Path Integral

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1 Before the PI

1.1 Schrödinger, Heisenberg & interaction

We denote $U = \exp(Ht/i\hbar)$, and similarly with $H_0 \to U_0$, $V \to U_V$.

Schrödinger

1. State kets are $\left|\psi(t)\right\rangle = U \left|\psi(t=0)\right\rangle$;
2. observables are $A(t) \equiv A(t=0)$;
3. base kets are defined by $A\left|a\right\rangle = a\left|a\right\rangle$, therefore $\left|a(t)\right\rangle \equiv \left|a(t=0)\right\rangle$.

Heisenberg

1. State kets are $\left|\psi(t)\right\rangle \equiv \left|\psi(t=0)\right\rangle$;
2. observables are $A(t) = U^\dagger A(t=0) U$;
3. base kets are $\left|a(t)\right\rangle = U^\dagger \left|a(t=0)\right\rangle$.

Interaction We denote by a subscript S or I objects in the Schrödinger or interaction system. In the

1. State kets are defined as $\left|\psi(t)\right\rangle_I = U_0^\dagger \left|\psi(t)\right\rangle_S$;
2. observables are defined as $A_I(t) = U_0^\dagger A_S U_0$;
3. as base kets we use eigenstates of H_0: $H_0\left|n\right\rangle = E_n\left|n\right\rangle$. These evolve like $\left|n(t)\right\rangle = U_0\left|n(t=0)\right\rangle$.

Then, we can generically write the evolution of a Schrödinger ket as

$$\left|\psi(t)\right\rangle_S = \sum_n c_n(t) \exp(E_n t/i\hbar) \left|n\right\rangle \,, \tag{1}$$

therefore the evolution of the interaction ket is

$$\left|\psi(t)\right\rangle_I = U_0^\dagger \left|\psi(t)\right\rangle_S = \sum_n c_n(t) \left|n\right\rangle \,. \tag{2}$$

We can write an equation for the evolution of the $c_n(t)$:

$$i\hbar \dot{c_n}(t) = \sum_m V_{nm} \exp(i\omega_{nm} t) c_m(t) \,, \tag{3}$$

where $\omega_{nm} = (E_n - E_m)/\hbar$ and $V_{nm} = \left\langle n\right| V \left|m\right\rangle$. This is a matrix equation for the coefficient vector.

Time-dep perturbations If we define the interaction-picture evolution operator as $\left|\alpha, t\right\rangle = U_I(t) \left|\alpha, 0\right\rangle$ we have its evolution as $i\hbar\partial_t U_I = V_I U_I$.

For small times $U_I \approx \mathbb{1}$, so we can integrate the Schrödinger equation:

$$\begin{aligned}
U_I &= \mathbb{1} + \frac{1}{i\hbar} \int_0^t V_I(t') U_I(t') \,\mathrm{d}t' && \text{(4a)} \\
&= \mathbb{1} + \frac{1}{i\hbar} \int_0^t V_I(t') \left(\mathbb{1} + \frac{1}{i\hbar} \int_0^{t'} V_I(t'') U_I(t'') \,\mathrm{d}t'' \right) \mathrm{d}t' && \text{(4b)} \\
&= \mathbb{1} + \frac{1}{i\hbar} \int_0^t V_I(t') \,\mathrm{d}t' + \frac{1}{(i\hbar)^2} \int_0^t \int_0^{t'} V_I(t') V_I(t'') \,\mathrm{d}t' \,\mathrm{d}t'' + o(V_I^2) && \text{(4c)}
\end{aligned}$$

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Now, if we start on a base ket $\left|i\right\rangle$, the evolution coefficients $c_n(t)$ will be given by the matrix elements $\left\langle n\right| U_I(t) \left|i\right\rangle$. We can compute these to any order in V_I, by taking the components of the previous equation and applying the following computation any time we have the components of V_I:

$$\left\langle n\right| V_I \left|i\right\rangle = \left\langle n\right| U_0^\dagger V U_0 \left|i\right\rangle = \exp(i\omega_{ni} t) V_{ni} \,, \tag{5}$$

since the $\left|n\right\rangle$ are eigenstates of the unperturbed Hamiltonian.

1.2 The propagator

If $H\left|\alpha'\right\rangle = \alpha' \left|\alpha'\right\rangle$, then the evolution operator can be decomposed as

$$U(t) = \sum_{\alpha'} \exp\left(\frac{E_{\alpha'} t}{i\hbar}\right) \left|\alpha'\right\rangle\!\left\langle\alpha'\right| \,. \tag{6}$$

This can be written in the position basis as a Green function by contracting with two position vectors:

$$\langle x' | U(t) | x'' \rangle = \sum_{\alpha'} \exp\left(\frac{E_{\alpha'} t}{i\hbar}\right) \langle x' | \alpha' \rangle \langle \alpha' | x'' \rangle \stackrel{\text{def}}{=} K(x', x''; t) , \tag{7}$$

and with this we can directly compute the evolution at a generic time: $\psi(x'', t) = \int \mathrm{d}^3 x' \, K(x'', x'; t) \psi(x')$. It is effectively the transition amplitude: $K = \langle x'', t | x', 0 \rangle$ when seen in the Heisenberg picture (since we are evolving a base ket).

1. $K(x', x'', t)$ satisfies the Schrödinger equation, since it is a sum of terms which do;
2. $\lim_{t \to 0} K(x', x'', t) = \delta^3(x', x'')$.

1.3 Some useful propagators

Free particle We consider $H = p^2/2m$; the momentum eigenstates are $p | p' \rangle = p' | p' \rangle$, and they are also energy eigenstates with $H | p' \rangle = ((p')^2/2m) | p' \rangle$.

We compute:

$$K(x', x'', t) = \int \mathrm{d}p' \, \langle x'' | p' \rangle \langle p' | x' \rangle \exp\left(\frac{(p')^2 t}{i\hbar 2m}\right) , \tag{8}$$

and recall that $\langle x | p \rangle = \exp(-px/i\hbar)/\sqrt{2\pi\hbar}$. We simplify the exponent to get a Gaussian integral: it is known that

$$\int_{\mathbb{R}} \mathrm{d}x \exp\left(-i\alpha x^2\right) = \sqrt{\frac{\pi}{i\alpha}} , \tag{9}$$

therefore in the end we get:

$$K(x', x'', t) = \frac{1}{2\pi\hbar} \exp\left(\frac{im(x'' - x')^2}{2\hbar t}\right) \sqrt{\frac{2m\pi\hbar}{it}} , \tag{10}$$

which for $t \to 0$ is in the form $\exp\left((x' - x'')^2/t\right) \sqrt{t} \to \delta(x'' - x')$.

Harmonic oscillator We consider $H = p^2/2m + m\omega^2 x^2/2$. It is known that the eigenfunctions are given by the Hermite polynomials:

$$\langle x' | n \rangle = \frac{1}{\pi^{1/4} \sqrt{2^n n!}} \frac{1}{x_0^{1/2}} H_n y \exp\left(-\frac{(x/x_0)^2}{2}\right) , \tag{11}$$

where $x_0 = \sqrt{\hbar/(m\omega)}$ (both masses and frequencies are inverse lengths in natural units!). We also know the eigenenergies, $E_n = \hbar\omega(n + 1/2)$. We can then compute away, to finally get:

$$K = \sqrt{\frac{m\omega}{2i\pi\hbar\sin(\omega t))}} \exp\left(\frac{im\omega\Big(((x'')^2 + (x')^2)\cos(\omega t) - 2x'x''\Big)}{2\hbar\sin(\omega t)}\right). \tag{12}$$

2 The Path Integral

We can time-slice the interval between a certain time $0 = t_0$ and another time $t = t_N$ in N parts. Then, evolving the system with a the propagator for each one, we get:

$$K(x_N, x_0, t) = \int \left(\prod_{i=1}^{N-1} \mathrm{d}x_i\right)\left(\prod_{i=0}^{N-1} \langle x_{i+1}, t_{i+1}|x_i, t_i\rangle\right). \tag{13}$$

We call the time-slice $\epsilon = t/N$. We will expand the in ϵ up to first order the evolution operator $\exp(H\epsilon/i\hbar)$.