b = -u_a v^b \nabla_b n^a$.

As an example, take $M = \mathbb{R}^3$ and $\Sigma = \mathbb{R}^2$; the metric on M is the identity and the Riemann tensor vanishes, so we get $\mathcal{R} = K = 0$.

If, instead, we take $\Sigma = C^2$, the surface of a cylinder, we will have $\mathcal{R}_{ijkl} = 0$ since we can deform it into a plane, while

Claim 3.1. *the extrinsic curvature of this objects has a nonvanishing component $k_{\varphi\varphi} = -a$, where a is the radius, while the trace reads $k = k^i_i = -1/a$.*

On the other hand, if we take a sphere we have both $\mathcal{R}_{ijkl} \neq 0$ and $k_{ij} = 0$; also, the trace reads $k = -2/a$.

We can use **projectors** to decompose tensors on M into tensors on Σ and "pieces" along n. This is because we can decompose the tangent space of the ambient manifold into

$$T_P(M) = V_P(n) \oplus T_P(\Sigma)\,, \tag{3.4}$$

so that a vector v^α is written as $v_\perp n^a + v^a_\parallel$, where $v_\parallel \in T_P(\Sigma)$.

We can define a projector P that maps a vector v^a to $v^a_\parallel$. Explicitly, this will look like

$$P^a{}_b = \delta^a{}_b + n^a n_b\,. \tag{3.5}$$

The induced metric can alternatively be written as the projection of the four-metric into Σ:

$$\gamma_{ab} = P^c_a P^d_b g_{cd} = g_{ab} + n_a n_b\,. \tag{3.6}$$

This is coordinate independent! Also, we can write the projectors as

$$P^a_b = \gamma^a_b = g^{ac}\gamma_{cb}\,, \tag{3.7}$$

therefore we will not write P anymore — we can just use γ.

We also need the covariant derivative: schematically, it is written as

$$DT = P \dots P \nabla T\,. \tag{3.8}$$

Also, the extrinsic curvature reads

$$K_{ab} = -\gamma^c_a \gamma^d_b \nabla_{(c} n_{d)}\,. \tag{3.9}$$

Using P on every tensor is what "putting into 3+1 form" means.

Eulerian observers

These are observers whose worldlines are defined by n: Σ_t is the set of all events which are simultaneous to the Eulerian observers.

Definition 3.1. *The acceleration of E. observers is defined as*

$$a_a = n^b \nabla_b n_a \,, \tag{3.10}$$

therefore $a_a n^a = 0$.

Definition 3.2. *The normal evolution vector* $m^a = \alpha n^a$ *is defined so that*

$$\nabla_m t = m^a (\mathrm{d}t)_a = +1 \,. \tag{3.11}$$

Claim 3.2. *3+1 geometry defines the kinematic of 3+1 GR. Specifically, the claims we make are:*

1. *The vector* m *carries points from* Σ_t *to* $\Sigma_{t+\delta t}$.
2. *The lapse function* α *relates* t *to the proper time of Eulerian observers.*
3. *The Lie derivative along* m, $\mathscr{L}_m$, *transports tensors from* Σ_t *to* $\Sigma_{t+\delta t}$.
4. $\mathscr{L}_m \gamma = -2\alpha K$.

Let us give some hints: if P is on Σ_t and P' is on $\Sigma_{t+\delta t}$, then

$$t(P') = t(P + \delta t \cdot m) = t(p) + \delta t \underbrace{m^a (\mathrm{d}t)_a}_{=1} = t(P) + \delta t \,, \tag{3.12}$$

while for the second point, the change in proper time reads

$$\mathrm{d}\tau^2 = -g(\delta t m, \delta t m) = -m^a m_a \, \mathrm{d}t^2 = \alpha^2 \, \mathrm{d}t^2 \,, \tag{3.13}$$

which is the reason for the term “lapse function”.

The third point follows from the first and the definition of the Lie derivative.

As for the fourth, we have

$$\begin{aligned}
\mathscr{L}_n \gamma_{ab} &= \mathscr{L}_n \big(g_{ab} + n_a n_b \big) && (3.14)\\
&= 2\nabla_{(a} n_{b)} + n_a \mathscr{L}_n n_b + n_b \mathscr{L}_n n_a && (3.15)\\
&= 2 \Big[\nabla_{(a} n_{b)} + n_{(a} a_{b)} \Big] = -2K_{ab} \,. && (3.16)
\end{aligned}$$

The last passage is left as an exercise: we start by carrying out the projections, then simplify terms in the form $nn\nabla n$ by substituting the acceleration.

The Lie derivative along n can always be written as

$$\mathscr{L}_n \gamma_{ab} = \phi^{-1} \mathscr{L}_{\phi n} \gamma_{ab} \tag{3.17}$$

for any scalar field.

Now we have three expressions for K_{ab} — each of them can be taken to be the definition, and the others can be derived from it.

What is the physical interpretation of these expressions? They tell us several things:

1. Σ_t and $\Sigma_{t+\delta t}$ are identified by the diffeomorphism generated by m;
2. the spacetime (M, g) is the "time" development of (Σ, γ), where the "time evolution" is governed by $\mathscr{L}_m$;
3. we can identify γ as the "main variable" of (3+1) GR, and also we identify K as the "velocity" of γ: the equation $\mathscr{L}_m \gamma = -2\alpha K$ is in the form "time derivative of variable = velocity", a kinematic equation.

This all hinges on the possibility to define these non-intersecting hypersurfaces.

Tuesday 2021-5-4, compiled 2021-06-22

The part which is missing from the discussion of the 3+1 formalism are the **Gauss-Codazzi-Ricci** equations: projections and contractions of the Riemann tensor. They connect the four-dimensional Riemann tensor to the three-dimension Riemann and extrinsic curvature tensors. They are:

$$
\begin{aligned}
\gamma_a^p \gamma_b^q \gamma_c^r \gamma_d^{s\,(4)} R_{pqrs} &= R_{abcd} + K_{ac} K_{bd} - K_{ad} K_{bc} && (3.18)\\
\gamma_a^p \gamma_b^q \gamma_c^r n^{s\,(4)} R_{pqrs} &= D_a K_{bc} - D_b K_{ac} && (3.19)\\
\gamma_{ap} n^r \,, & && (3.20)
\end{aligned}
$$

And more, also in contracted versions — see the notes.

We will then use these in order to write the EFE in terms of γ, K with spatial covariant derivatives and Lie derivatives along "time".

The final answer will be that the EFE can be into constraints and evolution equations — the latter contain the time derivative, and can be divided into kinematic and dynamic equations.

The final equation we will find is commonly called the ADM formulation of the EFE, and we will use an expression of these in a form which is due to York — we will denote these as ADMY.

We start by looking at the Hamiltonian constraint: we contract the EFE with temporal vectors and find

$$
\begin{aligned}
{}^4G_{ab} n^a n^b &= {}^4 R_{ab} n^a n^b - \frac{1}{2} {}^4R \underbrace{g_{ab} n^a n^b}_{= n^a n^b (\gamma_{ab} - n_a n_b)} && (3.21)\\
&= {}^4 R_{ab} n^a n^b + \frac{1}{2} {}^4R \,, && (3.22)
\end{aligned}
$$

which is convenient since what we find is just half of the Gauss equations: using those, we get

$$
{}^4G_{ab} n^a n^b = \frac{1}{2} \left(R + K^2 - K_{ab} K^{ab} \right) , \tag{3.23}
$$

where R is the three-dimensional Ricci trace, while K is the trace of K_{ab}. The other side of the equation reads

$$
T_{ab} n^a n^b = E \,, \tag{3.24}
$$

where E is the energy density measured by Eulerian observers. The constraint then reads

$$C_0 = R + K^2 - K_{ab}K^{ab} - 16\pi E = 0\,. \tag{3.25}$$

This is a scalar equation defined on Σ which only contains spatial quantities and derivatives.

As for the **momentum constraint**, we take one spatial and one temporal component:

$${}^4G_{pq}n^q\gamma^p_a = {}^4R_{pq}\gamma^p_a n^q - \frac{1}{2}{}^4R g_{qp}\gamma^p_a n^q\,, \tag{3.26}$$

where the second term can be simplified with the help of

$$g_{qp}\gamma^p_a n^q = \left(\gamma_{qp} - n_q n_p\right)\gamma^p_a n^q = 0\,, \tag{3.27}$$

since contractions of γ and n always yield zero. We are then left with the four-dimensional Ricci tensor, which is precisely the left-hand side of the contracted Codazzi equations: this means that we can write it as

$${}^4R_{pq}\gamma^p_a n^q = D_a K - D_b K^b_a\,, \tag{3.28}$$

while for the matter term we get

$$T_{pq}\gamma^p_a n^q = P_a\,, \tag{3.29}$$

where P_a is the momentum density measured by Eulerian observers.

This constraint can then be written as

$$\boxed{C_a = D_b K^b_a - D_a K - 8\pi P_a = 0\,,} \tag{3.30}$$

which is called the ADM momentum. This is a rank-1 tensor equation in three-space, and it only contains spatial derivatives.

Let us then move to the dynamical evolution equation: we consider the trace-reversed EFE

$${}^4R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2}T g_{ab}\right)\,, \tag{3.31}$$

which is helpful because it will allow us to use the identities we have. Contracting twice in the spatial direction gives us

$${}^4R_{pq}\gamma^p_a\gamma^q_b\,, \tag{3.32}$$

which is the left-hand side of the contracted Ricci equations combined with the contracted Gauss equations. What is then left is only to deal with the right-hand side:

$$T_{pq}\gamma^p_a\gamma^q_b = -S_{ab}\,, \tag{3.33}$$

where S_{ab} is the stress tensor measured by Eulerian observers, a purely spatial term, while the other term is the trace

$$S = g^{ab} S_{ab} , \tag{3.34}$$

and we can verify that

$$T_{ab} = S_{ab} + 2n_{(a} P_{b)} + n_a n_b E , \tag{3.35}$$

whose trace reads $T = g^{ab} T_{ab} = S - E$.

Clarify the minus sign: wouldn't we have

$$\gamma^a_p \gamma^b_q \Big(S_{ab} + 2n_{(a} P_{b)} + n_a n_b E \Big) = \gamma^a_p \gamma^b_q S_{ab} , \tag{3.36}$$

with no minus sign?

At this point we have all the parts for the kinematic equation:

$$\mathscr{L}_m K_{ab} = -D_a D_b \alpha + \alpha \Big\{ R_{ab} + K K_{ab} - 2 K_{ac} K^c_b + 4\pi \big[(S - E) \gamma_{ab} - 2 S_{ab} \big] \Big\} , \tag{3.37}$$

which could be called $ADM - K$. It contains $\mathscr{L}_m$, it is a rank 2 symmetric tensor equation (6 components).

In this ADMY formalism we then have the equations $C_0 = 0$ (ADM-H), $C_a = 0$ (ADM-M), $\mathscr{L}_m \gamma_{ab} = -2\alpha K_{ab}$ (ADM-γ) (kinematic equation), and the equation $\mathscr{L}_m K_{ab} = \dots$ (a dynamical equation). In total we have 6+6 dynamical equations, plus the constaints.

These equations involve γ and K as fundamental variables, various derivatives ($\mathscr{L}_m$, D, DD).

Also, we have lapse α and shift β^a (the latter is inside the expression for the Lie derivative) appearing in the equations, but there are no equations specifying them: this reflects the gauge freedom for the choice of foliation, and the three spatial coordinates on Σ. These must be **prescribed**.

One choice is to use **adapted coordinates**: if we take $x^\mu = (t, x^i)$ we find a natural basis for the tangent space, $e_\mu = \partial_\mu = (\partial_t, \partial_i)$.

The vector m is not necessarily aligned with ∂_t: if we start from a point $x^i(t) = c^i$ and move a point along ∂_t from Σ_t to $\Sigma_{t+\delta t}$ it will land on $x^i(t + \delta t) = c^i$ again.

insert figure

In general, though,

$$(\partial_t)^a = m^a + \beta^a , \tag{3.38}$$

where the vector β^a is spatial; if we move from $x^i(t) = c^i$ along m^a we will land on $x^i(t + \delta t) = c^i - \beta^i \delta t$.

The norm of this vector will read

$$(\partial_t)^2 = m^2 + \beta^2 = -\alpha^2 + \beta^2 , \tag{3.39}$$

since $m^a = n^a \alpha$. Therefore, ∂_t is not necessarily timelike: depending on our choice of α and β we can make it timelike, null or spacelike.

What does the possibility to have superluminal shift mean? We can make a foliation choice which avoids singularities — for example, in order to evolve the Schwarzschild metric. This is not a problem: we can change our *labels* of points arbitrarily fast.

We can specify everything in adapted coordinates:

$$(\partial_t)^a = (1, 0, 0, 0) \tag{3.40}$$

$$\beta^a = (0, \beta^i) \tag{3.41}$$

$$n^a = \alpha^{-1}\big((\partial_t)^a - \beta^a\big) = (1/\alpha, -\beta^i/\alpha) \tag{3.42}$$

$$n_a = (-\alpha, \vec{0})\,. \tag{3.43}$$

The metric components, on the other hand, read

$$g_{00} = g(\partial_t, \partial_t) = -\alpha^2 - \beta^2 \tag{3.44}$$

$$g_{0i} = g(\partial_t, \partial_i) = \beta_i \tag{3.45}$$

$$g_{ij} = g(\partial_i, \partial_j) = \gamma_{ij}\,. \tag{3.46}$$

Typically, the metric determinants are denoted as $g = \det g_{\mu\nu}$ and $\gamma = \det \gamma_{ij}$.

The inverse component $g^{00} = -\alpha^{-2} = \det g_{ij} / \det g_{\mu\nu} = \gamma/g$: this is a convenient way to show that $\sqrt{-g} = \alpha\sqrt{\gamma}$.

The full inverse metric reads

$$g^{\mu\nu} = \begin{bmatrix} -\alpha^{-2} & \alpha^{-2}\beta^i \\ \alpha^{-2}\beta^j & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{bmatrix}. \tag{3.47}$$

Note that matrices are not inverted by block: $g^{ij} \neq \gamma^{ij}$, even though $g_{ij} = \gamma_{ij}$.

The derivatives can also be expressed in coordinate form:

$$\mathscr{L}_m = \mathscr{L}_{\partial_t} - \mathscr{L}_\beta = \partial_t - \mathscr{L}_\beta \tag{3.48}$$

$$\mathscr{L}_\beta \gamma_{ij} = \beta^k D_k \gamma_{ij} + 2D_{(i}\beta_{j)} = 2\partial_{(i}\beta_{j)} - 2\Gamma^k_{ij}\beta_k\,, \tag{3.49}$$

while

$$\mathscr{L}_\beta K_{ij} = \beta^k \partial_k K_{ij} + K_{ik}\partial_j \beta^k + K_{jk}\partial_i\beta^k\,. \tag{3.50}$$

Now we can finally write the ADMY as PDEs: we make a choice, and write them in **geodesic gauge**, which means $\alpha = 1$ and $\beta^i = 0$. This means that in this gauge $\tau = t$, the proper time is the time measured by Eulerian observers, and $\partial_t = m$.

Recall: the acceleration of Eulerian observers is

$$a_a = n^b \nabla_b n_a = D_a \log \alpha\,, \tag{3.51}$$

which is zero in geodesic gauge, since we set α to be constant. This means that the worldlines of Eulerian observers are geodesics (in geodesic gauge).

The system is then

$$\partial_t \gamma_{ij} = -2K_{ij} \tag{3.52}$$

$$\partial_t K_{ij} = R_{ij} + KK_{ij} - 2K_{ik}K^k_j + \text{matter term} \tag{3.53}$$

$$C_0 = R + K^2 - K_{ij}K^{ij} - 16\pi E = 0 \tag{3.54}$$

$$C_i = D_j K^j_i - D_i K - 8\pi P_i = 0\,. \tag{3.55}$$

What type of PDEs are these? The evolution equations are first-order in time and second-order in space, wave-like equations; if we try to linearize them with $\gamma_{ij} \approx f_{ij} + h_{ij}$ we get

$$\begin{cases} \partial_t h_{ij} & = -2K_{ij} \\ \partial_t K_{ij} & \sim R_{ij} \sim -\frac{1}{2}\partial_k \partial^k \gamma_{ij} + \dots \end{cases}, \tag{3.56}$$

with some other terms, which we can safely ignore. This is the symmetric 2-tensor version of the wave equation! The Dalambertian is $\Box\phi = \left(-\partial_t\partial_t + \partial_k\partial^k\right)\phi$, which can be put into the form

$$\begin{cases} \partial_t \phi & = -\pi \\ \partial_t \pi & = \partial_k \partial^k \phi\,. \end{cases} \tag{3.57}$$

Of course, because of the terms we ignored this is not formal, but it captures the spirit of the equations.

We can rewrite the ADMY in geodesic gauge as a "wave equation" for γ_{ij}, by substituting back $K_{ij} = -\partial_t \gamma_{ij}$. In principal part (highest order derivatives), this yields

$$-\ddot{\gamma}_{ij} + \gamma^{kl}\left(\partial_k\partial_l\gamma_{ij} + \partial_i\partial_j\gamma_{kl} - \partial_i\partial_k\gamma_{jl} - \partial_j\partial_k\gamma_{il}\right) \approx 0\,. \tag{3.58}$$

Is it relevant that this looks similar to the Riemann tensor in Riemann normal coordinates?

This is not just an exercise: it is important since it is useful in order to classify the PDEs. The inverse spatial metric $\gamma^{kl} = \gamma^{kl}(\gamma_{ij})$ is just some rational polynomial of γ_{ij}; we then have a second order in both time and space, **quasilinear** equation system (this means that it is linear in the highest derivative).

If we had written the non-principal part as well, we'd see that it is quadratic in the first order spatial metric derivatives.

Tuesday 2021-5-11, compiled 2021-06-22

We continue our discussion of 3+1 GR.

The Euler equations read

$$-\ddot{\gamma}_{ij} + \gamma^{kl}(\gamma_{ij})\left[\partial_k\partial_l\gamma_{ij} + \partial_i\partial_j\gamma_{kl} - 2\partial_{(i|}\partial_k\gamma_{|j)k}\right] \simeq 0 \tag{3.59}$$

in principal part, which is a quasilinear hyperbolic equation.

We start out by discussing what kind of equations the constraints are. They read

$$0 = C_0 \approx \gamma^{ik}\gamma^{jl}\partial_k\partial_l\gamma_{ij} - \gamma^{ij}\gamma^{kl}\partial_k\partial_l\gamma_{ij} \tag{3.60}$$

$$0 = C_i = \gamma^{jk}\partial_j\dot{\gamma}_{ki} - \gamma^{kl}\partial_i\dot{\gamma}_{kl} \,. \tag{3.61}$$

These only contain second spatial derivatives or γ_{ij} or $\dot{\gamma}_{ij}$. The first one kind of looks like a Laplacian, so we might be tempted to say it is elliptic-like, but actually they are of **no standard type**, although they have been studied and classified this has not been done in this formulation.

Can we solve them by prescribing some boundary data? This leads to the so-called *initial data problem* in GR.

We have $6+6 = 12$ unknowns, and 4 equations: how can we make this initial data? We must *prescribe* 8 quantities on Σ_0, the first surface — these are called *free data* — and then solve $C_\mu = 0$ for the remaining 4 quantities.

In order to prescribe the free data we will need the **conformal decomposition**, which will be discussed later. Assuming we have a solution, do we need to then solve $C_\mu = 0$ for later times? **No**, since the constraints are **transported along the dynamics** as a consequence of the Bianchi identities $\nabla^b G_{ab} = 0$.

We will prove this using a simple approach from the $Z4$ system, which is a bit more elegant than differentiating the constraints and working everything out.

Consider the following extended theory, which we could introduce from an action but do not need to:

$$^4G_{ab} + 2\nabla_{(a}Z_{b)} - g_{ab}\nabla_c Z^c = 8\pi T_{ab} \,, \tag{3.62}$$

which reduces to GR when $Z^a = 0$ or if Z^a is a Killing vector.

Is $\nabla_c Z^c = 0$ for a Killing vector?

So, we can do a 3+1 split of this $Z4$ theory: this yields

$$\mathscr{L}_m\gamma_{ij} = -2\alpha K_{ij} \tag{3.63}$$

$$\mathscr{L}_m K_{ij} = (\text{ADM eqs}) + \alpha D_{(i}Z_{j)} \tag{3.64}$$

$$\mathscr{L}_m\theta = \frac{1}{2}C_0 - -\theta K + D_k Z^k - Z^k D_k \log\alpha \tag{3.65}$$

$$\mathscr{L}_m Z_i = C_i + D_i\theta - \theta D_i \log\alpha - ZK_i^k Z_k \,, \tag{3.66}$$

where $-\theta = n^a Z_a$.

Observe that the constraints became evolution equations! So, this is just a system of hyperbolic equations with no constraints, and GR is obtained from this extended theory under the *algebraic* constraint $Z^a = 0$.

Now the Bianchi identities read

$$\nabla^{a4}G_{ab} = 0 \implies \square \,, \tag{3.67}$$

a wave equation for Z^a. If $Z^a = 0 = \dot{Z}^a = C^a$, then $Z^a = 0$ for all times. This approach is rather general, and it can be applied for other theories or sets of equations which have a mixed character, with some hyperbolic and some elliptic equations.

An example is MHD: hydrodynamics with a perfectly conducting fluid.
In this case we have $0 = \partial_\mu T^{\mu\alpha} = \partial_\mu \left(T^{\mu\alpha}_{\mathrm{PF}} + T^{\mu\alpha}_{\mathrm{EM}} \right)$.
These can be written in terms of the magnetic field B^i as

$$\partial_t B^i + \partial_k \left(B^k v^i - B^i v^k \right) = 0 \tag{3.68}$$

$$C = \partial_i B^i = 0\,. \tag{3.69}$$

The question is the same: if we set $C = 0$ initially and then evolve the induction equation, is it guaranteed that the constraint will be preserved? In this case we only need a scalar field ψ:

$$\partial_i B^i + \partial_k \left(B^k v^i - B^i v^k \right) + \partial^i \psi = 0 \tag{3.70}$$

$$\partial_t \psi + \partial_i B^i = 0\,, \tag{3.71}$$

which like before reduces to MHD if $\psi = 0$.
Note that if we take a derivative of the second equation we find $\partial_{tt}\psi + \partial_t \partial_i B^i = 0$, into which we can substitute the first to get

$$\partial_{tt}\psi - \partial_i \partial^i \psi + \partial_i \partial_k B^{[k} v^{i]} = -\Box \psi = 0\,. \tag{3.72}$$

It can be shown that the constraint C satisfies the same equation.

Show!

This is a method which can be used to control the divergence of B.
If we are solving MHD, variations from $C = 0$ can appear due to numerical errors: if we are solving the extended system, we can see that the violation evolves according to the wave equation, so it will propagate away to the domain boundary because of $\Box C = 0$. Can we add a **damping term** to this propagation, so that the evolution reads $\Box C + \delta \dot{C} = 0$ for some small $\delta > 0$?
This corresponds to adding a term in the constraint equation with ψ:

$$\partial_t + \partial_i B^i + \delta \psi = 0\,. \tag{3.73}$$

The name of this strategy is the **divergence cleaning method**, it was proposed in 2012.
A similar thing can be done in the $Z4$ term. This was historically one of the things that allowed for the first successful BBH simulations.
The Powell method corresponds to the alternative approach of projecting the equation onto the constraint at each step. There are general formulations for this.
We use this specific method because it gives very *uniform* equations. PDE theory deals a lot with full hyperbolic systems, and not to mixed systems. So, it is very convenient to have everything be hyperbolic.
This, however, is by no means the only way to do things. We could also try to have as many elliptic equations as possible.
Nowadays these **free evolution schemes** are most often used, as opposed to the **fully constrained schemes** in which elliptic equations are maximized.

There still must be 2 hyperbolic equations in the fully constrained scheme, since they represent the two physical degrees of freedom of the GW.

What about δ choice? It seems like we would prefer to have it as large as possible, are there problems with that?

4 ADM Hamiltonian formulation of GR

The GR action is

$$S_{GR} = \int {}^4R\sqrt{-g} + \text{boundary terms}\,, \tag{4.1}$$

the boundary terms are important, but they will not be discussed today. This can be written in our 3+1 formalism as

$$\int \mathrm{d}t \int_{\Sigma_t} \underbrace{\left(R + K_{ij}K^{ij} - K^2\right)}_{\text{Lagrangian density}} \alpha\sqrt{\gamma}\,, \tag{4.2}$$

so we can introduce conjugate momenta:

$$\pi^{ij} = \frac{\partial L}{\partial \dot{\gamma}_{ij}} = \sqrt{\gamma}\left(K\gamma^{ij} - K^{ij}\right), \tag{4.3}$$

so the Hamiltonian density reads

$$\mathcal{H} = \pi^{ij}\dot{\gamma}_{ij} - L = \tag{4.4}$$

$$= \sqrt{\gamma}\left[\alpha C_0 + 2\beta^i C_i + 2D_j\left(K\beta^i - K^j_i\beta^i\right)\right], \tag{4.5}$$

understood as a function of γ_{ij} and π^{ij}. The Hamiltonian is then

$$H = \int_{\Sigma_t} \mathcal{H} = \int_{\Sigma_t} \sqrt{\gamma}\left(\alpha C_0 + 2\beta^i C_i\right). \tag{4.6}$$

The corresponding EoM read

$$\dot{\gamma}_{ij} = \frac{\partial \mathcal{H}}{\partial \pi^{ij}} = -2\frac{\alpha}{\sqrt{\gamma}}\underbrace{\left(\pi^{ij} - \frac{1}{2}\pi\gamma_{ij}\right)}_{\sim K_{ij}} + 2D_{(i}\beta_{j)}\,, \tag{4.7}$$

while

$$-\dot{\pi}_{ij} = \frac{\partial \mathcal{H}}{\partial \gamma_{ij}} = \cdots \sim \mathscr{L}_m K_{ij} \text{ from ADM } + C_0\,, \tag{4.8}$$

and we can see that lapse and shift appear in these equations: if we differentiate with respect to them we find

$$0 = \frac{\partial \mathcal{H}}{\partial \alpha} = C_0 \qquad \text{and} \qquad 0 = \frac{\partial \mathcal{H}}{\partial \beta^i} = C_i\,. \tag{4.9}$$

In the original ADM formulation they used γ_{ij} and π^{ij}; the York formulation used γ_{ij} and

$$K_{ij} = -\frac{1}{\sqrt{\gamma}}\left(\pi_{ij} - \frac{1}{2}\pi\gamma_{ij}\right). \tag{4.10}$$

Lapse and shift can then simply be understood to be Lagrange multipliers which enforce the constraint.

Next time we will approach the conformal decomposition of 3+1 GR and the initial data problem: how do we solve it? There are some free evolution schemes like BSSNOK or Z4C.

Also, once we get the initial data we will need to solve the Cauchy IVP, discussing hyperbolicity and well-posedness.

5 Conformal decomposition of 3+1 GR

Tuesday 2021-5-18, compiled 2021-06-22

Lichnerowicz in 1944 already used this to deal with the initial data problem.

York in 1971 showed that the dynamical degrees of freedom of GR (GWs) are encoded in the conformal equivalence class of the three-metric:

$$\gamma_{ij} = \psi^4 \widetilde{\gamma}_{ij}\,, \tag{5.1}$$

which can be shown using ADM + the Cotton-York tensor. As an example, if $\widetilde{\gamma}_{ij}$ is the flat 3-metric, we can directly say that there are no GW.

A common choice for conformal metrics is to take the uni-determinant ones: $\det \gamma_{ij} = \gamma = 1$. Now, we know that $\det(cA) = c^m \det A$, therefore

$$\det \gamma_{ij} = \gamma = \psi^{12}\widetilde{\gamma}\,. \tag{5.2}$$

Since $\widetilde{\gamma} = 1$, this means that $\psi = \gamma^{1/12}$.

With this choice, ψ is not a scalar! Therefore, $\widetilde{\gamma}_{ij}$ is also not a tensor but instead a tensor density, and it has no Levi-Civita connection associated to it.

The solution to this issue is to introduce an unphysical background three-metric on Σ_t, so that

1. f_{ij} has signature $+, +, +$ and $\det f_{ij} = f$;
2. it is time independent:

$$\mathscr{L}_{\partial_t} f_{ij} = \frac{\partial f_{ij}}{\partial t} = 0\,, \tag{5.3}$$

3. it has an inverse: $f^{ij} f_{jk} = \delta^i_k$;
4. it has a Levi-Civita connection $\mathcal{D}_k f_{ij} = 0$, whose raised-index version is $\mathcal{D}^i = f^{ij}\mathcal{D}_j$;
5. its Christoffel symbols are

$$F_{ij}{}^k = \frac{1}{2} f^{kl}\left(\partial_i f_{lk} + \partial_j f_{il} - \partial_l f_{ij}\right), \tag{5.4}$$

Now $\psi = (\gamma/f)^{1/12}$ is a scalar. The conformal metric will be

$$\widetilde{\gamma}_{ij} = \psi^{-4}\gamma_{ij} \qquad \text{and} \qquad \widetilde{\gamma}^{ij} = \psi^4\gamma^{ij}\,, \tag{5.5}$$

and its determinant will be $\widetilde{\gamma} = f$.

This conformal metric is a "good" metric. What do we use as this metric? We can take $f_{ij} = \mathbb{1}_3$, which has vanishing Christoffel symbols; if we are in Cartesian coordinates this is the natural choice.

If, instead, we are in spherical coordinates we might want to use $f_{ij} = \operatorname{diag}\Big(1, r^2, r^2\sin^2\theta\Big)$, which has $f = r^4\sin^2\theta$.

The idea is to take the simplest flat metric in the coordinates we are using.

In the weak-field limit, we have $\psi^4 = (1 - 2\phi)$, so $\psi \approx 1 - \phi/2$. The conformal factor is heuristically analogous to the Newtonian potential.

As an example, the Schwarzschild metric in isotropic coordinates can be written as

$$\mathrm{d}s^2 = -\frac{(1 - M/2r)^2}{(1 + M/2r)^2}\,\mathrm{d}t^2 + \underbrace{\left(1 + \frac{M}{2r}\right)^4}_{\psi^4}\underbrace{\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\right)}_{f_{ij}}\,. \tag{5.6}$$

Now we have to define the **conformal connection and the Ricci tensor**, and specifically the relationship between $\mathcal{D}_i$ and $\widetilde{\mathcal{D}}_i$, which both live on Σ_t.

The expression reads:

$$\mathcal{D}_j T^{\cdots}_{\cdots} \sim \widetilde{\mathcal{D}}_j T^{\cdots}_{\cdots} + \Sigma C T^{\cdots}_{\cdots} - \Sigma C T^{\cdots}_{\cdots}\,, \tag{5.7}$$

where $C^k_{ij} = \Gamma^k_{ij} - \widetilde{\Gamma}^k_{ij}$. Using this, the Riccis are related by

$$\begin{aligned} R_{ij} &= \widetilde{R}_{ij} + \widetilde{\mathcal{D}}C - \widetilde{\mathcal{D}}C + CC - CC & (5.8)\\ &= \widetilde{R}_{ij} + R^{\psi}_{ij}\,, & (5.9) \end{aligned}$$

where R^{ψ}_{ij} is a function which contains second covariant derivatives of the conformal factor, something like $\widetilde{D}\widetilde{D}\psi + \widetilde{D}\psi\widetilde{D}\psi$.

The Ricci scalar, on the other hand, reads

$$R = \gamma^{ij}R_{ij} = \cdots = \psi^{-4}\widetilde{R} - 8\psi^{-5}\widetilde{D}_i\widetilde{D}^i\psi\,. \tag{5.10}$$

CD of extrinsic curvature

We start by performing a trace-traceless decomposition:

$$K_{ij} = A_{ij} + \frac{1}{3}K\gamma_{ij}\,. \tag{5.11}$$

Then, we decompose the traceless part A^{ij}: $A^{ij} = \psi^p\overline{A}^{ij}$, and $\overline{A}$ is called the conformal traceless part of K^{ij}.

There are two possible choices for p: sometimes we take $p = -4$, which is based on $\mathscr{L}_m \gamma_{ij}$, which is useful for evolution schemes, and the hyperbolic part of the ADMY equations.

The alternative is $p = -10$, based on the constraint equations $C^i = 0$.

For the first choice, we take the kinematic equation for K_{ij}, and compute

$$\mathscr{L}_m \left(\psi^4 \widetilde{\gamma}_{ij} \right) = -2\alpha A_{ij} - \frac{2}{3} \alpha K \psi^4 \widetilde{\gamma}_{ij} \tag{5.12}$$

$$\psi^4 \mathscr{L}_m \widetilde{\gamma}_{ij} = -4\psi^{-5} \psi^3 \mathscr{L}_m \widetilde{\gamma}_{ij} - 2\alpha\psi^{-4} A_{ij} - \frac{2}{3} \alpha K \psi^{-4} \psi^4 \widetilde{\gamma}_{ij} \,. \tag{5.13}$$

We trace this equation with $\widetilde{\gamma}^{ij}$, and get

$$\widetilde{\gamma}^{ij} \mathscr{L}_m \widetilde{\gamma}_{ij} = -4 \underbrace{\widetilde{\gamma}^{ij} \widetilde{\gamma}_{ij}}_{=3} \mathscr{L}_m \log \psi - 2\alpha \psi^{-4} \widetilde{\gamma}^{ij} A_{ij} - \frac{2}{3} \alpha K \widetilde{\gamma}^{ij} \widetilde{\gamma}_{ij} \tag{5.14}$$

$$= -12 \mathscr{L}_m \log \psi - 2\alpha K \,. \tag{5.15}$$

Alternatively, we can write

$$\widetilde{\gamma}^{ij} \mathscr{L}_m \widetilde{\gamma}_{ij} \sim \operatorname{Tr} \left(M^{-1} \delta M \right) = \delta (\log \det M) \tag{5.16}$$

$$= \mathscr{L}_m \log \widetilde{\gamma} = \left(\partial_t \mathscr{L}_\beta \right) \log f = -\mathscr{L}_\beta \log f \tag{5.17}$$

$$= -\widetilde{\gamma}^{ij} \mathscr{L}_\beta \widetilde{\gamma}_{ij} = -\widetilde{\gamma}^{ij} \left(\beta^k \widetilde{D}_k \widetilde{\gamma}_{ij} + 2 \widetilde{\gamma}_{k(i} \widetilde{D}_{j)} \beta^k \right) \tag{5.18}$$

$$= -2 \widetilde{D}_i \beta^i \,. \tag{5.19}$$

Both ways, we see that

$$6 \mathscr{L}_m \log \psi + \alpha K = \widetilde{D}_i \beta^i \,, \tag{5.20}$$

so we have conformally decomposed the kinematic equation $\mathscr{L}_m \gamma = -2\alpha K$. We get

$$\mathscr{L}_m \log \psi = \frac{1}{6} \widetilde{D}_i \beta^i - \frac{1}{6} \alpha K \tag{5.21}$$

$$\mathscr{L}_m \widetilde{\gamma}_{ij} = -2\alpha \psi^{-4} A_{ij} - \frac{2}{3} \widetilde{D}_k \beta^k \widetilde{\gamma}_{ij} \,. \tag{5.22}$$

A natural choice which simplifies this is $p = -4$, since it simplifies $\psi^{-4} A_{ij} \to \widetilde{A}_{ij}$. This is commonly denoted as $\overline{A}$.

The second choice yields equations in the foorm

$$D_j A^{ij} = \widetilde{D}_j A^{ij} + C^i_{jk} A^{kj} + C^i_{jk} A^{ik} = \dots \tag{5.23}$$

$$= \psi^{-10} \widetilde{D}_j \left(\psi^{10} A^{ij} \right) . \tag{5.24}$$

This is commonly denoted as $\hat{A}$.

With this choice we find

$$0 = C^i = \widetilde{D}_j \hat{A}^{ij} - \frac{2}{3} \psi^6 \widetilde{D}^i K - 8\pi \psi^{10} p^i \,. \tag{5.25}$$

The Hamiltonian constraint, on the other hand, reads

$$0 = C^0 = R + K^2 - K_{ij}K^{ij} - 16\pi E\,. \tag{5.26}$$

This K is already a conformal variable! As for the other term, we have

$$K_{ij}K^{ij} = \left(A_{ij} + \frac{1}{2}K\gamma_{ij}\right)\left(A^{ij} + \frac{1}{2}K\gamma^{ij}\right) \tag{5.27}$$

$$= A_{ij}A^{ij} + \underbrace{\frac{2}{3}K\gamma_{ij}A^{ij}}_{=0} + \frac{1}{3}K^2\,. \tag{5.28}$$

Putting this together yields, for $p = -4$,

$$0 = C^0 = \widetilde{D}_i\widetilde{D}^i\psi - \frac{1}{6}\widetilde{R}\psi + \left(\frac{1}{8}\widetilde{A}_{ij}\widetilde{A}^{ij} - \frac{1}{12}K^2 + 2\pi E\right)\psi^5\,, \tag{5.29}$$

and for $p = -10$:

$$0 = C^0 = \widetilde{D}_i\widetilde{D}^i\psi - \frac{1}{6}\widetilde{R}\psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-7} + \left(2\pi E - \frac{1}{12}K^2\right)\psi^5\,. \tag{5.30}$$

The latter is called the **Lichnerovicz equation**: the Hamiltonian constraint in conformal variables. This is a complicated nonlinear equation, but under certain prescriptions we can make the $\widetilde{D}_i\widetilde{D}^i$ term look like a Laplacian.

The other CD ADMY equations are the dynamical ones, we have one for the evolution of K, and one for the evolution of $\widetilde{A}_{ij}$. See the notes for more details on these equations.

6 Asymptotic Flatness & Global Quantities

We have discussed a sufficient set of formalism, and we are ready to move to more physical topics. Tuesday 2021-5-25, compiled 2021-06-22

How do we characterize the spacetime? We'd like to define mass, momentum and such.

A notion we can give for these works for Asymptotically Flat spacetimes: ADM energy and momentum.

If the spacetime is not asymptotically flat but we have some symmetries described by Killing vectors, we can define Komar charges.

There are more details on these topics in the book by Wald.

Asymptotic flatness

Definition 6.1. *A globally hyperbolic spacetime is* ***asymptotically flat*** *iff:*

$$\forall \Sigma_t \colon \exists f_{ij} \quad \text{“background metric”}\,, \tag{6.1}$$

such that

1. *f_{ij} is flat except at most for a compact domain (the strong-field region, typically);*

2. *there exists a chart* $\left\{x^i\right\}$ *such that* f_{ij} *approaches* $\mathbb{1}_3$ *for* $r = \sqrt{x^i x_i} \to \infty$;

3. *for* $r \to \infty$ *the following conditions hold:*

   (a) $\gamma_{ij} = f_{ij} + \mathcal{O}\left(r^{-1}\right)$;

   (b) $\partial_k \gamma_{ij} = \mathcal{O}\left(r^{-2}\right)$;

   (c) $K_{ij} = \mathcal{O}\left(r^{-2}\right)$;

   (d) $\partial_k K_{ij} = \mathcal{O}\left(r^{-3}\right)$.

The region $r \to \infty$ is called **spatial infinity**, i_0.

A first example of a spacetime which is asymptotically flat is Schwarzschild; a counterexample on the other hand is a flat spacetime which contains a gravitational wave.

Let us show this: in this case the spatial metric (in the TT gauge) is

$$\gamma_{ij} = f_{ij} + \frac{1}{r} h_{ij}(t - r) + \mathcal{O}\left(r^{-2}\right) . \tag{6.2}$$

This is compatible with condition 3a, but the derivative is

$$\partial_k \gamma_{ij} = -\frac{h'(u)}{r} \frac{x^k}{r} - \frac{h_{ij}(u)}{r^2} \frac{x^k}{r} + \mathcal{O}\left(r^{-3}\right) , \tag{6.3}$$

so we have an $\mathcal{O}\left(r^{-1}\right)$ term: condition 3b is broken.

Also, cosmological spacetimes are typically not AF.

A note: asymptotic flatness depends on the foliation Σ_t (and on x^i, which however we can change); the transformation group which preserves these properties is the spin group: rotations, translations and boosts, generically

$$\Lambda^i_j + \dots , \tag{6.4}$$

to complete

If out spacetime is AF we can define ADM energy and mass: the full GR action reads

$$S_{GR} = \int_{\mathcal{V} \subset \mathcal{M}} {}^4R \sqrt{-g} \, \mathrm{d}^4 x + \oint_{\partial \mathcal{V}} (Y - Y_0) \sqrt{h} \, \mathrm{d}^3 y , \tag{6.5}$$

where the first term is the usual Hilbert action, while the second term is a boundary one, which is needed in order to obtain the correct EFE on the boundary of the region which is considered.

This would not be needed if we knew that the derivatives of the metric were zero at the boundaries, but this will not be true in general.

An alternative approach, by Levi-Civita, is to vary with respect to the connection which is not assumed to be metric-compatible.

Here, $\mathcal{V}$ is our 3D region, $\partial\mathcal{V}$ is its (timelike) boundary and h is the induced metric on $\partial\mathcal{V}$ if we see the boundary as embedded in the larger manifold $(\mathcal{M}, g)$.

The tensor Y_{ab} is the extrinsic curvature of $\partial\mathcal{V}$ in $(\mathcal{M}, g)$; the tensor $Y_{0,ab}$ is the extrinsic curvature of $\partial\mathcal{V}$ as embedded in $(\mathcal{M}, \eta)$, and similarly $h_{0,ab}$ (η is the flat 4D metric).

Y and Y_0 are the respective traces.

We will not complete the calculation, but imposing

$$\frac{\delta S}{\delta g^{ab}} = 0\,, \tag{6.6}$$

with $\partial g^{ab}\Big|_{\text{boundary}} = 0$ leads to the EFE.

This way, we can construct the Hamiltonian:

$$H = -\int_{\Sigma_t} \sqrt{\gamma}\Big(\alpha C_0 + 2\beta^i C_i\Big) - 2\oint_{\partial\Sigma_t} \sqrt{q}\Big[\alpha(\kappa - \kappa_0) + \beta^i S^j (K_{ij} - K\gamma_{ij})\Big]\,, \tag{6.7}$$

where we have the lapse α, the Hamiltonian constraint C_0, the shift β^i, the momentum constraint C_i, while in the other term we find the 2D intersection $\partial\Sigma_t = \partial\mathcal{V} \cap \Sigma_t$.

The vector S^i is the normal vector to $\partial\Sigma_t$; the metric q_{ij} is the induced metric on $\partial\Sigma_t$ as embedded in (Σ, γ). On the other hand, κ_{ij} is the extrinsic curvature.

The quantity κ_0, on the other hand, is the extrinsic curvature of $\partial\Sigma_t$ as embedded in (Σ, f).

Here we have a key observation: the first term in the Hamiltonian is identically zero on a solution (because the constraints are zero), therefore the boundary terms give us an "energy".

With this, the **ADM mass** is calculated on i_0 by taking $\alpha = 0 = \beta^i$:

$$\begin{aligned}
M_{\text{ADM}} &= -\frac{1}{8\pi}\lim_{r\to+\infty}\oint_{\partial\Sigma_t}\sqrt{q}(\kappa - \kappa_0) &(6.8)\\
&= +\frac{1}{16\pi}\lim_{r\to+\infty}\oint_{\partial\Sigma_t}\sqrt{q}S^i\Big[D^i\gamma_{ij} - D_i\Big(f^{kl}\gamma_{kl}\Big)\Big] &(6.9)\\
&= +\frac{1}{16\pi}\lim_{r\to+\infty}\oint\sqrt{q}S^i\Big(\partial_j\gamma_{ij} - \partial_i\gamma^k_k\Big) &(6.10)\\
&= -\frac{1}{2\pi}\lim_{r\to\infty}\oint\sqrt{q}S^i\left(D_i\psi - \frac{1}{8}D^j\widetilde{\gamma}_{ij}\right). &(6.11)
\end{aligned}$$

The second specifies to the background metric, the third specifies to Cartesian coordinates, the last is written in terms of conformal variables.

If we look at the third expression, asymptotic flatness guarantees that the integral exists!

Let us give an example, in Cartesian coordinates and for the weak-field metric. Here, $\widetilde{\gamma}_{ij} = f_{ij}$, $D^i\widetilde{\gamma}_{ij} = 0$, and $\psi = 1 - \phi/2$, and $D_i\psi = -D_i\phi/2$.

Let us calculate with the last definition:

$$M_{\text{ADM}} = \frac{1}{4\pi}\lim_{r\to\infty}\oint_{S_r}\sqrt{f}S^i\big(-D_i\phi - 0\big)\,, \tag{6.12}$$

where the last term vanishes because it is the divergence of the conformal metric, which is flat. Now we can use Green's theorem:

$$M_{\mathrm{ADM}} = +\frac{1}{4\pi}\int_{\Sigma_t}\sqrt{f}\,\mathrm{d}^3x\,D^iD_i\phi \tag{6.13}$$

$$= \frac{1}{4\pi}\int_{\Sigma_t}\sqrt{f}\,\mathrm{d}^3f\,4\pi\rho = \int_{\Sigma_t}\sqrt{f}\,\mathrm{d}^3x\,\rho\,. \tag{6.14}$$

Let us try the same for Schwarzschild (in isotropic coordinates): here, the conformal factor is $\psi = 1 + M/2r$, the conformal metric is $\widetilde{\gamma}_{ij} = f_{ij} = \operatorname{diag}\left(1, r^2, r^2\sin^2\theta\right)$. So, the combination we need is

$$\sqrt{q}\,\mathrm{d}^2y = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\ , \tag{6.15}$$

which we compute at i_0, with $\partial\Sigma_r \sim S_r$. We also have $S^iD_i = \partial_r$ at i_0; so

$$M_{\mathrm{ADM}} = -\frac{1}{2\pi}\lim_{r\to\infty}\oint_{S_r}\underbrace{\frac{\partial\psi}{\partial r}r^2}_{-(M/2r^2)r^2}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = M\,. \tag{6.16}$$

A useful theorem by Schoen and Yau in '79 and '81, and also by Witten in '81 says that is T_{ab} obeys the dominant energy condition (stating that for Eulerian observers $E^2 \geq p^2$) then:

1. $M_{\mathrm{ADM}} \geq 0$;
2. $M_{\mathrm{ADM}} = 0$ iff Σ_t is Minkowski;
3. $\mathrm{d}M_{\mathrm{ADM}}/\mathrm{d}t = 0$, which we could have already stated by seeing that the Hamiltonian is not time dependent.

So, this is a good definition for the energy of the spacetime.

If we do simulations, we are not extracting the ADM mass at infinity, and we must approximate for some finite radius.

There is a heuristic derivation: we start from Newton,

$$M = \int_V \rho = \frac{1}{4\pi}\int_V \Delta\phi = \oint_S S^i\partial_i\phi\,, \tag{6.17}$$

and now we must make an assumption to move to GR: $\phi \sim h_{00}$, and $\partial\phi \sim \partial h_{ij}$.

The Hamiltonian constraint $R +$ curvature $=$ energy reads

$$\partial\partial h_{ij} + 0 = E \approx r\,, \tag{6.18}$$

therefore the integral of ϕ becomes $\int_V R \sim \oint S^i\left(\partial_j h_{ij} + \partial_i h^k_k\right)$.

Clarify this derivation

Can we also have ADM momentum and angular momentum? Kind of — linear momentum, yes, not angular momentum.

Linear momentum is typically associated to spatial translations.

So, we define the ADM momentum from the boundary term at i_0 by taking $\alpha = 0$ and $\beta^i = (\partial_k)^i$:

$$P_k^{\mathrm{ADM}} = \frac{1}{8\pi} \lim_{r\to+\infty} \oint_{\partial\Sigma_t} \sqrt{q} (\partial_k)^i S^j \left(K_{ij} - K\gamma_{ij}\right) . \tag{6.19}$$

Like in the other case, AF guarantees the existence of these 3 quantities.

This P_k^{ADM} transforms like a 1-form — it is a vector at i_0.

Now, $P_\alpha^{\mathrm{ADM}} \propto (-M_{\mathrm{ADM}} P_k^{\mathrm{ADM}})$ transforms properly as a four-vector under the spin group.

As for "ADM angular momentum", we must look at spatial rotations at i_0 about the 3 axes. They will be in the form $\phi_x = -z\partial_y + y\partial_z$ and so on. Note that these scale like $\mathcal{O}(r)$.

We can take, at i_0, $\alpha = 0$ and $\beta^i = (\phi_k)^i$:

$$J_k^{\mathrm{ADM}} = \frac{1}{8\pi} \lim_{r\to\infty} \oint_{\partial\Sigma_t} \sqrt{q} (\phi_k)^i S^j \left(K_{ij} - K\gamma_{ij}\right) . \tag{6.20}$$

In this case, AF does not guarantee anything: the integrand is $(K_{ij} - K\gamma_{ij})(\phi_k)^j = \mathcal{O}\left(r^{-1}\right)$, but sometimes the contraction with S^i can solve the problem.

This is what happens in Kerr.

A big problem is the fact that J_k^{ADM} does *not* transform like a 1-form at i_0. A working definition of $\vec{J}$ exists only within a restricted class of transformations — specifically, we need the stronger decay conditions

$$\partial_k \widetilde{\gamma}_{ij} = \mathcal{O}\left(r^{-3}\right) \qquad \text{and} \qquad K = \mathcal{O}\left(r^{-3}\right) . \tag{6.21}$$

This amounts to imposing an isotropic gauge condition and an asymptotically maximal gauge.

Komar masses

In GR, *conserved charges* appear when one has symmetries: if K^a is a Killing vector

$$\mathscr{L}_K g_{ab} = 2\nabla_{(a} K_{b)} = 0 , \tag{6.22}$$

then the vector defined as

$$J^a = T^{ab} K_b \tag{6.23}$$

is a *conserved current*: $\nabla_a J^a = 0$. Here T^{ab} is any symmetric, divergence-free rank-2 tensor, not necessarily the stress-energy one. This can be proven in a rather trivial way:

$$\nabla_a J^a = \nabla_a \left(T^{ab} K_b\right) = K_b \nabla_a T^{ab} + T^{ab} \nabla_a K_b = 0 . \tag{6.24}$$

An important current is constructed from the four-Ricci:

$$J_0^a =^4 R^{ab} K_b = \nabla_b \nabla^a K^b = -\nabla_b \nabla^b K^a , \tag{6.25}$$

where the indices look wrong by they are in fact not — this does not hold in general, but it does for a Killing vector. The vector $\nabla^b K^a = A^{ba}$ is antisymmetric.

Now, using Stokes' theorem we can write

$$\int_{\Sigma_t} \mathrm{d}^3 x \sqrt{\gamma} n_a \nabla_a A^{ab} = \oint_{\partial\Sigma_t} \mathrm{d}^2 y \sqrt{q} (S_a n_b - n_a S_b) A^{ab} = Q_k \,. \tag{6.26}$$

This Q_k is the conserved Komar charge. The vectors n_a and S_b are normal and parallel to the surface of Σ_t respectively.

As an example, in a stationary spacetime $K^a = (\partial_t)^a$ we find the so-called Komar mass:

$$Q_k = M_k \,, \tag{6.27}$$

and a useful theorem in this regard states that if $K^a = n^a$ at i_0, then the Komar mass M_k and the ADM mass M_{ADM} are the same.

In a cosmological context everything there is more messy. We cannot even assume global hyperbolicity.

7 Initial Data Problem

Tuesday 2021-6-1, compiled 2021-06-22

It is perhaps the most difficult problem currently.

The problem is stated as

$$C_0 = R + K^2 - K_{ij} K^{ij} - 16\pi E = 0 \tag{7.1}$$

$$C_i = D_j K^j_i - D_i K - 8\pi P_i = 0 \,, \tag{7.2}$$

and we want to calculate γ_{ij} and K_{ij} on Σ_0 such that

1. the constraints are satisfied;
2. the solution is physically meaningful: it really describes a BH, BBH, n-BH, NS spacetime.

We will assume for today that the matter fields are given or zero — in general one would have to solve the hydrodynamic equations for them as well.

We have 4 equations for 12 unknowns! We will need to prescribe 8 quantities.

The problem is split in two: a problem of constrained data (4) and one of free data (8).

The choice of free data is guided by a few principles:

1. we need to define our (astro)physical expectations for Σ_0 and verify that they are matched;
2. we need some heuristic or intuition for the various fields;
3. mathematical (/ computational?) necessity: the equations $C_\alpha = 0$ should be as nice as possible — linearity, decoupling, well-posedness are all desirable.

The **Conformal Decomposition** allows us to implement these. Within it, there are two main formalisms: one is the "conformal/transverse traceless" formalism (CTT), due to York in 1973; the other is the "conformal/thin sandwich" (CTS), also due to York in 1999. Both are based off of early work by Lichnerowicz.

The CTS formalism is often used to generate binary systems — it makes it easier to implement the quasi-symmetries there.

We start from the Lichnerowicz equation (the Hamiltonian constraint):

$$C_0 = \widetilde{D}_i \widetilde{D}^i \psi - \frac{1}{8} \hat{A}_{ij} \hat{A}^{ij} \psi^{-7} + \left(-\frac{K^2}{12} + 2\pi E \right) \psi^5 \qquad = 0\,, \tag{7.3}$$

with $p = -10$. Let us solve it as a standalone equation: it is a Poisson-like one, since $\widetilde{D}_i \widetilde{D}^i$ is similar to a Laplacian. Then, this is some sort of elliptic operator applied to ψ equated to some powers of ψ.

If $K^2 = 0$ the equation simplifies, and the BVP can be studied: there are several known results about the well-posedness of this equation under the hypothesis $K = \text{const}$.

These are known as Constant Mean Curvature (CMC) spacetimes.

If we take an Asymptotically Flat, CMC spacetime with $K = 0$, and with $E = 0$, then the BVP with the Lichnerowicz equation is *solvable* for a "large" class of metrics $\widetilde{\gamma}_{ij}$ (a "positive Yamabi class").

There is a "prototype equation" for the Lichnerowicz one if we set $K = 0$:

$$\triangle u = f^2 u^p\,, \tag{7.4}$$

in flat spacetime, for some function f.

These theorems are found in somewhat specialized PDE literature. In the linear case $p = 1$, and we take $u\big|_{\partial\Omega} = 0$, then the solution, identically zero, is unique.

In the nonlinear case, the solution is unique iff $p > 0$ (or specifically, the same sign of the coefficient of the $f^2 u^p$ term).

So, what about the Lichnerowicz equation? The easiest thing to do is to linearize the equation: we write it as

$$\widetilde{\triangle} \psi + H(\psi) = 0\,. \tag{7.5}$$

We write $\psi = \psi_0 + \epsilon$, and expand

$$H(\psi) = H(\psi_0) + \left.\frac{\partial H}{\partial \epsilon}\right|_0 \epsilon + \mathcal{O}\left(\epsilon^2\right)\,, \tag{7.6}$$

so we find an equation in the form $\widetilde{\triangle} \epsilon = f \epsilon$ with:

$$f = \frac{1}{8} \widetilde{R} + \frac{7}{8} \hat{A}_{ij} \hat{A}^{ij} \psi_0^{-8} - 10\pi E \psi_0^4\,. \tag{7.7}$$

This immediately shows us the problem: if $K = 0$ then $\widetilde{R} > 0$, but we can check that f is not positive, because of the "matter term"!

However, we insist on solving this equation, and we can do so as long as we *rescale* the matter term.

If we define $\widetilde{E} = \psi^s E$, then, the term becomes

$$-5 \times 2\pi E \psi_0^4 \to -(s \times 5) 2\pi \widetilde{E} \psi^{4-s}\,, \tag{7.8}$$

so this is > 0 for any $s > 5$. This is *not* a trick: in practice, these elliptic equations are solved by iteration — specifying a guess, and then iterating.

Specifying the "correct" RHS therefore is key if we want to find a solution. So, what do we use? The best answer turns out to be 8: if we do this, we can express the dominant energy condition in conformal variables: if

$$\widetilde{E} = \psi^8 E \qquad \text{and} \qquad \widetilde{P^i} = \psi^{10} P^i\,, \tag{7.9}$$

then $\widetilde{E}^2 \geq \widetilde{P}^2$ implies $E^2 \geq P^2$. In general this would read

$$\psi^{2s} E^2 \geq \psi^{-4} \psi^{10} \psi^{10} P^2\,. \tag{7.10}$$

Maximal Slicing

What does it mean to impose $K = 0$? As we will see shortly, this is a gauge condition which extremizes the volume of Σ_t.

We just need to remember the definition of the trace: it is an identity that

$$K = \gamma^{ab} K_{ab} = -\frac{1}{2\alpha} \gamma^{ij} \mathscr{L}_m \gamma_{ij} = -\frac{1}{2\alpha} \mathscr{L}_m \log \gamma\,, \tag{7.11}$$

which can be nicely written as

$$K = -\frac{1}{\alpha} \mathscr{L}_m \log \sqrt{\gamma}\,, \tag{7.12}$$

so we recover the volume element on Σ_t, $\sqrt{\gamma}$!

The volume in a certain region is $V = \int \sqrt{\gamma}\, \mathrm{d}^3 x$, so if we perform a variation $v^a = \delta t \big(\alpha n^a + \beta^a\big)$, so that $v^a\big|_S = 0$ (the deformation is zero at the boundary), the volume changes as such:

$$\frac{\delta V}{\delta t} = \int \partial_t \sqrt{\gamma}\, \mathrm{d}^3 x = \int \Big(-\alpha K + D_i \beta^i\Big)\, \mathrm{d}^3 x \tag{7.13}$$

Using the kinematic equation.

$$= -\int_V \alpha K \sqrt{\gamma}\, \mathrm{d}^3 x + \underbrace{\oint_S \beta^i S_i}_{=0} = -\int \alpha K \sqrt{\gamma}\, \mathrm{d}^3 x\,, \tag{7.14}$$

therefore $K = 0$ yields $\delta V = 0$: this is a maximum.

It is the exact same geometric problem as a film of soap. If the soap is held at a ring, it stays flat.

The only difference from that case is that here we are in a Lorentzian geometry: in the Euclidean case we have a minimum, here we have a maximum.

This is an example of a singularity-avoiding gauge, used by many works.

The Conformal Transverse Traceless formalism

We use the $p = -10$ scaling of K_{ij}, and on top of the Lichnerowicz equation we derive an equation from the momentum constraint $C_i = 0$, for a vector x^i which is obtained with a further decomposition of $\hat{A}^{ij}$:

$$\hat{A}^{ij} = \hat{A}^{ij}_L + \hat{A}^{ij}_{TT}\,, \tag{7.15}$$

where the TT part is transverse and traceless: $\widetilde{\gamma}_{ij}\hat{A}^{ij}_{TT} = 0$ and $\widetilde{D}_j\hat{A}^{ij}_{TT} = 0$.

The longitudinal part on the other hand is expressed as follows:

$$\left(\widetilde{L}x\right)^{ij} = \widetilde{D}^i x^j + \widetilde{D}^j x^i - \frac{2}{3}\widetilde{D}_k x^k \widetilde{\gamma}^{ij}\,. \tag{7.16}$$

This seems like a rather strange expression: $\widetilde{L}$ is called the **Conformal Killing operator** on x^i. It is determined as follows:

$$\widetilde{D}_j\hat{A}^{ij} = \widetilde{D}_j\left(\widetilde{L}x\right)^{ij} = \widetilde{D}_j\widetilde{D}^j x^i + \frac{1}{3}\widetilde{D}^i\widetilde{D}_j x^i + \widetilde{R}^i{}_j x^i = \widetilde{\triangle}_L x^i\,, \tag{7.17}$$

where $\widetilde{\triangle}_L x^i$ is the **conformal vector Laplacian operator**.

There exists a unique L + TT decomposition of $\hat{A}^{ij}$ iff there exists a unique solution of the conformal Laplacian equation:

$$\widetilde{\triangle}_L x^i = \widetilde{D}_j\hat{A}^{ij}\,. \tag{7.18}$$

This is useful because there is a theorem by Cantor in 1979 which tells us that if Σ is asymptotically flat and we have $\partial_k\partial_l\widetilde{\gamma}_{ij} = \mathcal{O}\left(r^{-3}\right)$, then existence and uniqueness are guaranteed.

With this decomposition, we obtain

$$\widetilde{\triangle}_L x^i - \frac{2}{3}\widetilde{D}^i K\psi^6 - 8\pi\widetilde{P}^i = 0\,, \tag{7.19}$$

which can be solved together with the Lichnerowicz equation. This allows us to constrain ψ and x^i, as long as we determine the free data: $\widetilde{\gamma}_{ij}$, $\hat{A}^{ij}_{TT}$ and K.

Under maximal slicing, $K = 0$, the two equations decouple! They also partially decouple if we take $K = \text{const}$, since we can solve one and then the other.

Also, it is nice that the free data are the conformal metric: $\widetilde{\gamma}_{ij}$ and $\hat{A}^{ij}_{TT}$ are useful in heuristically determining the GW content of Σ.

Let us give an example of CTT data under the four hypotheses of conformal flatness, asymptotic flatness, and maximal slicing.

Asymptotic flatness is chosen because we want the solution to exist. This is the simplest choice of free data: we take $\hat{\gamma}_{ij} = f_{ij}$ (conformal flatness) and $K = 0$ (maximal slicing).

Further, we consider $E = 0 = P^i$ (vacuum).

Without justification we also assume that $\hat{A}^{ij}_{TT} = 0$.

Under all these assumptions we have

$$\widetilde{D}_i = D_i\,, \tag{7.20}$$

therefore

$$\widetilde{D}_i \widetilde{D}^i = D_i D^i = \triangle \tag{7.21}$$

is simply the flat elliptic operator, also $\widetilde{R} = 0$ and $\widetilde{L} = L$ is the one calculated in the flat case.

So, the CTT equations become:

$$\triangle\psi + \frac{1}{8}(Lx)_{ij}(Lx)^{ij}\psi^{-7} = 0 \tag{7.22}$$

$$\triangle_L x^i = \triangle x^i + \frac{1}{3} D_j D^i x^j = 0\,, \tag{7.23}$$

which are decoupled, so they can be solved independently of one another. They are "easy", since they use the flat Euclidean operators. If we want a boundary value problem we also need to specify the boundaries: we specify asymptotic flatness: $\psi = 1$ and $x^i = 0$ at ι_0.

Optionally we can impose strong-field inner boundary conditions: for example we can fix the topology.

Our **case 1** is to take $\Sigma_0 = \mathbb{R}^3$ (no inner boundary condition): then, the solution of the second CTT equation is $x^i = 0$, and the first one reads $\triangle\psi = 0$ together with $\psi = 1$ at ι_0: this implies $\psi \equiv 1$. As expected, we recover flat spacetime.

Can we also get a nontrivial solution? For that, we need an inner boundary. Tuesday 2021-6-8, compiled 2021-06-22

We are discussing the initial data problem, and we can adopt two formalisms: CTT and CTS.

In the CTT formalism, we made a longitudinal-transverse split of $\hat{A}^{ij}$.

We constructed asymptotically flat, conformally flat, time-symmetric initial data for the vacuum.

The equations almost decouple: they can be solved one after the other. We then get the simplified system

$$\triangle_L x^i = 0 \tag{7.24}$$

$$\triangle\psi + \frac{1}{8}(Lx)_{ij}(Lx)^{ij}\psi^{-7} = 0\,, \tag{7.25}$$

with outer boundary conditions $\psi \to 1$, $x^i \to 0$ as $r \to \infty$. This is a second-order elliptic problem, so we need a new set of boundary conditions: these inner boundary conditions determine the topology of the solution. If we suppose $\Sigma_0 \sim \mathbb{R}^3$, we get flat spacetime, but its slices are not stationary.

If we suppose $\Sigma_0 \sim \mathbb{R}^3 \setminus B$ for some ball B, we must impose some condition on $S = \partial B$: the simplest choice is to demand $x^i\Big|_S = 0$. Also, we can fix the conformal factor to 1. This yields to a flat Σ_0.

An alternative is to ask that S be a closed *minimal* surface: we ask that $D_i S^i = 0$, where S^i is the normal vector to S.

This is written as

$$0 = D_i S^i\Big|_S = \psi^{-6}\widetilde{D}_i\left(\psi^6 S^i\right) = \psi^{-6}\widetilde{D}_i\left(\psi^4 \widetilde{S}^i\right), \tag{7.26}$$

where $\widetilde{s}^i = \psi^{-2} s^i$ is the unit normal vector with respect to the conformal metric $\widetilde{\gamma}$:

$$\widetilde{\gamma}(\widetilde{S}, \widetilde{S}) = \psi^4 \widetilde{\gamma}(S, S) = \gamma(S, S) = 1 = \psi^{-6} \frac{1}{\sqrt{f}} \partial_i \left(\sqrt{f} \psi^4 \widetilde{S}^i \right) . \tag{7.27}$$

If we take B to be a sphere with radius $r = a$, then $\widetilde{S}^i = (1, 0, 0)$ everywhere, and $f_{ij} = (1, r^2, r^2 \sin\theta)$. Therefore,

$$\frac{1}{r^2} \partial_r \left(r^2 \psi^4 \right) \Big|_{r=a} = 0 \iff \left(\partial_r \psi + \frac{\psi}{2r} \right) \Big|_{r=a} = 0 \,, \tag{7.28}$$

which is a boundary condition of mixed Dirichlet-Neumann type for $\triangle \psi + \dots = 0$.

A solution is $\psi = 1 + a/r$, but what is the constant a? The standard way to determine it is to try to calculate global quantities.

For example, let us try to calculate the ADM mass:

$$\begin{aligned} M_{\mathrm{ADM}} &= -\frac{1}{2\pi} \lim_{r \to \infty} \int \frac{\partial \psi}{\partial r} r^2 \sin\theta \, \mathrm{d}^2\Omega & (7.29) \\ &= 2a \,. & (7.30) \end{aligned}$$

Therefore, $a = M_{\mathrm{ADM}}/2$.

Putting things together,

$$\gamma_{ij} = \psi \hat{\gamma}_{ij} = \psi f_{ij} = \left(1 + \frac{M_{\mathrm{ADM}}}{2r} \right) f_{ij} \,, \tag{7.31}$$

Schwarzschild in isotropic coordinates!

This also proves that the Einstein-Rosen bridge is a minimal surface.

???

We can do a transformation $r \to r' = M^4/4r$, which sends $[M/2, \infty)$ to $(0, M/2)$.

The metric is invariant under this transformation: with a bit of abuse of notation, $\gamma(r, \theta, \varphi) = \gamma(r', \theta, \varphi)$. This means that the transformation is an isometry. This means that we can attach a copy of the upper part of the embedding diagram to its lower part.

This means that we attach the I and III regions of the Kruskal diagram. Here, $r = 0$ does not correspond to a singularity but instead to the other asymptotically flat end.

What about the time symmetry condition? The solution we have here is a slice of Schwarzschild in isotropic coordinates. It is characterized by $\hat{A}^{ij}_{TT} = 0$ and $x^i = 0$. If we reconstruct the full $\hat{A}^{ij}$ we find that it is also zero, and we also know that $K = 0$ because of maximal slicing; this implies that $K_{ij} = 0$.

This means that $\mathscr{L}_m g_{ij} = 0$ at $t = 0$. Therefore, we say that Σ_0 is *momentarily static*.

If we map $t \to -t$, however, the line element is invariant: this is what we mean by time symmetry.

We do not have the full Schwarzschild in isotropic coordinates, we only have a slice.

There could be a nontrivial evolution of Σ_0! An example is when we use geodesic gauge.

We could insert multiple Black Hole initial conditions by superposition:

$$\psi = 1 + \sum_{p=1}^{N} \underbrace{\frac{M_P}{\left|x^i - c_P^i\right|}}_{\psi_{BL}}, \tag{7.32}$$

where the BL stands for Brill-Lindquist. This is only for Schwarzschild BHs, Kerr is much more complicated and will be discussed later.

More complicated initial data is Misner data. This can be constructed by suitable inner boundary conditions.

He draws ER bridges — why? do we need the other end of the bridge?

This construction does not allow for spin, nor does it include boosts.

Let us move to the work by Bowen and York in 1980. They proposed an ansatz for $\triangle_L x^i + \cdots = 0$:

$$x^i = -\frac{1}{4r}\left(7f^{ij}P_j + \frac{1}{r^2}P_j x^i x^j\right) - \frac{1}{r^3}\epsilon^{ij}_k S_h x^k, \tag{7.33}$$

with six parameters P^i and S^i.

This yields

$$\hat{A}^{ij} = (Lx)^{ij} = \underbrace{\frac{3}{2}\frac{1}{r^3}\left[x^i P^j + x^j P^i + (f^{ij} - \frac{x^i x^j}{r^2})P^k x_k\right]}_{\mathcal{O}(1/r^2)} + \underbrace{\frac{3}{r^5}\left(\epsilon^{ik}_{\ell} S_k x^{\ell} x^j + \epsilon^{jk}_{\ell} S_K x^{\ell} x^i\right)}_{\mathcal{O}(1/r^3)}. \tag{7.34}$$

P^i and S^i are the ADM momentum and angular momentum components in the isotropic gauge.

Insert sketch of proof for ADM momentum.

With the BY ansatz one can find nonrotating BHs with $S^i = 0 = P^i$; boosted BHs if $P^i \neq 0$ and $S^i = 0$; spinning BHs wih $S^i \neq 0$ and $S^i = 0$.

Did we find the Kerr solution? No. In 2000 Gasat and Price proved that there is no Kerr foliation which

1. has axial symmetry;
2. is conformally flat;
3. reduces to Schwarzschild in the nonrotating limit $S^z = 0$.

This is often simply stated as "there is no conformally flat slicing of Kerr".

If there is a moment of time symmetry, then the BY data with $S^z \neq 0$ are *nonstationary*! The evolution must therefore be nontrivial — something will happen. Typically, during the evolution GWs are produced.

These are typically called *puncture solutions*: with a BY ansatz and a solution of the Lichnerowicz equation (a numerical one, since the equation is nonlinear in ψ).

A recent approach, which is used in many simulations, is called Generalized Bill-Lindquist data (due to Brand and Brügmann in 1997).

We solve the Lichnerowicz equation on $\mathbb{R}^3$ by analytically separating the singular behaviour: we take the ansatz $\psi = \psi_{BL} + u$ where u is a correction.

We know that $\triangle \psi_{BL} = 0$: so, if we plug in the ansats in the nonlinear Lichnerowicz equation we get an equation in the form $\triangle u + \dots = 0$: specifically,

$$\triangle u + \frac{\hat{A}^{ij}\hat{A}_{ij}}{8\psi_{BL}^7}\left(1 + \frac{u}{\psi_{BL}}\right)^{-7} = 0\,, \tag{7.35}$$

with outer boundary conditions $u \sim 1 + \mathcal{O}(1/r)$. Crucially, no inner Boundary Conditions are needed!

By evaluating the fields near the punctures, one sees that at $x^i = c^i_P$ the correction u satisfies $\triangle u = 0$: it is regular! Thus, it can be solved on $\mathbb{R}^3$.

Next, we will discuss CTS and XCTS, which are not well-posed but are useful when discussing stationary slices. These

We continue discussing the initial data problem.

Tuesday 2021-6-15, compiled 2021-06-22

Conformal Thin Sandwich

We can give a different decomposition of $\widetilde{A}_{ij}$:

$$\begin{aligned}\mathscr{L}_m \widetilde{\gamma}_{ij} = \partial_t \widetilde{\gamma}_{ij} - \mathscr{L}_\beta \widetilde{\gamma}_{ij} &= \partial_t \widetilde{\gamma}_{ij} + \left(\widetilde{L}\beta\right)^{ij} + \frac{2}{3}\widetilde{D}_k \beta^k \widetilde{\gamma}_{ij} &(7.36)\\ &= 2\alpha \hat{A}_{ij} + \frac{2}{3}\widetilde{D}_k \beta^k \hat{\gamma}_{ij}\,. &(7.37)\end{aligned}$$

The idea is to combine these two equations in order to express $\hat{A}_{ij}$ as follows:

$$\hat{A}_{ij} = (2\alpha)^{-1}\left[\widetilde{\dot{\gamma}}_{ij} + \left(\widetilde{L}\beta\right)_{ij}\right], \tag{7.38}$$

and similarly for $\overline{A}_{ij}$ if we wanted to use the conformal rescaling.

This equation replaces the $L + TT$ decomposition of $\hat{A}_{ij}$ in CTS. We can plug this into the momentum constraint $C_i = 0$, and we find

$$\widetilde{D}_j\left[\widetilde{a}^{-1}\left(\widetilde{L}\beta\right)^{ij}\right] + \widetilde{D}_j\left[\widetilde{\alpha}^{-1}\widetilde{\dot{\gamma}}^{ij}\right] - \frac{4}{3}\psi^6 \widetilde{D}^i K - 16\pi \widetilde{P}^i = 0\,. \tag{7.39}$$

This must be paired with the Lichnerowicz equation:

$$\widetilde{D}^i \widetilde{D}_i \psi + \dots = 0\,. \tag{7.40}$$

The free data will be $\widetilde{\gamma}_{ij}$, K, $\widetilde{\dot{\gamma}}_{ij}$, E and P^i. The constrained data can be ψ and β^i.

The difference from before is that we have substituted this TT part with the time derivative of the conformal metric; in some sense specifying $\widetilde{\dot{\gamma}}$ can help in specifying free data, for example stationary data.

The term $\widetilde{\alpha} = \psi^{-6}\alpha$ is a conformally rescaled lapse. The maximal slicing condition $K = 0$ means that the CTS equations decouple. The momentum constraint for P^i turns out to be *linear* in this case.

York and Pfeiffer in 2003 proposed a formalism called XCTS, extended CTS, where the idea is to specify an equation for the conformal lapse $\widetilde{\alpha}$ in order not to have to specify it.

Let us consider

$$\mathscr{L}_m K = \dot{K} - \beta^i \widetilde{D}_i K \tag{7.41}$$

$$= -\psi^{-4}\left(\widetilde{D}_i \widetilde{D}^i K + 2\widetilde{D}_i \log \psi \widetilde{D}^i \alpha\right) + \alpha[\dots] \ : \tag{7.42}$$

the piece multiplied by ψ^{-4} can be written as

$$\widetilde{D}_i \widetilde{D}^i K + 2\widetilde{D}_i \log \psi \widetilde{D}^i \alpha = \psi^{-1}\left[\widetilde{D}_i \widetilde{D}^i (\alpha\psi) + \alpha \widetilde{D}^i \widetilde{D}_i \psi\right], \tag{7.43}$$

but using the Hamiltonian constraint $C_0 = 0$ we can write this as

$$\widetilde{D}_i \widetilde{D}^i \left(\hat{\alpha}\psi^7\right) - \left(\hat{\alpha}\psi^7\right)\left[\frac{\hat{K}}{8} + \frac{5}{12}K^2\psi^4 + \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-8} + 2\pi\left(E + 25\psi^8\right)\psi^{-4}\right] + \left(\dot{K} - \beta^i \widetilde{D}_i K\right)\psi^5 = 0\,. \tag{7.44}$$

The XCTS scheme uses this equation, the Lichnerowicz equation and the one previously dubbed equation 1.

Under $K = 0$ these do not decouple; however there is a nice feature: free data includes $\widetilde{\gamma}_{ij}$, $\dot{\widetilde{\gamma}}$, K and $\dot{K}$. Being able to specify these is very convenient.

There exist examples for which XCTS yields non-unique solutions. Let us give a simple example: the solution of these for conformally, asymptotically flat & zero-derivative vacuum initial data.

The equation reads

$$\triangle\psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-7} = 0 \tag{7.45}$$

$$D_j\left[\widetilde{\alpha}^{-1}(L\beta)^{ij}\right] = 0 iln \triangle\left(\widetilde{\alpha}\psi^7\right) - \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\widetilde{\alpha}\psi^{-7} \qquad = 0\,. \tag{7.46}$$

Asymptotically flat means $\psi = 1$, $\beta^i = 0$, $\alpha = 1$ as $r \to \infty$. Then, $\beta^i = 0$ is a solution we can take, together with $\dot{\widetilde{\gamma}} = 0$, which means $\hat{A}^{ij} = 0$.

The equations then read

$$\triangle\psi = 0 \tag{7.47}$$

$$\triangle(\hat{\alpha}\psi^7) = 0\,. \tag{7.48}$$

The inner boundary condition as before is a punctured space: $\Sigma_0 = \mathbb{R}^3 \setminus 0$. This yields $\psi = 1 + M/2r$.

The solution of the second equation is in the same form: $\phi = \widetilde{\alpha}\psi^7 = 1 + a/r$ for some constant a. Now, $\alpha = \psi^6\widetilde{\alpha} = \phi\psi^{-1}$; therefore the lapse reads

$$\alpha = \left(1 + \frac{a}{r}\right)\left(1 + \frac{M}{2r}\right)^{-1} = \frac{r + a}{r + M/2}\,. \tag{7.49}$$

What is the constant a, though? Remember that we are solving this equation on the punctured $\mathbb{R}^3$: a is fixed by the choice of the value of the lapse α at a certain radius, say $r \to 0$. The simplest choice is $\alpha_0 = +1$: this yields $a = M/2$, which also means that the lapse is identically equal to 1.

Therefore, we have re-found Schwarzschild in isotropic coordinates and geodesic gauge,

$$g = -\,\mathrm{d}t^2 + \psi^4 \left(\mathrm{d}r^2 + r^2 \,\mathrm{d}\Omega^2 \right) . \tag{7.50}$$

What if we set $\alpha_0 = -1$? We know that $\alpha > 0$, but let us explore this crazy condition anyways. This yields

$$a = -\frac{M}{2} , \tag{7.51}$$

which implies

$$\alpha = \left(1 - \frac{M}{r} \right) \left(1 + \frac{M}{2r} \right) . \tag{7.52}$$

This is the lapse in isotropic coordinates.

We have a moment of time symmetry, but the time development of Σ_0 are different! If we take $\alpha_0 = +1$, Σ_0 evolves in a nontrivial way! If, instead, $\alpha_0 = -1$ then Σ_0 does not evolve, and ∂_t is a Killing vector.

What do we do with this negative lapse? In some points the $\vec{n}$ vector will remain future-pointing, but we will be in a weird coordinate system such that the time starts to run backwards at small radii. We have to learn to live with it.

Where is XCTS useful? Imagine a binary system: here there is no ∂_t timelike Killing vector, and despite the rotation there is no rotational Killing vector ∂_φ.

Suppose we are considering a circular orbit with frequency Ω: we can define

$$n^a = (\partial_t)^a + \Omega \left(\partial_\varphi \right)^a , \tag{7.53}$$

which is conserved! This is because the orbit looks like a helix in t, r, φ space: the direction of n is always along this helix.

This is an idealized, approximate situation, in which there is no radiation.

In a comoving frame with $\dot{\gamma}_{ij} = 0 = \dot{K}$ we can use a CTT + Bowman-Yosk puncture method.

The other part of the discussion for today is about gauge conditions.

Gauge condition

We need to make a choice of the foliation and a choice of the spatial coordinates. We want our choice to

1. avoid singularities;

2. enhance symmetries;

3. minimize grid distortions.

Something useful is to have our slicing be as close as possible to some Kerr coordinates. If some Killing vector is present we want to stay close to it.

Let us start with **slicing** (α). We have already discussed *geodesic slicing* in which $\alpha = 1$ and $\beta^i = 0$: the coordinates follow freely-falling observers.

Here, $n^a = (\partial_t)^a$, and $\alpha^a = D_a \log \alpha$. This also means that $t = \tau$.

This is certainly not singularity-avoiding: free-falling observers fall into the singularity. Also, it does not seek symmetries, and it distorts the grid a lot.

In this gauge, the ADM equation reads

$$\partial_t K = K_{ij} K^{ij} + 4\pi (E - S) \tag{7.54}$$

$$\partial_t \log \sqrt{\gamma} = -K \,. \tag{7.55}$$

The behaviour of this gauge can be seen from these equations if we think about gravitational collapse: we can draw a r versus t diagram and see that matter eventually falls into the horizon.

$E - S$ gets large, $\partial_t K$ gets large, and $\partial_t \log \sqrt{\gamma}$ gets small. The curvature increases, and the coordinate volume element decreases. These are all characteristics of the *bad gauges*.

Another gauge we have seen is **maximal slicing**: $0 = K$, and we already discussed its geometric meaning.

If we look at $0 = K = -\nabla_a n^a$, the picture we can draw suggests the motion of an incompressible fluid: the coordinate volume element cannot be squeezed.

The Maximal Slicing equation can be also written as

$$D_i D^i \alpha = -\alpha \left[4\pi (E - S) + K_{ij} K^{ij} \right] = 0 \,, \tag{7.56}$$

to be compared with the previous geodesic slicing ones. Here the acceleration of an Eulerian observer will be different from 0, and the lapse α will evolve in a certain way which counteracts the effect of this volume element compression, as well as counteracting the "singularity seeking" property of geodesic slicing.

8 Gauge conditions

Tuesday 2021-6-22, compiled 2021-06-22

So far we have only discussed maximal slicing, $k = -\nabla_a n^a = 0$.

Now let us see Schwarzschild foliations: in (t, R) Schwarzschild coordinates we can consider $t = \text{const}$ hypersurfaces.

For $R > 2M$ time t is a temporal coordinate. Here, these constant-t hypersurfaces satisfy $k_{ab} = 0$, so we also have $k = 0$.

If we draw a Kruskal U, V diagram the horizon $R = 2M$ is a diagonal line $U = V$, while the singularity is a hyperbola.

This slicing is therefore *maximal*, and *not horizon penetrating*.

If we use the isotropic radius r as a coordinate, the lapse reads

$$\alpha = \left(1 - \frac{M}{2r} \right) \left(1 + \frac{M}{2r} \right)^{-1} . \tag{8.1}$$

α is antisymetric, and negative in the III Kruskal region.

In the region $R < 2M$, the foliation is given by $R = \text{const}$ because there R is a timelike coordinate.

Now, $\alpha = (2M/R - 1)^{-1}$, while the metric determinant is

$$\gamma = R^4 \sin^2\theta \left(\frac{2M}{R} - 1 \right), \tag{8.2}$$

and we can use

$$k = -\frac{1}{\alpha} \mathscr{L}_m \log \sqrt{\gamma} = \frac{3M - 2R}{R^2 (2M/R - 1)^{1/2}}, \tag{8.3}$$

therefore the $R = 3M/2$ slice is maximal.

It can be proven that this is a limit slice of a certain maximal foliation of stuff which looks like hyperbolas in a Kruskal diagram, which is maximal and symmetric with respect to $R = 2M$.

These correspond to solutions of the maximal equation the lapse:

$$D_i D^i \alpha - \alpha() \cdots = 0 \tag{8.4}$$

with boundary conditions such that $\alpha \to 1$ at ι_0, and $\alpha(2M) = 0$ (case 1, a Dirichlet boundary condition) or $\partial_R \alpha(3M/2) = 0$ (case 2, a Neumann boundary condition).

Different inner BCs determine different foliations and different properties. Both of the foliations we discussed are singularity-avoiding.

In Eddington-Finkelstein (Kerr-Schild) coordinates, we can set

$$t_{KS} = t + 2M \log \left(\frac{r}{2M} - 1 \right), \tag{8.5}$$

which yields a non-singularity avoiding foliation. How do we construct, in general, maximal foliations of Schwarzschild?

We consider the transformation

$$t \to \widetilde{t} = t + h(R), \tag{8.6}$$

where $h(R)$ is a *height function*, which (because of $0 = k$) satisfies the equation

$$h'(R) = \frac{c^2}{A^2(R) \left[A(R) R^4 + c^2 s \right]}, \tag{8.7}$$

where $A(R) = (1 - 2M/R)$ and c is a constant of integration.

The metric is given in terms of

$$\alpha = f(R) = 1 + \frac{2M}{R} + \frac{c^2}{R^4}, \tag{8.8}$$

while the shift is

$$\beta^r = \frac{c}{R^2} \sqrt{f(R)}, \tag{8.9}$$

and

$$\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = f^{-1}(R)\,\mathrm{d}R^2 + R^2\,\mathrm{d}\Omega\ , \tag{8.10}$$

which yields a family of foliations for different choices of c: $c = 0$ is standard Schwarszchild spacetime, but other choices are possible, and $c = (3/4)\sqrt{3}M^2$.

The main property of maximal slicing, which is quite general, is that the lapse α goes to zero in the regions of highest curvature.

This is called "lapse freezing" or "lapse collapse", and it is a generic property indicating the fact that the gauge is singularity-avoiding.

This is defined by $\Box x^\mu = 0$ (Harmonic gauge), so for $\mu = 0$ we have $\Box t = 0$.

This means

$$0 = \Box t = \sqrt{-g}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu t\right) = \sqrt{-g}\partial_\mu\left(\sqrt{-g}g^{\mu 0}\right). \tag{8.11}$$

This means that

$$0 = \partial_t\left(\alpha\sqrt{\gamma}g^{00}\right) + \partial_i\left(\alpha\sqrt{\gamma}g^{0i}\right), \tag{8.12}$$

but $g^{00} = \alpha^{-2}$ and $g^{0i} = \beta^i/\alpha^2$.

The equation then reads

$$0 = \partial_t\alpha - \beta^i\partial_i\alpha - \underbrace{\alpha\left[\frac{1}{\sqrt{\gamma}}\partial_t\sqrt{\gamma} - \frac{1}{\sqrt{\gamma}}\partial_i\left(\sqrt{\gamma}\beta^i\right)\right]}_{-\alpha k}, \tag{8.13}$$

so the equation for α reads $\mathscr{L}_m\alpha = -\alpha^2 k$.

If we set $\beta^i = 0$, we get $\alpha = C(x^i)\sqrt{\gamma}$. In the case of Schwarzschild, we have $\partial_t\alpha = 0$, $\beta^i = 0$ and $k = 0$.

Constant-t slices of Schwarzschild are harmonic slices.

The Bona-Masso family, also known as $1 + \log$ slicing, is determined by

$$\mathscr{L}_m\alpha = -\alpha^2 f(\alpha)k\,. \tag{8.14}$$

We can fall back to geodesic slicing with $f(\alpha) = 0$, or harmonic slicing with $f(\alpha) = 1$, or we can set $f(\alpha) = 2/\alpha$: this is $1 + \log$ slicing.

If we set $\beta^i = 0$, we have $\partial_t\alpha = \partial_t \log\gamma$, a solution to which is $\alpha = 1 + \log\gamma$.

For Schwarzschild, this is the Height function method:

$$\alpha^2 = 1 - \frac{2M}{R} + \frac{c^2}{R^4}e^\alpha\,, \tag{8.15}$$

which is very close to maximal slicing; however this is now an implicit equation with this new exponential term.

These are called "trumpet slices": in an embedding diagram they look like a wormhole, however they end at $R = 3M/2$.

This $1 + \log$ slicing is what is typically used today to evolve astrophysical black holes.

What about spatial gauge and the choice of shift β^i?

One item in our wishlist was "minimal distortion": we can quantify it through a distortion tensor,

$$Q_{ij} = \partial_t \gamma_{ij} = -2\alpha k_{ij} + \mathscr{L}_\beta \gamma_{ij}\,. \tag{8.16}$$

From this tensor we can consider

$$\Sigma_{ij} = Q_{ij} - \frac{1}{3} Q \gamma_{ij} = \cdots = -2\alpha A_{ij} + (L\beta)_{ij} = \psi^4 \partial_t \widetilde{\gamma}_{ij}\,. \tag{8.17}$$

We can use this to define a functional:

$$\begin{aligned} I[\beta^i] &= \int_{\Sigma_t} \Sigma_{ij}\Sigma^{ij}\sqrt{\gamma}\,\mathrm{d}^3x = && (8.18)\\ &= \int_{\Sigma_t} \left[4\alpha^2 A_{ij}A^{ij} - 4\alpha A_{ij}(L\beta)^{ij} + (L\beta)_{ij}(L\beta)^{ij}\right]\sqrt{\gamma}\,\mathrm{d}^3x\ , && (8.19)\end{aligned}$$

which we can extremize: we set $0 = \delta I[\beta^i]$, which yields

$$\begin{aligned} \delta I[\beta^i] &= \int_{\Sigma_t} 2\delta\left[\Sigma_{ij}(L\beta)^{ij}\right]\sqrt{\gamma}\,\mathrm{d}^3x && (8.20)\\ &= 2\int_{\Sigma_t} \Sigma_{ij}\left(D^i\delta\beta^j + D^j\delta\beta^i - \underbrace{\frac{2}{3}D_k\delta\beta^k\gamma^{ij}}_{\Sigma=0}\right)\sqrt{\gamma}\,\mathrm{d}^3x && (8.21)\\ &= 4\int_{\Sigma_t} \Sigma_{ij}D^i\delta\beta^j\sqrt{\gamma}\,\mathrm{d}^3x && (8.22)\\ &= 4\int_{\Sigma_t} \left[D^i\left(\Sigma_{ij}\delta\beta^j\right) - D^i\Sigma_{ij}\delta\beta^j\right]\sqrt{\gamma}\,\mathrm{d}^3x && (8.23)\\ &= \underbrace{4\int_{\partial\Sigma_t} \Sigma_{ij}\delta\beta^j\sqrt{\gamma}\,\mathrm{d}^3x}_{=0,\,\delta\beta^j\big|_\partial=0} - 4\int_{\Sigma_t} D^i\Sigma_{ij}\delta\beta^j\sqrt{\gamma}\,\mathrm{d}^3x = 0\,, && (8.24)\end{aligned}$$

therefore our condition, by the usual lemma of funcitonal variational calculus, is $D^i\Sigma_{ij} = 0$. In terms of β^i, this defines what is knowns as *minimal distortion shift*:

$$\triangle_L \beta^i = 2D_j\left(\alpha A^{ij}\right) - 16\pi\alpha P^i + \frac{4}{3}\alpha D^i k + 2A^{ij}D_j\alpha\,. \tag{8.25}$$

This gauge, as written, is not really used, however it is the starting point for other ones which better achieve the distortion minimization.

An observation: both Q_{ij} and Σ_{ij} are 0 if ∂_t is a Killing vector, so in that case minimal distortion is satisfied automatically for stationary spacetimes in adapted coordinates.

How do we use this then? We can use approximate minimal distortion equations:

$$0 = D^i\Sigma_{ij} = D^i\left(\psi^4\partial_t\widetilde{\gamma}_{ij}\right) \approx \widetilde{D}^i\left(\partial_t\widetilde{\gamma}_{ij}\right). \tag{8.26}$$

this elliptic equation for β^i is easier to implement numerically.

This is called Γ freezing: we can write the equation as

$$0 = D_j \dot{\tilde{\gamma}} = \partial_t D_j \tilde{\gamma}_{ij} = \cdots = -\partial_t \tilde{\Gamma}^i \,, \tag{8.27}$$

(see equation 9.32 in the notes), where

$$\tilde{\Gamma}^i = -D_j \tilde{\Gamma}^{ij} = \left(\tilde{\Gamma}^i_{jk} - F^i_{jk} \right) \tilde{\gamma}^{jk} \,. \tag{8.28}$$

We get an elliptic equation for β^i, which is written as

$$\tilde{\gamma}^{jk} D_j D_k \beta^i + \cdots = 0 \,, \tag{8.29}$$

which is nice because it is written in terms of partial derivatives.

Alcubierre and colleagues have proposed a **parabolic** Γ **driver**: the idea is to write *evolution* equations for β^i, such that the solutions "asymptote" to the "equilibrium" solution of the Γ-freezing.

This equation will look like

$$\partial_t \beta^i = k \partial_t \tilde{\Gamma}^i \approx k \left(\tilde{\gamma}^{jk} D_j D_k \beta^i + \dots \right) , \tag{8.30}$$

a parabolic equation.

For $t \to \infty$ this asymptotes to the Γ-freezing solution, however parabolic PDEs are known to be *stiff*, which severely constrains the number of timesteps which can be taken.

Therefore, people have also considered *hyperbolic drivers*:

$$\begin{aligned} \partial_{tt} \beta^i &= k \partial_t \tilde{\Gamma}^i - \left(\eta \partial_t \log k \right) \partial_t \beta^i & (8.31) \\ &= k \left(\tilde{\gamma}^{jk} \partial_j \partial_k \beta^i + \dots \right) - \left(\eta - \partial_t \log k \right) \partial_t \beta^i \,, & (8.32) \end{aligned}$$

and we can discard some term to get a damped wave equation:

$$\partial_{tt} \beta^i = k \tilde{\gamma}^{ij} \partial_j \partial_k \beta^i - \eta \partial_t \beta^i \,. \tag{8.33}$$

This equation transports things and damps them.

A simpler first-order version is written as an advection-like equation:

$$\partial_t \beta^i = \mu_s \tilde{\Gamma}^i - \eta \beta^i + \beta^j \partial_j \beta^i \,, \tag{8.34}$$

which has speed μ_s (which can be chosen freely) and a damping term $\eta > 0$, which can be also chosen.

These equations should be compared to the hyperbolic Γ-driver with the harmonic shift equation.

In summary: we have seen, for α:

1. geodesic gauge $\alpha = 1$, $\beta^i = 0$;
2. maximal slicing $k = 0$;
3. harmonic slicing $\Box t = 0$;

4. the Bona-Masso family, which includes harmonic slicing as well as $1 + \log$ slicing;

and for spatial slicing

1. minimal distortion;
2. Γ-drivers.

In geodesic gauge, the simulation crashes at $t = \pi$ for $M = 1$: this is because that is the point at which the observer falls into the singularity.

With $1 + \log$ slicing as well as a Γ driver we reach a sort of stationary configuration and the simulation does not crash.