Path Integral

Alessandro Lovo, Alberto Facheris, Francesco Gentile,
Giorgio Mentasti, Jacopo Tissino, Leonardo Zampieri

Autumn 2019

1 Before the PI

1.1 Schrödinger, Heisenberg & interaction

We denote $U = \exp(Ht/i\hbar)$, and similarly with $H_0 \to U_0$, $V \to U_V$.

Schrödinger

1. State kets are $\left|\psi(t)\right\rangle = U \left|\psi(t=0)\right\rangle$;
2. observables are $A(t) \equiv A(t=0)$;
3. base kets are defined by $A\left|a\right\rangle = a\left|a\right\rangle$, therefore $\left|a(t)\right\rangle \equiv \left|a(t=0)\right\rangle$.

Heisenberg

1. State kets are $\left|\psi(t)\right\rangle \equiv \left|\psi(t=0)\right\rangle$;
2. observables are $A(t) = U^\dagger A(t=0) U$;
3. base kets are $\left|a(t)\right\rangle = U^\dagger \left|a(t=0)\right\rangle$.

Interaction We denote by a subscript S or I objects in the Schrödinger or interaction system. In the

1. State kets are defined as $\left|\psi(t)\right\rangle_I = U_0^\dagger \left|\psi(t)\right\rangle_S$;
2. observables are defined as $A_I(t) = U_0^\dagger A_S U_0$;
3. as base kets we use eigenstates of H_0: $H_0\left|n\right\rangle = E_n\left|n\right\rangle$. These evolve like $\left|n(t)\right\rangle = U_0\left|n(t=0)\right\rangle$.

Then, we can generically write the evolution of a Schrödinger ket as

$$\left|\psi(t)\right\rangle_S = \sum_n c_n(t)\exp(E_n t/i\hbar)\left|n\right\rangle \,, \tag{1}$$

therefore the evolution of the interaction ket is

$$\left|\psi(t)\right\rangle_I = U_0^\dagger \left|\psi(t)\right\rangle_S = \sum_n c_n(t)\left|n\right\rangle \,. \tag{2}$$

We can write an equation for the evolution of the $c_n(t)$:

$$i\hbar\dot{c_n}(t) = \sum_m V_{nm}\exp(i\omega_{nm}t)c_m(t)\,, \tag{3}$$

where $\omega_{nm} = (E_n - E_m)/\hbar$ and $V_{nm} = \langle n| V |m\rangle$. This is a matrix equation for the coefficient vector.

Time-dep perturbations If we define the interaction-picture evolution operator as $|\alpha, t\rangle = U_I(t)|\alpha, 0\rangle$ we have its evolution as $i\hbar\partial_t U_I = V_I U_I$.

For small times $U_I \approx \mathbb{1}$, so we can integrate the Schrödinger equation:

$$U_I = \mathbb{1} + \frac{1}{i\hbar}\int_0^t V_I(t')U_I(t')\,\mathrm{d}t' \tag{4a}$$

$$= \mathbb{1} + \frac{1}{i\hbar}\int_0^t V_I(t')\left(\mathbb{1} + \frac{1}{i\hbar}\int_0^{t'} V_I(t'')U_I(t'')\,\mathrm{d}t''\right)\mathrm{d}t' \tag{4b}$$

$$= \mathbb{1} + \frac{1}{i\hbar}\int_0^t V_I(t')\,\mathrm{d}t' + \frac{1}{(i\hbar)^2}\int_0^t\int_0^{t'} V_I(t')V_I(t'')\,\mathrm{d}t'\,\mathrm{d}t'' + o(V_I^2) \tag{4c}$$

.

Now, if we start on a base ket $|i\rangle$, the evolution coefficients $c_n(t)$ will be given by the matrix elements $\langle n| U_I(t) |i\rangle$. We can compute these to any order in V_I, by taking the components of the previous equation and applying the following computation any time we have the components of V_I:

$$\langle n| V_I |i\rangle = \langle n| U_0^\dagger V U_0 |i\rangle = \exp(i\omega_{ni}t)V_{ni}\,, \tag{5}$$

since the $|n\rangle$ are eigenstates of the unperturbed Hamiltonian.

1.2 The propagator

If $H\left|\alpha'\right\rangle = \alpha'\left|\alpha'\right\rangle$, then the evolution operator can be decomposed as

$$U(t) = \sum_{\alpha'}\exp\left(\frac{E_{\alpha'}t}{i\hbar}\right)\left|\alpha'\right\rangle\!\left\langle\alpha'\right| \,. \tag{6}$$