Low Energy Theoretical Astroparticle Physics

Jacopo Tissino

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1 Neutrino physics

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Neutrinos are important because:

1. being neutral, they can trace sources;

   What does that mean?

2. they are connected with new physics;
3. there are open questions about their nature (Dirac or Majorana);
4. there could be a CP violation in the lepton sector;
5. they affect cosmology, with Baryon Acoustic Oscillations in the CMB, as well as the large scale structure.

nu.to.infn.it, "Neutrino Unbound"; Giunti et al, Giunti and Kim.
Outline of the course:

1. Dirac equation;
2. gauge theories;
3. standard EW model;
4. fermion masses and mixing;
5. neutrino oscillations in vacuo;
6. neutrino oscillations in matter;
7. current neutrino phenomenology;
8. extra on statistics and data analysis.

We know that neutrino masses are less than an eV; there are (at least) three flavours: ν_e, ν_μ and ν_τ, typically organized in doublets with the corresponding charged lepton.

These neutrinos can have charged interactions with a $W^\pm$ boson, as well as neutral interactions with a Z boson (which is connected with photon production).

Looking at these decays tells us about the number of (interacting) neutrino families: $N_\nu = 3$.

There are also bounds from cosmology: both in the CMB spectrum and in primordial nucleosynthesis.

Neutrinos are chiral in nature: only left-handed neutrinos (with negative helicity) seem to interact.

We know that a left-handed particle has a right-handed component of order $\beta \sim m/E$.

At order m^2/E there are neutrino flavor oscillations.

The Dirac equation

We start from Lorentz transformations: linear transformations which preserve the Minkowski metric. They are rotations and boosts. They can be written in terms of infinitesimal transformations: for example,

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \approx \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} \right) \begin{bmatrix} y \\ z \end{bmatrix}, \tag{1.1}$$

therefore

$$\Lambda = \mathbb{1} + i\,\mathrm{d}\theta \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \mathcal{O}\left(\theta^2\right) = e^{i\theta J_1} + \mathcal{O}\left(\theta^2\right). \tag{1.2}$$

The same holds for the boosts:

$$\Lambda = \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix} = e^{iuk_1} \qquad \text{where} \qquad K_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{1.3}$$

With these generators we can make any transformation we want: a Lorentz transformation will be in general given by the composition of a rotation around $\vec{\omega} = \theta\hat{n}$ and a boost in $\vec{u} = u\hat{v}$. This will then read

$$\Lambda = \exp\left(i\left(\vec{\omega}\cdot\vec{J} + \vec{u}\cdot\vec{K}\right)\right). \tag{1.4}$$

In general these do not commute: $[J_i, J_k] \neq 0$, $[K_i, K_j] \neq 0$, $[J_i, K_j] \neq 0$. This, however, is a closed algebra: all these commutators are given in terms of other J, K matrices.

This construction is not only useful if we need to make complicated transformations; it is also useful as a theoretical mean to parametrize a general transformation.

Pauli was trying to generalize the Schrödinger equation for a spin-1/2 particle. It can be found by mapping $\vec{p} \to -i\vec{\nabla}$ and $E \to i\partial_t$ in the eigenvalue equation for a classical kinematic Hamiltonian $H = p^2/2m + V$.

This is classical, so we forget boosts: let us try to at least have the 4D J_i rotation algebra $[J_i, J_j] = i\epsilon_{ijk}J_k$ for two-component vectors: this is also obeyed by the 2D matrices $\sigma_i/2$, where

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.5}$$

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{1.6}$$

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{1.7}$$

So, we could think to have a spinor transform under

$$\xi' = \lambda\xi \qquad \text{where} \qquad \lambda = \exp\left(i\vec{\omega}\cdot\frac{\vec{\sigma}}{2}\right) . \tag{1.8}$$

In order for the momentum $\vec{p}$ to act on 2D objects we can use $\vec{p}\cdot\vec{\sigma}$.

The idea is to use minimal coupling: $\vec{p} \to \vec{p} - q\vec{A}$, and $\vec{V} \to \vec{V} + q\Phi$.

How can we generalize this to boosts? Not only is it possible to do, but there are two ways to do it.

The Lorentz group, to which the matrices Λ in $x' = \Lambda x$ belong, satisfies

$$\Lambda = \exp\left(i\left(\vec{\omega}\cdot\vec{J} + \vec{u}\cdot\vec{K}\right)\right) \tag{1.9}$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{1.10}$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k \tag{1.11}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k . \tag{1.12}$$

Spinors will transform with

$$\xi' = \Lambda_\xi \xi \tag{1.13}$$

$$\Lambda_\xi = \exp\left(i\frac{\vec{\sigma}}{2}\cdot\vec{J} + ?\right) \tag{1.14}$$

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2} , \tag{1.15}$$

so what do we need to add? We can do either $\vec{k} = \pm i\vec{\sigma}/2$, so we get

$$\Lambda_\xi = \exp\left(i\frac{\vec{\sigma}}{2}\cdot\vec{\omega} \pm \vec{u}\cdot\frac{\vec{\sigma}}{2}\right) . \tag{1.16}$$

The plus sign is for right-handed spinors, the minus sign is for left-handed ones. If we have a spinor at rest, we cannot determine its helicity. If we boost it, we still cannot determine it!

But, it cannot be in a superposition of things which transform in different ways.

The idea is then to have a direct sum, an object which contains both the left and right-handed components:

$$\xi = \begin{bmatrix} \xi_L \\ \xi_R \end{bmatrix} . \tag{1.17}$$

These will be given by boosting the rest-frame spinor ξ in two different ways:

$$\xi_R' = \exp\left(\vec{u} \cdot \frac{\vec{\sigma}}{2}\right)\xi \tag{1.18}$$

$$\xi_L' = \exp\left(-\vec{u} \cdot \frac{\vec{\sigma}}{2}\right)\xi . \tag{1.19}$$

The Dirac equation is a relation between the left- and right-handed components of a free spinor. In order to derive it, we need some relations:

$$\exp\left(\vec{u} \cdot \frac{\vec{\sigma}}{2}\right) = \cosh(u/2) + \hat{u} \cdot \vec{\sigma} \sinh(u/2) \tag{1.20}$$

$$(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot \left(\vec{a} \times \vec{b}\right) \tag{1.21}$$

$$\exp\left(i\vec{\omega} \cdot \frac{\vec{\sigma}}{2}\right) = \cos(\theta/2) + i\hat{u} \cdot \vec{\sigma} \sin(\theta/2) . \tag{1.22}$$

With these, we get

$$\exp\left(\pm\vec{u} \cdot \frac{\vec{\sigma}}{2}\right) = \frac{E + m \pm \vec{p} \cdot \vec{\sigma}}{\sqrt{2m(E+m)}} \tag{1.23}$$

$$\xi_{R,L} = \frac{E + m \pm \vec{p} \cdot \vec{\sigma}}{\sqrt{2m(E+m)}}\xi \tag{1.24}$$

$$\xi = \frac{E + m \mp \vec{p} \cdot \vec{\sigma}}{\sqrt{2m(E+m)}}\xi_{R,L} , \tag{1.25}$$

therefore we can put these equations together into

$$-m\xi_{R,L} + \left(E \pm \vec{p} \cdot \vec{\sigma}\right) \xi_{L,R} = 0 , \tag{1.26}$$

or

$$\begin{bmatrix} -m & E + \vec{p} \cdot \vec{\sigma} \\ E - \vec{p} \cdot \vec{\sigma} & -m \end{bmatrix} \begin{bmatrix} \xi_R \\ \xi_L \end{bmatrix} = 0 . \tag{1.27}$$

If $m = 0$, these decouple into two equations:

$$(p \pm \vec{p} \cdot \vec{\sigma})\xi_{L,R} = 0 . \tag{1.28}$$

Therefore, for a massless particle

$$\left(\hat{p} \cdot \sigma\right) = \pm\xi_{R,L} . \tag{1.29}$$

This is not the case when the particle is massive.

Helicity is the expectation value of $\hat{p} \cdot \vec{\sigma}$, chirality is "being ξ_L or ξ_R".

This is all in momentum space, but it can also be written in position space by switching the momentum to a derivative: we introduce the Dirac matrices

$$\gamma^0 = \begin{bmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{bmatrix} \qquad \text{and} \qquad \vec{\gamma} = \begin{bmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}, \tag{1.30}$$

so that the Dirac equation becomes

$$(\gamma^\mu p_\mu - m)\psi = 0\,, \tag{1.31}$$

where $\psi = (\xi_L, \xi_R)^\top$.

This can then also be written in terms of derivatives as $(i\gamma^\mu \partial_\mu - m)\psi = 0$.

Left- and right-handed spinors transform in the same way under rotations; they differ under boosts.

It is useful to introduce

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ : \tag{1.32}$$

if we introduce

$$P_{L,R} = \frac{1 \mp \gamma^5}{2} \tag{1.33}$$

this will select the left- or right-handed component of a spinor.

We have derived this result using the Weyl basis; Dirac used a different one.

If the spinor transforms with $\psi \to T\psi$ the Dirac matrices transform with $\gamma^\mu = T\gamma^\mu T^{-1}$.

In the Dirac basis, which is useful when one studies non-relatvistic particles, the matrices look like

$$\gamma^0 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{bmatrix} \qquad \text{and} \qquad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}, \tag{1.34}$$

while

$$\gamma^5 = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}. \tag{1.35}$$

In the nonrelativistic limit $\vec{p} \to 0$ the equation decouples into

$$\begin{bmatrix} E - m & 0 \\ 0 & E + m \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0\,, \tag{1.36}$$

so we have $E = \pm m$ solutions. Note that ξ and η are not left- and right-handed, this is a different basis. In Dirac's time this was seen as a tragedy.

The fact that these are actually particles and antiparticle can be properly explained in QFT; in the end, the four degrees of freedom correspond to the left- and right-handed components of particles and antiparticles.

We can write the particle and antiparticle solutions as

$$\psi_p = \begin{bmatrix} \xi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\xi \end{bmatrix} e^{-ip_\mu x^\mu} \qquad \text{and} \qquad \psi_A = \begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\eta \\ \eta \end{bmatrix} e^{+ip_\mu x^\mu} . \tag{1.37}$$

It is convenient to introduce a charge-conjugation operator:

$$\mathsf{C}(\psi) = \psi^c = i\gamma^2\psi^* . \tag{1.38}$$

This can be seen by proving that $C(\psi_P) = \psi_A$. Also, if we look at the minimally-coupled Dirac equation for a particle in an external EM field, and we apply the conjugation operator we find that the particle will obey the same equation, but with an opposite charge.

The last thing to introduce here is the adjoint spinor: we know how a 4-spinor ψ behaves under a Lorentz transformation. What is the object which transforms with the inverse transformation? If we denote it as $\overline{\psi}$, we will be able to write *invariant* objects like $\overline{\psi}\psi$.

It takes some time to prove, but it comes out to be

$$\overline{\psi} = \psi^\dagger\gamma^0 . \tag{1.39}$$

We can make invariant objects like $\overline{\psi}\psi$, or objects like $\overline{\psi}\gamma^\mu\psi$: it can be proven that the latter is *divergenceless*, $\partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0$, so it can be used to describe currents.

2 The standard electroweak model

Gauge theories

Tuesday
2021-12-7

Yesterday we wrote the equation for a free fermion:

$$\xi'_{R,L} = \exp\left(-i\vec{\omega}\cdot\frac{\vec{\sigma}}{2} \pm \vec{u}\cdot\frac{\vec{\sigma}}{2}\right)\xi_{R,L} . \tag{2.1}$$

The Dirac equation tells us that these are coupled:

$$\begin{bmatrix} -m & E+\vec{p}\cdot\vec{\sigma} \\ E-\vec{p}\cdot\vec{\sigma} & m \end{bmatrix}\begin{bmatrix} \xi_R \\ \xi_L \end{bmatrix} = 0 . \tag{2.2}$$

For nonzero mass these components are coupled, while in the zero-mass case they are decoupled and helicity is equal to chirality.

Helicity is the eigenvalue under $\hat{p}\cdot\vec{\sigma}$, chirality is the eigenvalue under γ^5.

What is the meaning of γ^5 in position space? Is it a parity transformation?

The coupling of a fermion to an external EM field can be represented with $\partial_\mu \to \partial_\mu + iqA_\mu$.

The SM Lagrangian can be written on a mug as

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\psi}\not{D}\psi + \text{h.c.} + i\psi_i y_{ij}\psi_j\phi + \text{h.c.} + \left|\not{D}\phi\right|^2 - V(\phi) . \tag{2.3}$$

We have scalars ϕ, fermions ψ and spin-1 gauge fields A_μ.

From a Lagrangian $L(q, \dot{q})$ we get Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0\,. \tag{2.4}$$

If the Lagrangian is, say, $L = m\dot{q}^2/2 - V(q)$ we get Newton's law $m\ddot{q} = -\vec{\nabla}V = F$.

This is classical; in field theory we have Lagrangians in the form

$$\partial_\mu \frac{\partial \mathscr{L}(\phi, \partial\phi)}{\partial\partial_\mu\phi_i} - \frac{\partial \mathscr{L}}{\partial \phi_i} = 0\,. \tag{2.5}$$

We will write the Yukawa Lagrangian with terms like $\psi\phi$, and the QED Lagrangian with terms like ψA_μ.

The equation of motion for a scalar field is the Klein-Gordon equation: $(\partial_\mu\partial^\mu + m^2)\phi = 0$. The Lagrangian giving rise to this is

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m\phi^2\,. \tag{2.6}$$

We have written the EOM for a free fermion, the Dirac equation $\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0$; the Lagrangian giving it is the Dirac Lagrangian

$$\mathscr{L} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi\,. \tag{2.7}$$

Again, we have a mass-like term and a kinetic-like term.

The term $m\overline{\psi}\psi$ can be expanded into left and right components:

$$m\overline{\psi}\psi = m\left(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R\right)\,. \tag{2.8}$$

What about the EM field? A_μ is gauge-dependent, but

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} \tag{2.9}$$

is not. The Lagrangian giving Maxwell's equations in a vacuum (so, $\Box A^\mu = 0$ in the appropriate gauge) is

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \tag{2.10}$$

The absence of a mass term means that the photon is massless.

What about interactions? We start from the Yukawa interaction between a fermion and a scalar. The Lagrangian, for starters, must contain the free terms for both:

$$\mathscr{L} = \underbrace{\overline{\psi}\left(i\gamma^\mu\partial_\mu - m_\psi\right)\psi + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_\phi\phi^2\right)}_{\text{free terms}} - \underbrace{g\overline{\psi}\psi\phi}_{\text{interaction}}\,. \tag{2.11}$$

The pictorial way to represent this is to draw quadratic terms for fermions like straight lines with an arrow, scalar fields as dashed lines, and cubic interaction terms like vertices.

What do fermion-photon interactions look like?

$$\mathscr{L}_{\text{QED}} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m_\psi\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - q\overline{\psi}\gamma^\mu\psi A_\mu\,. \tag{2.12}$$

The EOM for the electromagnetic field are Maxwell's equations with an external current $j^\mu = q\overline{\psi}\gamma^\mu\psi$.

This is the same Lagrangian we get if we do a minimal coupling substitution $\partial \rightarrow \partial + iqA$. It is invariant under gauge transformations.

These diagrams can be used to compute scattering amplitudes perturbatively.

The way to get a cross-section is to multiply the square modulus of the amplitude by the phase space term. The idea to get decay rates is similar.

The guiding principle to describe EW interactions is gauge invariance. The Yukawa Lagrangian has global phase invariance; what happens if we try to make a transformation $\psi \rightarrow e^{-iq\alpha(a)}\psi$? The term q is only introduced here for later convenience.

The timezone analogy for local gauge invariance! If we have the freedom to choose a phase locally, we must have carriers of information moving at the maximum possible speed, otherwise processes would be disrupted.

We start with $U(1)$ gauge invariance, as written above, but we will also need $SU(2)$ invariance, written as

$$\exp\left(-ig\vec{\theta}\cdot\vec{T}\right), \tag{2.13}$$

where $\vec{T} = \vec{\sigma}/2$, but we write them differently to not confuse them with spacetime rotations.

Introducing gauge invariance for a term $\overline{\psi}\left(i\not{\partial} - m\right)\psi$ will take us to the QED Lagrangian.

The mass term is already invariant, the problem is the kinetic one. The trick is to introduce a covariant derivative

$$\mathrm{D}_\mu = \partial_\mu + iqA_\mu\,. \tag{2.14}$$

If $\psi \rightarrow e^{-iq\alpha(x)}\psi$, also $\mathrm{D}_\mu\psi \rightarrow e^{-iq\alpha(x)}\mathrm{D}_\mu\psi$, as long as A_μ also transforms like $A_\mu \rightarrow A_\mu + \partial_\mu\alpha$.

Then, the Lagrangian

$$\mathscr{L} = \overline{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,. \tag{2.15}$$

A term like $m^2A_\mu A^\mu$ would violate gauge invariance, so this principle also gives an explanation as to why the photon is massless if we accept the gauge invariance principle.

Protons and neutrons were thought to be part of a doublet under isospin SU(2) transformations.

But, we have interactions like protons and neutrons interacting with electrons and neutrinos. So, one might think that electrons + neutrinos have isospin-like SU(2) symmetry as well.

We want to make the doublet ψ_1, ψ_2 invariant under SU(2) transformations as written above: how do we do it? Again, we redefine the derivative:

$$\partial_\mu \to \mathrm{D}_\mu - ig\vec{T} \cdot \vec{A}_\mu \,, \tag{2.16}$$

so we need to introduce three fields $\vec{A}_\mu$. These are the gauge bosons related to SU(2) gauge invariance.

These now transform like

$$\vec{A}_\mu \to \vec{A}_\mu - \partial_\mu \vec{\theta}(x) + g\vec{\theta} \times \vec{A}_\mu \,. \tag{2.17}$$

The presence of this coupling means that the field is charged.

The field tensor here is

$$\vec{F}_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} + g\vec{A}_\mu \times \vec{A}_\nu \,. \tag{2.18}$$

We then get a Lagrangian like

$$\mathscr{L} = \overline{\psi} \left(i\gamma^\mu \mathrm{D}_\mu - m \right) \psi - \frac{1}{4} \vec{F}^{\mu\nu} \cdot \vec{F}_{\mu\nu} \,. \tag{2.19}$$

In the ψ term we have $\psi\psi$ terms, as well as $\psi\psi A$ interactions. In this F^2 term we have quadratic, trilinear and quadrilinear terms in A!

The actual group for the EW theory is $SU(2)_L \otimes U(1)_Y$. The simplest EW world contains

1. 1 massive electron, with both right- and left-handed components;
2. 1 massless ν_e, with only the left-handed component;
3. EM interactions (we want to have a diagram describing how the electron interacts with a massless photon);
4. weak chiral interactions like $\nu_L e_L W^\pm$, where $W^\pm$ is massive.

The Higgs mechanism accomplishes this, and it also gives mass to fermions. If neutrinos have Majorana masses, the Higgs mechanism cannot give them mass.

Let us define a doublet $L = (e_L, \nu_{e,L})$, and a singlet $R = e_R$. The theory must then be invariant if we redefine

$$L \to L' = \exp\left(-ig\vec{\theta}(x) \cdot \vec{T} \right) L \tag{2.20}$$

$$R \to R' = R \,. \tag{2.21}$$

The doublet L corresponds to the $T_3 = \pm 1/2$ quantum numbers, while $T_3 = 0$ for the singlet.

Could the three bosons be some combinations of the photon and the two $W^\pm$ bosons? We know that the charge operator Q has eigenvalues 0, -1 for the doublet, but $\operatorname{Tr} Q \neq 0$ while $\operatorname{Tr} T_i = 0$ for all T_i.

Why would Q need to be able to be written as a function of the T_i? The proper answer lies in Nöther's theorem.

We therefore introduce a further hypercharge symmmetry:

$$L' = \exp\left(-ig'\beta(x)\frac{Y}{2}\right)L \tag{2.22}$$

$$R' = \exp\left(-ig'\beta(x)\frac{Y}{2}\right)R\,, \tag{2.23}$$

so that this generator commutes with the T_i, and indeed $[T_i, Y] = 0$. The only way to have this is $Y \propto Q - T_3$, and indeed $Y = 2(Q - T_3)$.

In the end, the Lagrangian must be chargeless.

The full transformation is therefore

$$L' = \exp\left(-ig\vec{\theta}(x)\cdot\vec{T} - ig'\beta(x)\frac{Y}{2}\right)L \tag{2.24}$$

$$R' = \exp\left(-ig'\beta(x)\frac{Y}{2}\right)R\,. \tag{2.25}$$

The procedure is then like before: we need to redefine the derivative, as

$$\mathrm{D}_\mu L = \left(\partial_\mu + ig\vec{T}\cdot\vec{A}_\mu - ig'\frac{Y}{2}B_\mu\right)L\,, \tag{2.26}$$

so we have four fields, the three $\vec{A}_\mu$ with their $\vec{F}_{\mu\nu}$ and B_μ with its $G_{\mu\nu}$.

The Lagrangian will then read

$$\mathscr{L} = \overline{L}i\gamma^\mu \mathrm{D}_\mu L + \overline{R}i\gamma^\mu \mathrm{D}_\mu R + \text{kinetic}\,, \tag{2.27}$$

but we cannot write mass terms like $m\overline{\psi}\psi$, which would be $\overline{L}R$ or $L\overline{R}$: the matrix dimensions don't match up!

So, everything's massless: the Higgs mechanism comes to the rescue. It gives masses to all the bosons except the photon, as well as giving mass to the leptons.

Suppose we have a $U(1)\otimes U(1)$ symmetry, spontaneously broken to $U(1)$. The thing we think of is a pencil about to fall on a table.

Thursday 2021-12-9

We have discussed the Dirac equation and SU(2) gauge invariance; now we will look at the Higgs mechanism and SSB, with the goal in mind to understand the Lagrangian written on the CERN mug.

We study this model in the context of an L doublet (ν_{eL}, e_L) and an R singlet (e_R).

The Higgs mechanism yields neutral currents as a "bonus".

So far, we have written the gauge field Lagrangian:

$$\mathscr{L}_{\mathrm{GF}} = -\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}\,, \tag{2.28}$$

and the lepton kinetic Lagrangian:

$$\mathscr{L}_{\mathrm{F}} = \overline{L}(i\gamma^\mu \mathrm{D}_\mu)L + \overline{R}(i\gamma^\mu \mathrm{D}_\mu)R \tag{2.29}$$

$$\mathrm{D}_\mu L = \left(\partial_\mu - ig\vec{T}\cdot\vec{A}_\mu - ig'\frac{Y}{2}B_\mu\right)L \tag{2.30}$$

$$D_\mu R = \left(\partial_\mu - ig'\frac{Y}{2}B_\mu\right) R\,. \tag{2.31}$$

We want to have a breaking like $SU(2)_L \otimes U(1)_Y \to U(1)_Q$, where we want the ground state Q to be chargeless (since it represents the photon).

In order to have a Dirac-like mass term, $m\overline{\psi}_L\psi_R$, we need to saturate a doublet with a singlet: it cannot be done! So, we need a new field ϕ which is at least a doublet, and then we can make a term $\overline{L}\phi R$.

A complex field $\phi \in \mathbb{C}^2$ works, and it must be chargeless for things to work. The Yukawa Lagrangian will then be

$$\mathscr{L}_{\text{YUK}} = -y_e\left(\overline{L}\phi R + \overline{R}\phi^\dagger L\right)\,. \tag{2.32}$$

A mass term for the field ϕ in the form $m^2\phi^\dagger\phi$ would be stable, but we want to make it *un*stable at $\phi = 0$, o we need a $\phi^\dagger\phi$ term with a negative coefficient and a $(\phi^\dagger\phi)^2$ term with a positive coefficient:

$$V(\phi) = -\mu^2(\phi\phi^\dagger) + \lambda(\phi\phi^\dagger)^2\,. \tag{2.33}$$

The minimum of this potential is reached when $\phi\phi^\dagger = (1/2)\mu^2/\lambda = (1/2)v^2$.

Why $\phi\phi^\dagger$? That's 2x2...

So far, we have introduced the couplings g and g', the parameters μ and v for the Higgs field potential, and the Yukawa coupling y_e.

The connection to experiment will be to write in terms of these parameters e, G_F, M_Z, M_H, m_e.

So a 5 parameter fit for 5 measurements?

The ϕ field near the minimum can be parametrized as

$$\phi = \begin{bmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{bmatrix}\,. \tag{2.34}$$

The upper, charged part is equal to zero. In the Yukawa Lagrangian we find

$$\begin{aligned}
\mathscr{L}_{\text{YUK}} &= -y_e\left[\begin{bmatrix} \overline{\nu}_{eL} & e_L \end{bmatrix}\begin{bmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{bmatrix} e_R + \bar{e}_R \begin{bmatrix} 0 & \frac{v+H}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \nu_{eL} \\ e_L \end{bmatrix}\right] && (2.35)\\
&= -\frac{y_e}{\sqrt{2}}\,(v+H)\,(\bar{e}_L e_R + \bar{e}_R e_L) && (2.36)\\
&= -m_e\bar{e}e - \frac{y_e v}{\sqrt{2}}H\bar{e}e\,. && (2.37)
\end{aligned}$$

We have not obtained any mass for the neutrino, as we expected by not having a ν_R term.

The proportionality of the Yukawa coupling to the masses has been seen experimentally.

In general the degrees of freedom of the ϕ can be written as

$$\phi = \exp\left(i\vec{T} \cdot \frac{\vec{\sigma}}{2}\right) \begin{bmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{bmatrix}, \tag{2.38}$$

so we can use SU(2) gauge invariance to select a ground state; the three remaining degrees of freedom become Goldstone bosons and are "eaten" by the three $W_\mu^\pm$, Z_μ^0 bosons to give them a third, longitudinal polarization.

The Higgs sector of the Lagrangian becomes

$$\mathscr{L}_{\mathrm{H}} = \left(\mathrm{D}_\mu \phi^\dagger\right)\left(\mathrm{D}_\mu \phi\right) - V(\phi), \tag{2.39}$$

which contains the bosons. What we get if we diagonalize this Lagrangian is terms written in terms of the new fields

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(A_\mu^1 \pm A_\mu^2\right), \tag{2.40}$$

and

$$\begin{bmatrix} Z_\mu \\ A_\mu \end{bmatrix} = \begin{bmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{bmatrix} \begin{bmatrix} A_\mu^3 \\ B_\mu \end{bmatrix}, \tag{2.41}$$

where θ_W is the Weinberg angle, chosen such that $\tan\theta_W = g'/g$.

The masses of these bosons come out to be

$$M_W^2 = \frac{v^2}{4} g^2 \qquad \text{and} \qquad M_Z = \frac{v^2}{4}(g^2 + g'^2), \tag{2.42}$$

while $m_\gamma = 0$.

We get cubic couplings like HZZ but also quartic ones like $HHZZ$ or $HHWW$.

The mass terms for the Higgs is $m_H = \sqrt{2}\mu$.

The photon does not couple directly to the Higgs, but we can have a $H \to \gamma\gamma$ process through loops.

When we rewrite the gauge field Lagrangian in terms of the physical fields A_μ, W_μ and Z_μ, we get quadratic, cubic and quartic terms in the gauge terms.

The terms we get are γWW, ZWW, as well as the quartic $\gamma\gamma WW$, $WWWW$, $ZZWW$, γZWW.

What about the fermion field Lagrangian? We expect to get terms in the form $X^\mu J_\mu$, where X^μ is a gauge field while J_μ is a fermion current.

The result is

$$\mathscr{L}_{\mathrm{F}} = \mathscr{L}_{\text{electromagnetic}} + \mathscr{L}_{\text{charged current}} + \mathscr{L}_{\text{neutral current}}. \tag{2.43}$$

The electromagnetic term is

$$\mathscr{L}_{\text{electromagnetic}} = \underbrace{\frac{gg'}{\sqrt{g^2 + g'^2}}}_{e} J^\mu_{EM} A_\mu, \tag{2.44}$$

where e is the electric charge; the other terms read

$$\mathscr{L}_{\text{charged current}} = \frac{g}{\sqrt{2}} J^\mu_\pm W^\mp \tag{2.45}$$

$$\mathscr{L}_{\text{neutral current}} = \frac{g}{\cos\theta_W} J^\mu_{NC} Z_\mu \,. \tag{2.46}$$

The electromagnetic current will be $J^\mu_{EM} = \bar{e}\gamma^\mu Q e$ (by construction: we made the theory to reproduce Maxwell's equations).

The charged current is $J^\mu_+ = \bar{e}_L \gamma^\mu \nu_{eL}$ and $J^\mu_- = \bar{\nu}_{eL}\gamma^\mu e_L$.

These were the already-observed parts, while the new one is

$$J^\mu_{NC} = \bar{\nu}_{eL}\gamma^\mu T_3 \nu_{eL} + \bar{e}_L \gamma^\mu \left(T_3 - Q s_W^2 \right) e_L + \bar{e}_R \gamma^\mu (-Q s_W^2) e_R \,. \tag{2.47}$$

This term is carrying a current $T_3 - Q s_W^2$, where $s_W^2 = \sin^2\theta_W$. The generator T_3 is just half of the σ_3 Pauli matrix.

The phenomenology of this prediction is quite rich.

We have couplings in the form $\gamma^\mu P_{L,R}$, there is specific jargon to describe these interactions.

1. Vector (V) currents are those in the form $\overline{\psi}\gamma^\mu\psi$, which do not change sign under a parity transformation;

2. Axial (A) currents are those in the form $\overline{\psi}\gamma^\mu\gamma^5\psi$, which *do* change sign under a parity transformation.

EM interactions are of type V, charged currents are of type $V - A$ (or, $1 \pm \gamma^5$), while neutral currents are of the form $g_V V + g_A A$.

We can define left-handed couplings in the neutral current $g_L = T_3 - Q s_W^2$, as well as right-handed ones like $g_R = -Q s_W^2$ (since the action of T_3 on an R singlet is zero).

The vector and axial couplings are then defined as

$$g_V = g_L + g_R = T_3 - 2 Q s_W^2 \tag{2.48}$$

$$g_A = g_L - g_R = T_3 \,. \tag{2.49}$$

When a short-range gauge boson mediates an interaction, its contribution will be approximately $1/M^2$ at low energies.

So, both processes which are mediated by $W^\pm$ and Z must give the correct low-energy limit.

Specifically, we find

$$G_F \sim \frac{g^2}{M^2} \qquad \text{or} \qquad \frac{g^2}{8 M_Z^2 \cos^2\theta_W} = \frac{g^2}{8 M_W^2} = \frac{G_F}{\sqrt{2}} \,. \tag{2.50}$$

In the notes there are qualitative considerations about the amplitudes in β decay, getting the energy spectrum of the electron.

A charged current process is $\mu \to \nu_\mu \overline{\nu}_e e$. The Fermi constant is best estimated through this process, since it is very well-known both theoretically and experimentally.

The decay $\pi \to \ell \overline{\nu}_e$ probes the $V - A$ nature of the interaction.

Another interesting process is $\nu_\mu e$ scattering, which allows us to estimate s_W^2.

We can have the recoil of an entire nucleus with a low-energy neutrino, which is mediated by a Z boson, Coherent Elastic ν Nucleus Scattering, CEνNS.

This process turns out to be proportional to the number of neutrons squared, which goes up quickly — this is the only tabletop neutrino detector.

The idea is that the neutrino's wavelength is very long, so it cannot see the specific nucleons or quarks.

We separate out the parameters g, g', μ and ν from the Yukawa coupling of the electron — the first four remain the same, while we will need to introduce more Yukawa coupling for the different families.

The data are e, G_F, M_H, M_Z, as well as the mass of the electron.

The relations are

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} \tag{2.51}$$

$$G_F = \frac{1}{\sqrt{2} v^2} \tag{2.52}$$

$$M_H = \sqrt{2} \mu \tag{2.53}$$

$$M_Z = \frac{v}{2} \sqrt{g^2 + g'^2} \,, \tag{2.54}$$

as well as

$$m_e = \frac{y_e v}{\sqrt{2}} \,. \tag{2.55}$$

These are all computed at tree-level; at more loops there can be corrections of the order of a few parts in a hundred. There are proposals to replace e with M_W, so that all the terms refer to the same energy scale $\sim 100\,\text{GeV}$.

In natural units,

$$e = 0.3 \tag{2.56}$$

$$G_F = 1.17 \times 10^{-5}\,\text{GeV}^{-2} \tag{2.57}$$

$$M_H = 125\,\text{GeV} \tag{2.58}$$

$$M_Z = 91\,\text{GeV} \tag{2.59}$$

$$m_e = 0.51 \times 10^{-3}\,\text{GeV} \,, \tag{2.60}$$

$e = .3$ corresponds to $e^2/4\pi = 1/137 = \alpha$; why does this not match with the astropy code?

so

$$g = 0.64 \tag{2.61}$$

$$
\begin{aligned}
g' &= 0.34 && (2.62)\\
\mu &= 88\,\text{GeV} && (2.63)\\
v &= 246\,\text{GeV} && (2.64)\\
y_e &= 2.9 \times 10^{-6}\,, && (2.65)
\end{aligned}
$$

so

$$
\begin{aligned}
\lambda &= 0.13 && (2.66)\\
\sin^2\theta_W &= 0.22 && (2.67)\\
M_W &= 80\,\text{GeV}\,. && (2.68)
\end{aligned}
$$

In terms of naturalness, it is weird that $y_e \ll 1$, which tells us that $m_e \ll 100\,\text{GeV}$.

Fermion masses and mixing

Friday 2021-12-10

The full symmetry group of the standard model is $\mathrm{SU}(3)_c \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$.

The building blocks are doublets like (ν_{eL}, e_L) with charges respectively $Q = 0$ and $Q = -1$, as well as $T_3 = +1/2$ and $T_3 = -1/2$.

The right-handed singlets, on the other hand, also have $Q = 0$ and $Q = -1$ but $T = 0$.

Any right-handed neutrinos would be *sterile*, since they do not interact.

This also holds for muonic and tauonic flavors, as well as for quark doublets as long as we add $2/3$ to all charges.

The relation between the quantum numbers is $Y = 2(Q - T_3)$.

In general the doublets look like

$$
\begin{bmatrix} U^\alpha \\ D^\alpha \end{bmatrix}_L\,, \tag{2.69}
$$

where α is a generation index.

The most general mass term looks like $m\overline{\psi}_L\psi_R$ — we should also consider its Hermitian conjugate, but it works analogously.

As usual, we insert the Higgs field to saturate the doublets and diagonalize. We will need to look at the effect of this change of basis on the mixing matrix.

The most general currents look like

$$
\begin{aligned}
J^\mu_{EM} &= \sum_\alpha \overline{U}^\alpha \gamma^\mu U^\alpha + (U \to D) && (2.70)\\
J^\mu_{NC} &= \sum_\alpha \overline{U}^\alpha_L \gamma^\mu (T_3 - Q s_W^2) U^\alpha_L + \sum_\alpha \overline{U}^\alpha_R \gamma^\mu (-Q s_W^2) U^\alpha_R + (U \to D) && (2.71)\\
J^\mu_{-} &= \sum_\alpha \overline{U}^\alpha_L \gamma^\mu D^\alpha_L (U \to D \text{ yields } J^\mu_+)\,. && (2.72)
\end{aligned}
$$

The coupling terms then look like $J^\mu_{EM} A_\mu$, $J^\mu_{NC} Z_\mu$ and $J^\mu_{\mp} W^\pm$.

The various components will transform according to some unitary matrices:

$$
D^\alpha_R = W^{\alpha\beta} D^\beta_R \qquad \text{and} \qquad D^\alpha_L = S^{\alpha\beta} D^\beta_L \tag{2.73}
$$

$$U_R^\alpha = T^{\alpha\beta} D_R^\beta \qquad \text{and} \qquad U_L^\alpha = R^{\alpha\beta} D_L^\beta \,. \tag{2.74}$$

Most of the currents are left unchanged by these transformations; the charged current however is the one which changes:

$$J_-^\mu = \overline{U}_L \gamma^\mu \underbrace{R^\dagger S}_{\neq 1 \text{ in general}} D_L \,. \tag{2.75}$$

The Yukawa Lagrangian will then include a term like

$$\mathscr{L} \ni \underbrace{\sum_{\alpha\beta} y_D^{\alpha\beta} \begin{bmatrix} \overline{U}^\alpha & \overline{D}^\alpha \end{bmatrix}_L \begin{bmatrix} 0 \\ v/\sqrt{2} \end{bmatrix} D_R^\beta}_{\text{D-type couplings}} + \underbrace{\sum_{\alpha\beta} y_U^{\alpha\beta} \begin{bmatrix} \overline{U}^\alpha & \overline{D}^\alpha \end{bmatrix}_L \begin{bmatrix} v/\sqrt{2} \\ 0 \end{bmatrix} U_R^\beta}_{\text{U-type couplings}} \,. \tag{2.76}$$

We define a mass matrix

$$M_{U,D}^{\alpha\beta} = y_{U,D}^{\alpha\beta} \frac{v}{\sqrt{2}} \,, \tag{2.77}$$

so that

$$\mathscr{L} \ni \overline{D}_L M_D D_R + \overline{U}_L M_U U_R + \text{h. c.} \,\,. \tag{2.78}$$

How can we put the Higgs vacuum on the upper component? We can define $\widetilde{\phi} = i\sigma_2 \phi^*$. For this field, the 1 component has zero charge while the 2 component has -1 charge.

If we introduce this field, the charge of the Yukawa Lagrangian is zero.

Each of the $M_{U,D}$ matrices will be diagonalized like $S^\dagger M T = M_{\text{diag}}$, where S and T are both unitary.

We then have, for down-type fields: $\overline{D}_L M_D D_R$ going into

$$D_R \to W D_R \qquad D_L \to S D_L \qquad M_D \to M_D^{\text{diag}} \,, \tag{2.79}$$

while $\overline{U}_R M_U U_R$ becomes

$$U_R \to T U_R \qquad U_L \to R U_L \qquad M_U \to M_U^{\text{diag}} \,, \tag{2.80}$$

In general, these will be different matrices for leptons and for quarks.

The charged current then reads $J_-^\mu = \overline{U}_L \gamma^\mu V D_L$, where $V = R^\dagger S$.

More explicitly, we get

$$J_-^\mu \overline{U}_L^\alpha \gamma^\mu V^{\alpha\beta} D_L^\beta \,. \tag{2.81}$$

The interaction of a $\overline{U}_L^\alpha$ and D_L^β is then modulated by a factor $(g/\sqrt{2})V^{\alpha\beta}$.

If V were 2x2 it would be a real matrix, if it were 3x3 it would be a complex matrix.

For quarks, the values on the diagonal are close to 1, while the largest values off-diagonal are of the order 0.22 for the CKM matrix, the mass-mixing matrix for quarks.

For example, in β decay we get a modulation by a factor V_{ud}, which is slightly less than 1.

For leptons, the matrix is called PMNS.

In the usual construction of the Standard Model, there were no mass terms for neutrinos or right-handed $\nu_{\alpha R}$. Therefore, in this case there could be no mixing among leptons, since the mixing matrix only ever appears "sandwitched".

The basic definition of neutrino flavor is "which lepton does it have charged current interactions with".

For example, ν_e is the ν produced in β^+ decay, and $\overline{\nu}_e$ is the ν produced in β^- decay.

Let us then consider $\nu_e \to \nu_\mu$ oscillations. We always need to consider the amplitudes for production, propagation, detection.

However, it is convenient to isolate the propagation into the matrix

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} . \tag{2.82}$$

Forgetting the fact that this is a simplification can lead to paradoxes; for example, ether leptons do not oscillate.

There is a basis for the Dirac equation for which $i\gamma^\mu$ is real — this leads to the fact that $\psi = \psi^*$.

This cannot work for electrons and positrons, since they are charged. A neutral fermion, on the other hand, might be its own antiparticle.

If we have a β^+ decay, we will produce a ν_e which is left-handed. In a β^- decay, on the other hand, we will produce a right-handed $\overline{\nu}_e$.

If the neutrino is massless, this cannot change. Formally, the Dirac equation decouples.

This "Weyl" case is not completely excluded by current phenomenology; the lightest of the neutrinos could be completely massless.

If we have massive neutrinos, though, there will be an $\mathcal{O}(m_\nu/E)$ component with the opposite chirality.

However, the neutrino and antineutrino are still distinguishable.

If they are Majorana, on the other hand, $\nu_e = \overline{\nu}_e$; this must mean that they are neutral not just for electric charge but for all charges.

We have observed inverse beta decay:

$$\nu_e + \mathrm{n} \to \mathrm{p} + e^- \qquad \text{and} \qquad \overline{\nu}_e + \mathrm{p} \to \mathrm{n} + e^+ , \tag{2.83}$$

but not the same reaction for $\nu_e \leftrightarrow \overline{\nu}_e$. Is this an indication that neutrinos are indeed Dirac?

Well, if neutrinos are indeed Majorana these reactions are possible, but suppressed at order m_ν/E!

The experiment which saw the largest number of neutrinos collected about 2 million events. . .

The way to have good statistics is to look at decays, specifically neutrinoless double-beta decay.

The decay looks like

$$\{2n\} \to \{2p\} + 2e^- . \tag{2.84}$$

The decay of the first proton looks like the emission of a right-handed $\overline{\nu}_e$ with an e^-, through a charged-current interaction.

IF $m_\nu \neq 0$, the neutrino has a left-handed component (not possible if the neutrino is Weyl); if $\nu = \overline{\nu}$, this left-handed component is a left-handed ν_e (not possible if the neutrino is Dirac)!

At the second proton, the left-handed ν_e is absorbed and an e^- is emitted.

This process is suppressed by m_ν / E: it probes absolute neutrino mass.

Current neutrinoless double beta decay are probing lifetimes of the order of 10^{26} to 10^{27} yr.

It can be shown that if this process takes place, then neutrinos must be Majorana.

Pictorially, one can join the legs of the $0\nu\beta\beta$ diagram to find a diagram which turns a ν_e into a $\overline{\nu}_e$.

Mass terms for neutrinos

Monday 2021-12-13

We will discuss the difference between Weyl, Dirac and Majorana neutrinos for the single-generation case. Remember that under a Lorentz transformation $\Lambda(\vec{\omega}, \vec{n})$ they change as

$$\phi_R \to \exp\left(i\vec{\omega} \cdot \frac{\vec{\sigma}}{2} + \vec{n} \cdot \frac{\vec{\sigma}}{2} \right) \phi_R \tag{2.85}$$

$$\phi_L \to \exp\left(i\vec{\omega} \cdot \frac{\vec{\sigma}}{2} - \vec{n} \cdot \frac{\vec{\sigma}}{2} \right) \phi_L \,. \tag{2.86}$$

It can be shown that if ϕ_R is right-handed, $i\sigma_2 \phi_R^*$ is left-handed (and similarly with $L \leftrightarrow R$).

We can therefore parametrize our full spinor in terms of two right-handed spinors, as

$$\Psi = \begin{bmatrix} \phi_R \\ \phi_L \end{bmatrix} = \begin{bmatrix} u \\ i\sigma_2 v^* \end{bmatrix} . \tag{2.87}$$

Recall also the projectors

$$\psi_{L,R} = P_{L,R} \psi \qquad \text{where} \qquad P_{L,R} = \frac{1 \mp \gamma^5}{2} \,. \tag{2.88}$$

We also have $\overline{\psi} = \psi^\dagger \gamma^0$ and $\psi^c = C(\psi) = i\gamma^2 \psi^*$.

The convention for the order of operations is as follows:

1. projectors $P_{L,R}$ act before
2. charge conjugation $C(\cdot)$ which acts before
3. conjugation $\bar{\cdot}$.

In terms of our parametrization $\psi(u, v)$ the charge conjugate swaps u and v:

$$\psi^c = \begin{bmatrix} v \\ i\sigma_2 u^* \end{bmatrix} . \tag{2.89}$$

The Dirac case is the one in which $u \neq v$, the Majorana case is the one in which $u \simeq v$. The equality in the Majorana case can actually be relaxed including an arbitrary phase: $\psi = \psi^c e^{i\phi}$ for some fixed ϕ, called the *Majorana phase*.

The Weyl case is the one in which $m = 0$, meaning that $\psi = \psi_R$ or $\psi = \psi_L$. The spinor just has two degrees of freedom.

The Weyl-Majorana case would then be $\psi = \psi_R + \psi_R^c = \psi^c$.

The Dirac case is the general, massive, 4 degrees of freedom one.

A Dirac mass term looks like $m\overline{\psi}\psi$, a Majorana one would look like $(1/2)m\overline{\psi}\psi$.

In the Dirac case, this would read

$$\overline{\psi}\psi = \overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L\,, \tag{2.90}$$

while in the Majorana, left-handed case we would have

$$\overline{\psi}\psi = \overline{\psi}_L\psi_L^c + \overline{\psi}_L^c\psi_L\,, \tag{2.91}$$

and the right-handed one is analogous.

The most general Lagrangian would then include a term

$$\mathscr{L} \ni m_D\left(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L\right) + \frac{1}{2}m_L\left(\overline{\psi}_L^c\psi_L + \overline{\psi}_L\psi_L^c\right) + \frac{1}{2}m_R\left(\overline{\psi}_R^c\psi_R + \overline{\psi}_R\psi_R^c\right) \tag{2.92}$$

$$= \frac{1}{2}\begin{bmatrix} \overline{\psi}_L + \overline{\psi}_L^c, & \overline{\psi}_R + \overline{\psi}_R^c \end{bmatrix}\begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix}\begin{bmatrix} \psi_L + \psi_L^c \\ \psi_R + \psi_R^c \end{bmatrix}, \tag{2.93}$$

which we can diagonalize! The eigenvectors will in general be linear combinations of Majorana fields.

What does the Standard Model tell us about these masses? The m_D term is allowed, is comes from the SSB of the Yukawa terms as long as we have right-handed neutrinos.

Specifically, it will be $m_D \sim y_\nu v/\sqrt{2}$ where $v \approx 200\,\text{GeV}$. In order for this to work we must have $y_\nu \lesssim 10^{-11}$.

In the SM we cannot have Majorana mass terms m_L, since they break the electroweak symmetry.

Clarify this point

The term m_R, on the other hand ($m_R\overline{\psi}_R\psi_R^c + \text{h. c.}$), is allowed by the SM gauge group, but the m_R term is a new mass scale of the theory.

The mass matrix can be thought to look like

$$\begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix} = \begin{bmatrix} 0 & \sim v \\ \sim v & \sim \Lambda \end{bmatrix}, \tag{2.94}$$

where Λ is a scale from Beyond-the-Standard-Model physics.

These ideas came out when people were considering very large gauge groups.

The diagonalization of that matrix yields a light neutrino $\nu \sim \nu_L + \nu_L^c$ with mass of the order $\mathcal{O}\left(v^2/\Lambda\right) \ll v$, as well as a massive neutrino $\nu \sim \nu_R + \nu_R^c$ with mass of the order $\mathcal{O}(\Lambda)$.

This is the see-saw mechanism.

A technical detail: one mass comes out negative, but if ψ has a negative mass $\gamma^5\psi$ has a positive mass.

What happens if we have more than one neutrino generation?

In the SM, we have determined that the flavor eigenstates are linear combinations of the mass eigenstates with a unitary mixing matrix U.

We can also have a number N_s of sterile neutrino states which do not couple to the $W^\pm$ and Z bosons. The number N_s is not limited to the right-handed components of the three neutrinos, there could be more. The full mixing matrix will then be

$$\begin{bmatrix} 3\times 3 & 3\times N_s \\ N_s\times 3 & N_s\times N_s \end{bmatrix} \sim \begin{bmatrix} \sim U_{\mathrm{PMNS}} & \sim \mathcal{O}(v/\Lambda) \\ \sim \mathcal{O}(v/\Lambda) & U_{N\times N} \end{bmatrix}. \tag{2.95}$$

A phenomenologically motivated situation is already the $N_s = 1$ one, with a fourth sterile neutrino with mass $\sim 1\,\mathrm{eV}$. At present, though, the evidence for a sterile neutrino is very controversial.

The process of leptogenesis, by some process which satisfies Sakharov's conditions, is relevant for this discussion.

We need both C and CP violation. If we can find CP violation at low energy scales, it would be a good indication!

CP violation in the quark sector, $V_{\mathrm{CKM}} = V^*_{\mathrm{CKM}}$, cannot explain matter-antimatter asymmetry.

It is possible that a CP violating mechanism in the weak sector may give an imprint in the primordial GW background.

If the CP violating decay of heavy neutrinos is to blame, there might be a connection between the low- and high-energy manifestations of CP violation.

The effect of neutrino masses and mixing in propagation

We have

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = U_{\mathrm{PMNS}} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}, \tag{2.96}$$

where $U_{\mathrm{PMNS}} U^\dagger_{\mathrm{PMNS}} = 1$, while in general $U \neq U^\dagger$. We will assume $E \gg m$.

The convention used for this matrix is as follows, since it can be shown that it contains three angles and a CP-violating phase:

$$\begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix}}_{R(\theta_{23})} \underbrace{\begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ s_{13}e^{i\delta} & 0 & c_{13} \end{bmatrix}}_{R(\theta_{13})} \underbrace{\begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R(\theta_{12})} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\psi'} & 0 \\ 0 & 0 & e^{i\psi''} \end{bmatrix}}_{\text{Majorana}}. \tag{2.97}$$

If we have Majorana neutrinos the spinor is equal to its conjugate up to a phase, but only two of the phases are physically meaningful. However, these play no role in oscillations.

We have measured these angles: for θ_{23} there is (almost) maximal mixing, $\theta_{23} \approx \pi/4$ or $s_{23}^2 \approx 0.5$. This is called *octant ambiguity*, since we do not know the precise value.

The value of s_{12}^2 is approximately 0.3.

The value of s_{13}^2 is approximately 0.02.

What about the mass spectrum? Well, the energy is approximately

$$E = \sqrt{p^2 + m^2} \approx p + \frac{m^2}{2p} \approx p + \frac{m^2}{2E} . \tag{2.98}$$

The masses are numbered 1, 2 and 3; by convention we always take $m_2^2 > m_1^2$.

We have $\delta m^2 = m_2^2 - m_1^2 \approx 7.5 \times 10^{-5}\,\mathrm{eV}^2$, while $\left|\Delta m^2\right| = \left|m_3^2 - m_1^2\right| \approx 2.5 \times 10^{-3}\,\mathrm{eV}^2$. We do not know the sign of Δm^2!

As long as we deal with three neutrino masses and mixing, we have 5 known quantities and 5 unknowns: we know

1. δm^2;
2. $\left|\Delta m^2\right|$;
3. s_{23}^2;
4. s_{12}^2;
5. s_{13}^2,

while we do not know some parameters which are hard or impossible to measure with oscillations:

1. the value of the CP-violating δ — but we do have some weak $\sim 2\sigma$ indications that it might be nonzero;
2. the sign of Δm^2;
3. whether $\theta_{13} \lesseqgtr \pi/4$;
4. the absolute mass scale (we only have some bounds);
5. whether neutrinos are Dirac or Majorana.

There is a search for right-handed neutrinos at many scales, even at LHC.

Tomorrow we will start studying neutrino oscillations, dealing with wavefunctions as opposed to spinors, approximating time with position $t \sim x$; and using the fact that $E \approx p + m^2/2E$.

The Schrödinger equation will read

$$i \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} = H \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} , \tag{2.99}$$

where, in a vacuum, H is diagonal, with the three energies of the neutrinos. There will be corrections in matter.

We will have

$$H = p + \frac{1}{2E} \underbrace{\begin{bmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{bmatrix}}_{M^2} . \tag{2.100}$$

Neutrino oscillations in vacuum

Tuesday 2021-12-14

A "vacuum" in this context is a situation with very low fermion density, air is a good approximation for it for example. We will later quantify the effects of interactions with matter.

The solution of Schrödinger's equation for that Hamiltonian is

$$\begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} = e^{-iH_m x} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}_0 , \tag{2.101}$$

but the interesting thing is to rewrite this in the flavor basis:

$$H_f = UH_m U^\dagger = p\mathbb{1} + \frac{1}{2E} UM^2U^\dagger , \tag{2.102}$$

where M^2 is the diagonal matrix with the square masses on the diagonal.

This is not a diagonal Hamiltonian anymore!

Factoring out the evolution according to $p\mathbb{1}$ we get

$$i\frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \frac{1}{2E} UM^2U^\dagger \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} . \tag{2.103}$$

Since this is not diagonal, we can have both flavor *appearance* and flavor *disappearance*.

The evolution is given by the exponential

$$S = \exp\Big(-iH_f x\Big) = \exp\left(-\frac{i}{2E} UM^2U^\dagger x\right) = U \exp\left(-i\frac{M^2}{2E}x\right) U^\dagger , \tag{2.104}$$

because of the properties of unitary matrices in exponentials.

It is convenient to write this in components, $\nu_\beta = S_{\beta\alpha}\nu_\alpha$:

$$S_{\beta\alpha} = \sum_i U_{\beta i} \exp\left(-i\frac{m_i^2}{2E}\right) U^\dagger_{\alpha i} , \tag{2.105}$$

and the probability for this will be $P(\nu_\alpha \to \nu_\beta) = P_{\beta\alpha} = \left|S_{\beta\alpha}\right|^2$.

This modulus can be explicitly rewritten as

$$P_{\alpha\beta} = \underbrace{\delta_{\alpha\beta} - 4\sum_{i<j} \operatorname{Re} J^{ij}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij} x}{4E}\right)}_{\text{CP-conserving}} \underbrace{- 2\sum_{i<j} \operatorname{Im} J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij} x}{2E}\right)}_{\text{CP-violating}}, \tag{2.106}$$

where $\Delta m^2_{ij} = m_i^2 - m_j^2$ while $J^{ij}_{\alpha\beta} = U_{\alpha i} U^\dagger_{\beta i} U^\dagger_{\alpha j} U_{\beta j}$. This is called the Jarlskog invariant (Cecilia Jarlskog studied this first in the context of the CKM matrix).

This all is in natural units; the SI units version is

$$\frac{\Delta m^2_{ij} x}{4E} \approx 1.267 \left(\frac{\Delta m^2_{ij}}{\mathrm{eV}}\right) \left(\frac{x}{\mathrm{m}}\right) \left(\frac{\mathrm{MeV}}{E}\right). \tag{2.107}$$

Interchanging ν and $\overline{\nu}$ is equivalent to interchanging U and $U*$; interchanging α and β is equivalent to interchanging U and $U^\dagger$.

What is the behavior of this under CP and T symmetries?

Let us call S the source of neutrinos, and D their detector. We have ν_α at S, and detect ν_β at D.

A CP transformation means we swap S and D, and have $\overline{\nu}_\alpha$ travelling backward to be detected as $\overline{\nu}_\beta$.

A T transformation swaps S and D again.

If CP is conserved,

$$P(\nu_\alpha \to \nu_\beta) = P(\overline{\nu}_\alpha \to \overline{\nu}_\beta), \tag{2.108}$$

meaning that we must have $U = U^*$; while if T is conserved we must have

$$P(\nu_\alpha \to \nu_\beta) = P(\nu_\beta \to \nu_\alpha) \qquad \text{and} \qquad P(\overline{\nu}_\alpha \to \overline{\nu}_\beta) = P(\overline{\nu}_\beta \to \overline{\nu}_\alpha), \tag{2.109}$$

meaning that we must have $U = U^*$.

Say we observed $P(\nu_e \to \nu_\mu) \neq P(\overline{\nu}_e \to \overline{\nu}_\mu)$. Would that not prove that $\nu_e \neq \overline{\nu}_e$ and/or $\nu_\mu \neq \overline{\nu}_\mu$, meaning that neutrinos are not Majorana?

A CPT transformation, therefore, must satisfy

$$P(\nu_\alpha \to \nu_\beta) = P(\overline{\nu}_\beta \to \overline{\nu}_\alpha). \tag{2.110}$$

This surely holds! It can be seen by the requirement $U = U^{**}$, but it also holds in general as a theorem about field theories.

The CP-violating part can be isolated as shown in equation (2.106).

In the parametrization of the U_{PMNS} the CP-violation is parametrized as δ; it is currently compatible with 0.

In order to observe CP violation we must have $\delta \neq 0, \pi$ (meaning that $U \neq U^*$); the second condition is that we must be looking at $\alpha \neq \beta$! In disappearance experiments the CP-violating part cancels.

All the mixing angles must be nonzero: $\theta_{ij} \neq 0$.

The fourth condition is that all square mass differences are nonzero: $\Delta m^2_{ij} \neq 0$.

CP violation is a genuine 3-neutrino phenomenon, it cannot be observed with 2! Current experimental results are compatible with all conditions being realized, but it's hard since δm^2 is about 30 times smaller than Δm^2.

Most current experiments are only sensitive to a submatrix

$$\begin{bmatrix} \nu_\alpha \\ \nu_\beta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \nu_i \\ \nu_j \end{bmatrix} , \tag{2.111}$$

where $\theta = \theta_{ij}$.

In this case, the probability simply reads

$$P_{\alpha\beta} = \sin^2 2\theta \sin^2 \left(\frac{\Delta m^2 L}{4E} \right) , \tag{2.112}$$

while $P_{\alpha\alpha} = 1 - P_{\alpha\beta}$.

We have no information on the sign of Δm^2 since it is inside a $\sin^2$; there is no information on the absolute masses, there is no information on $\nu \leftrightarrow \overline{\nu}$.

There is an analogy with a double-slit experiment.

Practically, we do not observe probabilities but fluxes:

$$R_\beta = \int \Phi_\alpha \otimes P_{\alpha\beta} \otimes \sigma_\beta \otimes \epsilon_\beta , \tag{2.113}$$

which will be a multidimensional integral in general, a marginalization over all the parameters we do not care about — the flux at the source, the interaction cross-section, the detector efficiency.

Let's say we have a $\nu_\alpha \to \nu_\beta$ appearance experiment, and we do not observe anything.

What is typically done is to draw an *exclusion zone* in the $\Delta m^2, \sin^2 2\theta$ plane. The range for θ is therefore 0 to $\theta = \pi/4$.

Suppose, instead, that we observe a certain rate of β with $R_\beta = R \pm \sigma_\beta$.

In that case as well we can draw an allowed band. If we also have some spectral information, we can also measure L/E, therefore we can intersect several of these bands and measure Δm^2 and θ simultaneously.

We typically have *octant ambiguity*: the respective $\pi/4 < \theta < \pi/2$ value is typically also allowed.

This ambiguity has been removed for θ_{12} and θ_{23}, not for θ_{13}.

Experiments which are sensitive to $\Delta m^2 = m_3^2 - m_{1,2}^2$:

1. short baseline reactors, which attempt to observe $\overline{\nu}_e \to \overline{\nu}_e$ with $L \sim 1\,\text{km}$ and $E \sim$ few MeV;

2. long baseline accelerators, which attempt to observe $\nu_\mu \to \nu_\mu$ or ν_e or ν_τ (or antineutrinos), with $L \sim 100 \div 1000\,\text{km}$ and $E \sim 1 \div 10\,\text{GeV}$;[1]

[1] These are able to observe appearance as well as disappearance since the energy is so high!

3. atmospheric neutrino experiments, which attempt to observe $\nu_\mu \to \nu_\mu$ or ν_e (or antineutrinos) with $L \sim 10 \div 10^4\,\mathrm{km}$ and $E \geq 1\,\mathrm{GeV}$.

For these experiments, we typically have $\delta m^2 L/4E \ll 1$, therefore we can take $\delta m^2 \approx 0$. The probability can then be written as

$$P_{\alpha\beta} = 4|U_{\alpha 3}|^2 \left|U_{\beta 3}\right|^2 \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \tag{2.114}$$

$$P_{\alpha\alpha} = 1 - 4|U_{\alpha 3}|^2 (1 - |U_{\alpha 3}|)^2 \sin^2\left(\frac{\Delta m^2 L}{4E}\right), \tag{2.115}$$

so these are able to probe a whole column of the mixing matrix; therefore, we can use these to measure θ_{13} and θ_{23}, as well as $\left|\Delta m^2\right|$.

Explicitly,

$$P_{ee} = 1 - \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \tag{2.116}$$

$$P_{\mu e} = s_{23}^2 \sin 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \tag{2.117}$$

$$P_{\mu\mu} = 1 - c_{13}^2 s_{23}^2 \left(1 - c_{13}^2 s_{23}^2\right) \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \tag{2.118}$$

$$P_{\mu\tau} = c_{13}^4 \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right), \tag{2.119}$$

the first three have been probed by atmospheric neutrinos, the first has been probed by short baseline experiments, the second two have been probed by long baseline experiments, the last has only been probed by OPERA (CERN to LNGS).

Atmospheric neutrinos have been probed by SuperKamiokande and IceCube.

The octant ambiguity for θ_{13} from P_{ee} is resolved by $P_{\mu\mu}$: it is indeed small and $< \pi/4$.

On the other hand, θ_{23} is close to $\pi/4$, and it is not determined in which octant it lies.

In the opposite case, we have $\delta m^2 L/4E$ of order 1, while $\Delta m^2 L/4E \gg 1$, meaning that those oscillations are averaged away.

One can do this with hundreds of kilometers L but low energies, $E \sim$ few MeV.

One finds, for KamLAND as well as for solar neutrinos:

$$P_{ee}^{2\nu} = c_{13}^4 P_{2\nu} + s_{13}^4 \qquad \text{where} \qquad P_{2\nu} = 1 - \sin^2 2\theta_{12} \sin^2\left(\frac{\delta m^2 L}{4E}\right). \tag{2.120}$$

These experiments are sensitive to the first row of the mixing matrix.

They measure $\delta m^2 \sim 7.5 \times 10^{-5}\,\mathrm{eV}^2$.

Luckily there is a different possibility: solar neutrinos. These break the ambiguity, thanks to matter effects.