Gravitational Waves @ Jena University

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Introduction

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The syllabus can be found here.

Interesting things on the Indico server of Jena university.

In this first lecture, a basic introduction to the theory of gravitational waves: Einstein's first papers, the sticky bead argument by Bondi & Feynman, the quadrupole formula:

$$\overline{h}_{ij}(t,r) = \frac{2G}{c^4 r} \ddot{I}_{ij}(t-r)\,. \tag{0.1}$$

The idea behind the multipole expansion is that we are solving the Poisson equation $\nabla^2 \phi = \rho$, so

$$\phi(\vec{r}) = \int \frac{\rho(\vec{x})\, \mathrm{d}^3 x}{|\vec{r} - \vec{x}|}\,, \tag{0.2}$$

so as long as we are far away from the source we will see

$$\phi(\vec{r}) = -\frac{q}{r} - \frac{p_i n^i}{r^2} - \frac{Q_{ij} n^i n^j}{r^3} + \dots\,. \tag{0.3}$$

Quiz: which of these are GW sources?

1. spherical star: no, its quadrupole is vanishing;
2. rotating star: no, its quadrupole is constant;
3. star with a mountain: yes, its quadrupole evolves (potential source of continuous GW);
4. supernova explosion: yes, if there is asymmetry (potential source of burst GW);
5. binary system: yes, already detected!

Claim 0.1. *Order of magnitude expression:*

$$h \lesssim \frac{GM}{c^2 D} \frac{v^2}{c^2} = \frac{R}{D} \frac{GM}{c^2 R} \left(\frac{v}{c} \right)^2\,, \tag{0.4}$$

where D is the distance to the object, R is the characteristic scale of the object (so that $GM/c^2 R$ is the compactness), while v is the characteristic velocity. The quantity we calculate is $h \sim \delta L / L$, the strain.

Proof. To do. □

The Hulse-Taylor pulsar. The two-body problem in GR is difficult.

The typical waveform in the PN region looks like:

$$h_+(t) \approx \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}(t)}{c}\right)^{2/3}\cos\Big(2\pi f_{\mathrm{gw}}(t)t\Big)\,, \tag{0.5}$$

then we need numerical relativity to simulate the plunge and merger, and finally the ring-down is simulated using BH perturbation methods. The mass scale is

$$h(t) \sim \nu\frac{1}{r/M}(Mf_{\mathrm{gw}})^{2/3}\,, \tag{0.6}$$

while

$$\phi_{\mathrm{gw}}(t) \sim 2\phi_{\mathrm{orb}}(t) = 2M_c^{-5/8}t^{5/8} = 2\nu^{-3/8}\left(\frac{t}{M}\right)^{5/8}\,, \tag{0.7}$$

where $\nu = \mu/M$, and $\mu = 1/(1/M_1 + 1/M_2)$.

Multiple detectors are crucial for sky localization, as well as for the measurement of polarization.

At leading order, the two-body problem in GR is scale-invariant: the length of the signal can be estimated simply from the mass of the stars involved.

R-process nucleosynthesis might have something to do with BNS mergers, if the stars are torn apart by the collision.

1 Weak-field GR

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This is the limit of GR for weak gravitational fields: the metric is assumed to be in the form of the Minkowski one plus a perturbation. We are seeking the equations of motion under this assumption.

How do we quantify the term "small"? We assume that there is a **global inertial coordinate system** such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{1.1}$$

where, like in the rest of the course, we will use the letters α, β, γ or μ, ν for the coordinates x^μ; while letters like a, b represent the abstract notation.

The term "small", then, means that each component of $h_{\mu\nu}$ has an absolute value which is much smaller than 1. We are using the metric signature $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$.

What does this approximation describe?

1. Newtonian gravity;

2. gravito-electric / magnetic effects (this will be discussed in more detail later, an example is the Lense-Thirring effect);

3. propagation of gravitational waves.

In the case of the gravitational field around the Sun, in terms of orders of magnitude we have[1]

$$\left|h_{\mu\nu}\right| \sim \frac{\phi}{c^2} \sim \frac{GM_\odot}{c^2 R_\odot} \sim 10^{-6} . \tag{1.2}$$

From a field-theoretic point of view:

1. η is a background metric;

2. h is the "main" field;

3. the metric does *not* backreact on the matter ($T_{\mu\nu}$).

The metric perturbation h transform like a tensor on flat spacetime under Lorentz transformations: if $\Lambda^\top \eta \Lambda = \eta$, then the coordinates change like $x = \Lambda x'$, then the full metric transforms like

$$\begin{aligned} g_{\mu'\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} && (1.3) \\ &= \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \left(\eta_{\mu\nu} + h_{\mu\nu} \right) && (1.4) \\ &= \eta_{\mu'\nu'} + \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} h_{\mu\nu} , && (1.5) \end{aligned}$$

therefore the transformation for h is

$$h_{\mu\nu} \to h_{\mu'\nu'} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} h_{\mu\nu} . \tag{1.6}$$

Mind the notation: the meaning of $h_{\mu'\nu'}$ is $h_{\mu\nu}(x')$.

Symmetry of linearized GR

Full GR is diffeomorphism invariant, while linearized GR is *infinitesimal* diffeomorphism invariant. The relevant transformations are

$$x^\mu \to x^{\mu'} = x^\mu + \xi^\mu(x^\alpha) , \tag{1.7}$$

where the vector field ξ is selected so that $\left|\partial_\mu \xi^\alpha\right| \sim \left|h_{\mu\nu}\right| \ll 1$.

The Jacobian of this transformation is

$$\frac{\partial x^{\mu'}}{\partial x^\mu} = \delta^{\mu'}{}_\mu + \partial_\mu \xi^{\mu'} , \tag{1.8}$$

while the inverse Jacobian is

$$\frac{\partial x^\mu}{\partial x^{\mu'}} + \delta^\mu{}_{\mu'} - \partial_{\mu'} \xi^\mu + \mathcal{O}\left(|\partial \xi|^2 \right) , \tag{1.9}$$

[1] We make the c explicit here for clarity, but we will use geometric units $c = G = 1$ for the rest of the course.

since $(\mathbb{1}+\delta)(\mathbb{1}-\delta)=\mathbb{1}+\mathcal{O}\left(\delta^2\right)$.

Under this change of coordinates, we have

$$
\begin{aligned}
g_{\mu'\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} && (1.10)\\
&\sim (\delta\delta-\partial\delta-\partial\delta+\partial\partial)(\eta+h) = \delta\delta\eta + h - \partial\delta - \partial\delta + \mathcal{O}\left(\delta^2\right) && (1.11)\\
&= \delta^\mu_{\mu'}\delta^\nu_{\nu'}\eta_{\mu\nu} - \partial_{\mu'}\xi^\mu\delta^\nu_{\nu'}\eta_{\mu\nu} - \partial_{\nu'}\xi^\nu\delta^\mu_{\mu'} + \delta^\mu_{\mu'}\delta^\nu_{\nu'}h_{\mu\nu} && (1.12)\\
&= \eta_{\mu'\nu'} + h_{\mu'\nu'} - 2\partial_{(\mu'}\xi_{\nu')}\,, && (1.13)
\end{aligned}
$$

therefore we have our transformation law:

$$
h_{\mu'\nu'} = h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}\,. \tag{1.14}
$$

This can also be written in terms of the Lie derivative as

$$
h_{\mu\nu} \to h_{\mu\nu} + \mathscr{L}_\xi \eta_{\mu\nu}\,. \tag{1.15}
$$

This is the analogous of a gauge transformation in electromagnetism: $A_\alpha \to A_\alpha + \partial_\alpha\chi$, where A is the vector potential.

Equations of motion

The equations of motion will come through plugging $g=\eta+h$ into the EFE $G_{ab}=8\pi T_{ab}$ and keeping only the linear order in h.

We will need the following quantities:

$$
\begin{aligned}
g^{\mu\nu} &= \eta^{\mu\nu} + h^{\mu\nu} + \mathcal{O}\left(h^2\right) && (1.16)\\
\Gamma^\mu_{\alpha\beta} &= \frac{1}{2}\eta^{\mu\lambda}\left(\partial_\alpha h_{\lambda\beta} + \partial_\beta h_{\lambda\alpha} - \partial_\lambda h_{\alpha\beta}\right) + \mathcal{O}\left(h^2\right) && (1.17)\\
R_{\mu\nu} &= \partial\Gamma - \partial\Gamma + \mathcal{O}\left(h^2\right), && (1.18)
\end{aligned}
$$

where we already simplified the expressions by removing the higher-order terms. The result is

$$
R_{\mu\nu} = \partial^\alpha\partial_{(\mu}h_{\nu)\alpha} - \frac{1}{2}\partial_\lambda\partial^\lambda h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h + \mathcal{O}\left(h^2\right), \tag{1.19}
$$

where $h=h^\alpha_\alpha=\eta^{\alpha\beta}h_{\alpha\beta}$. Note that we are allowed to use η instead of g to lower indices. The Einstein tensor reads

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R && (1.20)\\
&= \partial^\alpha\partial_{(\mu}h_{\nu)} - \frac{1}{2}\partial_\lambda\partial^\lambda h_{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \frac{1}{2}\eta_{\mu\nu}\partial_\lambda\partial^\lambda h\,, && (1.21)
\end{aligned}
$$

which can be simplified if we consider the trace-reversed metric

$$
\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,, \tag{1.22}
$$

so that $\bar{h} = \eta^{\mu\nu} h_{\mu\nu} - \eta^{\mu\nu}\eta_{\mu\nu} h/2 = -h$. See equation 2.13 in the notes for a full explanation, but the idea is to insert $\bar{h}_{\mu\nu}$ and $\bar{h}$ into $G_{\alpha\beta}$ and to make some simplifications. We get

$$G_{\mu\nu} = -\frac{1}{2}\eta_{\alpha\beta}\partial^\alpha\partial^\beta \bar{h}_{\mu\nu} + \partial^\alpha \partial_{(\mu}\bar{h}_{\nu)\alpha} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta \bar{h}_{\alpha\beta}\,, \tag{1.23}$$

which is in the form $\Box_\eta \bar{h}_{\mu\nu} + \ldots \partial^\alpha \bar{h}_{\alpha\beta}$. We still have gauge freedom, so we can simplify the equation a great deal by setting $\partial^\alpha \bar{h}_{\alpha\beta} = 0$ — the Hilbert, or Lorentz gauge.

With this choice, we have

$$\boxed{\Box_\eta \bar{h}_{\mu\nu} = -\frac{16G}{c^4} T_{\mu\nu}\,,} \tag{1.24}$$

a relatively simple tensor wave equation.

Is is always possible to impose the Hilbert gauge? Yes: we can make an infinitesimal coordinate transformation to send a generic $h_{\mu\nu}$ to $h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$, so that $h \to h + 2\eta^{\alpha\beta}\partial_{(\alpha}\xi_{\beta)}$. Therefore,

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + 2\partial_{(\mu} h_{\nu)} - \eta_{\mu\nu}\partial_\alpha \xi^\alpha\,, \tag{1.25}$$

and we can send

check indices here

$$\partial^\alpha \bar{h}_{\mu\alpha} \to \partial^\alpha \bar{h}_{\mu\alpha} + \Box\xi_\mu + \partial^\mu\partial_\nu\xi_\mu + \partial_\nu\partial^\lambda\xi_\lambda\,, \tag{1.26}$$

so if we set $\Box\xi_\mu = -\partial^\alpha \bar{h}_{\mu\alpha} = v_\mu$ we can reduce ourselves to the Hilbert gauge from any starting point. All we need to do is solve the wave equation $\Box\xi_\mu = v_\mu$.

Now, to linear order $T_{\mu\nu}$ does not depend on h. So, we can find formal solutions using Green's functions, like in electromagnetism.

The Bianchi identities are now given by $\partial_\nu G^{\mu\nu} = 0$, so $\partial_\nu T^{\mu\nu} = 0$, which gives us the EOM for matter — note that this is a partial, not a covariant derivative! This means that there is no backreaction on the metric.

The linear EFE correspond to the equations of motion of a massless spin-2 field.

Weak-field solutions

Let us consider a *static source*: suppose that $T_{\mu\nu} = \rho t_\mu t_\nu$, where $t^\mu = (\partial_t)^\mu$ is the time vector along the time direction of the global inertial coordinate system while ρ is an energy density.

If $t^\mu = (1,0,0,0)$ then $T_{00} = \rho$ while $T_{0i} = 0 = T_{ij}$.

In this case, then, the stress-energy tensor is time-independent: therefore also on the other side we will have $\partial_t \bar{h}_{\mu\nu} = 0$.

Therefore, the left-hand side of the equation will read $\nabla^2 \bar{h}_{\mu\nu} = -16\pi\rho$ for $\mu = \nu = 0$ and $\nabla^2 h_{\mu\nu} = 0$ for all the other components.

These Poisson equations can be solved as boundary-value problems if we assume that $h_{\mu\nu} \to 0$ for $r \gg R$.

This looks very similar to the Newton equation $\nabla^2 \phi = 4\pi\rho$; therefore $\overline{h}_{00} = -4\phi$, while $\overline{h}_{\mu\nu} = 0$ for all other components.

We can reconstruct the metric using the fact that $\overline{h} = 4\phi$, so

$$h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{h} = -4\phi t_\mu t_\nu - \frac{1}{2}\eta_{\mu\nu} 4\phi \,, \tag{1.27}$$

so the metric reads

$$g_{\mu\nu} = \eta_{\mu\nu}(1 - 2\phi) - 4\phi t_\mu t_\nu \,, \tag{1.28}$$

therefore

$$g = -(1 + 2\phi)\, \mathrm{d}t^2 + (1 - 2\phi)\delta_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j \,. \tag{1.29}$$

We know that far away from the source, the Newtonian field decays like $\phi \approx -M/r + \mathcal{O}\left(r^{-2}\right)$.

Therefore, this metric approximation already includes special relativity as well: we have $g \to \eta$ for large r, but also $g = \eta$ for $M = 0$.

The geodesic equation for this weak field metric reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\partial^i \phi \,. \tag{1.30}$$

However, these Newtonian equations of motion are *not* consistent with $\partial_\mu T^{\mu\nu} = 0$. These describe the motion of the source which generates gravity, whereas the Newtonian EOM describe the motion of test particles in the weak-field metric.

The dual meaning of the full EFE — matter deforms the spacetime, the spacetime shapes the trajectories of matter — *cannot* be realized at linear order.

No-stress source

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We considered a source in the form $T_{\mu\nu} = \rho t_\mu t_\nu$, where $t^\mu = (1, \vec{0})$.

Now we will consider a source in the form

$$T_{\mu\nu} = -2\rho t_\mu t_\nu + 2J_{(\mu} t_{\nu)} \,, \tag{1.31}$$

where $J^\mu = \rho u^\mu = \rho(\gamma, \gamma v^i / c)$.

Probably the first 2 is not there.

The static source from before can be recovered from this expression in the low-velocity limit $v^i/c \to 0$. In that case, $T_{ij} = 0$: we can see that T_{ij} is of order v^2/c^2, so to first order they vanish.

In this situation, we get the system

$$\begin{cases} \Box \overline{h}_{0\mu} &= -16\pi T_{0\mu} \\ \Box \overline{h}_{ij} &= 0 \,. \end{cases} \tag{1.32}$$

In order to simplify, let us assume that $\partial_t \overline{h}_{ij} = 0$: then, the solution to the second of these becomes $\nabla^2 \overline{h}_{ij} = 0$, with flat boundary conditions at large distance. By linearity, this leads to $\overline{h}_{ij} = 0$.

Claim 1.1. *If we define $A_\mu = -(1/4)\overline{h}_{0\mu} = -(1/4)\overline{h}_{\mu\nu} t^\nu$, then the metric becomes*

$$g_{00} = -1 + 2A_0 \tag{1.33}$$

$$g_{0i} = 4A_i \tag{1.34}$$

$$g_{ij} = (1 + 2A_0)\delta_{ij} . \tag{1.35}$$

In terms of this A_μ, the Dalambertian equation from before reads

$$\Box A_\mu = -\frac{16}{4}\pi J_\mu = -4\pi J_\mu , \tag{1.36}$$

which are formally identical to the Maxwell equations! Therefore, we can employ known techniques from electromagnetism.

For example, if $\partial_t A_\mu = 0$ then

$$\begin{cases} A_0 & = -\phi \\ A_i & = \int \mathrm{d}^3 x^i \, \frac{J_i}{|x - x^i|} , \end{cases} \tag{1.37}$$

which is the reason why the phenomena which can be described through this formalism are known as gravito-electric and gravito-magnetic effects.

Claim 1.2. *For example, geodesics in a weak-field stationary (no stress) spacetime are described by a Lagrangian*

$$\mathscr{L} = -mc\left(-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right)^{1/2} \tag{1.38}$$

$$= -mc^2\left(-g_{00} - 2g_{0i}\frac{v^i}{c} - g_{ij}\frac{v^i v^j}{c^2}\right)^{1/2} \tag{1.39}$$

$$\approx -mc^2 + \frac{m}{2}v^2 + m\phi + 4mcA_i v^i . \tag{1.40}$$

We have a mass term, a kinetic term, a gravitational term, and a contribution to the Lorentz force.

The corresponding equations of motion read

$$\ddot{\vec{x}} = \vec{E} + 4\vec{v} \times \vec{B} , \tag{1.41}$$

where $\vec{E}$ and $\vec{B}$ are the gravitoelectric and gravitomagnetic fields derived from our A_μ. The differences from EM are: the absence of charge, and the factor of 4 before the magnetic term.

An example of a gravito-electromagnetic effect is the Lense-Thirring effect: a magnetic moment $\vec{s}$ in a magnetic field precesses, according to

$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = \vec{s} \times \vec{\Omega} \qquad \text{where} \qquad \vec{\Omega} = -\frac{q}{m}\vec{B}_{EM} , \tag{1.42}$$

so in order to generalize to the precession of a gyroscope in an EM field we need to map $q \to m$ and $\vec{B}_{EM} \to 4\vec{B}$.

This way, we see for example that $\Omega_g = -4B$. A mission called Gravity Probe B measured this effect: they found precession with $\Omega_g \sim 0.22\,\text{arcsec/yr}(R_\oplus/r)^3$. This is a 20 % accurate test of GR in the weak field.

What does that mean?

Another example is **frame dragging**, which applies in full GR: if we put the gyroscope around a BH a similar effect emerges. Around a Kerr BH we have

$$g_{0i}^{\text{Kerr}} \sim \Omega_{BH}\,, \tag{1.43}$$

and if the particle is close to the BH a particle is "locked" to the BH rotation.

2 Gravitational Waves in linear GR

GW are solutions of weak-field GR in a vacuum. There, the wave equation reads $0 = \Box_\eta \overline{h}_{\mu\nu}$. What are the properties of the solutions of these equations? The simplest thing we can do is look for plane wave solutions. We take a wave vector $k^\mu = (\omega, k^i)$ and an amplitude $A_{\mu\nu}$; then

$$\overline{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\mu x^\mu} = A_{\mu\nu} e^{i(-\omega t + \vec{k}\cdot\vec{x})}\,, \tag{2.1}$$

so $\partial_\mu \overline{h}_{\alpha\beta} = (ik_\mu)\overline{h}_{\alpha\beta}$.

Substituting the plane wave ansatz yields

$$0 = \Box \overline{h}_{\alpha\beta} = -\eta^{\mu\nu} k_\mu k_\nu \overline{h}_{\alpha\beta}\,, \tag{2.2}$$

therefore $k_\mu k^\mu = 0$. The wavevector is null.

This implies that the GW propagates at the speed of light: $\omega s^2 = \left|\vec{k}\right|^2$.

How do we completely specify a gauge? Any infinitesimal transformation such that $\Box \xi^\mu = 0$ preserves the Hilbert gauge, so we can make a residual gauge transformation.

The harmonic gauge implies that

$$0 = -\partial^\alpha \overline{h}_{\mu\alpha} = ik^\alpha \overline{h}_{\mu\alpha}\,, \tag{2.3}$$

which yields $k^\alpha A_{\alpha\mu} = 0$. This means that GWs are **transverse** to the propagation direction.

We know that $\overline{h}_{\mu\nu}$ maps to $\overline{h}_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} + \eta_{\mu\nu}\partial_\alpha \xi^\alpha$.

Let us use $\xi^\mu = B^\mu e^{ik_\alpha x^\alpha}$ as an ansatz for our residual gauge transformation, since it automatically harmonic: we get

$$A_{\mu\nu} \to A_{\mu\nu} - 2ik_{(\mu}B_{\nu)} + i\eta_{\mu\nu} k_\alpha B^\alpha\,, \tag{2.4}$$

and since we can pick B^μ arbitrarily we can impose $\overline{h} = A^\mu_\mu = 0$, the **traceless condition**, as well as $\overline{h}_{\mu 0} = 0$, the **transverse condition**. The second is suggested by the previously found result $k_\alpha A^{\alpha\beta} = 0$.

In terms of B, this is a linear algebraic system, and it is invertible.

In summary, we start from 10 variables, we use 4 equations to impose the Hilbert gauge, and 4 more to impose the TT gauge. The two degrees of freedom which are left are the true degrees of freedom of a GW.

More explicitly, if we have $k^\mu = (\omega, 0, 0, k_z)$ this means

1. $k^2 = 0$ implies $-\omega = k_z$;
2. the phase reads $k_\alpha x^\alpha = \omega(t - z)$;
3. the Hilbert gauge $k^\mu A_{\mu\nu} = 0$ tells us that $A_{0\nu} = A_{3\nu}$;
4. the transverse condition tells us that $A_{0\mu} = 0$ (so also $A_{3\mu} = 0$);
5. the traceless condition tells us that $A^\mu_\mu = 0$.

This leads to the usual formulation

$$A^{TT}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{2.5}$$

Therefore,

$$h^{TT}_{\mu\nu} = A^{TT}_{\mu\nu} \exp\big(i\omega(t - z)\big) . \tag{2.6}$$

In TT gauge we have $\overline{h}_{\mu\nu} = h_{\mu\nu}$ since the trace is zero. Importantly, the TT gauge can only be defined in vacuo! This is because in that case $\Box \overline{h}_{\mu\nu} \neq 0$, so while we can still exploit gauge freedom we cannot set components to zero inside the source.

The metric in TT gauge reads

$$g = -\,\mathrm{d}t^2 + \mathrm{d}z^2 + (1 + h_+)\,\mathrm{d}x^2\,(1 - h_\times)\,\mathrm{d}y^2 + 2h_\times\,\mathrm{d}x\,\mathrm{d}y\,g \quad = -\,\mathrm{d}t^2 + (\delta_{ij} + h^{TT}_{ij})\,\mathrm{d}x^i\,\mathrm{d}x^j\ . \tag{2.7}$$

How do we identify the GW degrees of freedom in general? We can impose the TT gauge outside the source (far away from the $T_{\mu\nu}$).

In general,

$$h^{TT}_{\mu\nu} = \Lambda_{\mu\nu}{}^{\alpha\beta}\overline{h}_{\alpha\beta}\,, \tag{2.8}$$

where Λ is a projection operator, defined as

$$\Lambda_{\mu\nu}{}^{\alpha\beta} = P^\alpha_\mu P^\beta_\nu - \frac{1}{2} P_{\mu\nu} P^{\alpha\beta} \tag{2.9}$$

$$P_{\mu\nu} = \delta_{\mu\nu} - n_\mu n_\nu\,, \tag{2.10}$$

where n^μ is the propagation direction.

The projection tensor $P_{\mu\nu}$ is symmetric, it is transverse ($P_{\mu\nu}n^{\nu} = 0$), it is idempotent ($P_{\mu\alpha}P_{\alpha\nu} = P_{\mu\nu}$), and its trace is equal to 2.

The tensor $\Lambda_{\mu\nu\alpha\beta}$ is also idempotent, transverse in all indices, traceless in $\mu\nu$ and $\alpha\beta$ separately, and symmetric in the swap of $\mu\nu$ and $\alpha\beta$.

In summary, we have found GW solutions, they propagate with c, they are transverse, they have two degrees of freedom.

Symmetric, Transverse, Trace-Free tensors play an important role in GW theory. They can be used to obtain the **Multipolar expansion**.

"Living review of relativity" (see webpage) describes all the tests of GR.