Low Energy Theoretical Astroparticle Physics

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2022-02-01

1 Neutrino physics

Monday 2021-12-6

Neutrinos are important because:

1. being neutral, they can trace sources;

   What does that mean?

2. they are connected with new physics;
3. there are open questions about their nature (Dirac or Majorana);
4. there could be a CP violation in the lepton sector;
5. they affect cosmology, with Baryon Acoustic Oscillations in the CMB, as well as the large scale structure.

`nu.to.infn.it`, "Neutrino Unbound"; Giunti et al, Giunti and Kim.
Outline of the course:

1. Dirac equation;
2. gauge theories;
3. standard EW model;
4. fermion masses and mixing;
5. neutrino oscillations in vacuo;
6. neutrino oscillations in matter;
7. current neutrino phenomenology;
8. extra on statistics and data analysis.

We know that neutrino masses are less than an eV; there are (at least) three flavours: ν_e, ν_μ and ν_τ, typically organized in doublets with the corresponding charged lepton.

These neutrinos can have charged interactions with a $W^\pm$ boson, as well as neutral interactions with a Z boson (which is connected with photon production).

Looking at these decays tells us about the number of (interacting) neutrino families: $N_\nu = 3$.

There are also bounds from cosmology: both in the CMB spectrum and in primordial nucleosynthesis.

Neutrinos are chiral in nature: only left-handed neutrinos (with negative helicity) seem to interact.

We know that a left-handed particle has a right-handed component of order $\beta \sim m/E$.

At order m^2/E there are neutrino flavor oscillations.

The Dirac equation

We start from Lorentz transformations: linear transformations which preserve the Minkowski metric. They are rotations and boosts. They can be written in terms of infinitesimal transformations: for example,

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \approx \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} \right) \begin{bmatrix} y \\ z \end{bmatrix}, \tag{1.1}$$

therefore

$$\Lambda = \mathbb{1} + i\,\mathrm{d}\theta \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \mathcal{O}\left(\theta^2\right) = e^{i\theta J_1} + \mathcal{O}\left(\theta^2\right). \tag{1.2}$$

The same holds for the boosts:

$$\Lambda = \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix} = e^{iuk_1} \qquad \text{where} \qquad K_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{1.3}$$

With these generators we can make any transformation we want: a Lorentz transformation will be in general given by the composition of a rotation around $\vec{\omega} = \theta\hat{n}$ and a boost in $\vec{u} = u\hat{v}$. This will then read

$$\Lambda = \exp\left(i \left(\vec{\omega} \cdot \vec{J} + \vec{u} \cdot \vec{K} \right) \right). \tag{1.4}$$

In general these do not commute: $[J_i, J_k] \neq 0$, $[K_i, K_j] \neq 0$, $[J_i, K_j] \neq 0$. This, however, is a closed algebra: all these commutators are given in terms of other J, K matrices.

This construction is not only useful if we need to make complicated transformations; it is also useful as a theoretical mean to parametrize a general transformation.

Pauli was trying to generalize the Schrödinger equation for a spin-1/2 particle. It can be found by mapping $\vec{p} \to -i\vec{\nabla}$ and $E \to i\partial_t$ in the eigenvalue equation for a classical kinematic Hamiltonian $H = p^2/2m + V$.

This is classical, so we forget boosts: let us try to at least have the 4D J_i rotation algebra $[J_i, J_j] = i\epsilon_{ijk}J_k$ for two-component vectors: this is also obeyed by the 2D matrices $\sigma_i/2$, where

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.5}$$

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{1.6}$$

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{1.7}$$

So, we could think to have a spinor transform under

$$\xi' = \lambda\xi \qquad \text{where} \qquad \lambda = \exp\left(i\vec{\omega} \cdot \frac{\vec{\sigma}}{2}\right). \tag{1.8}$$

In order for the momentum $\vec{p}$ to act on 2D objects we can use $\vec{p} \cdot \vec{\sigma}$.

The idea is to use minimal coupling: $\vec{p} \to \vec{p} - q\vec{A}$, and $\vec{V} \to \vec{V} + q\Phi$.

How can we generalize this to boosts? Not only is it possible to do, but there are two ways to do it.

The Lorentz group, to which the matrices Λ in $x' = \Lambda x$ belong, satisfies

$$\Lambda = \exp\left(i\left(\vec{\omega} \cdot \vec{J} + \vec{u} \cdot \vec{K}\right)\right) \tag{1.9}$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{1.10}$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k \tag{1.11}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k . \tag{1.12}$$

Spinors will transform with

$$\xi' = \Lambda_\xi \xi \tag{1.13}$$

$$\Lambda_\xi = \exp\left(i\frac{\vec{\sigma}}{2} \cdot \vec{J} + ?\right) \tag{1.14}$$

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2} , \tag{1.15}$$

so what do we need to add? We can do either $\vec{k} = \pm i\vec{\sigma}/2$, so we get

$$\Lambda_\xi = \exp\left(i\frac{\vec{\sigma}}{2} \cdot \vec{\omega} \pm \vec{u} \cdot \frac{\vec{\sigma}}{2}\right). \tag{1.16}$$

The plus sign is for right-handed spinors, the minus sign is for left-handed ones. If we have a spinor at rest, we cannot determine its helicity. If we boost it, we still cannot determine it!

But, it cannot be in a superposition of things which transform in different ways.

The idea is then to have a direct sum, an object which contains both the left and right-handed components:

$$\xi = \begin{bmatrix} \xi_L \\ \xi_R \end{bmatrix} . \tag{1.17}$$

These will be given by boosting the rest-frame spinor ξ in two different ways:

$$\xi'_R = \exp\left(\vec{u} \cdot \frac{\vec{\sigma}}{2}\right)\xi \tag{1.18}$$

$$\xi'_L = \exp\left(-\vec{u} \cdot \frac{\vec{\sigma}}{2}\right)\xi . \tag{1.19}$$

The Dirac equation is a relation between the left- and right-handed components of a free spinor. In order to derive it, we need some relations:

$$\exp\left(\vec{u} \cdot \frac{\vec{\sigma}}{2}\right) = \cosh(u/2) + \hat{u} \cdot \vec{\sigma} \sinh(u/2) \tag{1.20}$$

$$(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot \left(\vec{a} \times \vec{b}\right) \tag{1.21}$$

$$\exp\left(i\vec{\omega} \cdot \frac{\vec{\sigma}}{2}\right) = \cos(\theta/2) + i\hat{u} \cdot \vec{\sigma} \sin(\theta/2) . \tag{1.22}$$

With these, we get

$$\exp\left(\pm\vec{u} \cdot \frac{\vec{\sigma}}{2}\right) = \frac{E + m \pm \vec{p} \cdot \vec{\sigma}}{\sqrt{2m(E+m)}} \tag{1.23}$$

$$\xi_{R,L} = \frac{E + m \pm \vec{p} \cdot \vec{\sigma}}{\sqrt{2m(E+m)}}\xi \tag{1.24}$$

$$\xi = \frac{E + m \mp \vec{p} \cdot \vec{\sigma}}{\sqrt{2m(E+m)}}\xi_{R,L} , \tag{1.25}$$

therefore we can put these equations together into

$$-m\xi_{R,L} + \left(E \pm \vec{p} \cdot \vec{\sigma}\right) \xi_{L,R} = 0 , \tag{1.26}$$

or

$$\begin{bmatrix} -m & E + \vec{p} \cdot \vec{\sigma} \\ E - \vec{p} \cdot \vec{\sigma} & -m \end{bmatrix} \begin{bmatrix} \xi_R \\ \xi_L \end{bmatrix} = 0 . \tag{1.27}$$

If $m = 0$, these decouple into two equations:

$$(p \pm \vec{p} \cdot \vec{\sigma})\xi_{L,R} = 0 . \tag{1.28}$$

Therefore, for a massless particle

$$\left(\hat{p} \cdot \sigma\right) = \pm\xi_{R,L} . \tag{1.29}$$

This is not the case when the particle is massive.

Helicity is the expectation value of $\hat{p} \cdot \vec{\sigma}$, chirality is "being ξ_L or ξ_R".

This is all in momentum space, but it can also be written in position space by switching the momentum to a derivative: we introduce the Dirac matrices

$$\gamma^0 = \begin{bmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{bmatrix} \qquad \text{and} \qquad \vec{\gamma} = \begin{bmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}, \tag{1.30}$$

so that the Dirac equation becomes

$$(\gamma^\mu p_\mu - m)\psi = 0\,, \tag{1.31}$$

where $\psi = (\xi_L, \xi_R)^\top$.

This can then also be written in terms of derivatives as $(i\gamma^\mu \partial_\mu - m)\psi = 0$.

Left- and right-handed spinors transform in the same way under rotations; they differ under boosts.

It is useful to introduce

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ : \tag{1.32}$$

if we introduce

$$P_{L,R} = \frac{1 \mp \gamma^5}{2} \tag{1.33}$$

this will select the left- or right-handed component of a spinor.

We have derived this result using the Weyl basis; Dirac used a different one.

If the spinor transforms with $\psi \to T\psi$ the Dirac matrices transform with $\gamma^\mu = T\gamma^\mu T^{-1}$.

In the Dirac basis, which is useful when one studies non-relatvistic particles, the matrices look like

$$\gamma^0 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{bmatrix} \qquad \text{and} \qquad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}, \tag{1.34}$$

while

$$\gamma^5 = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}. \tag{1.35}$$

In the nonrelativistic limit $\vec{p} \to 0$ the equation decouples into

$$\begin{bmatrix} E - m & 0 \\ 0 & E + m \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0\,, \tag{1.36}$$

so we have $E = \pm m$ solutions. Note that ξ and η are not left- and right-handed, this is a different basis. In Dirac's time this was seen as a tragedy.

The fact that these are actually particles and antiparticle can be properly explained in QFT; in the end, the four degrees of freedom correspond to the left- and right-handed components of particles and antiparticles.

We can write the particle and antiparticle solutions as

$$\psi_p = \begin{bmatrix} \xi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\xi \end{bmatrix} e^{-ip_\mu x^\mu} \qquad \text{and} \qquad \psi_A = \begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\eta \\ \eta \end{bmatrix} e^{+ip_\mu x^\mu} . \tag{1.37}$$

It is convenient to introduce a charge-conjugation operator:

$$\mathsf{C}(\psi) = \psi^c = i\gamma^2\psi^* . \tag{1.38}$$

This can be seen by proving that $C(\psi_P) = \psi_A$. Also, if we look at the minimally-coupled Dirac equation for a particle in an external EM field, and we apply the conjugation operator we find that the particle will obey the same equation, but with an opposite charge.

The last thing to introduce here is the adjoint spinor: we know how a 4-spinor ψ behaves under a Lorentz transformation. What is the object which transforms with the inverse transformation? If we denote it as $\overline{\psi}$, we will be able to write *invariant* objects like $\overline{\psi}\psi$.

It takes some time to prove, but it comes out to be

$$\overline{\psi} = \psi^\dagger\gamma^0 . \tag{1.39}$$

We can make invariant objects like $\overline{\psi}\psi$, or objects like $\overline{\psi}\gamma^\mu\psi$: it can be proven that the latter is *divergenceless*, $\partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0$, so it can be used to describe currents.

2 The standard electroweak model

Gauge theories

Tuesday 2021-12-7

Yesterday we wrote the equation for a free fermion:

$$\xi'_{R,L} = \exp\left(-i\vec{\omega}\cdot\frac{\vec{\sigma}}{2} \pm \vec{u}\cdot\frac{\vec{\sigma}}{2}\right)\xi_{R,L} . \tag{2.1}$$

The Dirac equation tells us that these are coupled:

$$\begin{bmatrix} -m & E+\vec{p}\cdot\vec{\sigma} \\ E-\vec{p}\cdot\vec{\sigma} & m \end{bmatrix} \begin{bmatrix} \xi_R \\ \xi_L \end{bmatrix} = 0 . \tag{2.2}$$

For nonzero mass these components are coupled, while in the zero-mass case they are decoupled and helicity is equal to chirality.

Helicity is the eigenvalue under $\hat{p}\cdot\vec{\sigma}$, chirality is the eigenvalue under γ^5.

What is the meaning of γ^5 in position space? Is it a parity transformation?

The coupling of a fermion to an external EM field can be represented with $\partial_\mu \to \partial_\mu + iqA_\mu$.

The SM Lagrangian can be written on a mug as

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\psi}\not{D}\psi + \text{h.c.} + i\psi_i y_{ij}\psi_j\phi + \text{h.c.} + \left|\not{D}\phi\right|^2 - V(\phi) . \tag{2.3}$$

We have scalars ϕ, fermions ψ and spin-1 gauge fields A_μ.

From a Lagrangian $L(q, \dot{q})$ we get Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0\,. \tag{2.4}$$

If the Lagrangian is, say, $L = m\dot{q}^2/2 - V(q)$ we get Newton's law $m\ddot{q} = -\vec{\nabla} V = F$.

This is classical; in field theory we have Lagrangians in the form

$$\partial_\mu \frac{\partial \mathscr{L}(\phi, \partial\phi)}{\partial \partial_\mu \phi_i} - \frac{\partial \mathscr{L}}{\partial \phi_i} = 0\,. \tag{2.5}$$

We will write the Yukawa Lagrangian with terms like $\psi\phi$, and the QED Lagrangian with terms like ψA_μ.

The equation of motion for a scalar field is the Klein-Gordon equation: $(\partial_\mu \partial^\mu + m^2)\phi = 0$. The Lagrangian giving rise to this is

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m \phi^2\,. \tag{2.6}$$

We have written the EOM for a free fermion, the Dirac equation $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$; the Lagrangian giving it is the Dirac Lagrangian

$$\mathscr{L} = \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi\,. \tag{2.7}$$

Again, we have a mass-like term and a kinetic-like term.

The term $m\overline{\psi}\psi$ can be expanded into left and right components:

$$m\overline{\psi}\psi = m\left(\overline{\psi}_R \psi_L + \overline{\psi}_L \psi_R\right)\,. \tag{2.8}$$

What about the EM field? A_μ is gauge-dependent, but

$$F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} \tag{2.9}$$

is not. The Lagrangian giving Maxwell's equations in a vacuum (so, $\Box A^\mu = 0$ in the appropriate gauge) is

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\,. \tag{2.10}$$

The absence of a mass term means that the photon is massless.

What about interactions? We start from the Yukawa interaction between a fermion and a scalar. The Lagrangian, for starters, must contain the free terms for both:

$$\mathscr{L} = \underbrace{\overline{\psi}\left(i\gamma^\mu \partial_\mu - m_\psi\right)\psi + \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m_\phi \phi^2\right)}_{\text{free terms}} - \underbrace{g\overline{\psi}\psi\phi}_{\text{interaction}}\,. \tag{2.11}$$

The pictorial way to represent this is to draw quadratic terms for fermions like straight lines with an arrow, scalar fields as dashed lines, and cubic interaction terms like vertices.

What do fermion-photon interactions look like?

$$\mathscr{L}_{\text{QED}} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m_\psi\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - q\overline{\psi}\gamma^\mu\psi A_\mu\,. \tag{2.12}$$

The EOM for the electromagnetic field are Maxwell's equations with an external current $j^\mu = q\overline{\psi}\gamma^\mu\psi$.

This is the same Lagrangian we get if we do a minimal coupling substitution $\partial \to \partial + iqA$. It is invariant under gauge transformations.

These diagrams can be used to compute scattering amplitudes perturbatively.

The way to get a cross-section is to multiply the square modulus of the amplitude by the phase space term. The idea to get decay rates is similar.

The guiding principle to describe EW interactions is gauge invariance. The Yukawa Lagrangian has global phase invariance; what happens if we try to make a transformation $\psi \to e^{-iq\alpha(a)}\psi$? The term q is only introduced here for later convenience.

The timezone analogy for local gauge invariance! If we have the freedom to choose a phase locally, we must have carriers of information moving at the maximum possible speed, otherwise processes would be disrupted.

We start with $U(1)$ gauge invariance, as written above, but we will also need $SU(2)$ invariance, written as

$$\exp\left(-ig\vec{\theta}\cdot\vec{T}\right), \tag{2.13}$$

where $\vec{T} = \vec{\sigma}/2$, but we write them differently to not confuse them with spacetime rotations.

Introducing gauge invariance for a term $\overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi$ will take us to the QED Lagrangian.

The mass term is already invariant, the problem is the kinetic one. The trick is to introduce a covariant derivative

$$\mathrm{D}_\mu = \partial_\mu + iqA_\mu\,. \tag{2.14}$$

If $\psi \to e^{-iq\alpha(x)}\psi$, also $\mathrm{D}_\mu\psi \to e^{-iq\alpha(x)}\mathrm{D}_\mu\psi$, as long as A_μ also transforms like $A_\mu \to A_\mu + \partial_\mu\alpha$.

Then, the Lagrangian

$$\mathscr{L} = \overline{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,. \tag{2.15}$$

A term like $m^2 A_\mu A^\mu$ would violate gauge invariance, so this principle also gives an explanation as to why the photon is massless if we accept the gauge invariance principle.

Protons and neutrons were thought to be part of a doublet under isospin SU(2) transformations.

But, we have interactions like protons and neutrons interacting with electrons and neutrinos. So, one might think that electrons + neutrinos have isospin-like SU(2) symmetry as well.

We want to make the doublet ψ_1, ψ_2 invariant under SU(2) transformations as written above: how do we do it? Again, we redefine the derivative:

$$\partial_\mu \to \mathrm{D}_\mu - ig\vec{T} \cdot \vec{A}_\mu \,, \tag{2.16}$$

so we need to introduce three fields $\vec{A}_\mu$. These are the gauge bosons related to SU(2) gauge invariance.

These now transform like

$$\vec{A}_\mu \to \vec{A}_\mu - \partial_\mu \vec{\theta}(x) + g\vec{\theta} \times \vec{A}_\mu \,. \tag{2.17}$$

The presence of this coupling means that the field is charged.

The field tensor here is

$$\vec{F}_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} + g\vec{A}_\mu \times \vec{A}_\nu \,. \tag{2.18}$$

We then get a Lagrangian like

$$\mathscr{L} = \overline{\psi} \left(i\gamma^\mu \mathrm{D}_\mu - m \right) \psi - \frac{1}{4} \vec{F}^{\mu\nu} \cdot \vec{F}_{\mu\nu} \,. \tag{2.19}$$

In the ψ term we have $\psi\psi$ terms, as well as $\psi\psi A$ interactions. In this F^2 term we have quadratic, trilinear and quadrilinear terms in A!

The actual group for the EW theory is $SU(2)_L \otimes U(1)_Y$. The simplest EW world contains

1. 1 massive electron, with both right- and left-handed components;
2. 1 massless ν_e, with only the left-handed component;
3. EM interactions (we want to have a diagram describing how the electron interacts with a massless photon);
4. weak chiral interactions like $\nu_L e_L W^\pm$, where $W^\pm$ is massive.

The Higgs mechanism accomplishes this, and it also gives mass to fermions. If neutrinos have Majorana masses, the Higgs mechanism cannot give them mass.

Let us define a doublet $L = (e_L, \nu_{e,L})$, and a singlet $R = e_R$. The theory must then be invariant if we redefine

$$L \to L' = \exp\left(-ig\vec{\theta}(x) \cdot \vec{T} \right) L \tag{2.20}$$

$$R \to R' = R \,. \tag{2.21}$$

The doublet L corresponds to the $T_3 = \pm 1/2$ quantum numbers, while $T_3 = 0$ for the singlet.

Could the three bosons be some combinations of the photon and the two $W^\pm$ bosons? We know that the charge operator Q has eigenvalues 0, -1 for the doublet, but $\operatorname{Tr} Q \neq 0$ while $\operatorname{Tr} T_i = 0$ for all T_i.

Why would Q need to be able to be written as a function of the T_i? The proper answer lies in Nöther's theorem.

We therefore introduce a further hypercharge symmmetry:

$$L' = \exp\left(-ig'\beta(x)\frac{Y}{2}\right)L \tag{2.22}$$

$$R' = \exp\left(-ig'\beta(x)\frac{Y}{2}\right)R\,, \tag{2.23}$$

so that this generator commutes with the T_i, and indeed $[T_i, Y] = 0$. The only way to have this is $Y \propto Q - T_3$, and indeed $Y = 2(Q - T_3)$.

In the end, the Lagrangian must be chargeless.

The full transformation is therefore

$$L' = \exp\left(-ig\vec{\theta}(x)\cdot\vec{T} - ig'\beta(x)\frac{Y}{2}\right)L \tag{2.24}$$

$$R' = \exp\left(-ig'\beta(x)\frac{Y}{2}\right)R\,. \tag{2.25}$$

The procedure is then like before: we need to redefine the derivative, as

$$\mathrm{D}_\mu L = \left(\partial_\mu + ig\vec{T}\cdot\vec{A}_\mu - ig'\frac{Y}{2}B_\mu\right)L\,, \tag{2.26}$$

so we have four fields, the three $\vec{A}_\mu$ with their $\vec{F}_{\mu\nu}$ and B_μ with its $G_{\mu\nu}$.

The Lagrangian will then read

$$\mathscr{L} = \overline{L}i\gamma^\mu\mathrm{D}_\mu L + \overline{R}i\gamma^\mu\mathrm{D}_\mu R + \text{kinetic}\,, \tag{2.27}$$

but we cannot write mass terms like $m\overline{\psi}\psi$, which would be $\overline{L}R$ or $L\overline{R}$: the matrix dimensions don't match up!

So, everything's massless: the Higgs mechanism comes to the rescue. It gives masses to all the bosons except the photon, as well as giving mass to the leptons.

Suppose we have a $U(1)\otimes U(1)$ symmetry, spontaneously broken to $U(1)$. The thing we think of is a pencil about to fall on a table.

Thursday 2021-12-9

We have discussed the Dirac equation and SU(2) gauge invariance; now we will look at the Higgs mechanism and SSB, with the goal in mind to understand the Lagrangian written on the CERN mug.

We study this model in the context of an L doublet (ν_{eL}, e_L) and an R singlet (e_R).

The Higgs mechanism yields neutral currents as a "bonus".

So far, we have written the gauge field Lagrangian:

$$\mathscr{L}_{\mathrm{GF}} = -\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}\,, \tag{2.28}$$

and the lepton kinetic Lagrangian:

$$\mathscr{L}_{\mathrm{F}} = \overline{L}(i\gamma^\mu\mathrm{D}_\mu)L + \overline{R}(i\gamma^\mu\mathrm{D}_\mu)R \tag{2.29}$$

$$\mathrm{D}_\mu L = \left(\partial_\mu - ig\vec{T}\cdot\vec{A}_\mu - ig'\frac{Y}{2}B_\mu\right)L \tag{2.30}$$

$$\mathrm{D}_\mu R = \left(\partial_\mu - ig' \frac{Y}{2} B_\mu \right) R \,. \tag{2.31}$$

We want to have a breaking like $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y \to \mathrm{U}(1)_Q$, where we want the ground state Q to be chargeless (since it represents the photon).

In order to have a Dirac-like mass term, $m\overline{\psi}_L \psi_R$, we need to saturate a doublet with a singlet: it cannot be done! So, we need a new field ϕ which is at least a doublet, and then we can make a term $\overline{L}\phi R$.

A complex field $\phi \in \mathbb{C}^2$ works, and it must be chargeless for things to work. The Yukawa Lagrangian will then be

$$\mathscr{L}_{\mathrm{YUK}} = -y_e \left(\overline{L}\phi R + \overline{R}\phi^\dagger L \right) \,. \tag{2.32}$$

A mass term for the field ϕ in the form $m^2 \phi^\dagger \phi$ would be stable, but we want to make it *un*stable at $\phi = 0$, o we need a $\phi^\dagger \phi$ term with a negative coefficient and a $(\phi^\dagger \phi)^2$ term with a positive coefficient:

$$V(\phi) = -\mu^2 (\phi \phi^\dagger) + \lambda (\phi \phi^\dagger)^2 \,. \tag{2.33}$$

The minimum of this potential is reached when $\phi\phi^\dagger = (1/2)\mu^2/\lambda = (1/2)v^2$.

Why $\phi\phi^\dagger$? That's 2x2...

So far, we have introduced the couplings g and g', the parameters μ and v for the Higgs field potential, and the Yukawa coupling y_e.

The connection to experiment will be to write in terms of these parameters e, G_F, M_Z, M_H, m_e.

So a 5 parameter fit for 5 measurements?

The ϕ field near the minimum can be parametrized as

$$\phi = \begin{bmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{bmatrix} \,. \tag{2.34}$$

The upper, charged part is equal to zero. In the Yukawa Lagrangian we find

$$\begin{aligned}
\mathscr{L}_{\mathrm{YUK}} &= -y_e \left[\begin{bmatrix} \overline{\nu}_{eL} & e_L \end{bmatrix} \begin{bmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{bmatrix} e_R + \overline{e}_R \begin{bmatrix} 0 & \frac{v+H}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \nu_{eL} \\ e_L \end{bmatrix} \right] &(2.35)\\
&= -\frac{y_e}{\sqrt{2}} (v+H) \left(\overline{e}_L e_R + \overline{e}_R e_L \right) &(2.36)\\
&= -m_e \overline{e}e - \frac{y_e v}{\sqrt{2}} H \overline{e}e \,. &(2.37)
\end{aligned}$$

We have not obtained any mass for the neutrino, as we expected by not having a ν_R term.

The proportionality of the Yukawa coupling to the masses has been seen experimentally.

In general the degrees of freedom of the ϕ can be written as

$$\phi = \exp\left(i\vec{T}\cdot\frac{\vec{\sigma}}{2}\right)\begin{bmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{bmatrix}, \tag{2.38}$$

so we can use SU(2) gauge invariance to select a ground state; the three remaining degrees of freedom become Goldstone bosons and are "eaten" by the three $W_\mu^\pm$, Z_μ^0 bosons to give them a third, longitudinal polarization.

The Higgs sector of the Lagrangian becomes

$$\mathscr{L}_\mathrm{H} = \left(\mathrm{D}_\mu\phi^\dagger\right)\left(\mathrm{D}_\mu\phi\right) - V(\phi)\,, \tag{2.39}$$

which contains the bosons. What we get if we diagonalize this Lagrangian is terms written in terms of the new fields

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(A_\mu^1 \pm A_\mu^2\right), \tag{2.40}$$

and

$$\begin{bmatrix} Z_\mu \\ A_\mu \end{bmatrix} = \begin{bmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{bmatrix}\begin{bmatrix} A_\mu^3 \\ B_\mu \end{bmatrix}, \tag{2.41}$$

where θ_W is the Weinberg angle, chosen such that $\tan\theta_W = g'/g$.

The masses of these bosons come out to be

$$M_W^2 = \frac{v^2}{4}g^2 \qquad \text{and} \qquad M_Z = \frac{v^2}{4}(g^2 + g'^2)\,, \tag{2.42}$$

while $m_\gamma = 0$.

We get cubic couplings like HZZ but also quartic ones like $HHZZ$ or $HHWW$.

The mass terms for the Higgs is $m_H = \sqrt{2}\mu$.

The photon does not couple directly to the Higgs, but we can have a $H \to \gamma\gamma$ process through loops.

When we rewrite the gauge field Lagrangian in terms of the physical fields A_μ, W_μ and Z_μ, we get quadratic, cubic and quartic terms in the gauge terms.

The terms we get are γWW, ZWW, as well as the quartic $\gamma\gamma WW$, $WWWW$, $ZZWW$, γZWW.

What about the fermion field Lagrangian? We expect to get terms in the form $X^\mu J_\mu$, where X^μ is a gauge field while J_μ is a fermion current.

The result is

$$\mathscr{L}_\mathrm{F} = \mathscr{L}_\text{electromagnetic} + \mathscr{L}_\text{charged current} + \mathscr{L}_\text{neutral current}\,. \tag{2.43}$$

The electromagnetic term is

$$\mathscr{L}_\text{electromagnetic} = \underbrace{\frac{gg'}{\sqrt{g^2+g'^2}}}_{e} J^\mu_{EM} A_\mu\,, \tag{2.44}$$

where e is the electric charge; the other terms read

$$\mathscr{L}_{\text{charged current}} = \frac{g}{\sqrt{2}} J^{\mu}_{\pm} W^{\mp} \tag{2.45}$$

$$\mathscr{L}_{\text{neutral current}} = \frac{g}{\cos\theta_W} J^{\mu}_{NC} Z_{\mu} \,. \tag{2.46}$$

The electromagnetic current will be $J^{\mu}_{EM} = \bar{e}\gamma^{\mu} Q e$ (by construction: we made the theory to reproduce Maxwell's equations).

The charged current is $J^{\mu}_{+} = \bar{e}_L \gamma^{\mu} \nu_{eL}$ and $J^{\mu}_{-} = \bar{\nu}_{eL}\gamma^{\mu} e_L$.

These were the already-observed parts, while the new one is

$$J^{\mu}_{NC} = \bar{\nu}_{eL}\gamma^{\mu} T_3 \nu_{eL} + \bar{e}_L \gamma^{\mu} \left(T_3 - Q s^2_W \right) e_L + \bar{e}_R \gamma^{\mu} (-Q s^2_W) e_R \,. \tag{2.47}$$

This term is carrying a current $T_3 - Q s^2_W$, where $s^2_W = \sin^2\theta_W$. The generator T_3 is just half of the σ_3 Pauli matrix.

The phenomenology of this prediction is quite rich.

We have couplings in the form $\gamma^{\mu} P_{L,R}$, there is specific jargon to describe these interactions.

1. Vector (V) currents are those in the form $\overline{\psi}\gamma^{\mu}\psi$, which do not change sign under a parity transformation;

2. Axial (A) currents are those in the form $\overline{\psi}\gamma^{\mu}\gamma^5\psi$, which *do* change sign under a parity transformation.

EM interactions are of type V, charged currents are of type $V - A$ (or, $1 \pm \gamma^5$), while neutral currents are of the form $g_V V + g_A A$.

We can define left-handed couplings in the neutral current $g_L = T_3 - Q s^2_W$, as well as right-handed ones like $g_R = -Q s^2_W$ (since the action of T_3 on an R singlet is zero).

The vector and axial couplings are then defined as

$$g_V = g_L + g_R = T_3 - 2 Q s^2_W \tag{2.48}$$

$$g_A = g_L - g_R = T_3 \,. \tag{2.49}$$

When a short-range gauge boson mediates an interaction, its contribution will be approximately $1/M^2$ at low energies.

So, both processes which are mediated by $W^{\pm}$ and Z must give the correct low-energy limit.

Specifically, we find

$$G_F \sim \frac{g^2}{M^2} \qquad \text{or} \qquad \frac{g^2}{8 M_Z^2 \cos^2\theta_W} = \frac{g^2}{8 M_W^2} = \frac{G_F}{\sqrt{2}} \,. \tag{2.50}$$

In the notes there are qualitative considerations about the amplitudes in β decay, getting the energy spectrum of the electron.

A charged current process is $\mu \to \nu_\mu \overline{\nu}_e e$. The Fermi constant is best estimated through this process, since it is very well-known both theoretically and experimentally.

The decay $\pi \to \ell \overline{\nu}_e$ probes the $V - A$ nature of the interaction.

Another interesting process is $\nu_\mu e$ scattering, which allows us to estimate s_W^2.

We can have the recoil of an entire nucleus with a low-energy neutrino, which is mediated by a Z boson, Coherent Elastic ν Nucleus Scattering, CEνNS.

This process turns out to be proportional to the number of neutrons squared, which goes up quickly — this is the only tabletop neutrino detector.

The idea is that the neutrino's wavelength is very long, so it cannot see the specific nucleons or quarks.

We separate out the parameters g, g', μ and v from the Yukawa coupling of the electron — the first four remain the same, while we will need to introduce more Yukawa coupling for the different families.

The data are e, G_F, M_H, M_Z, as well as the mass of the electron.

The relations are

$$
\begin{aligned}
e &= \frac{g g'}{\sqrt{g^2 + g'^2}} && (2.51) \\
G_F &= \frac{1}{\sqrt{2} v^2} && (2.52) \\
M_H &= \sqrt{2} \mu && (2.53) \\
M_Z &= \frac{v}{2} \sqrt{g^2 + g'^2} \,, && (2.54)
\end{aligned}
$$

as well as

$$
m_e = \frac{y_e v}{\sqrt{2}} \,. \qquad (2.55)
$$

These are all computed at tree-level; at more loops there can be corrections of the order of a few parts in a hundred. There are proposals to replace e with M_W, so that all the terms refer to the same energy scale $\sim 100\,\text{GeV}$.

In natural units,

$$
\begin{aligned}
e &= 0.3 && (2.56) \\
G_F &= 1.17 \times 10^{-5}\,\text{GeV}^{-2} && (2.57) \\
M_H &= 125\,\text{GeV} && (2.58) \\
M_Z &= 91\,\text{GeV} && (2.59) \\
m_e &= 0.51 \times 10^{-3}\,\text{GeV} \,, && (2.60)
\end{aligned}
$$

$e = .3$ corresponds to $e^2/4\pi = 1/137 = \alpha$; why does this not match with the astropy code?

so

$$
g = 0.64 \qquad (2.61)
$$

$$g' = 0.34 \tag{2.62}$$
$$\mu = 88\,\text{GeV} \tag{2.63}$$
$$v = 246\,\text{GeV} \tag{2.64}$$
$$y_e = 2.9 \times 10^{-6}\,, \tag{2.65}$$

so

$$\lambda = 0.13 \tag{2.66}$$
$$\sin^2\theta_W = 0.22 \tag{2.67}$$
$$M_W = 80\,\text{GeV}\,. \tag{2.68}$$

In terms of naturalness, it is weird that $y_e \ll 1$, which tells us that $m_e \ll 100\,\text{GeV}$.

Fermion masses and mixing

Friday 2021-12-10

The full symmetry group of the standard model is $\mathrm{SU}(3)_c \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$.

The building blocks are doublets like (ν_{eL}, e_L) with charges respectively $Q = 0$ and $Q = -1$, as well as $T_3 = +1/2$ and $T_3 = -1/2$.

The right-handed singlets, on the other hand, also have $Q = 0$ and $Q = -1$ but $T = 0$.

Any right-handed neutrinos would be *sterile*, since they do not interact.

This also holds for muonic and tauonic flavors, as well as for quark doublets as long as we add $2/3$ to all charges.

The relation between the quantum numbers is $Y = 2(Q - T_3)$.

In general the doublets look like

$$\begin{bmatrix} U^\alpha \\ D^\alpha \end{bmatrix}_L\,, \tag{2.69}$$

where α is a generation index.

The most general mass term looks like $m\overline{\psi}_L\psi_R$ — we should also consider its Hermitian conjugate, but it works analogously.

As usual, we insert the Higgs field to saturate the doublets and diagonalize. We will need to look at the effect of this change of basis on the mixing matrix.

The most general currents look like

$$J^\mu_{EM} = \sum_\alpha \overline{U}^\alpha \gamma^\mu U^\alpha + (U \to D) \tag{2.70}$$
$$J^\mu_{NC} = \sum_\alpha \overline{U}^\alpha_L \gamma^\mu (T_3 - Q s_W^2) U^\alpha_L + \sum_\alpha \overline{U}^\alpha_R \gamma^\mu (-Q s_W^2) U^\alpha_R + (U \to D) \tag{2.71}$$
$$J^\mu_- = \sum_\alpha \overline{U}^\alpha_L \gamma^\mu D^\alpha_L (U \to D \text{ yields } J^\mu_+)\,. \tag{2.72}$$

The coupling terms then look like $J^\mu_{EM} A_\mu$, $J^\mu_{NC} Z_\mu$ and $J^\mu_\mp W^\pm$.

The various components will transform according to some unitary matrices:

$$D^\alpha_R = W^{\alpha\beta} D^\beta_R \qquad \text{and} \qquad D^\alpha_L = S^{\alpha\beta} D^\beta_L \tag{2.73}$$

$$U_R^\alpha = T^{\alpha\beta} D_R^\beta \qquad \text{and} \qquad U_L^\alpha = R^{\alpha\beta} D_L^\beta \,. \tag{2.74}$$

Most of the currents are left unchanged by these transformations; the charged current however is the one which changes:

$$J_-^\mu = \overline{U}_L \gamma^\mu \underbrace{R^\dagger S}_{\neq 1 \text{ in general}} D_L \,. \tag{2.75}$$

The Yukawa Lagrangian will then include a term like

$$\mathscr{L} \ni \underbrace{\sum_{\alpha\beta} y_D^{\alpha\beta} \begin{bmatrix} \overline{U}^\alpha & \overline{D}^\alpha \end{bmatrix}_L \begin{bmatrix} 0 \\ v/\sqrt{2} \end{bmatrix} D_R^\beta}_{\text{D-type couplings}} + \underbrace{\sum_{\alpha\beta} y_U^{\alpha\beta} \begin{bmatrix} \overline{U}^\alpha & \overline{D}^\alpha \end{bmatrix}_L \begin{bmatrix} v/\sqrt{2} \\ 0 \end{bmatrix} U_R^\beta}_{\text{U-type couplings}} \,. \tag{2.76}$$

We define a mass matrix

$$M_{U,D}^{\alpha\beta} = y_{U,D}^{\alpha\beta} \frac{v}{\sqrt{2}} \,, \tag{2.77}$$

so that

$$\mathscr{L} \ni \overline{D}_L M_D D_R + \overline{U}_L M_U U_R + \text{h. c. } . \tag{2.78}$$

How can we put the Higgs vacuum on the upper component? We can define $\widetilde{\phi} = i\sigma_2 \phi^*$. For this field, the 1 component has zero charge while the 2 component has -1 charge.

If we introduce this field, the charge of the Yukawa Lagrangian is zero.

Each of the $M_{U,D}$ matrices will be diagonalized like $S^\dagger M T = M_{\text{diag}}$, where S and T are both unitary.

We then have, for down-type fields: $\overline{D}_L M_D D_R$ going into

$$D_R \to W D_R \qquad D_L \to S D_L \qquad M_D \to M_D^{\text{diag}} \,, \tag{2.79}$$

while $\overline{U}_R M_U U_R$ becomes

$$U_R \to T U_R \qquad U_L \to R U_L \qquad M_U \to M_U^{\text{diag}} \,, \tag{2.80}$$

In general, these will be different matrices for leptons and for quarks.

The charged current then reads $J_-^\mu = \overline{U}_L \gamma^\mu V D_L$, where $V = R^\dagger S$.

More explicitly, we get

$$J_-^\mu \overline{U}_L^\alpha \gamma^\mu V^{\alpha\beta} D_L^\beta \,. \tag{2.81}$$

The interaction of a $\overline{U}_L^\alpha$ and D_L^β is then modulated by a factor $(g/\sqrt{2}) V^{\alpha\beta}$.

If V were 2x2 it would be a real matrix, if it were 3x3 it would be a complex matrix.

For quarks, the values on the diagonal are close to 1, while the largest values off-diagonal are of the order 0.22 for the CKM matrix, the mass-mixing matrix for quarks.

For example, in β decay we get a modulation by a factor V_{ud}, which is slightly less than 1.

For leptons, the matrix is called PMNS.

In the usual construction of the Standard Model, there were no mass terms for neutrinos or right-handed $\nu_{\alpha R}$. Therefore, in this case there could be no mixing among leptons, since the mixing matrix only ever appears "sandwitched".

The basic definition of neutrino flavor is "which lepton does it have charged current interactions with".

For example, ν_e is the ν produced in β^+ decay, and $\overline{\nu}_e$ is the ν produced in β^- decay.

Let us then consider $\nu_e \to \nu_\mu$ oscillations. We always need to consider the amplitudes for production, propagation, detection.

However, it is convenient to isolate the propagation into the matrix

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} . \tag{2.82}$$

Forgetting the fact that this is a simplification can lead to paradoxes; for example, ether leptons do not oscillate.

There is a basis for the Dirac equation for which $i\gamma^\mu$ is real — this leads to the fact that $\psi = \psi^*$.

This cannot work for electrons and positrons, since they are charged. A neutral fermion, on the other hand, might be its own antiparticle.

If we have a β^+ decay, we will produce a ν_e which is left-handed. In a β^- decay, on the other hand, we will produce a right-handed $\overline{\nu}_e$.

If the neutrino is massless, this cannot change. Formally, the Dirac equation decouples.

This "Weyl" case is not completely excluded by current phenomenology; the lightest of the neutrinos could be completely massless.

If we have massive neutrinos, though, there will be an $\mathcal{O}(m_\nu/E)$ component with the opposite chirality.

However, the neutrino and antineutrino are still distinguishable.

If they are Majorana, on the other hand, $\nu_e = \overline{\nu}_e$; this must mean that they are neutral not just for electric charge but for all charges.

We have observed inverse beta decay:

$$\nu_e + \mathrm{n} \to \mathrm{p} + e^- \qquad \text{and} \qquad \overline{\nu}_e + \mathrm{p} \to \mathrm{n} + e^+ , \tag{2.83}$$

but not the same reaction for $\nu_e \leftrightarrow \overline{\nu}_e$. Is this an indication that neutrinos are indeed Dirac?

Well, if neutrinos are indeed Majorana these reactions are possible, but suppressed at order m_ν/E!

The experiment which saw the largest number of neutrinos collected about 2 million events…

The way to have good statistics is to look at decays, specifically neutrinoless double-beta decay.

The decay looks like

$$\{2n\} \to \{2p\} + 2e^- . \tag{2.84}$$

The decay of the first proton looks like the emission of a right-handed $\overline{\nu}_e$ with an e^-, through a charged-current interaction.

IF $m_\nu \neq 0$, the neutrino has a left-handed component (not possible if the neutrino is Weyl); if $\nu = \overline{\nu}$, this left-handed component is a left-handed ν_e (not possible if the neutrino is Dirac)!

At the second proton, the left-handed ν_e is absorbed and an e^- is emitted.

This process is suppressed by m_ν / E: it probes absolute neutrino mass.

Current neutrinoless double beta decay are probing lifetimes of the order of 10^{26} to 10^{27} yr.

It can be shown that if this process takes place, then neutrinos must be Majorana.

Pictorially, one can join the legs of the $0\nu\beta\beta$ diagram to find a diagram which turns a ν_e into a $\overline{\nu}_e$.

Mass terms for neutrinos

Monday 2021-12-13

We will discuss the difference between Weyl, Dirac and Majorana neutrinos for the single-generation case. Remember that under a Lorentz transformation $\Lambda(\vec{\omega}, \vec{n})$ they change as

$$\phi_R \to \exp\left(i\vec{\omega} \cdot \frac{\vec{\sigma}}{2} + \vec{n} \cdot \frac{\vec{\sigma}}{2} \right) \phi_R \tag{2.85}$$

$$\phi_L \to \exp\left(i\vec{\omega} \cdot \frac{\vec{\sigma}}{2} - \vec{n} \cdot \frac{\vec{\sigma}}{2} \right) \phi_L \,. \tag{2.86}$$

It can be shown that if ϕ_R is right-handed, $i\sigma_2 \phi_R^*$ is left-handed (and similarly with $L \leftrightarrow R$).

We can therefore parametrize our full spinor in terms of two right-handed spinors, as

$$\Psi = \begin{bmatrix} \phi_R \\ \phi_L \end{bmatrix} = \begin{bmatrix} u \\ i\sigma_2 v^* \end{bmatrix} . \tag{2.87}$$

Recall also the projectors

$$\psi_{L,R} = P_{L,R}\psi \qquad \text{where} \qquad P_{L,R} = \frac{1 \mp \gamma^5}{2} \,. \tag{2.88}$$

We also have $\overline{\psi} = \psi^\dagger \gamma^0$ and $\psi^c = C(\psi) = i\gamma^2 \psi^*$.

The convention for the order of operations is as follows:

1. projectors $P_{L,R}$ act before
2. charge conjugation $C(\cdot)$ which acts before
3. conjugation $\bar{\cdot}$.

In terms of our parametrization $\psi(u, v)$ the charge conjugate swaps u and v:

$$\psi^c = \begin{bmatrix} v \\ i\sigma_2 u^* \end{bmatrix} . \tag{2.89}$$

The Dirac case is the one in which $u \neq v$, the Majorana case is the one in which $u \simeq v$. The equality in the Majorana case can actually be relaxed including an arbitrary phase: $\psi = \psi^c e^{i\phi}$ for some fixed ϕ, called the *Majorana phase*.

The Weyl case is the one in which $m = 0$, meaning that $\psi = \psi_R$ or $\psi = \psi_L$. The spinor just has two degrees of freedom.

The Weyl-Majorana case would then be $\psi = \psi_R + \psi_R^c = \psi^c$.

The Dirac case is the general, massive, 4 degrees of freedom one.

A Dirac mass term looks like $m\overline{\psi}\psi$, a Majorana one would look like $(1/2)m\overline{\psi}\psi$.

In the Dirac case, this would read

$$\overline{\psi}\psi = \overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L\,, \tag{2.90}$$

while in the Majorana, left-handed case we would have

$$\overline{\psi}\psi = \overline{\psi}_L\psi_L^c + \overline{\psi}_L^c\psi_L\,, \tag{2.91}$$

and the right-handed one is analogous.

The most general Lagrangian would then include a term

$$\mathscr{L} \ni m_D\left(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L\right) + \frac{1}{2}m_L\left(\overline{\psi}_L^c\psi_L + \overline{\psi}_L\psi_L^c\right) + \frac{1}{2}m_R\left(\overline{\psi}_R^c\psi_R + \overline{\psi}_R\psi_R^c\right) \tag{2.92}$$

$$= \frac{1}{2}\begin{bmatrix} \overline{\psi}_L + \overline{\psi}_L^c, & \overline{\psi}_R + \overline{\psi}_R^c \end{bmatrix}\begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix}\begin{bmatrix} \psi_L + \psi_L^c \\ \psi_R + \psi_R^c \end{bmatrix}, \tag{2.93}$$

which we can diagonalize! The eigenvectors will in general be linear combinations of Majorana fields.

What does the Standard Model tell us about these masses? The m_D term is allowed, is comes from the SSB of the Yukawa terms as long as we have right-handed neutrinos.

Specifically, it will be $m_D \sim y_\nu v/\sqrt{2}$ where $v \approx 200\,\text{GeV}$. In order for this to work we must have $y_\nu \lesssim 10^{-11}$.

In the SM we cannot have Majorana mass terms m_L, since they break the electroweak symmetry.

Clarify this point

The term m_R, on the other hand ($m_R\overline{\psi}_R\psi_R^c + \text{h. c.}$), is allowed by the SM gauge group, but the m_R term is a new mass scale of the theory.

The mass matrix can be thought to look like

$$\begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix} = \begin{bmatrix} 0 & \sim v \\ \sim v & \sim \Lambda \end{bmatrix}, \tag{2.94}$$

where Λ is a scale from Beyond-the-Standard-Model physics.

These ideas came out when people were considering very large gauge groups.

The diagonalization of that matrix yields a light neutrino $\nu \sim \nu_L + \nu_L^c$ with mass of the order $\mathcal{O}\left(v^2/\Lambda\right) \ll v$, as well as a massive neutrino $\nu \sim \nu_R + \nu_R^c$ with mass of the order $\mathcal{O}(\Lambda)$.

This is the see-saw mechanism.

A technical detail: one mass comes out negative, but if ψ has a negative mass $\gamma^5\psi$ has a positive mass.

What happens if we have more than one neutrino generation?

In the SM, we have determined that the flavor eigenstates are linear combinations of the mass eigenstates with a unitary mixing matrix U.

We can also have a number N_s of sterile neutrino states which do not couple to the $W^\pm$ and Z bosons. The number N_s is not limited to the right-handed components of the three neutrinos, there could be more. The full mixing matrix will then be

$$\begin{bmatrix} 3\times 3 & 3\times N_s \\ N_s\times 3 & N_s\times N_s \end{bmatrix} \sim \begin{bmatrix} \sim U_{\text{PMNS}} & \sim \mathcal{O}(v/\Lambda) \\ \sim \mathcal{O}(v/\Lambda) & U_{N\times N} \end{bmatrix} . \tag{2.95}$$

A phenomenologically motivated situation is already the $N_s = 1$ one, with a fourth sterile neutrino with mass $\sim 1\,\text{eV}$. At present, though, the evidence for a sterile neutrino is very controversial.

The process of leptogenesis, by some process which satisfies Sakharov's conditions, is relevant for this discussion.

We need both C and CP violation. If we can find CP violation at low energy scales, it would be a good indication!

CP violation in the quark sector, $V_{\text{CKM}} = V^*_{\text{CKM}}$, cannot explain matter-antimatter asymmetry.

It is possible that a CP violating mechanism in the weak sector may give an imprint in the primordial GW background.

If the CP violating decay of heavy neutrinos is to blame, there might be a connection between the low- and high-energy manifestations of CP violation.

The effect of neutrino masses and mixing in propagation

We have

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = U_{\text{PMNS}} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} , \tag{2.96}$$

where $U_{\text{PMNS}}U^\dagger_{\text{PMNS}} = 1$, while in general $U \neq U^\dagger$. We will assume $E \gg m$.

The convention used for this matrix is as follows, since it can be shown that it contains three angles and a CP-violating phase:

$$\begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix}}_{R(\theta_{23})} \underbrace{\begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ s_{13}e^{i\delta} & 0 & c_{13} \end{bmatrix}}_{R(\theta_{13})} \underbrace{\begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R(\theta_{12})} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\psi'} & 0 \\ 0 & 0 & e^{i\psi''} \end{bmatrix}}_{\text{Majorana}} . \tag{2.97}$$

If we have Majorana neutrinos the spinor is equal to its conjugate up to a phase, but only two of the phases are physically meaningful. However, these play no role in oscillations.

We have measured these angles: for θ_{23} there is (almost) maximal mixing, $\theta_{23} \approx \pi/4$ or $s_{23}^2 \approx 0.5$. This is called *octant ambiguity*, since we do not know the precise value.

The value of s_{12}^2 is approximately 0.3.

The value of s_{13}^2 is approximately 0.02.

What about the mass spectrum? Well, the energy is approximately

$$E = \sqrt{p^2 + m^2} \approx p + \frac{m^2}{2p} \approx p + \frac{m^2}{2E} \,. \tag{2.98}$$

The masses are numbered 1, 2 and 3; by convention we always take $m_2^2 > m_1^2$.

We have $\delta m^2 = m_2^2 - m_1^2 \approx 7.5 \times 10^{-5}\,\mathrm{eV}^2$, while $\left|\Delta m^2\right| = \left|m_3^2 - m_1^2\right| \approx 2.5 \times 10^{-3}\,\mathrm{eV}^2$. We do not know the sign of Δm^2!

As long as we deal with three neutrino masses and mixing, we have 5 known quantities and 5 unknowns: we know

1. δm^2;
2. $\left|\Delta m^2\right|$;
3. s_{23}^2;
4. s_{12}^2;
5. s_{13}^2,

while we do not know some parameters which are hard or impossible to measure with oscillations:

1. the value of the CP-violating δ — but we do have some weak $\sim 2\sigma$ indications that it might be nonzero;
2. the sign of Δm^2;
3. whether $\theta_{13} \lesseqgtr \pi/4$;
4. the absolute mass scale (we only have some bounds);
5. whether neutrinos are Dirac or Majorana.

There is a search for right-handed neutrinos at many scales, even at LHC.

Tomorrow we will start studying neutrino oscillations, dealing with wavefunctions as opposed to spinors, approximating time with position $t \sim x$; and using the fact that $E \approx p + m^2/2E$.

The Schrödinger equation will read

$$i\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} = H \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} , \tag{2.99}$$

where, in a vacuum, H is diagonal, with the three energies of the neutrinos. There will be corrections in matter.

We will have

$$H = p + \frac{1}{2E} \underbrace{\begin{bmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{bmatrix}}_{M^2} . \tag{2.100}$$

Neutrino oscillations in vacuum

Tuesday 2021-12-14

A "vacuum" in this context is a situation with very low fermion density, air is a good approximation for it for example. We will later quantify the effects of interactions with matter.

The solution of Schrödinger's equation for that Hamiltonian is

$$\begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} = e^{-iH_m x} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}_0 , \tag{2.101}$$

but the interesting thing is to rewrite this in the flavor basis:

$$H_f = UH_mU^\dagger = p\mathbb{1} + \frac{1}{2E} UM^2U^\dagger , \tag{2.102}$$

where M^2 is the diagonal matrix with the square masses on the diagonal.

This is not a diagonal Hamiltonian anymore!

Factoring out the evolution according to $p\mathbb{1}$ we get

$$i\frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \frac{1}{2E} UM^2U^\dagger \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} . \tag{2.103}$$

Since this is not diagonal, we can have both flavor *appearance* and flavor *disappearance*.

The evolution is given by the exponential

$$S = \exp\Big(-iH_f x\Big) = \exp\left(-\frac{i}{2E} UM^2U^\dagger x\right) = U \exp\left(-i\frac{M^2}{2E} x\right) U^\dagger , \tag{2.104}$$

because of the properties of unitary matrices in exponentials.

It is convenient to write this in components, $\nu_\beta = S_{\beta\alpha}\nu_\alpha$:

$$S_{\beta\alpha} = \sum_i U_{\beta i} \exp\left(-i\frac{m_i^2}{2E}\right) U^\dagger_{\alpha i} , \tag{2.105}$$

and the probability for this will be $P(\nu_\alpha \to \nu_\beta) = P_{\beta\alpha} = \left|S_{\beta\alpha}\right|^2$.

This modulus can be explicitly rewritten as

$$P_{\alpha\beta} = \underbrace{\delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\, J^{ij}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij} x}{4E}\right)}_{\text{CP-conserving}} \underbrace{- 2\sum_{i<j} \mathrm{Im}\, J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij} x}{2E}\right)}_{\text{CP-violating}}, \tag{2.106}$$

where $\Delta m^2_{ij} = m^2_i - m^2_j$ while $J^{ij}_{\alpha\beta} = U_{\alpha i} U^\dagger_{\beta i} U^\dagger_{\alpha j} U_{\beta j}$. This is called the Jarlskog invariant (Cecilia Jarlskog studied this first in the context of the CKM matrix).

This all is in natural units; the SI units version is

$$\frac{\Delta m^2_{ij} x}{4E} \approx 1.267 \left(\frac{\Delta m^2_{ij}}{\mathrm{eV}}\right) \left(\frac{x}{\mathrm{m}}\right) \left(\frac{\mathrm{MeV}}{E}\right). \tag{2.107}$$

Interchanging ν and $\overline{\nu}$ is equivalent to interchanging U and $U*$; interchanging α and β is equivalent to interchanging U and $U^\dagger$.

What is the behavior of this under CP and T symmetries?

Let us call S the source of neutrinos, and D their detector. We have ν_α at S, and detect ν_β at D.

A CP transformation means we swap S and D, and have $\overline{\nu}_\alpha$ travelling backward to be detected as $\overline{\nu}_\beta$.

A T transformation swaps S and D again.

If CP is conserved,

$$P(\nu_\alpha \to \nu_\beta) = P(\overline{\nu}_\alpha \to \overline{\nu}_\beta), \tag{2.108}$$

meaning that we must have $U = U^*$; while if T is conserved we must have

$$P(\nu_\alpha \to \nu_\beta) = P(\nu_\beta \to \nu_\alpha) \qquad \text{and} \qquad P(\overline{\nu}_\alpha \to \overline{\nu}_\beta) = P(\overline{\nu}_\beta \to \overline{\nu}_\alpha), \tag{2.109}$$

meaning that we must have $U = U^*$.

Say we observed $P(\nu_e \to \nu_\mu) \neq P(\overline{\nu}_e \to \overline{\nu}_\mu)$. Would that not prove that $\nu_e \neq \overline{\nu}_e$ and/or $\nu_\mu \neq \overline{\nu}_\mu$, meaning that neutrinos are not Majorana?

A CPT transformation, therefore, must satisfy

$$P(\nu_\alpha \to \nu_\beta) = P(\overline{\nu}_\beta \to \overline{\nu}_\alpha). \tag{2.110}$$

This surely holds! It can be seen by the requirement $U = U^{**}$, but it also holds in general as a theorem about field theories.

The CP-violating part can be isolated as shown in equation (2.106).

In the parametrization of the U_{PMNS} the CP-violation is parametrized as δ; it is currently compatible with 0.

In order to observe CP violation we must have $\delta \neq 0, \pi$ (meaning that $U \neq U^*$); the second condition is that we must be looking at $\alpha \neq \beta$! In disappearance experiments the CP-violating part cancels.

All the mixing angles must be nonzero: $\theta_{ij} \neq 0$.

The fourth condition is that all square mass differences are nonzero: $\Delta m^2_{ij} \neq 0$.

CP violation is a genuine 3-neutrino phenomenon, it cannot be observed with 2! Current experimental results are compatible with all conditions being realized, but it's hard since δm^2 is about 30 times smaller than Δm^2.

Most current experiments are only sensitive to a submatrix

$$\begin{bmatrix} \nu_\alpha \\ \nu_\beta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \nu_i \\ \nu_j \end{bmatrix}, \tag{2.111}$$

where $\theta = \theta_{ij}$.

In this case, the probability simply reads

$$P_{\alpha\beta} = \sin^2 2\theta \sin^2 \left(\frac{\Delta m^2 L}{4E} \right), \tag{2.112}$$

while $P_{\alpha\alpha} = 1 - P_{\alpha\beta}$.

We have no information on the sign of Δm^2 since it is inside a $\sin^2$; there is no information on the absolute masses, there is no information on $\nu \leftrightarrow \overline{\nu}$.

There is an analogy with a double-slit experiment.

Practically, we do not observe probabilities but fluxes:

$$R_\beta = \int \Phi_\alpha \otimes P_{\alpha\beta} \otimes \sigma_\beta \otimes \epsilon_\beta, \tag{2.113}$$

which will be a multidimensional integral in general, a marginalization over all the parameters we do not care about — the flux at the source, the interaction cross-section, the detector efficiency.

Let's say we have a $\nu_\alpha \to \nu_\beta$ appearance experiment, and we do not observe anything.

What is typically done is to draw an *exclusion zone* in the $\Delta m^2, \sin^2 2\theta$ plane. The range for θ is therefore 0 to $\theta = \pi/4$.

Suppose, instead, that we observe a certain rate of β with $R_\beta = R \pm \sigma_\beta$.

In that case as well we can draw an allowed band. If we also have some spectral information, we can also measure L/E, therefore we can intersect several of these bands and measure Δm^2 and θ simultaneously.

We typically have *octant ambiguity*: the respective $\pi/4 < \theta < \pi/2$ value is typically also allowed.

This ambiguity has been removed for θ_{12} and θ_{23}, not for θ_{13}.

Experiments which are sensitive to $\Delta m^2 = m_3^2 - m_{1,2}^2$:

1. short baseline reactors, which attempt to observe $\overline{\nu}_e \to \overline{\nu}_e$ with $L \sim 1\,\mathrm{km}$ and $E \sim$ few MeV;

2. long baseline accelerators, which attempt to observe $\nu_\mu \to \nu_\mu$ or ν_e or ν_τ (or antineutrinos), with $L \sim 100 \div 1000\,\mathrm{km}$ and $E \sim 1 \div 10\,\mathrm{GeV}$;[1]

[1] These are able to observe appearance as well as disappearance since the energy is so high!

3. atmospheric neutrino experiments, which attempt to observe $\nu_\mu \to \nu_\mu$ or ν_e (or antineutrinos) with $L \sim 10 \div 10^4\,\text{km}$ and $E \geq 1\,\text{GeV}$.

For these experiments, we typically have $\delta m^2 L/4E \ll 1$, therefore we can take $\delta m^2 \approx 0$. The probability can then be written as

$$P_{\alpha\beta} = 4|U_{\alpha 3}|^2 \left|U_{\beta 3}\right|^2 \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \tag{2.114}$$

$$P_{\alpha\alpha} = 1 - 4|U_{\alpha 3}|^2 (1 - |U_{\alpha 3}|)^2 \sin^2\left(\frac{\Delta m^2 L}{4E}\right), \tag{2.115}$$

so these are able to probe a whole column of the mixing matrix; therefore, we can use these to measure θ_{13} and θ_{23}, as well as $\left|\Delta m^2\right|$.

Explicitly,

$$P_{ee} = 1 - \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \tag{2.116}$$

$$P_{\mu e} = s_{23}^2 \sin 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \tag{2.117}$$

$$P_{\mu\mu} = 1 - c_{13}^2 s_{23}^2 \left(1 - c_{13}^2 s_{23}^2\right) \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \tag{2.118}$$

$$P_{\mu\tau} = c_{13}^4 \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right), \tag{2.119}$$

the first three have been probed by atmospheric neutrinos, the first has been probed by short baseline experiments, the second two have been probed by long baseline experiments, the last has only been probed by OPERA (CERN to LNGS).

Atmospheric neutrinos have been probed by SuperKamiokande and IceCube.

The octant ambiguity for θ_{13} from P_{ee} is resolved by $P_{\mu\mu}$: it is indeed small and $< \pi/4$.

On the other hand, θ_{23} is close to $\pi/4$, and it is not determined in which octant it lies.

In the opposite case, we have $\delta m^2 L/4E$ of order 1, while $\Delta m^2 L/4E \gg 1$, meaning that those oscillations are averaged away.

One can do this with hundreds of kilometers L but low energies, $E \sim$ few MeV.

One finds, for KamLAND as well as for solar neutrinos:

$$P_{ee}^{2\nu} = c_{13}^4 P_{2\nu} + s_{13}^4 \qquad \text{where} \qquad P_{2\nu} = 1 - \sin^2 2\theta_{12} \sin^2\left(\frac{\delta m^2 L}{4E}\right). \tag{2.120}$$

These experiments are sensitive to the first row of the mixing matrix.

They measure $\delta m^2 \sim 7.5 \times 10^{-5}\,\text{eV}^2$.

Luckily there is a different possibility: solar neutrinos. These break the ambiguity, thanks to matter effects.

Matter effects in neutrino oscillations

Wednesday 2021-12-15

This is also called the MSW effect. Neutrinos interact weakly with matter, meaning that they are rarely absorbed; however, neutrinos may feel the presence of matter even without changing direction.

Through some interaction, a propagating neutrino may feel the presence of background fermions (matter). This idea is called "coherent forward scattering".

The full Hamiltonian will include a neutrino "interaction energy potential"

$$H = \frac{1}{2E} U M^2 U^\dagger + \begin{bmatrix} V_{ee} & V_{e\mu} & V_{e\tau} \\ V_{\mu e} & V_{\mu\mu} & V_{\mu\tau} \\ V_{\tau e} & V_{\tau\mu} & V_{\tau\tau} \end{bmatrix} + \text{terms proportional to } \mathbb{1}_3 \,. \tag{2.121}$$

Pictorially, this matrix V has a diagonal term driven by neutral current interactions, with any neutrino interacting through a Z boson. The coupling of the Z does not depend on the flavor, so that component is proportional to $\mathbb{1}$, meaning it does not affect oscillations.

A second term is the one corresponding to the W boson: for this, we only have an effect for electron neutrinos, in the V_{ee} component.

At tree level, this is proportional to the Fermi constant G_F. Also, it will linearly depend on the electron number density.

The calculation yields

$$V_{ee} = \sqrt{2} G_F N_e \qquad \text{for neutrinos} \tag{2.122}$$

$$V_{ee} = -\sqrt{2} G_F N_e \qquad \text{for antineutrinos} \,. \tag{2.123}$$

This effect is quite analogous to the propagation of light within a medium with a non-1 index of refraction. It alters the oscillation pattern both in amplitude and in frequency.

The main oscillation channel for atmospheric neutrinos is $\nu_\mu \to \nu_\tau$, meaning that these matter effect did not need to be included to study it.

It is convenient to introduce a term $A = 2EV_{ee} = 2\sqrt{2} G_F N_e E$; then the Hamiltonian reads

$$H = \frac{1}{2E} \left(U M^2 U^\dagger + \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \,. \tag{2.124}$$

Qualitatively, we can understand that this term will be important when A is $\gtrsim \delta m^2$ or Δm^2.

It turns out that

$$\frac{A}{\Delta m^2_{ij}} \approx 1.526 \times 10^{-7} \left(\frac{N_e}{\text{mol/cm}^3} \right) \left(\frac{E}{\text{MeV}} \right) \left(\frac{\text{eV}^2}{\Delta m^2_{ij}} \right) \,. \tag{2.125}$$

For the Sun, in the core, $N_e \sim 10^2\,\text{mol/cm}^3$, so we get that as an order of magnitude $A \sim \delta m^2$.

Since the number density of neutrinos depends on position, we need to integrate the Schrödinger equation numerically.

Integrating these oscillatory functions, however, can be tricky! Errors accumulate.

For the crust of the Earth, say, the LHC to LNGS beam, approximating the density as constant can work quite well.

Let us consider a two-neutrino case:

$$H = \frac{1}{2E} U \begin{bmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{bmatrix} U^\dagger + \frac{1}{2E} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2E} \widetilde{U} \widetilde{M}^2 \widetilde{U}^\dagger \,, \tag{2.126}$$

so we find a corrected diagonal matrix $\widetilde{M}^2$ as well as corrected mixing angles! They read:

$$\sin 2\widetilde{\theta}_{12} = \frac{\sin 2\theta_{12}}{\sqrt{(\cos 2\theta_{12} - A/\delta m^2)^2 + \sin^2 2\theta_{12}}} \tag{2.127}$$

$$\widetilde{\delta m}^2 = \delta m^2 \frac{\sin 2\theta_{12}}{\sin 2\widetilde{\theta}_{12}} \,, \tag{2.128}$$

which looks like a Breit-Wigner resonance.

The corrected $\sin 2\widetilde{\theta}_{12}$ has a maximum when $\cos 2\theta_{12} = A/\delta m^2$, at which point we also have a minimum for $\widetilde{\delta m}^2$.

This resonance does not happen for antineutrinos, which have a different sign for A.

For the Sun, we make two approximations: the density is slowly changing, $\mathrm{d}N_e/\mathrm{d}x$ is nonzero but small, and there are many oscillations.

The transition is "adiabatic", so everything will only depend on the initial and final mixing angles, not on δm^2.

The result is

$$P_{ee}^{2\nu}(\text{solar}) \approx \cos^2 \widetilde{\theta}_i \cos^2 \theta + \sin^2 \widetilde{\theta}_i \sin^2 \theta \,. \tag{2.129}$$

This probability has been tested! It looks a bit like a decreasing "sigmoid", with a transition at few MeV.

This oscillation probability is no longer octant symmetric! It solves the degeneracy of the mixing angles in the KamLand experiment.

Uncertainties in the solar model do not really create problems in this regard; our models work well enough.

Thursday 2021-12-16

The last lecture for this course!

The **normal ordering** for neutrino masses is $m_1 < m_2 < m_3$, while **inverted ordering** is $m_3 < m_1 < m_2$.

We always denote the lightest neutrino mass as m_0 — it could be either m_1 or m_3. It could be zero! that is not excluded by current data.

We can plot m_i as a function of m_0, with the constraint of fixing δm^2 and Δm^2. We need to precisely define what the large Δm^2 is: the convention used by the professor is $\Delta m^2 = (\Delta m_{31}^2 + \Delta m_{32}^2)/2$.

Absolute neutrino mass observables are:

1. endpoint of β decay spectrum;
2. precision cosmology;
3. neutrinoless double beta decay (iff neutrinos are Majorana);
4. time-of-flight: neutrino masses cause a delay of the order

$$\Delta t \approx D\frac{1}{2}\left(\frac{m_i}{E}\right)^2 . \tag{2.130}$$

Unfortunately, the time-of-flight can be done with supernovae only constraining down to $\sim 100\,\text{eV}$ since the signal is extended in time.

What about β decay? Looking at the endpoint of the spectrum could indicate what the mass looks like. However, there are problems from both energy resolution and low statistics. The experiment KATRIN is trying to measure this!

Which mass would such an experiment be measuring? There is no "mass of the electron neutrino"!

One would theoretically expect three different kinks corresponding to the three different neutrino masses. The leading-order net effect of the superposition of these is

$$m_\beta^2 = \sum_i |U_{e1}|^2 m_i^2 . \tag{2.131}$$

We can rewrite this in terms of the mixing angles; this also changes according to normal and inverted ordering.

What about cosmology? The net effect measured there is $\Sigma = m_1 + m_2 + m_3$.

The way to measure this is to look at CMB anisotropies and the Large Scale Structure.

The contribution of neutrinos is peculiar in that they are "free-streaming".

The ΛCDM model is rather unsatifactory! Dark matter and dark energy are just parametrized ignorance.

Depending on how much one trusts the data (and ignores the inconsistencies such as the measurement of H_0) the bound comes out to be $\Sigma \lesssim 100\,\text{meV}$ or a few hundreds.

Cosmology is reaching into the region where the masses are quite degenerate.

The decay amplitude for $0\nu\beta\beta$ is written like

$$\Gamma \propto \left|\sum_i U_{ei}^2 e^{i\phi_i} m_i\right|^2 , \tag{2.132}$$

so the three phases can make the terms interfere! One can then define an effective neutrino mass

$$m_{\beta\beta} = \left| |U_{e1}|^2 m_i + |U_{e2}|^2 e^{i\alpha} m_2 + |U_{e3}|^2 e^{i\beta} m_3 \right| . \tag{2.133}$$

One can compute an allowed band for $m_{\beta\beta}$, both in normal and inverted ordering. The idea is to make these kinds of plots for $m_\beta(m_0)$, $m_{\beta\beta}(m_0)$ and $\Sigma(m_0)$ for NO and IO; however the alternative is to plot these against each other!

We can add exclusion bands to these.

The possibility of having a fourth, almost-sterile neutrino with $\Delta M^2 \sim 1\,\text{eV}$ can be probed with short baseline experiments, where $\Delta M^2 L/4E$ is close to 1.

There is some controversial evidence for this happening. Theory kind-of tells us that these effects will be too small to be observable.

If the oscillation probability for ΔM^2 is of significant magnitude, we can approximate the others as zero, so we get

$$P_{\alpha\alpha} = 1 - 4|U_{\alpha 4}|^2 \left(1 - |U_{\alpha 4}|^2\right) \sin^2\left(\frac{\Delta M^2 L}{4E}\right) \tag{2.134}$$

$$P_{\alpha\beta} = 4|U_{\alpha 4}|^2 \left|U_{\beta 4}\right|^2 \sin^2\left(\frac{\Delta M^2 L}{4E}\right) . \tag{2.135}$$

However, this must be small, since the $U_{\alpha 4} \sim \epsilon$ terms of the mixing matrix are small! If they were large we would observe large deviations from unitarity in the PMNS matrix.

The expectation is therefore that appearance effects are $\mathcal{O}\left(\epsilon^2\right)$ while disappearance effects are $\mathcal{O}(\epsilon)$, while current data seem to be measuring the same effect size for both.

What about a possible new flavor changing vertex ν_α + fermions $\to \nu_\beta$ + fermions, proportional to $\epsilon_{\alpha\beta} G_F$?

It would produce some kind of new matter effect, $V = \sqrt{2}\epsilon_{\alpha\beta} G_F N_f$ where N_f is the charged fermion density.

The bounds on the parameters $\epsilon_{\alpha\beta}$ are rather weak, of the order of 10^{-1} to 1. These are "non-standard interactions".

The last lecture, in the notes, is about two more advanced topics.

The first is how one can probe the mass ordering: in order to find out the sign of Δm^2, one must make it interfere with something which has a known sign: δm^2, or matter effects $Q = \sqrt{2} G_F N_e$, or the more exotic $Q = \sqrt{2} G_F N_\nu$ — this can be measurable only in supernovae.

Juno probes the first: it works with a medium baseline of $L \sim 50\,\text{km}$ and a very good energy resolution.

One can probe the full $P_{ee}(\delta m^2, \pm\Delta m^2, \theta_{12}, \theta_{13})$, and by the precise phase of the Δm^2 short oscillations.

Long baseline experiments, on the other hand, probe $P_{\mu e}$ considering matter effects. There is an additional dependence on δ and V. DUNE tries to do this.

The Romans used to say "nomen omen". The name "neutrino" was invented by Enrico Fermi, and it means "little neutral one". It is inspired by the Italian "neutro", which in turn comes from the Latin ne-uter, which includes "uter" which means "either".

Some other words which have the root "uoter" are in Greek, German (whether, weather).

The old Ionic Greek word Koteros means "which of the two".

The old Sanskrit word Kataras also means "which of the two".

Many Indo-European words began with Ka, and Kwa. Also "Quantus", "how much", originates from this.

This prefix is a sort of interrogative case, a "question mark".

The destiny of neutrinos is to raise questions!
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3 Dark Matter

Tuesday 2022-1-25

This part of the course is given by Piero Ullio ullio@sissa.it.

We know that $\Omega_{\rm DM} h^2 = 0.1200 \pm 0.0012$. This is about 26.7% of the critical density and 84.4% of the total matter density.

Bertone and Hooper wrote a good historical review of early DM work.

Zwicky was the first to write about the problem of DM: the virial mass of the Coma cluster did not match its luminous mass.

The mass of the Coma cluster came out to be around $M \sim 5 \times 10^{14} M_\odot$.

Current estimates give $M/L \sim 160 M_\odot / L_\odot$.

We can go further and measure the baryonic mass through X-rays! Assuming the gas is in hydrostatic equilibrium, it will emit through bremsstrahlung. We get the electron number density and the electron temperature, with this we then get the mass profile.

In the Abell 2029 cluster we get a baryonic mass fraction of the order of 14 %.

We can get better estimates through strong gravitational lensing.

We can do inference on the mass distribution which causes lensing.

How does one do inference on such a high-dimensional space?

Rotation curves in spiral galaxies are further evidence of DM. In order to explain them, we need DM with $\rho \propto 1/r^2$.

Now we are using more sophisticated galactic dynamics, with a set of tracers (observable galaxies, stars etc.) in an underlying potential with a distribution function $f(\vec{x}, \vec{v}, t)$.

We write a collisionless Boltzmann equation for the conservation of the phase-space probability of this problem.

This can be solved directly assuming stationarity and some symmetries: in terms of the total density ν, the average velocity $\overline{v}_i$ and the average stress tensor $\overline{\sigma}_{ij} = \overline{v_i v_j} - \overline{v}_i \overline{v}_j$: the equation is called the Jeans equation

$$\nu \frac{\partial \overline{v}_j}{\partial t} + \nu \overline{v}_i \frac{\partial \overline{v}_j}{\partial x_i} = -\nu \frac{\partial \phi}{\partial x_j} - \frac{\partial (\nu \sigma_{ij}^2)}{\partial x_j} \,. \tag{3.1}$$

The collisionless Boltzmann equation reads

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{\partial \phi}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{v}} = 0 \,, \tag{3.2}$$

where ϕ is the potential the galaxies are moving in.

The Jeans equation is the first moment of the Boltzmann equation. The Euler equation in fluid dynamics,

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\rho \vec{\nabla} \phi - \vec{\nabla} P \,, \tag{3.3}$$

is analogous to it.

A typical example is measuring the density of dark matter in pressure-supported systems such as dwarf galaxies.

For spherical symmetry we get the Spherical Jeans equation.

These dwarf spheroidal galaxies are nice DM laboratories: they have radii of just about 1 kpc.

There is a general circular velocity versus radius relation.

In the inner part: mapping how the gas is moving, with the 21 cm line. Near us, the way is to finely track the velocities of local object.

Just outside, we mostly measure in the vertical component of the oscillation around the galactic disk. The systematics start to grow at larger radii.

Far from the center, we look at halo stars; we can measure the velocity dispersion of those objects just like with dwarf spheroidal galaxies.

DM is included as an "ingredient" in the ΛCDM model. Together with the Ωs, we have perturbation and couplings, such as the baryon-photon fluid before decoupling, with Compton and Coulomb scattering.

The formation of LSS requires Dark Matter:

$$\frac{\delta\rho_b}{\rho_b} \propto 1 + z\,, \tag{3.4}$$

therefore it would not have had time to reach the collapse $\delta\rho \sim \rho$.

There are a few glitches in the concordance with ΛCDM; one is the H_0 tension, then we also have the σ_8 tension: power spectrum normalization from the CMB and weak lensing.

The EDGES detection of the 21 cm line seem to measure a larger absorption than was thought possible.

Wednesday 2022-1-26

The density profile of Dark Matter is uncertain: the NFW profile gives a $1/r$ scaling at low radii, the Einasto profile is smoother in that region.

This is the cusp-core problem, a mismatch between the results of simulations and observations.

For Dark Matter we introduce only gravity: there is, therefore, no characteristic scale, we expect self-similarity; a Harrison-Zel'dovich spectrum, once the system is virialized.

This is the root of the missing satellite problem, the number of substructures in CDM simulations is much larger than the satellites observed in the Milky Way.

The too-big-to-fail problem is a reframing of this issue: the satellites of the Milky Way are not massive enough, the tail of the distribution.

There seems to be too much power in ΛCDM on small scales. A possibility is also that baryonic components are not being properly treated.

Reformulating the DM problem in terms of elementary particles is an *assumption*; there are other possibilities, such as primordial black holes for examples.

The dilute limit is the one in which 2-body interactions dominate over multi-body ones.

How do we build a Lagrangian for DM? We would like to have its mass, its

We have **5 golden rules** which we know for sure cannot be (strongly) violated:

1. it is **optically dark**:

(a) it does not couple to photons before recombination;

(b) it does not contribute significantly to background radiation at any frequency;

(c) it cannot cool by radiating photons (like baryons do when they form galaxies);

2. it is **collisionless**;

3. it is in a **fluid limit**;

4. it is **classical**;

5. it is **not hot** (or, cold).

Suppose we build a Lagrangian where the DM particle χ couples to the photon with a milli-charge ϵe: $L_{\gamma\chi} = i\epsilon e \overline{\chi} \gamma^\mu \chi A_\mu$. This leads to a cross-section for interaction (Compton-like) with the photon, proportional to ϵ^4 / m_χ^2, and with baryons (Coulomb-like) proportional to ϵ^2 / μ_χ^2.

Since we know that in the CMB DM cannot behave like baryons, these cross-sections are bound to be small, so we find $\epsilon \lesssim 10^{-7}$.

Does one run a whole analysis of the CMB peaks with a different likelihood?

For other couplings we even have laboratory constraints.

The fact that DM is optically dark means that it is dissipation-less. It cannot dissipate into visible photons, but not into any other dark particle either!

We have simulations of tidal effects stripping apart satellite galaxies.

The collisionless nature of DM is shown very well in the Bullet cluster. A mapping of gravitational lensing and one from X-rays show a displacement between the DM and the baryons.

How do we measure the peculiar velocity in the tangential direction?

From it, we can infer a limit on the self-interaction cross-section for DM:

$$\frac{\sigma}{m} < 1.25\,\mathrm{cm}^2/\mathrm{g}\,. \tag{3.5}$$

Another limit can be computed from the fact that dwarf galaxies survive their orbit around their host.

We also have constraints from ellipticity: self-interactions tend to isotropize DM halos, so when we measure triaxial halos we can use that as a constraint.

In particle physics units, we get a rather loose bound:

$$\frac{\sigma}{m} \lesssim 2 \times 10^{-24}\,\mathrm{cm}^2 \frac{m_\chi}{\mathrm{GeV}}\,. \tag{3.6}$$

If DM were made of particles they would have accreted and re-emitted energy, from these effects we can find that $M_{\mathrm{DM}} \lesssim 43 M_\odot$. If DM were accreting we would see it in microlensing events.

This kind of DM would be called MACHO, Massive compact halo object.

We have upper limits for these going from $10^{-6} M_\odot$ up to a few solar masses.

There is still a window for primordial black holes, there is the possibility to detect them with future GW observatories.

We know that DM is confined on galactic scales, of 1 kpc, with densities like a $\mathrm{GeV}/\mathrm{cm}^3$, and velocities like 100 km/s.

If DM were bosonic, its De Broglie wavelength must be smaller than 1 kpc, so

$$m \gtrsim \frac{h}{1\,\mathrm{kpc}v} \approx 10^{-22}\,\mathrm{eV}\,. \tag{3.7}$$

Fuzzy DM is an ultralight scalar, approaching this limit.

For fermions, we must require that the phase space density does not exceed that of a completely degenerate Fermi gas, which should also not have a maximal momentum larger than the escape velocity of the gravitational well, $v_{\mathrm{esc}} = \sqrt{2G_N M/R}$. This yields

$$m > \sqrt[4]{\frac{9\pi}{2^{5/2}M^{1/2}R^{3/2}G_N^{5/2}}}\,. \tag{3.8}$$

The relation we get this from is

$$\frac{M}{4/3\pi R^3}\frac{1}{m} < \frac{g}{(2\pi)^3}\frac{4\pi}{3}p^3 < \frac{g}{(2\pi)^3}\frac{4\pi}{3}(mv_{\mathrm{esc}})^3\,. \tag{3.9}$$

A more sophisticated way to do this is the Gunn-Tremaine bound. In a collisionless system, the phase-space density is conserved: supposing that the DM was thermal initially with $f = g/h^3$ we get

$$m > 101\,\mathrm{eV}\left(\frac{100\,\mathrm{km/s}}{g\sigma}\right)^{1/4}\left(\frac{1\,\mathrm{kpc}}{r_e}\right)^{1/2}\,. \tag{3.10}$$

This already allows us to reject SM neutrinos as DM candidates.

Why is DM cold? If the kinetic term of the system were not negligible gravitational collapse would not work.

The free-streaming scale comes out to be

$$\lambda_{\mathrm{FS}} = 0.4\,\mathrm{Mpc}\frac{\mathrm{keV}}{m_X}\frac{T_X}{T}\left(1+\log\left(t_{\mathrm{eq}}/t_{\mathrm{NR}}\right)/2\right)\,. \tag{3.11}$$

Here t_{eq} is the time when the DM becomes non-relativistic.

Why is that the definition of the free-streaming scale?

The 5 golden rules imply that baryonic DM and hot DM are excluded, on the other hand non-baryonic CDM is favored.

We can remove power on small scales by introducing a new length scale:

1. a free-streaming scale, with warm DM;

2. a self-interaction scale;

3. a “quantum” scale with fuzzy DM or DM forming a Bose-Einstein condensate;

4. a (yet inaccessible) DM-baryon or DM-photon interaction scale.

People are trying to reproduce these with N-body simulations.
Particle physics people have mostly explored the possibilities of

1. DM as a **thermal relic**, thermal production and a freeze-in mechanism (WIMPs and sterile neutrinos, for example);

2. DM as a **condensate** (axions and ALPs);

3. DM generated **at a large temperature** (very massive candidates).

MOND is falsified by

1. we have evidence for DM from other sources (e.g. cosmology);

2. some galaxies do not have a flat rotation curve;

3. the bullet cluster.

4 The WIMP paradigm

Thursday 2022-1-27

We will be assuming homogeneity and isotropy, without considering perturbations at any point.

The fluids used in the standard model of cosmology can be written in terms of their microstates; in order to this however we need a very large number of particles.

Because of this, instead of a microphysical description we will use a thermodynamic one.

The thermodynamics here will, however, be quite different from what we use classically, in lab conditions: for the universe, there can be no concept of an infinite thermalizer, or of global equilibrium.

However, we can talk of *local* equilibrium: we introduce a notion of entropy S, and discuss when it reaches a stable maximum. The physical interpretation of this is to say that the local patch of microstates “talk” to each other more effectively than the patches’ “change rate”.

The collision rate is denoted as $\Gamma = \sigma v n$, and it defines a collision timescale $t_c = 1/\Gamma$. Also, we have an expansion timescale $t_U = H^{-1}$.

We will have local equilibrium when $t_c \ll t_U$, or $\Gamma \gg H$.

Suppose we have a particle χ with a mass m_χ, a number of internal degrees of freedom g_χ, and a set of statistics, either Fermi-Dirac or Bose-Einstein.

The phase space distribution function at equilibrium will be given by

$$\frac{\mathrm{d}^3 n_\chi^{\mathrm{eq}}}{\mathrm{d}k^3} = f_\chi^{\mathrm{eq}}(x^\mu, k^\mu) = f_\chi^{\mathrm{eq}}(t, E)\,, \tag{4.1}$$

where $E^2 = m_\chi^2 + \left|\vec{k}\right|^2$. This will be given by

$$f_\chi^{\text{eq}}(t, E) = \frac{g_\chi}{(2\pi)^3} \left(\exp\left(\frac{E_\chi - \mu}{T} \right) \mp 1 \right)^{-1} , \tag{4.2}$$

where we have a $-$ for bosons, and a $+$ for fermions. Also, $T(t)$ is the temperature, and $\mu_\chi(t)$ is the chemical potential for χ.

Since there can be reactions in the form $e^- + \gamma \to e^- + 2\gamma$, the chemical potential of the photon must be zero.

If we were to observe μ_γ to be different from zero (for example, in the CMB) we should think that the assumption of local equilibrium is the one which was violated.

The number density as a function of time is given by

$$n_\chi^{\text{eq}}(t) = \int \mathrm{d}^3k \, f_\chi^{\text{eq}} . \tag{4.3}$$

The integral can be computed analytically in the $m_\chi \ll T$ and $m_\chi \gg T$ limits. If $m_\chi \ll T$, we have

$$n_\chi^{\text{eq}} = f \frac{\zeta(3)}{\pi^2} g_\chi T^3 , \tag{4.4}$$

where the factor f is equal to 1 for bosons, and $3/4$ for fermions.

If, on the other hand, $m_\chi \gtrsim T$, as well as larger than $\left|\mu_\chi - T\right|$, we get

$$n_\chi^{texteq} = g_\chi \left(\frac{m_\chi T}{2\pi} \right)^{3/2} \exp\left(-\frac{m_\chi - \mu_\chi}{T} \right) . \tag{4.5}$$

For the energy density we integrate $\int \mathrm{d}^3k \, E f$, so we get

$$\rho_\chi^{\text{eq}}(t) = \begin{cases} f_2 \frac{\pi^2}{30} g_\chi T^4 & \text{if } m_\chi \ll T \\ n_\chi^{\text{eq}}(m_\chi + \frac{3}{2}T) & \text{if } m_\chi \gg T . \end{cases} \tag{4.6}$$

The coefficient f_2 is 1 for bosons and $7/8$ for fermions.

The mean value of the energy per particle, $\langle E \rangle$, in the relativistic case is about 2.70 (bosons) or 3.15 (fermions) times T.

Within the Standard Model, we expect $\mu_\gamma = 0$ and that is indeed what we see.

A little exercise: consider any SM state, charged under any group with a structure constant α, and look at its relativistic limit.
From dimensional analysis, we can show that local equilibrium happens when

$$T \lesssim \frac{\alpha^2 M_{\text{Pl}}}{g_*^{1/2}} . \tag{4.7}$$

The idea is to use $t_U = H^{-1}$, where $H = \sqrt{8\pi G\rho/3M_{\text{Pl}}^2}$, where $\rho \sim g_* T^4$, with

$$g_* = \sum_{i\in \text{BE}} g_i + \frac{7}{8}\sum_{i\in\text{FD}} g_i\,. \tag{4.8}$$

The sum is taken over all relativistic degrees of freedom.
For the SM at the highest temperatures we find $g_* \approx 106.75$.

In practice, we do not use the full phase space distribution function but only describe its moments, which account for its normalization and shape.

We will work in the *dilute limit*: only considering 2-body scatterings. Really, we should be thinking of ensembles of particles.

We think of a situation in which we have a stable DM particle χ, and some SM particle p in a "thermal bath" state. Stability means we do not have processes in the form $\chi \to 2p$.

Instead, we will have processes in the form $\chi + \overline{\chi} \leftrightarrow p + \overline{p}$ as well as $\chi + p \leftrightarrow \chi + p$.

The first are number-changing for χ, the second are not. The first, therefore, affect *chemical equilibrium*, while the second only affect the energy and the *kinetic equilibrium*.

Equilibrium for the first process is corresponds to $\Gamma_{\text{ann}} > H$, equilibrium for the second corresponds to $\Gamma_{\text{scatt}} > H$.

The cross-sections for scattering and annihilation could be different, but crossing symmetry makes them similar; on the other hand in the scattering rate the number density will account for all the particles in the heat bath, therefore we typically have $\Gamma_{\text{scatt}} \gg \Gamma_{\text{ann}}$.

This implies that chemical equilibrium is lost a long time before kinetic equilibrium is.

The Boltzmann equation

Its basic structure, for a particle χ with all its properties, reads

$$\hat{L}[f_\chi] = \hat{C}[f_\chi]\,. \tag{4.9}$$

The Liouville operator $\hat{L}$ describes the free, geodesic propagation of the particle: it will read

$$\hat{L} = k^\alpha \frac{\partial}{\partial x^\alpha} \underbrace{-\Gamma^\alpha_{\beta\gamma} k^\beta k^\gamma}_{= \mathrm{d}^2 x^\alpha / \mathrm{d}\tau^2} \frac{\partial}{\partial k^\alpha}\,. \tag{4.10}$$

Since we are only looking at the time dependence we only need the $\alpha = 0$ component, and the relevant Christoffel symbols read

$$\Gamma^0_{ij} = \frac{\dot{a}}{a}\gamma_{ij}\,, \tag{4.11}$$

where γ_{ij} is a flat 3D metric.

Then, the Liouville operator applied to f_χ reads

$$\hat{L}[f_\chi] = E\frac{\partial f_\chi}{\partial t} - H\left|\vec{k}\right|^2 \frac{\partial f_\chi}{\partial E}\,. \tag{4.12}$$

In order to track chemical equilibrium we just need the normalization of this equation:

$$n_\chi = \int \mathrm{d}^3k\, f_\chi\,, \tag{4.13}$$

and the kinetic integral

$$T_\chi = \frac{2}{3}\frac{1}{n_\chi}\int \mathrm{d}^3k\,\frac{\left|\vec{k}\right|^2}{2m_\chi}f_\chi\,. \tag{4.14}$$

We will always assume that this temperature is equal to the overall temperature of the thermal bath T.

Integrating the Boltzmann equation over d^3k we find

$$\frac{\mathrm{d}n_\chi}{\mathrm{d}t} - H\int \mathrm{d}^3k\,\frac{k^2}{E}\frac{\partial f_\chi}{\partial E}\,, \tag{4.15}$$

and we can integrate by parts to get, since $E\,\mathrm{d}E = k\,\mathrm{d}k$:

$$\begin{aligned}
4\pi\int \mathrm{d}k\,k^2\frac{k^2}{E}\frac{\partial f_\chi}{\partial E} &= 4\pi\int \mathrm{d}E\,k^3\frac{\partial f_\chi}{\partial E} && (4.16)\\
&= 4\pi k^3 f_\chi\Big|_{k=0}^{k\to\infty} - \int \mathrm{d}E\,f_\chi 3k^2\frac{\partial k}{\partial E} && (4.17)\\
&= -3Hn_\chi\,. && (4.18)
\end{aligned}$$

Therefore, the equation reads

$$\frac{\mathrm{d}n_\chi}{\mathrm{d}t} + 3Hn_\chi = \int\frac{\mathrm{d}^3k}{E}\hat{C}[f_\chi]\,. \tag{4.19}$$

If there are no collisions then $n_\chi a^3$, the number of particles in a comoving volume, is conserved.

The collisional term is written as

$$\int\frac{\mathrm{d}^3k}{E}\hat{C}[f_\chi] = -\int \mathrm{d}\Pi_\chi\,\mathrm{d}\Pi_{\overline{\chi}}\,\mathrm{d}\Pi_p\,\mathrm{d}\Pi_{\overline{p}}\,\Big(|M_{\text{forward}}|^2 h_\chi h_{\overline{\chi}}(1\mp h_p)(1\mp h_{\overline{p}}) - |M_{\text{reverse}}|^2 h_p h_{\overline{p}}(1\mp h_\chi)(1\mp h_{\overline{\chi}})\Big)\,(2\pi)^4\delta^{(4} \tag{4.20}$$

Here, $h_i = f_i/(g_i/(2\pi)^3)$, while

$$\mathrm{d}\Pi_i = \frac{g_i}{(2\pi)^3}\frac{\mathrm{d}^3k_i}{2E_i}\,. \tag{4.21}$$

The minus signs refer to fermions, the plus signs to bosons.

Why to include two different matrix elements? Well, if we have CP symmetry we will get

$$|M_{\text{forward}}|^2 = |M_{\text{backward}}|^2\,. \tag{4.22}$$

It is not a given that this will be the case.

In the dilute limit, besides considering only two body interactions we also assume $h_i \ll 1$, therefore $1 \pm h_i \approx 1$.

We can approximate these h_i as

$$h_i^{\text{eq}} \approx \exp\left(-\frac{E_i - \mu_i}{T}\right), \tag{4.23}$$

so with all this we get

$$\int \frac{\mathrm{d}^3 k}{E} \hat{C}[f_\chi] - \int \{\mathrm{d}\Pi_i\}_i |M|^2 \left(h_\chi h_{\overline{\chi}} - h_p^{\text{eq}} h_{\overline{p}}^{\text{eq}}\right) (2\pi)^4 \delta^{(4)}(\dots). \tag{4.24}$$

Further, we will have kinetic equilibrium, thereby

$$h_\chi(E,t) = \frac{n_\chi(t)}{n_\chi^{\text{eq}}(t)} h_\chi^{\text{eq}}(t,E) \ : \tag{4.25}$$

the shape of the distribution is the same at all times.

The product of these two distribution functions will read

$$h_p^{\text{eq}} h_{\overline{p}}^{\text{eq}} = \exp\left(-\frac{E_p + E_{\overline{p}}}{T}\right) = \exp\left(-\frac{E_\chi + E_{\overline{\chi}}}{T}\right) = h_\chi^{\text{eq}} h_{\overline{\chi}}^{\text{eq}}. \tag{4.26}$$

Thanks to this, we get the full Boltzmann equation:

$$\frac{\mathrm{d}n_\chi}{\mathrm{d}t} + 3Hn_\chi = -\langle \sigma v \rangle_T \left(n_\chi n_{\overline{\chi}} - n_\chi^{\text{eq}} n_{\overline{\chi}}^{\text{eq}}\right), \tag{4.27}$$

where

$$\langle \sigma v \rangle_T = \frac{1}{n_\chi^{\text{eq}} n_{\overline{\chi}}^{\text{eq}}} \int \{\mathrm{d}\Pi_i\}_i |M|^2 \exp\left(-\frac{E_\chi + E_{\overline{\chi}}}{T}\right) (2\pi)^4 \delta^{(4)}(\dots). \tag{4.28}$$

We reached an expression for the evolution of the number density of particle χ. Friday 2022-1-28

We now want to heuristically follow it up until the moment of decoupling.

When $\mu_\chi = 0$, we get

$$n_\chi^{\text{eq}} - n_{\overline{\chi}}^{\text{eq}} = 2g_\chi \left(\frac{m_\chi T}{2\pi}\right)^{3/2} \exp\left(-\frac{m_\chi}{T}\right) \sinh\left(\frac{\mu_\chi}{T}\right). \tag{4.29}$$

If $\mu_\chi = \mu_{\overline{\chi}}$, we get $n_\chi = n_{\overline{\chi}}$.

The situation in which $\mu_\chi \neq 0$ is called *asymmetric DM*.

With this requirement, our equation simplifies to

$$\frac{\mathrm{d}n_\chi}{\mathrm{d}t} + 3Hn_\chi = -\langle \sigma v \rangle_T \left(n_\chi^2 - \left(n_\chi^{\text{eq}}\right)^2\right). \tag{4.30}$$

We want to remove the expansion rate of the universe from the equation: therefore, we try to normalize n_χ to something which is conserved in a comoving volume.

It's convenient to normalize to the entropy density:

$$s = \frac{\rho + P}{T} \approx \frac{\rho_{\text{rel}} + P_{\text{rel}}}{T} = \frac{4}{3}\frac{\rho_{\text{rel}}}{T} = \frac{4}{3}\frac{\pi^2}{45} g_* T^3 \,. \tag{4.31}$$

We can compute this at any time: right now we have $s_0 \approx 2900\,\text{cm}^{-3}$.
The quantity is called $Y_\chi = n_\chi / s$, whose derivative is

$$\frac{\mathrm{d}Y_\chi}{\mathrm{d}t} = \frac{1}{s}\frac{\mathrm{d}n_\chi}{\mathrm{d}t} - \frac{1}{s^2} n_\chi \frac{\mathrm{d}s}{\mathrm{d}t} \,. \tag{4.32}$$

Since $a^3 s$ is constant, we get

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -3Hss \,. \tag{4.33}$$

This means that

$$\frac{\mathrm{d}Y_\chi}{\mathrm{d}t} = \frac{1}{s}\left(\frac{\mathrm{d}n_\chi}{\mathrm{d}t} + 3Hn_\chi\right) \,. \tag{4.34}$$

The variation of s with temperature reads

$$\frac{\mathrm{d}s}{\mathrm{d}T} = \frac{3s}{T}\left(1 + \frac{1}{3}\frac{T}{g_*}\frac{\mathrm{d}g_*}{\mathrm{d}T}\right) = \frac{3s}{TZ} \,, \tag{4.35}$$

where $Z \approx 1$ takes account of the correction by the variation of g_*.
Therefore, the variation of Y_χ with temperature reads

$$\frac{\mathrm{d}Y_\chi}{\mathrm{d}T} = \frac{\mathrm{d}Y_\chi}{\mathrm{d}t}\frac{\mathrm{d}s}{\mathrm{d}T}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-1} = \frac{\langle \sigma v \rangle_T}{sHT}\left(n_\chi^2 - \left(n_\chi^{\text{eq}}\right)^2\right) \,. \tag{4.36}$$

In the end, therefore, the Boltzmann equation reads

$$\frac{\mathrm{d}Y_\chi}{\mathrm{d}T} = \frac{\langle \sigma v \rangle_T \, s}{HT}\left(Y_\chi^2 - \left(Y_\chi^{\text{eq}}\right)^2\right) \tag{4.37}$$

$$\frac{T}{Y_\chi^{\text{eq}}}\frac{\mathrm{d}Y_\chi}{\mathrm{d}T} = \frac{\langle \sigma v \rangle_T \, n_\chi^{\text{eq}}}{HT}\left(\frac{Y_\chi^2}{(Y_\chi^{\text{eq}})^2} - 1\right) \tag{4.38}$$

$$= \frac{\Gamma}{H}\left(\frac{Y_\chi^2}{(Y_\chi^{\text{eq}})^2} - 1\right) \,. \tag{4.39}$$

This clarifies what is meant by Γ when comparing Γ and H: $\Gamma(T) = \langle \sigma v \rangle_T \, n_\chi^{\text{eq}}$.
On the other hand,

$$H = \sqrt{\frac{8\pi}{3m_{\text{Pl}}^2} g_* \frac{\pi^2}{30} T^4} \,. \tag{4.40}$$

While the species is relativistic, $\Gamma \propto T^3$ while $H \propto T^2 / M_{\text{Pl}}$. Therefore, there will be a configuration at high T such that $\Gamma(T) \gg H(T)$.

The opposite is true at low temperatures. So, at high temperatures $Y_\chi \approx Y_\chi^{\mathrm{eq}}$. On the other hand, there will be a freeze-out temperature T_{freeze} such that $\Gamma \approx H$, and at smaller T we will remain at $Y_\chi(T) \approx Y_\chi(T_{\mathrm{freeze}})$.

This will also be the case up until the current time.

We can use this to compute the current relic density Ω_χ for a cold relic, such that $T_{\mathrm{freeze}} \lesssim m_\chi$.

Cold relic is not a synonym for CDM: that means that χ is non-relativistic at the time of matter-radiation equality, which is much smaller than the freezeout temperature typically.

Cold relic implies CDM, but not the other way around.

We get:

$$\Omega_\chi h^2 = \frac{\rho_\chi(T_0)}{\rho_{\mathrm{crit}}(T_0) h^{-2}} \tag{4.41}$$

$$= \frac{m_\chi n_\chi(T_0)}{\rho^0_{\mathrm{crit}} h^{-2}} \tag{4.42}$$

$$= \frac{m_\chi Y_\chi(T_0) s_0}{\rho_{\mathrm{crit}} h^{-2}} \,. \tag{4.43}$$

However, we also know that $\Gamma(T_{\mathrm{freeze}}) = \langle \sigma v \rangle_T \, Y_\chi^{\mathrm{eq}}(T_{\mathrm{freeze}}) s(T_{\mathrm{freeze}})$. This all yields

$$\Omega_\chi h^2 = \frac{m_\chi s_0}{\rho^0_{\mathrm{crit}} h^{-2}} \tag{4.44}$$

$$\approx \frac{m_\chi s_0}{\rho^0_{\mathrm{crit}}} \frac{H(T_{\mathrm{freeze}})}{s(65536 T_{\mathrm{freeze}}) \langle \sigma v \rangle_T} \,. \tag{4.45}$$

This comes out to be

$$\Omega_\chi h^2 = \frac{1}{g_*^{1/2}(T_{\mathrm{freeze}})} \frac{m_\chi}{T_{\mathrm{freeze}}} \frac{10^{-27}\,\mathrm{cm^3 s^{-1}}}{\langle \sigma v \rangle_{T_{\mathrm{freeze}}}} \,. \tag{4.46}$$

What is the value of the thermally-averaged cross-section $\langle \sigma v \rangle$?

We get

$$\exp\left(\frac{m_\chi}{T_{\mathrm{freeze}}} \right) \left(\frac{m_\chi}{T_{\mathrm{freeze}}} \right)^{-1/2} = K = \frac{3\sqrt{5}}{4\sqrt{2}\pi^3} \frac{g_\chi}{g_*^{1/2}} \sigma_0 m_\chi m_{\mathrm{Pl}} \,. \tag{4.47}$$

This can be solved iteratively. For a mass $m_\chi \approx 100\,\mathrm{GeV}$, and a cross-section $\sigma_0 \approx 2.5 \times 10^{-27}\,\mathrm{cm^3 s^{-1}}$, with $g_* \approx 60$, we find $m_\chi / T_{\mathrm{freeze}} \approx 20$.

This yields

$$\Omega_\chi h^2 \approx \frac{3 \times 10^{-27}\,\mathrm{cm^3 s^{-1}}}{\langle \sigma v \rangle_{T_{\mathrm{freeze}}}} \,. \tag{4.48}$$

That's the WIMP miracle: we get $\Omega_{\mathrm{DM}} h^2 \approx 0.1$ with a value for σv compatible with the weak interaction.

The recipe for WIMPs is

1. we need a massive particle, which is non-relativistic at freeze-out;
2. we need it to be stable (within the current lifetime of the universe);
3. we need it to have a weak-interaction-like coupling to the SM.

This third point also implicitly constrains the mass scale: a lower limit comes from the fact that the reaction $\chi\overline{\chi} \leftrightarrow F\overline{F}$ must be at equilibrium for some SM particle. This typically does not work within the SM unless $m_\chi \gtrsim 1\,\text{GeV}$; if there is a beyond-the-standard model thermalizer, we could go down to $m_\chi \gtrsim 1\,\text{MeV}$.

There is also an upper limit, which comes from unitarity when making a partial wave decomposition of the cross-section $\sigma = \sum_J \sigma_J$. These are bounded by

$$\sigma_J \lesssim \frac{\pi(2J+1)}{p_i^2} \qquad \text{where} \qquad p_i \approx m_\chi \frac{v_{\text{rel}}}{2}\,, \tag{4.49}$$

therefore

$$(\sigma_J v_{\text{rel}})_{\max} \approx \frac{4\pi(2J+1)}{m_\chi^2 v_{\text{rel}}}\,. \tag{4.50}$$

The relative velocity is fixed by equipartition;

$$\frac{1}{3} m_\chi v_{\text{rel}}^2 = 3T\,, \tag{4.51}$$

which comes out to be $v_{\text{rel}} \approx \sqrt{6T_{\text{freeze}}/m_\chi} \approx \sqrt{6/20}$.

The various σ_J with $J > 0$ are smaller by a factor v_{rel}^2 each; this is an upper bound on $\langle \sigma v \rangle$, combining which with what we know about DM abundance leads to a lower limit on the mass of χ in the form

$$m_\chi \lesssim 90\,\text{TeV}\,. \tag{4.52}$$

The Higgs portal for a scalar singet

Consider the SM plus a single real scalar field S, which is a SM singlet but which couples to the SM Higgs. This is based on early work by Silvera and Zee in 1985.

This is the most minimal model outside the SM.

The Lagrangian includes some interactions, which we choose so that S is stable:

$$\mathscr{L} = \mathscr{L}_{\text{SM}} + \frac{1}{2}\partial_\mu S \partial^\mu S - \frac{m_S^2}{2} S^2 - \frac{\lambda_s}{4} S^4 - \lambda S^2 (H^\dagger H)\,, \tag{4.53}$$

where we introduce a $\mathbb{Z}_2$ symmetry: we write a Lagrangian which is symmetric under $S \to -S$.

A trick to make a renormalizable Lagrangian is to not have any term with a dimension which is larger than 4 in the field.

What is the potential for S? In the unitarity gauge we can write $H^\dagger = (h, 0)/\sqrt{2}$, so we get

$$V = \frac{m_0^2}{2} S^2 + \lambda_s S^4 + \frac{\lambda}{2} S^2 h^2 + \lambda_h (h^2 - v_{EW}^2)^2 \,. \tag{4.54}$$

We need to impose that the minimum of V is the same as the minimum of V_{SM}: the configuration $h = v_{EW}$. This is used to put constraints on the various couplings.

After EW symmetry breaking, and mapping $h \to h + v_{EW}$ we get

$$V = V_{\text{SM}} + \frac{1}{2}(m_0^2 + \lambda v_{EW}^2) S^2 + \frac{\lambda_S}{4} S^4 + \frac{\lambda}{2} h^2 S^2 + \lambda v_{EW} h S^2 \,. \tag{4.55}$$

Exam: oral or presentation on a given topic. Monday 2022-1-31

The Lagrangian therefore reads

$$\mathscr{L} = \mathscr{L}_{\text{SM}} + \frac{1}{2} \partial_\mu S \partial^\mu S - \frac{1}{2} m_S^2 S^2 - \lambda v_{\text{EW}} h S^2 - \frac{\lambda}{2} h^2 S^2 \,. \tag{4.56}$$

We have a coupling λ between a Higgs h and two Ss, then this h can couple to two fermions like $hF\overline{F}$ with a coupling constant y_F.

What happens with thermal equilibrium? We need to consider the cases $T \gg m_h$ and $T \ll m_h$.

If $T \gg m_h$, the Higgs mass is irrelevant, so when we estimate $\Gamma = n_T \sigma_{\text{ann}} v$ we get something proportional to $\Gamma \propto T^3 \times \lambda^2 / T^2 \times 1 = \lambda^2 T$.

On the other hand, if $T \ll m_H$ we get $\Gamma \sim T^3 \lambda^2 T^2 / m_h^4 \times 1 \sim \lambda^2 T^5 / m_h^4$.

The velocity is still relativistic, therefore we are assuming that $m_S \ll T \ll m_h$.

The expansion of the Universe scales like

$$H \sim g_*^{1/2} \frac{T^2}{m_{\text{Pl}}} \,. \tag{4.57}$$

We get decoupling when $\max(\Gamma/H) \gtrsim 1$: this means

$$\left. \frac{\Gamma}{H} \right|_{T \sim m_P} \sim \lambda^2 \frac{M_p}{m_h} g_*^{-1/2} \,. \tag{4.58}$$

This requires $\lambda \gtrsim g_*^{1/4} \sqrt{m_h / M_p} \sim 10^{-8}$.

Therefore, we can have thermalization as long as $\lambda \gtrsim 10^{-8}$ — the coupling can be rather weak while still yielding thermalization.

We now want to compute the relic density: we get

$$\Omega_S h^2 \sim \frac{3 \times 10^{-27}\,\text{cm}^3\text{s}^{-1}}{\langle \sigma_s v \rangle_T} \,. \tag{4.59}$$

Let us make an estimate for this thermally averaged cross-section:

$$\langle \sigma_S v \rangle = \frac{g \lambda^2 v_{\text{EW}}^2}{(4 m_S^2 - m_h^2)^2 + m_h^2 \Gamma_h^2} F_h(m_s) \,, \tag{4.60}$$

where we split the initial Feynman diagram in two.

The width for the Higgs transition is roughly $\Gamma_h \sim 4.2\,\mathrm{MeV}$.

The factor $F_h(m_s)$ is given by

$$F_h(m_s) = \sum_X \lim_{m_{\widetilde{h}} \to 2m_s} \frac{\Gamma(\widetilde{h} \to X)}{m_{\widetilde{h}}}\,, \tag{4.61}$$

where X is any allowed final state.

Since $m_S \ll m_h$, we will have

$$\sigma_S v \sim \frac{\lambda^2 m_s^2}{m_h^2}\,. \tag{4.62}$$

Since this is constrained by observation of Ω_S, we must have λm_s equal to some fixed value.

When, however, we reach $m_s \approx m_h/2$ we get a huge enhancement due to resonance.

In a $\log \lambda$ against $\log m_s$ plot this corresponds to a large *dip*, since we need a much smaller value of λ in order to balance such a natively high cross-section.

The width of this dip will not just be $\Gamma_h \sim 4.2\,\mathrm{MeV}$ because of Doppler broadening.

After the dip, more final states will open up. So, if $m_s \gg m_h$ then the term $\lambda h^2 S^2$ in the Lagrangian also starts to become relevant.

Then, we get $\sigma_S v \approx \lambda^2/m_S^2$.

This leads to a high-energy regime where, roughly, λ/m_s is constant. Before that, we get a region of roughly constant λ.

This curve corresponds to a fixed value of $\Omega_s h^2$; above it we get under-production and below it we get over-production.

Can we get other constraints? If $m_S < m_h/2$, we could have $h \to SS$ decay: this would contribute to the "invisible width" in Higgs decay.

The rest of the invisible width is accounted for with neutrinos.

This excludes a portion of the parameter space in the low m_s region.

Most of the rest of the line is excluded by other reasons, what remains as a viable candidate is only the resonance dip.

Supersymmetric WIMPs

Just a sketch.

A supersymmetric version of the Standard Model can be given, for example, by the Minimal SUSY SM, in which to any SM degree of freedom we associate one degree of freedom in such a way that the setup is symmetric under the exchange of fermionic and bosonic degrees of freedom.

R-parity is in the form $R = (-1)^{3(B-L)+2S}$. SM particles have $R = +1$, and the SUSY partners have $R = -1$.

SUSY particles appear in pairs in scatterings and annihilations, the result of this is that the lightest SUSY particle is stable.

Is this LSP a WIMP DM candidate? We want it to have no electric charge, no $SU(3)_c$ charge, and no $SU(2)_L$ charge.

One possibility is the neutrino's couterpart: this has been excluded by direct detection.

Another possibility is to look at $SU(2)_L \times U(1)_Y$ gauge, Higgs with 0 electric charge. The gluino, on the other hand, is excluded because it would be too strongly interacting.

We could have the SUSY $U(1)_Y$ partner $\widetilde{B}$, the neutral partner of the $SU(2)$ gauge bosons $\widetilde{W}^3$, the SUSY partner of the Higgs doublet $\widetilde{H}_u$ or $\widetilde{H}_d$.

These are 4 spin 1/2 particles which are Majorana fermions and which are collectively called *neutralinos*.

There must be some soft SUSY breaking, since we didn't observe any of them yet. So, there must be a mass matrix with off-diagonal terms.

Depending on the hierarchy between these masses, the LSP could be the Higgsino (an $SU(2)_L$ doublet), the Wino (an $SU(2)_L$ triplet), or the Bino (an $SU(2)_L$ singlet).

The states $\widetilde{\chi}^0$ and $\widetilde{\chi}^\pm$ are nearly degenerate in mass, but it's hard to measure this at LHC.

These particles χ_i are all sharing a quantum number, so they all "talk" to each other. Suppose that the masses are ordered such that $M_1 < M_2 < \dots$; then if the temperature of freeze-out is given by

$$T_{\text{FO}} \sim \frac{M_1}{20} \tag{4.63}$$

the particles with $M_i - M_1 \lesssim T_{\text{FO}}$ will remain non-relativistic until they decouple.

There are many processes causing a change in χ_i abundance.

The full Boltzmann equation will read

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} + 3Hn_i = -\sum_j \left\langle \sigma_{ij} v \right\rangle \left(n_i n_j - n_i^{\text{eq}} n_j^{\text{eq}} \right) - \sum_{j,P,P'} \left\langle \sigma'_{ij} v \right\rangle \left(n_i n_P^{\text{eq}} - n_j n_{P'}^{\text{eq}} \right) - \sum_{j<i} \Gamma_{i\to j} n_i + \sum_{j>i} \Gamma_{j \to i} n_j \,. \tag{4.64}$$

This will be the case for any i between 1 and N. At late times, however, all the $i > 1$ will have decayed into 1: therefore, the relic density will be given by $n_{\text{tot}} = \sum_i n_i$, which constrains Ω_1.

The result is then

$$\frac{\mathrm{d}n}{\mathrm{d}t} + 3Hn = -\sum_{i,j} \left\langle \sigma_{ij} v \right\rangle \left(n_i n_j - n_i^{\text{eq}} n_j^{\text{eq}} \right) \,. \tag{4.65}$$

We still have the individual densities on the right, but we can give an estimate for $\Gamma \sim n_j \left\langle \sigma_{ij} v \right\rangle$, while $\Gamma_{\text{scatt}} \sim n_P^{\text{eq}} \left\langle \sigma'_{ij} v \right\rangle$.

Since $n_i^{\text{eq}} \gg n_j$, we expect n_i/n to scale like $n_i^{\text{eq}}/n^{\text{eq}}$.

This then yields an equation in the form

$$\frac{\mathrm{d}n}{\mathrm{d}t} + 3Hn = - \left\langle \sigma_{\text{eff}} v \right\rangle \left(n^2 - n_{\text{eq}}^2 \right) \tag{4.66}$$

$$\langle \sigma_{\text{eff}} v \rangle = \sum_{ij} \left\langle \sigma_{ij} v \right\rangle \frac{n_i^{\text{eq}} n_j^{\text{eq}}}{(n^{\text{eq}})^2} . \tag{4.67}$$

We were discussing the Boltzmann equation for a set of N particles χ_i, which satisfy a collective Boltzmann equation for $n = \sum_i n_i$. Tuesday 2022-2-1

The ratio of the equilibrium density of each species to the overall equilibrium density appears: what will be its value?

$$\frac{n_i^{\text{eq}}}{n^{\text{eq}}} = \frac{g_i (M_i/M_1)^{3/2} \exp(-M_i/T) \exp(M_1/T)}{\sum_j g_j (M_i/M_j)^{3/2} \exp\left(-M_j/T\right) \exp(M_1/T)} \tag{4.68}$$

$$= \frac{g_i (1+\Delta_i)^{3/2} \exp(-M_1 \Delta_i/T)}{\sum_j g_j (1+\Delta_i)^{3/2} \exp\left(-M_1 \Delta_j/T\right)} , \tag{4.69}$$

where we added the exponential of M_1/T since we want to get factors of $\Delta_i = (M_i - M_1)/M_1$.

We can call the denominator g_{eff}: then,

$$\langle \sigma_{\text{eff}} v \rangle = \sum_{i,j} \left\langle \sigma_{ij} v \right\rangle \frac{n_i^{\text{eq}} n_j^{\text{eq}}}{(n^{\text{eq}})^2} \tag{4.70}$$

$$= \sum_{ij} \left\langle \sigma_{ij} v \right\rangle \frac{g_i g_j}{g_{\text{eff}}^2} (1+\Delta_i)^{3/2} (1+\Delta_j)^{3/2} \exp\left(-\frac{M_1}{T} (\Delta_i + \Delta_j) \right) . \tag{4.71}$$

Suppose we only have two particles. Then, we can denote

$$g_{\text{eff}} = g_1 + g_2 (1+\Delta)^{3/2} \exp\left(-\frac{M_1}{T} \Delta \right) = g_1 (1+w) . \tag{4.72}$$

The annihilation cross-sections will read

$$\sigma_{22} = \alpha \sigma_{21} = \alpha^2 \sigma_{11} . \tag{4.73}$$

Then, we get

$$\sigma_{\text{eff}} = \sigma_{11} \frac{g_1^2}{g_1^2 (1+w)^2} + 2\sigma_{12} \frac{w}{(1+w)^2} + \sigma_{22} \frac{w^2}{(1+w)^2} \approx \sigma_{11} \frac{(1+\alpha w)^2}{(1+w)^2} . \tag{4.74}$$

In the limit where $\Delta \to 0$, we get $w \to g_2/g_1$ and so

$$\sigma_{\text{eff}} \to \sigma_{11} \frac{(1+\alpha (g_2/g_1)^2}{(1+g_2/g_1)} , \tag{4.75}$$

therefore, assuming $\Delta \sim 1/20$, if $\alpha \gtrsim 1$ we reduce Ω, while if $\alpha \lesssim 1$ we enhance it!

Consider the case of a pure Higgsino: $\widetilde{\chi}^0_{1,2}$ and $\widetilde{\chi}^+$.

Some considerations are then made regarding the possibility of the lightest DM particle being a Bino or a Higgsino.

Indirect detection is disfavored in this case, since today we only have $\sigma_0 v$ for the lightest state.

Could we have the opposite scenario, $\sigma_0 v \gg \langle \sigma_{\text{eff}} v \rangle$?

This is the case for Sommerfeld enhancement.

Now, the cross-section is computed in the context of empty space! If we compare the "size" of the particle, $\sim \sqrt{\sigma}$, to the separation of the particles, $1/\sqrt[3]{n}$, we still get that they are quite far away from each other.

In the regime where long-range forces become active, the range of interaction will not be $\sqrt{\sigma}$ anymore!

The Coulomb potential, for example, can be written as $V = -\alpha_{\text{em}}/r$. It has an infinite range!

Really, we should solve the Schrödinger equation in the non-relativistic limit,

$$\left(\frac{1}{2m_\chi} \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{|k|^2}{2m_\chi} - V_\phi(r) \right) \psi_k(r) = 0 \,. \tag{4.76}$$

There is then an enhancement for the annihilation rate in the form

$$S = \frac{\sigma_{\text{ann}}}{\sigma_0} = \frac{\left|\psi_k(0)\right|^2}{\left|\psi_k^{(0)}(0)\right|} \,, \tag{4.77}$$

where $V_\phi = -\alpha/r$ and the denominator in S is in the case where we have no field.

In the Coulomb case, this can be computed exactly:

$$S = \frac{\pi \alpha_\phi}{v} \frac{1}{1 - \exp\left(-\pi\alpha_\phi / v\right)} \,, \tag{4.78}$$

where v is the relative velocity.

In the case where $m_\phi \neq 0$ we get a Yukawa interaction, suppressed by $\exp\left(-m_\phi r\right)$; there, we can also compute S analytically.

$$S = \frac{\pi\alpha_\phi}{v} \frac{\sinh\left(12v/\pi\alpha_\phi\xi\right)}{\cosh\left(12v/\pi\alpha_\phi\xi\right) - \cos\left(2\pi\sqrt{6/\pi^2\xi - (6v/\pi^2\alpha_\phi\xi)^2}\right)} \,, \tag{4.79}$$

If v is large, or if $m_\phi \gtrsim m_\chi$, we can expand in $1/\xi$.

If, on the other hand, $\xi \ll 1$, then $S = \pi\alpha_\phi/v$.

There are resonances when $\xi = 6/\pi^2 n^2$: then, $S = \alpha_\phi^2/v^2 n^2$, for $n \in \mathbb{Z}$.

We can plot different curves for $\log S$ as a function of ξ, with different values of v.

Feebly Interacting Massive Particles

This is DM which is not in thermal equilibrium in the Early Universe.

For example, these could be sterile neutrinos. These would be SM singlets, but which mix with SM neutrinos.

Let us look at the 1+1 configuration: we have $|\nu_\alpha\rangle$ with $\alpha \in \{e, \mu, \tau\}$, plus a $|\nu_s\rangle$. Further suppose that

$$|\nu_\alpha\rangle = \cos\theta_\alpha |\nu_1\rangle + \sin\theta_\alpha |\nu_2\rangle \tag{4.80}$$

$$|\nu_s\rangle = -\sin\theta_\alpha |\nu_1\rangle + \cos\theta_\alpha |\nu_2\rangle \ , \tag{4.81}$$

and we assume that $m_2 \gg m_1$. Since we want these neutrinos on the order of the keV at least, but we know that the masses of SM neutrinos are very small, therefore we need $\theta_\alpha \ll 1$.

How do neutrino oscillations work in the early universe? we get a transition amplitude in the form

$$A(\nu\alpha \to \nu_s) = c_s \langle \nu_s | e^{-i\frac{m_i^2 t}{2E_i}} |\nu_\alpha\rangle \ . \tag{4.82}$$

Therefore, the final probability reads

$$\mathbb{P}(\nu_\alpha \to \nu_s) = \sin^2(2\theta_\alpha)\sin^2(\frac{\Delta m^2}{2E_i}\frac{L}{2})\,. \tag{4.83}$$

We can write a similar expression in the Early Universe with $t_\alpha = 2E_\nu/\Delta m^2$.

The evolution can be described by a Hamiltonian

$$H = U\,\mathrm{diag}(\frac{m_1^2}{2E_\nu}, \frac{m_j^2}{2E_\nu})U^\dagger + V_{\mathrm{int}}\,. \tag{4.84}$$

The interaction potential only has a single nonzero component $V_{\alpha\alpha}$, which accounts for the interactions with the heat bath.

The relevant interaction rate is $\Gamma \sim 100\,\mathrm{MeV}$, so we get

$$V_{\alpha\alpha} \approx C_\alpha \times 25 G_F^2 T^4 E_\nu\,, \tag{4.85}$$

where $C_e \approx 3.5$.

The probability of a ν_α becoming a ν_s is therefore

$$P(\nu_\alpha \to \nu_s) = \sin^2(2\theta_\alpha^m)\sin^2(t/2t_\alpha^m)\,, \tag{4.86}$$

where

$$t_\alpha^m = \frac{t_\alpha^{\mathrm{vac}}}{\sqrt{\sin^2 2\theta_\alpha + (\cos(2\theta_\alpha) - V_{\alpha\alpha}t_\alpha^{\mathrm{vac}})^2}}\,, \tag{4.87}$$

and

$$\sin^2(2\theta_\alpha^m) = \left(\frac{t_\alpha^m}{t_\alpha^{\mathrm{vac}}}\right)^2 \sin^2(2\theta_\alpha)\,. \tag{4.88}$$