

# Path Integral

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## 1 Before the PI

### 1.1 Schrödinger, Heisenberg & interaction

We denote  $U = \exp(Ht/i\hbar)$ , and similarly with  $H_0 \rightarrow U_0$ ,  $V \rightarrow U_V$ .

#### Schrödinger

1. State kets are  $|\psi(t)\rangle = U |\psi(t=0)\rangle$ ;
2. observables are  $A(t) \equiv A(t=0)$ ;
3. base kets are defined by  $A|a\rangle = a|a\rangle$ , therefore  $|a(t)\rangle \equiv |a(t=0)\rangle$ .

#### Heisenberg

1. State kets are  $|\psi(t)\rangle \equiv |\psi(t=0)\rangle$ ;
2. observables are  $A(t) = U^\dagger A(t=0)U$ ;
3. base kets are  $|a(t)\rangle = U^\dagger |a(t=0)\rangle$ .

**Interaction** We denote by a subscript  $S$  or  $I$  objects in the Schrödinger or interaction system. In the

1. State kets are defined as  $|\psi(t)\rangle_I = U_0^\dagger |\psi(t)\rangle_S$ ;
2. observables are defined as  $A_I(t) = U_0^\dagger A_S U_0$ ;
3. as base kets we use eigenstates of  $H_0$ :  $H_0 |n\rangle = E_n |n\rangle$ . These evolve like  $|n(t)\rangle = U_0 |n(t=0)\rangle$ .

Then, we can generically write the evolution of a Schrödinger ket as

$$|\psi(t)\rangle_S = \sum_n c_n(t) \exp(E_n t / i\hbar) |n\rangle, \quad (1)$$

therefore the evolution of the interaction ket is

$$|\psi(t)\rangle_I = U_0^\dagger |\psi(t)\rangle_S = \sum_n c_n(t) |n\rangle. \quad (2)$$

We can write an equation for the evolution of the  $c_n(t)$ :

$$i\hbar \dot{c}_n(t) = \sum_m V_{nm} \exp(i\omega_{nm}t) c_m(t), \quad (3)$$

where  $\omega_{nm} = (E_n - E_m)/\hbar$  and  $V_{nm} = \langle n | V | m \rangle$ . This is a matrix equation for the coefficient vector.

**Time-dep perturbations** If we define the interaction-picture evolution operator as  $|\alpha, t\rangle = U_I(t) |\alpha, 0\rangle$  we have its evolution as  $i\hbar \partial_t U_I = V_I U_I$ .

For small times  $U_I \approx \mathbb{1}$ , so we can integrate the Schrödinger equation:

$$U_I = \mathbb{1} + \frac{1}{i\hbar} \int_0^t V_I(t') U_I(t') dt' \quad (4a)$$

$$= \mathbb{1} + \frac{1}{i\hbar} \int_0^t V_I(t') \left( \mathbb{1} + \frac{1}{i\hbar} \int_0^{t'} V_I(t'') U_I(t'') dt'' \right) dt' \quad (4b)$$

$$= \mathbb{1} + \frac{1}{i\hbar} \int_0^t V_I(t') dt' + \frac{1}{(i\hbar)^2} \int_0^t \int_0^{t'} V_I(t') V_I(t'') dt' dt'' + o(V_I^2) \quad (4c)$$

Now, if we start on a base ket  $|i\rangle$ , the evolution coefficients  $c_n(t)$  will be given by the matrix elements  $\langle n | U_I(t) | i \rangle$ . We can compute these to any order in  $V_I$ , by taking the components of the previous equation and applying the following computation any time we have the components of  $V_I$ :

$$\langle n | V_I | i \rangle = \langle n | U_0^\dagger V U_0 | i \rangle = \exp(i\omega_{ni}t) V_{ni}, \quad (5)$$

since the  $|n\rangle$  are eigenstates of the unperturbed Hamiltonian.

## 1.2 The propagator

If  $H |\alpha'\rangle = \alpha' |\alpha'\rangle$ , then the evolution operator can be decomposed as

$$U(t) = \sum_{\alpha'} \exp\left(\frac{E_{\alpha'} t}{i\hbar}\right) |\alpha'\rangle \langle \alpha'|. \quad (6)$$

This can be written in the position basis as a Green function by contracting with two position vectors:

$$\langle x' | U(t) | x'' \rangle = \sum_{\alpha'} \exp\left(\frac{E_{\alpha'} t}{i\hbar}\right) \langle x' | \alpha' \rangle \langle \alpha' | x'' \rangle \stackrel{\text{def}}{=} K(x', x''; t), \quad (7)$$

and with this we can directly compute the evolution at a generic time:  $\psi(x'', t) = \int d^3x' K(x'', x'; t) \psi(x')$ . It is effectively the transition amplitude:  $K = \langle x'', t | x', 0 \rangle$  when seen in the Heisenberg picture (since we are evolving a base ket).

1.  $K(x', x'', t)$  satisfies the Schrödinger equation, since it is a sum of terms which do;
2.  $\lim_{t \rightarrow 0} K(x', x'', t) = \delta^3(x', x'')$ .

### 1.3 Some useful propagators

**Free particle** We consider  $H = p^2/2m$ ; the momentum eigenstates are  $p |p'\rangle = p' |p'\rangle$ , and they are also energy eigenstates with  $H |p'\rangle = ((p')^2/2m) |p'\rangle$ .

We compute:

$$K(x', x'', t) = \int dp' \langle x'' | p' \rangle \langle p' | x' \rangle \exp\left(\frac{(p')^2 t}{i\hbar 2m}\right), \quad (8)$$

and recall that  $\langle x | p \rangle = \exp(-px/i\hbar) / \sqrt{2\pi\hbar}$ . We simplify the exponent to get a Gaussian integral: it is known that

$$\int_{\mathbb{R}} dx \exp(-i\alpha x^2) = \sqrt{\frac{\pi}{i\alpha}}, \quad (9)$$

therefore in the end we get:

$$K(x', x'', t) = \frac{1}{2\pi\hbar} \exp\left(\frac{im(x'' - x')^2}{2\hbar t}\right) \sqrt{\frac{2m\pi\hbar}{it}}, \quad (10)$$

which for  $t \rightarrow 0$  is in the form  $\exp((x' - x'')^2/t) \sqrt{t} \rightarrow \delta(x'' - x')$ .

**Harmonic oscillator** We consider  $H = p^2/2m + m\omega^2 x^2/2$ . It is known that the eigenfunctions are given by the Hermite polynomials:

$$\langle x' | n \rangle = \frac{1}{\pi^{1/4} \sqrt{2^n n!}} \frac{1}{x_0^{1/2}} H_n y \exp\left(-\frac{(x/x_0)^2}{2}\right), \quad (11)$$

where  $x_0 = \sqrt{\hbar/(m\omega)}$  (both masses and frequencies are inverse lengths in natural units!). We also know the eigenenergies,  $E_n = \hbar\omega(n + 1/2)$ . We can then compute away, to finally get:

$$K = \sqrt{\frac{m\omega}{2i\pi\hbar \sin(\omega t)}} \exp\left(\frac{im\omega\left((x'')^2 + (x')^2\right)\cos(\omega t) - 2x'x''}{2\hbar \sin(\omega t)}\right). \quad (12)$$

## 2 The Path Integral

We can time-slice the interval between a certain time  $0 = t_0$  and another time  $t = t_N$  in  $N$  parts. Then, evolving the system with a the propagator for each one, we get:

$$K(x_N, x_0, t) = \int \left( \prod_{i=1}^{N-1} dx_i \right) \left( \prod_{i=0}^{N-1} \langle x_{i+1}, t_{i+1} | x_i, t_i \rangle \right). \quad (13)$$

We call the time-slice  $\epsilon = t/N$ . We will expand the in  $\epsilon$  up to first order the evolution operator  $\exp(H\epsilon/i\hbar)$ .

The bracket  $\langle x_{i+1}, t_{i+1} | x_i, t_i \rangle$ , if we assume  $t_{i+1} - t_i = \epsilon$ , can be written as

$$\langle x_{i+1} | \exp\left(\frac{H\epsilon}{i\hbar}\right) | x_i \rangle. \quad (14)$$

The first case we consider is a free particle Hamiltonian,  $X = p^2/2m + V(x)$ ; we insert a momentum completeness

$$\langle x_{i+1} | \exp\left(\frac{H\epsilon}{i\hbar}\right) | x_i \rangle = \int dp' \langle x_{i+1} | p' \rangle \langle p' | \exp\left(\frac{H\epsilon}{i\hbar}\right) | x_i \rangle, \quad (15)$$

and now make a

**Claim 1.**

$$\langle p' | \exp\left(\frac{H\epsilon}{i\hbar}\right) | x_i \rangle = \exp\left(\frac{\epsilon}{i\hbar} \left(p'^2/2m + v(x_i)\right)\right) + O(\epsilon^{3/2}). \quad (16)$$

*Proof.* We rewrite the expression as

$$\langle p' | \exp(\epsilon(T + U)) | x \rangle, \quad (17)$$

where  $T = \frac{p'^2/2m}{i\hbar}$ ,  $U = \frac{V}{i\hbar}$ . If we multiply and divide by  $e^{\epsilon U}$  (respectively  $T$ ) we get that proving our identity is equivalent to proving that:

$$\langle p | \exp(-\epsilon T) \exp(\epsilon H/i\hbar) \exp(-\epsilon U) | x \rangle = \langle p' | x \rangle, \quad (18)$$

and we call the product of exponentials  $\exp(-\epsilon^2 C)$ .

The CBH formula yields a solution  $C$  to the equation  $e^C = e^A e^B$  in terms of commutators of  $A$  and  $B$ . With it, we arrive at:

$$\exp(-\epsilon^2 C) = \exp\left(-\frac{\epsilon^2}{2}[T, U] + O(\epsilon^3)\right), \quad (19)$$

...

□

Then, using the fact that  $\langle x|p\rangle = \exp(-xp/i\hbar)/\sqrt{2\pi\hbar}$ , we get:

$$\int dp' \langle x_{i+1}|p'\rangle \langle p'| \exp\left(\frac{H\epsilon}{i\hbar}\right) |x_i\rangle = \quad (20a)$$

$$= \frac{1}{2\pi\hbar} \int dp' \exp\left(-\frac{(x_{i+1} - x_i)p'}{i\hbar}\right) \exp\left(\frac{\epsilon}{i\hbar} \left(\frac{p'^2}{2m} - V(x_i)\right)\right) \quad (20b)$$

,

## 2.1 Path integral of a quadratic Lagrangian

Let  $L(x, \dot{x})$  be a quadratic Lagrangian:

$$L = a\dot{x}^2 + bx\dot{x} + cx^2 + d\dot{x} + ex + f. \quad (21)$$

The propagator for  $L$ , computed via path integral, is:

$$K = \int D[x] \exp\left(\frac{i}{\hbar} \int dt L\right) \quad (22a)$$

$$= \int D[x] \exp\left(\frac{i}{\hbar} \int dt \underbrace{(a\dot{x}^2 + bx\dot{x} + cx^2 + d\dot{x} + ex + f)}_F\right). \quad (22b)$$

Let  $x_c$  be the classical path; let's write  $x = x_c + y$  and therefore  $D[x] = D[y]$ . Properly replacing  $x$  and  $\dot{x}$ , the propagator can be written as:

$$F = a(\dot{x}_c^2 + \dot{y}^2 + 2\dot{x}_c\dot{y}) + b(x_c\dot{x}_c + x_c\dot{y} + y\dot{x}_c + y\dot{y}) + c(x_c^2 + y^2 + 2x_cy) + d(\dot{x}_c + \dot{y}) + e(x_c + y) + f \quad (23a)$$

$$= (a\dot{x}_c^2 + bx_c\dot{x}_c + cx_c^2 + d\dot{x}_c + ex_c + f) + (a\dot{y}^2 + by\dot{y} + cy^2) + \dot{y}(2a\dot{x}_c + bx_c + d) + y(b\dot{x}_c + 2cx_c + e) \quad (23b)$$

$$= L_c + (a\dot{y}^2 + by\dot{y} + cy^2) + \dot{y} \left. \frac{\partial L}{\partial \dot{x}} \right|_{x_c} + y \left. \frac{\partial L}{\partial x} \right|_{x_c}, \quad (23c)$$

and so

$$K = \int D[y] \exp \left( \frac{i}{\hbar} \int dt \left( L_c + (a\dot{y}^2 + by\dot{y} + cy^2) + \dot{y} \frac{\partial L}{\partial \dot{x}} \Big|_{x_c} + y \frac{\partial L}{\partial x} \Big|_{x_c} \right) \right). \quad (24)$$

Evaluating by parts the last part of the integral:

$$I = \int dt \left( \dot{y} \frac{\partial L}{\partial \dot{x}} \Big|_{x_c} + y \frac{\partial L}{\partial x} \Big|_{x_c} \right) \quad (25)$$

$$= y \Big|_i^f \frac{\partial L}{\partial \dot{x}} \Big|_{x_c} + \int dt \left( -y \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \Big|_{x_c} - \frac{\partial L}{\partial x} \Big|_{x_c} \right) \right) \quad (26)$$

that, by the equation of motion and the definition of  $y$ , is 0. Therefore,

$$K = e^{\frac{i}{\hbar} S_c} \int D[y] \exp \left( \frac{i}{\hbar} \int dt (a\dot{y}^2 + by\dot{y} + cy^2) \right) \quad (27)$$

Only the quadratic terms contribute to the propagator prefactor; the linear terms affect only the classical action computing.

### 3 Adding a source term to the action

#### 3.1 The free particle case

Let's suppose to have a free particle, and let's add a source term:

$$L = \frac{m}{2} \dot{x}^2 + Jx \quad (28)$$

As showed for the quadratic Lagrangians, the source factor affect only the classic action term:

$$K = e^{\frac{i}{\hbar} S_c} \int D[y] \exp \left( \frac{i}{\hbar} \int dt \left( \frac{m}{2} \dot{y}^2 \right) \right) \quad (29)$$

and, remembering the already computed path integral for free particle,

$$K = \sqrt{\frac{m}{2\pi i \hbar (t_f - t_i)}} e^{\frac{i}{\hbar} S_c} \quad (30)$$

### 3.2 Generic Lagrangian and functional derivatives

Let  $L$  be a generic Lagrangian, and let's add a source term

$$L' = L + Jx \quad (31)$$

where  $J = J(t)$ . Considering time and space quantized, the propagator is:

$$K = \int dx_1 \dots dx_n \exp \left( \frac{i}{\hbar} \sum_i (L(\dot{x}_i, x_i) + J_i x_i) \right) \quad (32)$$

Deriving  $K$  by a generic  $J_i$ ,

$$\frac{\partial K}{\partial J_i} = \frac{i}{\hbar} \int D[x] x_i \exp \left( \frac{i}{\hbar} \sum_i (L(\dot{x}_i, x_i) + J_i x_i) \right) \quad (33)$$

and going to the continuous limit,

$$\frac{\delta K[J]}{\delta J(t^*)} = \frac{i}{\hbar} \int D[x] x(t^*) \exp \left( \frac{i}{\hbar} \int dt (L(\dot{x}, x) + Jx) \right) \quad (34)$$

where  $\frac{\delta K[J]}{\delta J(t^*)}$  is the functional derivative.

To find the meaning of this expression the path integral can be split in three integral, wrt the position at time  $t^*$ . Let  $D[x_{i*}]$  be the differential on all the path from  $x_i$  to  $x^*$ , and  $D[x_{*f}]$  be the differential on all the path from  $x^*$  to  $x_f$ . So:

$$\frac{\delta K[J]}{\delta J(t^*)} = \frac{i}{\hbar} \int D[x] x(t^*) \exp \left( \frac{i}{\hbar} \int dt (L(\dot{x}, x) + Jx) \right) \quad (35)$$

$$= \frac{i}{\hbar} \int D[x_{i*}] dx^* D[x_{*f}] x^* \exp \left( \frac{i}{\hbar} \left( \int_{t_i}^{t^*} + \int_{t^*}^{t_f} \right) dt (L(\dot{x}, x) + Jx) \right) \quad (36)$$

$$= \frac{i}{\hbar} \int dx^* K(f, *) x^* K(*, i) \quad (37)$$

Remembering that  $K(f, i) = \langle x_f, t_f | x_i, t_i \rangle$ :

$$\frac{\delta K[J]}{\delta J(t^*)} = \frac{i}{\hbar} \int dx^* \langle x_f, t_f | x^*, t^* \rangle x^* \langle x^*, t^* | x_i, t_i \rangle \quad (38)$$

$$= \frac{i}{\hbar} \int dx^* \langle x_f | e^{-\frac{i}{\hbar} H(t_f - t^*)} | x^* \rangle x^* \langle x^* | e^{-\frac{i}{\hbar} H(t^* - t_i)} | x_i \rangle \quad (39)$$

Therefore, having  $\int dx |x\rangle x \langle x| = \hat{x}$ :

$$\frac{\delta K[J]}{\delta J(t^*)} = \frac{i}{\hbar} \langle x_f | e^{-\frac{i}{\hbar} H(t_f - t^*)} \hat{x} e^{-\frac{i}{\hbar} H(t^* - t_i)} | x_i \rangle \quad (40)$$

$$= \frac{i}{\hbar} \langle x_f, t_f | \hat{x}(t^*) | x_i, t_i \rangle \quad (41)$$

that means, the functional derivative allow us to compute the expectation value for  $\hat{x}$  at a fixed time  $t^*$ .

### 3.3 Smart way to solve perturbed harmonic oscillator

Let's consider a perturbed harmonic oscillator:

$$L = \underbrace{\frac{m}{2}\dot{x}^2 + \frac{m}{2}\omega^2 x^2}_{L_{HO}} - \lambda x^4 \quad (42)$$

To solve the path integral, let's add a source term to the Lagrangian and compute  $K[J]$ :

$$K[J] = \int D[x] \exp\left(\frac{i}{\hbar} \int dt (L_{HO} - \lambda x^4 + Jx)\right) \quad (43)$$

$$= \int D[x] \exp\left\{-\frac{i}{\hbar} \int dt \lambda x^4\right\} \exp\left(\frac{i}{\hbar} \int dt (L_{HO} + Jx)\right) \quad (44)$$

$$= \int D[x] \sum_n \frac{1}{n!} \left(-\frac{i\lambda}{\hbar} \int dt x^4\right)^n \exp\left(\frac{i}{\hbar} \int dt (L_{HO} + Jx)\right) \quad (45)$$

$$= \sum_n \frac{1}{n!} \left(-\frac{i\lambda}{\hbar}\right)^n \int D[x] dt_1 \dots dt_n x_1^4 \dots x_n^4 \exp\left(\frac{i}{\hbar} \int dt (L_{HO} + Jx)\right) \quad (46)$$

$$= \sum_n \frac{1}{n!} \left(-\frac{i\lambda}{\hbar}\right)^n \frac{\hbar^4}{i^4} \frac{\delta^4}{\delta J_1^4} \dots \frac{\hbar^4}{i^4} \frac{\delta^4}{\delta J_n^4} \int D[x] \exp\left(\frac{i}{\hbar} \int dt (L_{HO} + Jx)\right) \quad (47)$$

where when writing a product that involves a functional derivative the time integral is understood, i.e.

$$\frac{\partial F[J]}{\partial J} y = \int dt \frac{\partial F[J]}{\partial J(t)} y(t) \quad (48)$$

In the equation (47) can be recognized the path integral for the un-perturbed harmonic oscillator:

$$K[J] = \sum_n \frac{1}{n!} \left(-i\lambda\hbar^3\right)^n \frac{\delta^4}{\delta J_1^4} \dots \frac{\delta^4}{\delta J_n^4} K_{HO}[J] = \exp\left(-i\lambda\hbar^3 \frac{\delta^4}{\delta J^4}\right) K_{HO}[J] \quad (49)$$

Finally, to return to the initial problem without source term, is enough to compute everything in  $J = 0$ :

$$K = \exp\left(-i\lambda\hbar^3 \frac{\delta^4}{\delta J^4}\right) K_{HO}[J] \Big|_{J=0} \quad (50)$$