Path Integral

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Autumn 2019

1 Before the PI

1.1 Schrödinger, Heisenberg & interaction

We denote $U = \exp(Ht/i\hbar)$, and similarly with $H_0 \to U_0$, $V \to U_V$.

Schrödinger

1. State kets are $\left|\psi(t)\right\rangle = U\left|\psi(t=0)\right\rangle$;
2. observables are $A(t) \equiv A(t=0)$;
3. base kets are defined by $A\left|a\right\rangle = a\left|a\right\rangle$, therefore $\left|a(t)\right\rangle \equiv \left|a(t=0)\right\rangle$.

Heisenberg

1. State kets are $\left|\psi(t)\right\rangle \equiv \left|\psi(t=0)\right\rangle$;
2. observables are $A(t) = U^\dagger A(t=0) U$;
3. base kets are $\left|a(t)\right\rangle = U^\dagger \left|a(t=0)\right\rangle$.

Interaction We denote by a subscript S or I objects in the Schrödinger or interaction system. In the

1. State kets are defined as $\left|\psi(t)\right\rangle_I = U_0^\dagger \left|\psi(t)\right\rangle_S$;
2. observables are defined as $A_I(t) = U_0^\dagger A_S U_0$;
3. as base kets we use eigenstates of H_0: $H_0\left|n\right\rangle = E_n\left|n\right\rangle$. These evolve like $\left|n(t)\right\rangle = U_0\left|n(t=0)\right\rangle$.

Then, we can generically write the evolution of a Schrödinger ket as

$$\left|\psi(t)\right\rangle_S = \sum_n c_n(t) \exp(E_n t / i\hbar) \left|n\right\rangle \,, \tag{1}$$

therefore the evolution of the interaction ket is

$$\left|\psi(t)\right\rangle_I = U_0^\dagger \left|\psi(t)\right\rangle_S = \sum_n c_n(t) \left|n\right\rangle \,. \tag{2}$$

We can write an equation for the evolution of the $c_n(t)$:

$$i\hbar \dot{c_n}(t) = \sum_m V_{nm} \exp(i\omega_{nm} t) c_m(t) \,, \tag{3}$$

where $\omega_{nm} = (E_n - E_m)/\hbar$ and $V_{nm} = \left\langle n\right| V \left|m\right\rangle$. This is a matrix equation for the coefficient vector.

Time-dep perturbations If we define the interaction-picture evolution operator as $\left|\alpha, t\right\rangle = U_I(t) \left|\alpha, 0\right\rangle$ we have its evolution as $i\hbar \partial_t U_I = V_I U_I$.

For small times $U_I \approx \mathbb{1}$, so we can integrate the Schrödinger equation:

$$U_I = \mathbb{1} + \frac{1}{i\hbar} \int_0^t V_I(t') U_I(t') \, \mathrm{d}t' \tag{4a}$$

$$= \mathbb{1} + \frac{1}{i\hbar} \int_0^t V_I(t') \left(\mathbb{1} + \frac{1}{i\hbar} \int_0^{t'} V_I(t'') U_I(t'') \, \mathrm{d}t'' \right) \mathrm{d}t' \tag{4b}$$

$$= \mathbb{1} + \frac{1}{i\hbar} \int_0^t V_I(t') \, \mathrm{d}t' + \frac{1}{(i\hbar)^2} \int_0^t \int_0^{t'} V_I(t') V_I(t'') \, \mathrm{d}t' \, \mathrm{d}t'' + o(V_I^2) \tag{4c}$$

.

Now, if we start on a base ket $\left|i\right\rangle$, the evolution coefficients $c_n(t)$ will be given by the matrix elements $\left\langle n\right| U_I(t) \left|i\right\rangle$. We can compute these to any order in V_I, by taking the components of the previous equation and applying the following computation any time we have the components of V_I:

$$\left\langle n\right| V_I \left|i\right\rangle = \left\langle n\right| U_0^\dagger V U_0 \left|i\right\rangle = \exp(i\omega_{ni} t) V_{ni} \,, \tag{5}$$

since the $\left|n\right\rangle$ are eigenstates of the unperturbed Hamiltonian.

1.2 The propagator

If $H \left|\alpha'\right\rangle = \alpha' \left|\alpha'\right\rangle$, then the evolution operator can be decomposed as

$$U(t) = \sum_{\alpha'} \exp\left(\frac{E_{\alpha'} t}{i\hbar} \right) \left|\alpha'\right\rangle\!\left\langle\alpha'\right| \,. \tag{6}$$

This can be written in the position basis as a Green function by contracting with two position vectors:

$$\langle x'| U(t) |x''\rangle = \sum_{\alpha'} \exp\left(\frac{E_{\alpha'} t}{i\hbar}\right) \langle x'|\alpha'\rangle \langle \alpha'|x''\rangle \stackrel{\text{def}}{=} K(x', x''; t), \tag{7}$$

and with this we can directly compute the evolution at a generic time: $\psi(x'', t) = \int \mathrm{d}^3 x' \, K(x'', x'; t)\psi(x')$. It is effectively the transition amplitude: $K = \langle x'', t|x', 0\rangle$ when seen in the Heisenberg picture (since we are evolving a base ket).

1. $K(x', x'', t)$ satisfies the Schrödinger equation, since it is a sum of terms which do;
2. $\lim_{t\to 0} K(x', x'', t) = \delta^3(x', x'')$.

1.3 Some useful propagators

Free particle We consider $H = p^2/2m$; the momentum eigenstates are $p\,|p'\rangle = p'\,|p'\rangle$, and they are also energy eigenstates with $H\,|p'\rangle = ((p')^2/2m)\,|p'\rangle$.

We compute:

$$K(x', x'', t) = \int \mathrm{d}p' \, \langle x''|p'\rangle \langle p'|x'\rangle \exp\left(\frac{(p')^2 t}{i\hbar 2m}\right), \tag{8}$$

and recall that $\langle x|p\rangle = \exp\left(-px/i\hbar\right)/\sqrt{2\pi\hbar}$. We simplify the exponent to get a Gaussian integral: it is known that

$$\int_{\mathbb{R}} \mathrm{d}x \exp\left(-i\alpha x^2\right) = \sqrt{\frac{\pi}{i\alpha}}, \tag{9}$$

therefore in the end we get:

$$K(x', x'', t) = \frac{1}{2\pi\hbar} \exp\left(\frac{im(x''-x')^2}{2\hbar t}\right)\sqrt{\frac{2m\pi\hbar}{it}}, \tag{10}$$

which for $t \to 0$ is in the form $\exp\left((x'-x'')^2/t\right)\sqrt{t} \to \delta(x''-x')$.

Harmonic oscillator We consider $H = p^2/2m + m\omega^2 x^2/2$. It is known that the eigenfunctions are given by the Hermite polynomials:

$$\langle x'|n\rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} \frac{1}{x_0^{1/2}} H_n y \exp\left(-\frac{(x/x_0)^2}{2}\right), \tag{11}$$

where $x_0 = \sqrt{\hbar/(m\omega)}$ (both masses and frequencies are inverse lengths in natural units!). We also know the eigenenergies, $E_n = \hbar\omega(n + 1/2)$. We can then compute away, to finally get:

$$K = \sqrt{\frac{m\omega}{2i\pi\hbar\sin(\omega t))}} \exp\left(\frac{im\omega\Big(((x'')^2 + (x')^2)\cos(\omega t) - 2x'x''\Big)}{2\hbar\sin(\omega t)}\right). \tag{12}$$

2 The Path Integral

We can time-slice the interval between a certain time $0 = t_0$ and another time $t = t_N$ in N parts. Then, evolving the system with a the propagator for each one, we get:

$$K(x_N, x_0, t) = \int \left(\prod_{i=1}^{N-1} \mathrm{d}x_i\right)\left(\prod_{i=0}^{N-1} \langle x_{i+1}, t_{i+1}|x_i, t_i\rangle\right). \tag{13}$$

We call the time-slice $\epsilon = t/N$. We will expand the in ϵ up to first order the evolution operator $\exp(H\epsilon/i\hbar)$.

The braket $\langle x_{i+1}, t_{i+1}|x_i, t_i\rangle$ can be written as

Timeslicing & introduction of the PI (JJJ)

Trotter product formula (JJ)

Classical path decomposition in the PI (JJ)

Orders of magnitude (JJJ)

Schrödinger equation from the PI (GIO)

PI for quadratic Lagrangians (LEO)

Math appendix: (complex) gaussian integrals (JJ)

PI for the harmonic oscillator (GIO in caso)

PI for general lagrangians (nonquadratic) ()

Math appendix: functional derivatives (JJ)

Semiclassical approximation (ALBE)

Operator Matrix Elements (J)

Adding a source term to the action (LEO)

Application to the forced harmonic oscillator (LEO)

Perturbation theory ()

Imaginary time & the Euclidean PI (ALE)

Instantons (FRANCÉ)

Kinks (J)

2.1 Path integral of a quadratic Lagrangian

Let $L(x, \dot{x})$ be a quadratic Lagrangian:

$$L = a\dot{x}^2 + bx\dot{x} + cx^2 + d\dot{x} + ex + f\,. \tag{14}$$

The propagator for L, computed via path integral, is:

$$K = \int \mathrm{D}[x] \exp\left(\frac{i}{\hbar}\int \mathrm{d}t\, L\right) \tag{15a}$$

$$= \int \mathrm{D}[x] \exp\left(\frac{i}{\hbar}\int \mathrm{d}t\, \underbrace{(a\dot{x}^2 + bx\dot{x} + cx^2 + d\dot{x} + ex + f)}_{F}\right). \tag{15b}$$

Let x_c be the classical path; let's write $x = x_c + y$ and therefore $\mathrm{D}[x] = \mathrm{D}[y]$. Properly replacing x and $\dot{x}$, the propagator can be written as:

$$F = a(\dot{x}_c^2 + \dot{y}^2 + 2\dot{x}_c\dot{y}) + b(x_c\dot{x}_c + x_c\dot{y} + y\dot{x}_c + y\dot{y}) + c(x_c^2 + y^2 + 2x_c y) + \\ + d(\dot{x}_c + \dot{y}) + e(x_c + y) + f \tag{16a}$$

$$= (a\dot{x}_c^2 + bx_c\dot{x}_c + cx_c^2 + d\dot{x}_c + ex_c + f) + (a\dot{y}^2 + by\dot{y} + cy^2) + \\ + \dot{y}(2a\dot{x}_c + bx_c + d) + y(b\dot{x}_c + 2cx_c + e) \tag{16b}$$

$$= L_c + (a\dot{y}^2 + by\dot{y} + cy^2) + \dot{y}\left.\frac{\partial L}{\partial \dot{x}}\right|_{x_c} + y\left.\frac{\partial L}{\partial x}\right|_{x_c}, \tag{16c}$$

and so

$$K = \int \mathrm{D}[y] \exp\left(\frac{i}{\hbar}\int \mathrm{d}t \left(L_c + (a\dot{y}^2 + by\dot{y} + cy^2) + \dot{y}\left.\frac{\partial L}{\partial \dot{x}}\right|_{x_c} + y\left.\frac{\partial L}{\partial x}\right|_{x_c}\right)\right). \tag{17}$$

Evaluating by parts the last part of the integral:

$$I = \int \mathrm{d}t \left(\dot{y}\left.\frac{\partial L}{\partial \dot{x}}\right|_{x_c} + y\left.\frac{\partial L}{\partial x}\right|_{x_c}\right) \tag{18}$$

$$= y\Big|_i^f \left.\frac{\partial L}{\partial \dot{x}}\right|_{x_c} + \int \mathrm{d}t \left(-y\left(\frac{\mathrm{d}}{\mathrm{d}t}\left.\frac{\partial L}{\partial \dot{x}}\right|_{x_c} - \left.\frac{\partial L}{\partial x}\right|_{x_c}\right)\right) \tag{19}$$

that, by the equation of motion and the definition of y, is 0. Therefore,

$$K = e^{\frac{i}{\hbar}S_c} \int \mathrm{D}[y] \exp\left(\frac{i}{\hbar}\int \mathrm{d}t\, (a\dot{y}^2 + by\dot{y} + cy^2)\right) \tag{20}$$

Only the quadratic terms contribute to the propagator prefactor; the linear terms affect only the classical action computing.

3 Adding a source term to the action

3.1 The free particle case

Let's suppose to have a free particle, and let's add a source term:

$$L = \frac{m}{2}\dot{x}^2 + Jx \tag{21}$$

As showed for the quadratic Lagrangians, the source factor affect only the classic action term:

$$K = e^{\frac{i}{\hbar}S_c} \int \mathrm{D}[y] \exp\left(\frac{i}{\hbar}\int \mathrm{d}t \, (\frac{m}{2}\dot{y}^2)\right) \tag{22}$$

and, remembering the already computed path integral for free particle,

$$K = \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}} e^{\frac{i}{\hbar}S_c} \tag{23}$$

3.2 Generic Lagrangian and functional derivatives

Let L be a generic Lagrangian, and let's add a source term

$$L' = L + Jx \tag{24}$$

where $J = J(t)$. Considering time and space quantized, the propagator is:

$$K = \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \exp\left(\frac{i}{\hbar}\sum_i (L(\dot{x}_i, x_i) + J_i x_i)\right) \tag{25}$$

Deriving K by a generic J_i,

$$\frac{\partial K}{\partial J_i} = \frac{i}{\hbar}\int \mathrm{D}[x] \, x_i \exp\left(\frac{i}{\hbar}\sum_i (L(\dot{x}_i, x_i) + J_i x_i)\right) \tag{26}$$

and going to the continuous limit,

$$\frac{\delta K[J]}{\delta J(t^*)} = \frac{i}{\hbar}\int \mathrm{D}[x] \, x(t^*) \exp\left(\frac{i}{\hbar}\int \mathrm{d}t \, (L(\dot{x}, x) + Jx)\right) \tag{27}$$

where $\frac{\delta K[J]}{\delta J(t^*)}$ is the functional derivative.

To find the meaning of this expression the path integral can be split in three integral, wrt the position at time t^*. Let $\mathrm{D}[x_{i*}]$ be the differential on all the path from x_i to x^*, and $\mathrm{D}[x_{*f}]$ be the differential on all the path from x^* to x_f. So:

$$\frac{\delta K[J]}{\delta J(t^*)} = \frac{i}{\hbar}\int \mathrm{D}[x]\, x(t^*)\exp\left(\frac{i}{\hbar}\int \mathrm{d}t\,(L(\dot{x},x)+Jx)\right) \tag{28}$$

$$= \frac{i}{\hbar}\int \mathrm{D}[x_{i*}]\ \mathrm{d}x^*\,\mathrm{D}[x_{*f}]\, x^*\exp\left(\frac{i}{\hbar}\left(\int_{t_i}^{t^*}+\int_{t^*}^{t_f}\right)\mathrm{d}t\,(L(\dot{x},x)+Jx)\right) \tag{29}$$

$$= \frac{i}{\hbar}\int \mathrm{d}x^*\, K(f,*)x^*K(*,i) \tag{30}$$

Remembering that $K(f,i) = \left\langle x_f, t_f \middle| x_i, t_i\right\rangle$:

$$\frac{\delta K[J]}{\delta J(t^*)} = \frac{i}{\hbar}\int \mathrm{d}x^*\left\langle x_f, t_f \middle| x^*, t^*\right\rangle x^*\left\langle x^*, t^* \middle| x_i, t_i\right\rangle \tag{31}$$

$$= \frac{i}{\hbar}\int \mathrm{d}x^*\ \left\langle x_f \middle| e^{-\frac{i}{\hbar}H(t_f-t^*)} \middle| x^*\right\rangle x^*\left\langle x_* \middle| e^{-\frac{i}{\hbar}H(t^*-t_i)} \middle| x_i\right\rangle \tag{32}$$

Therefore, having $\int \mathrm{d}x\,|x\rangle\, x\,\langle x| = \hat{x}$:

$$\frac{\delta K[J]}{\delta J(t^*)} = \frac{i}{\hbar}\left\langle x_f \middle|\, e^{-\frac{i}{\hbar}H(t_f-t^*)}\hat{x}^* e^{-\frac{i}{\hbar}H(t^*-t_i)}\,|x_i\rangle\right. \tag{33}$$

$$= \frac{i}{\hbar}\left\langle x_f, t_f \middle|\, \hat{x}(t^*)\,|x_i, t_i\rangle\right. \tag{34}$$

that means, the functional derivative allow us to compute the expectation value for $\hat{x}$ at a fixed time t^*.

3.3 Smart way to solve perturbed harmonic oscillator

Let's consider a perturbed harmonic oscillator:

$$L = \underbrace{\frac{m}{2}\dot{x}^2 + \frac{m}{2}\omega^2 x^2}_{L_{HO}} - \lambda x^4 \tag{35}$$

To solve the path integral, let's add a source term to the Lagrangian and compute

$K[J]$:

$$K[J] = \int \mathrm{D}[x] \, \exp\left(\frac{i}{\hbar}\int \mathrm{d}t \,(L_{HO} - \lambda x^4 + Jx)\right) \tag{36}$$

$$= \int \mathrm{D}[x] \, \exp\left\{-\frac{i}{\hbar}\int \mathrm{d}t \,\lambda x^4\right\} \exp\left(\frac{i}{\hbar}\int \mathrm{d}t \,(L_{HO} + Jx)\right) \tag{37}$$

$$= \int \mathrm{D}[x] \sum_n \frac{1}{n!}\left(-\frac{i\lambda}{\hbar}\int \mathrm{d}t \, x^4\right)^n \exp\left(\frac{i}{\hbar}\int \mathrm{d}t \,(L_{HO} + Jx)\right) \tag{38}$$

$$= \sum_n \frac{1}{n!}\left(-\frac{i\lambda}{\hbar}\right)^n \int \mathrm{D}[x] \, \mathrm{d}t_1 \ldots \mathrm{d}t_n \, x_1^4 \ldots x_n^4 \exp\left(\frac{i}{\hbar}\int \mathrm{d}t \,(L_{HO} + Jx)\right) \tag{39}$$

$$= \sum_n \frac{1}{n!}\left(-\frac{i\lambda}{\hbar}\right)^n \frac{\hbar^4}{i^4}\frac{\delta^4}{\delta J_1^4} \cdots \frac{\hbar^4}{i^4}\frac{\delta^4}{\delta J_n^4} \int \mathrm{D}[x] \, \exp\left(\frac{i}{\hbar}\int \mathrm{d}t \,(L_{HO} + Jx)\right) \tag{40}$$

where when writing a product that involves a functional derivative the time integral is understood, i.e.

$$\frac{\partial F[J]}{\partial J} y = \int \mathrm{d}t \, \frac{\partial F[J]}{\partial J(t)} y(t) \tag{41}$$

In the equation (**??**) can be recognized the path integral for the un-perturbed harmonic oscillator:

$$K[J] = \sum_n \frac{1}{n!}\left(-i\lambda\hbar^3\right)^n \frac{\delta^4}{\delta J_1^4} \cdots \frac{\delta^4}{\delta J_n^4} K_{HO}[J] \qquad = \exp\left(-i\lambda\hbar^3 \frac{\delta^4}{\delta J^4}\right) K_{HO}[J] \tag{42}$$

Finally, to return to the initial problem without source term, is enough to compute everything in $J = 0$:

$$K = \exp\left(-i\lambda\hbar^3 \frac{\delta^4}{\delta J^4}\right) K_{HO}[J]\Bigg|_{J=0} \tag{43}$$