Gravitational Waves @ Jena University

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Introduction

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The syllabus can be found here.

Interesting things on the Indico server of Jena university.

In this first lecture, a basic introduction to the theory of gravitational waves: Einstein's first papers, the sticky bead argument by Bondi & Feynman, the quadrupole formula:

$$\overline{h}_{ij}(t,r) = \frac{2G}{c^4 r}\ddot{I}_{ij}(t-r)\,. \tag{0.1}$$

The idea behind the multipole expansion is that we are solving the Poisson equation $\nabla^2\phi = \rho$, so

$$\phi(\vec{r}) = \int \frac{\rho(\vec{x})\,\mathrm{d}^3x}{|\vec{r}-\vec{x}|}\,, \tag{0.2}$$

so as long as we are far away from the source we will see

$$\phi(\vec{r}) = -\frac{q}{r} - \frac{p_i n^i}{r^2} - \frac{Q_{ij}n^i n^j}{r^3} + \dots\,. \tag{0.3}$$

Quiz: which of these are GW sources?

1. spherical star: no, its quadrupole is vanishing;
2. rotating star: no, its quadrupole is constant;
3. star with a mountain: yes, its quadrupole evolves (potential source of continuous GW);
4. supernova explosion: yes, if there is asymmetry (potential source of burst GW);
5. binary system: yes, already detected!

Claim 0.1. *Order of magnitude expression:*

$$h \lesssim \frac{GM}{c^2 D}\frac{v^2}{c^2} = \frac{R}{D}\frac{GM}{c^2 R}\left(\frac{v}{c}\right)^2, \tag{0.4}$$

where D is the distance to the object, R is the characteristic scale of the object (so that GM/c^2R is the compactness), while v is the characteristic velocity. The quantity we calculate is $h \sim \delta L/L$, the strain.

Proof. To do. □

The Hulse-Taylor pulsar. The two-body problem in GR is difficult.

The typical waveform in the PN region looks like:

$$h_+(t) \approx \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\text{gw}}(t)}{c}\right)^{2/3}\cos\Big(2\pi f_{\text{gw}}(t)t\Big)\,, \tag{0.5}$$

then we need numerical relativity to simulate the plunge and merger, and finally the ring-down is simulated using BH perturbation methods. The mass scale is

$$h(t) \sim \nu\frac{1}{r/M}(Mf_{\text{gw}})^{2/3}\,, \tag{0.6}$$

while

$$\phi_{\text{gw}}(t) \sim 2\phi_{\text{orb}}(t) = 2M_c^{-5/8}t^{5/8} = 2\nu^{-3/8}\left(\frac{t}{M}\right)^{5/8}\,, \tag{0.7}$$

where $\nu = \mu/M$, and $\mu = 1/(1/M_1 + 1/M_2)$.

Multiple detectors are crucial for sky localization, as well as for the measurement of polarization.

At leading order, the two-body problem in GR is scale-invariant: the length of the signal can be estimated simply from the mass of the stars involved.

R-process nucleosynthesis might have something to do with BNS mergers, if the stars are torn apart by the collision.

1 Weak-field GR

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This is the limit of GR for weak gravitational fields: the metric is assumed to be in the form of the Minkowski one plus a perturbation. We are seeking the equations of motion under this assumption.

How do we quantify the term "small"? We assume that there is a **global inertial coordinate system** such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{1.1}$$

where, like in the rest of the course, we will use the letters α, β, γ or μ, ν for the coordinates x^μ; while letters like a, b represent the abstract notation.

The term "small", then, means that each component of $h_{\mu\nu}$ has an absolute value which is much smaller than 1. We are using the metric signature $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.

What does this approximation describe?

1. Newtonian gravity;

2. gravito-electric / magnetic effects (this will be discussed in more detail later, an example is the Lense-Thirring effect);

3. propagation of gravitational waves.

In the case of the gravitational field around the Sun, in terms of orders of magnitude we have[1]

$$\left|h_{\mu\nu}\right| \sim \frac{\phi}{c^2} \sim \frac{GM_\odot}{c^2 R_\odot} \sim 10^{-6}\,. \tag{1.2}$$

From a field-theoretic point of view:

1. η is a background metric;
2. h is the "main" field;
3. the metric does *not* backreact on the matter ($T_{\mu\nu}$).

The metric perturbation h transform like a tensor on flat spacetime under Lorentz transformations: if $\Lambda^\top \eta \Lambda = \eta$, then the coordinates change like $x = \Lambda x'$, then the full metric transforms like

$$\begin{aligned} g_{\mu'\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} && (1.3)\\ &= \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \left(\eta_{\mu\nu} + h_{\mu\nu}\right) && (1.4)\\ &= \eta_{\mu'\nu'} + \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} h_{\mu\nu}\,, && (1.5) \end{aligned}$$

therefore the transformation for h is

$$h_{\mu\nu} \to h_{\mu'\nu'} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} h_{\mu\nu}\,. \tag{1.6}$$

Mind the notation: the meaning of $h_{\mu'\nu'}$ is $h_{\mu\nu}(x')$.

Symmetry of linearized GR

Full GR is diffeomorphism invariant, while linearized GR is *infinitesimal* diffeomorphism invariant. The relevant transformations are

$$x^\mu \to x^{\mu'} = x^\mu + \xi^\mu(x^\alpha)\,, \tag{1.7}$$

where the vector field ξ is selected so that $\left|\partial_\mu \xi^\alpha\right| \sim \left|h_{\mu\nu}\right| \ll 1$.

The Jacobian of this transformation is

$$\frac{\partial x^{\mu'}}{\partial x^\mu} = \delta^{\mu'}{}_\mu + \partial_\mu \xi^{\mu'}\,, \tag{1.8}$$

while the inverse Jacobian is

$$\frac{\partial x^\mu}{\partial x^{\mu'}} + \delta^\mu{}_{\mu'} - \partial_{\mu'} \xi^\mu + \mathcal{O}\left(|\partial\xi|^2\right), \tag{1.9}$$

[1] We make the c explicit here for clarity, but we will use geometric units $c = G = 1$ for the rest of the course.

since $(\mathbb{1}+\delta)(\mathbb{1}-\delta) = \mathbb{1} + \mathcal{O}\left(\delta^2\right)$.

Under this change of coordinates, we have

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} \tag{1.10}$$

$$\sim (\delta\delta - \partial\delta - \partial\delta + \partial\partial)(\eta + h) = \delta\delta\eta + h - \partial\delta - \partial\delta + \mathcal{O}\left(\delta^2\right) \tag{1.11}$$

$$= \delta^\mu_{\mu'}\delta^\nu_{\nu'}\eta_{\mu\nu} - \partial_{\mu'}\xi^\mu\delta^\nu_{\nu'}\eta_{\mu\nu} - \partial_{\nu'}\xi^\nu\delta^\mu_{\mu'} + \delta^\mu_{\mu'}\delta^\nu_{\nu'}h_{\mu\nu} \tag{1.12}$$

$$= \eta_{\mu'\nu'} + h_{\mu'\nu'} - 2\partial_{(\mu'}\xi_{\nu')}\,, \tag{1.13}$$

therefore we have our transformation law:

$$h_{\mu'\nu'} = h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}\,. \tag{1.14}$$

This can also be written in terms of the Lie derivative as

$$h_{\mu\nu} \to h_{\mu\nu} + \mathscr{L}_\xi \eta_{\mu\nu}\,. \tag{1.15}$$

This is the analogous of a gauge transformation in electromagnetism: $A_\alpha \to A_\alpha + \partial_\alpha\chi$, where A is the vector potential.

Equations of motion

The equations of motion will come through plugging $g = \eta + h$ into the EFE $G_{ab} = 8\pi T_{ab}$ and keeping only the linear order in h.

We will need the following quantities:

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} + \mathcal{O}\left(h^2\right) \tag{1.16}$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\eta^{\mu\lambda}\left(\partial_\alpha h_{\lambda\beta} + \partial_\beta h_{\lambda\alpha} - \partial_\lambda h_{\alpha\beta}\right) + \mathcal{O}\left(h^2\right) \tag{1.17}$$

$$R_{\mu\nu} = \partial\Gamma - \partial\Gamma + \mathcal{O}\left(h^2\right), \tag{1.18}$$

where we already simplified the expressions by removing the higher-order terms. The result is

$$R_{\mu\nu} = \partial^\alpha\partial_{(\mu}h_{\nu)\alpha} - \frac{1}{2}\partial_\lambda\partial^\lambda h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h + \mathcal{O}\left(h^2\right), \tag{1.19}$$

where $h = h^\alpha_\alpha = \eta^{\alpha\beta}h_{\alpha\beta}$. Note that we are allowed to use η instead of g to lower indices. The Einstein tensor reads

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \tag{1.20}$$

$$= \partial^\alpha\partial_{(\mu}h_{\nu)} - \frac{1}{2}\partial_\lambda\partial^\lambda h_{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \frac{1}{2}\eta_{\mu\nu}\partial_\lambda\partial^\lambda h\,, \tag{1.21}$$

which can be simplified if we consider the trace-reversed metric

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,, \tag{1.22}$$

so that $\bar{h} = \eta^{\mu\nu}h_{\mu\nu} - \eta^{\mu\nu}\eta_{\mu\nu}h/2 = -h$. See equation 2.13 in the notes for a full explanation, but the idea is to insert $\bar{h}_{\mu\nu}$ and $\bar{h}$ into $G_{\alpha\beta}$ and to make some simplifications. We get

$$G_{\mu\nu} = -\frac{1}{2}\eta_{\alpha\beta}\partial^\alpha\partial^\beta\bar{h}_{\mu\nu} + \partial^\alpha\partial_{(\mu}\bar{h}_{\nu)\alpha} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta}\,, \tag{1.23}$$

which is in the form $\Box_\eta\bar{h}_{\mu\nu} + \dots\partial^\alpha\bar{h}_{\alpha\beta}$. We still have gauge freedom, so we can simplify the equation a great deal by setting $\partial^\alpha\bar{h}_{\alpha\beta} = 0$ — the Hilbert, or Lorentz gauge.

With this choice, we have

$$\boxed{\Box_\eta\bar{h}_{\mu\nu} = -\frac{16G}{c^4}T_{\mu\nu}\,,} \tag{1.24}$$

a relatively simple tensor wave equation.

Is is always possible to impose the Hilbert gauge? Yes: we can make an infinitesimal coordinate transformation to send a generic $h_{\mu\nu}$ to $h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$, so that $h \rightarrow h + 2\eta^{\alpha\beta}\partial_{(\alpha}\xi_{\beta)}$. Therefore,

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} + 2\partial_{(\mu}h_{\nu)} - \eta_{\mu\nu}\partial_\alpha\xi^\alpha\,, \tag{1.25}$$

and we can send

check indices here

$$\partial^\alpha\bar{h}_{\mu\alpha} \rightarrow \partial^\alpha\bar{h}_{\mu\alpha} + \Box\xi_\mu + \partial^\mu\partial_\nu\xi_\mu + \partial_\nu\partial^\lambda\xi_\lambda\,, \tag{1.26}$$

so if we set $\Box\xi_\mu = -\partial^\alpha\bar{h}_{\mu\alpha} = v_\mu$ we can reduce ourselves to the Hilbert gauge from any starting point. All we need to do is solve the wave equation $\Box\xi_\mu = v_\mu$.

Now, to linear order $T_{\mu\nu}$ does not depend on h. So, we can find formal solutions using Green's functions, like in electromagnetism.

The Bianchi identities are now given by $\partial_\nu G^{\mu\nu} = 0$, so $\partial_\nu T^{\mu\nu} = 0$, which gives us the EOM for matter — note that this is a partial, not a covariant derivative! This means that there is no backreaction on the metric.

The linear EFE correspond to the equations of motion of a massless spin-2 field.

Weak-field solutions

Let us consider a *static source*: suppose that $T_{\mu\nu} = \rho t_\mu t_\nu$, where $t^\mu = (\partial_t)^\mu$ is the time vector along the time direction of the global inertial coordinate system while ρ is an energy density.

If $t^\mu = (1, 0, 0, 0)$ then $T_{00} = \rho$ while $T_{0i} = 0 = T_{ij}$.

In this case, then, the stress-energy tensor is time-independent: therefore also on the other side we will have $\partial_t\bar{h}_{\mu\nu} = 0$.

Therefore, the left-hand side of the equation will read $\nabla^2\bar{h}_{\mu\nu} = -16\pi\rho$ for $\mu = \nu = 0$ and $\nabla^2 h_{\mu\nu} = 0$ for all the other components.

These Poisson equations can be solved as boundary-value problems if we assume that $h_{\mu\nu} \to 0$ for $r \gg R$.

This looks very similar to the Newton equation $\nabla^2 \phi = 4\pi\rho$; therefore $\bar{h}_{00} = -4\phi$, while $\bar{h}_{\mu\nu} = 0$ for all other components.

We can reconstruct the metric using the fact that $\bar{h} = 4\phi$, so

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} = -4\phi t_\mu t_\nu - \frac{1}{2}\eta_{\mu\nu} 4\phi \,, \tag{1.27}$$

so the metric reads

$$g_{\mu\nu} = \eta_{\mu\nu}(1 - 2\phi) - 4\phi t_\mu t_\nu \,, \tag{1.28}$$

therefore

$$g = -(1 + 2\phi)\, \mathrm{d}t^2 + (1 - 2\phi)\delta_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j \,. \tag{1.29}$$

We know that far away from the source, the Newtonian field decays like $\phi \approx -M/r + \mathcal{O}\left(r^{-2}\right)$.

Therefore, this metric approximation already includes special relativity as well: we have $g \to \eta$ for large r, but also $g = \eta$ for $M = 0$.

The geodesic equation for this weak field metric reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\partial^i \phi \,. \tag{1.30}$$

However, these Newtonian equations of motion are *not* consistent with $\partial_\mu T^{\mu\nu} = 0$. These describe the motion of the source which generates gravity, whereas the Newtonian EOM describe the motion of test particles in the weak-field metric.

The dual meaning of the full EFE — matter deforms the spacetime, the spacetime shapes the trajectories of matter — *cannot* be realized at linear order.

No-stress source

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We considered a source in the form $T_{\mu\nu} = \rho t_\mu t_\nu$, where $t^\mu = (1, \vec{0})$.

Now we will consider a source in the form

$$T_{\mu\nu} = -2\rho t_\mu t_\nu + 2J_{(\mu} t_{\nu)} \,, \tag{1.31}$$

where $J^\mu = \rho u^\mu = \rho(\gamma, \gamma v^i / c)$.

Probably the first 2 is not there.

The static source from before can be recovered from this expression in the low-velocity limit $v^i/c \to 0$. In that case, $T_{ij} = 0$: we can see that T_{ij} is of order v^2/c^2, so to first order they vanish.

In this situation, we get the system

$$\begin{cases} \Box \bar{h}_{0\mu} &= -16\pi T_{0\mu} \\ \Box \bar{h}_{ij} &= 0 \,. \end{cases} \tag{1.32}$$

In order to simplify, let us assume that $\partial_t \overline{h}_{ij} = 0$: then, the solution to the second of these becomes $\nabla^2 \overline{h}_{ij} = 0$, with flat boundary conditions at large distance. By linearity, this leads to $\overline{h}_{ij} = 0$.

Claim 1.1. *If we define $A_\mu = -(1/4)\overline{h}_{0\mu} = -(1/4)\overline{h}_{\mu\nu} t^\nu$, then the metric becomes*

$$g_{00} = -1 + 2A_0 \tag{1.33}$$

$$g_{0i} = 4A_i \tag{1.34}$$

$$g_{ij} = (1 + 2A_0)\delta_{ij}\,. \tag{1.35}$$

In terms of this A_μ, the Dalambertian equation from before reads

$$\Box A_\mu = -\frac{16}{4}\pi J_\mu = -4\pi J_\mu\,, \tag{1.36}$$

which are formally identical to the Maxwell equations! Therefore, we can employ known techniques from electromagnetism.

For example, if $\partial_t A_\mu = 0$ then

$$\begin{cases} A_0 & = -\phi \\ A_i & = \int \mathrm{d}^3 x^i \, \frac{J_i}{|x - x^i|}\,, \end{cases} \tag{1.37}$$

which is the reason why the phenomena which can be described through this formalism are known as gravito-electric and gravito-magnetic effects.

Claim 1.2. *For example, geodesics in a weak-field stationary (no stress) spacetime are described by a Lagrangian*

$$\mathscr{L} = -mc\left(-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right)^{1/2} \tag{1.38}$$

$$= -mc^2\left(-g_{00} - 2g_{0i}\frac{v^i}{c} - g_{ij}\frac{v^i v^j}{c^2}\right)^{1/2} \tag{1.39}$$

$$\approx -mc^2 + \frac{m}{2}v^2 + m\phi + 4mcA_i v^i\,. \tag{1.40}$$

We have a mass term, a kinetic term, a gravitational term, and a contribution to the Lorentz force.

The corresponding equations of motion read

$$\ddot{\vec{x}} = \vec{E} + 4\vec{v} \times \vec{B}\,, \tag{1.41}$$

where $\vec{E}$ and $\vec{B}$ are the gravitoelectric and gravitomagnetic fields derived from our A_μ. The differences from EM are: the absence of charge, and the factor of 4 before the magnetic term.

An example of a gravito-electromagnetic effect is the Lense-Thirring effect: a magnetic moment $\vec{s}$ in a magnetic field precesses, according to

$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = \vec{s} \times \vec{\Omega} \qquad \text{where} \qquad \vec{\Omega} = -\frac{q}{m}\vec{B}_{EM}\,, \tag{1.42}$$

so in order to generalize to the precession of a gyroscope in an EM field we need to map $q \to m$ and $\vec{B}_{EM} \to 4\vec{B}$.

This way, we see for example that $\Omega_g = -4B$. A mission called Gravity Probe B measured this effect: they found precession with $\Omega_g \sim 0.22\,\text{arcsec}/\text{yr}(R_\oplus/r)^3$. This is a 20 % accurate test of GR in the weak field.

What does that mean?

Another example is **frame dragging**, which applies in full GR: if we put the gyroscope around a BH a similar effect emerges. Around a Kerr BH we have

$$g_{0i}^{\text{Kerr}} \sim \Omega_{BH}\,, \tag{1.43}$$

and if the particle is close to the BH a particle is "locked" to the BH rotation.

2 Gravitational Waves in linear GR

GW are solutions of weak-field GR in a vacuum. There, the wave equation reads $0 = \Box_\eta \bar{h}_{\mu\nu}$. What are the properties of the solutions of these equations? The simplest thing we can do is look for plane wave solutions. We take a wave vector $k^\mu = (\omega, k^i)$ and an amplitude $A_{\mu\nu}$; then

$$\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\mu x^\mu} = A_{\mu\nu} e^{i(-\omega t + \vec{k}\cdot\vec{x})}\,, \tag{2.1}$$

so $\partial_\mu \bar{h}_{\alpha\beta} = (ik_\mu)\bar{h}_{\alpha\beta}$.

Substituting the plane wave ansatz yields

$$0 = \Box \bar{h}_{\alpha\beta} = -\eta^{\mu\nu} k_\mu k_\nu \bar{h}_{\alpha\beta}\,, \tag{2.2}$$

therefore $k_\mu k^\mu = 0$. The wavevector is null.

This implies that the GW propagates at the speed of light: $\omega s^2 = \left|\vec{k}\right|^2$.

How do we completely specify a gauge? Any infinitesimal transformation such that $\Box \xi^\mu = 0$ preserves the Hilbert gauge, so we can make a residual gauge transformation.

The harmonic gauge implies that

$$0 = -\partial^\alpha \bar{h}_{\mu\alpha} = ik^\alpha \bar{h}_{\mu\alpha}\,, \tag{2.3}$$

which yields $k^\alpha A_{\alpha\mu} = 0$. This means that GWs are **transverse** to the propagation direction.

We know that $\bar{h}_{\mu\nu}$ maps to $\bar{h}_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} + \eta_{\mu\nu}\partial_\alpha \xi^\alpha$.

Let us use $\xi^\mu = B^\mu e^{ik_\alpha x^\alpha}$ as an ansatz for our residual gauge transformation, since it automatically harmonic: we get

$$A_{\mu\nu} \to A_{\mu\nu} - 2ik_{(\mu}B_{\nu)} + i\eta_{\mu\nu}k_\alpha B^\alpha\,, \tag{2.4}$$

and since we can pick B^μ arbitrarily we can impose $\bar{h} = A^\mu_\mu = 0$, the **traceless condition**, as well as $\bar{h}_{\mu 0} = 0$, the **transverse condition**. The second is suggested by the previously found result $k_\alpha A^{\alpha\beta} = 0$.

In terms of B, this is a linear algebraic system, and it is invertible.

In summary, we start from 10 variables, we use 4 equations to impose the Hilbert gauge, and 4 more to impose the TT gauge. The two degrees of freedom which are left are the true degrees of freedom of a GW.

More explicitly, if we have $k^\mu = (\omega, 0, 0, k_z)$ this means

1. $k^2 = 0$ implies $-\omega = k_z$;
2. the phase reads $k_\alpha x^\alpha = \omega(t - z)$;
3. the Hilbert gauge $k^\mu A_{\mu\nu} = 0$ tells us that $A_{0\nu} = A_{3\nu}$;
4. the transverse condition tells us that $A_{0\mu} = 0$ (so also $A_{3\mu} = 0$);
5. the traceless condition tells us that $A_\mu^\mu = 0$.

This leads to the usual formulation

$$A_{\mu\nu}^{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{2.5}$$

Therefore,

$$h_{\mu\nu}^{TT} = A_{\mu\nu}^{TT} \exp\big(i\omega(t - z)\big) . \tag{2.6}$$

In TT gauge we have $\bar{h}_{\mu\nu} = h_{\mu\nu}$ since the trace is zero. Importantly, the TT gauge can only be defined in vacuo! This is because in that case $\Box \bar{h}_{\mu\nu} \neq 0$, so while we can still exploit gauge freedom we cannot set components to zero inside the source.

The metric in TT gauge reads

$$g = -\,\mathrm{d}t^2 + \mathrm{d}z^2 + (1 + h_+)\,\mathrm{d}x^2\,(1 - h_\times)\,\mathrm{d}y^2 + 2h_\times\,\mathrm{d}x\,\mathrm{d}y\, g \quad = -\,\mathrm{d}t^2 + (\delta_{ij} + h_{ij}^{TT})\,\mathrm{d}x^i\,\mathrm{d}x^j . \tag{2.7}$$

How do we identify the GW degrees of freedom in general? We can impose the TT gauge outside the source (far away from the $T_{\mu\nu}$).

In general,

$$h_{\mu\nu}^{TT} = \Lambda_{\mu\nu}{}^{\alpha\beta} \bar{h}_{\alpha\beta} , \tag{2.8}$$

where Λ is a projection operator, defined as

$$\Lambda_{\mu\nu}{}^{\alpha\beta} = P_\mu^\alpha P_\nu^\beta - \frac{1}{2} P_{\mu\nu} P^{\alpha\beta} \tag{2.9}$$

$$P_{\mu\nu} = \delta_{\mu\nu} - n_\mu n_\nu , \tag{2.10}$$

where n^μ is the propagation direction.

The projection tensor $P_{\mu\nu}$ is symmetric, it is transverse ($P_{\mu\nu}n^{\nu} = 0$), it is idempotent ($P_{\mu\alpha}P_{\alpha\nu} = P_{\mu\nu}$), and its trace is equal to 2.

The tensor $\Lambda_{\mu\nu\alpha\beta}$ is also idempotent, transverse in all indices, traceless in $\mu\nu$ and $\alpha\beta$ separately, and symmetric in the swap of $\mu\nu$ and $\alpha\beta$.

In summary, we have found GW solutions, they propagate with c, they are transverse, they have two degrees of freedom.

Symmetric, Transverse, Trace-Free tensors play an important role in GW theory. They can be used to obtain the **Multipolar expansion**.

"Living review of relativity" (see webpage) describes all the tests of GR.

3 Effects of GW on test masses

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Consider two masses, separated by a distance L. The distance between them can be measured as $L = (c/2)\Delta t_{PP''}$, where PP'' is the trajectory of a light beam moving from P to Q and back to P.

In flat spacetime, the separation vector between them can then be written as $h^i = L_0 n^i$ for some unit vector n^i.

In the presence of a GW, the length will read

$$\begin{aligned} L^2 &= g_{\mu\nu}\left(x^{\mu}_{q'} - x^{\mu}_{p'}\right)\left(x^{\nu}_{q'} - x^{\nu}_{p'}\right) && (3.1)\\ &\to g_{\mu\nu}\left(x^{i}_{q'} - x^{i}_{p'}\right)\left(x^{j}_{q'} - x^{j}_{p'}\right) && (3.2)\\ &\to g_{\mu\nu}x^{i}_{q'}x^{j}_{q'} && (3.3)\\ &\to \left(\delta_{ij} + h^{TT}_{ij}\right)x^{i}_{q'}x^{j}_{q'} = L_0^2\left(\delta_{ij} + h^{TT}_{ij}\right)n^i n^j\,, && (3.4) \end{aligned}$$

and so we can compute

$$\frac{\delta L}{L_0} = \frac{L}{L_0} - 1 = \sqrt{1 + h^{TT}_{ij}n^i n^j} - 1 \approx \frac{1}{2}h^{TT}_{ij}n^i n^j\,. \tag{3.5}$$

This justifies the heuristic formula $\delta L/L_0 \sim h$.

A more formal treatment can be given through the **geodesic deviation** formula: we can show that if u^{μ} is the tangent vector of a family of geodesics and s^{μ} is the displacement between geodesics, then

$$u^{\mu}\nabla_{\mu}(u^{\nu}\nabla_{\nu}s^{\alpha}) = R^{\alpha}_{\lambda\rho\sigma}u^{\lambda}u^{\rho}s^{\sigma}\,, \tag{3.6}$$

and in the weak field limit the Riemann tensor is in the form $\partial^2 h$.

If we plug in everything (as we did in the exercise) we get

$$\frac{\mathrm{d}^2 s_{\alpha}}{\mathrm{d}t^2} = R_{\alpha 00\mu}s^{\mu} \qquad \text{with} \qquad R_{\alpha 00\mu} = \frac{1}{2}\ddot{h}^{TT}_{\alpha\mu}\,. \tag{3.7}$$

The temporal evolution of the spatial vector then reads

$$\ddot{s}^i(t) = \frac{1}{2}\ddot{h}^{TT}_{ij}(t)s^j_0 + \mathcal{O}\left(h^2\right), \tag{3.8}$$

so if initially $\dot{s}^i(t=0)$ and $s^i(t=0) = s_0^i$ we get

$$s^i(t) = s_0^i + \frac{1}{2} h_{ij}^{TT}(t) s_0^j = \left(\delta_{ij} + \frac{1}{2} h_{ij}^{TT}(t)\right) s_0^j\,. \tag{3.9}$$

A ring of particles in the xy plane is deformed by a wave travelling along the z axis: it becomes an ellipse with axes along the x and y direction for the h_+ polarization,

$$\frac{\delta x^2}{r_0^2(1+h_+)^2} + \frac{\delta y^2}{r_0^2(1-h_+)^2} = 1\,. \tag{3.10}$$

The effect of the cross polarization is similar but rotated by 45°.

4 Sources of GW

We start from a formal solution of $\Box \overline{h}_{\mu\nu} = -16\pi T_{\mu\nu}$: like in electromagnetism, we use Green functions,

$$\overline{h}_{\mu\nu}(t, \vec{x}) = -16\pi \int G_R(x^\alpha - x^{\alpha\prime}) T_{\mu\nu}(x^{\alpha\prime})\, \mathrm{d}^4 x'\ , \tag{4.1}$$

where $\Box G_R(x) = \delta^{(4)}(x)$; explicitly

$$G_R(x) = -\frac{1}{4\pi} \frac{\delta(u-t)}{|\vec{x}|}\,. \tag{4.2}$$

where u is the retarded time. (to be defined…check)

With this, we get

$$\overline{h}_{\mu\nu}(t, \vec{x}) = 4 \int \frac{T_{\mu\nu}(u, \vec{x})}{|\vec{x} - \vec{x}'|}\, \mathrm{d}^3 x'\ . \tag{4.3}$$

The assumptions we can make are the following: a mean-field approach, negligible self-gravity (this means that the quantity $2GM/c^2R = R_S/R \ll 1$).

Also, in order to derive the quadrupole formula, we assume that the distance from us to the source is large compared to the scale of the source and that the velocity of the source is slow compared to c.

The result we will find is that we can compute

$$\overline{h}_{ij}(t, \vec{x}) = \frac{4}{r} \int \mathrm{d}^3 x'\, T_{ij}\left(t - \frac{r}{c} + \frac{\hat{n} \cdot \vec{x}}{r}, \vec{x}'\right), \tag{4.4}$$

and we can further simplify the integrand by expanding in x'/r.