Kolmogorov's 5/3 law

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"Andrei Nikolaevich Kolmogorov's work in 1941 remains a major source of inspiration for turbulence research... Is it by accident that the deepest insight into turbulence came from Andrei Nikolaevich Kolmogorov, a mathematician with a keen interest in the real world?"

Uriel Frisch, Turbulence: the legacy of A.N. Kolmogorov, Cambridge University Press, 1995

Overview

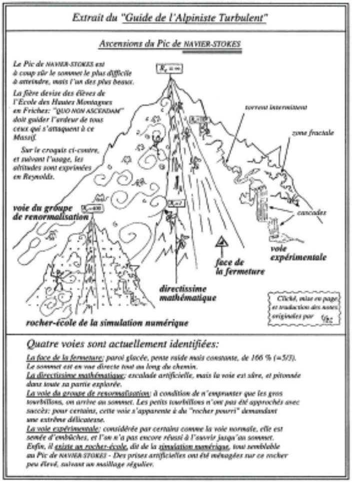
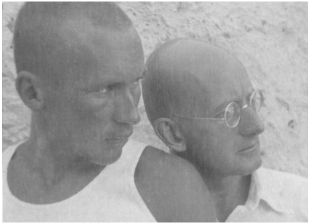


Fig. 9.1. Cartoon drawn in 1977 by the astronomer Philippe Delache, a penetrating observer of the turbulence community. He was the author's friend and died prematurely in 1994. The figure shows the 'Navier–Stokes peak' and four explored faces: experimentation, closure, mathematics and renormalization. It also shows a reduced model, the rock-climbing school of numerical simulation.

Overview

- About Kolmogorov.
- Introduction: laminar vs turbulent, Navier-Stokes equations, Reynolds number, Richardson's cascade.
- Reynolds stress and the spectrum of turbulence.
- Kolmogorov's 1941 theory: hypothesis, Kolmogorov's scales, Kolmogorov's spectrum.
- Concluding remarks.
- References.

1. About Kolmogorov

Born: April 25, 1903, Tambov, Imperial Russia
Died: October 20, 1987 (aged 84), Moscow, USSR
Institution: Moscow State University, Advisor: Nikolai Luzin
Students: Vladimir Arnold, Grigory Barenblatt, Roland Dobrushin, Eugene B. Dynkin, Israil Gelfand, Boris V. Gnedenko, Leonid Levin, Per Martin-Löf, Yuri Prokhorov, Vladimir A. Rokhlin, Yakov G. Sinai, Albert N. Shiryaev, Anatoli G. Vitushkin, and many others
Research areas: probability theory, topology, logic, turbulence, dynamical systems, analysis, etc. ... almost everything but number theory
Awards: USSR State Prize (1941), Balzan Prize (1963), Lenin Prize (1965), Wolf Prize (1980), Lobachevsky Prize (1987)

2. Introduction

- "...*turbulence* or *turbulent flow* is a fluid regime characterized by chaotic, stochastic property changes. This includes low momentum diffusion, high momentum convection, and rapid variation of pressure and velocity in space and time. Flow that is not turbulent is called *laminar flow*." (Wikipedia)
- *Navier-Stokes equations* for incompressible fluid of constant density (1823 - 1845):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \nu \, \triangle \mathbf{v},$$

$$\nabla \mathbf{v} = 0,$$

+initial and boundary conditions.

  Here, ν - kinematic viscosity ($10^{-6} m^2/s$ for water and $1.5 \cdot 10^{-5} m^2/s$ for air).

- *Reynolds number* (control parameter): $Re = \frac{LV}{\nu}$, where L is a characteristic length, V is a characteristic velocity of the flow.
- *Similarity principle* for incompressible flow: for a given geometrical shape of the boundaries, Re is the only control parameter of the flow.

2. Introduction

What happens when increasing the Reynolds number in flow past a cylinder? (Van Dyke (1982) An Album of Fluid Motion)

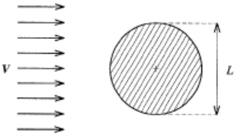


Fig. 1.1. Uniform flow with velocity V, incident on a cylinder of diameter L.

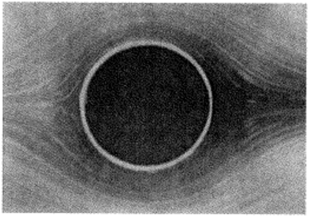


Fig. 1.2. Uniform flow past a cylinder at $R = 0.16$ (Van Dyke 1982). Photograph S. Taneda.

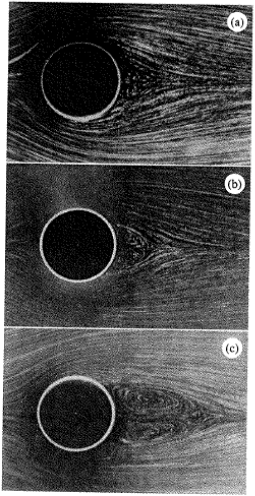


Fig. 1.4. Circular cylinder at $R = 9.6$ (a), $R = 13.1$ (b) and $R = 26$ (c) (Van Dyke 1982). Photograph S. Taneda.

2. Introduction

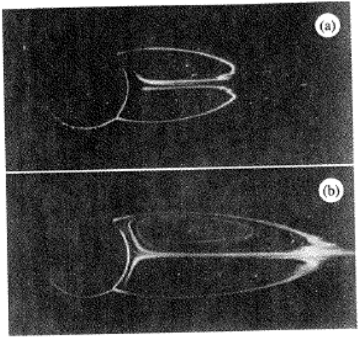


Fig. 1.5. Circular cylinder at $R = 28.4$ (a) and $R = 41.0$ (b) (Van Dyke 1982). Photograph S. Taneda.

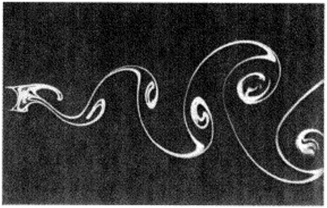


Fig. 1.6. Kármán vortex street behind a circular cylinder at $R = 140$ (Van Dyke 1982). Photograph S. Taneda.

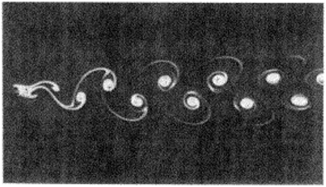


Fig. 1.7. Kármán vortex street behind a circular cylinder at $R = 105$ (Van Dyke 1982). Photograph S. Taneda.

2. Introduction

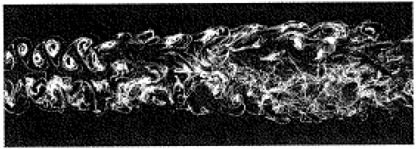


Fig. 1.9. Wake behind two identical cylinders at $R = 240$. Courtesy R. Dumas.

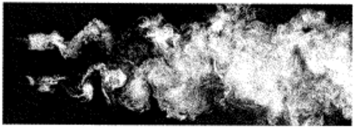


Fig. 1.10. Wake behind two identical cylinders at $R = 1800$. Courtesy R. Dumas.

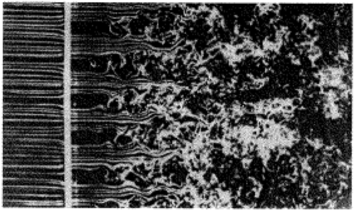


Fig. 1.11. Homogeneous turbulence behind a grid. Photograph T. Corke and H. Nagib.

2. Introduction

Richardson's picture of turbulence (1922):

- a turbulent flow is composed by *'eddies'* of different size
- the large eddies are unstable and eventually break up into smaller eddies, and so on
- the energy is passed down from the large scales of the motion to smaller scales until reaching a sufficiently small length scale such that the viscosity of the fluid can effectively dissipate the kinetic energy.

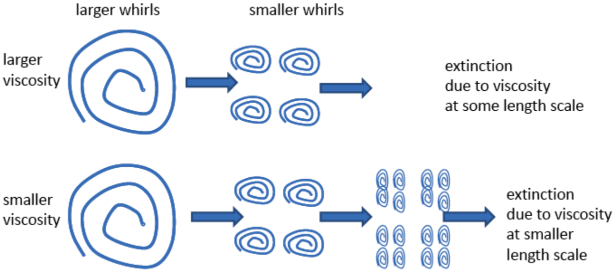


Figure: Richardson's energy cascade.

2. Introduction

Lewis Fry Richardson described this process in a verse:

"Big whirls have little whirls,
Which feed on their velocity,
And little whirls have lesser whirls,
And so on to viscosity."

He took an inspiration from the *Jonathan Swift's* verse:

So, nat'ralists observe, a flea
Hath smaller fleas that on him prey,
And these have smaller yet to bite 'em,
And so proceed ad infinitum.
Thus every poet, in his kind,
Is bit by him that comes behind.

Remark by U. Frish: "The last two lines, which are not usually quoted, may also be relevant, if ..."

3. Reynolds stress and the spectrum of turbulence

Even under controlled laboratory conditions the detailed velocity field in such motions is not reproducible. Only the average (ensemble average) properties can be reproduced and predicted by a theory. Therefore, the motions are represented as a mean and fluctuating parts:

$$\mathbf{v} = \mathbf{U} + \mathbf{u}, \quad \text{where} \quad \overline{\mathbf{u}} = 0.$$

Then, the momentum equation becomes

$$\frac{\partial}{\partial t}(\mathbf{U}+\mathbf{u}) + ((\mathbf{U}+\mathbf{u})\nabla)(\mathbf{U}+\mathbf{u}) = -\nabla(P+p')/\rho_0 + \nu\triangle(\mathbf{U}+\mathbf{u})$$

The average reduces to

$$\frac{\partial}{\partial t}\mathbf{U} + (\mathbf{U}\nabla)\mathbf{U} = -\nabla P/\rho_0 + \nu\triangle\mathbf{U} + \Phi,$$

where

$$\Phi_i = -\overline{u_j\frac{\partial u_i}{\partial x_j}} = -\frac{\partial}{\partial x_j}\overline{u_i u_j} = \frac{1}{\rho_0}\frac{\partial}{\partial x_j}\tau_{ij}$$

and

$$\tau_{ij} = -\rho_0\overline{u_i u_j}$$

is the *'Reynolds stress'*.

3. Reynolds stress and the spectrum of turbulence

One of the most important quantities describing the turbulent motion is the *covariance tensor of the velocity field*

$$R_{ij}(\mathbf{r}, \mathbf{x}, t) = \overline{u_i(\mathbf{x} - \frac{1}{2}\mathbf{r}, t)u_j(\mathbf{x} + \frac{1}{2}\mathbf{r}, t)}.$$

In particular,

$$-\rho_0 R_{ij}(0, \mathbf{x}, t) = -\rho_0 \overline{u_i u_j} = \tau_{ij}.$$

The *spectrum tensor* is the Fourier transform of R_{ij} with respect to separation vector $\mathbf{r}$:

$$\Psi_{ij}(\mathbf{k}, \mathbf{x}, t) = (2\pi)^{-3} \int_{R^3} R_{ij}(\mathbf{r}, \mathbf{x}, t) e^{-i\mathbf{k}\mathbf{r}} d\mathbf{r},$$

the inverse relation being

$$R_{ij}(\mathbf{r}, \mathbf{x}, t) = \int_{R^3} \psi_{ij}(\mathbf{k}, \mathbf{x}, t) e^{i\mathbf{k}\mathbf{r}} d\mathbf{k}.$$

3. Reynolds stress and the spectrum of turbulence

Then, $\psi_{ii}(\mathbf{k})$ specifies the distribution of kinetic energy among the vector wave numbers $\mathbf{k}$ of the motion:

$$\begin{aligned}\frac{1}{2}\overline{u^2} &= \frac{1}{2}\int_{R^3}\psi_{ii}(\mathbf{k})d\mathbf{k} \\ &= \frac{1}{2}\int_0^\infty\int_{S^2}\psi_{ii}(\mathbf{k})dS(k)dk \\ &= \int_0^\infty E(k)dk.\end{aligned}$$

Here,

$$E(\mathbf{k}) = \frac{1}{2}\int_{S^2}\psi_{ii}(\mathbf{k})dS(k)$$

is the *scalar energy spectrum*, obtained by integrating $\psi_{ii}(\mathbf{k})$ over the spherical shell of radius k is wave number space. Thus, $E(k)$ represents the density of contributions to the kinetic energy per unit scalar wave number, regardless of direction.

3. Reynolds stress and the spectrum of turbulence

A dynamical equation for the rate of change of $E(k)$ can be derived from the momentum equation by

- taking the scalar product of the momentum equation for $\mathbf{x}'$ with the velocity at $\mathbf{x}''$
- rewriting the resulting equation with the roles of $\mathbf{x}'$ and $\mathbf{x}''$ reversed and adding the two
- averaging and taking the Fourier transform
- integrating over the spherical surface of radius k in wave number space

Result:

$$\frac{\partial}{\partial t}E(k) = -\frac{\partial}{\partial k}\epsilon(k) - 2\nu k^2 E(k) + \dots,$$

where $\epsilon(k)$ represents the rate of spectral energy transfer from all wave numbers whose magnitude is smaller than k to those with magnitude greater than k (*'energy dissipation rate'*).

4. Kolmogorov's 1941 theory

Are there any universal aspects of the turbulence?

Two important length scales:

- energy is supplied to the turbulent fluctuations at length scales of order l (*'energy-containing scale'*)
- energy is transferred in scale by nonlinear process, as a spectral cascade, until it is dissipated by viscosity at length scales of order η (*'dissipation scale'*)

Hypothesis 1

For very high Re, the turbulent motions with length scales much smaller than l are statistically independent of the components of the motion at the energy-containing scales.

The energy-containing scales of the motion may be inhomogeneous and anisotropic, but this information is lost in the cascade so that at much smaller scales the motion is locally homogeneous and isotropic.

4. Kolmogorov's 1941 theory

If the energy is transferred over many stages to the large wave numbers where it is dissipated, then the time scales characteristic of the interactions at large wave numbers must be very much smaller than the time scale of the energy-containing eddies. The motion of these large wave numbers is close to a state of statistical equilibrium (*'equilibrium range'*).

Thus, for $k \gg l^{-1}$

$$\frac{\partial}{\partial t} E(k) = -\frac{\partial}{\partial k}\epsilon(k) - 2\nu k^2 E(k) \approx 0.$$

Then one can introduce

$$\epsilon_0 = 2\nu \int_0^\infty k^2 E(k) dk,$$

the energy dissipation rate in the equilibrium range.

4. Kolmogorov's 1941 theory

Hypothesis 2 (first similarity hypothesis)

For very high *Re*, the statistics of components in the equilibrium range, being independent of the larger scales, is universally and uniquely determined by the viscosity ν and the rate of energy dissipation ϵ_0.

From these two quantities, we can define length and velocity scales that are characteristic of the motion in the equilibrium range. They are, respectively,

$$\eta = \left(\frac{\nu^3}{\epsilon_0}\right)^{1/4} \quad \textit{`Kolmogorov's length'}$$

$$v = (\nu\epsilon_0)^{1/4} \quad \textit{`Kolmogorov's velocity.'}$$

Here, $[\nu] = \frac{m^2}{s}, \quad [\epsilon_0] = \frac{m^2}{s^3}$.

Time-scale in the equilibrium range

$$\frac{\eta}{v} = \left(\frac{\nu}{\epsilon_0}\right)^{1/2} \quad \textit{`Kolmogorov's time'}$$

4. Kolmogorov's 1941 theory

How much energy in each k?

Since $[E(k)] = \frac{m^3}{s^2}$, then on dimensional grounds

$$\begin{aligned} E(k) &= v^2 \eta f(k\eta), \\ &= \epsilon_0^{2/3} k^{-5/3} F(k\eta) \end{aligned}$$

Hypothesis 3 (second similarity hypothesis)

At very high Re the statistics of scales in the range $l^{-1} \ll k \ll \eta^{-1}$ (called *'inertial subrange'*) are universally and uniquely determined by the scale k and the rate of energy dissipation ϵ_0.

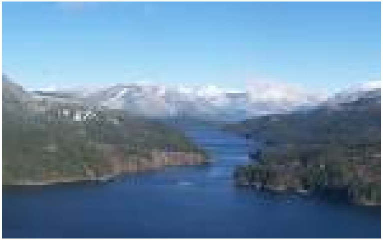
Then, in the inertial subrange the energy spectrum $E(k)$ of the turbulence must be of the form

$$E(k) = C\epsilon_0^{2/3} k^{-5/3} \quad (l^{-1} \ll k \ll \eta^{-1}),$$

where C is a constant. This is the famous *'Kolomogorov's 5/3 law'* (*'Kolmogorov-Obukhov 5/3 law'*).

4. Kolmogorov's 1941 theory

Several experimental studies have been made of the spectrum of turbulence at large *Re*. First convincing data was obtained in observations by Grant et. al (1962) on the turbulence generated in a tidal stream in Seymour Narrows, between Vancouver Island and Quadra Island, Britih Columbia.



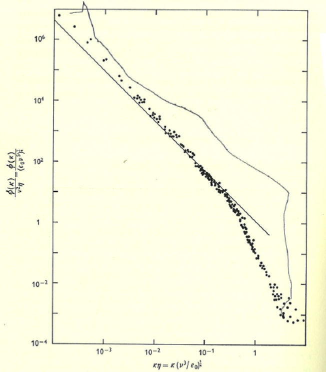


Fig. 6.2. The turbulence spectra, measured by Grant, Stewart and Moilliet (1962) and scaled according to the Kolmogorov parameters. The viscous dissipation rate ϵ_0 varied over a range of values of the order 100. The straight line represents variation as $\kappa^{-\frac{5}{3}}$. The top few points are believed to be rather high on account of the low frequency heaving motions of the ship.

4. Kolmogorov's 1941 theory

More experimental observations:

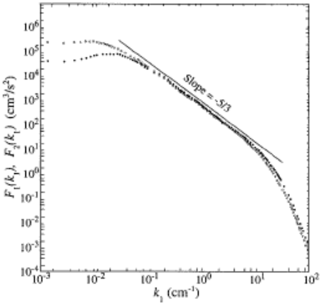


Fig. 5.7. log–log plot of the energy spectra of the streamwise component (white circles) and lateral component (black circles) of the velocity fluctuations in the time domain in a jet with $R_\lambda = 626$ (Champagne 1978).

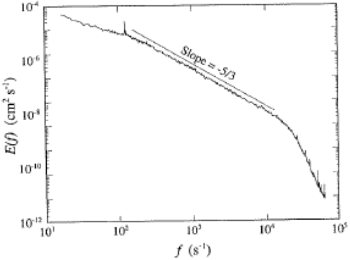


Fig. 5.8. log–log plot of the energy spectrum in the time domain in a low-temperature helium gas flow between counter-rotating cylinders with $R_\lambda = 1200$ (Maurer, Tabeling and Zocchi 1994).

5. Concluding remarks

- There is presently no fully deductive theory which starts from the Navier-Stokes equations and leads to the Kolmogorov's law.
- There is no natural closure for the averaged equations (*'closure problem'*).
- Modern viewpoint (U.Frish) postulates symmetries and self-similarity at small scales rather than universality (independence on the particular mechanisms generating the turbulence).
- One of the existing problems is *'intermittency'*, i.e. activity during only a fraction of the time, which decreases with the scale under consideration. Kolmogorov 1962 - an attempt to deal with that problem.
- Considerable progress in the area of *'weak wave turbulence'*, where Zakharov in 1965 has shown that wave kinetic equations (closed integro-differential equations for the spectrum) have exact power-law solutions which are similar to Kolmogorov's spectrum of hydrodynamic turbulence.

5. Concluding remarks

Kolmogorov's quotations (in Russian, from http://www.kolmogorov.info/) :

"Mathematicians always wish mathematics to be as 'pure' as possible, i.e. rigorous, provable. But usually most interesting real problems that are offered to us are inaccessible in this way. And then it is very important for a mathematician to be able to find himself approximate, non-rigorous but effective ways of solving problems."

"Mathematics is vast. One person is unable to study all its branches. In this sense specialization is inevitable. But at the same time mathematics is a united science. More and more links appear between its areas, sometimes in a most unexpected way. Some areas serve as tools for other areas. Therefore an isolation of mathematicians in too narrow borders should be destructive for our science."

"I always imagined humanity as the myriad of lights wandering in a fog, only just being able to sense the shining of other lights, but connected by a net of distinctive fiery threads, in one, two, three... directions. And the emergence of such connections through the fog to another light can be very easily called a 'miracle'."

6. References

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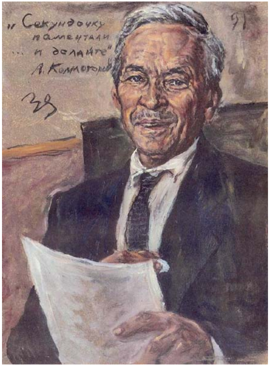
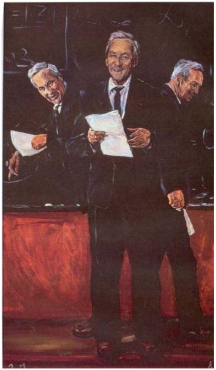
The end

Portraits of A.N. Kolmogorov by D. Gordeev
(http://www.kolmogorov.info/photo.html)





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Handwritten by Kolmogorov on a blackboard in his home:
"Men are cruel, but Man is kind" (Rabindranath Tagore)