Gravitational Waves @ Jena University

Jacopo Tissino

2021-06-21

Introduction

Monday 2021-4-12, compiled 2021-06-21

The syllabus can be found here.

Interesting things on the Indico server of Jena university.

In this first lecture, a basic introduction to the theory of gravitational waves: Einstein's first papers, the sticky bead argument by Bondi & Feynman, the quadrupole formula:

$$\overline{h}_{ij}(t, r) = \frac{2G}{c^4 r} \ddot{I}_{ij}(t - r) \,. \tag{0.1}$$

The idea behind the multipole expansion is that we are solving the Poisson equation $\nabla^2 \phi = \rho$, so

$$\phi(\vec{r}) = \int \frac{\rho(\vec{x}) \, \mathrm{d}^3 x}{|\vec{r} - \vec{x}|} \,, \tag{0.2}$$

so as long as we are far away from the source we will see

$$\phi(\vec{r}) = -\frac{q}{r} - \frac{p_i n^i}{r^2} - \frac{Q_{ij} n^i n^j}{r^3} + \dots \,. \tag{0.3}$$

Quiz: which of these are GW sources?

1. spherical star: no, its quadrupole is vanishing;
2. rotating star: no, its quadrupole is constant;
3. star with a mountain: yes, its quadrupole evolves (potential source of continuous GW);
4. supernova explosion: yes, if there is asymmetry (potential source of burst GW);
5. binary system: yes, already detected!

Claim 0.1. *Order of magnitude expression:*

$$h \lesssim \frac{GM}{c^2 D} \frac{v^2}{c^2} = \frac{R}{D} \frac{GM}{c^2 R} \left(\frac{v}{c} \right)^2 \,, \tag{0.4}$$

where D is the distance to the object, R is the characteristic scale of the object (so that $GM/c^2 R$ is the compactness), while v is the characteristic velocity. The quantity we calculate is $h \sim \delta L / L$, the strain.

Proof. To do. □

The Hulse-Taylor pulsar. The two-body problem in GR is difficult.

The typical waveform in the PN region looks like:

$$h_+(t) \approx \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}(t)}{c}\right)^{2/3}\cos\Big(2\pi f_{\mathrm{gw}}(t)t\Big)\,, \tag{0.5}$$

then we need numerical relativity to simulate the plunge and merger, and finally the ring-down is simulated using BH perturbation methods. The mass scale is

$$h(t) \sim \nu\frac{1}{r/M}(Mf_{\mathrm{gw}})^{2/3}\,, \tag{0.6}$$

while

$$\phi_{\mathrm{gw}}(t) \sim 2\phi_{\mathrm{orb}}(t) = 2M_c^{-5/8}t^{5/8} = 2\nu^{-3/8}\left(\frac{t}{M}\right)^{5/8}\,, \tag{0.7}$$

where $\nu = \mu/M$, and $\mu = 1/(1/M_1 + 1/M_2)$.

Multiple detectors are crucial for sky localization, as well as for the measurement of polarization.

At leading order, the two-body problem in GR is scale-invariant: the length of the signal can be estimated simply from the mass of the stars involved.

R-process nucleosynthesis might have something to do with BNS mergers, if the stars are torn apart by the collision.

1 Weak-field GR

Monday 2021-4-19, compiled 2021-06-21

This is the limit of GR for weak gravitational fields: the metric is assumed to be in the form of the Minkowski one plus a perturbation. We are seeking the equations of motion under this assumption.

How do we quantify the term "small"? We assume that there is a **global inertial coordinate system** such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{1.1}$$

where, like in the rest of the course, we will use the letters α, β, γ or μ, ν for the coordinates x^μ; while letters like a, b represent the abstract notation.

The term "small", then, means that each component of $h_{\mu\nu}$ has an absolute value which is much smaller than 1. We are using the metric signature $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$.

What does this approximation describe?

1. Newtonian gravity;
2. gravito-electric / magnetic effects (this will be discussed in more detail later, an example is the Lense-Thirring effect);

3. propagation of gravitational waves.

In the case of the gravitational field around the Sun, in terms of orders of magnitude we have[1]

$$\left|h_{\mu\nu}\right| \sim \frac{\phi}{c^2} \sim \frac{GM_\odot}{c^2 R_\odot} \sim 10^{-6} . \tag{1.2}$$

From a field-theoretic point of view:

1. η is a background metric;
2. h is the "main" field;
3. the metric does *not* backreact on the matter ($T_{\mu\nu}$).

The metric perturbation h transform like a tensor on flat spacetime under Lorentz transformations: if $\Lambda^\top \eta \Lambda = \eta$, then the coordinates change like $x = \Lambda x'$, then the full metric transforms like

$$\begin{aligned} g_{\mu'\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} & (1.3) \\ &= \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \left(\eta_{\mu\nu} + h_{\mu\nu} \right) & (1.4) \\ &= \eta_{\mu'\nu'} + \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} h_{\mu\nu} , & (1.5) \end{aligned}$$

therefore the transformation for h is

$$h_{\mu\nu} \to h_{\mu'\nu'} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} h_{\mu\nu} . \tag{1.6}$$

Mind the notation: the meaning of $h_{\mu'\nu'}$ is $h_{\mu\nu}(x')$.

Symmetry of linearized GR

Full GR is diffeomorphism invariant, while linearized GR is *infinitesimal* diffeomorphism invariant. The relevant transformations are

$$x^\mu \to x^{\mu'} = x^\mu + \xi^\mu(x^\alpha) , \tag{1.7}$$

where the vector field ξ is selected so that $\left|\partial_\mu \xi^\alpha\right| \sim \left|h_{\mu\nu}\right| \ll 1$.

The Jacobian of this transformation is

$$\frac{\partial x^{\mu'}}{\partial x^\mu} = \delta^{\mu'}{}_\mu + \partial_\mu \xi^{\mu'} , \tag{1.8}$$

while the inverse Jacobian is

$$\frac{\partial x^\mu}{\partial x^{\mu'}} + \delta^\mu{}_{\mu'} - \partial_{\mu'} \xi^\mu + \mathcal{O}\left(|\partial\xi|^2 \right) , \tag{1.9}$$

[1] We make the c explicit here for clarity, but we will use geometric units $c = G = 1$ for the rest of the course.

since $(\mathbb{1}+\delta)(\mathbb{1}-\delta) = \mathbb{1} + \mathcal{O}\left(\delta^2\right)$.

Under this change of coordinates, we have

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} \tag{1.10}$$

$$\sim (\delta\delta - \partial\delta - \partial\delta + \partial\partial)(\eta + h) = \delta\delta\eta + h - \partial\delta - \partial\delta + \mathcal{O}\left(\delta^2\right) \tag{1.11}$$

$$= \delta^\mu_{\mu'}\delta^\nu_{\nu'}\eta_{\mu\nu} - \partial_{\mu'}\xi^\mu\delta^\nu_{\nu'}\eta_{\mu\nu} - \partial_{\nu'}\xi^\nu\delta^\mu_{\mu'} + \delta^\mu_{\mu'}\delta^\nu_{\nu'}h_{\mu\nu} \tag{1.12}$$

$$= \eta_{\mu'\nu'} + h_{\mu'\nu'} - 2\partial_{(\mu'}\xi_{\nu')}\,, \tag{1.13}$$

therefore we have our transformation law:

$$h_{\mu'\nu'} = h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}\,. \tag{1.14}$$

This can also be written in terms of the Lie derivative as

$$h_{\mu\nu} \to h_{\mu\nu} + \mathscr{L}_\xi \eta_{\mu\nu}\,. \tag{1.15}$$

This is the analogous of a gauge transformation in electromagnetism: $A_\alpha \to A_\alpha + \partial_\alpha\chi$, where A is the vector potential.

Equations of motion

The equations of motion will come through plugging $g = \eta + h$ into the EFE $G_{ab} = 8\pi T_{ab}$ and keeping only the linear order in h.

We will need the following quantities:

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} + \mathcal{O}\left(h^2\right) \tag{1.16}$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\eta^{\mu\lambda}\left(\partial_\alpha h_{\lambda\beta} + \partial_\beta h_{\lambda\alpha} - \partial_\lambda h_{\alpha\beta}\right) + \mathcal{O}\left(h^2\right) \tag{1.17}$$

$$R_{\mu\nu} = \partial\Gamma - \partial\Gamma + \mathcal{O}\left(h^2\right), \tag{1.18}$$

where we already simplified the expressions by removing the higher-order terms. The result is

$$R_{\mu\nu} = \partial^\alpha\partial_{(\mu}h_{\nu)\alpha} - \frac{1}{2}\partial_\lambda\partial^\lambda h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h + \mathcal{O}\left(h^2\right), \tag{1.19}$$

where $h = h^\alpha_\alpha = \eta^{\alpha\beta}h_{\alpha\beta}$. Note that we are allowed to use η instead of g to lower indices. The Einstein tensor reads

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \tag{1.20}$$

$$= \partial^\alpha\partial_{(\mu}h_{\nu)} - \frac{1}{2}\partial_\lambda\partial^\lambda h_{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \frac{1}{2}\eta_{\mu\nu}\partial_\lambda\partial^\lambda h\,, \tag{1.21}$$

which can be simplified if we consider the trace-reversed metric

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,, \tag{1.22}$$

so that $\bar{h} = \eta^{\mu\nu} h_{\mu\nu} - \eta^{\mu\nu}\eta_{\mu\nu} h/2 = -h$. See equation 2.13 in the notes for a full explanation, but the idea is to insert $\bar{h}_{\mu\nu}$ and $\bar{h}$ into $G_{\alpha\beta}$ and to make some simplifications. We get

$$G_{\mu\nu} = -\frac{1}{2}\eta_{\alpha\beta}\partial^\alpha\partial^\beta \bar{h}_{\mu\nu} + \partial^\alpha \partial_{(\mu}\bar{h}_{\nu)\alpha} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta \bar{h}_{\alpha\beta}\,, \tag{1.23}$$

which is in the form $\Box_\eta \bar{h}_{\mu\nu} + \ldots \partial^\alpha \bar{h}_{\alpha\beta}$. We still have gauge freedom, so we can simplify the equation a great deal by setting $\partial^\alpha \bar{h}_{\alpha\beta} = 0$ — the Hilbert, or Lorentz gauge.

With this choice, we have

$$\boxed{\Box_\eta \bar{h}_{\mu\nu} = -\frac{16G}{c^4} T_{\mu\nu}\,,} \tag{1.24}$$

a relatively simple tensor wave equation.

Is is always possible to impose the Hilbert gauge? Yes: we can make an infinitesimal coordinate transformation to send a generic $h_{\mu\nu}$ to $h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$, so that $h \to h + 2\eta^{\alpha\beta}\partial_{(\alpha}\xi_{\beta)}$. Therefore,

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + 2\partial_{(\mu}h_{\nu)} - \eta_{\mu\nu}\partial_\alpha \xi^\alpha\,, \tag{1.25}$$

and we can send

check indices here

$$\partial^\alpha \bar{h}_{\mu\alpha} \to \partial^\alpha \bar{h}_{\mu\alpha} + \Box \xi_\mu + \partial^\mu \partial_\nu \xi_\mu + \partial_\nu \partial^\lambda \xi_\lambda\,, \tag{1.26}$$

so if we set $\Box \xi_\mu = -\partial^\alpha \bar{h}_{\mu\alpha} = v_\mu$ we can reduce ourselves to the Hilbert gauge from any starting point. All we need to do is solve the wave equation $\Box \xi_\mu = v_\mu$.

Now, to linear order $T_{\mu\nu}$ does not depend on h. So, we can find formal solutions using Green's functions, like in electromagnetism.

The Bianchi identities are now given by $\partial_\nu G^{\mu\nu} = 0$, so $\partial_\nu T^{\mu\nu} = 0$, which gives us the EOM for matter — note that this is a partial, not a covariant derivative! This means that there is no backreaction on the metric.

The linear EFE correspond to the equations of motion of a massless spin-2 field.

Weak-field solutions

Let us consider a *static source*: suppose that $T_{\mu\nu} = \rho t_\mu t_\nu$, where $t^\mu = (\partial_t)^\mu$ is the time vector along the time direction of the global inertial coordinate system while ρ is an energy density.

If $t^\mu = (1, 0, 0, 0)$ then $T_{00} = \rho$ while $T_{0i} = 0 = T_{ij}$.

In this case, then, the stress-energy tensor is time-independent: therefore also on the other side we will have $\partial_t \bar{h}_{\mu\nu} = 0$.

Therefore, the left-hand side of the equation will read $\nabla^2 \bar{h}_{\mu\nu} = -16\pi\rho$ for $\mu = \nu = 0$ and $\nabla^2 h_{\mu\nu} = 0$ for all the other components.

These Poisson equations can be solved as boundary-value problems if we assume that $h_{\mu\nu} \to 0$ for $r \gg R$.

This looks very similar to the Newton equation $\nabla^2 \phi = 4\pi\rho$; therefore $\overline{h}_{00} = -4\phi$, while $\overline{h}_{\mu\nu} = 0$ for all other components.

We can reconstruct the metric using the fact that $\overline{h} = 4\phi$, so

$$h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{h} = -4\phi t_\mu t_\nu - \frac{1}{2}\eta_{\mu\nu}4\phi \,, \tag{1.27}$$

so the metric reads

$$g_{\mu\nu} = \eta_{\mu\nu}(1 - 2\phi) - 4\phi t_\mu t_\nu \,, \tag{1.28}$$

therefore

$$g = -(1 + 2\phi)\,\mathrm{d}t^2 + (1 - 2\phi)\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \,. \tag{1.29}$$

We know that far away from the source, the Newtonian field decays like $\phi \approx -M/r + \mathcal{O}\left(r^{-2}\right)$.

Therefore, this metric approximation already includes special relativity as well: we have $g \to \eta$ for large r, but also $g = \eta$ for $M = 0$.

The geodesic equation for this weak field metric reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\partial^i \phi \,. \tag{1.30}$$

However, these Newtonian equations of motion are *not* consistent with $\partial_\mu T^{\mu\nu} = 0$. These describe the motion of the source which generates gravity, whereas the Newtonian EOM describe the motion of test particles in the weak-field metric.

The dual meaning of the full EFE — matter deforms the spacetime, the spacetime shapes the trajectories of matter — *cannot* be realized at linear order.

No-stress source

Monday 2021-4-26, compiled 2021-06-21

We considered a source in the form $T_{\mu\nu} = \rho t_\mu t_\nu$, where $t^\mu = (1, \vec{0})$.

Now we will consider a source in the form

$$T_{\mu\nu} = -2\rho t_\mu t_\nu + 2J_{(\mu}t_{\nu)} \,, \tag{1.31}$$

where $J^\mu = \rho u^\mu = \rho(\gamma, \gamma v^i/c)$.

Probably the first 2 is not there.

The static source from before can be recovered from this expression in the low-velocity limit $v^i/c \to 0$. In that case, $T_{ij} = 0$: we can see that T_{ij} is of order v^2/c^2, so to first order they vanish.

In this situation, we get the system

$$\begin{cases} \square\overline{h}_{0\mu} &= -16\pi T_{0\mu} \\ \square\overline{h}_{ij} &= 0 \,. \end{cases} \tag{1.32}$$

In order to simplify, let us assume that $\partial_t \overline{h}_{ij} = 0$: then, the solution to the second of these becomes $\nabla^2 \overline{h}_{ij} = 0$, with flat boundary conditions at large distance. By linearity, this leads to $\overline{h}_{ij} = 0$.

Claim 1.1. *If we define $A_\mu = -(1/4)\overline{h}_{0\mu} = -(1/4)\overline{h}_{\mu\nu}t^\nu$, then the metric becomes*

$$g_{00} = -1 + 2A_0 \tag{1.33}$$

$$g_{0i} = 4A_i \tag{1.34}$$

$$g_{ij} = (1 + 2A_0)\delta_{ij} \,. \tag{1.35}$$

In terms of this A_μ, the Dalambertian equation from before reads

$$\Box A_\mu = -\frac{16}{4}\pi J_\mu = -4\pi J_\mu \,, \tag{1.36}$$

which are formally identical to the Maxwell equations! Therefore, we can employ known techniques from electromagnetism.

For example, if $\partial_t A_\mu = 0$ then

$$\begin{cases} A_0 & = -\phi \\ A_i & = \int \mathrm{d}^3 x^i \, \frac{J_i}{\left|x - x^i\right|} \,, \end{cases} \tag{1.37}$$

which is the reason why the phenomena which can be described through this formalism are known as gravito-electric and gravito-magnetic effects.

Claim 1.2. *For example, geodesics in a weak-field stationary (no stress) spacetime are described by a Lagrangian*

$$\mathscr{L} = -mc\left(-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right)^{1/2} \tag{1.38}$$

$$= -mc^2\left(-g_{00} - 2g_{0i}\frac{v^i}{c} - g_{ij}\frac{v^i v^j}{c^2}\right)^{1/2} \tag{1.39}$$

$$\approx -mc^2 + \frac{m}{2}v^2 + m\phi + 4mcA_i v^i \,. \tag{1.40}$$

We have a mass term, a kinetic term, a gravitational term, and a contribution to the Lorentz force.

The corresponding equations of motion read

$$\ddot{\vec{x}} = \vec{E} + 4\vec{v} \times \vec{B} \,, \tag{1.41}$$

where $\vec{E}$ and $\vec{B}$ are the gravitoelectric and gravitomagnetic fields derived from our A_μ. The differences from EM are: the absence of charge, and the factor of 4 before the magnetic term.

An example of a gravito-electromagnetic effect is the Lense-Thirring effect: a magnetic moment $\vec{s}$ in a magnetic field precesses, according to

$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = \vec{s} \times \vec{\Omega} \qquad \text{where} \qquad \vec{\Omega} = -\frac{q}{m}\vec{B}_{EM} \,, \tag{1.42}$$

so in order to generalize to the precession of a gyroscope in an EM field we need to map $q \to m$ and $\vec{B}_{EM} \to 4\vec{B}$.

This way, we see for example that $\Omega_g = -4B$. A mission called Gravity Probe B measured this effect: they found precession with $\Omega_g \sim 0.22\,\text{arcsec}/\text{yr}(R_\oplus/r)^3$. This is a 20 % accurate test of GR in the weak field.

What does that mean?

Another example is **frame dragging**, which applies in full GR: if we put the gyroscope around a BH a similar effect emerges. Around a Kerr BH we have

$$g_{0i}^{\text{Kerr}} \sim \Omega_{BH}\,, \tag{1.43}$$

and if the particle is close to the BH a particle is "locked" to the BH rotation.

2 Gravitational Waves in linear GR

GW are solutions of weak-field GR in a vacuum. There, the wave equation reads $0 = \Box_\eta \bar{h}_{\mu\nu}$. What are the properties of the solutions of these equations? The simplest thing we can do is look for plane wave solutions. We take a wave vector $k^\mu = (\omega, k^i)$ and an amplitude $A_{\mu\nu}$; then

$$\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\mu x^\mu} = A_{\mu\nu} e^{i(-\omega t + \vec{k}\cdot\vec{x})}\,, \tag{2.1}$$

so $\partial_\mu \bar{h}_{\alpha\beta} = (ik_\mu)\bar{h}_{\alpha\beta}$.

Substituting the plane wave ansatz yields

$$0 = \Box \bar{h}_{\alpha\beta} = -\eta^{\mu\nu} k_\mu k_\nu \bar{h}_{\alpha\beta}\,, \tag{2.2}$$

therefore $k_\mu k^\mu = 0$. The wavevector is null.

This implies that the GW propagates at the speed of light: $\omega s^2 = \left|\vec{k}\right|^2$.

How do we completely specify a gauge? Any infinitesimal transformation such that $\Box \xi^\mu = 0$ preserves the Hilbert gauge, so we can make a residual gauge transformation.

The harmonic gauge implies that

$$0 = -\partial^\alpha \bar{h}_{\mu\alpha} = ik^\alpha \bar{h}_{\mu\alpha}\,, \tag{2.3}$$

which yields $k^\alpha A_{\alpha\mu} = 0$. This means that GWs are **transverse** to the propagation direction.

We know that $\bar{h}_{\mu\nu}$ maps to $\bar{h}_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} + \eta_{\mu\nu}\partial_\alpha \xi^\alpha$.

Let us use $\xi^\mu = B^\mu e^{ik_\alpha x^\alpha}$ as an ansatz for our residual gauge transformation, since it automatically harmonic: we get

$$A_{\mu\nu} \to A_{\mu\nu} - 2ik_{(\mu}B_{\nu)} + i\eta_{\mu\nu}k_\alpha B^\alpha\,, \tag{2.4}$$

and since we can pick B^μ arbitrarily we can impose $\bar{h} = A^\mu_\mu = 0$, the **traceless condition**, as well as $\bar{h}_{\mu 0} = 0$, the **transverse condition**. The second is suggested by the previously found result $k_\alpha A^{\alpha\beta} = 0$.

In terms of B, this is a linear algebraic system, and it is invertible.

In summary, we start from 10 variables, we use 4 equations to impose the Hilbert gauge, and 4 more to impose the TT gauge. The two degrees of freedom which are left are the true degrees of freedom of a GW.

More explicitly, if we have $k^\mu = (\omega, 0, 0, k_z)$ this means

1. $k^2 = 0$ implies $-\omega = k_z$;
2. the phase reads $k_\alpha x^\alpha = \omega(t - z)$;
3. the Hilbert gauge $k^\mu A_{\mu\nu} = 0$ tells us that $A_{0\nu} = A_{3\nu}$;
4. the transverse condition tells us that $A_{0\mu} = 0$ (so also $A_{3\mu} = 0$);
5. the traceless condition tells us that $A_\mu^\mu = 0$.

This leads to the usual formulation

$$A_{\mu\nu}^{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{2.5}$$

Therefore,

$$h_{\mu\nu}^{TT} = A_{\mu\nu}^{TT} \exp\big(i\omega(t - z)\big) . \tag{2.6}$$

In TT gauge we have $\overline{h}_{\mu\nu} = h_{\mu\nu}$ since the trace is zero. Importantly, the TT gauge can only be defined in vacuo! This is because in that case $\Box \overline{h}_{\mu\nu} \neq 0$, so while we can still exploit gauge freedom we cannot set components to zero inside the source.

The metric in TT gauge reads

$$g = -\,\mathrm{d}t^2 + \mathrm{d}z^2 + (1 + h_+)\,\mathrm{d}x^2\,(1 - h_\times)\,\mathrm{d}y^2 + 2h_\times\,\mathrm{d}x\,\mathrm{d}y\, g \quad = -\,\mathrm{d}t^2 + (\delta_{ij} + h_{ij}^{TT})\,\mathrm{d}x^i\,\mathrm{d}x^j \ . \tag{2.7}$$

How do we identify the GW degrees of freedom in general? We can impose the TT gauge outside the source (far away from the $T_{\mu\nu}$).

In general,

$$h_{\mu\nu}^{TT} = \Lambda_{\mu\nu}{}^{\alpha\beta} \overline{h}_{\alpha\beta} \,, \tag{2.8}$$

where Λ is a projection operator, defined as

$$\Lambda_{\mu\nu}{}^{\alpha\beta} = P_\mu^\alpha P_\nu^\beta - \frac{1}{2} P_{\mu\nu} P^{\alpha\beta} \tag{2.9}$$

$$P_{\mu\nu} = \delta_{\mu\nu} - n_\mu n_\nu \,, \tag{2.10}$$

where n^μ is the propagation direction.

The projection tensor $P_{\mu\nu}$ is symmetric, it is transverse ($P_{\mu\nu}n^\nu = 0$), it is idempotent ($P_{\mu\alpha}P_{\alpha\nu} = P_{\mu\nu}$), and its trace is equal to 2.

The tensor $\Lambda_{\mu\nu\alpha\beta}$ is also idempotent, transverse in all indices, traceless in $\mu\nu$ and $\alpha\beta$ separately, and symmetric in the swap of $\mu\nu$ and $\alpha\beta$.

In summary, we have found GW solutions, they propagate with c, they are transverse, they have two degrees of freedom.

Symmetric, Transverse, Trace-Free tensors play an important role in GW theory. They can be used to obtain the **Multipolar expansion**.

"Living review of relativity" (see webpage) describes all the tests of GR.

3 Effects of GW on test masses

Monday 2021-5-3, compiled 2021-06-21

Consider two masses, separated by a distance L. The distance between them can be measured as $L = (c/2)\Delta t_{PP''}$, where PP'' is the trajectory of a light beam moving from P to Q and back to P.

In flat spacetime, the separation vector between them can then be written as $h^i = L_0 n^i$ for some unit vector n^i.

In the presence of a GW, the length will read

$$
\begin{aligned}
L^2 &= g_{\mu\nu}\left(x^\mu_{q'} - x^\mu_{p'}\right)\left(x^\nu_{q'} - x^\nu_{p'}\right) && (3.1)\\
&\to g_{\mu\nu}\left(x^i_{q'} - x^i_{p'}\right)\left(x^j_{q'} - x^j_{p'}\right) && (3.2)\\
&\to g_{\mu\nu}x^i_{q'}x^j_{q'} && (3.3)\\
&\to \left(\delta_{ij} + h^{TT}_{ij}\right)x^i_{q'}x^j_{q'} = L_0^2\left(\delta_{ij} + h^{TT}_{ij}\right)n^i n^j\,, && (3.4)
\end{aligned}
$$

and so we can compute

$$
\frac{\delta L}{L_0} = \frac{L}{L_0} - 1 = \sqrt{1 + h^{TT}_{ij}n^i n^j} - 1 \approx \frac{1}{2}h^{TT}_{ij}n^i n^j\,. \tag{3.5}
$$

This justifies the heuristic formula $\delta L/L_0 \sim h$.

A more formal treatment can be given through the **geodesic deviation** formula: we can show that if u^μ is the tangent vector of a family of geodesics and s^μ is the displacement between geodesics, then

$$
u^\mu \nabla_\mu (u^\nu \nabla_\nu s^\alpha) = R^\alpha_{\lambda\rho\sigma}u^\lambda u^\rho s^\sigma\,, \tag{3.6}
$$

and in the weak field limit the Riemann tensor is in the form $\partial^2 h$.

If we plug in everything (as we did in the exercise) we get

$$
\frac{\mathrm{d}^2 s_\alpha}{\mathrm{d}t^2} = R_{\alpha 00\mu}s^\mu \qquad \text{with} \qquad R_{\alpha 00\mu} = \frac{1}{2}\ddot{h}^{TT}_{\alpha\mu}\,. \tag{3.7}
$$

The temporal evolution of the spatial vector then reads

$$
\ddot{s}^i(t) = \frac{1}{2}\ddot{h}^{TT}_{ij}(t)s^j_0 + \mathcal{O}\left(h^2\right), \tag{3.8}
$$

so if initially $\dot{s}^i(t=0)$ and $s^i(t=0)=s^i_0$ we get

$$s^i(t) = s^i_0 + \frac{1}{2}h^{TT}_{ij}(t)s^j_0 = \left(\delta_{ij} + \frac{1}{2}h^{TT}_{ij}(t)\right)s^j_0 . \tag{3.9}$$

A ring of particles in the xy plane is deformed by a wave travelling along the z axis: it becomes an ellipse with axes along the x and y direction for the h_+ polarization,

$$\frac{\delta x^2}{r_0^2(1+h_+)^2} + \frac{\delta y^2}{r_0^2(1-h_+)^2} = 1 . \tag{3.10}$$

The effect of the cross polarization is similar but rotated by 45°.

4 Sources of GW

We start from a formal solution of $\Box\overline{h}_{\mu\nu} = -16\pi T_{\mu\nu}$: like in electromagnetism, we use Green functions,

$$\overline{h}_{\mu\nu}(t,\vec{x}) = -16\pi \int G_R(x^\alpha - x^{\alpha\prime})T_{\mu\nu}(x^{\alpha\prime})\,\mathrm{d}^4x' \ , \tag{4.1}$$

where $\Box G_R(x) = \delta^{(4)}(x)$; explicitly

$$G_R(x) = -\frac{1}{4\pi}\frac{\delta(u-t)}{|\vec{x}|} . \tag{4.2}$$

where u is the retarded time. (to be defined... check)

With this, we get

$$\overline{h}_{\mu\nu}(t,\vec{x}) = 4\int \frac{T_{\mu\nu}(u,\vec{x})}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x' \ . \tag{4.3}$$

The assumptions we can make are the following: a mean-field approach, negligible self-gravity (this means that the quantity $2GM/c^2R = R_S/R \ll 1$).

Also, in order to derive the quadrupole formula, we assume that the distance from us to the source is large compared to the scale of the source and that the velocity of the source is slow compared to c.

The result we will find is that we can compute

$$\overline{h}_{ij}(t,\vec{x}) = \frac{4}{r}\int \mathrm{d}^3x'\, T_{ij}\left(t - \frac{r}{c} + \frac{\hat{n}\cdot\vec{x}}{r}, \vec{x}'\right) , \tag{4.4}$$

and we can further simplify the integrand by expanding in x'/r.

5 Quadrupole formula

Monday 2021-5-10, compiled 2021-06-21

The self-gravity is denoted as $\sigma = 2GM/c^2R = R_S/R$. In order for the quadrupole formula to work, we need $\sigma \ll 1$ (negligible self-gravity) as well as the weak-field limit.

Hypothesis 1 is that we are far away from the source, compared to the source's linear scale.

Hypothesis 2 is that the source is slow:

$$v \sim \frac{|T_{0i}|}{|T_{00}|} \ll c = 1\,. \tag{5.1}$$

The process is as follows:

1. we use the two hypotheses to simplify the Green equation;
2. we use the conservation law of $T_{\mu\nu}$ in flat spacetime $0 = \partial^\mu T_{\mu\nu}$ to further express T_{ij} in the Green equation in terms of $T_{00} = \rho c^2$
3. we project into the TT gauge.

The retarded time is

$$u = t - \left|\vec{x} - \vec{x}'\right| \approx t - r + \hat{n} \cdot \vec{x}'\,. \tag{5.2}$$

The far-field approximation is given by expanding $T_{\mu\nu}(t_r + \hat{n} \cdot \vec{x}'/c)$ around t_r.

As for the slow-velocity expansion, we do a Fourier transform and expand the exponential

$$e^{i\omega\left(t_r + \hat{n}\cdot\vec{x}'/c\right)} \approx e^{-i\omega t_r}\left(1 - \frac{i\omega}{c} n_i x^i + \dots\right). \tag{5.3}$$

Then, in the time domain time derivatives correspond to $i\omega$ terms.

6 GW energy

Monday 2021-5-17, compiled 2021-06-21

Because of the equivalence principle, there can be no local definition of energy density or of a stress-energy tensor for $g_{\mu\nu}$.

However, we can define the Landau-Lifshitz pseudotensor (which is actually a tensor density): from the gothic metric

$$\mathfrak{g}^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta} \tag{6.1}$$

we define

$$\Lambda^{\alpha\mu\beta\nu} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}\,, \tag{6.2}$$

a four-tensor density which has the same symmetries as the Riemann tensor. Also, we have the properties that

$$\partial_\mu \partial_\nu \Lambda^{\alpha\mu\beta\nu} = 2(-g)G^{\alpha\beta} + 16\pi(-g)\tau_{LL}^{\alpha\beta}\,, \tag{6.3}$$

where $G^{\alpha\beta}$ is the Einstein tensor, while τ_{LL} (defined to contain whatever remains) is called the Landau-Lifshitz tensor: this can also be expressed as

$$\partial_\mu \partial_\nu \Lambda^{\alpha\mu\beta\nu} = 16\pi(-g)\left(T^{\alpha\beta} + \tau_{LL}^{\alpha\beta}\right). \tag{6.4}$$

If we take a further derivative with respect to α, in a vacuum, we find

$$\partial_\alpha \partial_\mu \partial_\nu \Lambda^{\alpha\mu\beta\nu} = \partial_\alpha \tau_{LL}^{\alpha\beta} = 0\,, \tag{6.5}$$

where the last equality is due to the symmetries of Λ. Therefore, the tensor τ_{LL} is conserved, and we can try to interpret it as the "energy of the gravitational field".

In normal coordinates at any given point τ_{LL} is identically zero.

A global concept of energy does not exist for certain spacetimes — see ADM and the Hamiltonian formulation of GR.

However, asymptotically flat spacetimes have some nice properties.

If we foliate a spacetime (M, g) into 3D spacelike hypersurfaces Σ, we say it is asymptotically flat if the reduced metric $\gamma_{ij} \to f_{ij}$ tends towards the Minkowski metric at radial infinity.

In this case, we can associate an energy to Σ. Using these concepts, there is hope to identify some quantity which represents GW energy.

We explore directions at infinity which are **asymptotically null**, and suppose that at $\Sigma_{t_{1,2}}$ we have stationary states, with dynamics in between.

Then, if there is a difference of energy $\Delta E = E_2 - E_1$, we expect there to be a flux of "some $\tau_{\mu\nu}$" from the surface.

We can work in linearized theory: $g = f + h$. The characteristics we expect a definition of energy to have are, at the very least

1. quadratic in h,
2. generate curvature via a stress-energy tensor $\tau_{\alpha\beta}$,
3. gauge invariant under infinitesimal coordinate transformations.

We start by specifying the LL tensor to the weak field case. The metric reads $g = \eta + h^{(1)} + h^{(2)} + \mathcal{O}(3)$, where by the indices in parentheses we mean orders in some expansion parameter.

The Ricci reads

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \mathcal{O}(3)\,. \tag{6.6}$$

Now, $R^{(0)}_{\mu\nu} = (\mathrm{Ric}[\eta])_{\mu\nu} \sim \eta \partial^2 \eta = 0$, but if we had a non-flat background this would not hold. On the other hand,

$$R^{(1)}_{\mu\nu} = \left(\mathrm{Ric}^{(1)}[h^1]\right)_{\mu\nu} \sim \eta \partial^2 h^{(1)}\,, \tag{6.7}$$

where $\mathrm{Ric}^{(i)}$ is the Ricci operator expanded up to i-th order. This is relevant in the second-order term:

$$R^{(2)}_{\mu\nu} = \left(\mathrm{Ric}^{(1)}[h^{(2)}]\right)_{\mu\nu}\left(\mathrm{Ric}^{(2)}[h^{(1)}]\right)_{\mu\nu} \sim \eta\partial^2 h^{(2)} + h^{(1)}\partial^2 h^{(1)} . \tag{6.8}$$

Then, the EFE in vacuo read

$$0 = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \mathcal{O}(3) , \tag{6.9}$$

therefore to order 0 we have the η metric; to order 1 we can calculate $h^{(1)}$, while to order 2 we can assume the $h^{(1)}$ metric is already calculated and solve $R^{(2)}_{\mu\nu} = 0$ for h^2.

The second order equation reads

$$G^{(1)}_{\mu\nu}[h^{(2)}] = \left(\mathrm{Ric}^{(1)}[h^2] - \frac{1}{2}R^{(1)}[h^2]\eta\right)_{\mu\nu} \tag{6.10}$$

$$= \left(-\mathrm{Ric}^{(2)}[h^1] + \frac{1}{2}R^{(2)}[h^1]\eta\right)_{\mu\nu} \tag{6.11}$$

$$\stackrel{?}{=} 8\pi\tau_{\mu\nu} , \tag{6.12}$$

but can this actually be our GW stress-energy tensor? It is symmetric, it is quadratic in h, and due to the Bianchi identities it is also conserved. However, it is not gauge invariant, and it is not unique!

If $h^{(1)}$ is **asymptotically flat**, then $h \sim \mathcal{O}(1/r)$, $\partial_i h \sim \mathcal{O}\left(1/r^2\right)$ and $\partial_i\partial_j h \sim \mathcal{O}\left(1/r^3\right)$.

In this case, $E = \int_\Sigma \mathrm{d}^3x\, \tau_{00}$ is both **gauge invariant** and **unique**: $E[h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}] = E[h_{\mu\nu}]$.

We can then write an energy flux in the form

$$\Delta E = -\int_S \mathrm{d}^2y\, \tau_{\mu 0} n^\mu , \tag{6.13}$$

across a surface S which determines the spatial boundary of Σ over time.

Weak-field GR can be interpreted as a field theory on η for the field h.

Physically speaking, it is true that we can "eliminate" the gravitational field at a point, but here we are more interested in the following question: can the GW in a neighborhood of that point contribute to the curvature via a gauge invariant tensor?

The issue comes down to: what is the distinction between "waves" and background?

One can prove that a suitable average of $\mathrm{Ric}^{(2)}[h^{(1)}]$ is actually gauge-invariant: this is the **Isaacson tensor**,

$$\tau_{\alpha\beta} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\right\rangle , \tag{6.14}$$

due to the fact that $\mathrm{Ric}^{(2)}[h] \sim h\partial\partial h + \partial h\partial h + \partial(h\partial h)$.

Let us take the derivative $\partial_\mu \tau^{\mu\nu}$:

$$\int_V \mathrm{d}^3x \left(\partial_0\tau^{00} + \partial_i\tau^{0i}\right) = \dot{E} + \oint_S \mathrm{d}^2y\, \tau_{0i}n^i \tag{6.15}$$

$$= \dot{E} + r^2 \oint \mathrm{d}^2 y \left\langle \partial^0 h_{ij} \partial_r h^{ij} \right\rangle \,, \tag{6.16}$$

which yields the quadrupole formula:

$$\frac{\mathrm{d}E_{GW}}{\mathrm{d}t} = \frac{c^3}{32\pi G} r^2 \int \mathrm{d}\Omega \left\langle \partial_t h_{ij}^{TT} \partial_t h_{ij}^{TT} \right\rangle \tag{6.17}$$

$$= \frac{G}{5c^5} \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle \,. \tag{6.18}$$

Let us look at some dimensional analysis:

$$[Q] = aML^2 \tag{6.19}$$

$$[\dddot{Q}] = aML^2 T^{-3} \sim a\Omega^3 ML^2 = E^2 T^{-2} \tag{6.20}$$

$$[\frac{G}{c^5}] = TE^{-1} \,. \tag{6.21}$$

The factor G/c^5 is the inverse of a power: $c^5/G \approx 10^{52}\,\mathrm{W}$, a very large power, and we are dividing by it. This suggests a potential rewriting of the formula, which was an idea of Weber's:

$$\dot{E} \sim \frac{G}{c^5} a^2 \Omega^6 M^2 R^4 = \cdots = a^2 \frac{c^5}{G} \left(\frac{v}{c} \right)^6 \frac{GM}{c^2 R} \,. \tag{6.22}$$

The power is therefore huge if $v \sim c$ and $R \sim R_S$!

The above formula for $\dot{E}$ is *correct* for a leading-order description of a binary system, if we take $a^2 = 32/5$, Ω to be the orbital frequency, M to be μ (the reduced mass), and R the orbital radius. So,

$$\dot{E} = \frac{32}{5} \frac{G\mu^2}{c^5} R^4 \Omega^3 \,. \tag{6.23}$$

Of course, $\dot{E}_{\text{orbital energy}} = -\dot{E}_{\text{gw}}$.

7 GW energy

Monday 2021-5-31, compiled 2021-06-21

The thing that remains to clarify in this expression is what kind of energy we are talking about:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5c^5} \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle \,. \tag{7.1}$$

If we try to do dimensional analysis, we find

$$\dot{E} \sim \frac{G}{c^5} a^2 \Omega^6 M^2 R^4 \,. \tag{7.2}$$

The term G/c^5 has units of $[T]/[E]$, and it is the inverse of $c^5/G \approx 10^{52}\,\mathrm{W}$. Therefore, the expression has a very small number in front of it.

However, we can rewrite the expression in terms of c^5/G:

$$\dot{E} = a^2 \frac{c^5}{G} \left(\frac{v}{c}\right)^6 \left(\frac{GM}{c^2 R}\right). \tag{7.3}$$

The first person who did this, Weber, was also the first to claim a detection of GW, with bar detectors. This triggered the field of experimental GW detection.

If $v/c \to 1$, and $\sigma = GM/c^2R \sim 1/2$, then $\dot{E} \sim 10^{50}\,\mathrm{W}$.

We expect such a source to be the most luminous in the Universe, in GWs at least!

Dyson predicted a "flash" of intense radiation as two NS merge.

The factor a is dimensionless — what is its numerical value? It is $32/5$. With this, the formula

$$\dot{E}_{GW} = \frac{32}{5} \frac{G\mu^2}{c^5} R^4 \Omega^3 \tag{7.4}$$

is actually correct.

Using Kepler's law, $\Omega^2 = GMR^{-3}$, we can remove the radial dependence and calculate $P = \dot{E}_{GW} = \Omega^{10/3}$.

The reservoir of energy is the orbital energy:

$$E = \frac{1}{2}\mu(\Omega R)^2 - \frac{m\mu}{R} \sim -\Omega^{2/3}. \tag{7.5}$$

So, as the energy decreases, the frequency increases. We can keep approximating the orbits as circular, with ever smaller radii. This is known as the adiabatic approximation, $T_{\text{orbit}} \gg 1/\nu_{GW} \sim 1/2\Omega$.

Really, the orbit is not circular but instead an inspiral.

In the final stages, this is a runaway process: the merger. This was demonstrated before the detection of GW, with the Hulse-Taylor pulsar.

Let us discuss frequency evolution according to radiation reaction uses these equations. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \log E = \frac{\mathrm{d}}{\mathrm{d}t} \log \Omega^{2/3} \implies \frac{2}{3}\frac{\dot{\Omega}}{\Omega} = \frac{\dot{E}}{E} = \frac{P_{\text{gw}}}{E}. \tag{7.6}$$

Alternatively we could work with energy balance:

$$-P_{\text{gw}} = \dot{E} = \frac{\mathrm{d}E}{\mathrm{d}\Omega}\dot{\Omega}. \tag{7.7}$$

Keeping all the factors leads to

$$\frac{\dot{\omega}}{\omega} = \frac{36}{5}\left(\frac{1}{2}\frac{GM_c}{c^3}\omega\right)^{5/3} = \frac{96}{5}\nu\left(\frac{1}{2}\frac{Gm}{c^3}\omega\right)^{5/3}. \tag{7.8}$$

The frequency $\omega = 2\Omega$ is the GW frequency — a general result from the quadrupole formula — while M_c is the chirp mass. Its expression is due to the combination of masses m_1, m_2 appearing in the quadrupole formula applied to a binary.

The quantity $m = m_1 + m_2$ is the total binary mass, while $\nu = m_1 m_2/m = \mu/m$ is the *symmetric mass ratio*.

The equation holds under the following:

$$\hat{\omega} \to m\omega \qquad \text{and} \qquad t \to t/m\,, \tag{7.9}$$

since $\dot{\hat{\omega}}/\hat{\omega} \propto \hat{\omega}^{5/3}$. This is a general feature of the two-body problem in GR (in vacuum, for point particles).

It is suggestive to have $\dot{\omega}/\omega$ on the LHS of the equation: Kepler's law yields

$$\dot{R} = -\frac{2}{3}\frac{\dot{\Omega}}{GMR^{-3}}R\Omega = -\frac{2}{3}\frac{\dot{\Omega}}{\Omega}\Omega R\,, \tag{7.10}$$

so the quantity

$$Q_\omega^{-1} = \frac{\dot{\omega}}{\omega^2} = -3\frac{\dot{R}}{R\Omega} \tag{7.11}$$

is, up to a constant factor, the ratio between the radial velocity and the tangential velocity: it quantifies "how much" the motion is approximated by a circular orbit, "how much" the orbit is adiabatic.

If $Q_\omega \gg 1$, we are in the adiabatic regime. This is called the **adiabaticity parameter**.

In terms of the chirp mas, we have

$$\mathcal{M}_c = \frac{c^3}{G}\left(\left(\frac{5}{96}\right)^3 \pi^{-8} f^{-11} \dot{f}^3\right)^{1/5}. \tag{7.12}$$

If we have a detection of a very long signal, the frequency increases in a runaway process.

Therefore, if we measure the frequency evolution we can estimate the chirp mass. The signal `GW170817` was so long that the chirp mass was measured up to the fourth digit.

8 Short wavelength approximation

How do we define a GW on an arbitrary background? We want to use $g = \eta + h$, where now η is not necessarily flat.

The questions are:

1. How do we define the decomposition $g = \eta + h$?

2. How do we interpret h as a GW in general?

Rigorously speaking, this is **not possible** in general. We have one metric, we need to make some arbitrary choice.

There are situations in which one can attempt to identify different relevant scales. There are situations in which one can attempt to identify different scales of variation, for example if the η metric varies with typical scale L and the h metric perturbation has a typical length λ in space.

We take GWs with $\lambda = c/f$, and $f \approx 10^3\,\mathrm{Hz}$, therefore $\lambda \sim 50\,\mathrm{km}$. Also, $|h| \sim 10^{-21}$.

On the Earth, $\phi_\oplus \sim GM_\oplus/c^2R_\oplus \sim 10^{-6}$.

Now, $\phi_\oplus$ is not smooth at the scale λ: it is **not possible** to distinguish $\eta_\oplus$ from h!

How do we do it, then? The Earth's gravitational field is quasi-static for frequencies $f \ll f_{\mathrm{gw}}$. In this sense, it is possible to distinguish the background from the perturbation.

We assume that $|h| = \epsilon$ is very small, and that $\lambda \ll L$ or $f \gg F$, the maximum frequency of the evolution background.

We start from $R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - Tg_{\mu\nu}/2\right) = 8\pi\overline{T}_{\mu\nu}$. We then expand in ϵ:

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + \mathcal{O}\left(\epsilon^2\right). \tag{8.1}$$

In terms of ϵ and $k = 1/\lambda$ we will have

$$\eta \sim \mathcal{O}\left(\epsilon^0\right) \tag{8.2}$$

$$\partial\eta \sim \mathcal{O}(1/L) \tag{8.3}$$

$$\partial^2\eta \sim \mathcal{O}(1/L) \tag{8.4}$$

$$h^n \sim \mathcal{O}(\epsilon^n) \tag{8.5}$$

$$\partial h^1 \sim \mathcal{O}(\epsilon k) \tag{8.6}$$

$$\partial^2 h^1 \sim \mathcal{O}\left(\epsilon k^2\right). \tag{8.7}$$

The term $R^{(0)}_{\mu\nu} \sim \eta\partial^2\eta$ is a long-wavelength one; the term $R^{(1)}_{\mu\nu} \sim \eta\partial^2\eta^{(1)}\mathcal{O}\left(\epsilon k^2\right)$ is a short wavelength one; the term

$$R^{(2)}_{\mu\nu} \sim \eta\partial^2 h^{(2)} + h^{(1)}\partial^2 h^{(1)} \sim \mathcal{O}\left(\epsilon^2 k^2\right) \tag{8.8}$$

has components that are both short and long wavelength.

We can then formally separate the EFE: the long-wavelength part reads

$$R^{(0)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\text{long}} + 8\pi\left[\overline{T}_{\mu\nu}\right]\text{long} \tag{8.9}$$

$$, \tag{8.10}$$

while the short wavelength part reads

$$R^{(1)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\text{short}} + 8\pi\left[\overline{T}_{\mu\nu}\right]\text{short} \tag{8.11}$$

$$. \tag{8.12}$$

The term $-\left[R^{(2)}_{\mu\nu}\right]^{\text{long}}$ can be interpreted as $\tau_{\mu\nu}$, the feedback of the GWs on the background curvature.

On the other hand, the short-wavelength part will describe the propagation of GWs on a generic background.

How do we separate the scales? As discussed by Brill and Hartle, as well as Isaacson, we use the averaging procedure:

$$\left\langle S_{\mu\nu} \right\rangle = \int \mathrm{d}^4 x \, \eta^{\mu'}_{\alpha}(x, x') \eta^{\nu'}_{\beta}(x, x') S_{\mu'\nu'}(x') f(x, x') \sqrt{\eta(x')} \,, \tag{8.13}$$

where f is a filter function which vanishes for $\lambda \gg L$; the functions η transport the $S_{\mu\nu}$ around a point.

This is described in some detail in **[misnerGravitation1973]**. It generalizes the Isaacson tensor:

$$\tau_{\alpha\beta} = -\left[R^{(2)}_{\mu\nu} \right]^{\text{long}} = -\left\langle R^{(2)}_{\mu\nu} \right\rangle \,. \tag{8.14}$$

We have two expansion parameters: ϵ and λ, where $\lambda/L \ll 1$. (Note that I write λ but it should be a barred lambda).

The equation equates terms with different powers of ϵ: we need some *consistency conditions*.

If $\overline{T}_{\mu\nu} = 0$, the GWs contribute to the background metric: $1/L^2 \sim \epsilon^2 k^2 = \epsilon^2/\lambda^2$. Therefore, $\epsilon = \lambda/L$.

With $\left[R^{(2)}_{\mu\nu} \right]^{\text{long}} \approx 0$, we have

$$\frac{1}{L^2} \sim \epsilon^2 k^2 + (\text{matter}) \gg \epsilon^2 k^2 \,, \tag{8.15}$$

so $\epsilon \ll \lambda/L$. In this sense we want ϵ to be small. Note that for a flat background, $1/L = 0$ strictly!

So, no gravitational waves with finite amplitude exist as a perturbation of flat spacetime.

About the short wavelengths: GW propagation in general is given by

$$\begin{aligned} 0 &= \left[R^{(1)}_{\mu\nu} \right]^{\text{short}} + \text{`matter terms'} \qquad (8.16) \\ &\approx \Box_\eta \overline{h}_{\mu\nu} + \text{`matter terms'} \qquad (8.17) \end{aligned}$$

in a "generalized" Hilbert gauge.

9 Multipolar expansion and STF formalism

Monday 2021-6-7, compiled 2021-06-21

Suppose we have a static potential, which obeys $\triangle\phi = 4\pi\rho$ for some static source with energy density ρ which is localized within a radius R. The procedure for solving this kind of problem, like in electromagnetism, is through Green's functions. If we take a function G such that $\triangle G(x - y) = 4\pi\delta(x - y)$, then $G(x - y) = 1/\left|x - y\right|$; then

$$\phi = \triangle^{-1}\rho = \int \mathrm{d}^3 y \, G(x - y)\rho(y) = \int \mathrm{d}^3 y \, \frac{\rho(y)}{\left|x - y\right|} \,. \tag{9.1}$$

We can make this formal solution clearer by introducing spherical harmonics and expressing the solution as a series of multipoles of the source:

$$L^2 Y_{\ell m} = -\ell(\ell + 1) Y_{\ell m} \tag{9.2}$$

are eigenfunctions of L^2, where

$$\triangle = r^{-2}\partial_r\left(r^2\partial_r\right) + r^{-2}L^2(\theta,\varphi)\,. \tag{9.3}$$

In terms of these,

$$\frac{1}{\left|x-y\right|} = \sum_{\ell=0}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r'^{\ell}}{r^{\ell+1}}Y^*_{\ell m}(\theta',\varphi')Y_{\ell m}(\theta,\varphi)\,, \tag{9.4}$$

where $r' = \left|y\right|$ and $r = |x|$, and similarly for the angles.

For the exterior solution, a function in the form $Y_{\ell m}r^{-(\ell+1)}$ is a solution of the equation $\triangle(Y_{\ell m}r^{-(\ell+1)}) = 0$. Therefore, the exterior solution reads

$$\phi^{\text{ext}} = \sum_{\ell=0}\sum_{m=-\ell}^{\ell}\frac{Q_{\ell m}}{2\ell+1}\frac{Y_{\ell m}}{r^{\ell+1}}\,, \tag{9.5}$$

a linear combination with coefficients $Q_{\ell m}$.

The coefficients are determined by **matching** the interior solution:

$$\triangle^{-1}\rho = \sum_{\ell}\sum_{m}\int \mathrm{d}^3y\,\frac{r'^{\ell}}{2\ell+1}\rho(y)Y^*_{\ell m}(\theta',\varphi')\frac{Y_{\ell m}}{r^{\ell+1}}\,. \tag{9.6}$$

Therefore, the coefficients

$$Q_{\ell m} = \int \mathrm{d}^3y\, r'^{\ell}Y^*_{\ell m}\rho \tag{9.7}$$

are the **multipoles**. For example, with $\ell = m = 0$ we recover $Q_{00} = \int \mathrm{d}^3y\,\rho$, the "mass".

This procedure can be applied in Cartesian coordinates, which leads us to introduce the STF tensors.

We start by Taylor expanding the Green function in y:

$$\frac{1}{\left|x-y\right|} = f(y) = f(0) + y^i\left.\frac{\partial f}{\partial y^i}\right|_0 + y^iy^j\left.\frac{\partial^2 f}{\partial y^i\partial y^j}\right|_0 + \mathcal{O}\left(y^3\right)\,. \tag{9.8}$$

Now,

$$\left.\frac{\partial f}{\partial y^i}\right|_{y=0} = \frac{\partial}{\partial y^i}\left.\left(\frac{1}{\left|x-y\right|}\right)\right|_{y=0} = -\frac{\partial}{\partial x^i}\left.\left(\frac{1}{\left|x-y\right|}\right)\right|_{y=0}\,, \tag{9.9}$$

which generalizes to

$$\frac{\partial^L f}{y^{i_1}\dots y^{i_L}} = (-)^L\frac{\partial^L}{x^{i_1}\dots x^{i_L}}\frac{1}{|x|}\,, \tag{9.10}$$

so we can rewrite $1/\left|x-y\right|$ in terms of derivatives of $1/|x|$:

$$\frac{1}{\left|x-y\right|} = \sum_{\ell=0}\frac{(-)^{\ell}}{\ell!}y^{i_1}\dots y^{i_L}\underbrace{\frac{\partial}{\partial x^{i_1}}\dots\frac{\partial}{\partial x^{i_L}}\frac{1}{|x|}}_{\text{STF}}\,, \tag{9.11}$$

and the highlighted terms are STF in the sense that they are

1. symmetric in the indices $i_1 \dots i_L$;
2. trace-free, in that the contraction of that object with $\delta^{i_a i_b}$ is a bunch of derivatives acting on $\triangle(1/|x|) = 0$.

If we plug this into the initial equation, we get

$$\triangle^{-1}\rho = \int \mathrm{d}^3y\,\rho \sum_\ell \frac{(-)^\ell}{\ell!} y^{i_1} \dots y^{i_L} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_L}} \frac{1}{|x|} \tag{9.12}$$

$$= \sum_{\ell=0} \frac{(-)^\ell}{\ell!} \underbrace{\int \mathrm{d}^3y \left(\rho y^{i_1} \dots y^{i_L}\right)}_{\text{Cartesian multipoles of } \rho} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_L}} \frac{1}{|x|}\,. \tag{9.13}$$

These multipoles read

$$Q_{i_1 \dots i_L} = \int \mathrm{d}^3y\,\rho y^{<i_1} \dots y^{i_L>}\,, \tag{9.14}$$

where the index brackets $<>$ denote trace-free symmetrization — this can be done since any traceful or antisymmetric part vanishes by contraction with the STF object.

There must be a parallel between the spherical harmonics $Y_{\ell m}$ and this object $y^{<i_1} \dots y^{i_L>}$! We can construct such an object like

$$T^{<ij>} = T^{(ij)} - \frac{1}{3} T \delta^{ij}\,. \tag{9.15}$$

In order to get the relation we seek, we need the relation

$$\partial_i r = \partial_i (x^j x_j)^{1/2} = \frac{x_i}{r} = n_i\,, \tag{9.16}$$

while

$$\partial_i n_j = \frac{1}{r}(\delta_{ij} - n_i n_j)\,. \tag{9.17}$$

With these, we find that

$$\partial_i \frac{1}{r} = -\frac{n_i}{r^2} \qquad \text{and} \qquad \partial_i \partial_j \frac{1}{r} = -\frac{3}{r^3} n^{<i} n^{j>}\,, \tag{9.18}$$

and in general

$$\partial_{i_1} \dots \partial_{i_L} \frac{1}{r} = (-)^\ell \frac{(2\ell - 1)!!}{r^{\ell+1}} n^{<i_1} \dots n^{i_\ell>}\,, \tag{9.19}$$

therefore

$$y^i = r' n'^i\,, \tag{9.20}$$

so

$$\frac{1}{|x-y|} = \sum_\ell \frac{(2\ell-1)!!}{\ell!} \frac{r'^\ell}{r^{\ell+1}} n'^{i_1} \dots n'^{i_\ell} n^{<i_1} \dots n^{i_\ell>}\,. \tag{9.21}$$

This means that the Taylor expansion above can also be written as

$$\triangle^{-1}\rho = \sum_{\ell=0} \frac{(2\ell-1)!!}{\ell!} \frac{1}{r^{\ell+1}} Q_{i_1\dots i_\ell} n^{<i_1} \dots n^{i_\ell>} . \tag{9.22}$$

Why do we use unit vectors? We want a connection between the spherical coordinates and the Cartesian ones.

It is a theorem that the functions

$$F_\ell(\hat{n}) = Q_{i_1\dots i_\ell} n^{<i_1} \dots n^{i_\ell>} \tag{9.23}$$

are eigenfunctions of L^2 (the angular part of $\triangle$) with eigenvalues $\Lambda = -\ell(\ell+1)$. The proof of this can be found in the notes — it is a direct calculation.

This means that there exist some cartesian tensors

$$\mathcal{Y}^{\ell m}_{i_1\dots i_\ell} \tag{9.24}$$

such that

$$Y_{\ell m}(\theta, \varphi) = \mathcal{Y}^{\ell m}_{i_1\dots i_\ell} n^{<i_1} \dots n^{i_\ell>} . \tag{9.25}$$

The θ, φ dependence is contained inside of $\hat{n} = \hat{n}(\theta, \varphi)$.

What we have found is an alternative way of representing spherical harmonics and rotations.

It is a theorem that the functions $\mathcal{Y}^{\ell m}_{i_1\dots i_\ell}$ form a basis for rank-ℓ STF tensors:

$$T_{i_1\dots i_\ell} = \sum_{m=-\ell}^{\ell} T_{\ell m} \mathcal{Y}^{\ell m}_{i_1\dots i_\ell} . \tag{9.26}$$

The components $T_{\ell m}$ are called **spherical components** of $T_{i_1\dots i_2}$.

Some useful properties are:

$$T_{i_1\dots i_\ell} n^{i_1} \dots n^{i_\ell} = \sum_{m=-\ell}^{\ell} T_{\ell m} Y_{\ell m} ; \tag{9.27}$$

the components are calculated like

$$T_{\ell m} = \frac{4\pi\ell!}{(2\ell+1)!!} T_{i_1\dots i_\ell} \left(\mathcal{Y}^{i_1\dots i_\ell}_{\ell m} \right)^* . \tag{9.28}$$

Under rotations, $T_{\ell m}$ transform like $Y^*_{\ell m}$. This is because a change in $Y_{\ell m}$ must be compensated.

In summary, rank ℓ STF tensors have $2\ell+1$ components and are an irreducible representation of $SO(3)$.

In multi-index notation we have

$$\partial_L \frac{1}{r} \propto \frac{1}{r^{\ell+1}} n^L . \tag{9.29}$$

See the notes for the generalization to the full h_{ij} in linear theory. Do the green exercise:

In order to move to GR from linear theory we need to substitute $T_{\mu n}$ with the Isaacson tensor $\tau_{\mu\nu}$.

Monday 2021-6-14, compiled 2021-06-21

Last time we discussed the STF decomposition as a way to do spherical harmonics in Cartesian coordinates.

The very nice thing is that the general solutions we find in this way are not only valid for flat spacetime: if we substitute the stress-energy tensor appropriately (including the Isaacson tensor), we can apply them to any background.

10 Multipolar expansion and tensor spherical harmonics

The TT expression is

$$h_{ij}^{TT} = \frac{G}{c^4}\frac{1}{r}\Lambda_{ij}^{kl}\sum_{a=0}\frac{1}{a!}\partial_t^a S^{kli_1\dots i_a}(u)n_{i_1}\dots n_{i_a}\,, \tag{10.1}$$

where

$$S^{kli_1\dots i_a} \sim \int \mathrm{d}^3x\, T^{kl}x^{i_1}\dots x^{i_a}\,. \tag{10.2}$$

There is a STF formula which we discussed last time.

Finally, there is a tensor spherical harmonics expression:

$$h_{ij}^{TT} = \frac{G}{c^4}\frac{1}{r}\sum_{\ell=2} sum_{m=-\ell}^{\ell}\left[u_{\ell m}(n)\left(Y_{\ell m}^{E2}\right)_j(\theta,\varphi) + v_{\ell m}(n)\left(Y_{\ell m}^{B2}\right)_{ij}(\theta,\varphi)\right], \tag{10.3}$$

where the Y are called tensor spherical harmonics, of electric and magnetic type, while the u and v are coefficients.

The relation between these three expression is as follows: using the orthogonality property of the $Y_{\ell m}$ we can extract the coefficients. A sketch of the calculation is as follows, if $1a$, $1b$ and $1c$ are the three alternative expressions:

$$\int (1a)\left(Y_{\ell m}^{E2}\right)_{ij}^{*}\mathrm{d}\Omega = \int (1c)\left(Y_{\ell m}^{E2}\right)_{ij}^{*}\mathrm{d}\Omega = u_{\ell m}\,, \tag{10.4}$$

whose first component will read

$$\sum_{a=0}\frac{1}{a!}\partial_t^a S^{k\ell i_1\dots i_a}\underbrace{\int \mathrm{d}\Omega\,\left(Y_{\ell m}^{E2}\right)_{ij}^{*}\Lambda_{ij}^{k\ell}n_{i_1}\dots n_{i_a}}_{=\mathcal{Y}_{i_1\dots i_a}^{\ell m*}+\mathcal{O}\left(c^{-2}\right)}\,, \tag{10.5}$$

where the brace identification is the result of a long calculation.

The result is then

$$\sum_{a=0}\frac{1}{a!}\partial_t^a S^{k\ell i_1\dots i_a}\mathcal{Y}_{i_1\dots i_a}^{\ell m*} = \frac{\mathrm{d}^\ell}{\mathrm{d}u^\ell}M^{i_1\dots i_\ell}\mathcal{Y}_{i_1\dots i_\ell}^{\ell m*} \sim \int \mathrm{d}^3x\, r^\ell T^{00}Y_{\ell m}^{*}\,. \tag{10.6}$$

Thinking "quantum mechanically", we have $\vec{J} = \vec{L} + \vec{S}$. We define the Tensor Spherical Harmonics Y of a field of spin s as the simultaneous eigenfunctions of these operators: neglecting indices for the moment, we have

$$J^2 Y = j(j+1)Y \tag{10.7}$$
$$J_z Y = j_z Y \tag{10.8}$$
$$L^2 Y = L(L+1)Y \tag{10.9}$$
$$S^2 Y = s(s+1)Y\,. \tag{10.10}$$

In the spin-1/2 case this is a spinor.

These objects read

$$\left|jj_z\right\rangle = \sum_{L_z=-L}^{L} \sum_{s_z=-s}^{s} \left\langle LL_z s s_z \middle| j s_z \right\rangle \left|LL_z s s_z\right\rangle\,, \tag{10.11}$$

and the total angular momentum obeys $|L - s| \leq j \leq L + s$.

How do we construct these eigenfunctions? We take the scalar eigenfunctions of the angular momentum

$$L^2 Y_{LL_z} = L(L+1) Y_{LL_z}\,, \tag{10.12}$$

and the spin eigenfunctions (ignoring indices)

$$S^2 X_{ss_z} = s(s+1) X_{ss_z}\,, \tag{10.13}$$

we can make use of the Clebsh-Gordan coefficients:

$$Y = \sum_{L_z} \sum_{s_z} \left\langle LL_z s s_z \middle| j s_z \right\rangle Y_{LL_z} X_{SS_z}\,. \tag{10.14}$$

Let us particularize to the spin-1/2 case, in which $X_{1/2} = \left\{ (0,1)^\top, (1,0)^\top \right\}$. The vector spherical harmonics are

$$X_{1\pm1} = \mp \frac{1}{\sqrt{2}} \left(\hat{x} \pm i \hat{y} \right)\,, \tag{10.15}$$

while $X_{10} = \hat{z}$.

The solutions of $\Box v^i = 0$ can be represented as an expansion in $Y_{s=1}$. These harmonics are usually combined in a new orthonormal basis:

$$Y^R_{jj_z} = \sqrt{2j+1} \left[j^{1/2} Y^{j-1}_{jj_z} - (j+1)^{1/2} Y^{j+1}_{jj_z} \right] = Y_{jj_z} \hat{n} \tag{10.16}$$
$$Y^E_{jj_z} = \sqrt{2j+1} \left[(j+1)^{1/2} Y^{j-1}_{jj_z} + j^{1/2} Y^{j+1}_{jj_z} \right] = \sqrt{j(j+1)} r \cdot \nabla Y_{jj_z} \tag{10.17}$$
$$Y^B_{jj_z} = i Y^j_{jj_z} = \hat{n} \times Y^E_{jj_z}\,. \tag{10.18}$$

The direction along the propagation direction $\hat{n}$ is longitudinal, and orthogonal to it we have electric and magnetic types. Under parity the electric type transforms as $(-)^\ell$, the magnetic type transforms as $(-)^{\ell+1}$.

In terms of notation, we move from (j, j_z) to ℓ, m.

The solution of the vector wave operator reads

$$V^i(t,r,\theta,\varphi) = \sum_{\ell=0} \sum_{m=-\ell}^{\ell} R_{\ell m}(t,r) \left(Y^R_{\ell m}(\theta,\varphi) \right)^i + \sum_{\ell} \sum_{m} E_{\ell m}(t,r) \left(Y^E_{\ell m}(\theta,\varphi) \right)^i + \sum_{\ell} \sum_{m} B_{\ell m} \left(Y^B_{\ell m}(\theta,\varphi) \right)^i . \tag{10.19}$$

How do we use this for electromagnetism? The vector potential A^i is described by such a vector SH decomposition, with $R_{\ell m} = 0$, while $E_{\ell m}$ and $B_{\ell m}$ are the components of the electromagnetic field.

For the spin-2 case we have harmonics going from Y^{j-2} to Y^{j+2}. These can be used to make a new orthornormal basis:

$$Y^{S0}_{\ell m}, Y^{E1}_{\ell m}, Y^{E2}_{\ell m}, Y^{B1}_{\ell m}, Y^{B2}_{\ell m}, . \tag{10.20}$$

The generic 2-tensor is a combination of these, but since the graviton is massless only the two transverse ones matter — E2 and B2.

GW have a h_+ and $h_\times$ polarization — these are precisely related to these two fundamental transverse modes.

In alternative theories of gravity this might not be the case; there can be up to 6 multipole polarizations.

The geodesic deviation equation reads

$$\ddot{x}_i = -R_{0i0j} x^j = S_{ij} x^i = \begin{bmatrix} A_S + A_+ & A_\times & A_1 \\ 0 & A_S - A_+ & A_2 \\ 0 & 0 & A_L \end{bmatrix} x^j . \tag{10.21}$$

In the regular GR case we only have A_+ and $A_\times$; in an alternative metric theory of gravity the other 4 polarizations may be nonzero.

This is something which can be tested with a network of interferometers!

11 Post-Newtonian formalism

Monday 2021-6-21, compiled 2021-06-21

We know that the scaling of the perturbation is typically

$$h \sim \frac{R}{D} \frac{GM}{c^2 R} \left(\frac{v}{c} \right)^2 . \tag{11.1}$$

The assumptions for this linearized gravity quadrupole formula are:

1. fixed background;
2. v is small;
3. background and source are independent (this is true for non-selfgravitating sources).

However: this formula tells us that the most significant sources of GW are indeed self-gravitating, and not described by it!

So, how do we move forward, and get a formalism for compactness GM/c^2R approaching $1/2$, such as BHs or NSs?

A strongly self-gravitating source will also typically have high velocity, because of the virial theorem:

$$\left(\frac{v}{c}\right)^2 \sim \frac{2GM}{c^2R} = \frac{R_s}{R} \,. \tag{11.2}$$

The "background" for a strongly self-gravitating source is dependent on the source! It is not really a background.

The idea behind the PN formalism is to iteratively solve the Einstein equations, as an expansion in powers of $\epsilon = v/c$.

We will still refer to sources with *compact support* $r < R$, and assume that T^{00} is the dominant component: $\left|T^{00}/T^{0i}\right| = \mathcal{O}(\epsilon)$ and $\left|T^{00}/T^{ij}\right| = \mathcal{O}\left(\epsilon^2\right)$.

The expansion to order $\mathcal{O}\left(\epsilon^{2n}\right) = \mathcal{O}\left((v/c)^{2n}\right)$ is called the n-PN order expansion.

So we can talk about 1PN, 1.5 PN, 2.5PN and so on.

A *historical remark*: the PN formalism started with Einstein; Landau and Lifshitz already included it in their book, also De Sitter and Chandrasekhar worked on it.

Einsten-Infeld-Hoffmann developed the 1PN Lagrangian for the motion of N particles. Peter and Hathews (1963).

The formalism was really nailed down systematically by Blanchet and Damour. Also, Will and Wisemann worked on it in the 90s.

Naîvely, we'd work as such: calculate the motion to ϵ^n, and then use linear theory to calculate the GWs emitted. This is wrong: the background is nontrivial, there is a complicated dependence through the Einstein equations.

WE emission subtracts energy to the motion, and GWs at a certain PN order are sources of GW at higher orders through the Isaacson tensor.

The equation $\mathrm{d}E/\mathrm{d}t = P_{gw}$ is an ansatz: this is used but it not guaranteed to hold at each PN order.

Let us start with the metric at 1PN order:

$$g_{00} = -1 - 2\phi = -1 + \frac{2U}{c^2} \tag{11.3}$$

$$g_{0i} = 0 \tag{11.4}$$

$$g_{ij} = \delta_{ij} \,, \tag{11.5}$$

where U is the solution to the Newtonian Poisson equation:

$$U = \frac{G}{c^2} \int \mathrm{d}^3y \, \frac{T^{00}(y)}{|x-y|} \,. \tag{11.6}$$

Now, we have seen that different components of $T_{\mu\nu}$ have different orders of c, so different metric coefficients must be expanded up to different powers, determined by how they appear in the EFE.

Component	Newton (0PN)	1PN	2PN
g_{00}	$-1+g_{00}^{(2)}$	$g_{00}^{(4)}$	$g_{00}^{(6)}$
g_{0i}	0	$g_{0i}^{(3)}$	$g_{0i}^{(5)}$
g_{ij}	δ_{ij}	$g_{ij}^{(2)}$	$g_{ij}^{(4)}$

Figure 1:

If g_{00} is expanded to order ϵ^n, then g_{0i} must be expanded to order ϵ^{n-1} because it has a power ϵ intrinsically in it, and similarly g_{ij} must be expanded to the power ϵ^{n-2}.

Understand this better. Look at Maggiore.

In terms of time reversal symmetry, g_{00} and g_{ij} are even while g_{0i} are odd.

This is correct up to when we must include radiation reaction, but it turns out that that comes in at 2.5PN order.

So, in order to write the metric at 1PN order, we can match the EFE at each order taking into account the power counting.

Derivatives have orders $\partial_t = \mathcal{O}(v)\partial_i$, while

$$-\frac{1}{c^2}\partial_{tt} + \triangle = \left[1 + \mathcal{O}(\epsilon)\right]\triangle\,, \tag{11.7}$$

therefore the retardation effects are "higher order".

Therefore, the lowest PN solutions are typically given in terms of *instantaneous potentials* (solution of Poisson equations).

This is expected: the PN expansion reads

$$F(n) = F(t - \frac{r}{c}) = F(t) - \frac{r}{c}\dot{F}(t) + \frac{r^2}{2c^2}\ddot{F}(t) + \dots\,. \tag{11.8}$$

In Fourier terms:

$$\widetilde{F} \approx \widetilde{F}\left(1 - \frac{r\omega}{c} + \frac{r^2\omega^2}{c^2} + \dots\right) \tag{11.9}$$

$$= \widetilde{F}\left(1 - \frac{r}{\lambda} - \frac{r^2}{2\lambda^2}\right)\,. \tag{11.10}$$

the lambdas are barred — characteristic length.

This means that this is also an expansion for nearby fields: the expansion is in small velocities $\epsilon = v/c \ll 1$, as well as in the *near-zone* $r/\lambda \ll 1$.

Therefore, the PN expansion is **not valid to compute GWs far away from the source**!

Let us define a **near zone** $r \ll L$, but $L \gg R$, such that $r/\lambda \ll 1$, such that the aforementioned expansion makes sense.

We also have a **far zone** $R < r < \infty$, and an **overlap zone** $R < r < L$ in which the near and far zones overlap.

Back to the 1PN metric: we impose Harmonic gauge, so that $\partial_\mu \left(\sqrt{-g} g^{\mu\nu}\right) = 0$, and we start from the 0PN equation $\triangle U = -(8\pi G/c^4) T^{00}_{(0)}$. Then, to 1PN order we get:

$$\triangle g^{(2)}_{ij} = -\frac{8\pi G}{c^4} \delta_{ij} T^{00}_{(0)} \tag{11.11}$$

$$\triangle g^{(3)}_{0i} = \frac{16\pi G}{c^4} T^{0i}_{(1)} \tag{11.12}$$

$$\triangle g^{(4)}_{00} = \dots , \tag{11.13}$$

so, assuming that $T^{\mu\nu}$ is known, these can be solved in terms of Green's functions. After all the calculations one finds

$$\Box_\eta V = -\frac{4\pi G}{c^4} \underbrace{\left(T^{00} + T^{ii}\right)}_{\text{active gravitational mass density}} \tag{11.14}$$

$$\Box_\eta V^i = \dots , \tag{11.15}$$

where we have the 1PN potential V which defines the metric:

$$g_{00} = -1 + \frac{2U}{c^2} + \frac{2V^2}{c^4} \tag{11.16}$$

$$g_{0i} = -\frac{4V_i}{c^3} \tag{11.17}$$

$$g_{ij} = \delta_{ij} \left(1 + \frac{2V}{c^2}\right) . \tag{11.18}$$

The Einstein-Infeld-Hoffmann Lagrangian is the N-particle one for this metric: the corresponding stress-energy tensor is

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \sum_a \frac{\mathrm{d}\tau_a}{\mathrm{d}t} m_a \frac{\mathrm{d}x^\mu_a}{\mathrm{d}t} \frac{\mathrm{d}x^\nu_a}{\mathrm{d}t} \delta(x - x_a) . \tag{11.19}$$

The action reads

$$S = -mc^2 \int \mathrm{d}t \sqrt{-g_{00} - 2g_{0i}\frac{v^i}{c} - g_{ij}\frac{v^i v^j}{c^2}} . \tag{11.20}$$

This can be written as $L = L_{\text{Newt}} + L_{1PN}$.

If one takes $N = 1$ they get the two-body problem, the EOM for a relativistic binary at 1PN order. This solution can be mapped to a Newtonian problem and solved: this has been done by Damour and Deruelle in 86, which is important since it was applied to the Hulse-Taylor pulsar: the equation they found was

$$\langle \dot\omega \rangle = 2.113\,53 \left(\frac{m_1 + m_2}{M_\odot}\right)^{2/3} \text{deg/yr} . \tag{11.21}$$

The orbital decay has been observed very accurately, which can be a way to determine the binary mass.

What are the difficulties? We get Poisson equations in the form

$$\triangle g_{\mu\nu}^{(n)} = \text{matter source} + \text{metric source } g_{\mu\nu}^{(n-1)} \,. \tag{11.22}$$

The second term does not have compact support! The Poisson integrals used so far are not good solutions. We need different boundary conditions.

For example, consider $\triangle u = \rho = \text{const}$. The Poisson integral is not a solution since $u \neq 0$ at infinity: it diverges.

However, a solution exists: $u(r) = -(1/6)\rho r^2$. We need other formal solutions to the Poisson equation.

How is this solved? We do it by inverting the Poisson equation with boundary conditions by a suitable procedure which involves regularization of the divergences and analytic continuation. We will not explore this in detail, but it is good to know that the problem exists.

Another difficulty is the following: the PN expansion of the metric potential is valid in the near zone but it blows up for large $r \to \infty$. It is in the form

$$F(\epsilon, r) = \sum_n c_n(r)\epsilon^n \,. \tag{11.23}$$

This is an expansion with two scales, and the series is *not* uniformly convergent. This is analogous to perturbation theory in quantum mechanics: asymptotic series.

The residuals eventually blow up, but there is an optimum n for the expansion. The solution is to perform a post-Minkowskian expansion in the far zone:

$$F = \sum_n G^n F_n \,, \tag{11.24}$$

and match the PN expansion in the near zone.

Also, there is the problem of **backreaction**, GW influence on the motion. This breaks the power-counting.

When does it enter the game? We know that

$$E = K + V \approx -\frac{V}{2} + V = \frac{V}{2} = -K = -\frac{M}{2}v^2 \,, \tag{11.25}$$

therefore $\dot{E} \sim -Mv\dot{v}$.

Further, we know that

$$P_{gw} = \dot{E}_{gw} \approx \frac{GM^2}{c^5}\frac{v^6}{r} \,, \tag{11.26}$$

therefore

$$\dot{v} \approx \frac{GM}{r^2}\left(\frac{v}{c}\right)^5 \,, \tag{11.27}$$

which gives the correct order of 5, meaning 2.5PN.

What are the **Relaxed EFE**? We define

$$\mathfrak{h}^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}\,, \tag{11.28}$$

a new field in the full theory. We then rewrite the EFE in terms of it, using the gauge $\partial_\mu \mathfrak{h}^{\mu\nu} = 0$ (equivalent to harmonic gauge).

The EFE then read

$$\Box \mathfrak{h}^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \tag{11.29}$$

$$= \frac{16\pi G}{c^4}\left[-gT^{\mu\nu} + \tau_u^{\mu\nu} + \tau^{\mu\nu}[\mathfrak{h}^{\alpha\beta}]\right], \tag{11.30}$$

where τ_u is the Landau-Liftschitz pseudotensor. These are formally similar to the linearized equations, and in the weak-field limit $\mathfrak{h}^{\mu\nu} = -\overline{h}^{\mu\nu}$.

All the nonlinearity is still there though, this makes no approximations: they are just made to look that way.

This τ satisfies $\partial_\mu \tau^{\mu\nu} = 0$ (notice the flat derivative). Also, these are called "relaxed" because a solution of the relaxed equations alone does not imply the conservation of $\tau^{\mu\nu}$: one needs to add the gauge.

The formal solution of these REFE is

$$\mathfrak{h}^{\mu\nu} = -\frac{4G}{c^2}\int \mathrm{d}^3y\, \frac{\tau^{\mu\nu}(t - |x-y|/c, y)}{|x-y|}\,, \tag{11.31}$$

but we do not truly know τ; however we can solve this iteratively.

We start by finding a solution for the REFE in the far region with the multipolar formula:

$$\mathfrak{h}^{\mu\nu} = \sum_n \mathfrak{h}^{\mu\nu}_{(n)} G^n\,, \tag{11.32}$$

there for $r > R$ we have the unspecified source multipole moments $\mathfrak{h}^{\mu\nu}_{(n)}$.

Then we find a solution for the REFE in the near zone:

$$\mathfrak{h}^{\mu\nu} = \sum_n \frac{1}{c^n}\mathfrak{h}^{\mu\nu}_{(n)}\,. \tag{11.33}$$

Here, for $r < L$, we have ${}^{(n)}\mathfrak{h}^{\mu\nu}$, multipoles of $\tau^{\mu\nu}$. We match the two in the overlapping zone to determine τ.

Finally, we compute the TT solution far away, with a gauge transformation from harmonic to radiative coordinates: from $(I_L, J_L, W_L, \dots)$ to (U_L, V_L), the radiative multipoles of h_{ij}^{TT}.

The good thing is that this can all be done with a finite amount of terms, however in general one does not know how far the expansion can be pushed.

What does the overlap region look like?