Path Integral

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1 Lesson 1

In lesson 1...

2 Lesson 4

In the previous lessons we introduced the Dirac representation, where we defined:

$$
\begin{aligned}
|\alpha t_0 t\rangle &= U(t-t_0)\,|\alpha t_0\rangle && \text{(1a)}\\
&= e^{-\frac{iH(t-t_0)}{\hbar}}\,|\alpha\rangle && \text{(1b)}\\
&= \sum_{a'} e^{-\frac{iE_{a'}(t-t_0)}{\hbar}} \left|a'\right\rangle \left\langle a'\middle|a t_0\right\rangle && \text{(1c)}\\
&= \sum c_a'(t_0) e^{-\frac{iE_{a'}(t-t_0)}{\hbar}} \left|a'\right\rangle && \text{(1d)}\\
&= \sum c_a'(t) \left|a'\right\rangle && \text{(1e)}
\end{aligned}
$$

and

$$\left\langle \bar{x'}\middle|a' t_0 t\right\rangle = \sum c_a'(t_0) e^{-\frac{iE_{a'}(t-t_0)}{\hbar}} \left\langle \bar{x'}\middle|a'\right\rangle\ , \tag{2}$$

such that $\psi(\bar{x'},t) = \sum c_a'(t_0) e^{-\frac{iE_{a'}(t-t_0)}{\hbar}} u_{a'}(\bar{x'})$. Than from Dirac completeness relation,

$$c_{a'}(t_0) = \left\langle a'\middle|\alpha t_0\right\rangle = \int d^3x' \left\langle a'\middle|\bar{x'}\right\rangle \left\langle \bar{x'}\middle|\alpha t_0\right\rangle , \tag{3}$$

and for this reason

$$\left\langle x'' \middle| \alpha t_0 t \right\rangle = \sum e^{-\frac{iE_{a'}(t-t_0)}{\hbar}} \left\langle x'' \middle| a' \right\rangle \int d^3x' \left\langle a' \middle| x' \right\rangle \left\langle x' \middle| a t_0 \right\rangle \tag{4a}$$

$$= \int d^3x' \sum \left\langle x'' \middle| a' \right\rangle \left\langle a' \middle| x' \right\rangle e^{-\frac{iE_{a'}(t-t_0)}{\hbar}} \left\langle x' \middle| a t_0 \right\rangle \tag{4b}$$

$$= \int d^3x' K \left\langle x' \middle| a t_0 \right\rangle \tag{4c}$$

where we defined the propagator K. We write otherwise $\psi(x'', t) = \int d^3x' K(x'', t, x', t_0)\psi(x', t_0)$, such that the propagator satisfies:

- K respects Schrodinger equation, since $i\hbar \left\langle x' \middle| \alpha \dot{t_0} t \right\rangle = \left\langle x' \right| H \left| \alpha t_0 t \right\rangle$ and $\left\langle x' \middle| a' t_0 t \right\rangle = e^{-\frac{iE_{a'}(t-t_0)}{\hbar}} \left\langle x'' \middle| a' \right\rangle$
- $\lim_{t \to t_0} K = \delta^3(x'' - x')$, i.e. the evolution of a particle which was in x' at $t = 0$

2.1 Free particle propagator

Now we want to obtain the propagator K aforementioned for the particular case of a free particle hamiltonian. In the space of momenta we have

$$P \left| p' \right\rangle = p' \left| p' \right\rangle, \left\langle x' \middle| p' \right\rangle = \frac{e^{\frac{ip'x'}{\hbar}}}{\sqrt{2\pi\hbar}}, \qquad H \left| p' \right\rangle = \frac{p'^2}{2m} \left| p' \right\rangle. \tag{5}$$

In this case we have

$$K = \int dp' \left\langle x'' \middle| p' \right\rangle \left\langle p' \middle| x' \right\rangle e^{-\frac{ip^2(t-t_0)}{\hbar 2m}} = \int_{-\infty}^{+\infty} dp' \frac{e^{\frac{i}{\hbar}\left(p'(x''-x') - \frac{p'^2}{2m}(t-t_0)\right)}}{2\pi\hbar} \tag{6}$$

Considering that

$$\frac{i}{\hbar}\left(p'(x''-x') - \frac{p'^2}{2m}(t-t_0)\right) = -\frac{i(t-t_0)}{2m\hbar}\left(p'^2 - \frac{2mp'(x''-x')}{t-t_0}\right) = -\frac{i(t-t_0)}{2m\hbar}\left(p' - \frac{m(x''-x')}{t-t_0}\right)^2 + \frac{i(t \tag{7}$$

Now, defining the new variable $\xi = p' - \frac{m(x''-x')}{t-t_0}$, we finally obtain

$$K = \frac{e^{\frac{im(x''-x')^2}{2m\hbar}}}{2\pi\hbar} \int_{-\infty}^{+\infty} d\xi e^{-\frac{i(t-t_0)\xi^2}{2m\hbar}} = \frac{e^{\frac{im(x''-x')^2}{2m\hbar}}}{2\pi\hbar} \sqrt{\frac{2m\pi\hbar}{i(t-t_0)}} \tag{8}$$

2.2 Harmonic oscillator

In the same way we studied the free particle problem, we want to obtain the propagator for the harmonic oscillator, i.e. a single particle system under the evolution with the hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}, \tag{9}$$

where we know that defining the operator distruction $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$, we obtain an explicit writing for the eigenvalues $E_N = \hbar\omega(N + \frac{1}{2})$ and the eigenfunctions respect the relations $|n+1\rangle = \frac{a^+}{\sqrt{n+1|n\rangle}}$, in such a way that

$$\left\langle x' \middle| a \left|0\right\rangle = \sqrt{\frac{m\omega}{2\hbar}} \left\langle x' \middle| \left(x + \frac{ip}{m\omega}\right)\left|0\right\rangle . \tag{10}$$

It follows that $\left(x' + \frac{i}{m\omega}(-i\hbar\frac{d}{dx'})\right)\left\langle x' \middle| 0 \right\rangle$ and, defining $x_0 = \sqrt{\frac{\hbar}{m\omega}}$, we have $\left\langle x' \middle| 0 \right\rangle = \eta e^{-\frac{x'^2}{2x_0^2}}$ with $\eta = \frac{1}{(x_0\pi)^{\frac{1}{4}}}$. In this case we obtain

$$\left\langle x' \middle| n \right\rangle = \frac{1}{\pi^{\frac{1}{4}}\sqrt{2^n n}} \frac{\left(x' - x_0^2\frac{d}{dx'}\right)^n}{x_0^{n+\frac{1}{2}}} e^{-\frac{x'^2}{2x_0^2}} = \frac{1}{\pi^{\frac{1}{4}}\sqrt{2^n n!}} H_n(y) e^{-\frac{x'^2}{2x_0^2}}, \tag{11}$$

where we defined $y = \frac{x'}{x_0}$, and the propagator

$$K = \sum_{n=0}^{\infty} \left\langle x' \middle| n \right\rangle \left\langle n \middle| x'' \right\rangle e^{-i\omega(n+\frac{1}{2})(t-t_0)} = \frac{e^{-\frac{i\omega\delta t}{2}}}{\sqrt{\pi}x_0} e^{-\frac{1}{2}\left(\frac{x''^2}{x_0^2} + \frac{x'^2}{x_0^2}\right)} \sum \frac{e^{-i\omega\delta t n}}{2^n n!} H_n(\frac{x''}{x_0}) H_n(\frac{x'}{x_0}). \tag{12}$$

Now, considering that, for $\zeta = e^{-i\omega\delta t}$, $\xi = \frac{x''}{x_0}$ and $\eta = \frac{x'}{x_0}$

$$e^{-(\xi^2+\eta^2)} \sum \frac{\zeta^n}{2^n n!} H_n\xi H_n\eta = \frac{e^{-\frac{\xi^2+\eta^2-2\xi\eta\zeta}{1-\zeta^2}}}{\sqrt{i-\zeta^2}}, \tag{13}$$

we have

$$K = \frac{e^{-\frac{i\omega\delta t}{2}}}{\sqrt{\pi}x_0} e^{\frac{1}{2}(\xi^2+\eta^2)} \frac{e^{-\frac{\xi^2+\eta^2-2\xi\eta\zeta}{1-\zeta^2}}}{\sqrt{i-\zeta^2}} = \frac{e^{\left(-\frac{(\xi^2+\eta^2)e^{i\omega\delta t}-2\xi\eta}{e^{i\omega\delta t}-e^{-i\omega\delta t}} + \frac{\xi^2+\eta^2}{2}\right)}}{x_0\sqrt{2\pi i \sin(\omega\delta t)}} = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega\delta t)}} e^{-\frac{(\xi^2+\eta^2)(\cos(\omega\delta t)+i\sin(\omega\delta t))-2\xi\eta-is(\xi^2+}{2i\sin(\omega\delta t)}} \tag{14}$$