

Gravitational Wave Exercises @ Jena

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Exercises for the Gravitational Waves course. The exercise sheets can be found on the [webpage](#).

1 Quadrupole approximation

The procedure to find the quadrupole approximation for the GW emitted by a Newtonian system is in the form:

1. determine the density $\rho(\vec{x}, t)$;
2. calculate the trace-free inertia tensor $Q^{ij}(t)$;
3. calculate the gravitational wave strain h_{ij} .

Two point particles

In order to model the point-like nature of the particles we can write an expression for the density as a sum of two delta-functions, whose locations rotate around an axis — let us fix it to be the z axis for simplicity, and also let us suppose that we are on the $z = 0$ plane:

$$\rho(\vec{x}, t) = m\delta(\vec{x} - \vec{x}_1(t)) + m\delta(\vec{x} - \vec{x}_2(t)), \quad (1.1)$$

where

$$\vec{x}_1(t) = r \begin{bmatrix} \cos(\omega t + \phi) \\ -\sin(\omega t + \phi) \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{x}_2(t) = -\vec{x}_1(t). \quad (1.2)$$

For simplicity we will also assume that the radius r is constant — this cannot be precisely the case, since there will be some amount of power lost because of the GW emission, but it is a reasonable approximation.

So, the trace-free inertia tensor will be given by

$$Q^{ij}(t) = \int \rho(\vec{x}, t) \left(x^i x^j - \frac{1}{3} \delta^{ij} r^2 \right) d^3x \quad (1.3)$$

$$= m \sum_{k=1,2} \left(x_k^i x_k^j - \frac{1}{3} \delta^{ij} r^2 \right) \quad (1.4)$$

$$= 2mr^2 \begin{bmatrix} \cos^2 - 1/3 & -\cos \sin \\ -\cos \sin & \sin^2 - 1/3 \end{bmatrix}, \quad (1.5)$$

where we write only the upper-left 2x2 submatrix in Q^{ij} , since the other entries are zero, and we omit the argument of the sines and cosines (which is always $\omega t + \phi$).

The second derivative of this tensor will be given by

$$\ddot{Q}^{ij}(t) = 4mr^2\omega^2 \begin{bmatrix} \sin^2 - \cos^2 & 2\sin \cos \\ 2\sin \cos & \cos^2 - \sin^2 \end{bmatrix}, \quad (1.6)$$

Now, in order to compute the gravitational wave strain we need the projection tensor $\Lambda_{ij,kl}$. If the propagation direction we are interested in is \vec{k} , then the tensor

$$P_{ij} = \delta_{ij} - \frac{k_i k_j}{|k|^2} \quad (1.7)$$

will project a vector onto the subspace orthogonal to \vec{k} ; and the tensor

$$\Lambda_{ij,kl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \quad (1.8)$$

$$=, \quad (1.9)$$

will project a rank-2 tensor onto the corresponding subspace.