Early Universe Cosmology

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Contents

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Introduction

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Live lectures will be in room P1A, usually at the blackboard, sometimes with slides. There will be notes uploaded to Moodle.

Course content An up-to-date overview of the physics of the Early Universe. The goal is to be able to understand and analyze the problems from both a theoretical and observational point of view.

There are three main parts:

1. inflationary models: the issue of the inital conditions;
2. cosmological perturbations, GW of cosmological origin;
3. baryogenesis, production of DM particles.

All of these will be connected with observations: nowadays cosmology is data-driven.

There are connections with: cosmology, astroparticle physics, astrophysics, GR, theoretical physics, field theory, multimessenger astrophysics, GW.

We will do "blended learning", with shifts in the classroom. We will do 48 hours of lectures.

Textbooks: Lyth and Liddle [LL09] "The primordial Density Perturbation", [LL00] "Cosmological inflation and Large-Scale Structure".

There will be 2 exam dates for each session. The exam is an oral one. Office hours can be arranged anytime.

We should limit questions in the break, prefer asking them during the lecture itself.

Chapter 1

Inflationary models

1.1 A general introduction

The basic issue is to find what initial conditions would produce the universe as we currently observe it.

Observational probes of the Hot Big Bang model include the Hubble diagram (redshift against distance for galaxies), Big Bang nucleosynthesis, the CMB.

On large scales we observe a **smooth universe**. However, that is a "zeroth-order" approximation: there are structures and anisotropies. All the structures need initial conditions to start from and then grow through gravitational instability.

We have several observables to probe the anisotropies: CMB, LSS, clusters of galaxies, weak gravitational lensing. There are **initial fluctuations** on the order of

$$\frac{\delta\rho}{\rho} \sim \frac{\delta T}{T} \sim 10^{-5} \,. \tag{1.1.1}$$

What is the initial time and temperature at which these perturbations start? Is there a **dynamical** mechanism which produces the perturbations? How do the perturbations evolve exactly? How do they relate to baryogenesis?

Under a Newtonian treatment, relative density perturbations grow like $\delta_m \propto a(t)$. The problem we will address here is how the initial value of δ_m comes about.

The CMB is a very good blackbody, without spectral distortions except for the Sunyaev-Zel'dovich effect (inverse Compton scattering from high-energy electrons in galaxies up-scattering the CMB photons).

We recall some basic concepts about the smooth model of the universe: critical density, Hubble parameter and so on.

The standard ΛCDM model does predict a small deviation, $\mu/T \sim 1.9 \times 10^{-8}$, from the Planckian, whose phase space distribution is:

$$f = \left[\exp\left(\frac{h\nu - \mu}{k_B T}\right) - 1\right]^{-1} . \tag{1.1.2}$$

The number density of photons can be extracted from this distribution:

$$n_\gamma = \int f \, \mathrm{d}^3 p = \frac{4\pi}{c} \int \frac{I_\nu}{h\nu} \, \mathrm{d}\nu \approx 422 \, \mathrm{cm}^{-3} \,. \tag{1.1.3}$$

Currently we have upper bounds: $\mu / T < 9 \times 10^{-5}$ at 95 % CL.

The CMB radiation is also highly, but not perfectly, isotropic. the scale of the temperature angular anisotropies are of the order $\Delta T / T \sim 10^{-5}$ (the quoted value for $\Delta T / T$ is a root-mean-square, since the average of ΔT is zero). This is to say: in each direction we observe a very good blackbody, whose characteristic temperature changes slightly depending on the direction.

Planck 2018 had an angular resolution of 5 arcminutes, and it also measured the polarization of the CMB.

We also have redshift galaxy surveys like the Sloan Digital Sky Survey. We map galaxies in redshift space. There is a statistical pattern of the galaxies, which is connected to the origin of the inhomogeneities.

The idea is that the seeds of the perturbations are quantum mechanical, coming from the inflaton scalar field, which are made into galaxies and galaxy clusters from gravitational instabilities.

At $z \sim 20$ the DM distribution was quite smooth, it then clustered.[1]

The components of the ΛCDM model are:

1. dark energy 68 %;
2. dark matter 26 %;
3. hydrogen and helium gas 4 %;
4. stars 0.5 %;
5. neutrinos 0.26 %;
6. metals 0.025 %;
7. radiation 0.005 %.

These numbers are expressed as fractions of the critical energy density,

$$\rho_{0,\,\mathrm{crit}} = \frac{3 H_0^2}{8 \pi G} \,, \tag{1.1.4}$$

and the fact that they approximately add up to 1 means that the total energy budget of the universe is compatible with spatial flatness.

We also need seed perturbations and baryo-leptogenesis. We will see phases in which the universe is not in thermal equilibrium.

We want to find information about energies up to 10^{16} GeV: we will see that the inflationary phase corresponds to this epoch.

[1] https://www.youtube.com/watch?v=FBkYIqtYb0I.

GW from inflation travel basically unimpeded from inflation to us.

Today, we have **radiation** with $w = 1/3$, $\rho \propto a^{-4} \propto T^4$, so, Tolman's law $Ta = \text{const}$.

Baryonic matter has $\Omega_b h^2 = 0.0224 \pm 0.0001$. Its equation of state is $P = nT \ll nm$. So, $\rho \propto a^{-3}$. Dark matter is also nonrelativistic, with $P \approx 0$, and $\Omega_{DM} \approx h^2 0.120 \pm 0.001$.

The **cosmological constant** has $P = -\rho$, and $\Omega_\Lambda = 0.6847 \pm 0.0073$.

Neutrinos have $\sum m_\nu < 0.12\,\text{eV}$, and $\Omega_\nu h^2 < 0.0012$. Both values are at 95 % CL.

Spatial curvature has $\Omega_k = 1 - \Omega_0 = 0.001 \pm 0.002$, from Planck, Baryon Acoustic Oscillations, local measurements.

The presence of discordance can surely signal systematics, but also new physics. There are certain discordances.

Beyond galactic rotation curves, we also have evidence for DM from the power spectrum of inhomogeneities. We Fourier-transform the density perturbation field δ to get $\delta_{\vec{k}}$; then we can calculate the power spectrum

$$\Delta^2(k) = \frac{\partial \sigma^2}{\partial \log k} \propto k^3 \left| \delta_{\vec{k}} \right|^2 \propto k^{3+n} T^2(k) \,. \tag{1.1.5}$$

The Poisson equation reads

$$4\pi G \bar{\rho} \delta = \nabla^2 \Phi \implies \delta_{\vec{k}} \propto k^2 \Phi_{\vec{k}} \,. \tag{1.1.6}$$

If the gravitational perturbation is written as

$$\Phi_{\vec{k}} = \Phi_{\vec{k}}^{\text{primordial}} T(k) \times \text{growth function} \,, \tag{1.1.7}$$

and the primordial field perturbation squared is $\left| \Phi_k^{\text{primordial}} \right|^2 \propto k^{n-4}$, where the power spectral index reads $n = 0.9600 \pm 0.0042$. The growth function is commonly denoted as $g(a)$. This explains the last proportionality sign we wrote earlier, the density power spectrum includes information about the power-spectral index of the initial conditions. This index measures the amplitude of the inhomogeneities in DM density.

Also, if we only had baryons **without dark matter** the power spectrum of the matter density perturbations would look **very different** from what we see.

Baryon Acoustic Oscillations are an oscillatory imprint in the power spectrum, they have been measured today.

Inflation is an early epoch in the history of the universe during which expansion is accelerated. The basic predictions of inflation are so far confirmed, however we have not detected the SGWB from it, which would be a "smoking gun".

Monday 2020-10-5, compiled 2021-01-25

These next few lectures, we will consider the motivations for inflationary models. The problems they solved were the **shortcomings of the Hot Big Bang** model.

1. The horizon problem;

2. the flatness problem;

3. unwanted relics / magnetic monopole problem.

We start by recalling some basic elements in cosmology. In order to describe a homogeneous and isotropic universe we use the FLRW metric:

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right], \tag{1.1.8}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$. The quantity $a(t)$ is called the *scale factor*. The coordinates r, θ and φ are called *comoving coordinates*.

Physical distances and comoving distances are related by

$$\lambda_{\mathrm{phys}} = a(t)\lambda_{\mathrm{comoving}}\,. \tag{1.1.9}$$

The constant k is the spatial curvature of the universe, which can always be rescaled so that it is equal to

1. $+1$ for a spatially closed universe;
2. 0 for a spatially flat universe;
3. -1 for a spatially open universe.

In terms of the scale factor we define the **Hubble parameter**

$$H = \frac{\dot{a}}{a}\,, \tag{1.1.10}$$

which describes the rate at which the universe expands.

The dynamics of gravity are described by the Einstein equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{1.1.11}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the particle species filling the universe, while $G_{\mu\nu} = R_{\mu\nu} - Rg_{\mu\nu}/2$ is the Einstein tensor, describing curvature.

These can be derived from an action principle through the action

$$S = \underbrace{\frac{1}{16\pi G}\int R\sqrt{-g}\,\mathrm{d}^4x}_{S_{EH}} + S_{\mathrm{matter}}\,. \tag{1.1.12}$$

Often we use an ideal fluid energy-momentum tensor:

$$T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu}\,, \tag{1.1.13}$$

where $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$ is a projector onto the space orthogonal to the four-velocity. This does not account for any anisotropy, it is the most symmetric energy-momentum tensor.

In order to solve the Einstein equations we can proceed with some assumptions, without needing to know the action for all the fundamental fields. The perfect fluid S-E tensor has all the FLRW symmetries, as long as ρ and P are only functions of time.

Requiring the FLRW symmetries means that the S-E tensor must be diagonal, however we can have viscosity as long as it is not *shear* but *bulk* viscosity, which adds onto the diagonal terms.

Inserting the FLRW metric into the Einstein equation yields the Friedmann equations:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.1.14}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right) \tag{1.1.15}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + P\right). \tag{1.1.16}$$

The first two can be derived from the Einstein equations directly, the third comes from the "conservation law" ${T_{\mu\nu}}^{;\nu} = 0$.

They are not independent, only two are. We have too many parameters: a, ρ and P, but only two independent equations, so we "close" the system of equations with an equation of state, commonly $P = P(\rho) = w\rho$.

These equations of state describe many kinds of fluids (approximately): dust with $w = 0$, which means $\rho \propto a^{-3}$; radiation with $w = 1/3$, which means $\rho \propto a^{-4}$; a cosmological constant with $w = -1$, which means $\rho = \text{const}$.

In general as long as $w \neq 1$ we have $\rho \propto a^{-3(1+w)}$ and $a \propto t^{2/3(1+w)}$.

1.1.1 The horizon problem

The **particle horizon**, denoted as $d_H(t)$, is given by

$$d_H(t) = a(t)\int_0^t \frac{c\,\mathrm{d}\tau}{a(\tau)}, \tag{1.1.17}$$

and it sets the radius of a sphere centered at an observer O. The points inside this sphere have been able to have causal interactions with observer O in the time from the Big Bang to t.

It is the proper distance (as measured today) which could have been travelled by light starting at the beginning and moving in a geodesic. It can be derived from the FLRW metric by assuming radial light-like motion

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a(t)\frac{\mathrm{d}r^2}{1-kr^2} = 0, \tag{1.1.18}$$

and setting $k = 0$:[2]

$$c\,\mathrm{d}t = \pm a(t)\,\mathrm{d}r\,, \tag{1.1.19}$$

which we can use to calculate the *comoving distance* from the point of emission to today, which we then multiply by the scale factor calculated at a chosen point.

[2] This is a good approximation for early times, even if the universe is not flat.

We know that the scale factor goes to 0 as t goes to 0, so the integral giving us r_H could diverge. We can show that d_H is finite as long as $\alpha = 2/3(1+w)$ is smaller than one, meaning that $w > -1/3$, which is equivalent to $\ddot{a} < 0$. In a decelerating universe, the particle horizon is finite.

In general, the calculation yields

$$d_H(t) = \frac{3(1+w)}{1+3w} ct \,. \tag{1.1.20}$$

With $w = 0$, a spatially flat matter-dominated universe, $d_H = 3ct$. This is called an Einstein-De Sitter universe. With $w = 1/3$, a spatially flat radiation-dominated universe, we have $d_H = 2ct$.

Another way to characterize causality is the Hubble radius:

$$r_C(t) = \frac{c}{H(t)} \,. \tag{1.1.21}$$

The characteristic time of expansion is $\tau(H) = H^{-1}$.

Claim 1.1.1. *In a FLRW universe, typically after a Hubble time the scale factor doubles.*

Proof. By solving the Friedmann equations we see that the scaling is as $a(t) \propto t^\alpha$, $H = \alpha/t$, so a Hubble time is just $\tau_H = H^{-1} = t/\alpha$. Then, the change in the scale factor in a Hubble time is

$$\frac{a(t+\tau_H)}{a(t)} = \frac{(t+t/\alpha)^\alpha}{t^\alpha} = \left(1 + \frac{1}{\alpha}\right)^\alpha \,. \tag{1.1.22}$$

We know that α is between 0 and 1: the value of this ratio of scale factors is 1 for $\alpha = 0$ (which would correspond to $w \to \infty$), and goes monotonically up to 2 for $\alpha = 1$ (which corresponds to $w = -1/3$). With $w = 1/3$, for example, we get $\alpha = 1/2$ and the ratio of the scale factors is ≈ 1.7, which is reasonably close to 2. $\square$

Since

$$H(t) = \frac{2}{3(1+w)} \frac{1}{t} = \frac{\alpha}{t} \,, \tag{1.1.23}$$

we can define

$$R_H = \frac{1+3w}{2} d_C(t) \approx d_H(t) \,. \tag{1.1.24}$$

Typically the quantity $(1+3w)/2$ is of order 1, so the Hubble radius r_C and the particle horizon r_H are similar. The two are similar in a regular FLRW universe, while they differ a lot if there is inflation.

Claim 1.1.2. *In an inflationary period, the particle horizon is exponentially larger than the Hubble radius.*

Proof. Let us assume that the Hubble parameter H is constant for simplicity. In a real scenario there will be deviations from this hypothesis, but they will not compromise the result.

Then, the Hubble radius is simply given by $r_H = H^{-1}$; while the particle horizon must be calculated by integrating the conformal time. We need a relation between the scale factor and time: we can integrate $\dot{a} = Ha$ to get $a(t) = e^{Ht}$. Then, we find

$$r_C(t) = a(t) \int_0^t \frac{\mathrm{d}\tau}{a(\tau)} = e^{Ht} \int_0^t e^{-H\tau}\, \mathrm{d}\tau = \underbrace{H^{-1}}_{r_H} \left(e^{Ht} - 1 \right), \tag{1.1.25}$$

which becomes exponentially larger than r_H for $t \gg H^{-1}$. $\square$

The particle horizon takes into account **all the past history of an observer**, the Hubble radius does not care about it: it only describes causal connections taking place in a Hubble time.

Let us introduce the *comoving Hubble radius*: $r_H(t)$, given by

$$r_H(t) = \frac{r_C(t)}{a(t)} = \frac{c}{aH}. \tag{1.1.26}$$

Let us plot this for a matter or radiation-dominated FLRW universe.

In radiation domination, $r_H \propto \sqrt{t}$, while in matter domination $a \sim t^{2/3}$ so $r_H \propto t^{1/3} \sim a^{1/2}$.

This comoving radius is then always increasing, initially faster and then slower.

Instead, consider the comoving particle horizon: $d_H(t)/a(t)$, so just the integral in the definition of $d_H(t)$:

$$\frac{d_H(t)}{a(t)} = \int \frac{c\, \mathrm{d}t}{a} = \int \frac{\mathrm{d}a}{a} \underbrace{\frac{c}{aH}}_{r_H}, \tag{1.1.27}$$

so we can see that the comoving *particle horizon* is the logarithmic integral over the scale factor of the comoving *Hubble radius*: as we mentioned before, this takes into account the whole past history.

In a matter dominated universe, $d_H = 2r_C \approx 5h^{-1}$Gpc.

1.1.2 Horizon problem

Now we discuss the horizon problem, which is best understood in a comoving plot: $\log r_H$ versus $\log t$. We neglect dark energy for simplicity.

If we choose a fixed comoving size λ, we get in our model that in early times λ is super-horizon, then at a certain point it crosses the horizon, becoming smaller than r_H. The time at which $r_H = \lambda$ is called the *horizon crossing* time, $t_H(\lambda)$.

For times earlier than $t_H(\lambda)$, by definition it is impossible for points at a distance λ to be causally connected. This happens for every scale, and it means that for many regions we are interested in there cannot have been causal connection in the early universe. But, today we observe the universe to exhibit the same properties across the whole sky, even though the regions were causally disconnected earlier.

This is most directly expressed in terms of CMB photons. They would have become causally connected at the quadrupole scale (separations of 90°) almost *today*.

We can compute the size of the horizon at the last scattering epoch: this subtends an angle in the sky of around 1°; however we observe photons with the same temperature on much larger scales, this was already seen by COBE with an angular resolution of 7°.

Claim 1.1.3. *Without inflation, the largest angular scale at which we would expect to see correlations on the Last Scattering Surface is on the order of* 1°*.*

Proof. Let us establish some notation: we define

$$E(z) = \frac{H(z)}{H_0} = \sqrt{\Omega_{0,\Lambda} + \Omega_{0,k}(1+z)^2 + \Omega_{0,m}(1+z)^3 + \Omega_{0,r}(1+z)^4}\,, \tag{1.1.28}$$

(the last expression follows from the first Friedmann equation), which allows us to compute distances, since the physical distance as measured today from a point at redshift z would be

$$d(z) = \frac{1}{H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')}\,, \tag{1.1.29}$$

since $d(z)$ is the integral over the appropriate range of $a_0\,\mathrm{d}\eta$ (this is the same integral as one in the definition of the particle horizon), and the integration element of conformal time can be written as

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a} = -\frac{\mathrm{d}z}{a_0 H(z)}\,, \tag{1.1.30}$$

as can be calculated by differentiating the definition of redshift:

$$1+z = \frac{a_0}{a} \implies \frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{a_0\dot{a}}{a^2} = -\frac{a_0 H(z)}{a} \implies \frac{\mathrm{d}z}{H(z)a_0} = -\frac{\mathrm{d}t}{a}\,. \tag{1.1.31}$$

The minus sign will be accounted for by the fact that we are integrating backward in conformal time if we do it for increasing z. Now, the physical distance from here to the last scattering surface as measured now reads

$$d(z_{LS}) = \int_{\text{emission}}^{\text{absorption}} a_0\,\mathrm{d}\eta = -a_0\int_{z_{LS}}^{0}\frac{\mathrm{d}z'}{a_0 H(z')} = \frac{1}{H_0}\int_0^{z_{LS}}\frac{\mathrm{d}z'}{E(z')}\,. \tag{1.1.32}$$

We, however, want a new notion of distance d_A which allows us to write the small-angle approximation:

$$d_A(z_{LS}) = \frac{\Delta x}{\Delta\theta}\,, \tag{1.1.33}$$

where $\Delta\theta$ is a (small) angle on the sky as measured from here, while Δx is a physical distance as measured at the time of last scattering. The small-angle relation holds in comoving coordinates:

$$r_{LS} = \frac{\Delta r}{\Delta\theta}\,, \tag{1.1.34}$$

where $r_{LS} = d(z_{LS})/a_0$ is the comoving distance to the last scattering surface, and Δr is the comoving size of the angular feature we are considering. The physical scale Δx is calculated as $a_{LS}\Delta r$, and $a_{LS} = a_0/(1+z_{LS})$, so the new distance measure must read

$$\underbrace{a_0 r_{LS}}_{d(z_{LS})} = a_0 \frac{\Delta r}{\Delta\theta} = (1+z_{LS})\frac{\Delta x}{\Delta\theta} \tag{1.1.35}$$

$$\frac{\Delta x}{\Delta\theta} = d_A(z_{LS}) = \frac{d(z_{LS})}{1+z_{LS}} = \frac{1}{H_0(1+z_{LS})}\int_0^{z_{LS}} \frac{\mathrm{d}z'}{E(z')}\,. \tag{1.1.36}$$

This new distance is called the *angular diameter distance*. Now, we need to find the appropriate Δx: what is the physical size of the largest correlations we can expect to see? This will just be the particle horizon (corresponding to light-speed communication), measured at that time:

$$\Delta x = a_{LS}\int \mathrm{d}\eta = \frac{a_{LS}}{a_0 H_0}\int_{z_{LS}}^{\infty}\frac{\mathrm{d}z'}{E(z')} = \frac{H_0^{-1}}{1+z_{LS}}\int_{z_{LS}}^{\infty}\frac{\mathrm{d}z'}{E(z')}\,. \tag{1.1.37}$$

Now we are almost done: we can invert the small-angle relation to get

$$\Delta\theta = \frac{\Delta x}{d_A(z_{LS})} = \frac{H_0^{-1}(1+z_{LS})^{-1}\int_{z_{LS}}^{\infty} E^{-1}(z')\,\mathrm{d}z'}{H_0^{-1}(1+z_{LS})^{-1}\int_0^{z_{LS}} E^{-1}(z')\,\mathrm{d}z'} \tag{1.1.38}$$

$$= \frac{\int_{z_{LS}}^{\infty} E^{-1}(z')\,\mathrm{d}z'}{\int_0^{z_{LS}} E^{-1}(z')\,\mathrm{d}z'} \approx 0.02 \approx 1.2^\circ\,. \tag{1.1.39}$$

The result was obtained by numerical integration of the best-fit values from the Planck mission, as provided by the "Planck18" class from the Astropy module[3] [Ast+18]. However, the same qualitative result can be obtained analytically with much simpler assumptions: taking full matter domination ($\Omega_{m,0} = 1$ and the others equal to zero) still yields $\Delta\theta \approx 1.8^\circ$, a very reasonable number, and the calculation can then be done analytically, since $E(z) = (1+z)^{3/2}$ is integrable with the usual techniques. □

Photons which could not have been in causal contact in the HBB model are observed to have the same temperature.

The inflationary solution to this issue is to think that, before the radiation-dominated epoch, the comoving Hubble radius decreased for a certain period of time.

This allows the parts of the sky to have been in causal contact in the early universe.

This means that $\ddot{a} > 0$ in the inflationary phase (which is equivalent to $w < -1/3$), since we are imposing $0 > \dot{r} = (\frac{\mathrm{d}}{\mathrm{d}t})(1/\dot{a}) = -\ddot{a}/(\dot{a})^2$. These are only the *kinematics* of inflation, we are not yet discussing how it might come about.

Wednesday 2020-10-7, compiled 2021-01-25

We come back to the horizon problem. [Plot of the comoving Hubble radius r_H as a function of time]

The problem is solved if there is an early epoch in which r_H decreases in time, due to accelerated expansion.

[3] https://docs.astropy.org/en/stable/cosmology/index.html

After the end of this inflation, the regular FLRW universe's history starts, with the radiation, then matter, then cosmological constant dominated phases. An accelerated expansion, however, is not enough to solve the horizon problem: what we need is for *every* observable scale, up to the largest ones, was causally connected in the early universe. In other words, the inflation phase must last *long enough*.

More specifically, our constraint on inflation is that it must start when the Hubble radius was at least as large as it is today. This can be expressed in terms of the *number of e-folds*:

$$N = \log\left(\frac{a_f}{a_{\text{in}}}\right) = \int_{t_{\text{in}}}^{t_f} H(t)\,\mathrm{d}t\ , \tag{1.1.40}$$

Since $H = \dot{a}/a = \frac{\mathrm{d}(\log a)}{\mathrm{d}t}$.

the ratio of the scale factor at the beginning and at the end of inflation. The number of elapsed *e*-folds is a natural measure of time in the epoch of inflation.

We can give the bound $N \gtrsim 60 \div 70$ in order to solve the horizon problem. This is a *huge* expansion! Typical atomic scales of $10^{-15}\,$m get stretched to the typical scales of the Solar System, $10^{11}\,$m.

The condition is $r_H(t_{\text{in}}) \gtrsim r_H(t_0)$. We can express this as

$$\frac{1}{a_{\text{in}} H_{\text{in}}} \gtrsim \frac{1}{a_0 H_0} \tag{1.1.41}$$

$$\frac{a_f}{a_{\text{in}}} = e^N \gtrsim \frac{H_{\text{in}}}{H_0}\frac{a_f}{a_0}\,. \tag{1.1.42}$$

See on Moodle: paper with the exact computation.

We want to bring on the left all the quantities in the inflationary epoch. Recall that $H^2 \propto \rho \propto a^{-3(1+w_i)}$, which we will apply to the inflationary epoch with an equation of state $w_i < -1/3$. This means that

$$\frac{H_i}{H_f} H_f = \left(\frac{a_i}{a_f}\right)^{-3(1+w_i)/2} H_f\,, \tag{1.1.43}$$

so

$$\frac{a_f}{a_i}\left(\frac{a_f}{a_i}\right)^{-3(1+w_i)/2} = \gtrsim \frac{H_f}{H_0}\frac{a_f}{a_0} \tag{1.1.44}$$

$$\left(\frac{a_f}{a_i}\right)^{\frac{-(1+3w_i)}{2}} \gtrsim \frac{T_0}{H_0}\frac{H_f}{T_f}\,, \tag{1.1.45}$$

where we applied Tolman's law, $T \sim 1/a$, neglecting the matter dominated phase — this is a reasonable approximation, we find a similar result to the complete calculation. This yields

$$N \gtrsim -\frac{2}{1+3w_i}\left[\log\frac{T_0}{H_0} + \log\frac{H_f}{T_f}\right], \tag{1.1.46}$$

where $T_0 = 2.7\,\mathrm{K} \approx 10^{-13}\,\mathrm{GeV}$, while $H_0 \sim 10^{-42}\,\mathrm{GeV}$ in natural units. Therefore, the first logarithm is of the order ~ 67. We also need the *pre-heating* temperature and Hubble parameter: H_f and T_f. This is model-dependent: it is what gives the theoretical uncertainty. The dependence, however, is weak: only logarithmic.

With current measurements, we are starting to be able to measure this term as well. Let us give an estimate for it:

$$H_f^2 \approx \frac{8\pi G}{3}\rho_{\mathrm{rad}}\,, \tag{1.1.47}$$

where $\rho_{\mathrm{rad}} = \frac{\pi^2}{30} g_* T^4$. This then yields

$$H_f^2 = \frac{8\pi G}{3}\frac{\pi^2}{30} g_* T^4 \sim \frac{T_f^4}{M_P^2}\,. \tag{1.1.48}$$

There is model dependence here, in g_*! If we go BDSM (beyond de standard model) it could change. We are giving a very rough estimate with the Planck mass. This then means $H_f \sim T_f^2/M_p$. So,

$$\log\left(\frac{H_f}{T_f}\right) \approx \log\frac{T_f}{M_p}\,. \tag{1.1.49}$$

Typically, models predict

$$10^{-5} < \frac{T_f}{M_p} < 1\,, \tag{1.1.50}$$

but this is not set in stone, we could have different predictions as well.

Now, let us assume that $w_i \sim -1$, something like a cosmological constant. Then, the prefactor is of the order 1, so the bound is $N \gtrsim 60 \div 70$ as was mentioned before.

We now discuss the causal structure of the FLRW metric:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right]. \tag{1.1.51}$$

Let us express this in different coordinates: we introduce χ, so that

$$r = S_k(\chi) = \begin{cases} \sinh\chi & k=-1 \\ \chi & k=0 \\ \sin\chi & k=+1 \end{cases}. \tag{1.1.52}$$

This allows the term $\mathrm{d}r^2/(1-kr^2)$ to become simply $\mathrm{d}\chi^2$.

Also, we introduce conformal time: $\mathrm{d}\eta = \mathrm{d}t/a(t)$, so that the metric becomes

$$\mathrm{d}s^2 = a^2(\eta)\left[-\,\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + S_k^2(\chi)\,\mathrm{d}\Omega^2\right]. \tag{1.1.53}$$

The meaning of χ is still a comoving distance. However, the interesting thing is that this metric is conformally related to ("is a time-dependent rescaling of") the Minkowski metric (if we consider radial motion, at least), we say that it is *conformally flat*.

In these coordinates, light propagates at 45° in the (η, χ) plane.

Then, we can draw a diagram for the horizon problem in these coordinates: the Big Bang singularity looks like a straight line at constant η. The last-scattering surface is also a straight line at constant η. We can then draw a past light-cone from a point in the last-scattering surface. Inflation pushes the BB surface back in conformal time, so that light has more time to propagate and light cones will intersect.

Claim 1.1.4. *The conformal time at the end of inflation looks like $\eta \propto 2/(1+3w)a^{2/(1+3w)}H_*^{-1}$, where H_* is the Hubble parameter at some reference time.*

Proof. Let us again make use of the fact that

$$d\eta = -\frac{dz}{a_0 H(z)} . \tag{1.1.54}$$

Here a_0 is usually taken to mean "now", however the analysis which follows does not really depend on that fact. Then, the conformal time reads

$$\eta = \int_0^\eta d\widetilde{\eta} = \int_z^\infty \frac{d\widetilde{z}}{a_0 H(\widetilde{z})} \tag{1.1.55}$$

Switched the integration margins: $[0, \eta]$ corresponds to $(\infty, z]$.

$$= \frac{1}{a_0 H_0} \int_z^\infty \frac{d\widetilde{z}}{E(\widetilde{z})} . \tag{1.1.56}$$

If we need to account for different fluids the integral cannot be done analytically, so we only account for one: the expression for the E function simplifies as

$$E(z) = \sqrt{\sum_i \Omega_{0,i}(1+z)^{3+3w_i}} = \sqrt{(1+z)^{3+3w}} \tag{1.1.57}$$

for a single component, with $\Omega_0 = 1$ and equation of state w. Then, the integral reads

$$\eta = \frac{1}{a_0 H_0} \int_z^\infty (1+\widetilde{z})^{-(3+3w)/2} \, d\widetilde{z} \tag{1.1.58}$$

$$= \frac{1}{a_0 H_0} \left(-\frac{1+3w}{2} \right)^{-1} \left[(1+\widetilde{z})^{-(1+3w)/2} \right]_{\widetilde{z}=z}^{\widetilde{z}=\infty} \tag{1.1.59}$$

$$= \frac{1}{a_0 H_0} \frac{2}{(1+3w)} (1+z)^{-(1+3w)/2} \tag{1.1.60}$$

$$= \frac{2}{(1+3w)H_0} a^{-(1+3w)/2} a_0^{-1-(1+3w)/2} \tag{1.1.61}$$

$$\propto \frac{2}{(1+3w)H_0} a^{-(1+3w)/2} . \tag{1.1.62}$$

This manipulation only works as long as $w > -1/3$: otherwise, the integral diverges. □

1.1.3 Flatness problem

Now, we move to the flatness problem. The HBB model is not intrinsically flawed, however the shortcomings we are discussing tell us that the initial conditions which would be required in order to yield the current universe would be very specific.

We should set initial conditions which are homogeneous and isotropic, with very specific small fluctuations. Inflation provides a dynamical solution to these problems, which is an attractor towards these initial conditions.[4]

The first Friedmann equation reads

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \tag{1.1.63}$$

which we can express through $\Omega = \rho/\rho_c$, where $\rho_c = 3H^2/(8\pi G)$:

$$\Omega - 1 = \frac{k}{a^2H^2} = kr_H^2(t)\,, \tag{1.1.64}$$

so if Ω differs from unity even by a small amount, this difference increases with time.

At 95 % CL, we know that $|\Omega - 1| = |\Omega_k| < 0.4\,\%$, so the universe we observe is consistent with flatness.

Specifically, in the Planck epoch we will have

$$\Omega(t_{\text{Pl}}) - 1 \approx (\Omega_0 - 1) \times 10^{-60}\,, \tag{1.1.65}$$

so $\left|\Omega(t_{\text{Pl}}) - 1\right| < 10^{-62}$.

Claim 1.1.5. *It can be shown that*

$$\left(\Omega^{-1} - 1\right)\rho a^2 = const = -\frac{3k}{8\pi G}\,. \tag{1.1.66}$$

Proof. The calculation is as follows:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.1.67}$$

$$\rho_c = \rho - \underbrace{\frac{3k}{8\pi G}}_{\text{const}}\frac{1}{a^2} \tag{1.1.68}$$

$$\rho\left(\underbrace{\frac{\rho_c}{\rho}}_{\Omega^{-1}} - 1\right)a^2 = \text{const}\,. \tag{1.1.69}$$

□

For times before the matter-radiation equivalence $\rho \propto a^{-4}$, so (neglecting the matter component) $\rho(t) = \rho_{\text{eq}}(a_{\text{eq}}/a)^4$. Also, during matter domination up to now (neglecting the cosmological constant)

$$\rho_0 = \rho_{\text{eq}}\left(\frac{a_{\text{eq}}}{a_0}\right)^3, \tag{1.1.70}$$

[4] See Hossenfelder [Hos19] for a critical discussion of this fine-tuning problem.

therefore

$$(\Omega^{-1}-1)\left(\frac{a_{\text{eq}}}{a}\right)^4 a^2\rho_{\text{eq}}\left(\frac{a_0}{a_{\text{eq}}}\right)^3\frac{1}{\rho_{\text{eq}}a_0^2} = (\Omega_0^{-1}-1) \tag{1.1.71}$$

$$\Omega^{-1}-1 = (\Omega_0^{-1}-1)\frac{a^2}{a_{\text{eq}}a_0} \tag{1.1.72}$$

$$\Omega^{-1}-1 = (\Omega_0^{-1}-1)(1+z_{\text{eq}})\frac{a^2}{a_0^2}\,, \tag{1.1.73}$$

since $1+z_{\text{eq}} = a_0/a_{\text{eq}}$. Also, we can approximate $a/a_0 \sim T_0/T_{\text{Pl}}$ by extending Tolman's law. Then,

$$\Omega^{-1}-1 = (\Omega_0^{-1}-1)\underbrace{(1+z_{\text{eq}})}_{\sim 10^4}\underbrace{\frac{T_0^2}{T_{\text{Pl}}^2}}_{\sim 10^{-64}}\,. \tag{1.1.74}$$

This proves the relation we wrote earlier. It is an extreme extrapolation to go back to the Planck time, but even if we only went back to Big Bang nucleosynthesis ($\sim 1\,\text{MeV}$) we would get

$$\left|\Omega(t_{\text{BBN}})-1\right| < 10^{-18}\,. \tag{1.1.75}$$

How does inflation solve the problem? Recall that $\Omega - 1 = kr_H^2$, and inflation is by definition a time in which r_H decreases. At the end of inflation, $\Omega - 1$ is very close to 0, meaning that r_H is small, but at the start of inflation it could have been relatively far from 1.

Next week we will discuss the proper mechanism of this process. During an inflationary phase, $a(t) \approx \exp(Ht)$. So, as long as H is approximately constant, we have

$$r_H^2 = \frac{1}{a^2H^2} \propto \frac{1}{a^2}\,. \tag{1.1.76}$$

Then, we have

$$\frac{|\Omega-1|_{t_f}}{|\Omega-1|_{t_i}} \sim \left(\frac{a_i}{a_f}\right)^2 \sim \exp(-2N)\,. \tag{1.1.77}$$

This means that, with very broad possible initial conditions, we find $\Omega - 1$ very close to zero at the end of inflation.

A De-Sitter phase is a reference example of a possible inflationary stage. It would correspond to $\rho = \text{const}$, $w = -1$: in general, since the "curvature energy density" scales like a^{-2}, curvature becomes negligible.

Note, however, that this is an unrealistic example: it does not include any method for the inflation to end.

As we saw last time, inflation provides an "attractor solution" to the flatness problem. Monday 2020-10-12, compiled 2021-01-25

We must impose

$$\frac{1-\Omega_i^{-1}}{1-\Omega_0^{-1}} \gtrsim 1 \tag{1.1.78}$$

in order to solve the flatnesss problem. Since $(1-\Omega^{-1})\rho a^2$ is a constant, this ratio is equal to

$$\frac{1-\Omega_i^{-1}}{1-\Omega_0^{-1}} = \frac{\rho_0 a_0^2}{\rho_i a_i^2} = \frac{\rho_0 a_0^2}{\rho_{\text{eq}} a_{\text{eq}}^2} \frac{\rho_{\text{eq}} a_{\text{eq}}^2}{\rho_f a_f^2} \frac{\rho_f a_f^2}{\rho_i a_i^2}\,, \tag{1.1.79}$$

and since $\rho \propto a^{-3(1+w)}$ the term we are considering scales like $\rho a^2 \propto a^{-(1+3w)}$. Substituting this in, accounting for the fact that the first term is in the matter-dominated epoch while the second is in the radiation-dominated one we have

$$\frac{1-\Omega_i^{-1}}{1-\Omega_0^{-1}} = \left(\frac{a_0}{a_{\text{eq}}}\right)^{-1} \left(\frac{a_{\text{eq}}}{a_f}\right)^{-2} \left(\frac{a_f}{a_i}\right)^{-(1+3w_f)} \tag{1.1.80}$$

$$e^{N\left|1+3w_f\right|} = \frac{1-\Omega_i^{-1}}{1-\Omega_0^{-1}} X\,, \tag{1.1.81}$$

where w_f is the equation of state in the inflationary phase, while

$$X = \frac{a_0}{a_{\text{eq}}} \left(\frac{a_{\text{eq}}}{a_f}\right)^2 \tag{1.1.82}$$

$$\approx 10^{60}\,. \tag{1.1.83}$$

Then,

$$e^{N\left|1+3w_f\right|} \gtrsim \underbrace{\frac{1-\Omega_i^{-1}}{1-\Omega_0}}_{\gtrsim 1} X \tag{1.1.84}$$

$$N_{\text{min}} = \frac{\log X}{\left|1+3w_f\right|} \approx 60 \div 70\,, \tag{1.1.85}$$

where we assumed $w_f \sim -1$. There is some model dependence, specifically regarding the transition from inflation to radiation domination.

It is interesting that this is similar to the number of e-folds needed to solve the horizon problem.

We can also write the expression, defining $N = pN_{\text{min}}$, as

$$1-\Omega_0^{-1} = \frac{1-\Omega_i^{-1}}{\exp\Big((p-1)N_{\text{min}}(1+3w_f)\Big)}\,, \tag{1.1.86}$$

so, taking $p=2$ and $w_i = -1$ we have

$$1-\Omega_0^{-1} = (1-\Omega_i^{-1})e^{-2N_{\text{min}}} = (1-\Omega_i^{-1})X^{-1}\,. \tag{1.1.87}$$

Constraining Ω_0 allows us to constrain $\Delta N = N - N_{\text{min}}$. Right now we have $|1-\Omega_0| < 0.4\,\%$ (at 95 % CL) we can say $\Delta N \gtrsim 5$.

1.1.4 The flatness problem as an age problem

Consider the HBB model, at $t < t_{\mathrm{eq}}$: the radiation-dominated epoch.

The Planck time, the only characteristic time of the universe a priori, is $t_{\mathrm{Pl}} \approx 10^{-43}\,\mathrm{s}$. If the universe is spatially closed, then $t_{\mathrm{collapse}} = 2t_m$. We would expect for the time of collapse to be of the order of the Planck time.

If the universe was spatially open, we would expect that for $t > t_* \sim t_{\mathrm{Pl}}$ we would have curvature domination: $a(t)/a(t_*) \sim t/t_*$.

Therefore, we would have $t_0 = t_{\mathrm{Pl}} T_{\mathrm{Pl}} / T_0 \sim 10^{-11}\,\mathrm{s}$.

In order for this to not happen, we need the energy density term to finely balance the curvature term in the first Friedmann equation.

1.1.5 The unwanted relics problem

This is also known as the "magnetic monopoles problem". Consider a massive particle X, such that $\Omega_{0X} \gg 1$.

Typically, $\Omega_{0X} \sim 1/\sigma_A$.

Historical examples are cosmic topological defects, arising from the SSB of some gauge theory. Also, we could have cosmic strings: one-dimensional defects, arising from the SSB of $U(1)$. Magnetic monopoles could come from a Grand Unified Theory.

Domain walls (which are 2D) arise from the SSB of discrete symmetries, 3D textures arise from the SSB of $SU(2)$.

Other examples of unwanted relics are gravitinos (spin 3/2 superpartners of gravitons, with $m \sim 100\,\mathrm{GeV}$) or spin-0 *moduli* from superstring theory.

We will explore how these can overclose the universe. We start with SSB in a cosmological context. SSB means that the ground state has less symmetry than the full theory. How do we describe it in the context of an expanding universe?

If go back in time to when the temperature was very high, we expect a restoration of the symmetry.

We start with a lot of symmetry, and as the universe expands we lose it. There are *phase transitions* accompanying this change.

Wednesday 2020-10-14, compiled 2021-01-25

We were talking about Spontaneous Symmetry Breaking: how does it work in a cosmological context? Specifically, we ask about its effect on cosmological phase transitions at high temperatures in the early universe.

We start out with a scalar field φ with Lagrangian[5]

$$\mathscr{L}_\varphi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi)\,, \tag{1.1.88}$$

where we choose a potential

$$V(\varphi) = \frac{\lambda}{4} \left(\varphi^2 - \sigma^2 \right)^2 . \tag{1.1.89}$$

[5] We can write it with partial derivatives instead of covariant ones since for a scalar φ they are equal: $\nabla_\mu \varphi = \partial_\mu \varphi$.

This is the typical example of a potential which exhibits SSB. Its vacuum (minimum) is a pair of points at $|\varphi| = \sigma$.

The Lagrangian is invariant under $\varphi \to -\varphi$; either of the vacuum states is not.

We need to consider finite-temperature effects on the propagator of the scalar field. The temperature corrections to this potentials yields a temperature-dependent mass term, which looks like

$$m_T^2 = \alpha\lambda T^2 \,, \tag{1.1.90}$$

where λ is the coupling of the field, while α is a dimensionless order-1 number.

Then, the potential reads

$$V_T(\varphi) = V_{T=0}(\varphi) + \frac{1}{2}\alpha\lambda\varphi^2 T^2 \,. \tag{1.1.91}$$

If the temperature is sufficiently high, the potential will have only one vacuum again. This means that if we go far enough back in time the symmetry is restored.

The moment at which the potential goes from one minimum to two is the one at which we have a **phase transition**. At which temperature does it happen? We can find out by considering the sign of the second derivative of the potential at $\varphi = 0$:

$$\left.\frac{\mathrm{d}^2 V}{\mathrm{d}\varphi^2}\right|_{\varphi=0} = -\lambda\sigma^2 + \lambda\alpha T^2 \,, \tag{1.1.92}$$

so we have a critical temperature $T \approx \sigma/\sqrt{\alpha}$ at which the symmetry is broken.

1.1.6 Topological defects

The defects are quite similar to the defects we find in regular phase transitions we know at our scales, like water to ice: the crystal which forms is not perfect.

The minima $\varphi = \pm\sigma$ are the *true vacuum* of the system, while $\varphi = 0$ is the *false vacuum*.

There will be regions in the universe in which the scalar field goes to $+\sigma$, and other regions in which it goes into $-\sigma$. This is because the two minima are equivalent: there are even odds for the field at any point to fall into either. In causally connected regions it will go into the same minimum. There will then be boundaries between the regions in which the field goes into $+\sigma$ and $-\sigma$.

In a mostly-plus metric signature, the equation of motion reads

$$\Box\varphi = \frac{\partial V}{\partial \varphi} \,, \tag{1.1.93}$$

so if we neglect the curvature and consider static solutions we will have

$$\nabla^2\varphi = \frac{\partial V}{\partial \varphi} \,. \tag{1.1.94}$$

Further, we consider an infinite **domain wall** in the xy plane, assuming that for $z \to \pm\infty$ we have $\varphi = \pm\sigma$. Also, we assume that the whole field has no x or y dependence. Let us then substitute into the equation:

$$-\frac{\partial^2 \varphi}{\partial z^2} = -\frac{\partial V}{\partial \varphi} = -\lambda \varphi(\varphi^2 - \sigma^2)\,. \tag{1.1.95}$$

Solving this yields

$$\varphi(z) = \sigma \tanh\left(\frac{z}{\Delta}\right), \tag{1.1.96}$$

where Δ is the *thickness* of the wall, which we can estimate through energetic configurations: the surface energy will have contributions through the gradient of the field: $\Delta(\partial_z \varphi)^2 \sim \Delta\sigma^2/\Delta^2 = \sigma^2/\Delta$; and a potential term $V(\varphi) \sim \Delta V(\varphi = 0) \sim \Delta\lambda\sigma^4/4$.

This is because the total Hamiltonian for a unit-area region of this stationary configuration will also look like

$$\begin{aligned}
H &= \int \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi)\right] \mathrm{d}z && (1.1.97)\\
&= \int \frac{1}{2}\left(\partial_z \varphi\right)^2 + V(\varphi)\,\mathrm{d}z && (1.1.98)\\
&= \frac{\Delta}{2} \int \left(\partial_z \varphi\right)^2 \mathrm{d}z + \frac{\lambda}{4}\sigma^4 \Delta \underbrace{\int \left(\tanh^2 z - 1\right)^2 \mathrm{d}z}_{\mathcal{O}(1)} && (1.1.99)\\
&\approx \Delta \frac{\sigma^2}{\Delta^2} + \frac{\lambda}{4}\sigma^2 \Delta\,. && (1.1.100)
\end{aligned}$$

The two contributions scale oppositely with Δ: the kinetic energy decreases for a wide wall, the potential energy decreases for a narrow wall. Then, we can find an optimum at $\Delta \sim 1/(\sigma\sqrt{\lambda})$.

This domain wall is not removable, the configuration is topologically stable.

Now we discuss the **Kibble mechanism**, which demonstrates how phase transitions always generate domain walls. Let us denote as ξ the typical linear size of the domains: it is called the *correlation length* of φ.

We know that in the radiation-dominated epoch there is a finite particle horizon $d_H(t) \approx 2t$, so we must have $\xi \lesssim d_H(t)$, since regions which were further apart could not have communicated yet at that time.

Then, we find a lower bound on the number density of the domain walls, $n_X \sim \xi^{-3} \gtrsim d_H^{-3}(t) = (2t)^{-3}$.

Recall from regular HBB cosmology that

$$H^2(t) = \frac{8\pi G}{3}\rho_r = \frac{8\pi G}{3}\frac{\pi^2}{30} g_* T^4\,, \tag{1.1.101}$$

so

$$t = \frac{1}{2H} \approx 0.3 \frac{1}{\sqrt{g_*}} \frac{M_{\text{Pl}}}{T^2}\,, \tag{1.1.102}$$

therefore

$$n_X \gtrsim \left(\frac{\sqrt{g_*}}{0.6}\frac{T}{M_{\rm Pl}}\right)^3 T^3 \sim \left(\frac{\sqrt{g_*}}{0.6}\frac{T}{M_{\rm Pl}}\right)^3 n_\gamma(T)\,, \tag{1.1.103}$$

since the number density of photons scales like $n_\gamma(T) \sim T^3$.

Let us evaluate this number density, for $T \sim T_{\rm GUT} \sim 10^{15}\,$GeV. Here, $g_* \sim 100$, meaning that we get

$$n_X(T_{\rm GUT}) > 10^{-9} \div 10^{-10} n_\gamma(T_{\rm GUT})\,. \tag{1.1.104}$$

Therefore, we have the ratio

$$\frac{n_X(T_{\rm GUT})}{n_\gamma(T_{\rm GUT})} > 10^{-9} \div 10^{-10}\,. \tag{1.1.105}$$

If we assume that after production these objects are stable, there are no processes which can modify their number. Then, for lower temperatures we keep the same ratio, both number densities will scale like $n_\gamma \sim T^3 \sim a^{-3}$.

This is a very similar number to the ratio of baryons to photons, $\eta = n_b/n_\gamma$! This means that

$$\Omega_{0x} = \frac{m_x \eta_x(t_0)}{\rho_{\rm crit}} \gtrsim \frac{m_x \eta_{0b}}{\rho_{\rm crit}} = \frac{m_x}{m_p}\Omega_{0b}\,, \tag{1.1.106}$$

which means that, since $m_x \sim T_{\rm GUT} \sim 10^{15}\,$GeV, we must have $\Omega_{0x} \gtrsim 10^{14}$! This definitely **over-closes** the universe.

How does inflation solve this problem? These objects are produced in these early stages, but their number density is very **diluted**. Each $\pm\sigma$ region is inflated to the size of the observable universe.

Now we will give some arguments as to why a scalar field makes sense, a characterization of different inflationary models, and discuss the generation of the first primordial density perturbations.

A De Sitter phase is one with a cosmological constant Λ:

$$H^2 = \frac{8\pi G}{3}\rho_\Lambda - \frac{k}{a^2}\,. \tag{1.1.107}$$

Here $P_\Lambda = -\rho_\Lambda$. Then, $a(t) \propto \exp(Ht)$, with $H = $ const. This ρ_Λ is constant as the universe expands.

A cosmological constant term can be written in terms of a vacuum energy density of the quantum system. It appears in the EFE as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}\,. \tag{1.1.108}$$

This can be calculated as $\langle 0|\, T_{\mu\nu}\,|0\rangle \propto -\langle 0|\,\varphi\,|0\rangle\, g_{\mu\nu} = -\langle\rho\rangle\, g_{\mu\nu}$.

Plugging this into the EFE we find

$$\Lambda = -8\pi G \langle \rho \rangle . \tag{1.1.109}$$

We cannot get rid of the vacuum energy density of the system, since energy gravitates.

Let us come back to the Lagrangian

$$\mathscr{L}_\varphi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) , \tag{1.1.110}$$

whose energy momentum tensor is

$$T^\varphi_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi + \mathscr{L}_\varphi g_{\mu\nu} . \tag{1.1.111}$$

Let us look at the Vacuum Expectation Value of the field: $\langle \varphi \rangle = \langle 0 | \varphi | 0 \rangle$. If this is a constant, it should correspond to the minimum of the classical potential: the ground state. This behaves like a cosmological constant. Since φ is a constant, we have

$$\left\langle T_{\mu\nu} \right\rangle = g_{\mu\nu} V(\langle \phi \rangle) , \tag{1.1.112}$$

since the derivatives of a constant vanish. This is an effective Λ.

A phase transition can move us away from this Vacuum Energy Value (VEV).

The VEV of φ can be a function of time.

Are there other options beyond a scalar field? Say, a vector field, or a spinor? The first reason we do not choose this is because it breaks isotropy.

1.2 Dynamics of inflation

Monday 2020-10-19, compiled 2021-01-25

Our action will be in the form

$$S = S_{EH} + S_\varphi + S_{\text{matter}} , \tag{1.2.1}$$

where S_{EH} is the Einstein-Hilbert action for the metric, S_φ is the action for the field φ, while "matter" encompasses all the other fields:

$$S = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} R + \int \mathrm{d}^4 x \sqrt{-g} \mathscr{L}_\varphi [\varphi, g_{\mu\nu}] + S_{\text{matter}} . \tag{1.2.2}$$

We are using the invariant volume element $\mathrm{d}^4 x \sqrt{-g}$, which represents the physical 4-volume regardless of the coordinates.

The simplest possibility for the Lagrangian of a scalar field which is able to drive inflation reads:

$$\mathscr{L}_\varphi = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) , \tag{1.2.3}$$

for a *real* scalar field φ. Also, note that we did not consider any explicit coupling of φ with gravity or other fields: these kinds of terms, which go by the name "nonminimal coupling", might look like $\xi \varphi^2 R$.

These kinds of nonminimal theories represent one of the simplest extensions of GR: they are *scalar-tensor* theories, and in them the field φ as well as $g_{\mu\nu}$ can mediate gravity. One of the theories which currently fits the cosmological data best is of this kind.

What could we put in the potential? The mass is given by $m_\varphi^2 = \partial^2 V \Big/ \partial\varphi^2$, so a simple mass term would look like $m^2\varphi^2/2$, but we could also have quartic terms like $\lambda\varphi^4/4$: these are self-interaction terms.

The other fields will typically be negligible during inflation since their energy density is quickly "redshifted away". Sometimes some of them are non-negligible: this happens if they are coupled to the scalar field, and we must consider them; we typically do so in an "effective" way, by inserting them into the potential $V(\varphi)$.

We can associate an energy-momentum tensor to the scalar field: in general, it is defined by

$$T_{\mu\nu}^{(\varphi)} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\varphi}{\delta g^{\mu\nu}} \,. \tag{1.2.4}$$

This comes from the way we write the Einstein equations from a variational principle. For our scalar field, integrating by parts inside the action, we find

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \left[-\frac{\partial(\sqrt{-g}\mathscr{L}_\varphi)}{\partial g^{\mu\nu}} + \partial_\alpha \frac{\partial(\sqrt{-g}\mathscr{L}_\varphi)}{\partial {g^{\mu\nu}}_{,\alpha}} + \text{higher order terms} \right] . \tag{1.2.5}$$

The reason for the alternating signs is that we must integrate by parts in order to get the expression in this form. We then get

$$T_{\mu\nu}^{(\varphi)} = -2\frac{\partial \mathscr{L}_\varphi}{\partial g^{\mu\nu}} + \frac{2}{\sqrt{-g}} \mathscr{L}_\varphi \frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} \,, \tag{1.2.6}$$

since there is no dependence on the derivative(s) of the metric in our case.

Claim 1.2.1. *The following expression holds:*

$$\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \,. \tag{1.2.7}$$

Proof. The expression can be derived from the general matrix relation $\operatorname{Tr}\log M = \log\det M$, applied taking $M = g^{\mu\nu}$: abusing the notation a bit with the indices

$$\operatorname{Tr}\log g^{\mu\nu} = \log\det g^{\mu\nu} \,, \tag{1.2.8}$$

which we then differentiate, using $g = \det g_{\mu\nu}$, the fact that the trace is linear, and the matrix logarithmic derivative expression $\delta\log M = M^{-1}\delta M$:[6]

$$\operatorname{Tr}\left(M^{-1}\delta M\right) = \frac{\delta\det g^{\mu\nu}}{\det g^{\mu\nu}} = \frac{\delta(1/g)}{1/g} \tag{1.2.9}$$

[6] Which holds as long as M^{-1} and δM commute.

$$\mathrm{Tr}\left(g_{\mu\nu}\delta g^{\nu\rho}\right) = -\frac{\delta g}{g} \tag{1.2.10}$$

$$g_{\mu\nu}\delta g^{\mu\nu} = -\frac{\delta g}{g}\,. \tag{1.2.11}$$

Symmetry of the metric's indices.

Let us now consider a variation of $\sqrt{-g}$:

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g = +\frac{1}{2}\underbrace{\frac{1}{\sqrt{-g}}g}_{-\sqrt{-g}}\,g_{\mu\nu}\delta g^{\mu\nu} \tag{1.2.12}$$

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}g_{\mu\nu}\,. \tag{1.2.13}$$

This is the final result. We used the fact that $g/\sqrt{-g} = -\sqrt{-g}$, which is the correct solution of $(g/\sqrt{-g})^2 = -g$ because we know g to be negative. □

This yields

$$T^{(\varphi)}_{\mu\nu} = -2\frac{\partial\mathscr{L}_\varphi}{\partial g^{\mu\nu}} + \mathscr{L}_\varphi g_{\mu\nu} \tag{1.2.14}$$

$$= \partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}\left[-\frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} - V(\varphi)\right]. \tag{1.2.15}$$

Claim 1.2.2. *In general, this is a perfect fluid stress energy tensor, with*

$$P = -\frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} - V(\varphi) \tag{1.2.16}$$

$$\rho = -\frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + V(\varphi) \tag{1.2.17}$$

$$u_\mu = \frac{\partial_\mu\varphi}{|\partial\varphi|} \tag{1.2.18}$$

$$|\partial\varphi| = \sqrt{-g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}}\,. \tag{1.2.19}$$

Proof. As the parameters are given, we only need to verify that the expressions are equivalent:

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu} \tag{1.2.20}$$

$$= \underbrace{(-g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta})}_{|\partial\varphi|^2}u_\mu u_\nu + \left(-\frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} - V(\varphi)\right)g_{\mu\nu} \tag{1.2.21}$$

$$= \varphi_{,\mu}\varphi_{,\nu} - \left(\frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + V(\varphi)\right)g_{\mu\nu}\,. \tag{1.2.22}$$

□

We start by considering a homogeneous and isotropic case, and then perturb it. This is done by splitting the field into the average classical background motion of φ, called φ_0

$$\varphi = \varphi(\vec{x}, t) = \varphi_0(t) + \delta\varphi(\vec{x}, t)\,, \tag{1.2.23}$$

where φ_0 will be the VEV of the field: $\varphi_0 = \langle 0|\, \varphi(\vec{x}, t)\, |0\rangle$, while $\delta\varphi$ encompasses the quantum fluctuations.

Are we allowed to do this kind of split? Formally yes, but we need to guarantee that the perturbations are indeed small compared to the classical trajectory: $\left\langle \delta\varphi^2 \right\rangle \ll \varphi_0^2(t)$. We consider the variance since it is the first nonzero moment, as $\left\langle \delta\varphi \right\rangle = 0$.

This will not always be the case, sometimes the fluctuation will be dominating; however usually for an inflationary model to work we expect that the condition is satisfied. The fluctuations are what generates the density fluctuations which create the anisotropies in the CMB photons: we know that the size of these anisotropies is of the order of one part in 10^5, so we can give a qualitative argument for the perturbations of the scalar field to be relatively small compared to the mean value.

If we do the explicit computation for the energy-momentum tensor of the classical background φ_0 only we find:

$$T^0_0 = -\left(\frac{1}{2}\dot{\varphi}_0(t)^2 + V(\varphi_0)\right) = -\rho_\varphi(t) \tag{1.2.24}$$

$$T^i_j = \left(\frac{1}{2}\dot{\varphi}_0^2(t) - V(\varphi_0)\right)\delta^i_j = P_\varphi \delta^i_j\,. \tag{1.2.25}$$

This is a perfect-fluid energy-momentum tensor. If we are in a regime for which

$$\frac{1}{2}\dot{\varphi}_0^2(t) \ll V(\varphi_0) \tag{1.2.26}$$

when we have $P_\varphi \approx -\rho_\varphi$: this is a *quasi De Sitter* expansion, with $w_\varphi \approx -1$.

This is achieved if the potential for the scalar field is "flat enough": then, we reach a *friction-domination* regime, which is commonly called *slow-roll* inflation.

Insert picture of flat potential

If $V(\varphi)$ is approximately a constant, then it mimics a cosmological constant. Inflation is driven by the vacuum energy density associated with the scalar field.

Let us look at the slow-roll dynamics in more detail. What is the equation of motion for this (quantum!) scalar field?

Claim 1.2.3. *It is just the Klein-Gordon equation, which can be derived by functional differentiation of the action with respect to φ:*

$$\Box\varphi = \frac{\partial V}{\partial \varphi}\,. \tag{1.2.27}$$

The Dalambertian operator here reads

$$\Box\varphi = \frac{1}{\sqrt{-g}}\left(g^{\mu\nu}\sqrt{-g}\varphi_{,\mu}\right)_{,\nu}\,. \tag{1.2.28}$$

Proof. Functionally differentiating the action yields

$$\frac{\delta S}{\delta \varphi} = \frac{\delta}{\delta \varphi} \int \mathrm{d}^4x \, \sqrt{-g} \left(-\frac{1}{2} g^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta} - V(\varphi) \right) \tag{1.2.29}$$

$$= \frac{\delta}{\delta \varphi} \int \mathrm{d}^4x \left(\left(\frac{\sqrt{-g}}{2} g^{\alpha\beta} \varphi_{,\alpha} \right)_{,\beta} \varphi - V(\varphi) \sqrt{-g} \right) \tag{1.2.30}$$

$$= \sqrt{-g} \left(\frac{1}{\sqrt{-g}} \partial_\beta \left(\sqrt{-g} g^{\alpha\beta} \partial_\alpha \varphi \right) - \frac{\partial V}{\partial \varphi} \right) = 0 \,. \tag{1.2.31}$$

□

Let us see what this reduces to in a **flat FLRW** metric: then $\sqrt{-g} = a^3$, so

$$\Box \varphi = \frac{1}{a^3} \left(g^{00} a^3 \varphi_{,0} \right)_{,0} + \frac{1}{a^3} \left(g^{ii} a^3 \varphi_{,i} \right)_{,i} = \frac{\partial V}{\partial \varphi} \tag{1.2.32}$$

$$-\ddot{\varphi} - \dot{\varphi} 3 \frac{\dot{a}}{a} + \frac{\nabla^2}{a^2} \varphi = \frac{\partial V}{\partial \varphi} \tag{1.2.33}$$

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{\nabla^2 \varphi}{a^2} = -\frac{\partial V}{\partial \varphi} \,. \tag{1.2.34}$$

The term $3H\dot{\varphi}$ is a kind of *friction* term: the propagation of the field is "held back" by the expansion. If we consider the background field, it will be constant in space, so it will evolve as

$$\ddot{\varphi}_0 + 3H\dot{\varphi}_0 = -\frac{\partial V}{\partial \varphi_0} \,. \tag{1.2.35}$$

We then must solve this equation combined with the Friedmann equation

$$H^2 = \frac{8\pi G}{3} \left(\rho_\varphi + \rho_m + \rho_r \right) - \frac{k}{a^2} \,. \tag{1.2.36}$$

The matter and radiation densities scale like a^{-3} for ρ_m, a^{-4} for ρ_r; in this early phase the scalar field will dominate the dynamics, so the equation will simplify to

$$H^2 \approx \frac{8\pi G}{3} V(\varphi_0) \,. \tag{1.2.37}$$

Under these slow-roll conditions, we also have $\ddot{\varphi}_0 \ll 3H\dot{\varphi}_0$: therefore equation (1.2.35) simplifies to

$$3H\dot{\varphi}_0 \approx -\frac{\partial V}{\partial \varphi_0} \,. \tag{1.2.38}$$

We also expect that V and all of its derivatives change very slowly with φ. This means that in this equation we have $\frac{\partial V}{\partial \varphi_0} \approx$ const, as well as $H \approx$ const: this is the same equation which is obeyed by a particle under a constant force and friction: it will then reach the asymptotic "**terminal velocity**" and move with a constant $\dot{\varphi}_0$. This solution is an attractor.

We typically write this as $\dot{\varphi} = -V'(\varphi_0)/3H$, and $H^2 = \frac{8\pi G}{3} V(\varphi)$.

1.2.1 Slow-roll parameters

These are parameters we need to quantify how much the potential indeed looks like we expected.

The first parameter we define is

$$\epsilon = -\frac{\dot{H}}{H^2} = +4\pi G\frac{\dot{\varphi}^2}{H^2} \approx \frac{3}{2}\frac{\dot{\varphi}^2}{V} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \tag{1.2.39}$$

where the manipulations follow from equations (1.2.38) and (1.2.37).

Proof. Let us show that the manipulations work. We start by differentiating equation (1.2.37) with respect to time:

$$2H\dot{H} = \frac{8\pi G}{3}\frac{\partial V}{\partial \varphi}\dot{\varphi} \tag{1.2.40}$$

$$\dot{H} = 4\pi G \underbrace{\frac{V'}{3H}}_{-\dot{\varphi}} \dot{\varphi} = -4\pi G\dot{\varphi}^2, \tag{1.2.41}$$

which, divided by H^2, yields the first equality.

The second equality comes from the substitution of the expression for H^2 from equation (1.2.37). Then, we can square the relation $\dot{\varphi} = -V'/3H$ to get

$$\dot{\varphi}^2 = \left(\frac{V'}{3H}\right)^2, \tag{1.2.42}$$

which allows us to manipulate

$$\frac{3}{2}\frac{\dot{\varphi}^2}{V} = \frac{3}{2V}\left(\frac{V'}{3H}\right)^2 \tag{1.2.43}$$

$$= \frac{3(V')^2}{2V\frac{8\pi G}{3}V} \tag{1.2.44}$$

$$= \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2. \tag{1.2.45}$$

□

Claim 1.2.4. *The relation $\dot{H} = -4\pi G\dot{\varphi}^2$ holds exactly in the case of a single scalar field with the action we have been using so far.*

Proof. We start from the Friedmann equation for flat spacetime:

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(V(\varphi) + \frac{1}{2}\dot{\varphi}^2\right). \tag{1.2.46}$$

Now, we proceed as we did previously, by differentiating everything with respect to time:

$$
\begin{aligned}
2H\dot{H} &= \frac{8\pi G}{3}\left(V'\dot{\varphi} + \ddot{\varphi}\dot{\varphi}\right) && (1.2.47)\\
&= \frac{8\pi G}{3}\dot{\varphi}\left(-3H\dot{\varphi}\right) && (1.2.48)\\
\dot{H} &= -4\pi G\dot{\varphi}^2\,, && (1.2.49)
\end{aligned}
$$

Equation of motion.

where we assumed homogeneity. The derivation seems not to work if we do not assume it. □

Requiring $\epsilon \ll 1$ can also be stated as asking that[7]

$$
\begin{aligned}
\frac{(V')^2}{16\pi G V^2} &\ll 1 && (1.2.50)\\
\frac{(V')^2}{V} &\ll 16\pi G V = \frac{2}{3}H^2 && (1.2.51)\\
\frac{1}{V}\left(\frac{\partial V}{\partial \varphi}\right)^2 &\ll H^2\,. && (1.2.52)
\end{aligned}
$$

So, ϵ gives a bound on the first derivative of the potential; the second derivative is controlled by the parameter

$$
\eta = -\frac{\ddot{\varphi}}{H\dot{\varphi}}\,, \tag{1.2.53}
$$

and we can also define

$$
\eta_V = \frac{1}{3}\frac{V''}{H^2} = \frac{1}{8\pi G}\frac{V''}{V}\,. \tag{1.2.54}
$$

Asking that $\eta_V \ll 1$ is equivalent to $V'' \ll H^2$.

Claim 1.2.5. *These three parameters are related by $\eta = \eta_V - \epsilon$.*

Proof. We start from the relation $\dot{\varphi} \approx -V'/(3H)$. Differentiating it with respect to time will yield the desired relation:

$$
\begin{aligned}
\ddot{\varphi} &\approx -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{V'}{3H}\right) && (1.2.55)\\
&= -\frac{1}{3H}V''\dot{\varphi} - \frac{V'}{3}\underbrace{\left(-\frac{\dot{H}}{H^2}\right)}_{\epsilon} && (1.2.56)\\
&= -\dot{\varphi}H\frac{V''}{3H^2} - \frac{V'}{3}\epsilon && (1.2.57)
\end{aligned}
$$

[7] A note on the dimensionality: the potential V has the dimensions of the Lagrangian: in natural units, m^{-4}. The field φ has the dimensions of an inverse length (a mass), just like the Hubble rate.

$$= -\dot{\varphi} H \eta_V - \frac{V'}{3}\epsilon \tag{1.2.58}$$

$$\underbrace{-\frac{\ddot{\varphi}}{H\dot{\varphi}}}_{\eta} = \eta_V - \epsilon \,. \tag{1.2.59}$$

□

Is it ok to differentiate this, even though we *know* that $\ddot{\varphi} = -V' - 3H\dot{\varphi}$ from the EOM?

Wednesday 2020-10-21, compiled 2021-01-25

We come back to the dynamics of slow-roll inflation. We defined $\epsilon = -\dot{H}/H^2$, which in our model corresponds to $\dot{\varphi}^2/V$. So, the conditions of the potential being flat (V' being small) and the kinetic energy being small compared to the potential are seen to correspond to $\epsilon \ll 1$.

On the other hand, $\eta = -\ddot{\varphi}/(H\dot{\varphi}) = \eta_V - \epsilon$, where

$$\eta_V = \frac{1}{3}\frac{V''}{H^2} = \frac{1}{8\pi G}\frac{V''}{V} \,. \tag{1.2.60}$$

What we ask is that all three of these parameters be small.

This means that H must change slowly over time. We can manipulate the second derivative of the scale factor so that ϵ appears:

$$\ddot{a} = \frac{\mathrm{d}}{\mathrm{d}t}(Ha) = a\dot{H} + \dot{a}H = a\left(\dot{H} + H^2\right) = aH^2\left(1 + \frac{\dot{H}}{H^2}\right) = aH^2(1-\epsilon) \,. \tag{1.2.61}$$

We can see that $\ddot{a} > 0$ only if $\epsilon < 1$. What is then the relevance of $\eta < 1$ then? We must have not only a phase of accelerating expansion, but a phase of accelerating expansion which lasts *sufficiently long*. In order for this to happen, we need $\epsilon \sim$ const. Since $\epsilon \sim \dot{\varphi}^2$ while $\eta \sim \ddot{\varphi}$, requiring $\eta \ll 1$ ensures this.

Also, $\eta \ll 1$ is needed in order to neglect the acceleration term in the Klein-Gordon equation, so that we move towards an attractor solution in the friction-dominated regime.

There could be arbitrarily many more slow-roll parameters, defined in terms of higher-order derivatives [Guz+16, pag. 406]:

$$\xi^2 = \frac{1}{8\pi G}\left(\frac{V'V'''}{V''}\right), \tag{1.2.62}$$

and we will always expand in these parameters; in this course we will usually stop at second order (keeping ϵ and η).

We will then be able to consider both ϵ and η as approximately constant: their time derivatives are of higher order in them, specifically $\dot{\epsilon}, \dot{\eta} = \mathcal{O}\left(\epsilon^2, \eta^2\right)$.

Claim 1.2.6. *An example: the following relation holds:*

$$\frac{\dot{\epsilon}}{H} = 2\epsilon(\epsilon - \eta) \,. \tag{1.2.63}$$

Proof. Let us take the derivative explicitly:

$$\frac{\dot{\epsilon}}{H} = \frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\dot{H}}{H^2}\right) = -\frac{\ddot{H}}{H^3} + 2\underbrace{\frac{\dot{H}^2}{H^3 H}}_{=\epsilon^2} . \tag{1.2.64}$$

Now, we need to use the fact that, as we have shown earlier (1.2.41), $\dot{H} \propto \dot{\varphi}^2$. In terms of logarithmic derivatives this can be written as

$$\frac{\ddot{H}}{\dot{H}} = 2\frac{\ddot{\varphi}}{\dot{\varphi}}, \tag{1.2.65}$$

which we can substitute in the aforementioned expression to get

$$-\frac{\ddot{H}}{H^3} = -\frac{1}{H^3}2\frac{\ddot{\varphi}}{\dot{\varphi}}\dot{H} = -2\left(-\frac{\dot{H}}{H^2}\right)\left(-\frac{\ddot{\varphi}}{H\dot{\varphi}}\right) = -2\epsilon\eta . \tag{1.2.66}$$

□

Forgetting about the precise coefficients, let us consider a model such that $V(\varphi) \propto \varphi^\alpha$: then, the parameter ϵ reads

$$\epsilon \sim \frac{1}{\pi G}\left(\frac{V'}{V}\right)^2 \sim \frac{\alpha^2 M_{\mathrm{Pl}}^2}{\varphi^2}, \tag{1.2.67}$$

meaning that in order to have $\epsilon \ll 1$ we need $\varphi \gtrsim M_{\mathrm{Pl}}$. Note that we are talking about the *background* solution, dropping the index 0 for simplicity.

These are then called *large-field* models, since the value of φ must be very large. Another kind of potential is in the form

$$V(\varphi) = V_0\left[1 - \left(\frac{\varphi}{\mu}\right)^p + \dots\right], \tag{1.2.68}$$

where $\varphi < \mu < M_{\mathrm{Pl}}$, while $p > 2$.

The dots (higher order terms) are important for the end of inflation, not for most of it. This potential has a plateau near $\varphi = 0$, and $V(\varphi = 0) = V_0$. In this region,

$$\epsilon \sim \frac{1}{\pi G}\left(\frac{V'}{V}\right)^2 \sim \frac{p^2}{\pi G}\frac{\varphi^{2p-2}}{\mu^{2p}} \sim \frac{p^2\varphi^{2p}M_{\mathrm{Pl}}^2}{\varphi^2\mu^{2p}}, \tag{1.2.69}$$

so $\epsilon \to 0$ as $\varphi \to 0$ (the exponent of φ is $2p - 2 > 0$): these are called *small-field models* of inflation, since we can have small ϵ even with small φ.

Why do we need the condition $\varphi < \mu < M_{\mathrm{Pl}}$?

Claim 1.2.7. *In the case $p = 2$ and in the case $\mu > M_{Pl}$ this model becomes a* large-field *one.*

Proof. In that case, the condition we must ask for $\epsilon < 1$ is

$$4\frac{M_{\text{Pl}}^2 \varphi^2}{\mu^4} \ll 1\,, \tag{1.2.70}$$

which can be satisfied by $\varphi \gtrsim M_{\text{Pl}}$. □

There is also a third category (high gravitation models), but it is mostly excluded by data.

The quantity we are interested in is the *excursion* $\Delta\varphi$ of the field in the *observable window*: the difference between its value when the horizon crosses the largest observable scales (φ_{CMB} — called so since we can measure it from CMB observations) and the value of the field at the end of inflation, φ_{end}: $\Delta\varphi = \varphi_{\text{CMB}} - \varphi_{\text{end}}$. This interval corresponds to the ~ 60 *e*-folds of inflation; inflation likely lasted much more, however the earliest parts of it, which correspond to scales much larger than the horizon today, are hardly observable.

We can compute $\Delta\varphi$ as

$$\Delta\varphi = \int_{\varphi_{\text{CMB}}}^{\varphi_{\text{end}}} \mathrm{d}\varphi = \int_{t_{\text{CMB}}}^{t_{\text{end}}} \dot\varphi \,\mathrm{d}t \approx \frac{\dot\varphi}{H}\int_{Ht_{\text{CMB}}}^{Ht_{\text{end}}} \mathrm{d}(Ht) = \frac{\dot\varphi}{H} N_{\text{CMB}} \sim N_{\text{CMB}}\sqrt{\epsilon} M_{\text{Pl}}\,, \tag{1.2.71}$$

where $N_{\text{CMB}} \sim 60 \div 70$ is the number of *e*-folds in the sub-horizon part of inflation. We used the fact that $\ddot\varphi$ is negligible compared to $\dot\varphi / H^{-1}$, so $\dot\varphi$ can be taken to be constant; also, we used the first alternative expression for ϵ in equation (1.2.39).

If ϵ is of the order of $1/N_{\text{CMB}}$, as happens in large-field models, the excursion of the field becomes $\Delta\varphi \sim \sqrt{N_{\text{CMB}}} M_{\text{Pl}} \gtrsim M_{\text{Pl}}$.

Where does this $\epsilon \sim 1/N$ come from?

On the other hand, if we have $\epsilon \to 0$ we can have smaller $\Delta\varphi \lesssim M_{\text{Pl}}$.

Do we have to account for quantum gravity if $\Delta\varphi \gtrsim M_{\text{Pl}}$? No, since the condition required is actually $V \leq M_{\text{Pl}}^4$. However, a large (transPlanckian) excursion of the scalar field can constitute a problem, especially if we try to include these models in a "UV-complete" theory, so we must be careful.

We have often taken a De Sitter or quasi-De Sitter phase of expansion, but this is not necessarily the case: recall that

$$\ddot a = aH^2\left(1 + \frac{\dot H}{H^2}\right) = aH^2(1-\epsilon)\,, \tag{1.2.72}$$

so if $\dot H = 0$ (which defines De Sitter expansion) we do indeed have accelerated expansion, but $\dot H \neq 0$ does not prevent it, as long as $\left|\dot H\right| < H^2$ for negative $\dot H$, or even more easily with $\dot H > 0$ (which corresponds to $\ddot a > \dot a^2/a > 0$).

In our models we usually had

$$\dot H = -4\pi G \dot\varphi^2 < 0\,. \tag{1.2.73}$$

This is to say that inflation is not De Sitter, although it can be close to it in certain phases: for starters, it must end at a certain point. What kind of fluid could correspond to these other two solutions?

In general, for a spatially flat FLRW universe, we have

$$a(t) = a_* \left(1 + \frac{1}{\alpha} H_* (t - t_*)\right)^{\alpha} \qquad \alpha = \frac{2}{3(1+w)}, \tag{1.2.74}$$

with $w = \text{const}$. This is a good approximation, at least for the subhorizon phase of inflation. Accelerated expansion is achieved for $-1 < w < -1/3$: for these values the scale factor expands like a powerlaw, this is called *powerlaw inflation*.

The De Sitter case is one we know. The case in which $\dot{H} > 0$ is realized with $w < -1$; this goes like

$$a(t) \propto (t - t_{\text{asymptote}})^{-\alpha}. \tag{1.2.75}$$

This is called *pole inflation*: it is *superexponential*.

In the simple models we are considering we always have $\dot{H} < 0$, so this is not relevant.

We have always worked with equations like $H^2 = \frac{8\pi G}{3} V(\varphi)$, which starts us off with an *unperturbed* FLRW metric.

How can we be sure that the solution we find is indeed an attractor? It seems like we are requiring very specific initial conditions, not general at all!

We need the **cosmic no-hair principle** (sometimes called "theorem", although it is not one). This tells us that starting from very general initial conditions inflation moves us towards a flat FLRW metric. This is discussed in Kolb and Turner [KT94].

We do not have a full GR solution to describe an anisotropic inhomogeneous universe; we consider instead a homogeneous but anisotropic spacetime: a *Bianchi model*. The Bianchi classification distinguishes between different kinds of universes like these.

Bianchi class I universes expand at different rates in different directions:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_i a_i^2(t)\, \mathrm{d}{x_i}^2. \tag{1.2.76}$$

We can then define a volume $V = a_1 a_2 a_3$, and an average scale factor $\overline{a}(t) \propto V^{1/3}$: this yields an *averaged* Friedmann equation like

$$\overline{H}^2 = \left(\frac{\dot{\overline{a}}}{\overline{a}}\right)^2 = \frac{1}{9}\left(\frac{\dot{V}}{V}\right)^2 = \frac{8\pi G}{3}\left[\rho_\varphi + \rho_m + \rho_r\right] + F(a_1, a_2, a_3), \tag{1.2.77}$$

where the term F represents the dynamical effects of the anisotropic expansion of the universe. It can also include the effects of curvature, with a term like $-k/\overline{a}^2$.

Note that we could also write a full set of equation from the Einstein ones, to describe the evolution of the three anisotropic scale factors a_i: the point is that writing the averaged equation we did we can explicitly see the effects of the anisotropy on the mean scale factor. This allows us

Suppose we have a model of inflation with a scalar field φ and a potential $V(\varphi)$, which drives inflation if it is considered without the anisotropic term F.

The question is: if we account for F, does it **"destroy" inflation**? This is to say: under these more general anisotropic initial conditions, can inflation move us towards a flat FLRW universe?

The evolution of φ will be governed by the Klein-Gordon equation:

$$\ddot{\varphi} + 3\overline{H}\dot{\varphi} = -\frac{\partial V}{\partial \varphi} \,. \tag{1.2.78}$$

We can then show that usually there is no problem from the anisotropy. Let us start with the simplest model, in which we take $\rho_\varphi = \text{const}$, a cosmological constant.

Typically, $F \propto \overline{a}^\alpha$, where $\alpha \leq -2$. This then means that, if ρ_φ is constant, eventually ρ_φ will dominate. Everything else will be exponentially suppressed.

Under certain assumptions this can be proven formally, and is called Wald's Theorem.

There can be exceptions: for example, a closed universe with high Ω_k initially can collapse before inflation starts.

We, however, do not have a cosmological constant but a field: does φ roll to the minimum *before* inflation starts, if the anisotropic term F is present? It does not. The effect of F is to give an **additional contribution to** H, usually increasing it, which modifies the "friction" term, which allows for the slow-roll inflation to still occur.

What about completely general *inhomogeneous* as well as anisotropic spacetimes? Various analyses have shown (still, not as a mathematical theorem!) that also in this case typically we evolve towards a flat FLRW metric.

Monday 2020-10-26, compiled 2021-01-25

Our scalar field is given by

$$\varphi(\vec{x}, t) = \varphi_0(t) + \delta\varphi(\vec{x}, t) \,, \tag{1.2.79}$$

a background value, which we have studied up to now, plus some quantum fluctuations.

The full field obeys the KG equation:

$$\ddot{\varphi}(\vec{x}, t) + 3H\varphi(\vec{x}, t) - \frac{\nabla^2 \varphi(\vec{x}, t)}{a^2} = -\frac{\partial V}{\partial \varphi} \,, \tag{1.2.80}$$

which we can Taylor expand up to linear order around the background value both the field $\varphi \approx \varphi_0 + \delta\varphi$ and the potential $V(\varphi) \approx V_0 + \delta\varphi \left(\frac{\partial V}{\partial \varphi}\right)\Big|_{\varphi_0}$ if we assume that the perturbation is indeed small:

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \frac{\nabla^2}{a^2}\delta\varphi = -\frac{\partial^2 V(\varphi)}{\partial \varphi^2}\delta\varphi \,. \tag{1.2.81}$$

We want to see how the dynamics of $\delta\varphi$ affect the super-horizon scales. The first order equation for $\delta\varphi$ is coupled to the background one:

$$\ddot{\varphi}_0(t) + 3H\dot{\varphi}_0(t) = -\left.\frac{\partial V}{\partial \varphi}\right|_{\varphi_0} \,. \tag{1.2.82}$$

Let us differentiate it with respect to time, assuming that we are in a phase of De Sitter expansion with $\dot{H} \approx 0$:

$$\dddot{\varphi}_0 + 3H\ddot{\varphi}_0 = -\left.\frac{\partial^2 V}{\partial \varphi^2}\right|_{\varphi_0} \dot{\varphi}_0 \,, \tag{1.2.83}$$

which looks similar to the equation for $\delta\varphi$, except for the Laplacian.

We can neglect the Laplacian on superhorizon scales: for one, on large scales the variations are negligible. More formally, in Fourier space that term is $k^2\delta\varphi/a^2$, to be compared with things like $3H\dot{\delta\varphi} \sim 3H^2\delta\varphi$, since the characteristic time of the variations of the field is $1/H$.

Now, on superhorizon scales $k^{-1} \gg r_H$, which means that $k \ll aH = r_H^{-1}$, therefore $k^2/a^2 \ll H^2$. We can then neglect the Laplacian term.

What we should do now is to consider the Wronskian of the two-equation system:

$$W(x, y) = \dot{x}y - x\dot{y} \tag{1.2.84}$$

$$W(\dot{\varphi}_0, \delta\varphi) = \ddot{\varphi}_0\delta\varphi - \dot{\varphi}_0\dot{\delta\varphi}\,. \tag{1.2.85}$$

It can be shown that if the Hubble parameter is approximately constant then $\dot{W} = -3HW$, meaning $W \sim \exp(-3Ht)$, which quickly goes to zero.

If $W = 0$, then the two variables $\dot{\varphi}_0$ and $\delta\varphi$ are dependent, then we can decompose $\delta\varphi$ like

$$\delta\varphi(\vec{x}, t) = -\delta t(\vec{x})\dot{\varphi}_0(t)\,. \tag{1.2.86}$$

Therefore, the full field is given by reverse-Taylor expanding as

$$\varphi(\vec{x}, t) = \varphi_0(t) - \delta t(\vec{x})\dot{\varphi}_0(t) \tag{1.2.87}$$

$$\approx \varphi_0(t - \delta t(\vec{x}))\,. \tag{1.2.88}$$

This expression has a very clear physical interpretation: point by point, the field goes through the **same evolution**, taking on the same values, only at **different times** at each point.

This holds as long as we have a single scalar field, in multi-field models of inflation this will not be the case anymore.

What we want to compute is the final effect of these fluctuations on super-horizon scales. The equation for $\delta\varphi$ reads

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} - \frac{\nabla^2\delta\varphi}{a^2} = V''(\varphi_0)\delta\varphi\,. \tag{1.2.89}$$

We move to Fourier space:

$$\delta\varphi(\vec{x}, t) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, e^{i\vec{k}\cdot\vec{x}}\delta\varphi_{\vec{k}}(t)\,. \tag{1.2.90}$$

Since the field is real, $\delta\varphi_{\vec{k}} = \delta\varphi^*_{-\vec{k}}$. These Fourier modes evolve independently of each other up to linear order.

Why do we make a 3D Fourier transform and not a 4D one? We have time-dependent coefficients in the equation, such as H. We Fourier transform to take account of the spatial symmetry: it is not useful to transform in time since there is **no time translation symmetry**.

By transforming we are implicitly using a stationary wave basis, which is natural in a spatially flat universe: in a non-flat universe instead of $e^{i\vec{k}\cdot\vec{x}}$ we should use generalized solution of the Helmholtz equation

$$\nabla^2 Q_{\vec{k}} + k^2 Q_{\vec{k}} = 0\,, \tag{1.2.91}$$

the eigenvalue equation for the Laplacian operator. These are *time-independent* waves, while propagating waves $\exp\left(-ik_\mu x^\mu\right)$ would not suit our needs. In Fourier space the equation reads

$$\ddot{\delta\varphi}_{\vec{k}} + 3H\dot{\delta\varphi}_{\vec{k}} + \frac{k^2}{a^2}\delta\varphi_{\vec{k}} = -V''(\varphi_0)\delta\varphi_{\vec{k}}\,. \tag{1.2.92}$$

We will need to use the standard tools of second quantization: we start by introducing $\hat{\delta\varphi} = a\delta\varphi$, and we use conformal time, $a\,\mathrm{d}\tau = \mathrm{d}t$. Then, we will have

$$\hat{\delta\varphi}(\vec{x},\tau) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[u_k(\tau)a_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + u_k^*(\tau)a_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}}\right]. \tag{1.2.93}$$

Is it the expression for the hatted field?

The functions u are classical functions of time, while a and $a^\dagger$ are the annihilation operators. They are defined so that $a_{\vec{k}}\left|0\right\rangle = 0$ for any $\vec{k}$, and similarly $\left\langle 0\right| a_{\vec{k}}^\dagger = 0$.

Here, $\left|0\right\rangle$ is called the *free vacuum state*.

We impose the normalization condition

$$u_k^* u_k' - u_k u_k^{*,\prime} = -i\,, \tag{1.2.94}$$

which is equivalent to requiring

$$\left[a_{\vec{k}}, a_{\vec{k}'}^\dagger\right] = \hbar\delta^{(3)}(\vec{k}-\vec{k}')\,, \tag{1.2.95}$$

while $[a_k, a_{k'}] = \left[a_k^\dagger, a_{k'}^\dagger\right] = 0$.

In flat spacetime, once the commutation relations are fixed we are done: the solutions are known to be plane waves, with $u_{\vec{k}} \sim \exp(-i\omega_k t)/\sqrt{2\omega_k}$, and $\omega_k = \sqrt{k^2+m^2}$.

Now, instead, we need to make some assumptions. Because of the **equivalence principle**, for small distances and time intervals the solutions should reproduce the flat-spacetime ones. From this guiding principle, we require

$$\frac{k}{aH} \to \infty \implies u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\,, \tag{1.2.96}$$

which is called the *Bunch-Davies vacuum choice*.

Note that $aH \propto dH$

In the denominator we only have k since $k^2 \gg m^2$ for the small scales we are considering.

What is the equation for u? In conformal time ($' = \frac{\mathrm{d}}{\mathrm{d}\tau}$) it is

$$u_k''(\tau) + \left[k^2 - \frac{a''}{a} + a^2 \frac{\partial^2 V}{\partial \varphi^2}\right] u_k(\tau) = 0\,, \tag{1.2.97}$$

which is associated to the rescaled variable $\hat{\delta\varphi} = a\delta\varphi$. We can also see the reason why we have chosen to use the rescaled $\delta\varphi$ instead of the regular one: what we have found with this ansatz is basically a harmonic oscillator with a time-dependent frequency, changing according to the accelerated expansion of the universe. The ansatz yields a canonically-normalized kinetic term in the harmonic oscillator.

In order to solve the equation, we will assume de-Sitter expansion with $H =$ const, a massless scalar field with $m_\varphi^2 = \partial^2 V \big/ \partial\varphi^2 = 0$. Then, we know that $\mathrm{d}\tau = \mathrm{d}t\,/a = \mathrm{d}t\, e^{-Ht}$, so

$$\tau = -\frac{1}{H} e^{-Ht} = -\frac{1}{aH}\,. \tag{1.2.98}$$

Conventionally we choose the integration bounds so that τ runs from negative infinity to zero. This is just a matter of convention, a constant time shift has no effect on the dynamics. We also have, since $H = a'/a^2 \approx$ const, that $a'' = 2a'^2/a$, which tells us that

check

$$\frac{a''}{a} = \frac{2}{\tau^2}\,, \tag{1.2.99}$$

so if $\lambda_{\text{phys}} = a\lambda \ll H^{-1}$, therefore $\lambda \gg aH$. Therefore, being in the subhorizon scale means that

$$\frac{a''}{a} = 2a^2 H^2 \ll k^2\,, \tag{1.2.100}$$

therefore on **subhorizon** (microscopic) scales the equation reads

$$u_k'' + k^2 u_k = 0\,, \tag{1.2.101}$$

solved by

$$u_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\,. \tag{1.2.102}$$

As expected, we recover the regular flat-spacetime solution. However, what is really interesting to us are the cosmological, **superhorizon** solutions: this means $k \ll aH$, so

$$u_k''(\tau) - \frac{a''}{a} u_k(\tau) = 0\,, \tag{1.2.103}$$

which is a second-order equation: it will have two independent solutions, and it can be shown that the generic solution is written as

$$u_k(\tau) = \underbrace{B(k)a(\tau)}_{\text{growing mode}} + \underbrace{C(k)a^{-2}(\tau)}_{\text{decaying mode}}\,. \tag{1.2.104}$$

We neglect the decaying mode, since even if it is excited it will quickly decay. The physical fluctuation $\delta\varphi$ is proportional to $u_k/a \sim B(k)$, therefore we can see that the "growing mode" is actually asymptotically constant on superhorizon scales.

We then want to determine the scale of $B(k)$.

We need to match the subhorizon and superhorizon solutions, at the point $k = aH$.

$$\left|B(k)\right|a = \left|\frac{e^{-ik\tau}}{\sqrt{2k}}\right| \tag{1.2.105}$$

$$\left|\delta\varphi_k\right| = \left|B(k)\right| = \frac{1}{a\sqrt{2k}} = \frac{H}{\sqrt{2k^3}}, \tag{1.2.106}$$

recover last bit

The fluctuation gets "frozen in" at horizon crossing:

$$\left|\delta\varphi_k\right| = \frac{H}{\sqrt{2k^3}}. \tag{1.2.107}$$

This is called a *gravitational amplification mechanism*. This is analogous to pair production from vacuum under a strong electrostatic field. The electric field separates the newly-formed electron-positron pair.

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Today we will see how to explicitly solve the equation

$$u_k''(\tau) + \left[k^2 - \frac{a''}{a} + a^2m^2\right]u_k(\tau) = 0, \tag{1.2.108}$$

where τ is the conformal time, while $m^2 = \partial^2 V \big/ \partial\varphi^2$. The stage in which $m^2 = 0$ is the quasi de-Sitter stage. Then, $\epsilon = -\dot{H}/H^2 \ll 1$. We have

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon). \tag{1.2.109}$$

Integrating the conformal time definition (we need to also integrate by parts) we can find

$$\tau = -\frac{1}{aH(1-\epsilon)}. \tag{1.2.110}$$

Using the facts that $a''/a = a^2H^2(2-\epsilon) = (2/\tau^2)(1+(3/2)\epsilon)$ and $\epsilon = 2 - a''a/a'^2$, which can be proven from the definition of H and ϵ, the equation becomes

$$u_k''(\tau) + \left[k^2 - \frac{\nu^2 - 1/4}{\tau^2}\right]u_k(\tau) = 0, \tag{1.2.111}$$

where $\nu^2 = 9/4 + 3\epsilon$. This is a Bessel equation: these equations are generally in the form

$$z^2y''(z) + zy'(z) + (z^2 - \nu^2)y(z) = 0. \tag{1.2.112}$$

The solutions are called Hankel functions, and the general solution will read

$$u_k(\tau) = \sqrt{-\tau}\Big[c_1(k)H_\nu^{(1)}(-k\tau) + c_2(k)H_\nu^{(2)}(-k\tau)\Big]\,, \tag{1.2.113}$$

where $H_\nu^{(2)} = H_\nu^{(1),*}$. We want to impose the asymptotic behaviour of the solution: let us start with the sub-horizon case $k/aH \gg 1$. This means that

$$u_k(\tau) = \frac{1}{\sqrt{2k}}e^{-ik\tau}\,, \tag{1.2.114}$$

and the asymptotics of the Hankel functions are

$$H_\nu^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}\exp\left(i(x - \frac{\pi}{2}\nu - \frac{\pi}{4})\right) \tag{1.2.115}$$

for $x \gg 1$. This works well for us: we can set $c_2(k) = 0$ and only use $c_1(k)$.

We must choose

$$c_1(k) = \frac{\sqrt{\pi}}{2}\exp\left(i\left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right) \tag{1.2.116}$$

in order to have the correct normalization. Then, our solution will read

$$u_k(\tau) = \frac{\sqrt{\pi}}{2}\exp\left(i(\nu + \frac{1}{2})\frac{\pi}{2}\right)\sqrt{-\tau}H_\nu^{(1)}(-k\tau)\,, \tag{1.2.117}$$

so in the **superhorizon** $k/aH \ll 1$ (or $-k\tau \ll 1$) regime we have the asymptotic $x \ll 1$ expansion of the Hankel function:

$$H_\nu^{(1)}(x) \sim \sqrt{\frac{2}{\pi}}e^{-i\pi/2}2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}x^{-\nu}\,, \tag{1.2.118}$$

therefore

$$u_k(\tau) \approx \exp\left(i(\nu - \frac{1}{2})\frac{\pi}{2}\right)2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}(-k\tau)^{1/2-\nu}\frac{1}{\sqrt{2k}}\,, \tag{1.2.119}$$

so if we want the modulus $\left|\delta\varphi_k\right| = |u_k|/a$ we can make use of the fact that $\tau \sim -1/aH \propto 1/a$, therefore $\left|\delta\varphi_k\right| \approx -H\tau|u_k|$. Inserting the expression we have for u_k yields

$$\begin{aligned}\left|\delta\varphi_k\right| &\approx 2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{3/2-\nu} &(1.2.120)\\ &\approx \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{3/2-\nu}\,, &(1.2.121)\end{aligned}$$

Expanded to first order in ϵ, so that $3/2 - \nu \approx -\epsilon$.

which holds at super-horizon scales. In the De Sitter case, the **exact** result reads

$$u_k(\tau) \propto \sqrt{-\tau}H_{3/2}^{(1)}(-k\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right). \tag{1.2.122}$$

The next step is to generalize to a scalar field which is not massless anymore: we shall consider a mass $m^2 = \partial^2 V \Big/ \partial\varphi^2 \ll H^2$, so a "light" scalar field. We require this because the slow-roll parameter is $\eta_V = m^2/(3H^2) \ll 1$: a massive field (compared to H^2) would not be able to drive slow-roll inflation.

Then, the m^2a^2 term in the equation is nothing but

$$m^2a^2 = 3\eta_V a^2 H^2 = \frac{3\eta_V}{\tau^2}\,, \tag{1.2.123}$$

but we also know that

$$\frac{a''}{a} = \frac{2}{\tau^2}\left[1 + \frac{3}{2}\epsilon\right], \tag{1.2.124}$$

so we can still recast the equation into a Bessel form: only the coefficient of the τ^{-2} term changes.

The equation will read

$$u_k''(\tau) + \left[k^2 - \frac{\nu^2 - 1/4}{\tau^2}\right] u_k(\tau) = 0\,, \tag{1.2.125}$$

with $\nu^2 = 9/4 + 3\epsilon - 3\eta_V$. Then, the solution has exactly the same form as before: on super-horizon scales, with $k \ll aH$, we have

$$\left|\delta\varphi_k\right| = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{3/2-\nu} \qquad\qquad \nu \approx \frac{3}{2} + \epsilon - \eta_V\,. \tag{1.2.126}$$

Note that we did not require the field to be anything but light: the computation we have done can apply to an inflaton field but also to any other scalar field evolving in this phase of the expansion of the universe. If it is another scalar field, why should we require it to be light? It can be shown that if m^2 is **large** compared to H^2 the superhorizon-scale fluctuations of that field have a very hard time being excited: the field basically remains in its vacuum state.

The Klein-Gordon equation reads

$$\ddot{\varphi}(\vec{x},t) + 3H\dot{\varphi}(\vec{x},t) - \frac{\nabla^2\varphi(\vec{x},t)}{a^2} = -\frac{\partial V}{\partial\varphi}\,, \tag{1.2.127}$$

and we have obtained is starting from the expression

$$\Box\varphi = \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi\right), \tag{1.2.128}$$

where the metric is taken to be a FLRW one. We have neglected a crucial component: we should also **perturb the spacetime**, beyond the FLRW metric, if we want to have a consistent discussion.

This is important to do for the inflaton especially since φ dominates the energy density of the universe. The Einstein equations,

$$G_{\mu\nu} = 8\pi G T^{(\varphi)}_{\mu\nu} \,, \tag{1.2.129}$$

are perturbed with a variation in the field, $\delta\varphi$, which perturbs the energy-momentum tensor $\delta T^{(\varphi)}_{\mu\nu}$, which means that we must also perturb the Einstein tensor $\delta G_{\mu\nu}$ and then finally the metric $\delta g_{\mu\nu}$.

So, we find an equation

$$u''_k(\tau) + \left[k^2 - \frac{a''}{a} + M^2 a^2\right] u_k(\tau) = 0 \,. \tag{1.2.130}$$

If we account for metric perturbations, we get

$$\frac{M^2}{H^2} \approx 3\eta_V - 6\epsilon \,. \tag{1.2.131}$$

Then, we get

$$\nu^2 = \frac{9}{4} + 9\epsilon - 3\eta_V \,. \tag{1.2.132}$$

Just like before we can calculate

$$\left|\delta\varphi_k\right| = \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{3/2-\nu} \,. \tag{1.2.133}$$

Then we find that in order to calculate Δu_k we can use the Sasaki-Mukanov equation:

$$Q_\varphi = \delta\varphi + \frac{\varphi}{H}\hat{\Phi} \,, \tag{1.2.134}$$

where $\hat{\Phi} = \Phi + (1/6)\nabla^2\chi^\parallel$, where Φ and $\chi^\parallel$ are scalar perturbations of g_{ij}.

What is Q?

We want to show how primordial GW are generated from the inflaton perturbations. We consider a tensor-perturbed FLRW metric

$$\mathrm{d}s^2 = a^2(\tau)\left[-\,\mathrm{d}\tau^2 + \left(\delta_{ij} + h_{ij}(\vec{x}, \tau)\right) \mathrm{d}x^i \, \mathrm{d}x^j\right] . \tag{1.2.135}$$

We are neglecting all scalar and vector perturbations for simplicity. The symmetric tensor h_{ij} can be chosen to be traceless: $h^i_i = \partial^{\mathrm{BG}}_i h^{ij} = 0$, where the derivative is that constructed from the background unperturbed spacetime. This is basically the TT-gauge, but in a cosmological context the conditions come about naturally.

The equations of motion read

$$h''_{ij} + 2\frac{a'}{a} - \nabla^2 h_{ij} = 0 \,, \tag{1.2.136}$$

where a prime denotes a derivative with respect to the conformal time τ. If we wanted to use cosmic time instead, the equation would read

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2 h_{ij}}{a^2} = 0\,. \tag{1.2.137}$$

We can see that this is the same as $\Box^{\mathrm{BG}} h_{ij} = 0$. The equation is the same as a **massless, minimally coupled scalar field**. We already know the solution for these equations: the mechanism of quantum vacuum amplification still works. We have a 0 on the right-hand side since we are working up to linear order, in full generality there will be other source terms, commonly denoted as π^T_{ij} (T for "source of tensor modes"). This term describes the tensor component of the anisotropic stresses, from the quadrupole up.

Measuring these primordial GWs would be the first probe of the quantum nature of gravity.

GWs start out with 6 degrees of freedom, but 4 are gauged away, so we are left with only 2. We can then expand h_{ij} in a plane wave basis:

$$h_{ij}(\vec{x}, \tau) = \int \sum_{\lambda=+,\times} \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{ik\cdot x} h_\lambda(\vec{k}, \tau) \epsilon^\lambda_{ij}(\vec{k})\,, \tag{1.2.138}$$

so that in Fourier space the equations of motion for the coefficients read

$$h''_\lambda + 2\frac{a'}{a} h'_\lambda + k^2 h_\lambda = 0\,. \tag{1.2.139}$$

This is exactly the same as the equations of motion for the scalar massless minimally coupled field. We can then see that if $k \gg aH$ we get

$$h_\lambda \sim \frac{e^{-ik\tau}}{a}\,, \tag{1.2.140}$$

while for $k \ll aH$ the gravitational perturbation has a constant mode and a decaying mode. We can treat the two coefficients like $h_{+,\times} = \phi_{+,\times}\sqrt{32\pi G}$, so that ϕ has the dimension of a mass while h is dimensionless. Then, like we saw before in the superhorizon regime $k \ll aH$ we get

The normalization comes from the action.

$$|h_{+,\times}| = \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{-\epsilon} \sqrt{32\pi G}\,. \tag{1.2.141}$$

We can distinguish GWs from inflation from those produced by single astrophysical events: inflation yields a *stochastic background* of GWs. These GWs are a crucial target of future experiments, both in interferometry and in CMB maps.

We need to introduce the power spectrum of perturbations: the two-point correlation function, in real space, is given in terms of a generic stochastic perturbation field $\delta(\vec{x})$: we can calculate

$$\left\langle \delta(\vec{x} + \vec{r})\delta(\vec{x}) \right\rangle = \xi(r)\,. \tag{1.2.142}$$

In Fourier space, we define the power spectrum as the Fourier transform of the two-point correlation function: the field is written as

$$\delta(\vec{x}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3k \, e^{i\vec{k}\cdot\vec{x}} \delta(\vec{k}) \,, \tag{1.2.143}$$

and the power spectrum $P(k)$ is defined by

$$\left\langle \delta(\vec{k})\delta(\vec{k}') \right\rangle = (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}') P(k) \,. \tag{1.2.144}$$

The plus is there since we wrote $\delta(k)\delta(k')$ instead of $\delta(k)\delta^*(k')$: if we included the conjugate than the argument of the δ would be $k-k'$, due to the fact that since $\delta(x)$ is real we have $\delta^*(k) = \delta(-k)$.

We have

$$\sigma^2 = \left\langle \delta^2(x) \right\rangle = \frac{1}{2\pi^2} \int \mathrm{d}k \, k^2 P(k) \tag{1.2.145}$$

$$= \frac{1}{2\pi^2} \int \frac{\mathrm{d}k}{k} k^3 P(k) \tag{1.2.146}$$

$$= \int \mathrm{d} \log k \, \Delta_\delta^2(k) \,, \tag{1.2.147}$$

where we define the dimensionless power spectrum:

$$\Delta_\delta^2(k) = \frac{k^3}{2\pi^2} P(k) \,. \tag{1.2.148}$$

We can show that $P(k)$ is indeed the Fourier transform of $\xi(r)$:

$$\left\langle \delta(x+r)\delta(x) \right\rangle = \frac{1}{(2\pi)^6} \int \mathrm{d}^3k_1 \, e^{ik_1(x+r)} \int \mathrm{d}^3k_2 \, e^{ik_2 x} \left\langle \delta(k_1)\delta(k_2) \right\rangle \tag{1.2.149}$$

$$= \frac{1}{(2\pi)^3} \int \mathrm{d}^3k_1 \, \mathrm{d}^3k_2 \, \delta^{(3)}(k_1+k_2) P(k_1) e^{ik_1(x+r)+ik_2 x} \tag{1.2.150}$$

$$= \frac{1}{(2\pi)^3} \int \mathrm{d}^3k_1 \, e^{ik_1 r} P(k_1) \,, \tag{1.2.151}$$

as we wanted to show. In the $r = 0$ case we recover the expression from above for the variance.

Let us continue with our discussion of the power spectrum: the **spectral index** n_s is defined as[8]

$$n_s - 1 = \frac{\mathrm{d} \log \Delta(k)}{\mathrm{d} \log k} \,. \tag{1.2.152}$$

The index s means "scalar". In general this will depend on the wavenumber k; it is a convenient description of the shape of the power spectrum.

[8] Note that the dimensionless power spectrum is sometimes denoted as Δ^2 and sometimes as Δ: here (in this section) we use the latter definition.

If n_s were a constant, then we would have a *powerlaw* spectrum: $\Delta(k) = \Delta(k_0)(k/k_0)^{n_s-1}$. If $n_s = 1$, we have the **Harrison-Zel'dovich** power spectrum, for which Δ does not depend on k. This would be a *scale-invariant* power spectrum.

In a quantum-mechanical formalism, we will calculate the power spectrum as

$$\langle 0| \delta\varphi_{\vec{k}_1} \delta\varphi_{\vec{k}_2} |0\rangle \ , \tag{1.2.153}$$

which will be written in terms of creation and annihilation operators: we have

$$\langle 0| aa |0\rangle = \langle 0| a^\dagger a |0\rangle = \langle 0| a^\dagger a^\dagger |0\rangle = 0 \, , \tag{1.2.154}$$

while

$$\langle 0| aa^\dagger |0\rangle = \underbrace{\langle 0| \left[a, a^\dagger\right] |0\rangle}_{\delta^{(3)}(\vec{k}_1 - \vec{k}_2)} - \underbrace{\langle 0| a^\dagger a |0\rangle}_{=0} , \tag{1.2.155}$$

so

$$\left\langle \delta\varphi_{\vec{k}_1} \delta\varphi_{\vec{k}_2} \right\rangle = (2\pi)^3 \left|\delta\varphi_{\vec{k}_1}\right|^2 \delta^3(\vec{k}_1 - \vec{k}_2) \, , \tag{1.2.156}$$

where $\delta\varphi_{kl} = u_{kl}/a$. Therefore, the power spectrum is given by

$$P(k) = \left|\delta\varphi_k\right|^2 . \tag{1.2.157}$$

Recall that in the superhorizon case we found

$$\left|\delta\varphi_k\right| = \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{3/2-\nu} , \tag{1.2.158}$$

where $\nu^2 = 9/4 + 9\epsilon - 3\eta_V$, so $\nu \approx 3/2 + 3\epsilon - \eta_V$, meaning that the index is $3/2 - \nu = \eta_V - 3\epsilon$.

Then, the power spectrum reads

$$\Delta_{\delta\varphi}(k) = \frac{k^3}{2\pi^2} \left|\delta\varphi_k\right|^2 = \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu} . \tag{1.2.159}$$

There will be a weak scale dependence proportional to the slow-roll parameters: $3 - 2\nu = 2\eta_V - 6\epsilon$.

1.3 From $\delta\varphi$ to primordial density perturbations

The first Friedmann equation will read

$$H^2 = \frac{8\pi G}{3}\rho_\varphi \approx \frac{8\pi G}{3} V(\varphi) \, , \tag{1.3.1}$$

so the density fluctuation can be written as

See equation (1.2.38)

$$\delta\rho_\varphi \approx V'(\varphi)\delta\varphi \approx -3H\dot{\varphi}\delta\varphi \, . \tag{1.3.2}$$

Recall that we can define a time shift $\delta t = -\delta\varphi/\dot{\varphi}$. This means that we will have perturbations in the expansion of the universe from place to place.

The number of e-folds is given by

$$N = \log\left(\frac{a(t)}{a(t_*)}\right) = \int_{t_*}^{t} H(\widetilde{t})\,\mathrm{d}\widetilde{t}\,. \tag{1.3.3}$$

The fluctuations will perturb the number of e-folds by

$$\zeta = \delta N = H\delta t = -H\frac{\delta\varphi}{\dot{\varphi}} \approx -H\frac{\delta\rho_\varphi}{\dot{\rho}_\varphi}\,. \tag{1.3.4}$$

This is called the "δN formalism" for the study of large-scale perturbations. The last equality in (1.3.4) comes from the fact that

$$\dot{\rho}_\varphi = -3H\Big(\rho_\varphi + P_\varphi\Big) = -3H\dot{\varphi}^2\,, \tag{1.3.5}$$

From equation (1.2.24).

so indeed

$$H\frac{\delta\rho_\varphi}{\dot{\rho}_\varphi} = \frac{-3H^2\varphi\delta\varphi}{-3H\dot{\varphi}^2} = H\frac{\delta\varphi}{\dot{\varphi}}\,. \tag{1.3.6}$$

The quantity $\delta N = \zeta$ is **gauge invariant**. It is written as

$$\zeta = -\hat{\Phi} - H\frac{\delta\rho}{\dot{\rho}}\,, \tag{1.3.7}$$

where $\hat{\Phi}$ is related to scalar perturbations of the spatial part of the metric, g_{ij}. We shall explore this later.

This ζ is called the **curvature perturbation on uniform energy density hypersurfaces**. Why did we write this with ρ instead of ρ_φ? This definition is completely general; it can be applied at any time and for a generic evolution of the universe. We can then specify it to

$$\zeta_\varphi \approx -H\frac{\delta\rho_\varphi}{\dot{\rho}_\varphi}\,. \tag{1.3.8}$$

What we will show is that on superhorizon scales ζ remains constant (for single-field inflation, at least). Therefore, this keeps a sort of "record" of what happened after horizon crossing.

Let us denote as $t_H^{(1)}(k)$ the time of horizon crossing during inflation for perturbations with wavenumber k, and $t_H^{(2)}(k)$ the time *after inflation* of the second horizon crossing, when the perturbation comes back inside the horizon. The value of ζ at these two times will be the same.

This can be seen since...

We know that $\delta\varphi \sim H/2\pi$ (statistically), and $H^2 \approx 8\pi G V(\varphi)/3$: then, specifying the potential gives a prediction for the power spectrum.

Suppose that the perturbation re-enters during the radiation-dominated epoch. Then,

$$H\frac{\delta\rho}{\dot\rho} \approx \frac{H\delta\rho_\gamma}{-4H\rho_\gamma} = -\frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}, \tag{1.3.9}$$

since

$$\dot\rho = \frac{\mathrm{d}\rho}{\mathrm{d}a}\dot a = \frac{\rho}{a}\frac{\mathrm{d}\log\rho}{\mathrm{d}\log a}\dot a = \frac{\rho}{a}(-4)\dot a = -4\rho H\,. \tag{1.3.10}$$

So, the dimensionless power spectrum is

$$\Delta_{\delta\rho/\rho}(k) = \frac{H^2}{\dot\varphi^2}\Delta_{\delta\varphi}(k)\bigg|_{t_H^{(1)}(k)}, \tag{1.3.11}$$

and

$$\Delta_{\delta\varphi}(k) = \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}, \tag{1.3.12}$$

so

Recover a few minutes

Wednesday 2020-11-4, compiled 2021-01-25

Last lecture we found

$$\Delta_{\delta\rho/r}(k) = \left(\frac{H^2}{2\pi\dot\varphi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}, \tag{1.3.13}$$

where $3-2\nu = 2\eta_V - 6\epsilon = n_s - 1$. The last equality comes from the definition

$$n_s - 1 = \frac{\mathrm{d}\log\Delta}{\mathrm{d}\log k}\,. \tag{1.3.14}$$

A power spectrum with $n_s = 1$ is called a Harrison-Zel'dovich spectrum, and it is scale-invariant. If $n_s > 1$ the spectrum is "blue", while if $n_s < 1$ the spectrum is "red" (or, respectively, red or blue "tilted"). In the two cases, we have either more energy at longer or shorter wavelengths.

This deviation from $n_s = 1$ is a specific prediction of single-field models. If we had $n_s = 1$ exactly, we would expect there to be some symmetry which prevents $n_s \neq 1$. The detection of a spectral index $n_s \neq 1$ is a good indication of an inflation-like process.

From CMB data [Col+19, eq. 21] we have $n_s = 0.9649 \pm 0.0042$ at a 68 % CL.

Recall the definition of ζ: for $k \ll aH$ it is constant, and it is given by

$$\zeta = \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}, \tag{1.3.15}$$

but we also know that $\rho_\gamma \propto T^4$, so $\zeta \sim \delta T/T$.

1.3.1 Power spectrum of primordial gravitational waves

The two polarization states evolve like two scalar fields:

$$h_{+,\times} = \sqrt{32\pi G}\phi_{+,\times}\,, \tag{1.3.16}$$

so

$$\Delta_{h_{+,\times}}(k) = 32\pi G \Delta_{\phi_{+,\times}}\,, \tag{1.3.17}$$

and we know that for a massless scalar field ϕ

$$\Delta_\phi = \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{-2\epsilon}\,, \tag{1.3.18}$$

See equation (1.2.159).

therefore

$$\Delta_{h_{+,\times}}(k) = \frac{32\pi}{M_P^2}\left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{-2\epsilon} \tag{1.3.19}$$

$$= \frac{8}{\pi M_P^2} H^2 \left(\frac{k}{aH}\right)^{-2\epsilon}\,. \tag{1.3.20}$$

Once again, we find a powerlaw as a function of k. The amplitude is of the order H^2/M_P^2, but H is determined by the vacuum energy density of the inflaton, so

$$\frac{H^2}{M_P^2} = \frac{8\pi}{3}\frac{V(\varphi)}{M_P^4}\,. \tag{1.3.21}$$

If we define the characteristic energy of inflation as $E_{\text{inf}} = V^{1/4}$ (recall that V is an energy density, with dimensions of a mass to the fourth power) we find that the amplitude of the GW spectrum is of the order E_{inf}/M_P. Measuring the primordial SGWB amplitude gives us directly the energy scale of inflation.

The "tensor spectral index" is defined as

$$n_T = \frac{\mathrm{d}\log \Delta_T}{\mathrm{d}\log k} = -2\epsilon\,, \tag{1.3.22}$$

where Δ_T is the total tensor amplitude: we just by 2 to account for the two polarizations,

$$\Delta_T = \frac{16}{\pi M_P^2} H^2 \left(\frac{k}{aH}\right)^{-2\epsilon}\,. \tag{1.3.23}$$

In single-field models $\epsilon > 0$ always, therefore n_T will always be red-tilted.

We can also write

$$\Delta_{h_{+,\times}} = \frac{8}{M_P^2 \pi} H_*^2\,, \tag{1.3.24}$$

where the star means we calculate the Hubble rate at the time of horizon crossing during inflation.

This is because the fluctuation remains constant when it is on superhorizon scales: so, its value is set by whatever it is at horizon crossing, $k = aH$. The term k/aH then simplifies, and we find the aforementioned expression.

We need to define a few observationally relevant quantities. The **tensor-to-scalar ratio** r is defined as

$$r = \frac{\Delta_T}{\Delta_\zeta} = \frac{\frac{16}{M_P^2 \pi} H_*^2}{\Delta_\zeta} \,. \tag{1.3.25}$$

This measures the level of primordial GWs compared to the amount of primordial scalar perturbations. From CMB temperature anisotropies we can measure ζ, since at large scales $\zeta \sim \delta T/T \sim 10^{-5}$. Therefore, $\Delta_\zeta \sim \zeta^2 \sim 10^{-10}$ (this is known to a better precision, we just give an order of magnitude here).

The energy scale of inflation then can be written as

$$E_{\text{inf}} \approx 10^{16}\,\text{GeV} \left(\frac{r}{0.01} \right)^{1/4} \,. \tag{1.3.26}$$

Measuring r can allow us to probe extremely high energy scales. The energy scale is reminiscent of Grand Unification Theories. In modern models the connection to GUT theories is less strong.

The latest measurements, using B-mode polarization of CMB photons, which is characteristic of tensor perturbations, give a bound $r < 0.044$ at a 95 % CL. This number also includes B-mode data. In the future we hope to be able to measure $r \sim 10^{-2} \div 10^{-3}$.

The overall amplitude of tensor perturbations is

$$\Delta_T = \frac{16}{\pi M_P^2} H_*^2 \,, \tag{1.3.27}$$

so

$$\Delta_\zeta = \frac{H^2}{4\pi^2} \frac{H^2}{\dot{\varphi}^2} \,, \tag{1.3.28}$$

See equation (1.3.13).

and $\dot{H} = -4\pi G \dot{\varphi}^2$ (see (1.2.73)).

Check: this is an exact expression, we just differentiate $H^2 = \frac{8\pi G}{3} \left(\dot{\varphi}/2 + V(\varphi) \right)$ and the use the equation of motion.

This means that

$$\epsilon = -\frac{\dot{H}}{H^2} = +4\pi G \frac{\dot{\varphi}^2}{H^2} \,. \tag{1.3.29}$$

So,

$$\frac{H^2}{\dot{\varphi}^2} = \frac{4\pi}{M_P^2} \frac{1}{\epsilon} \implies \Delta_\zeta = \frac{H^2}{4\pi} \frac{4\pi}{M_P^2} \frac{1}{\epsilon} = \frac{H^2}{\pi M_P^2} \frac{1}{\epsilon} \,. \tag{1.3.30}$$

This allows us to express the tensor-to-scalar ratio as

$$r = \frac{\Delta_T}{\Delta_\zeta} = 16\epsilon \,. \tag{1.3.31}$$

This gives us the *consistency relation* for single-field models: $n_T = -2\epsilon$, therefore $r = -8n_T$. It relates two observable variables: it can be used as a check. This is why it is called the "Holy Grail of cosmology". If we saw it holds, we would be rather sure that inflation actually occurred.

This is extremely difficult to measure, it would be very hard to see even r, n_T typically will have much larger errorbars.

1.3.2 Base observational predictions

We predict a scalar power spectrum

$$\Delta_\zeta(k) = \Delta_\zeta(k_0)\left(\frac{k}{k_0}\right)^{n_s-1}, \tag{1.3.32}$$

where k_0 is the *pivot scale*. In the Planck data release [Col+19] the value of r is reported as $r_{0.002}$, where the index corresponds to the pivot scale $k_0 = 0.002\,\mathrm{Mpc}^{-1}$. The spectral index as usual is $n_s - 1 = 2\eta_V - \epsilon$.

Also,

$$\Delta_T(k) = \Delta_T(k_0)\left(\frac{k}{k_0}\right)^{n_T}, \tag{1.3.33}$$

where $n_T = -2\epsilon$. At the pivot scale we have

$$\Delta_\zeta(k_0) = \left.\frac{H^2}{\pi M_P^2 \epsilon}\right|_{k_0} \tag{1.3.34}$$

$$\Delta_T(k_0) = \left.\frac{16H^2}{\pi M_P^2}\right|_{k_0} . \tag{1.3.35}$$

The value of H is determined by the potential $V(\varphi)$, while the value of ϵ depends on the derivative $V'(\varphi)$, and the value of η_V depends on the second derivative $V''(\varphi)$: recall

$$H^2 = \frac{8\pi G}{3} V(\varphi) \tag{1.3.36}$$

$$\epsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \tag{1.3.37}$$

$$\eta_V = \frac{1}{8\pi G}\left(\frac{V''}{V}\right) . \tag{1.3.38}$$

If we are able to constrain the two (scalar and tensor) amplitudes and spectral indices we can reconstruct the inflationary potential. We can reduce this four-parameter space: on the largest angular scales, $\Delta T/T \sim \Delta_\zeta + \Delta_T$. This is very imprecise, there are transfer functions involved, however the basic idea is that there are contributions from both terms.

But $\Delta T/T$ is well-known, so instead of the two amplitudes we can just measure $r = \Delta_T/\Delta_\zeta$. If we assume that the consistency relation holds, we have $r = -8n_T$ which reduces the parameter space to two parameters only: (r, n_s). This is commonly used in order to visualize inflationary models, since we can plot contours in a plane. There, we can show observational constraints as in the lower row of figure 26 in the Planck 2018 release [Col+19].

Large-field models have $0 < \eta_V < 2\epsilon$, while small field models have $\eta_V < 0$. Generally, small-field models predict low levels of GWs: we can express

$$n_s - 1 = 2\eta_V - 6\epsilon = 2\eta_V - \frac{3}{8}r\,, \tag{1.3.39}$$

so we can express $r = r(n_s, \eta_V)$. For more details see Kinney, Melchiorri, and Riotto [KMR00]. Small field models have $V'' < 0$, large field models have $V'' > 0$.

Some examples of inflationary models

Large field models could be $V \propto \phi^p$ (a chaotic inflation scenario) or $V \propto \exp(\phi/\mu)$ (powerlaw inflation).[9] These have $0 < \eta_V < 2\epsilon$.

Small field models could be $V \propto 1 - (\phi/\mu)^p$ (from SSB of, say, axion models) or $V \propto 1 + (\phi/\mu)^p$ (from SUSY, often involving a second field). These have $\eta_V < 0$.

There are also hybrid models, mixing small-field and large-field characteristics. These are among the earlier examples of two-field models: ϕ starts off similarly to large-field models, and moves towards a minimum with a nonzero VEV, provided by a second field ψ, which is stuck to the minimum of its potential.

This second field is necessary for the model but it is not dynamical in this first phase, however it becomes unstable at the end of inflation, triggering it. These models have $\eta_V > 2\epsilon$.

We can have these kinds of models arising from adding an R^2 term to the Einstein-Hilbert action. This corresponds to a new degree of freedom, with a transformation $g_{\mu\nu} \to e^{-2\omega} g_{\mu\nu}$ and a field $\phi \propto \omega$. The potential is in the form

$$V \propto \left(1 - e^{-2\omega}\right)^2. \tag{1.3.40}$$

We also have *natural inflation*, with $V \propto 1 - \cos(\phi/\mu)$: this is related to a shift symmetry $\phi \to \phi + c$. If this symmetry were exact we would have $V = \text{const}$, but if instead it is broken we have the aforementioned potential. These are axion-like models: the shift symmetry arises naturally if ϕ is a phase, corresponding to a $U(1)$ symmetry like the Peccei-Quinn one. The inflaton would not be the axion itself, but they would have similar properties.

1.4 An overview of early inflationary models

Monday 2020-11-9, compiled 2021-01-25

The main reason to study these despite their issues is that they informed the modern viewpoint on inflation, which we will understand better after studying them.

[9] Note that inflation cannot be exactly De Sitter! It would never end.

1.4.1 Old inflation

This is a model first proposed by Guth in 1981. The idea was to directly link inflationary dynamics to some high temperature SSB phase transition.

We consider the usual SSB potential

$$V(\varphi) = \frac{\lambda}{4}\left(\varphi^2 - \sigma^2\right)^2, \tag{1.4.1}$$

for a real scalar field φ: the symmetry $\varphi \to -\varphi$ is spontaneously broken in the vacuum for positive σ^2. Accounting for one-loop quantum corrections at a finite temperature T, we find the effective potential

$$V(\varphi, T) = V(\varphi) + \alpha\varphi^2 T^2 + \gamma(\varphi^2)^{3/2} T + \beta T^4 . \tag{1.4.2}$$

The potential is written in a funny way in order for it to still be explicitly symmetric under $\varphi \to -\varphi$. At high temperatures these corrections will dominate over $V(\varphi)$; at low temperatures the most important correction will be the quadratic one in T.

As the temperature decreases, beyond the minimum at $\varphi = 0$ we get another one at $\varphi = \pm\varphi_c$. This minimum gets lower and lower, and at $T = T_c$ it goes below $\varphi(0)$, becoming the true vacuum.

Insert figure(s)

The vacuum at $\varphi = 0$ is called the "false vacuum", while the one at $\varphi = \pm\sigma$ is called the "true vacuum".

Starting at a high temperature, we need to consider the energy density of radiation, which we know is given by

$$\rho_r(T) = \frac{\pi^2}{30} g_*(T) T^4 , \tag{1.4.3}$$

together with the energy density of the scalar field, which is initially trapped at $\varphi = 0$, so that $\rho_0 = \lambda\sigma^4/4$. Then, the first Friedmann equation will read

$$H^2 = \frac{8\pi G}{3}(\rho_0 + \rho_r) , \tag{1.4.4}$$

ant the contribution ρ_0, which acts like an effective cosmological constant, will eventually come to dominate as the temperature decreases: when $\rho_0 > \rho_r$ inflation starts its De Sitter phase. As this begins, the energy density of radiation will decrease exponentially.

When the potential barrier becomes low enough, the field φ can undergo quantum tunneling, moving to the true vacuum. This kind of phase transition is called a *first order* phase transition, as opposed to *second order* ones.[10]

These models have the so-called **graceful exit problem**. The transition to the true vacuum will generally take place in bubbles, which are moving in an exponentially-expanding

[10] Roughly speaking, a first-order transition involves a discontinuity in the energy, while in the second-order case it is merely nondifferentiable, while remaining continuous.

false-vacuum background. So, they may never meet, thus being unable to explain observations.

Also, after inflation we need reheating: this may be explained by the latent heat of the field having fallen in the true vacuum, $\Delta V = V(0) - V(\sigma)$. Still, though, this heat is trapped in the bubbles.

1.4.2 New inflation

These problems were solved by new inflation, proposed by Linde in 1982 and by Albrecht and Steinhardt in the same year. Here, the transition becomes a **second order** one: there is no potential barrier.

During the phase transition, as the derivative $V''(\varphi = 0)$ changes sign the field can start *slow-rolling* towards its minimum.

Since there is no barrier, there is no nucleation of bubbles either. What happens instead is described by **spinodal decomposition**: basically, there are different SSB in different domains which then inflate to encompass the universe.

What are the **problems** of new inflation? We would like to ask for $\left\langle \delta\varphi^2 \right\rangle \ll \varphi_0^2$: the fluctuations of the scalar field should be small, in order to be treated perturbatively. This is hard to ensure. This is reflected in the scale of density perturbations, $\delta\rho/\rho$.

For these models of inflation we find $\delta\rho/\rho \sim \mathcal{O}\left(\lambda^{1/2}\right)$, where λ is the coupling constant between the inflaton field and the thermal bath.

Typically people considered values for λ between 0.1 and 1, which means (also accounting for the constant in front) we would get anisotropies of the order of 100: way too large, since from CMB anisotropies we know that $\delta\rho/\rho \sim 10^{-5}$. In order to match this observational value we would need $\lambda \sim 10^{-10}$, however this is not compatible with us being sure to have thermal equilibrium between the inflaton reheating and its thermal bath.

Thermal equilibrium and sufficiently small density perturbations seem to be in conflict with each other.

This is the reason why **chaotic inflationary models** (Linde 1983): disconnecting inflation from phase transitions. Suppose we have a flat potential $V(\varphi) \equiv V_0$. This would lead to unending De Sitter inflation with $-\infty < \varphi < \infty$; so instead of it we consider a roughly flat potential $V(\varphi) = (\lambda/4)\varphi^4$, with $\lambda \ll 1$.

The constraint we need to require is that $V(\varphi) \lesssim M_P^4$, which ensures that quantum gravity corrections are not needed. Then, we must constrain φ to lie between plus and minus $M_P/\lambda^{1/4}$. But if $\lambda \ll 1$, φ will be able to attain initial values larger than the Planck mass: this allows for slow-roll inflation.

These models are called chaotic because the values of φ are taken to be randomly distributed, and in some regions inflation will not take place. At the Planck time $t = t_P$ this is required: there, $\Delta E \Delta T \gtrsim 1$, so the potential $V(\varphi)$ must have an uncertainty larger or equal than M_P.

Now having $\lambda \sim 10^{-10}$ is not an issue: the inflaton is not coupled, and we are fine with this.

The high temperature corrections to this potential are negligible.

As an example we consider the **Coleman-Weinberg potential**:

$$V(\varphi) = \frac{B\sigma^4}{2} + B\varphi^4 \left[\log\left(\frac{\varphi^2}{\sigma^2} \right) - \frac{1}{2} \right] , \tag{1.4.5}$$

a typical potential which can characterize symmetry breaking. The peculiarity of this potential is the logarithm: it comes from the one-loop corrections for the interactions between the field and other particles.

This has a minimum at $\varphi = \sigma$, while near the origin it is quite flat: we can approximate it as

$$V(\varphi) \approx \frac{B\sigma^4}{2} - \lambda \frac{\varphi^4}{4} , \tag{1.4.6}$$

where $\lambda = \left| 4B \log\left(\varphi^2/\sigma^2 \right) \right| \approx (10 \div 100)B$.

Very close to the origin, we will have $V(\varphi) \approx B\sigma^4/2 = \text{const}$, so

$$H^2 \approx \frac{8\pi G}{3} V(\varphi) \approx \frac{4\pi G}{3} B\sigma^4 . \tag{1.4.7}$$

In the original model of new inflation, this was considered with $B \approx 10^{-3}$, and $B = \frac{25}{16}\alpha_{\text{GUT}}^2$, and $\alpha_{\text{GUT}} = g_{\text{GUT}}^2/4\pi$.

Since it pertained to a GUT, we also had $\sigma \sim T_C \sim 10^{15}\,\text{GeV}$, therefore $H^2 \approx \left(10^{10}\,\text{GeV}\right)^2$.

why?

Do these kinds of models actually work? The first question is: can they solve the horizon problem? The number of e-folds is given by

$$N = \int_{t_i}^{t_F} H \,\mathrm{d}t = \int_{\varphi_i}^{\varphi_f} \frac{H}{\dot{\varphi}} \,\mathrm{d}\varphi , \tag{1.4.8}$$

and recalling $3H\dot{\varphi} \approx V'(\varphi)$, plus $H^2 = 8\pi G V(\varphi)/3$, we find

$$N = -8\pi G \int_{\varphi_i}^{\varphi_f} \frac{V(\varphi)}{V'(\varphi)} \,\mathrm{d}\varphi . \tag{1.4.9}$$

Using the near-origin approximation, we get $V'(\varphi) \approx -\lambda\varphi^3$, so

$$N \approx -3H^2 \int_{\varphi_i}^{\varphi_f} \frac{\mathrm{d}\varphi}{-\lambda\varphi^3} \tag{1.4.10}$$

$$\approx 3\lambda^{-1} H^2 \int_{\varphi_i}^{\varphi_f} \frac{\mathrm{d}\varphi}{\varphi^3} = \frac{3}{2}\frac{H^2}{\lambda} \left(\frac{1}{\varphi_i^2} - \frac{1}{\varphi_f^2} \right) . \tag{1.4.11}$$

When does inflation end? We know that slow-roll holds as long as $\left|V''\right| < H^2$, so we can say that φ_f is determined by $\left|V''(\varphi_f)\right| \sim 10H^2$, and we know that $V''(\varphi) = -3\lambda\varphi^2$: therefore, $\varphi_f = -3^{1/2} H/\lambda^{1/2}$.

If $B \approx 10^{-3}$, this means

$$\varphi_f^2 \approx \frac{3H^2}{\lambda} \approx (30 \div 300) H^2 \,, \tag{1.4.12}$$

and we know that $H^2 \approx \left(10^{10}\,\text{GeV}\right)^2$.

Then, an initial condition which works is $\varphi_i \approx 10^8 \div 10^9\,\text{GeV}$. This means that $\varphi_i \sim H/10$, which means that $N \gtrsim 1000$ easily. So, we can solve the horizon and flatness problems without issue.

However, we will see that there are indeed problems with this model. The amplitude of the primordial fluctuations is too large, the quantum fluctuations of the inflaton are too large, invalidating the semiclassical approach.

Wednesday 2020-11-11, compiled 2021-01-25

Last time we discussed the Coleman-Weinberg potential:

$$V(\varphi) = \frac{B\sigma^4}{2} - B\varphi^4 \left[\log\left(\frac{\varphi^2}{\sigma^2}\right) - \frac{1}{2}\right] , \tag{1.4.13}$$

where typically $B \sim 10^{-3}$. It can be approximated for $\varphi \ll \sigma$ as

$$V(\varphi) \approx \frac{B\sigma^4}{2} - \frac{\lambda}{4}\varphi^4 \,, \tag{1.4.14}$$

where $\lambda \sim (10 \div 100) B \sim 0.1$, and we have seen that if the initial value of φ is of the order of 10^8 to 10^9 GeV, corresponding to roughly $H/10$ then we can get a total number of e-folds of more than 1000.

Let us now move to **problems** with this model. We must guarantee that the variance of the fluctuations of the scalar field must be smaller than the classical trajectory:

$$\left\langle \delta\varphi^2 \right\rangle \ll \varphi_0^2(t) \,, \tag{1.4.15}$$

otherwise we cannot meaningfully treat the problem in a perturbative way. This is violated, since $\delta\varphi \sim H/2\pi$ is comparable to $\varphi_i \sim H/10$. This might destroy inflation.

We have

$$\frac{\delta\rho}{\rho} \approx \frac{H^2}{\dot\varphi} \approx \frac{3H^3}{V'(\varphi)} \approx \frac{3}{\lambda}\left(\frac{H}{\varphi}\right)^3 , \tag{1.4.16}$$

See equation (1.3.13).

since $\dot\varphi = -V'(\varphi)/3H$. When $\varphi \ll \sigma$, $V'(\varphi) \approx -\lambda\varphi^3$.

This will be of order unity at least!

Let try to make it so the number of e-folds $N_\lambda(\varphi \to \varphi_f)$ is at least 50 or 60: this is given by

$$N_\lambda(\varphi \to \varphi_f) = \frac{3}{2}\frac{H^2}{\lambda}\left(\frac{1}{\varphi^2} - \underbrace{\frac{1}{\varphi_f^2}}_{\text{negligible}}\right) \tag{1.4.17}$$

$$\frac{H}{\varphi} \approx \lambda^{1/2} \left(\frac{2}{3}\right)^{1/2} N_\lambda^{1/2} \,, \tag{1.4.18}$$

so the density perturbation will be

$$\frac{\delta\rho}{\rho} \approx \lambda^{1/2} N_\lambda^{3/2} (8/3)^{1/2} \,, \tag{1.4.19}$$

so if $\lambda \approx 0.1$ and indeed $N_\lambda \sim 50 \div 60 = N_{\mathrm{CMB}}$ we get

$$\frac{\delta\rho}{\rho} \sim 100 \,. \tag{1.4.20}$$

This is in stark contrast to the CMB observation: $\delta\rho/\rho \sim 10^{-5}$.

So, we get a contradiction: if λ is relatively large the density perturbations are huge, and in order to have the observed density perturbations we would need to have it tiny ($\lambda \sim 10^{-10}$), which would prohibit thermal equilibrium, which we assumed.

Chaotic models of inflation attempted to solve these problems: they used

$$V(\varphi) = \frac{\lambda}{4} \varphi^4 \,, \tag{1.4.21}$$

and they are *large-field* models. These models do not require coupling between the inflaton and other fields; however here we also find a relation

$$\frac{\delta\rho}{\rho} \sim \lambda^{1/2} N_\lambda^{3/2} \,. \tag{1.4.22}$$

Something about the Helmholtz free energy in the finite-temperature corrections to the potential $V(\varphi; T)$.

Hybrid models of inflation: we take a potential in the shape

$$V(\varphi) = V_0 \left[1 + \left(\frac{\varphi}{\mu}\right)^p \right] \,, \tag{1.4.23}$$

so that the potential is rolling down a potential towards a nonzero vacuum energy.

These were the first models for which the potential depends on two fields, so that the aforementioned potential for φ is an effective one: the full potential is

$$V(\varphi, \psi) = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} \left(\psi^2 - M^2\right)^2 + \frac{\lambda'}{2} \varphi^2 \psi^2 \,. \tag{1.4.24}$$

If we evaluate it at $\psi = 0$ we find

$$V(\varphi, \psi = 0) = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} M^4 \,, \tag{1.4.25}$$

in the same form as the aforementioned potential, with $V_0 = \lambda M^4/4$.

The effective mass of ψ, only considering the quadratic coupling, is

$$\frac{\partial^2 V}{\partial \psi^2} = m_\psi^2 = \lambda' \varphi^2 = m_\psi^2(\varphi)\,, \tag{1.4.26}$$

so the square mass of ψ reads

$$m_\psi^2 \bigg|_{\psi=0} = -\lambda M^2 + \lambda' \varphi^2\,, \tag{1.4.27}$$

and it can be positive or negative! The critical value of φ for which it has zero mass is given by

$$\varphi_c = \left(\frac{\lambda}{\lambda'}\right)^{1/2} M\,, \tag{1.4.28}$$

for $\varphi > \varphi_c$ we have $m_\psi^2 \Big|_{\psi=0} > 0$.

When φ becomes such that $\varphi < \varphi_c$ then the point $\psi = 0$ becomes unstable, so ψ moves from zero to either $+$ or $-M$.

We get a *second-order* phase transition for ψ.

Add graph for $V(\psi, \phi)$.

1.4.3 Some remarks on inflation model building

What are the loop contributions to the tree-level potential? The tree-level potential can look like

$$V(\varphi) = V_0 \pm \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} \varphi^4 + \frac{\lambda'}{\Lambda^2} \varphi^6 + \dots\,, \tag{1.4.29}$$

but the UV completion (the sixth-power term and beyond) must be suppressed with a large cutoff scale Λ.

Loop corrections can change the parameters, but it can also add whole new terms (like the log in the Coleman-Weinberg potential). These can be useful but also dangerous, for example spoiling the flatness needed for slow-roll inflation.

Typically, we will have something like

$$\Delta V^{\text{one-loop}}(\varphi) = \sum_i \frac{\pm N_i}{64\pi^2} M_i^4(\varphi) \log\left(\frac{M_i^2(\varphi)}{\mu^2}\right)\,, \tag{1.4.30}$$

where $M_i^2(\varphi)$ is the effective mass of the i-th particle species which interacts with φ. This logarithm generates the one in the Coleman-Weinberg potential.

The vacuum energy is given by

$$E^\psi_{\text{vacuum}} = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{p^2 + M_\psi^2(\varphi)}\,. \tag{1.4.31}$$

Let us make some examples: if we have two scalar fields 1 and 2 with which φ interacts we might get terms in the form

$$\Delta V(\varphi) \sim m_1^2 m_2^2 \log\left(\frac{\varphi}{\mu}\right), \tag{1.4.32}$$

so starting from a tree-level flat potential $V(\varphi) = V_0$ we might get

$$V(\varphi) = V_0 \left(1 + \alpha \log\left(\frac{\varphi}{\mu}\right)\right), \tag{1.4.33}$$

where

$$\alpha = \frac{m_1^2 m_2^2}{V_0}. \tag{1.4.34}$$

Another example which is similar to Coleman-Weinberg is

$$\Delta V(\varphi) \sim \frac{c}{2}\left(m_1^2 + m_2^2\right)\varphi^2 \log\frac{\varphi}{\mu}, \tag{1.4.35}$$

so

$$V(\varphi) = V_0 + \frac{\varphi^2}{2}\left[m^2 + c\widetilde{m}^2 \log\frac{\varphi}{\mu}\right], \tag{1.4.36}$$

which can arise from a tree-level potential like $V_0 + m^2\varphi^2/2$, where we defined $\widetilde{m}^2 = m_1^2 + m_2^2$.

This might be observationally detected through a **running of the spectral index**: $n_s = n_s(k)$.

The η problem Suppose we have a potential V depending on n complex scalar fields:

$$V = V(\varphi_n, \overline{\varphi}_n) = e^{k(\varphi_n,\overline{\varphi}_n)/M_{\text{Pl}}^2}\widetilde{V}(\varphi_n, \overline{\varphi}_n), \tag{1.4.37}$$

which is commonly called the Kähler potential — it arises in the context of supergravity theories. It arises from a Lagrangian in the form

$$\mathscr{L} = \partial_\mu \overline{\varphi}_m K_{\overline{m}n} \partial^\mu \varphi_n, \tag{1.4.38}$$

where

$$K_{\overline{m}} = \frac{\partial K}{\partial \overline{\varphi}_m} \qquad K_n = \frac{\partial K}{\partial \varphi_n}. \tag{1.4.39}$$

If $K = \sum_m \left|\varphi_m\right|^2$ we find simply $K_{\overline{m}n} = \delta_{nm}$; then

$$\eta_V = \frac{1}{3}\frac{m^2}{H^2} = \frac{1}{3}\frac{V''(\varphi)}{H^2}. \tag{1.4.40}$$

Let us identify φ, the inflaton, with the real part of one of the fields: $\varphi = \mathrm{Re}\, \varphi_n$. Then,

$$V = e^{K/M_P^2} \widetilde{V}\,, \tag{1.4.41}$$

so that

$$\frac{\partial V}{\partial \varphi} = e^{K/M_p^2} \frac{2\varphi}{M_P^2} \widetilde{V} + e^{k/M_P^2} \widetilde{V}'\,, \tag{1.4.42}$$

therefore the square mass of the field reads

$$m_\varphi^2 = \left. \frac{\partial^2 V}{\partial \varphi^2} \right|_{\varphi=0} = \frac{2V}{M_P^2} + e^{K/M_P^2} \widetilde{V}''\,. \tag{1.4.43}$$

Then, using $H^2 \approx \frac{8\pi G}{3} V(\varphi)$ we get

$$\eta_V = \frac{1}{3} \frac{m^2}{H^2} \sim \frac{m^2 M_P^2}{V}\,. \tag{1.4.44}$$

This yields

$$\eta_V = 1 + M_P^2 \frac{\widetilde{V}''}{\widetilde{V}}\,, \tag{1.4.45}$$

which is a problem if we want slow-roll inflation.

Possibly not a 1 here.

If we have a potential V_{sr} for the slow-roll, we might add a term

$$V = V_{\mathrm{sr}} + \frac{\varphi^2}{\Lambda^2} V_{\mathrm{sr}}\,, \tag{1.4.46}$$

so if $\Lambda \leq M_P$, we might get $\eta_V = V'' M_P^2 / V \approx 1$. The main point is that inflation is very sensitive to high-energy physics.

The inflationary potential can in general be an effective one, which works for energies $E \leq \Lambda$ for some cutoff Λ. We can use some additional symmetries imposed on the UV completion.

$$\mathscr{L}_{\mathrm{eff}} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4} \varphi^4 + \sum_{p=1}^{\infty} \left[\lambda_p \varphi^4 + \nu_p \partial_\mu \varphi \partial^\mu \varphi \right] \left(\frac{g}{\Lambda} \varphi \right)^{2p}\,, \tag{1.4.47}$$

which contains nonrenormalizable terms, but this is fine since it is an effective theory. We have employed the reflection symmetry $\varphi \to -\varphi$.

We can impose that $\varphi \ll \Lambda$, so that the tower of terms does not spoil the flatness. This means that we have a small-field model of inflation. Alternatively, we may employ some symmetry like a shift symmetry $\varphi \to \varphi + c$. In that case, the tower of terms cannot spoil the flatness, therefore we may see large-field models of inflation, which predict a relatively high tensor-to-scalar ratio.

Chapter 2

Reheating

2.1 Radiation from the inflaton

This is what happens in the transition between the inflationary phase and the usual radiation-dominated epoch. Monday 2020-11-16, compiled 2021-01-25

We will give a simplified treatment, which however captures the main characteristics of the model.

Consider the typical inflationary potential: flat at $\varphi \sim 0$, sloping down towards a minimum. At $\varphi \sim \varphi_f$ the field starts "falling down" towards the minimum quickly. The condition is $V''(\varphi) \gtrsim H^2$, meaning that $\eta_V \gtrsim 1$, which also implies that quickly we will find $\epsilon \gtrsim 1$.

When the field falls down it will **oscillate**, however its oscillations will be damped. This is due to two factors: the expansion of the universe and the coupling of the field to other particles.

The damped oscillations are described by a coupled Klein-Gordon equation:

$$\ddot{\varphi} + 3H\dot{\varphi} + \Gamma_\varphi \dot{\varphi} = -V'(\varphi) \,, \tag{2.1.1}$$

where Γ_φ is the **decay rate** of the inflaton field into other kinds of particles. This has the same form as the expansion term: it is a damping term as well.

The energy density of the scalar field can be differentiated:

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \tag{2.1.2}$$

$$\dot{\rho}_\varphi = \dot{\varphi}\ddot{\varphi} + V'(\varphi)\dot{\varphi} \,, \tag{2.1.3}$$

into which we can substitute into the KG equation to get

$$\dot{\rho}_\varphi + (3H + \Gamma_\varphi)\dot{\varphi}^2 = 0 \,. \tag{2.1.4}$$

The timescale of the oscillations of φ will be much smaller than H^{-1}, the timescale of the expansion of the universe.

Oscillations at $\varphi = \sigma$ will have a frequency $\omega^2 = V''(\sigma)$, which is also the effective mass of the inflaton there, $m_\varphi^2(\sigma)$.

Since $\omega^2 \gg H^2$ (this is true since $\eta_V \gg 1$) we can take averages over a period of the relevant quantities:

$$\left\langle \dot{\varphi}^2 \right\rangle_{\text{period}} = \rho_\varphi \,, \tag{2.1.5}$$

since in general $\left\langle \dot{\varphi}^2/2 \right\rangle = \langle V \rangle$ by the virial theorem: therefore, substituting into (2.1.2) we get $\left\langle \dot{\varphi}^2 \right\rangle = \rho_\varphi$. Then our equation becomes

$$\dot{\rho}_\varphi + 3H\rho_\varphi = -\Gamma_\varphi \rho_\varphi \,, \tag{2.1.6}$$

which, neglecting Γ_φ, looks like the continuity equation for nonrelativistic matter, which yields $\rho_\varphi \propto a^{-3}$. This is expected: we have used the fact that $m_\varphi^2(\sigma) \gg H^2$, which is saying that the scalar field is very massive.

In the spirit of keeping things simple, we will use a toy model and assume that all the decay products of φ are relativistic, whose continuity equation is

$$\dot{\rho}_R + 4H\rho_R = +\Gamma_\varphi \rho_\varphi \,, \tag{2.1.7}$$

by conservation of energy. This equation comes from $\nabla_\nu T^{\mu\nu} = 0$. Gravity will then be described by the first Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\Big(\rho_\varphi + \rho_R\Big) \,. \tag{2.1.8}$$

We have a simple and exact solution:

$$\rho_\varphi = M^4 \left(\frac{a}{a_{\text{osc}}}\right)^{-3} \exp\Big(-\Gamma_\varphi (t - t_{\text{osc}})\Big) \,, \tag{2.1.9}$$

See [KT94, eq. 8.30].

where t_{osc} and a_{osc} correspond to the time at which oscillations start, and $M^4 = \rho_\varphi(t_{\text{osc}})$. As a first approximation, this is the height of the potential the field is "falling from": $M^4 \sim V(\varphi = 0)$.

Let us consider the evolution up to the time $t \approx \Gamma_\varphi^{-1}$. Then, the decay will not have been very efficient yet, and the universe will still be nonrelativistic matter (φ) dominated, so we will have $a \propto t^{2/3}$.

The time of the start of oscillation will be

$$t_{\text{osc}} \approx H^{-1} = \frac{M_P}{M^2} \,, \tag{2.1.10}$$

since at the start of oscillations

$$H^2 \approx \frac{8\pi G}{3} M^4 \approx \frac{M^2}{M_P} \,. \tag{2.1.11}$$

So,

$$\dot{\rho}_R + 4H\rho_R = \Gamma_\varphi M^4 \left(\frac{a}{a_{\text{osc}}}\right)^{-3} \tag{2.1.12}$$

$$\dot{\rho}_R + \frac{8}{3}\frac{\rho_R}{t} = \Gamma_\varphi M^4 \left(\frac{t}{t_{\text{osc}}}\right)^{-2}, \tag{2.1.13}$$

since $a \propto t^{2/3}$ (we are in a radiation dominated phase), therefore $H = (2/3)t^{-1}$. With a powerlaw ansatz $\rho_R = Bt^\alpha$ we get

$$\alpha t^{\alpha-1} + \frac{8}{3}\frac{t^\alpha}{t} = \frac{\Gamma_\varphi}{B} M^4 \left(\frac{t}{t_{\text{osc}}}\right)^{-2}, \tag{2.1.14}$$

the homogeneous solution is given by $\alpha = -8/3$, while the particular has $\alpha = -1$.

The initial condition we set is $\rho_R(t_{\text{osc}}) = 0$, since before reheating inflation was taking place, diluting the energy density of radiation. This yields, in the pre-radiation-domination epoch:

$$\rho_R \approx \Gamma_\varphi M_P^2 \frac{9}{40\pi}\frac{1}{t}\left[1 - \left(\frac{t}{t_{\text{osc}}}\right)^{-5/3}\right]. \tag{2.1.15}$$

Where does the π come from?

Starting from this solution, and using $a \propto t^{2/3}$, we find

$$\rho_R = \frac{0.4}{\pi^{1/2}} \Gamma_\varphi M_P M^2 \left(\frac{a}{a_{\text{osc}}}\right)^{-3/2}\left[1 - \left(\frac{a}{a_{\text{osc}}}\right)^{-5/2}\right]. \tag{2.1.16}$$

The maximum energy density of radiation will be roughly given by $\rho_R^{\text{max}} \approx \Gamma_\varphi M_P M^2$.

The a-dependent part has a maximum value of roughly 0.35.

The radiation energy density will be given by $\rho_R = \frac{\pi^2}{30} g_* T^4$, so the maximum temperature will be

$$T^{\text{max}} = g_*^{-1/4} \rho_R^{\text{max},1/4} \sim g_*^{-1/4} \left(\Gamma_\varphi M_P M^2\right)^{1/4}. \tag{2.1.17}$$

In the reheating phase the energy density scales like $\rho_R \propto a^{-3/2}$: it is *decreasing*, but much *slower* than the usual $\rho_R \propto a^{-4}$.

Let us also discuss the entropy:

$$^*S = sa^3 \qquad\qquad s = \frac{2\pi^2}{45} g_{*s} T^3, \tag{2.1.18}$$

so, since $\rho_R \propto a^{-3/2}$ and $\rho_R \propto T^4$ we have $s \propto \rho_R^{3/4} \propto a^{-9/8}$, therefore

$$S \propto a^3 a^{-9/8} = a^{15/8}. \tag{2.1.19}$$

It makes sense that this is increasing.

What is the *reheating temperature*? We want to match the inflationary solution and the radiation-dominated solution.

Since we are in the radiation-dominated phase, we have $a \propto t^{1/2}$: so

$$H^2 = \frac{8\pi G}{3}\rho_R = \frac{8\pi}{3}\frac{1}{M_P^2}\frac{\pi^2}{30} g_* T^4 = \frac{1}{4t^2}. \tag{2.1.20}$$

The reheating temperature can be computed as the temperature at $T_{RH} = T(t \approx \Gamma_\varphi^{-1})$. Plugging this in, we get

$$T_{RH} \approx 0.55 g_*^{-1/4} \sqrt{\Gamma_\varphi M_P}\,. \tag{2.1.21}$$

What is interesting to note here is that there is no memory of the energy scale M, the vacuum energy scale of inflation.

Add comment about the redshift of the inflaton's energy

Only if $\Gamma_\varphi \gg H_{\text{osc}}$ we would have had $T_{RH} \sim M$; in that case the reheating would have been 100 % efficient, the harmonic oscillator would have been overdamped.

Note that the maximum temperature reached in the reheating phase is different from the reheating temperature.

See [KT94, fig. 8.3].

2.2 Boltzmann equation applications

A usual rule of thumb is given in terms of the interaction rate $\Gamma = n\sigma|v|$: if $\Gamma \gtrsim H$ thermal equilibrium can be established, while if $\Gamma < H$ interactions become inefficient.

If, roughly speaking, $T \propto a^{-1}$, then $\dot{T}/T = -H$.

When $\Gamma \sim H$, we need the Boltzmann equation in order to find out what exactly is going on. Let us give two examples.

$2 \leftrightarrow 2$ scattering between relativistic particles may be mediated by a massless boson, such as the photon, or by a massive boson, such as the $W^\pm$ or Z^0 boson.

In the first (massless boson) case, we have

$$\sigma \sim \frac{\alpha^2}{T^2} \qquad \alpha = \frac{g^2}{4\pi}\,, \tag{2.2.1}$$

while in the second (massive boson) case we have

$$\sigma \sim G_X^2 T^2 \qquad G_X = \frac{\alpha}{m_X^2}\,. \tag{2.2.2}$$

Roughly speaking,

$$\sigma \sim \alpha^2 |\text{propagator}|^2 \frac{q^4}{E^2}\,, \tag{2.2.3}$$

where q is the spatial momentum of the interacting particles, while E is the center of mass energy, so for relativistic particles $q^4/E^2 \sim E^2 \sim T^2$.

In the massless boson case, the propagator looks like

$$\text{propagator} \sim \frac{-ig_{\mu\nu}}{p^2}\,, \tag{2.2.4}$$

where $p^2 = (q_1 + q_2)^2$. Then, the cross-section looks like $\sigma \sim \alpha^2/T^2$ typically.

In the massive boson case, we have

$$\text{propagator} \sim \frac{-ig_{\mu\nu} + p_\mu p_\nu / m_X^2}{p^2 - m_X^2}\,, \tag{2.2.5}$$

so $\sigma \sim \alpha^2 T^2 / m_X^4$.

In the massless boson case, then,

$$\Gamma = n\sigma |v| \approx n\sigma\,, \tag{2.2.6}$$

so if $n \sim T^3$ (which is the case if we have radiation domination and equilibrium) we get $\Gamma = \alpha^2 T$. We want to know when this will be larger than H, using the fact that

$$H \sim \frac{T^2}{M_P} \qquad\qquad H^2 = \frac{8\pi G}{3} \underbrace{\rho_R}_{\propto T^4}\,, \tag{2.2.7}$$

so $\Gamma \gtrsim H$ when $T < \alpha^2 M_P \sim 10^{15}\,\text{GeV}$.

For the massive gauge boson, at temperatures $T \ll m_X$ we get $\Gamma \gtrsim H$ when

$$\Gamma = G_X^2 T^5 \gtrsim \frac{T^2}{M_P} \tag{2.2.8}$$

$$T \gtrsim G_X^{-2/3} M_P^{-1/3}\,. \tag{2.2.9}$$

With normalization reflecting the case of weak interactions, and using $G_X \sim \alpha / m_X^2$, we find

$$T \gtrsim \left(\frac{m_X}{100\,\text{GeV}} \right)^{4/3} 1\,\text{MeV}\,, \tag{2.2.10}$$

which can be used to figure out when neutrinos decouple.

2.3 The Boltzmann equation

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The phase space distribution function is denoted as $f(x^\mu, p^\mu)$; the Boltzmann equation reads

$$\hat{L}[f] = \hat{C}[f]\,. \tag{2.3.1}$$

If $\hat{C} = 0$, then Liouville's theorem tells us that $\mathrm{d}f/\mathrm{d}t = 0$. The nonrelativistic expression for $\hat{L}[p]$ is

$$\hat{L}[f] = \left[\frac{\partial \vec{x}}{\partial t} \cdot \nabla_{\vec{x}} + \frac{\partial \vec{v}}{\partial t} \cdot \nabla_{\vec{v}} + \frac{\partial}{\partial t} \right] f\,, \tag{2.3.2}$$

while the general relativistic expression is

$$\hat{L}[f] = \left[p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \right] f\,, \tag{2.3.3}$$

since by the geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = -\Gamma^\mu_{\alpha\beta}p^\alpha p^\beta\,. \tag{2.3.4}$$

In the homogeneous and isotropic case we will have $f = f(|p|, t)$, since the dependences on $|p|$ and E are related. The only nonzero Christoffel symbols for a flat FLRW metric are

$$\Gamma^0_{ij} = \delta_{ij} a\dot{a} \qquad\qquad \Gamma^i_{0j} = \Gamma^j_{i0}\delta^i_j\frac{\dot{a}}{a}\,. \tag{2.3.5}$$

Therefore, the Liouville operator in our case can be written in terms of $p^2 = g_{ij}p^i p^j = a^2\delta_{ij}p^i p^j$, the square of the *three*-momentum, which scales like $1/a$:

$$\hat{L}[f(E,t)] = E\frac{\partial f}{\partial t} - \frac{\dot{a}}{a}p^2\frac{\partial f}{\partial E}\,. \tag{2.3.6}$$

The number density reads

$$n = \frac{g}{(2\pi)^3}\int \mathrm{d}^3 p\, f\,, \tag{2.3.7}$$

so the Boltzmann equation can be manipulated by rewriting the Liouville operator as

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{g}{(2\pi)^3}\int \mathrm{d}^3 p\, f\right)}_{\dot{n}(t)} - \frac{\dot{a}}{a}\frac{g}{(2\pi)^3}\int \mathrm{d}^3 p\,\frac{p^2}{E}\frac{\partial f}{\partial E} = \frac{g}{(2\pi)^3}\int \mathrm{d}^3 p\,\frac{C[f]}{E}\,. \tag{2.3.8}$$

Manipulating the second term in the equation we find

$$\frac{g}{(2\pi)^3}\int \mathrm{d}^3 p\, p^2\frac{\partial f}{\partial E} = \frac{4\pi g}{(2\pi)^3}\int \mathrm{d}p\,\frac{p^4}{E}\frac{\partial f}{\partial E}\,, \tag{2.3.9}$$

and since $E^2 = p^2 + m^2 \implies E\,\mathrm{d}E = p\,\mathrm{d}p$ we get

Integrated by parts

$$\frac{4\pi g}{(2\pi)^3}\int \mathrm{d}p\, p^3\frac{\partial f}{\partial p} = -\frac{4\pi g}{(2\pi)^3}\int \mathrm{d}p\,(3p^2) f(p) = -3n(t)\,. \tag{2.3.10}$$

Then our equation reads

$$\dot{n}(t) + 3Hn(t) = \frac{g}{(2\pi)^3}\int \mathrm{d}^3 p\,\frac{\hat{C}[f]}{E}\,. \tag{2.3.11}$$

This, as expected, gives $n \propto a^{-3}$ if there are no collisions.

Let us apply this to a process $1+2 \leftrightarrow 3+4$, of which we are interested in the density $n_1(t)$: the collision terms reads

$$\begin{aligned}\frac{g_1}{(2\pi)^3}\int \frac{\hat{C}[f_1]}{E_1}\mathrm{d}^3 p_1 = \int \mathrm{d}\Pi_1\,\mathrm{d}\Pi_2\,\mathrm{d}\Pi_3\,\mathrm{d}\Pi_4\,(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\times \\ \left[|\mathcal{M}|^2_{3+4\to1+2} f_3 f_4(1\pm f_1)(1\pm f_2) - |\mathcal{M}|^2_{1+2\to3+4} f_1 f_2(1\pm f_3)(1\pm f_4)\right]\end{aligned}, \tag{2.3.12}$$

The minus in the second term is because in that case we are annihilating particle 1.

where

$$\mathrm{d}\Pi_i = \frac{g_i\,\mathrm{d}^3 p_i}{(2\pi)^3 2E_i}\,, \tag{2.3.13}$$

while $\mathcal{M}$ is the Feynman amplitude. The terms $1 \pm f_i$ are given by the Pauli blocking $(-)$ or Bose enhancement $(+)$ coming from the statistics of the chosen particle.

This equation must be written for all the particle species, in general yielding a complicated integrodifferential equation system. Because of time reversal symmetry (our first assumption) we will have $|\mathcal{M}|^2_{1+2\to 3+4} = |\mathcal{M}|^2_{3+4\to 1+2} \equiv |\mathcal{M}|^2$.

We can further postulate that all the particle species are in kinetic equilibrium (our second assumption): then, we say that they are distributed according to the Bose-Einstein or Fermi-Dirac distribution,

$$f^i_{BE-FD} = \left[\exp\left(\frac{E-\mu}{T} \mp 1\right)\right]^{-1}. \tag{2.3.14}$$

In this case we write $\mp$ since the $-$ corresponds to Bosons, while the $+$ is for Fermions.

This works well as long as the scatterings are fast enough. This parametrization is quite useful, since it makes our integrodifferential equations into simple(r) ODEs.

Kinetic equilibrium is not just local thermodynamic equilibrium: it also requires local *chemical* equilibrium, which implies the following for the chemical potentials:

$$\mu_1 + \mu_2 = \mu_3 + \mu_4\,. \tag{2.3.15}$$

Since we can find such linear relations for any possible reaction, the true number of independent chemical potentials will dramatically shrink. We expect these to be related to conserved quantities. Since these are mostly small, we can approximate $\mu_i = 0 \forall i$. For the CMB photons, this is confirmed with $\mu_\gamma / T \lesssim 10^{-4}$.

Further, we make the semiclassical approximation (our third assumption): $1 \pm f_i \approx 1$, so that the distribution just becomes $f = \exp\big((\mu - E)/T\big)$. With all these considerations, we find

$$\dot{n}_1(t) + 3Hn_1(t) = \int \mathrm{d}\Pi_1\,\mathrm{d}\Pi_2\,\mathrm{d}\Pi_3\,\mathrm{d}\Pi_4\,(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)|\mathcal{M}|^2\big(f_3f_4 - f_1f_2\big) \tag{2.3.16}$$

$$= \int \mathrm{d}\Pi_1\,\mathrm{d}\Pi_2\,\mathrm{d}\Pi_3\,\mathrm{d}\Pi_4\,(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4) \\ |\mathcal{M}|^2 \exp\left(-\frac{E_1+E_1}{T}\right)\left(\exp\left(\frac{\mu_3+\mu_4}{T}\right) - \exp\left(\frac{\mu_1+\mu_2}{T}\right)\right). \tag{2.3.17}$$

Used $E_1 + E_2 = E_3 + E_4$.

This yields

$$n_i(t) = \frac{g_i}{(2\pi)^3}\int \mathrm{d}^3 p_i\, f_i \tag{2.3.18}$$

$$= g_i \exp\left(\frac{\mu_i}{T}\right) \int \frac{d^3 p_i}{(2\pi)^3} \exp\left(-\frac{E_i}{T}\right). \tag{2.3.19}$$

At equilibrium, and using the approximation that $\mu_i = 0$, this yields

$$n_i^{\text{eq}} = g_i \int \frac{d^3 p_i}{(2\pi)^3} \exp\left(-\frac{E_i}{T}\right) \approx \begin{cases} g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(-\frac{m_i}{T}\right) & \text{nonrelativistic} \\ \frac{g_i}{\pi^2} T^3 & \text{relativistic} \end{cases}. \tag{2.3.20}$$

Also,

$$f_3 f_4 - f_1 f_2 = \exp\left(-\frac{E_1 + E_2}{T}\right) \left[e^{\frac{\mu_3}{T}} e^{\frac{\mu_4}{T}} - e^{\frac{\mu_1}{T}} e^{\frac{\mu_2}{T}}\right] = \exp\left(-\frac{E_1 + E_2}{T}\right) \left[\frac{n_3 n_4}{n_3^{\text{eq}} n_4^{\text{eq}}} - \frac{n_1 n_2}{n_1^{\text{eq}} n_2^{\text{eq}}}\right]. \tag{2.3.21}$$

The thermally averaged cross-section is

$$\langle \sigma |v| \rangle = \frac{1}{n_1^{\text{eq}} n_2^{\text{eq}}} \int d\Pi_1\, d\Pi_2\, d\Pi_3\, d\Pi_4\, (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) |\mathcal{M}|^2 \exp\left(-\frac{E_1 + E_2}{T}\right). \tag{2.3.22}$$

Then,

$$\dot{n}_1(t) + 3H n_1(t) = n_1^{\text{eq}} n_2^{\text{eq}} \langle \sigma |v| \rangle \left[\frac{n_3 n_4}{n_3^{\text{eq}} n_4^{\text{eq}}} - \frac{n_1 n_2}{n_1^{\text{eq}} n_2^{\text{eq}}}\right]. \tag{2.3.23}$$

For a standard DM particle — a process in the form $\psi\overline{\psi} \leftrightarrow X\overline{X}$, with ψ and $\overline{\psi}$ in local thermal equilibrium, and $n_\psi = n_{\overline{\psi}}$ as well as $n_X = n_{\overline{X}}$—, we get

$$\dot{n}_1(t) + 3H n_1(t) = \langle \sigma |v| \rangle \left[(n_1^{\text{eq}})^2 - n_1^2\right]. \tag{2.3.24}$$

The left-hand side is roughly $n_1/\tau \sim n_1 H$; while the right-hand side is roughly $n_1^2 \langle \sigma v \rangle$: then, we are comparing H and $\Gamma = n_1 \langle \sigma v \rangle$. If $\Gamma \gg H$, then n_1 must tend towards its equilibrium value in the limit.

Chapter 3

Baryogenesis and Dark Matter production

3.1 Baryogenesis

Monday 2020-11-23, compiled 2021-01-25

The net baryon number density is denoted as

$$n_B = n_b - n_{\overline{b}}\,, \tag{3.1.1}$$

where n_b and $n_{\overline{b}}$ are the number densities of baryons and antibaryons respectively. We can estimate it as

$$n_B = n_N \approx 1.38 \times 10^{-5}\,\mathrm{cm}^{-3}\Omega_{0b}h^2\,, \tag{3.1.2}$$

since

$$n_N = (m_p n_N)\frac{1}{m_p} = \frac{\rho_{0b}}{m_p} = \underbrace{\frac{\rho_{0b}}{\rho_{0,\,\mathrm{crit}}}}_{\Omega_{0b}}\frac{\rho_{0,\,\mathrm{crit}}}{m_p}\,, \tag{3.1.3}$$

and recalling that

$$\rho_{\mathrm{crit}} = \frac{H_0^2}{8\pi G} = \frac{1}{8\pi G}\left(2h10^{-42}\,\mathrm{GeV}\right). \tag{3.1.4}$$

Another useful quantity to have is

$$B = \frac{n_B}{s}\,, \tag{3.1.5}$$

where s is the entropy density. This is interesting to us since $s \propto a^{-3}$, just like n_B will do if there is no baryon annihilation or generation. What we are doing amounts to counting the amount of baryons in a comoving volume: if none are generated, B is constant.

This is given by

$$B \approx 7 \times 10^{-9}\Omega_{0b}h^2\,, \tag{3.1.6}$$

where $\Omega_{0b}h^2 \approx 0.0224 \pm 0.0001$ with the current measurements, therefore $B \approx 10^{-10}$.

Then, we can solve for s_0 in the expression: recall that in general the entropy is given in terms of the total relativistic degrees of freedom g_{*s}:

$$s = \frac{2\pi^2}{45} g_{*s} T^3 \,, \tag{3.1.7}$$

and

$$n_\gamma = \frac{\zeta(3) g_\gamma T^3}{\pi^2} \,, \tag{3.1.8}$$

where $g_\gamma = 2$ is the number of polarizations of a photon, therefore

$$s \approx 1.8 g_{*s} n_\gamma \tag{3.1.9}$$

$$s_0 \approx 1.8 g_{*s}(t_0) n_\gamma(t_0) \,. \tag{3.1.10}$$

The Kolb and Turner [KT94] does the following calculation with three massless neutrinos, however we now know that at least two neutrino species are nonrelativistic today. So, with one relativistic neutrino we have $n_\gamma \approx 422\,\mathrm{cm}^{-3}$, and we can give the conservative estimate:

$$g_{*s}(t_0) \approx 2.63 \,, \tag{3.1.11}$$

and using the fact that $T/T_\nu = (11/4)^{1/3}$ we get

$$s_0 = 4.7 n_\gamma(t_0) \,. \tag{3.1.12}$$

We can measure the baryon number Ω_{0b} from a different source from CMB anisotropies: primordial nucleosynthesis. This is in very good agreement with the CMB data.

The **annihilation catastrophe**: in a symmetric baryonic universe baryons and antibaryons remain in thermal equilibrium down to $T \sim 20\,\mathrm{MeV}$, and at these temperatures $n_b/s \sim 10^{-20}$. This is around 10 orders of magnitude smaller than what we observe. This occurs because baryons and antibaryons keep annihilating.

Therefore, at $T \gg 20\,\mathrm{MeV}$ an initial asymmetry between baryons and antibaryons must be present.

So, we are looking for the **mechanism** which, starting from symmetric initial conditions, is able to generate an initial $B \approx 10^{-10}$.

We have the **Sakharov conditions** telling us what we need in order for this to occur:

1. violation of baryon number conservation;

2. C and CP violation;

3. out of equilibrium conditions.

If we are in thermal equilibrium, the chemical potentials μ can all be taken to be zero; then the baryon and antibaryon distribution functions will look like

$$f_b = f_{\overline{b}} = \frac{1}{\exp\left(\frac{E \pm \mu}{T}\right) + 1} = \frac{1}{\exp\left(\frac{E}{T}\right) + 1} \,, \tag{3.1.13}$$

and the energies are indeed the same:

$$E_{b/\overline{b}} = \sqrt{p^2 + m^2_{b/\overline{b}}}, \tag{3.1.14}$$

but by CPT symmetry $m_b = m_{\overline{b}}$, so $E_b = E_{\overline{b}}$: then indeed $f_b = f_{\overline{b}}$, which means $n_b = n_{\overline{b}}$, so $B = 0$.

Consider a particle X which decays into either qq, with BR r, or into $\overline{q}\ell$ with branching ratio $1 - r$. These generate baryon number $2/3$ and $-1/3$ respectively.

The antiparticle $\overline{X}$ can decay into $\overline{qq}$ or $q\overline{\ell}$, with branching ratios $\overline{r}$ and $1 - \overline{r}$ respectively. C violation tells us that we can have $r \neq \overline{r}$.

We can then calculate the net baryon number generated by the decay of an X particle:

$$B_X = \frac{2}{3}r - \frac{1}{3}(1-r) = r - \frac{1}{3}, \tag{3.1.15}$$

while the net baryon number for the decay of an anti-X will be

$$B_{\overline{X}} = -\frac{2}{3}\overline{r} + \frac{1}{3}(1-\overline{r}) = -(\overline{r} - \frac{1}{3}). \tag{3.1.16}$$

The total, for an X-$\overline{X}$ pair, is

$$B_X + B_{\overline{X}} = r - \overline{r} = \epsilon. \tag{3.1.17}$$

This tells us that we need **CP violation**.

We also need departures from thermal equilibrium. Suppose we want to consider the decay of a particle X, and we start at $T \gg m_X$. At this stage, we expect $n_X \approx n_{\overline{X}} \approx n_\gamma$.

On the other hand, if $T \ll m_X$ we have $n_X = (m_X T)^{3/2} \exp(-m_X/T)$, which decays quickly.

Add plot of n_X/n_γ versus $z = m_X/T$.

If $H \gg \Gamma$ the particle can go out of equilibrium. We an have decay of X, or B-violating scattering, or processes mediated by X.

The annihilation rate in any case is $\Gamma_{\text{ann}} \propto n_X$.

Then, considering a process in the form $qq \to \overline{q} + \ell$ mediated by X (like $W^+ \to \ell^+ + \nu_e$, $H \to \ell\overline{\ell}$) we have

$$\Gamma_D = \begin{cases} \alpha m_X & T \lesssim m_x, \alpha = g^2/4\pi \\ \alpha m_X (\frac{m_X}{T}) & T \geq m_X \end{cases}, \tag{3.1.18}$$

or, for the Γ of Inverse Decay:

$$\Gamma_{ID} = \begin{cases} \Gamma_D & T \geq m_X \\ \sim \Gamma_D e^{-m_X/T} & T \lesssim m_X \end{cases}. \tag{3.1.19}$$

If we have $2 \leftrightarrow 2$ scattering, mediated by X, we will see

$|v| \approx 1.$

$$\Gamma_s \approx n\sigma|v| \approx T^3 \frac{\alpha^2 T^2}{(T^2 + m_X^2)^2}. \tag{3.1.20}$$

This interpolates between the α^2/T^2 case, in which X is massless, and the $G_X^2 T^2$ case, in which X is massive, and $G_X \sim \alpha/m_X^2$.

Now, in this early universe stage the Hubble rate will look like $H \sim \sqrt{g_*}T^2/m_P$, without any dependence on m_X.

So, the quantity Γ/H at the time in which $m_X/T = 1$ will tell us whether the decays are efficient. Let us calculate it explicitly:

$$K = \left(\frac{\Gamma_D}{H}\right)\bigg|_{T=m_X} = \frac{\alpha m_X}{m_x^2\sqrt{g_*}/m_P} = \frac{\alpha m_P}{m_X\sqrt{g_*}}. \tag{3.1.21}$$

Let us take the regime $T \lesssim m_X$. Here,

$$\frac{\Gamma_{ID}}{H} \sim \frac{e^{-m_X/T}\Gamma_D}{H} \sim Ke^{-m_X/T}, \tag{3.1.22}$$

therefore

$$\frac{\Gamma_S}{H} \sim \frac{\frac{\alpha^2}{m_X^4}T^5}{\frac{T^2}{m_P}\sqrt{g_*}} \sim \alpha\frac{\alpha m_P}{\sqrt{g_*}m_X}\left(\frac{T}{m_X}\right)^3 \sim \alpha K\left(\frac{T}{m_X}\right)^3. \tag{3.1.23}$$

If $K \ll 1$, then for sure all the possible processes will be inefficient, and the particle X will be out of equilibrium.

Then the net baryon number will be

$$n_B = \epsilon\, n_X\bigg|_{t_D} \approx \epsilon n_\gamma. \tag{3.1.24}$$

So, using the expression for the entropy given in (3.1.9) we get

$$B = \frac{n_B}{s} = \frac{\epsilon n_\gamma}{g_{*s}n_\gamma} \sim \frac{\epsilon}{g_{*s}}. \tag{3.1.25}$$

Since $B \sim 10^{-10}$ and $g_* \sim 10^2 \div 10^3$, we will need $\epsilon \sim 10^{-7} \div 10^{-8}$.

The condition $K \ll 1$ is equivalent to $m_X \gg g_*^{-1/2}\alpha m_P$. So, the particles we need to consider are very massive: on the order of 10^{16} GeV. These can be realized in some models of GUTs.

Plot the curve n_X/n_γ: roughly 1 for a large time, then when $m_X \sim T$ it drops.

If on the other hand $K \gg 1$, because of thermal equilibrium we will have $B \approx 0$. If, instead, we had $K \lesssim 1$ (close to 1), we need to use the full Boltzmann equation.

Recall that for $T \geq m_X$, we had

$$\Gamma_D = \alpha m_X\left(\frac{m_X}{T}\right). \tag{3.1.26}$$

Where does the suppression factor m_X/T come from? It is a time dilation effect: $t_D \sim \Gamma_D^{-1}$, but we need to multiply t_D by the Lorentz factor γ, so the real decay rate is $\Gamma_D\gamma^{-1}$, but

$$E_X \sim |q| = m_X|\vec{v}|\gamma \approx m_X\gamma, \tag{3.1.27}$$

so $\gamma \sim E_X / m_X \sim T / m_X$. This yields

$$\Gamma_D \gamma^{-1} = (\alpha m_X) \frac{m_X}{T} \,. \tag{3.1.28}$$

Again we consider the problem of X and $\overline{X}$ decaying into the relativistic particles b and $\overline{b}$, which carrying baryon number $\pm 1/2$ respectively. This is a simplification, but not a large one. Wednesday 2020-11-25, compiled 2021-01-25

The Feynman amplitudes can be parametrized as

$$① = \left|\mathcal{M}(X \to bb)\right|^2 = \left|\mathcal{M}(\overline{bb} \to X)\right|^2 = +\frac{1}{2}(1+\epsilon)|\mathcal{M}_0|^2 \tag{3.1.29}$$

$$② = \left|\mathcal{M}(X \to \overline{bb})\right|^2 = \left|\mathcal{M}(bb \to X)\right|^2 = +\frac{1}{2}(1-\epsilon)|\mathcal{M}_0|^2 \,, \tag{3.1.30}$$

which means that

$$\frac{① - ②}{① + ②} = \epsilon \,. \tag{3.1.31}$$

We suppose we are in kinetic equilibrium, and neglect quantum effects:

$$f_b(E) = \exp\left(-\frac{(E-\mu)}{T} \right) \tag{3.1.32}$$

$$f_{\overline{b}}(E) = \exp\left(-\frac{(E+\mu)}{T} \right) \tag{3.1.33}$$

$$f_X(E) = \exp\left(-\frac{(E-\mu_X)}{T} \right) . \tag{3.1.34}$$

Chemical equilibrium, together with the process $b + \overline{b} \leftrightarrow 2\gamma$ and $\mu_\gamma = 0$, would tell us $\mu_b = -\mu_{\overline{b}}$.

This yields, confusing g_{*s} for g_* (which works well in the high-energy regime):

$$B = \frac{n_B}{s} = \frac{1}{2} \frac{n_b - n_{\overline{b}}}{1.8 g_{*s} n_\gamma} = \frac{g}{2 \times 1.8 g_*} \times \sinh\left(\frac{\mu}{T} \right) , \tag{3.1.35}$$

where the hyperbolic sine comes from the fact that we must consider the distribution functions, which translates to

$$f_b - f_{\overline{b}} \sim \exp\left(+\frac{\mu}{T} \right) - \exp\left(-\frac{\mu}{T} \right) \sim \sinh\left(\frac{\mu}{T} \right) . \tag{3.1.36}$$

We can expand the hyperbolic sine as μ/T since μ/T is small.

Using the aforementioned reactions, we get $\mu_X = 2\mu_b = 2\mu$, and also $\mu_X = -2\mu$, so both are zero, which means that $B = 0$. This illustrates how…

The expression for B can also be written as

$$B = \frac{n_B}{1.8 g_* n_\gamma} \,, \tag{3.1.37}$$

where we used the approximated expression

$$n_\gamma \approx \frac{2T^3}{\pi^2}\,; \tag{3.1.38}$$

this tells us

$$\frac{\mu}{T} \approx \frac{n_B}{n_\gamma}\,. \tag{3.1.39}$$

Now, let us consider the processes $X \to bb$ and $X \to \overline{bb}$ using the Boltzmann equation in the semiclassical approximation:

$$\begin{aligned}\dot{n}_X + 3Hn_X = &\int \mathrm{d}\Pi_X\, \mathrm{d}\Pi_1\, \mathrm{d}\Pi_2\, (2\pi)^4 \delta^{(4)}(p_X - p_1 - p_2) \\ &\left[-f_X|\mathcal{M}_0|^2 + f^{(1)}_{\bar{b}} f^{(2)}_{\bar{b}} \frac{1}{2}(1+\epsilon)|\mathcal{M}_0|^2 + f^{(1)}_b f^{(2)}_b \frac{1}{2}(1-\epsilon)|\mathcal{M}_0|^2\right],\end{aligned} \tag{3.1.40}$$

where by $f^{(i)}$ we mean $f(p_i)$, and

$$\mathrm{d}\Pi_i = \frac{g_i\, \mathrm{d}^3 p_i}{(2\pi)^3 2E_i}\,. \tag{3.1.41}$$

The products of distribution functions read

$$f^{(1)}_{b/\bar{b}} f^{(2)}_{b/\bar{b}} = \exp\left(\frac{-(E_1+E_2)}{T}\right) \exp\left(\pm\frac{2\mu}{T}\right) \tag{3.1.42}$$

$$\approx f_X^{\mathrm{eq}}(E_X) \exp\left(\pm\frac{2\mu}{T}\right) \tag{3.1.43}$$

$$\approx f_X^{\mathrm{eq}}(E_X) \left(1 \pm \frac{2\mu}{T}\right) \tag{3.1.44}$$

$$\approx f_X^{\mathrm{eq}}(E_X) \left(1 \pm 2\frac{n_B}{n_\gamma}\right). \tag{3.1.45}$$

We are working to first order in both ϵ and n_B/n_γ, products of these would be higher order. The BE then reads

$$\begin{aligned}\dot{n}_X + 3Hn_X = &\int \mathrm{d}\Pi_X\, \mathrm{d}\Pi_1\, \mathrm{d}\Pi_2\, (2\pi)^4 \delta^{(4)}(p_X - p_1 - p_2) \\ &\left[-f_X + f_X^{\mathrm{eq}}\right]|\mathcal{M}_0|^2\end{aligned}, \tag{3.1.46}$$

but

$$f_X = \exp\left(\frac{\mu_X}{T}\right) f_X^{\mathrm{eq}} = \frac{n_X}{n_X^{\mathrm{eq}}} f_X^{\mathrm{eq}}\,. \tag{3.1.47}$$

The square bracket in the BE then reads

$$-\frac{f_X^{\mathrm{eq}}}{n_X^{\mathrm{eq}}}\left[-n_X + n_X^{\mathrm{eq}}\right] = -\frac{f_X^{\mathrm{eq}}}{n_X^{\mathrm{eq}}} \langle M_0 \rangle^2\,. \tag{3.1.48}$$

Recall that $n_B = (n_b - n_{\overline{b}})/2$ (since the baryon number they carry is $\pm 1/2$): then, we need to write another Boltzmann equation here, without a $1/2$ factor since there are two baryons

$$\begin{aligned} \dot{n}_b + 3Hn_b = \int \mathrm{d}\Pi_X \,\mathrm{d}\Pi_1 \,\mathrm{d}\Pi_2 \,(2\pi)^4 \delta^{(4)}(p_X - p_1 - p_2) \\ \left[-(1-\epsilon) f_b^{(1)} f_b^{(2)} + (1+\epsilon) f_X \right] |\mathcal{M}_0|^2 \end{aligned} . \tag{3.1.49}$$

However, we must also account for X-mediated scatterings $bb \to \overline{bb}$ and vice versa: this means we need to add a new term to the right-hand side of the BE,

$$2 \int \mathrm{d}\Pi_1 \,\mathrm{d}\Pi_2 \,\mathrm{d}\Pi_3 \,\mathrm{d}\Pi_4 \,\delta^{(4)}(p_1 + p_2 - p_3 - p_4) \left[-f_b^{(1)} f_b^{(2)} \left| \mathcal{M}'(bb \to \overline{bb}) \right|^2 + f_{\overline{b}}^{(1)} f_{\overline{b}}^{(2)} \left| \mathcal{M}'(\overline{bb} \to bb) \right|^2 \right] . \tag{3.1.50}$$

The quantity $\mathcal{M}'$ is the Feynman amplitude of the scattering process where the physical intermediate X has been removed; we have already counted processes in which a X was formed and then decayed. Instead, the processes we were missing and which we are now accounting for are the ones in which X is virtual (i. e. not on shell). Then,

$$\left| \mathcal{M}(bb \to \overline{bb}) \right|^2 = \left| \mathcal{M}'(bb \to \overline{bb}) \right|^2 + \left| \mathcal{M}_{\text{ris}}(bb \to \overline{bb}) \right|^2 . \tag{3.1.51}$$

In the end the equation for X is written as

$$\dot{n}_X + 3Hn_X = -\Gamma_D (n_X - n_X^{\text{eq}}) . \tag{3.1.52}$$

We then find that the bracket to be integrated in the BE for $n_B = (n_b - n_{\overline{b}})/2$ is

$$\frac{1}{2} \left[-(1-\epsilon) f_b^{(1)} f_b^{(2)} + (1+\epsilon) f_X + (1+\epsilon) f_{\overline{b}}^{(1)} f_{\overline{b}}^{(2)} \right] |\mathcal{M}_0|^2 , \tag{3.1.53}$$

and we can make the estimate

$$f_{b/\overline{b}}^{(1)} f_{b/\overline{b}}^{(2)} \approx f_X^{\text{eq}} \left(1 \pm 2 \frac{n_B}{n_\gamma} \right) . \tag{3.1.54}$$

This allows us to write the expression in (3.1.53) as

$$\left[\epsilon f_X^{\text{eq}} + \epsilon f_X \right] |\mathcal{M}_0|^2 - 2 f_X^{\text{eq}} \frac{n_B}{n_\gamma} |\mathcal{M}_0|^2 . \tag{3.1.55}$$

Every time n_X^{eq} appears, it is coming from inverse decay processes. What we have found is a Boltzmann equation for $n_B = (n_b / n_{\overline{b}})/2$. There seems to be a problem: in what we have found we have something like $n_X + n_X^{\text{eq}}$, which is weird! This is actually fixed if we account for scatterings.

The scattering term yields

$$4 \times \left[-f_b^{(1)} f_b^{(2)} \left| \mathcal{M}'(bb \to \overline{bb}) \right|^2 + f_{\overline{b}}^{(3)} f_{\overline{b}}^{(4)} \left| \mathcal{M}'(\overline{bb} \to bb) \right|^2 \right] . \tag{3.1.56}$$

There can be a certain part of scattering processes which do not violate CP; for these, the backward and forward amplitudes are equal:

$$\left|\mathcal{M}'(bb \to \overline{bb})\right|^2 = \left|\mathcal{M}'(\overline{bb} -> bb)\right|^2, \tag{3.1.57}$$

therefore, using equation (3.1.54), we get

$$-4\frac{n_B}{n_\gamma} f_X^{\text{eq}} \left|\mathcal{M}'(\overline{bb} -> bb)\right|^2. \tag{3.1.58}$$

Now, instead, let us consider the CP violating part of the processes: neglecting higher order terms, we find

$$-2f_X^{\text{eq}} \left(-\left|\mathcal{M}'(\overline{bb} \to bb)\right|^2 + \left|\mathcal{M}'(bb \to \overline{bb})\right|^2\right). \tag{3.1.59}$$

Finally then,

$$\left|\mathcal{M}'\right|^2 = |\mathcal{M}|^2 - |\mathcal{M}_{\text{ris}}|^2, \tag{3.1.60}$$

and using CPT plus unitarity we can say that for the *total amplitude* we have $\left|\mathcal{M}(bb \to \overline{bb})\right|^2 = \left|\mathcal{M}(\overline{bb} \to bb)\right|^2$.

Then we get

$$-2f_X^{\text{eq}} \left[\left(-\left|\mathcal{M}_{\text{ris}}(\overline{bb} \to bb)\right|^2 + \left|\mathcal{M}_{\text{ris}}(bb \to \overline{bb})\right|^2\right)\right]. \tag{3.1.61}$$

Here there would be another two pages of computations to do; what we find is

$$\left|\mathcal{M}_{\text{ris}}(bb \to \overline{bb})\right|^2 \sim |\mathcal{M}(bb \to X)|^2 \left|\mathcal{M}(X \to \overline{bb})\right|^2 \propto \frac{1}{4}(1-\epsilon)^2 |\mathcal{M}_0|^4, \tag{3.1.62}$$

so finally we will get

$$-2f_X^{\text{eq}} \epsilon |\mathcal{M}_0|^2. \tag{3.1.63}$$

The thermally averaged decay rate reads

$$\Gamma_D = \frac{1}{n_X^{\text{eq}}} \int \mathrm{d}\Pi_X \, \mathrm{d}\Pi_1 \, \mathrm{d}\Pi_2 \, \delta^{(4)}(p_X - p_1 - p_2) f_X^{\text{eq}} |\mathcal{M}0|^2. \tag{3.1.64}$$

So, the Boltzmann equation for n_B reads

$$\dot{n}_B + 3Hn_B = +\underbrace{\epsilon\Gamma_D \left(n_X - n_X^{\text{eq}}\right)}_{X \text{ decay}} - \underbrace{2\Gamma_D \left(\frac{n_X^{\text{eq}}}{n_\gamma}\right) n_B}_{X \text{ inverse decay}} - \underbrace{4n_B n_\gamma \left\langle \sigma |v| \right\rangle}_{\text{scattering}}, \tag{3.1.65}$$

where

$$\left\langle \sigma|v| \right\rangle = \frac{1}{n_\gamma^2} \int \mathrm{d}\Pi_1 \,\mathrm{d}\Pi_2 \,\mathrm{d}\Pi_3 \,\mathrm{d}\Pi_4 \, f_X^{\mathrm{eq}} \left| \mathcal{M}'(bb \to \overline{bb}) \right|^2 . \tag{3.1.66}$$

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We start by writing down the two Boltzmann equations we derived, using the notation of a dot for $\partial/\partial t$:

$$\dot{n}_X + 3Hn_X = -\Gamma_D \left(n_X - n_X^{\mathrm{eq}} \right) \tag{3.1.67}$$

$$\dot{n}_B + 3Hn_B = +\epsilon\Gamma_D \left(n_X - n_X^{\mathrm{eq}} \right) - 2\Gamma_D \left(\frac{n_X^{\mathrm{eq}}}{n_\gamma} \right) n_B - 4n_B n_\gamma \left\langle \sigma|v| \right\rangle . \tag{3.1.68}$$

The source term shows the need for all three of the Sacharov conditions: $\epsilon \neq 0$ quantifies C and CP violation (as well as baryon number violation), $n_X - n_X^{\mathrm{eq}} \neq 0$ quantifies out-of-equilibrium processes.

We define the variable $z = m_X/T$ (not redshift!) and $X = n_X/s$, where s is the entropy density: X measures the number of X particles in a comoving volume. Also, we define $B = n_B/s$. The Boltzmann equations then become:

$$\dot{n}_X + 3Hn_X = -\Gamma_D \left(n_X - n_X^{\mathrm{eq}} \right) \tag{3.1.69}$$

$$\underbrace{a^{-3} \frac{\partial}{\partial t} \left(n_X a^3 \right)}_{\propto s\dot{X}} = -\Gamma_D \left(n_X - n_X^{\mathrm{eq}} \right) \tag{3.1.70}$$

$$\dot{X} = -\Gamma_D (X - X^{\mathrm{eq}}) , \tag{3.1.71}$$

while the other can be written using a derivative with respect to z, denoted with a prime: since $\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}a} \frac{\mathrm{d}a}{\mathrm{d}t} = zH$, which comes from $T \propto 1/a$, we have

$$X' = -\frac{\Gamma_D}{zH} (X - X^{\mathrm{eq}}) \tag{3.1.72}$$

$$= -\frac{z\Gamma_D}{z^2 H} (X - X^{\mathrm{eq}}) , \tag{3.1.73}$$

but

$$z^2 H = \frac{m_X^2}{T^2} g_*^{1/2} \frac{T^2}{m_P} = H(z=1) , \tag{3.1.74}$$

therefore

$$X' = -\frac{z\Gamma_D}{H(z=1)} (X - X^{\mathrm{eq}}) \tag{3.1.75}$$

$$= -z \underbrace{\frac{\Gamma_D(z)}{\Gamma_D(z=1)}}_{\gamma_D(z)} \underbrace{\frac{\Gamma_D(z=1)}{H(z=1)}}_{K} (X - X^{\mathrm{eq}}) \tag{3.1.76}$$

$$= -z\gamma_D(z) K \underbrace{(X - X^{\mathrm{eq}})}_{\Delta(z)} \tag{3.1.77}$$

$$\Delta' = -X'_{\text{eq}} - z\gamma_D K\Delta\,. \tag{3.1.78}$$

K is defined in eq. (3.1.21).

The fact that we are in an expanding universe is crucial: without expansion $X'_{\text{eq}} \equiv 0$, therefore we obtain an exponential decay of the deviation from equilibrium of X.

For the baryon number, on the other hand, with similar steps as before we get

$$B' = +\epsilon z\gamma_D(z)K\Delta(z) - zK\gamma_B(z)B\,, \tag{3.1.79}$$

where we can approximate, using (3.1.9) and $2 \times 1.8 \approx 4$:

$$\gamma_B(z) \approx -4\gamma_D(z)X^{\text{eq}}g_* + 4\frac{n_\gamma \langle\sigma|v|\rangle}{\Gamma_D(z=1)}\,. \tag{3.1.80}$$

To summarize, the equations are

$$\Delta' = -X'_{\text{eq}} - z\gamma_D(z)K\Delta \tag{3.1.81}$$

$$B' = +\epsilon zK\gamma_D(z)\Delta - zK\gamma_B(z)B\,. \tag{3.1.82}$$

The variable X_{eq} can be expressed in the relativistic and nonrelativistic cases respectively as

$$X_{\text{eq}}(z) = \frac{n_X^{\text{eq}}}{s} = \begin{cases} g_*^{-1} & z \ll 1 \\ g_*^{-1}z^{3/2}e^{-z} & z \gg 1 \end{cases}, \tag{3.1.83}$$

See (2.3.20).

while

$$\gamma_D(z) = \frac{\Gamma_D(z)}{\Gamma_D(z=1)} = \begin{cases} z & z \ll 1 \\ 1 & z \gg 1 \end{cases}, \tag{3.1.84}$$

and

$$\langle\sigma|v|\rangle \sim \frac{\alpha^2 T^2 A}{(T^2 + m_x^2)^2}\,, \tag{3.1.85}$$

See (2.2.8).

and finally

$$\gamma_B(z) \approx -4\gamma_D(z)X^{\text{eq}}g_* + \frac{4n_\gamma \langle\sigma|v|\rangle}{\Gamma_D(z=1)} \tag{3.1.86}$$

$$= \begin{cases} z + \frac{A\alpha}{z} & z \ll 1 \\ z^{3/2}e^{-z} + A\alpha z^{-5} & z \gg 1 \end{cases}. \tag{3.1.87}$$

The two terms in the relativistic $z \gg 1$ regime account for inverse decay and $2 \leftrightarrow 2$ CP conserving scattering respectively.

The solutions read

$$\Delta(z) = \Delta_i \exp\left[-\int_0^z z'\gamma_D(z')K\,\mathrm{d}z'\right] - \int_0^z X'_{\text{eq}}(z')\exp\left[-\int_{z'}^z z''K\gamma_D(z'')\,\mathrm{d}z''\right]\mathrm{d}z' \tag{3.1.88}$$

$$B(z) = B_i \exp\left[-\int_0^z z'\gamma_B(z')K\,\mathrm{d}z'\right] + \epsilon K \int_0^z z'\gamma_D(z')\Delta(z')\exp\left[-\int_{z'}^z z''K\gamma_B(z'')\,\mathrm{d}z''\right]\mathrm{d}z' \,. \tag{3.1.89}$$

We start with a thermal distribution for both baryons and X: $\Delta_i = B_i = 0$. As long as $K \ll 1$, we are able to obtain a baryon number of the order of $B \approx \epsilon/g_*$.

We will consider the $K \ll 1$, $z \gg 1$ case: then, $\gamma_D(z) \approx 1$ and $\gamma_B \approx 0$, so

$$\Delta(z) \approx \int_0^z X'_{\mathrm{eq}}(z')\,\mathrm{d}z'\, e^{-kz^2/2} \tag{3.1.90}$$

$$= \Big[X_{\mathrm{eq}}(0) - X_{\mathrm{eq}}(z)\Big]e^{-kz^2/2} \,. \tag{3.1.91}$$

Then we also get

$$B(z) = \epsilon K \int_0^z z' \left[X_{\mathrm{eq}}(0) - \underbrace{X_{\mathrm{eq}}(z')}_{\approx 0}\right] e^{-Kz'^2/2}\,\mathrm{d}z' \tag{3.1.92}$$

Boltzmann suppression.

$$\approx \epsilon K X_{\mathrm{eq}}(0) \int_0^z z' e^{-kz'^2/2}\,\mathrm{d}z' \tag{3.1.93}$$

$$= \epsilon X_{\mathrm{eq}}(0) \int_0^z \frac{\mathrm{d}}{\mathrm{d}z'}\Big(-e^{-kz'^2/2}\Big)\,\mathrm{d}z' \tag{3.1.94}$$

$$= \epsilon X_{\mathrm{eq}}(0)\Big[1 - e^{-kz^2/2}\Big] \tag{3.1.95}$$

$$\to \epsilon X_{\mathrm{eq}}(0) = \frac{\epsilon}{g_*} \,, \tag{3.1.96}$$

See (3.1.25).

in the limit of $z \gg 1$. What do we expect in the opposite regime, $K \gg 1$?

Recall that the differential equation reads

$$\Delta' = -X'_{\mathrm{eq}} - zK\gamma_D\Delta \,, \tag{3.1.97}$$

therefore we expect Δ' to be small, since $K \gg 1$ means that the decays of X particles are very efficient (and it can be shown that $\Delta' < \Delta$): this means that we can take $\Delta' \approx 0$, so that

$$\Delta \approx -\frac{X'_{\mathrm{eq}}}{zK\gamma_D(z)} \,, \tag{3.1.98}$$

but we know

$$X_{\mathrm{eq}}(z) \sim z^{3/2}e^{-z} \tag{3.1.99}$$

$$X'_{\mathrm{eq}} \sim z^{1/2}e^{-z} - \underbrace{z^{3/2}e^{-z}}_{-X_{\mathrm{eq}}} \,, \tag{3.1.100}$$

and the last term is the leading one.

Then,

$$\Delta \approx \frac{X_{\mathrm{eq}}(z)}{zK} \propto \frac{1}{K} \,. \tag{3.1.101}$$

For $K \gtrsim 1$ we will have

$$B_f = B(z \to \infty) = \epsilon K \int_0^\infty z\Delta(z)\gamma_D(z) \exp\left(- \int_z^\infty z' K \gamma_B(z') \, \mathrm{d}z' \right) . \tag{3.1.102}$$

What is the physical significance of γ_B? It tends to **damp** the baryon asymmetry. We can define an epoch of freeze-out as the z_f at which $z_f K \gamma_B(z_f) \approx 1$, since after this epoch we get $zK\gamma_B(z) < 1$.

We can use the Laplace, or saddle-point approximation: if the function g has a global maximum at x_0, we have

$$\int_a^b h(x) e^{cg(x)} \, \mathrm{d}x \approx h(x_0) e^{cg(x_0)} \times \sqrt{\frac{2\pi}{|g''(x_0)|}} . \tag{3.1.103}$$

In our case,

$$cg(x) = - \int_z^\infty zK\gamma_B(z') \, \mathrm{d}z' \,, \tag{3.1.104}$$

so we will have our maximum at $z = z_f$, the freezeout epoch.

The approximation then yields

$$B_f \approx \epsilon K z_f \frac{X_{\mathrm{eq}}(z_f)\gamma_D}{z_f K \gamma_D(z_f)} \underbrace{\exp\left(- \int_{z_f}^\infty zK\gamma_B(z) \, \mathrm{d}z \right)}_{\approx 1} \sqrt{\frac{2\pi}{\left|zK\gamma_b\right|'\Big|_{z_f}}} . \tag{3.1.105}$$

The integrand is $\ll 1$ because of the suppression of γ_B.

Added a γ_D in the numerator, evaluated where?

so finally, using $X_{\mathrm{eq}}(z_f) = g_*^{-1} z^{3/2} e^{-z}$, we get

$$B_f \approx \frac{\epsilon}{g_*} z_f^{3/2} e^{-z_f} \frac{1}{\sqrt{(zK\gamma_B)'}} . \tag{3.1.106}$$

For $K \geq 1$ inverse decays dominate vs scattering processes.

Add comparison

In this case then the term under the square root looks like $z_f K \gamma_B(z_f) \approx 1$ (from the definition of z_f), which means $z_f K z_f^{3/2} e^{-z_f} \approx 1$, which can be solved to yield $z_f \approx 4(\log K)^{0.6}$.

skipped some steps

Making the derivative explicit we find

$$B_f \approx \frac{\epsilon}{g_*} z_f^{3/2} e^{-z_f} \underbrace{\frac{1}{\sqrt{z_f^{5/2} K e^{-z_f}}}}_{\approx 1} \tag{3.1.107}$$

$$\approx \frac{\epsilon}{g_*} \frac{1}{4K(\log K)^{0.6}} . \tag{3.1.108}$$

In the $K \gg 1$ case, on the other hand, scattering dominates and γ_B gets a contribution from only the second term in (3.1.86): then $K A \alpha z_f^{-4} \approx 1$, so $z_f \approx (KA\alpha)^{1/4}$. This gives us

$$B_f = \frac{\epsilon}{g_*} \frac{z_f^{3/2} e^{-z_f}}{\sqrt{\left|(zK\gamma_B)'\right|}} . \tag{3.1.109}$$

This time the denominator looks like

$$(zK\gamma_B)' = -4A\alpha z^{-5} \approx -\frac{4}{z} , \tag{3.1.110}$$

therefore

$$B_f = \frac{\epsilon}{g_*} \frac{z_f^{3/2} e^{-z_f}}{z_f^{-1/2}} \tag{3.1.111}$$

$$= \frac{\epsilon}{g_*} z_f^2 e^{-z_f} \tag{3.1.112}$$

$$= \frac{\epsilon}{g_*} (KA\alpha)^{1/2} \exp\Big(-(KA\alpha)^{1/4} \Big) . \tag{3.1.113}$$

We can find the z_f at which inverse decays and scatterings are balances:

$$z_f = 4\big(\log K\big)^{0.6} = (AK\alpha)^{1/4} , \tag{3.1.114}$$

which yields

$$K_c \approx \frac{300}{A\alpha} . \tag{3.1.115}$$

If $1 < K < K_c$, then inverse decays dominate over scatterings; if $K > K_c$ on the other hand scattering dominate, and B_f is exponentially suppressed.

The value of K is determined as

$$K = \frac{\alpha m_P}{g_*^{1/2} m_X} , \tag{3.1.116}$$

and it will depend on the specifics of the X particle we are interested in.

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We will consider another kind of baryogenesis mechanism, connected to the reheating phase.

The baryon asymmetry generated by the out-of-equilibrium decays of X particles is of the order of

$$B \approx \frac{\epsilon}{g_*} , \tag{3.1.117}$$

where $g_* \approx 10^2$ for the SM, or even $g_* \approx 10^3$ BTSM.

A possible approach is to identify the X particle as the inflaton φ.

We will use the usual notations: n_X, n_B, and ϵ for the amount of CP violation. The Boltzmann equation reads

$$\dot{n}_X + 3Hn_X = -\Gamma_X n_X \,, \tag{3.1.118}$$

which is equivalent to

$$\dot{\rho}_\varphi + 3H\rho_\varphi = -\Gamma_\varphi \rho_\varphi \,, \tag{3.1.119}$$

with the identification $\rho_\varphi = m_X n_X$, where m_X is the effective of the inflaton field before the start of oscillations.

The right-hand side should read $-\Gamma_X(n_X - n_X^{\mathrm{eq}})$, but we approximated it; the BE for baryon asymmetry on the other hand is

$$\dot{n}_B + 3Hn_B = +\epsilon\Gamma_X n_X \,, \tag{3.1.120}$$

and we also need to write the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\Big(\rho_\varphi + \rho_R\Big) \,. \tag{3.1.121}$$

The number density of X decays like

$$n_X = n_{X,i}\left(\frac{a_i}{a}\right)^3 e^{-\Gamma_\varphi(t-t_i)} \,, \tag{3.1.122}$$

where t_i is the "oscillation time".

As usual, we solve the BE for baryon asymmetry by writing the LHS as

$$\frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t} \,, \tag{3.1.123}$$

therefore

$$n_B a^3 = \epsilon\Gamma_X \int_{t_i}^{t} n_{X,i}\left(\frac{a_i}{a}\right)^3 a^3 e^{-\Gamma_\varphi(t-t_i)}\,\mathrm{d}t \;. \tag{3.1.124}$$

Doing the integral we get,

$$n_B a^3 = \epsilon n_{X,i} a_i^3\Big(1 - e^{-\Gamma_\varphi(t-t_i)}\Big) \,. \tag{3.1.125}$$

At late times, we will get

$$n_B a_f^3 = \epsilon n_{X,i} a_i^3 \,. \tag{3.1.126}$$

This has a very simple physical meaning: the left-hand side counts the number of net (signed) baryons in a comoving volume, while the right-hand side is ϵ multiplied by the number of X particles in a comoving volume.

The final B is given by

$$B_f = \left.\frac{n_B}{s}\right|_{t_f} = \frac{n_B a_f^3}{s a_f^3} = \frac{\epsilon n_{X,i} a_i^3}{S_f}\,, \tag{3.1.127}$$

but what is S_f, the total entropy at a late time? There are two ways to compute it: one is to solve the thermodynamic differential equations, and the alternative which we will adopt.

We write it as

$$S_f = s_f a_f^3\,, \tag{3.1.128}$$

and we know that in general for radiation domination

$$s = \frac{2\pi^2}{45} g_* T^3\,, \tag{3.1.129}$$

and also

$$\rho_R = \frac{\pi^2}{30} g_* T^4 \implies T = g_*^{-1/4} \rho_R^{1/4} \left(\frac{30}{\pi^2}\right)^{1/4}\,, \tag{3.1.130}$$

therefore

$$S = s a^3 = g_*^{1/4} \rho_R^{3/4} a^3\,. \tag{3.1.131}$$

Using this estimate, we can write

$$B_f = \frac{\epsilon n_{X,i} a_i^3}{g_{*f}^{1/4} \rho_{R,f}^{3/4} a_f^3}\,. \tag{3.1.132}$$

We can build a hierarchy of timescales, and consider a time t such that: $t_i = t_{\text{oscill}} < t \leq t_\varphi = \Gamma_\varphi^{-1}$. In this phase, φ dominates. Then, with the assumption that there is complete conversion of the energy between φ and radiation, we can write

$$\rho_{R,f}^{3/4} = \rho_{X,i}^{3/4} \left(\frac{a_i}{a_f}\right)^{9/4}\,, \tag{3.1.133}$$

so, since $a \propto t^{2/3}$ (matter domination):

$$B_f = \frac{\epsilon n_{X,i} a_i^{3/4}}{g_{*f}^{1/4} \rho_{X,i}^{3/4} a_f^{3/4}} \tag{3.1.134}$$

$$= \frac{\epsilon n_{X,i}}{g_*^{1/4} \rho_{X,i}^{3/4}} t_i^{1/2} \Gamma_\varphi^{1/2}\,, \tag{3.1.135}$$

therefore in Friedmann's equation we have

$$H_i^2 = \frac{8\pi}{3} \frac{\rho_{X,i}}{m_P^2} \approx \frac{1}{t_i^2}\,, \tag{3.1.136}$$

therefore $t_i \approx m_P \rho_{X,i}^{-1/2}$. Plugging this in and using $\rho_{X,i} = m_X n_{X,i}$, we get

$$B_f \approx \frac{\epsilon g_*^{-1/4} m_P^{1/2} \Gamma_\varphi^{1/2}}{m_X} \tag{3.1.137}$$

$$\approx \epsilon \frac{T_{RH}}{m_X}, \tag{3.1.138}$$

where we recognize the approximate expression for the reheating temperature (2.1.21).

This expression should be compared with $B_f = \epsilon / g_*$, the alternative. Which is larger? For $k \ll 1$, the mass of the bosons violating baryon number must be quite large: $m_X \gtrsim 10^{10}$ GeV. In order for these to be relativistic, the temperature must then also be very large: $T \gtrsim m_X \sim 10^{10}$ GeV, which constrains the reheating temperature.

In this alternative scenario, with the inflaton violating baryon number conservation, we are allowed to violate this constraint.

Are B, C and CP violated in the SM? Yes, in some nonperturbative electroweak processes, but not enough.

3.2 Dark matter production

We will start with the "freeze-out" mechanism. From the latest Planck data, we know that at a 68 % CL:

$$\Omega_{DM} h^2 = 0.120 \pm 0.001 \tag{3.2.1}$$

$$h = 0.674 \pm 0.005, \tag{3.2.2}$$

therefore $\Omega_{DM} \approx 26.4\,\%$. Dark Matter as measured here must be a non-relativistic pressureless fluid.

Consider a massive particle ψ, with mass m_ψ. Let us define

$$y = n_\psi / s. \tag{3.2.3}$$

As long as $\Gamma_\psi < H$, the abundance of ψ "freezes out" when $z = m_\psi / T$ reaches 1.

The Boltzmann equation for a process $\psi\overline{\psi} \leftrightarrow X\overline{X}$ where X is in equilibrium with the plasma reads

$$\dot{n}_\psi + 3H n_\psi = -\langle \sigma |v| \rangle \left[n_\psi^2 - (n_\psi^{\text{eq}})^2 \right]. \tag{3.2.4}$$

Then,

$$\dot{y} = -\langle \sigma |v| \rangle s \left[y^2 - y_{\text{eq}}^2 \right], \tag{3.2.5}$$

therefore, because

$$\frac{\mathrm{d}z}{\mathrm{d}t} = zH, \tag{3.2.6}$$

we can express the same with derivatives with respect to z, denoted with primes:

$$y' = -\frac{\langle \sigma |v| \rangle s}{zH}\left[y^2 - y_{\text{eq}}^2\right] \tag{3.2.7}$$

$$= -\frac{z \langle \sigma |v| \rangle s}{H(z=1)}\left[y^2 - y_{\text{eq}}^2\right], \tag{3.2.8}$$

since $z^2 H = H(z=1)$, which corresponds to the moment at which $m_\psi = T$.

$$y_{\text{eq}} = \frac{n_\psi^{\text{eq}}}{s} = \begin{cases} 0.278 \frac{g_{\text{eff}}}{g_{*s}} & z \ll 1 \\ 0.145 \frac{g_\psi}{g_{*s}} z^{3/2} e^{-z} & z \gg 1 \,. \end{cases} \tag{3.2.9}$$

In the $z \gg 1$ limit inverse processes are suppressed.
The effective number of degrees of freedom is given by

$$g_{\text{eff}} = \begin{cases} g_\psi & \text{boson} \\ \frac{3}{4} g_\psi & \text{fermion} \,. \end{cases} \tag{3.2.10}$$

We can write the derivative of y with respect to z as

$$y' = -y_{\text{eq}}^2 \frac{\langle \sigma |v| \rangle s}{zH}\left[\left(\frac{y}{y_{\text{eq}}}\right)^2 - 1\right] \tag{3.2.11}$$

$$\frac{z}{y_{\text{eq}}} y' = -\underbrace{n_{\text{eq}}^\psi \frac{\langle \sigma |v| \rangle s}{H}}_{\Gamma_A / H}\left[\left(\frac{y}{y_{\text{eq}}}\right)^2 - 1\right]. \tag{3.2.12}$$

If $\Gamma_A / H < 1$, then the right-hand side becomes small, so the left-hand side (which is basically $\Delta y / y$ in a comoving volume) must also be small, meaning that y is roughly constant (frozen out): we have produced a relic abundance of ψ particles.

3.2.1 Hot Dark Matter relics

These particles are called "hot" since they **decouple while still being relativistic**. This is what we would expect for neutrinos, for example.

This particle decouples in the upper plateau of the curve $y(z)$, so we expect little dependence for it on z_f.

Here we will have

$$y_\infty = y(z \to \infty) \approx y_{\text{eq}}(z_f) \tag{3.2.13}$$

$$\approx \frac{\zeta(3) g_{\text{eff}} T_f^3 / \pi^2}{(\pi^2/45) g_{*s}(z_f) T_f^3} \approx 0.278 \frac{g_{\text{eff}}}{g_{*s}(z_f)} \,. \tag{3.2.14}$$

Used (3.2.9).

Now,

$$\rho_{\psi 0} = m_\psi n_{\psi 0} = m_\psi y_\infty s_0 \,, \tag{3.2.15}$$

where $s_0 \approx 1.8 g_{*s}(t_0) n_{\gamma 0}$, where $n_{\gamma 0} \approx 422\,\text{cm}^{-3}$.

Assuming that there is one relativistic neutrino species (this is conservative, we know there to be at least 2), we can write

$$g_{*s}(t_0) \approx 2.63 \,, \tag{3.2.16}$$

therefore $s_0 \approx 2 \times 10^3$. This means that

$$\rho_{0,\text{crit}} = 1.054 \times 10^4\,\text{eVcm}^{-3} h^2 \,, \tag{3.2.17}$$

and

$$\Omega_{0\psi} h^2 \approx 5.56 \times 10^{-2} \times \frac{m_\psi}{\text{eV}} \frac{g_{\text{eff}}}{g_{*s}(z_f)} \,. \tag{3.2.18}$$

In our case, $g_{\text{eff}} = (3/4) \times 2$ since we have one neutrino species (with two polarizations), while $g_{*s}(z_f) = 10.75$, accounting for three neutrinos, electrons, positrons and photons.

From Planck data, including lensing of the CMB and BAO, we know that at a 95 % CL:

$$\sum_\nu m_\nu < 0.12\,\text{eV} \,. \tag{3.2.19}$$

They will surely be relativistic when they decouple at $T_f \sim$ MeV. This tells us that, if neutrinos are ψ,

$$\Omega_{0\nu\bar{\nu}} h^2 = \frac{(\sum_\nu m_\nu/\text{eV})}{128} \lesssim 10^{-3} \,. \tag{3.2.20}$$

Thus, we learn that neutrinos cannot constitute the majority of DM. Despite this, they have a characteristic signature effect on structure formation.

The free-stream scale is given by

$$\lambda_{\text{fs}}(t) = a(t) \int_0^t \frac{v_\nu}{a(\widetilde{t})} \,\mathrm{d}\widetilde{t} \,. \tag{3.2.21}$$

Below this scale ($\lambda < \lambda_{\text{fs}}$), neutrinos move too fast and damp the power spectrum of matter.

3.2.2 Cold Dark Matter relics

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CDM particles, definitionally, decouple when they are *not* relativistic, so $z_f = m_\psi/T > 1$ at freezeout; recall that freezeout is defined as the moment when the inequality $\Gamma_A/A \lesssim 1$ starts to be satisfied, where Γ_A refers to the annihilation process $\psi\overline{\psi} \leftrightarrow X\overline{X}$.

If we define $y = n_\psi / s$, we can draw the usual decay curve $y = y(z)$, which is constant and then has a decay (something like an inverted sigmoid); at decoupling the amount stops changing, and its value depends on z at that time. The Boltzmann equation reads

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{-zs \left\langle \sigma_A |v| \right\rangle}{H(z=1)} \left(y^2 - y_{\mathrm{eq}}^2 \right) . \tag{3.2.22}$$

Typically we can model the term $\left\langle \sigma_A |v| \right\rangle$ as a powerlaw:

$$\left\langle \sigma_A |v| \right\rangle \sim T^n \tag{3.2.23}$$

$$\left\langle \sigma_A |v| \right\rangle = \sigma_0 \left(\frac{T}{m} \right)^n = \sigma_0 \overline{z}^{-n} , \tag{3.2.24}$$

therefore the differential equation for the abundance y reads

$$\frac{\mathrm{d}y}{\mathrm{d}z} = -\frac{\lambda}{z^{2+n}} \left(y^2 - y_{\mathrm{eq}}^2 \right) \tag{3.2.25}$$

$$\lambda = \frac{z^3 s \sigma_0}{H(z=1)} , \tag{3.2.26}$$

but recall that

$$z^3 = \left(\frac{m}{T} \right)^3 \tag{3.2.27}$$

$$s = \frac{2\pi^2}{45} g_* T^3 \tag{3.2.28}$$

$$H(z=1) = \left(\frac{8\pi G}{3} \right)^{1/2} \left(\frac{\pi^2}{30} \right)^{1/2} g_*^{1/2} \underbrace{T^2 \Big|_{z=1}}_{=m_\psi^2} , \tag{3.2.29}$$

See (3.1.74).

and with all these substitutions we can write

$$\lambda \approx 0.264 m_P m_\psi \sigma_0 \frac{g_{*s}}{g_*^{1/2}} . \tag{3.2.30}$$

Further, we can model the equilibrium abundance as

$$y_{\mathrm{eq}} = 0.145 \left(\frac{g_*}{g_{*s}} \right) z^{3/2} e^z , \tag{3.2.31}$$

with $z > 1$.

Now, let us define $\Delta = y - y_{\mathrm{eq}}$; then, denoting derivatives with respect to z with primes we get

$$\Delta' = -y'_{\mathrm{eq}} - \frac{\lambda}{z^{2+n}} \left(\Delta + 2y_{\mathrm{eq}} \right) \Delta , \tag{3.2.32}$$

which can be solved by distinguishing two regimes: the first is for **early times**, defined by $1 < z < z_f$, meaning that the particle is relativistic but still coupled. This tells us that both Δ and Δ' are small; with this we can write

$$\begin{aligned} \Delta &\approx \frac{z^{2+n}}{\lambda} \frac{-y'_{\text{eq}}}{\Delta + 2y_{\text{eq}}} && (3.2.33) \\ &\approx \frac{z^{2+n}}{\lambda} \frac{-y'_{\text{eq}}}{2y_{\text{eq}}} && (3.2.34) \\ &\approx \frac{z^{2+n}}{\lambda} \frac{y_{\text{eq}}}{2y_{\text{eq}}} = \frac{z^{2+n}}{2\lambda}, && (3.2.35) \end{aligned}$$

since when we differentiate y_{eq} one term is negligible.

I think, to check.

On the other hand, at **late times** $z \gg z_f$ we get $y(z) \gg y_{\text{eq}}(z)$,[1] therefore $\Delta \approx y$. With this approximation, we get

$$\Delta' = -\frac{\lambda}{z^{2+n}} \Delta^2, \qquad (3.2.36)$$

and solving this we find

$$\begin{aligned} -\frac{\Delta'}{\Delta^2} &\approx \frac{\lambda}{z^{2+n}} && (3.2.37) \\ \frac{1}{\Delta_\infty} - \frac{1}{\Delta_f} &\approx -\frac{\lambda}{n+1} z^{-n-1} \Big|_{z=z_f}^{z=\infty} && (3.2.38) \\ \frac{1}{\Delta_\infty} - \frac{1}{\Delta_f} &\approx +\frac{\lambda}{z_f^{n+1}(n+1)}, && (3.2.39) \end{aligned}$$

but $y_\infty < y_f$, meaning that we can neglect the y_f term and so

$$y_\infty \approx \frac{z_f^{n+1}(n+1)}{\lambda} \approx \frac{3.79(n+1)z_f^{n+1}}{(g_{*s}/g_*^{1/2}) m_P m_\psi \sigma_0}. \qquad (3.2.40)$$

From this, using $n_\psi = s_0 y_\infty$ and the expression $s_0 \approx 2 \times 10^3 \,\text{cm}^{-3}$ (if one neutrino species is relativistic today), then we can calculate

$$\Omega_{\psi_0} h^2 = \frac{0.75 \times 10^9 (n+1) z_f^{n+1}}{(g_{*s}/g_*^{1/2}) m_P \sigma_0} \text{GeV}^{-1}. \qquad (3.2.41)$$

Here we used $\Omega_{\psi_0} = m_\psi n_{\psi_0} / \rho_{\text{crit}}$, with $\rho_{\text{crit}} \approx 10^4 \,\text{eVcm}^{-3}\text{h}^2$. We appear to have eliminated the dependence on m_ψ, but this is not completely the case: σ_0, z_f and g_* still depend (at least weakly) on it.

[1] This is true since y is the *actual* amount of particles in a comoving volume, while y_{eq} is the amount which *would* be reached if there were equilibrium, which cannot happen due to decoupling.

The crucial result from this manipulation is

$$\Omega_{\psi_0} h^2 \propto \frac{1}{\sigma_0} . \tag{3.2.42}$$

This is somewhat intuitive: if the particle interacts more then its abundance has more time to decrease with the Boltzmann suppression.

Manipulating the expression with some typical values, we get

$$\Omega_{\psi_0} h^2 \approx \mathcal{O}(1)(n+1)\left(\frac{z_f}{10}\right)^{n+1}\left(\frac{g_*}{100}\right)^{1/2}\left(\frac{100}{g_{*s}}\right)\frac{10^{-38}\,\mathrm{cm}^2}{\sigma_0} . \tag{3.2.43}$$

The natural normalization arising for σ_0 is 10^{-2} pbn (picobarns), a very small cross-section, which is the typical order of magnitude of weak-interaction cross-sections. This is what is called the **WIMP miracle**, where WIMP means Weakly-Interacting Massive Particle.

The freezout epoch is the one at which the abundance y departs from its equilibrium value: then, we ask that $\Delta(z_f) = y(z_f) - y_{\mathrm{eq}}(z_f) = c y_{\mathrm{eq}}$, with c being a coefficient of order unity.

Plugging in the expression we derived earlier into (3.2.33), we get

$$\Delta(z_f) = \lambda^{-1} \frac{z_f^{n+2} y_{\mathrm{eq}}(z_f)}{2 y_{\mathrm{eq}}(z_f) + \Delta(z_f)} = \frac{z_f^{n+2}}{\lambda}\frac{1}{2+c} , \tag{3.2.44}$$

therefore

$$\frac{z_f^{n+2}}{\lambda}\frac{1}{2+c} = c a z_f^{3/2} e^{-z_f} \tag{3.2.45}$$

$$a = 0.145 \frac{g_\psi}{g_{*s}} , \tag{3.2.46}$$

a transcendental equation which can be solved iteratively, yielding

$$z_f^{(0)} = \log\left[c(c+2) a \lambda\right] \tag{3.2.47}$$

$$= \log\left[0.038(n+1) m_{\mathrm{Pl}} m_\psi \sigma_0 \frac{g_\psi}{g_{*s}}\right] , \tag{3.2.48}$$

and a good approximation is found to be $c(c+2) = n+1$. The second iteration then yields

$$z_f^{(1)} = \log\left[c(c+2) a \lambda\right] - \left(\frac{1}{2} + n\right) \log\left[c(c+2) a \lambda\right] , \tag{3.2.49}$$

which means that z_f does indeed depend on the mass of the particle, although it is a weak dependence since it is in the logarithm.

As an example, if m_ψ were of the order of 100 GeV and if we had $\sigma_0 \sim 10^{-38}\,\mathrm{cm}^{-2}$, then we would get

$$z_f \approx \begin{cases} 17 & \text{for } n = 0 \\ 14 & \text{for } n = 1 \\ 12 & \text{for } n = 2 \end{cases} . \tag{3.2.50}$$

This means that $T_f \approx m_\psi/10$, therefore $T_f \approx 10 \div 100\,\mathrm{GeV}$ is the typical order of magnitude of the decoupling temperature. The relics we discussed are *thermal*, but **non-thermal** relics are also possible; their abundance is much more model-dependent. For example, axions might be produced through a "misalignment mechanism", and other relics might be produced through a "freeze-in mechanism".

See picture from the slides: FIMP.

Chapter 4

Cosmological perturbation within GR

4.1 The gauge issue

We will treat the **gauge issue** for perturbations. Einstein's equations are invariant under diffeomorphisms, so we must be careful and use invariant quantities. We already used

$$\mathcal{J} = -H\frac{\delta\rho}{\dot{\rho}} - \hat{\Phi}\,, \tag{4.1.1}$$

where $\hat{\Phi}$ is a scalar perturbation of the spatial part of the metric tensor, which will be discussed later.

Then, we will try to get a proper treatment of cosmological perturbations within GR. We have already seen how to treat them in a Newtonian way (i.e. Jeans instability). This is not ok when the wavelength of the perturbation is larger than the Hubble radius.

The Poisson equation, in terms of $\delta = \delta\rho/\rho$, reads

$$\nabla^2\varphi = 4\pi G\delta\rho = 4\pi G\bar{\rho}\delta\,, \tag{4.1.2}$$

so in terms of characteristic lengths we get

$$\frac{\varphi}{\lambda_{\text{phys}}^2} \sim \frac{3}{2}H^2\delta\,, \tag{4.1.3}$$

From the Friedmann equations.

therefore, in terms of $\lambda_H = 1/H$:

$$\varphi \sim \left(\frac{\lambda_{\text{phys}}}{\lambda_H}\right)^2\delta\,, \tag{4.1.4}$$

and reintroducing the speed of light

$$\frac{\varphi}{c^2} \sim \left(\frac{\lambda_{\text{phys}}}{\lambda_H}\right)^2\delta\,. \tag{4.1.5}$$

When φ/c^2 is of order 1 or larger we expect to see GR effects.

We know that typically $\delta \sim 10^{-5}$ for primordial perturbations, so when $\lambda_{\text{phys}} \ll \lambda_H$ the Newtonian treatment is fine; while when the perturbations are outside the horizon the perturbations are frozen.

We will adopt a perturbative approach, which in GR is a perturbation of the geometry itself (as well as the fields). Consider a generic tensor field T: a perturbation of this quantity is defined as

$$\Delta T = T - T_0 \,, \tag{4.1.6}$$

where T is evaluated in the perturbed universe, while T_0 is evaluated in the unperturbed FLRW spacetime. We are then comparing two tensors, but they will be defined at two different spacetime locations. This is not allowed in differential geometry: in order to make the comparison meaningful we need a one-to-one correspondence between the perturbed and unperturbed spacetimes (denoted respectively as $\mathcal{M}$ and $\mathcal{M}_0$); this amounts to making a specific gauge choice.

A gauge transformation is a change of the map $\mathcal{M}_0 \to \mathcal{M}$ (which we denote as ψ, such that for $p \in \mathcal{M}_0$ and $O \in \mathcal{M}$ we have $p \to O = \psi(p)$) which *keeps the coordinates on $\mathcal{M}_0$ fixed*; this is different from a coordinate transformation.

The gauge issue comes from the freedom of always making a gauge transformation: changing the map ψ to ψ': $p \to O' = \psi'(p)$.

A coordinate system is defined by a threading of spacetime into lines (corresponding to fixed spatial coordinates) and by a slicing of spacetime into hypersurfaces (corresponding to fixed time).

A gauge transformation can be defined in a completely coordinate-free way.

We can change our point of view: instead of changing the point in $\mathcal{M}$ to which our point in $\mathcal{M}_0$ maps, we can see the transformation as a change in the point in $\mathcal{M}_0$ from which we start to reach the fixed point in $\mathcal{M}$. In terms of the points we defined before, this amounts to finding a $q \in \mathcal{M}_0$ such that $\psi'(q) = O$, and then seeing the gauge transformation as a map which takes p to $q = \psi'^{,-1}(\psi(p))$.

If we have a quantity T which is a scalar under coordinate transformation, then it still *can* change if we make a gauge transformation.

A general infinitesimal coordinate transformation looks like

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$$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x) \,, \tag{4.1.7}$$

where both the vector field ξ and its derivative are infinitesimally small.

A scalar field ϕ will transform trivially, meaning that

$$\phi'(x') = \phi(x) \,, \tag{4.1.8}$$

while a covariant and a contravariant vector will transform like

$$V'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu(x) \tag{4.1.9}$$

$$V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x) \,, \tag{4.1.10}$$

and tensors with more indices will transform analogously, with a Jacobian for each contravariant index and an inverse Jacobian for each covariant index.

Active approach We denote the unperturbed FLRW manifold as $\mathcal{M}_0$, and the perturbed one as $\mathcal{M}_\lambda$. The parameter λ is a bookkeeping tool we can use to keep track of the perturbative order, at the end we will set $\lambda = 1$.

We denote the map ψ_λ as the one from $\mathcal{M}_0$ to $\mathcal{M}_\lambda$, and φ_λ as the one going backwards. Also, we denote our

Let us fix a coordinate system x^μ on $\mathcal{M}_0$, and consider a vector field ξ^μ: then we can define a **congruence of curves** by

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \xi^\mu(\lambda)\,. \tag{4.1.11}$$

Then, we can define a point Q which is at a parametric distance λ from P along these integral curves:

$$x^\mu(Q) = x^\mu(P) + \lambda\xi^\mu(x(P)) + \mathcal{O}\left(\lambda^2\right). \tag{4.1.12}$$

This is called an infinitesimal point transformation in the active approach.

Passive approach We can also introduce a "passive approach" to gauge transformations: the same relation we used can be seen as the one we would get by introducing a new coordinate system $y^\mu(Q)$, such that

$$\begin{aligned} y^\mu(Q) &= x^\mu(P) && (4.1.13)\\ &= x^\mu(Q) - \lambda\xi^\mu(x(P))\,. && (4.1.14) \end{aligned}$$

If $x(P) = x(Q) - \lambda\xi$, this can be expanded as

$$y^\mu(Q) = x^\mu(Q) - \lambda\xi^\mu(x^\mu(Q)) + \mathcal{O}\left(\lambda^2, \xi^2\right). \tag{4.1.15}$$

This is a "passive coordinate transformation", since we are simply changing the names we give to the points, in the form

$$y^\mu(\lambda) = x^\mu - \lambda\xi^\mu + \mathcal{O}\left(\lambda^2\right), \tag{4.1.16}$$

which will become $y^\mu = x^\mu - \xi^\mu$, a regular coordinate transformation.

Consider a vector field Z which has components Z^μ in the x coordinate system; then we can define a new vector field $\widetilde{Z}$ which has components $\widetilde{Z}^\mu$ in the x coordinate system, such that the components $\widetilde{Z}^\mu$ evaluated at the coordinate point $x^\mu(P)$ are equal to the components Z'^μ that the "old" vector field Z has in the y coordinates.

The relation is

$$\widetilde{Z}^\mu(x(P)) = Z'^\mu(y(Q)) = \left.\frac{\partial y^\mu}{\partial x^\nu}\right|_{x(Q)} Z^\nu(x(Q))\,. \tag{4.1.17}$$

In the active approach, instead, we would have related the first term to the third directly.

This provides a transportation law from the point Q to the point P, in the *same* x coordinate system. Looking at the first and last term in the equation is the "active approach", looking at the last two terms is the "passive approach".

The Jacobian at hand is

$$\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu - \lambda \frac{\partial \xi^\mu}{\partial x^\nu}\,, \tag{4.1.18}$$

therefore, Taylor expanding, we get

$$\begin{aligned}
\widetilde{Z}^\mu(x(P)) &= Z^\mu(x(Q)) - \lambda \frac{\partial \xi^\mu}{\partial x^\nu} Z^\nu(x(Q)) && (4.1.19)\\
&= Z^\mu(x(P)) + \frac{\partial Z^\mu}{\partial x^\nu}\lambda \xi^\nu(x(P)) - \lambda \frac{\partial \xi^\mu}{\partial x^\nu} Z^\nu x(P) && (4.1.20)\\
&= Z^\mu(x(P)) + \lambda \mathscr{L}_\xi Z^\mu + \mathcal{O}\left(\lambda^2\right). && (4.1.21)
\end{aligned}$$

Using (4.1.12).

This has given us the transformation law for the vector field, in terms of the Lie derivative:

$$\begin{aligned}
\mathscr{L}_\xi(Z^\mu) &= \frac{\partial Z^\mu}{\partial x^\nu}\xi^\nu - \frac{\partial \xi^\mu}{\partial x^\nu} Z^\nu && (4.1.22)\\
&= (\xi^\nu \partial_\nu) Z^\mu - (Z^\nu \partial_\nu)\xi^\mu\,. && (4.1.23)
\end{aligned}$$

The new Z is in the *same* coordinates as before. Setting $\lambda = 1$, we get

$$\widetilde{Z}^\mu = Z^\mu + \mathscr{L}_\xi Z^\mu\,. \tag{4.1.24}$$

The effect of a gauge transformation is that the new tensor is equal to the old one plus the Lie derivative (at the *same* coordinate point) of the vector field corresponding to the transformation. For scalars, we have $\mathscr{L}_\xi S = S_{,\mu}\xi^\mu$; for vectors and tensors we have

$$\begin{aligned}
\mathscr{L}_\xi V_\mu &= V_{\mu,\lambda}\xi^\lambda + \xi^\lambda_{,\mu} V_\lambda && (4.1.25)\\
\mathscr{L}_\xi T_{\mu\nu} &= T_{\mu\nu,\lambda}\xi^\lambda + \xi^\lambda_{,\mu} T_{\lambda\nu} + \xi^\lambda_{,\nu} T_{\mu\lambda}\,. && (4.1.26)
\end{aligned}$$

We would instead have a minus on the right if the vector V was contravariant.

We get the same result if we replace the partial derivatives with covariant derivatives, as can be found from a direct calculation, as long as the connection is torsion-free (which is equivalent to the Christoffel symbols' lower indices being symmetric).

If we consider the metric tensor $T_{\mu\nu} = g_{\mu\nu}$, which is covariantly constant ($g_{\mu\nu;\rho} = 0$) we get

$$\mathscr{L}_\xi g_{\mu\nu} = g_{\lambda\nu}\xi^\lambda_{;\mu} + \xi^\lambda_{;\nu} g_{\mu\lambda} = \xi_{\mu;\nu} + \xi_{\nu;\mu}\,. \tag{4.1.27}$$

In two different gauges we can find

$$\begin{aligned}
\Delta T &= T - T_0 && (4.1.28)\\
\widetilde{\Delta T} &= \widetilde{T} - T_0\,, && (4.1.29)
\end{aligned}$$

but, using the transformation law for tensors, we know that

$$\widetilde{T} = T_0 + \widetilde{\Delta T} = T + \mathscr{L}_\xi T = T_0 + \Delta T + \mathscr{L}_\xi T \tag{4.1.30}$$

$$\widetilde{\Delta T} = \Delta T + \mathscr{L}_\xi T\,, \tag{4.1.31}$$

at least at linear order. At this order, however, we can also substitute the Lie derivative of T with that of T_0:

$$\boxed{\widetilde{\Delta T} = \Delta T + \mathscr{L}_\xi T_0\,.} \tag{4.1.32}$$

This is the crux of the gauge problem: the quantity ΔT is gauge dependent.

4.1.1 Cosmological perturbations

We start from the background FLRW metric:

$$\mathrm{d}s^2 = g^{(0)}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{4.1.33}$$

$$= a^2(\eta)\left[-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right], \tag{4.1.34}$$

where, as usual, η is the conformal time, defined by $\mathrm{d}\eta = \mathrm{d}t\,/a(t)$.

The perturbed metric reads

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{4.1.35}$$

$$g_{00} = -a^2(\eta)\left[1 + 2\sum_{r=1}^{\infty}\frac{1}{r!}\psi^{(r)}(\vec{x},\eta)\right] \tag{4.1.36}$$

$$g_{0i} = g_{i0} = a^2(\eta)\sum_{r=1}^{\infty}\frac{1}{r!}\omega_i^{(r)}(\vec{x},\eta) \tag{4.1.37}$$

$$g_{ij} = a^2(\eta)\left(\left[1 - 2\sum_{r=1}^{\infty}\frac{1}{r!}\phi^{(r)}(\vec{x},\eta)\right]\delta_{ij} + \sum_{r=1}^{\infty}\frac{1}{r!}\chi_{ij}^{(r)}(\vec{x},\eta)\right), \tag{4.1.38}$$

where ψ is called the *lapse function*, ω_i is called the *shift function*, while we take χ_{ij} to be a *traceless* perturbation: $\chi^i_i = 0$.

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We will focus on the linear-order perturbations, thus setting $r = 1$. We can decompose our perturbation into its scalar, vector and tensor contributions. The scalar contributions are $\psi(\vec{x},\eta)$ and $\phi(\vec{x},\eta)$; they represent a GR extension of Newtonian gravitational potentials. The vector contributions are related to transverse (aka divergence-free, or vertical, or solenoidal) vector fields. We can make the decomposition $\omega_i(\vec{x},\eta) = \partial_i\omega^\parallel + \omega_i^\perp$, where $\partial^i\omega_i^\perp = 0$. This is known as the Helmholtz decomposition, and it amounts to considering modes $\omega(\vec{k})$ which are parallel and orthogonal to $\vec{k}$ in Fourier space. By counting the number of constraints or by reasoning geometrically in Fourier space one can see that the term $\omega^\parallel$ includes 1 degree of freedom, while the term $\omega_i^\perp$ includes two.

On the other hand, the tensor component can be decomposed into

$$\chi_{ij} = \underbrace{D_{ij}\chi^{\parallel}}_{\text{scalar}} + \underbrace{2\chi^{\perp}_{(i,j)}}_{\text{vector}} + \underbrace{\chi^{T}_{ij}}_{\text{tensor}} , \tag{4.1.39}$$

where $D_{ij} = \partial_i\partial_j - (1/3)\delta_{ij}\nabla^2$ is a traceless derivative operator. We assume that $\partial^i\chi^{\perp}_i = 0$, while the tensor χ^T_{ij} is both traceless and transverse: $\chi_i^{i,T} = 0 = \partial^i\chi^T_{ij}$.

The last term contains gravitational waves. We can write the stress-energy tensor as

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} , \tag{4.1.40}$$

where u_μ is the four-velocity of the fluid, while ρ and p are the energy density and isotropic pressure, and $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is a projection tensor onto the hypersurfaces orthogonal to the four-velocity.

The energy density is a function of position, which can be written as

$$\rho(\vec{x},\eta) = \rho^{(0)}(\eta) + \sum_{r=1}^{\infty}\frac{1}{r!}\delta^{(r)}\rho(\vec{x},\eta) , \tag{4.1.41}$$

and an analogous expression can be written for the isotropic pressure $p(\vec{x},\eta)$.

The perturbation of the pressure can be written as

$$\delta p = \underbrace{\frac{\partial p}{\partial \rho}\delta\rho}_{\text{adiabatic}} + \underbrace{\frac{\partial p}{\partial S}\delta S}_{\text{non-adiabatic}} \tag{4.1.42}$$

$$= c_s^2\delta\rho + \delta p_{\text{non-ad}} . \tag{4.1.43}$$

On the other hand, the four-velocity of a comoving observer is

$$u^\mu = \frac{1}{a}\left(\delta^\mu_0 + \sum_{r=1}^{\infty}\frac{1}{r!}v^\mu_{(r)}\right) , \tag{4.1.44}$$

since the unperturbed four-velocity is just $u^\mu_{(0)} = \delta^\mu_0/\sqrt{-g_{00}} = \delta^\mu_0/a$. As for the perturbations, it can be shown (from the normalization condition $u^2 = -1$) that at linear order $v^0 = -\psi$. Then, we can just consider the peculiar spatial velocity, and Helmholtz-decompose it:

$$v^i = \partial^i v^{\parallel} + v^i_{\perp} , \tag{4.1.45}$$

where, as usual, $\partial_i v^i_\perp = 0$. This latter term describes vorticity.

As we discussed earlier, any tensor T can be gauge-transformed at linear order as

$$\widetilde{\Delta T} = \Delta T + \mathscr{L}_\xi(T_0) , \tag{4.1.46}$$

along the gauge transformation represented by the vector field ξ. We write this vector field as

$$\xi^0 = \alpha(\vec{x},\eta) \tag{4.1.47}$$

$$\xi^i = \partial^i \beta + d^i\,, \tag{4.1.48}$$

where $\partial^i d_i = 0$.

Denoting derivatives with respect to conformal time with a prime, the perturbations transform like

$$\widetilde{\psi} = \psi + \alpha' + \frac{a'}{a}\alpha \tag{4.1.49}$$

$$\widetilde{\omega}_i = \omega_i - \alpha_{,i} + \beta'_{,i} + d'_i \tag{4.1.50}$$

$$\widetilde{\phi} = \phi - \frac{1}{3}\nabla^2\beta - \frac{a'}{a}\alpha d \tag{4.1.51}$$

$$\widetilde{\chi}_{ij} = \chi_{ij} + 2D_{ij}\beta + 2d_{(i,j)}\,, \tag{4.1.52}$$

which follows from the equation we derived earlier, $\widetilde{\Delta T} = \Delta T + \mathscr{L}_\xi T_0$ and we can decompose these as

$$\widetilde{\omega}^{\parallel} = \omega^{\parallel} - \alpha + \beta' \tag{4.1.53}$$

$$\widetilde{\omega}_i^{\perp} = \omega_i^{\perp} + d'_i \tag{4.1.54}$$

$$\widetilde{\chi}^{\parallel} = \chi^{\parallel} + 2\beta \tag{4.1.55}$$

$$\widetilde{\chi}_i^{\perp} = \chi_i^{\perp} + d_i \tag{4.1.56}$$

$$\widetilde{\chi}_{ij}^{T} = \chi_{ij}^{T}\,. \tag{4.1.57}$$

The traceless tensor part of the perturbation is gauge invariant.

The energy density perturbation can be shown to transform like

$$\widetilde{\delta\rho} = \delta\rho + \mathscr{L}_\xi \rho^{(0)}(\eta) \tag{4.1.58}$$

$$= \delta\rho + \rho^{(0),\prime}\alpha\,, \tag{4.1.59}$$

showing that $\delta\rho$ is *not* a scalar.

The velocity transforms like

$$\delta u^\mu = \frac{1}{a} v^\mu\,, \tag{4.1.60}$$

so

$$\widetilde{\delta u}^\mu = \delta u^\mu + \mathscr{L}_\xi u^{\mu,0}\,, \tag{4.1.61}$$

meaning that

$$\widetilde{v}^0 = v^0 - \frac{a'}{a}\alpha - \alpha' \tag{4.1.62}$$

$$\widetilde{v}^i = v^i - \beta^{,i\prime} - d^{i,\prime}\,, \tag{4.1.63}$$

so

$$\widetilde{v}_{\parallel} = v_{\parallel} - \beta' \tag{4.1.64}$$

$$\widetilde{v}^i_\perp = v^i_\perp - d^i\,. \tag{4.1.65}$$

The split of the cosmological perturbation is useful since (at linear order) the evolutions of the scalar, vector and tensor modes is decoupled.

The gauge is determined by ξ^μ, which amounts to choosing the scalars α and β, plus the vector d^i: a total of four degrees of freedom, since d^i has one constraint.

A possible gauge choice is the **Poisson gauge**:[1]

$$\widetilde{\omega}^\parallel = \widetilde{\chi}^\parallel = 0 = \widetilde{\chi}^T_i\,. \tag{4.1.66}$$

In this case, the lapse function quite closely matches the behaviour of Newtonian gravitational perturbations.

In order to impose these condition we need to set

$$0 = \widetilde{\chi}^\parallel = \chi^\parallel + 2\beta = 0 \implies \beta = -\frac{1}{2}\chi^\parallel\,. \tag{4.1.67}$$

This gauge is also known as the **orthogonal zero-shear gauge**.

4.2 Common gauge choices and gauge invariant quantities

If we consider constant-η spatial hypersurfaces, and take a vector $N_\mu \propto \mathrm{d}\eta/\mathrm{d}x^\mu$ perpendicular to them (this will often correspond to the four-velocity), we can make the following decomposition:

$$N_{\mu;\nu} = \frac{1}{3}\theta h_{\mu\nu} + \omega_{\mu\nu} + \sigma_{\mu\nu} - a_\mu N_\nu\,, \tag{4.2.1}$$

where $\theta = N^\mu_{;\mu}$ represents the local compression of the fluid, in FLRW $\theta = 3H$. The term $\omega_{\mu\nu}$ is given by

$$\omega_{\mu\nu} = h^\alpha_\mu h^\beta_\nu u_{[\alpha;\beta]} \tag{4.2.2}$$

and it is known as the *vorticity tensor*. If indeed $N_\mu = u_\mu$, then $\omega_{\mu\nu} = 0$.

The tensor $\sigma_{\mu\nu}$ instead represents shear:

$$\sigma_{\mu\nu} = h^\alpha_\mu h^\beta_\nu u_{(\alpha;\beta)} - \frac{1}{3}\theta h_{\mu\nu}\,, \tag{4.2.3}$$

and finally $a^\mu = u^\nu u_{\mu;\nu}$. If $u_\mu = N_\mu$, $\sigma_{\mu\nu}$ is the geometric shear, so $\sigma_{\mu 0} = 0$, meaning that we can only work with

$$\sigma_{ij} = D_{ij}\sigma + \sigma^\perp_{(i,j)} + \sigma^T_{ij} \tag{4.2.4}$$

$$\sigma = -\omega^\parallel + \frac{1}{2}\chi^\parallel\,. \tag{4.2.5}$$

[1] Also known as the longitudinal or Newtonian gauge, since it reproduces the Newtonian treatment of the gravitational potential perturbations.

This term vanishes in the Poisson gauge.

The vector N reads:

$$N^{\mu} = (1/a)\left[1 - \psi, -\omega^{i}\right]. \tag{4.2.6}$$

Another possible choice is the **synchronous**, or **time-orthogonal gauge**:

$$\widetilde{\psi} = \widetilde{\omega}^{\parallel} = \widetilde{\omega}_i^{\perp} = 0\,, \tag{4.2.7}$$

meaning that $\delta g_{0\mu} = 0$. This amounts to only perturbing the spatial part of the metric. If we take observers at fixed spatial coordinates, their subjective time in this gauge will be

$$\mathrm{d}\tau = \mathrm{d}t\left(1 + \psi\right) = \mathrm{d}t\;, \tag{4.2.8}$$

so they will all be synchronized.

We can write

$$\widetilde{\psi} = \psi + \alpha' + \frac{a'}{a}\alpha = \psi + \frac{(a\alpha)'}{a} = 0\,, \tag{4.2.9}$$

so we must have

$$a\alpha = -\int a\psi\,\mathrm{d}\eta + X(\vec{x})\,, \tag{4.2.10}$$

so we have not completely fixed the gauge. Similarly to electrodynamics, we have residual gauge freedom.

Another possible choice is the **comoving gauge**, defined by

$$v^{i} = 0\,, \tag{4.2.11}$$

meaning that $v^{\parallel} = v^{i}_{\perp} = 0$. This basically amounts to setting the coordinate system as comoving with the fluid.

We have an additional condition, which we can use to impose that constant η spatial hypersurfaces must be orthogonal to u^{μ}: this leads to the condition $v^{\parallel} + \omega^{\parallel} = 0$, which tells us that $\omega^{\parallel} = 0$ because of the previous condition.

This is because at leading order we can lower the index of N^{μ} and u^{μ}, and use $v^{0} = \psi$, to get

$$N_{\mu} = \left[-a(1+\psi), 0\right] \tag{4.2.12}$$

$$u_{\mu} = a\left[-(1+\psi), v_i + \omega_i\right], \tag{4.2.13}$$

therefore the equation $N_{\mu} = u_{\mu}$ reads $v_i + \omega_i = 0$. This condition also leads to $T^{0}_{i} = 0$: the momentum flux vanishes. This is also called the comoving orthogonal gauge.

Other options are the **spatially flat gauge**, also known as the uniform curvature gauge: this is found by selecting constant-η spatial hypersurfaces where the spatial metric is left unperturbed, at least for scalar and vector perturbations.

This means

$$\widetilde{\phi} = \widetilde{\chi}^{\parallel} = \widetilde{\chi}_i^{\perp} = 0\,, \tag{4.2.14}$$

and the reason for the name is that the intrinsic spatial curvature (Ricci scalar) of the constant-η hypersurfaces is

$$ {}^{(3)}R = \frac{6k}{a^2} + \frac{rk}{a^2}\hat{\phi} + \frac{4}{a^2}\nabla^2\hat{\phi}\,, \tag{4.2.15} $$

where $\hat{\phi} = \phi + (1/6)\nabla^2\chi^{\parallel}$. This is why $\hat{\phi}$ is sometimes called the *curvature perturbation* itself, as $R \propto \hat{\phi}$ in Fourier space.

The **uniform energy density gauge** is defined by $\delta\widetilde{\rho} = 0$.

There are two approaches we can take with regard to gauge freedom: we can choose a gauge and work within it in a self-consistent way. This is fine, but residual gauge freedom can create issues.

The alternative is to only use gauge-invariant perturbations. A paper by Bardeen [Bar80] showed how to do so. Starting with scalars, we would have 4 of them: $\psi, \phi, \omega^{\parallel}, \chi^{\parallel}$, however only 2 degrees of freedom can remain. We define

$$ 2\Psi_A = 2\psi + 2\omega^{\parallel,\prime} + 2\frac{a'}{a}\omega^{\parallel} - \left(\chi^{\parallel,\prime\prime} + \frac{a'}{a}\chi^{\parallel,\prime}\right) \tag{4.2.16} $$

$$ 2\Phi_H = -2\phi - \frac{1}{3}\nabla^2\chi^{\parallel} + 2\frac{a'}{a}\omega^{\parallel} - \frac{a'}{a}\chi^{\parallel,\prime}\,, \tag{4.2.17} $$

and it can be shown (exercise) that these are indeed gauge invariant. These quantities have a physical meaning: if we go to the zero-shear gauge (4.2.4), then $\Phi_H = -\hat{\phi}$, while $\Psi_A = \psi$. These are known as the Bardeen gauge-invariant potentials, since they mimic Newtonian potentials.

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Let us continue with gauge-invariant cosmological perturbations. Let us now give a gauge-invariant definition for the matter velocity:

$$ 2v_s = 2v^{\parallel} + \chi^{\parallel,\prime}\,, \tag{4.2.18} $$

where, as usual, by "matter" we mean anything which goes on the right-hand side of the Einstein equations, and s means "scalar".

This velocity is related to the amplitude of the **shear tensor** for the matter velocity: specifically, from the shear tensor $\sigma_{\mu\nu}$ we can define the quantity $\left(\sigma^{ij}\sigma_{ij}/2\right)^{1/2}$.

For the scalar part, we have (following the notation by Bardeen)

$$ \epsilon_m = \delta\rho + \rho_0'\left(v^{\parallel} + \omega^{\parallel}\right) \tag{4.2.19} $$

$$ \rho_0 = \rho_0(\eta)\,. \tag{4.2.20} $$

Note that energy density perturbations themselves are not gauge invariant. The quantity ϵ_m corresponds to $\delta\rho$ in the gauge where $v^{\parallel} + \omega^{\parallel} = 0$, which means that we are selecting constant-η hypersurfaces which are orthogonal to the worldlines of the fluid: the fluid's rest frame.

The quantity $v^{\parallel} + \omega^{\parallel}$ enters into the expression for T_i^0, which describes the momentum flux of the fluid. We also define

$$ 2E_g = 2\delta\rho + \rho_0'\left(2\omega^{\parallel} - \chi^{\parallel,\prime}\right), \tag{4.2.21} $$

which is also equal to $\delta\rho$ in the gauge in which $2\omega^{\parallel} - \chi^{\parallel,\prime} = 0$, which is the zero-shear or Poisson gauge.

We can also construct one vector perturbation, since we start with two and remove one with the gauge degree of freedom described as d^i:

$$\Psi_i = \omega_i^{\perp} - \chi_i^{\perp,\prime} . \tag{4.2.22}$$

This is related to the amplitude of the vector geometric component of the geometric shear $\sigma_{\mu\nu}$. This term describes frame-dragging effects.

The matter velocity can be described as

Compare to (4.2.18).

$$V_s^i = v_{\perp}^i + \chi_{\perp}^{i,\prime} . \tag{4.2.23}$$

Also, we can define

$$V_c^i = v_{\perp}^i + \omega_{\perp}^i . \tag{4.2.24}$$

This is related to the amplitude of the vorticity[2] tensor $\omega_{\mu\nu}$ (and, specifically, the quantity $\left(\omega^{ij}\omega_{ij}/2\right)^{1/2}$).

As for tensor perturbation modes, linear tensor perturbations are automatically gauge-invariant (at linear order, at least):

$$\chi_{ij}^{T,\prime} = \chi_{ij}^{T} . \tag{4.2.25}$$

4.3 Perturbed Einstein Equations

Let us write the equations of motion for these linear cosmological perturbations. We start from the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, and the Bianchi identities $T^{\mu\nu}_{;\nu} = 0$. The Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{4.3.1}$$

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \tag{4.3.2}$$

$$R^{\alpha}_{\mu\nu\beta} \sim \frac{\partial\Gamma}{\partial x} + \Gamma^2 \tag{4.3.3}$$

$$\Gamma^{\mu}_{\nu\rho} \sim g^{-1}\frac{\partial g}{\partial x} . \tag{4.3.4}$$

We can compute the nonvanishing Christoffel symbols in the unperturbed case (see, for example, the notes for Theoretical Cosmology), and then the perturbations of these; note that the inverse of a perturbed metric can be computed by inverting the perturbation according to the flat metric only (see [TML20, eq. 8.20]):

$$\delta\Gamma^0_{00} = \psi' \tag{4.3.5}$$

[2] Local angular velocity of fluid elements

$$\delta\Gamma^0_{0i} = \partial_i\psi + \frac{a'}{a}\partial_i\omega^{\parallel} \tag{4.3.6}$$

$$\delta\Gamma^i_{00} = \frac{a'}{a}\partial^i\omega^{\parallel} + \partial^i\omega^{\parallel,\prime} + \partial^i\psi \tag{4.3.7}$$

$$\delta\Gamma^0_{ij} = -2\frac{a'}{a}\psi\delta_{ij} - \partial_i\partial_j\omega^{\parallel} - 2\frac{a'}{a}\phi\delta_{ij} - \phi'\delta_{ij} + \frac{a'}{a}D_{ij}\chi^{\parallel} + \frac{1}{2}D_{ij}\chi^{\parallel,\prime} \tag{4.3.8}$$

$$\delta\Gamma^i_{0j} = -\phi\delta^i_j + \frac{1}{2}D^i_j\chi^{\parallel,\prime} \tag{4.3.9}$$

$$\delta\Gamma^i_{jk} = \dots . \tag{4.3.10}$$

The spatial components of the metric can be written as

$$g_{ij} = \underbrace{a^2(\eta)\left[1-2\phi\right]\delta_{ij}}_{a^2(\eta,\vec{x})} + \dots , \tag{4.3.11}$$

where we can express

$$a(\eta,\vec{x}) = a(\eta)\left(1-\phi(\eta,\vec{x})\right) , \tag{4.3.12}$$

therefore $\delta a = -a\phi$, which means that

$$\delta\left(\frac{a'}{a}\right) = -\phi' . \tag{4.3.13}$$

The unperturbed Ricci tensor reads

$$R_{00} = -3\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2 \tag{4.3.14}$$

$$R_{ij} = \left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right]\delta_{ij} . \tag{4.3.15}$$

Its perturbation is

$$\delta R_{00} = \frac{a'}{a}\nabla^2\omega^{\parallel} + \nabla^2\omega^{\parallel,\prime} + 3\phi^{\parallel} + 3\frac{a'}{a}\phi' + 3\frac{a'}{a}\psi' \tag{4.3.16}$$

$$\delta R_{0i} = \frac{a'}{a}\partial_i\omega^{\parallel} + \left(\frac{a'}{a}\right)^2\partial_i\omega^{\parallel} + 2\partial_i\phi' + 2\frac{a'}{a}\partial_i\psi + \frac{1}{2}\partial_k D^k_i\left(\chi_{\parallel}\right)' \tag{4.3.17}$$

$$\delta R_{ij} = \dots . \tag{4.3.18}$$

Copy full expressions from the notes.

The Ricci scalar R is given by $R = (6/a^2)(a'/a)$ in flat FRLW, while its perturbation is

$$\delta R = \frac{1}{a^2}\left(-6\frac{a'}{a}\nabla^2\omega^{\parallel} - 2\nabla^2\omega^{\parallel,\prime} - 2\nabla^2\psi - 6\phi^{\parallel} - 6\frac{a'}{a}\psi' - 18\frac{a'}{a}\phi' - 12\frac{a''}{a}\psi + 4\nabla^2\phi + \partial_k\partial^i D^k_i\chi^{\parallel}\right) . \tag{4.3.19}$$

This expression is fully general, in a specific gauge it can be significantly simplified. The stress-energy tensor we will use is given by

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + \Pi_{\mu\nu}\,, \tag{4.3.20}$$

where $\Pi^\mu_\mu = \Pi_{\mu\nu} u^\nu = 0$. This allows us to account for imperfections in the fluid. The only nonvanishing components of Π are the spatial ones Π_{ij}. This is true in any frame.

We can write it as

$$\Pi_{ij} = D_{ij}\Pi^{\parallel} + 2\Pi^{\perp}_{(i,j)} + \Pi^{T}_{ij}\,, \tag{4.3.21}$$

where, as usual, $D_{ij} = \partial_i\partial_j - (1/3)\delta_{ij}\nabla^2$. The component

$$-T^0_0 = \rho_0(\eta) + \delta\rho(\eta,\vec{x}) \tag{4.3.22}$$

$$= \rho_0(\eta)(1+\delta)\,, \tag{4.3.23}$$

while

$$p = p_0(\eta)(1+\Pi_L)\,, \tag{4.3.24}$$

where $\Pi_L = \delta p / p_0(\eta)$. The L stands for "longitudinal".

The unperturbed spatial components of the stress-energy tensor read:

$$T^i_j = p_0(\eta)\left[(1+\Pi_L)\delta^i_j + \Pi^i_{T,j}\right] \tag{4.3.25}$$

$$\Pi^i_{T,j} = \left.\frac{\Pi^i_j}{p_0(\eta)}\right|_{\text{traceless}}\,. \tag{4.3.26}$$

The perturbation reads

$$\delta T^0_i = \left(\rho_0 + p_0\right)(v_i + \omega_i)\text{or} \qquad \delta T^i_0 = -(\rho_0+p_0)v^i\,. \tag{4.3.27}$$

This is quite general: it is the density of the i-component of the fluid's momentum, or the flux of energy in the i-th direction.

The linearly perturbed 00 EFE for scalar perturbation (as they are decoupled from the vector and tensor ones) reads

$$\frac{3a'}{a}\left(\hat{\phi}' + \frac{a'}{a}\psi\right) - \nabla^2\left(\hat{\phi} + \frac{a'}{a}\sigma\right) = -4\pi G a^2 \delta\rho\,, \tag{4.3.28}$$

where $\hat{\phi} = \phi + (1/6)\nabla^2\chi^{\parallel}$ and $\sigma = -\omega^{\parallel} + (1/2)\chi^{\parallel\prime}$.

The $0i$ equation is

$$\hat{\phi}' + \frac{a'}{a}\psi = -4\pi G a^2\left(\rho_0 + p_0\right)V\,, \tag{4.3.29}$$

where $V = v^{\parallel} + \omega^{\parallel}$. These two are not really "evolution" equations, they should be interpreted as constraints (to relate with δT_{00} and δT_{0i}) for the evolution of the other components. On the other hand, from the trace of the ij equations we find

$$\hat{\phi}'' + 2\frac{a'}{a}\hat{\phi}' + \frac{a'}{a}\psi + \left[2\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\right]\psi = 4\pi G a^2 \left(\Pi_L + \frac{2}{3}\nabla^2\Pi_T\right)p_0\,, \tag{4.3.30}$$

where the T in $\Pi_T = \Pi^{\parallel}/p_0(\eta)$ means "traceless". From the traceless part of these equations we find

$$\sigma' + 2\frac{a'}{a}\sigma + \hat{\phi} - \psi = 8\pi G a^2 \Pi_T p_0\,. \tag{4.3.31}$$

So far we have not chosen a gauge; let us now put ourselves in the Poisson gauge. Here,

$$\omega^{\parallel} = 0 = \chi^{\parallel}\,. \tag{4.3.32}$$

Then, $\hat{\phi} = \phi = -\Phi_H$, and $\psi = \Psi_A$. Now $\sigma = -\omega^{\parallel} + (1/2)\chi^{\parallel\prime} = 0$. Then, the equations reads

$$\phi - \psi = 8\pi G a^2 \Pi_T p_0 \tag{4.3.33}$$

$$\Phi_H + \Psi_A = -8\pi G a^2 \Pi_T p_0\,. \tag{4.3.34}$$

If the anisotropic stress can be neglected, then $\phi = \psi$; this is the case for CDM, not so for nonisotropic relativistic particles or modified gravity. If this is the case, then we can write the evolution equation in a simpler way: Π_T vanishes, by definition $\Pi_L p_0 = \delta p$, which we can split into

$$\delta p = c_s^2 \delta\rho + \delta p_{\text{non-adiab}}\,, \tag{4.3.35}$$

and considering only the adiabatic part we find

$$\Phi_H'' + 3\left(1 + c_s^2\right)\frac{a'}{a}\Phi_H' + \left[2\left(\frac{a'}{a}\right)' + \left(1 + 3c_s^2\right)\left(\frac{a'}{a}\right)^2 - c_s^2\nabla^2\right]\Phi_H = 0\,, \tag{4.3.36}$$

which is an isolated evolution equation for Φ_H, a wave-like propagation equation.

The Poisson gauge is the most Newton-like one: the evolution equation becomes

$$-\nabla^2\Phi_H = 4\pi G a^2 \underbrace{\left(\delta\rho - \frac{3a'}{a}(\rho_0 + p_0)V\right)}_{\epsilon_m} \tag{4.3.37}$$

Inserting the $0i$ equation into the 00 one.

$$= 4\pi G a^2 \epsilon_m\,, \tag{4.3.38}$$

a Poisson equation.

For the vector perturbation Ψ_i, we find (from the $0i$ equation)

$$\nabla^2\Psi_i = 16\pi G a^2 (\rho_0 + p_0) V_{i,c}\,, \tag{4.3.39}$$

while for the tensor perturbations, starting from the traceless part of the ij EFE, we get

$$\chi_{ij}^{\prime\prime,T} + 2\frac{a'}{a}\chi_{ij}^{T,\prime} - \nabla^2\chi_{ij}^T = 16\pi G a^2 p_0(\eta)\Pi_{ij}^T \,. \tag{4.3.40}$$

We should also perturb for the matter source of the equations: $T^{\mu\nu}_{;\nu} = 0$. The energy density continuity equation ($\mu = 0$) reads

$$\delta\rho' + \frac{3a'}{a}(\delta p + \delta\rho) - 3(\rho_0 + p_0)\hat{\phi}' + (\rho_0 + p_0)\nabla^2(V + \sigma) = 0\,, \tag{4.3.41}$$

while for $\mu = i$ we get

$$V' + \left(1 + 3c_s^2\right)\frac{a'}{a}V + \psi + \frac{1}{(\rho_0 + p_0)}\left(\delta p + \frac{2}{3}p_0\nabla^2\Pi_T\right) = 0\,. \tag{4.3.42}$$

The curvature perturbation on uniform energy density hypersurfaces is

$$\zeta = -\hat{\phi} - H\frac{\delta\rho}{\dot{\rho}_0} = -\hat{\phi} - \frac{a'}{a}\frac{\delta\rho}{\rho_0'}\,, \tag{4.3.43}$$

where, as usual, $\hat{\phi} = \phi + (1/6)\nabla^2\chi^{\parallel}$.

In a uniform energy density gauge $\zeta = -\hat{\phi}$; also in the flat gauge we have $\hat{\phi} = 0$, which tells us that ζ is an energy density perturbation. On super-horizon scales and taking equation (4.3.41) in the gauge where $\delta\rho = 0$, the Laplacian in the energy density continuity equation can be taken to vanish ($k \ll 1$), and we can express it as

$$\zeta' = -\frac{a'}{a}\frac{\delta p}{\rho_0 + p_0}\,, \tag{4.3.44}$$

but we must evaluate this in the uniform energy density gauge the adiabatic contribution to the pressure perturbation vanishes, therefore we are left with

$$\zeta' = -\frac{a'}{a}\frac{\delta p_{\text{non-adiabatic}}}{\rho_0 + p_0}\,. \tag{4.3.45}$$

For single-field models of slow-roll inflation, we find that on super-horizon scales

$$\delta p_{\text{non-adiabatic}} \propto \frac{k^2\Phi_H}{a^2} \approx 0 \implies \zeta \approx \text{const}\,. \tag{4.3.46}$$

The continuity equation for vector perturbations reads

$$\left[(\rho_0 + p_0)V_{ic}\right]' + \frac{4a'}{a}(\rho_0 + p_0)V_{ic} = -\nabla_k\left(\Pi_{,i}^{\perp k} + \Pi_i^{\perp,k}\right). \tag{4.3.47}$$

By Kelvin's circulation theorem, vorticity is conserved along trajectories unless there are dissipative effects. Then, the divergence on the right-hand side of this equation vanishes, therefore we can write the left-hand side as

$$a^3(\rho_0 + p_0)V_{ic}a = \text{const}\,. \tag{4.3.48}$$

This amounts to a momentum times a, so the equation represents the conservation of the intrinsic angular momentum.

Tuesday 2020-12-22, compiled 2021-01-25

Today we will give a summary of certain topics we discussed in the course, mentioning some more advanced topics. We have mentioned the evolution of a curvature perturbation on uniform energy density hypersurfaces ζ. We defined it as the local fluctuation of the number of e-folds: $\zeta = \delta N = -H\delta\varphi/\dot{\varphi}$. This is the δN formalism.

It is an alternative method to study cosmological perturbations which works on super-horizon scales — alternative to standard perturbation theory.

Consider two disconnected regions with a typical length scale $\lambda_s \gtrsim H^{-1}(t)$, and take their separation to be $\lambda \gg \lambda_s$. In this viewpoint, ζ takes into account how much one of these regions has expanded relative to the other.

We want to recover the result $\zeta = \delta N$ and generalize it. The regions are separated by a distance much larger than the Hubble scale, so we expect them to be completely disconnected.

The number of e-folds at a spatial position $\vec{x}$ is given by

$$N = N(t, \vec{x}) = N(t, \varphi_*(\vec{x}))\,, \tag{4.3.49}$$

where $\varphi_*(\vec{x}) = \varphi_*^0 + \delta\varphi(\vec{x})$ is the inflaton field, evaluated at the same initial time t_* corresponding to the time of horizon crossing for a generic fluctuation mode $\lambda \sim 2\pi/k$.

We can Taylor expand:

$$\delta N = \left.\frac{\partial N}{\partial \varphi_*}\right|_{\varphi_*^0} \delta\varphi_* + \frac{1}{2}\frac{\partial^2 N}{\partial \varphi_*^2}^2 \delta\varphi_*^2\,. \tag{4.3.50}$$

We can write the number of e-folds as

$$N = \int_{t_0}^{t} H \,\mathrm{d}t = \int_{\varphi_*}^{\varphi(t)} \frac{H}{\dot{\varphi}}\,\mathrm{d}\varphi\,, \tag{4.3.51}$$

therefore

$$\frac{\partial N}{\partial \varphi_*} = -\left.\frac{H}{\dot{\varphi}}\right|_{\varphi_*} \implies \delta N = \underbrace{-\left.\frac{H}{\dot{\varphi}}\right|_{\varphi_*} \delta\varphi_*}_{\text{constant on superhorizon scales}}\,. \tag{4.3.52}$$

This way we see that the term we wrote is the first order one, but we can also go beyond: the second-order term can be useful to study primordial non-Gaussianity.

The lowest order statistic which can be used to characterize it is the three-point function, in the form $\langle\zeta\zeta\zeta\rangle$.

This can be computed in this formalism as

$$\left(\frac{\partial N}{\partial \varphi_*}\right)^3 \left\langle \delta\varphi^*_{k_1}\delta\varphi^*_{k_2}\delta\varphi^*_{k_3}\right\rangle + \left(\frac{\partial N}{\partial \varphi_*}\right)^2 \frac{\partial^2 N}{\partial \varphi_*^2}\left\langle \delta\varphi^*_{k_1}\delta\varphi^*_{k_2}(\delta\varphi_*\delta\varphi_*)_{k_3}\right\rangle + \dots \tag{4.3.53}$$

Expectation values like these can arise from self-interactions in the field itself, or from interactions with other fields; in any case, nonlinear physics. The notation $(\delta\varphi_* \delta\varphi_*)_{k_3}$ denotes convolution in Fourier space.

A period of accelerated expansion will necessarily yield gravitational waves: $\chi^T_{ij} = h_{ij}$, with two polarizations

$$h_{+,\times} = \sqrt{16\pi G}\phi_{+,\times}\,, \tag{4.3.54}$$

evolving according to

$$u_k'' + \left(k^2 - \frac{a''}{a}\right) u_k = 0\,. \tag{4.3.55}$$

The a''/a term sources GWs from the quantum vacuum. Inflationary GWs would show up as a B-mode on the CMB polarization. However, we must be careful since there could be other sources of early GWs, such as cosmic strings or phase transitions.

The excursion of ϕ is related to the tensor-to-scalar ratio:

$$\frac{\Delta\phi}{M_P} \approx \left(\frac{r}{0.01}\right)^{1/2}\,. \tag{4.3.56}$$

Large field models, which are superPlanckian, have small r, and it small field models have small r.

Add some more comments.

What about exams? In January and February they will still be online oral examinations. We have two options: we can do a standard orderly examination in which we are asked about the standard parts of the course, or (for those who have attended the course) choosing a macro-topic to prepare in detail — ideally even in more detail than was done in the course — and then talk mostly about it. Even if we choose this second option, a few broad questions will be asked about the rest of the course. The presentation should be a blackboard one.

Exams can be done outside the usual session.

There will be a couple of extra lectures in January, possibly not by professor Bartolo.

4.3.1 IN-IN formalism

Monday 2021-1-11, compiled 2021-01-25

In cosmology we directly observe correlation functions, as opposed to particle accelerators. In particle physics speak, they are VEVs of products of fields: here we will be interested in

$$Q(t) = \delta\phi_{\vec{k}_1}(t)\phi_{\vec{k}_2}(t)\phi_{\vec{k}_3}(t)\,, \tag{4.3.57}$$

where $\delta\phi$ is the inflaton perturbation. This is the **bispectrum**.

The IN-IN formalism gives us a formula like

$$\langle\Omega|Q(t)|\Omega\rangle = \langle 0| \left[\overline{T}\exp\left(i\int_{t_0}^t \mathrm{d}t'\, H'_{\text{int}}(t')\right)\right] Q'(t) \left[\overline{T}\exp\left(-i\int_{t_0}^t \mathrm{d}t'\, H'_{\text{int}}(t')\right)\right] |0\rangle\,. \tag{4.3.58}$$

Short description of the interaction picture for a field described by Hamiltonian $H = H_0 + H_{\text{int}}$. Fields evolve according to the free time-evolution operator only. Then, we get

$$\begin{aligned}\langle Q(t) \rangle &= \langle \Omega | Q(t) | \Omega \rangle = \langle \Omega | U^{-1}(t, t_0) Q(t_0) U(t, t_0) | \Omega \rangle && (4.3.59)\\ &= \langle \Omega | U^{-1}(t, t_0) U_{\text{free}}(t, t_0) Q'(t) U^{-1}_{\text{free}}(t, t_0) U(t, t_0) | \Omega \rangle && (4.3.60)\\ &= \langle \Omega | F^{-1}(t, t_0) Q'(t) F(t, t_0) | \Omega \rangle \,, && (4.3.61)\end{aligned}$$

where

$$\frac{\mathrm{d}F(t, t_0)}{\mathrm{d}t} = -iH_{\text{int}} F(t, t_0)\,. \tag{4.3.62}$$

We can get a formal solution as the exponential of a time-ordered integral; using Wick's theorem we can expand the exponential.

How do we have $|0\rangle$, the vacuum of the free theory, instead of $|\Omega\rangle$?

In QFT we usually have the S-matrix formalism, which is an "in-out" formalism; here instead we use an "in-in" one. We turn off the interaction theory in the far past; we can then replace $|\Omega\rangle$ with $|0\rangle$. We use the Bunch-Davies vacuum as $|0\rangle$: we say that the quantum fluctuation on small scales are approximately plane waves.

As an example, we will compute the bispectrum for a model

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi(t, \vec{x}) \partial_\nu (t, \vec{x}) - V[\phi(t, \vec{x})] \right], \tag{4.3.63}$$

where M_{Pl} is the reduced Planck mass. For simplicity, let us take $g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$.

We expand

$$\phi(t, \vec{x}) = \phi_0(t) + \delta\phi(t, \vec{x}) \tag{4.3.64}$$

$$V(\phi) = V(\phi_0) + \left.\frac{\mathrm{d}V}{\mathrm{d}\phi}\right|_{\phi_0} \delta\phi + \frac{1}{2} \left.\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2}\right|_{\phi_0} \delta\phi^2 + \frac{1}{3!} \left.\frac{\mathrm{d}^3V}{\mathrm{d}\phi^3}\right|_{\phi_0} \delta\phi^3\,. \tag{4.3.65}$$

We substitute these into the action (see slides) and the first term yields the unperturbed EOM:

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + \frac{\mathrm{d}V}{\mathrm{d}\phi_0} = 0\,. \tag{4.3.66}$$

The second term vanishes. We define the parameters

$$\left.\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2}\right|_{\phi_0} = m^2_{\delta\phi} \qquad\qquad \left.\frac{\mathrm{d}^3V}{\mathrm{d}\phi^3}\right|_{\phi_0} = -\lambda\,. \tag{4.3.67}$$

We impose $m_{\delta\phi} = 0$ to simplify our discussion. The Lagrangian density then becomes

$$\mathcal{L} = \underbrace{\frac{a}{2}\dot{\delta}_\chi^2 - \dot{a}\dot{\delta}_\chi\delta_\chi + \frac{1}{2}\frac{\dot{a}^2}{a}\delta_\chi^2 - \frac{\delta^{ij}}{2a}\partial_i\delta_\chi\partial_j\delta_\chi}_{\mathcal{L}_{\text{free}}} + \underbrace{\frac{\lambda}{3!}\delta_\chi^3}_{\mathcal{L}_{\text{int}}}, \tag{4.3.68}$$

in terms of $\delta_\chi = a\delta\phi$.

The EOM read

$$\dots . \tag{4.3.69}$$

The bispectrum is connected to the three-point function since we lose one independent momentum because of the requirement of momentum conservation.

What follows is a long discussion on how to perform the integrals involved in the computation of the bispectrum.

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