Effective Algorithms With Circulant-Block Matrices

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ABSTRACT

Effective numerical algorithms for circulant-block matrices A whose blocks are circulant are obtained. The eigenvalues of such matrices are determined in terms of the eigenvalues of matrices of reduced dimension, and systems of linear equations involving these matrices are solved efficiently using fast Fourier transforms.

1. INTRODUCTION

Circulant matrices arise in many applications in mathematics, physics, and other applied sciences in problems possessing a periodicity property [3, 10, 12–15, 19]. In this section we give the definition and the basic properties of circulant matrices [8] and define the special class of circulant-block matrices having circulant blocks with any arbitrary block structure. The circulant-block matrices with the circulant or factor circulant structure were considered in [1, 4–6, 17, 18].

In Section 2, we discuss the spectral properties of circulant-block matrices and obtain the result that any eigenvalue problem for a circulant-block matrix can be reduced to a number of eigenvalue problems for matrices of reduced dimension.

In Section 3, we give the algorithm for solving systems of linear equations with diagonalizable circulant-block matrices.

In Section 4, we consider some special classes of nondiagonalizable circulant-block matrices. Finally we present the results from numerical tests on some realistic data.

655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

DEFINITION 1. A square matrix $A \in \mathbb{R}^{n\times n}$ is called *circulant* if

$$a_{k+1,l+1} = a_{kl} \qquad \text{for } k = 1,\dots,n-1, \quad l = 1,\dots,n-1;$$

$$a_{k+1,1} = a_{kn} \qquad \text{for } k = 1,\dots,n-1.$$

It is clear that A is uniquely defined by its first row $(a_{11},\dots,a_{1n}) = a^T$. A simple circulant matrix is the permutation matrix J:

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Thus each circulant matrix can be written as

$$A = \sum_{l=1}^{n} a_{1l} J^{l-1}. \tag{1}$$

The most important property of circulant matrices [8] is that they are diagonalizable by the Fourier matrix F:

$$A = n^{-1} F \Lambda F^*,$$

where the Fourier matrix $F \in \mathbb{C}^{n\times n}$ is defined by

$$f_{kl} = \omega_k^{l-1}, \qquad \omega_k = e^{i(2\pi/n)(k-1)}, \quad i^2 = -1.$$

Λ is the diagonal matrix with the eigenvalues of A, computed from

$$\Lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_n) = \operatorname{diag}(Fa).$$

The eigenvalues of the matrix J are the roots of unity:

$$\begin{aligned} Jf_k &= \omega_k f_k, \qquad \omega_k = e^{i(2\pi/n)(k-1)}, \\ f_k &= \left(\omega_k^0, \omega_k^1, \dots, \omega_k^{n-1}\right)^T \in \mathbb{C}^n, \\ k &= 1,\dots,n. \end{aligned} \tag{2}$$

DEFINITION 2. A square block matrix $A \in \mathbb{R}^{N\times N}$, $N = nm$,

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \cdots & \cdots & \cdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}, \qquad A_{kl} \in \mathbb{R}^{n\times n},$$

is called a *circulant-block matrix* (CB matrix) if each block A_{kl}, $k, l, \ldots, m$, is circulant.

Each circulant-block matrix A may be written as [see (1)]

$$A = A(J) = \begin{pmatrix} a_{11}(J) & \cdots & a_{1m}(J) \\ \cdots & \cdots & \cdots \\ a_{m1}(J) & \cdots & a_{mm}(J) \end{pmatrix}, \tag{3}$$

where $a_{kl}(J)$ are the polynomials of degree $n - 1$, and is defined by nm^2 elements (the first rows of all blocks A_{kl}). The circulant-block matrices have the following property:

LEMMA 1. *The set of all nonsingular CB matrices of the same dimension $N = nm$ and of the same block dimension n is a group with respect to matrix multiplication, and the set of all CB matrices of the same dimension $N = nm$ and of the same block dimension n is an Abelian group with respect to matrix addition.*

2. SPECTRAL PROPERTIES

We proceed now to compute the spectrum of the CB matrix $A(J)$ of dimension $N = nm$ using the spectrum of the matrix $A(t)$ of dimension m.

LEMMA 2. *Let $A \in \mathbb{C}^{m\times m}$ have the structure*

$$A = A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1m}(t) \\ \cdots & \cdots & \cdots \\ a_{m1}(t) & \cdots & a_{mm}(t) \end{pmatrix}, \tag{4}$$

where $a_{kl}(t)$ are arbitrary functions. Let $B \in \mathbb{C}^{n\times n}$ have the eigenvalue ω with corresponding eigenvector x:

$$Bx = \omega x.$$

Furthermore, let $A(t)$ have eigenvalue $\lambda(t)$ with eigenvector $y(t)$:

$$A(t)\,y(t) = \lambda(t)\,y(t).$$

Then the matrix $A(B) \in \mathbb{C}^{N\times N}$, $N = mn$, has eigenvalue $\lambda(\omega)$ with eigenvector $y(\omega) \otimes x$, where $\otimes$ denotes the Kronecker product.

Proof. Using the property of the Kronecker product

$$(C \otimes D)(u \otimes v) = (Cu) \otimes (Dv),$$

we obtain, with $E_{kl} = e_k e_l^T \in \mathbb{R}^{m\times m}$,

$$\begin{aligned} A(B)(y(\omega) \otimes x) &= \sum_{k,l} [E_{kl} \otimes a_{kl}(B)][y(\omega) \otimes x] \\ &= \sum_{k,l} [E_{kl}\,y(\omega)] \otimes [a_{kl}(B)\,x] \\ &= \sum_{k,l} a_{kl}(\omega)[E_{kl}\,y(\omega)] \otimes x \\ &= \left(\sum_{k,l} [a_{kl}(\omega)\,E_{kl}]\,y(\omega)\right) \otimes x \\ &= [A(\omega)\,y(\omega)] \otimes x \\ &= \lambda(\omega)\,y(\omega) \otimes x. \end{aligned}$$

■

LEMMA 3. *Let $B \in \mathbb{C}^{n\times n}$ and $A = A(t) \in \mathbb{C}^{m\times m}$ of the form (4). Let ω be an eigenvalue of B with corresponding eigenvector x:*

$$Bx = \omega x.$$

If the matrix $A(\omega)$ is nondiagonalizable, then so is $A(B)$.

If $y(\omega)$ is a generalized eigenvector of the matrix $A(\omega)$, then $y(\omega) \otimes x$ is a generalized eigenvector of the matrix $A(B)$.

Proof. Let $\lambda(\omega)$ be an eigenvalue of multiplicity $k \geqslant 2$ of $A(\omega)$ such that

$$A(\omega)\,y_l(\omega) = \lambda(\omega)\,y_l(\omega) + y_{l-1}(\omega), \qquad l = 1, 2, \ldots,$$

where $y_0(\omega) = 0$, $y_1(\omega)$ is an eigenvector of $A(\omega)$, and $y_l(\omega)$, $L > 1$, are generalized eigenvectors of $A(\omega)$. Now we consider the equation derived in Lemma 2:

$$A(B)[y_l(\omega) \otimes x] = [A(\omega)y_l(\omega)] \otimes x$$

$$= \lambda(\omega)y_l(\omega) \otimes x + y_{l-1}(\omega) \otimes x, \qquad l = 1, \ldots .$$

The vector $y_0(\omega) \otimes x$ is zero, the vector $y_1(\omega) \otimes x$ is an eigenvector of the matrix $A(B)$ (see Lemma 12), and the vectors $y_l(\omega) \otimes x$, $l > 1$, are generalized eigenvectors of the matrix $A(B)$. Thus the matrix $A(B)$ is nondiagonalizable and the proof is complete. ■

The next theorem then follows immediately from Lemmas 1 and 2.

THEOREM 1. *Let $B \in \mathbb{C}^{n\times n}$ be diagonalizable, having the eigenvalues ω_k, $k = 1, \ldots, n$. The matrix $A(B)$ of the form (4) is diagonalizable if and only if all matrices $A(\omega_k)$, $k = 1, \ldots, n$, are diagonalizable.*

LEMMA 4. *Let $A, B \in \mathbb{C}^{m\times m}$ be two matrices of the form (4), and $C \in \mathbb{C}^{n\times n}$ be given. Let ω be an eigenvalue of C with corresponding eigenvector x:*

$$Cx = \omega x.$$

Furthermore, let the generalized eigenvalue problem $\{A(t), B(t)\}$ have eigenvalue $\lambda(t)$ with eigenvector $y(t)$:

$$A(t)y(t) = \lambda(t)B(t)y(t).$$

Then the generalized eigenvalue problem $\{A(C), B(C)\}$ has eigenvalue $\lambda(\omega)$ with corresponding eigenvector $y(\omega) \otimes x$.

Proof. As in Lemmas 12 and 13,

$$A(C)[y(\omega) \otimes x] = [A(\omega)y(\omega)] \otimes x$$

$$= \lambda(\omega)[B(\omega)y(\omega)] \otimes x$$

$$= \lambda(\omega)B(C)[y(\omega) \otimes x].$$ ■

Since the eigenvalues and eigenvectors of the matrix J are known [see (2)], we are now able to compute all the eigenvalues and eigenvectors of the matrix $A = A(J)$ if all the matrices $A(\omega_k) \in \mathbb{C}^{n\times n}$ are diagonalizable. If

$$\begin{aligned} A(\omega_k)x_l(\omega_k) &= \lambda_l(\omega_k)x_l(\omega_k), \\ A^*(\omega_k)y_l(\omega_k) &= \overline{\lambda_l(\omega_k)}\, y_l(\omega_k), \\ y_j(\omega_k)^* x_i(\omega_k) &= \delta_{ij} \qquad (\delta_{ij} = \text{Kronecker symbol}), \\ l &= 1,\dots,m,\ k = 1,\dots,n, \end{aligned} \tag{5}$$

then $A(J)$ is also diagonalizable and

$$\begin{aligned} A(J)(x_l(\omega_k) \otimes f_k) &= \lambda_l(\omega_k)(x_l(\omega_k) \otimes f_k), \\ A(J)^T(y_l(\omega_k) \otimes f_k) &= \overline{\lambda_l(\omega_k)}\,(y_l(\omega_k) \otimes f_k), \\ l &= 1,\dots,m, \qquad k = 1,\dots,n. \end{aligned} \tag{6}$$

Then $A(J)$ can be given with the help of (6) by the formula

$$A = A(J) = \frac{1}{n}\sum_{k,l} \lambda_l(\omega_k)[x_l(\omega_k) \otimes f_k][y_l(\omega_k) \otimes f_k]^*. \tag{7}$$

3. SYSTEMS OF LINEAR EQUATIONS

Let A be a $N \times N$ CB matrix. Consider the system of linear equations

$$Ay = b, \qquad A \in \mathbb{R}^{N\times N}, \quad y, b \in \mathbb{R}^N, \tag{8}$$

with a diagonalizable matrix A.

If we use Gaussian elimination for solving this linear system, then the number of arithmetic operations will be

$$O(N^3) = O(n^3m^3),$$

but the solution of (8) can be obtained with the help of (7) as

$$y = A^{-1}(J)b = \frac{1}{n}\sum_{k,l} \frac{1}{\lambda_l(\omega_k)}[x_l(\omega_k) \otimes f_k][y_l(\omega_k) \otimes f_k]^* b. \tag{9}$$

In order to describe the numerical algorithm for the evaluation of the right-hand side of (9), the solution of (8), we introduce the following notation:

$$X_l = (x_l(\omega_1) \vdots x_l(\omega_2) \vdots \cdots \vdots x_l(\omega_n)) \in \mathbb{C}^{m\times n}, \qquad l = 1, \ldots, m,$$

for the right eigenvectors of the matrices $A(\omega_k)$;

$$F = (f_1 \vdots \cdots \vdots f_m) \in \mathbb{C}^{n\times n}$$

for the matrix of the Fourier transform;

$$B = (b_1 \vdots \cdots \vdots b_m) \in \mathbb{R}^{n\times m}$$

for the given right-hand side of (8);

$$Y = (y_1 \vdots \cdots \vdots y_m) \in \mathbb{R}^{n\times m}$$

for the solution of (8);

$$e_k = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{R}^m$$

for the kth column of the unit matrix $I \in \mathbb{R}^{m\times m}$; and

$$\beta_{lk} = [n\lambda_l(\omega_k)]^{-1}, \qquad l = 1, \ldots, m, \quad k = 1, \ldots, m.$$

ALGORITHM 1.

1. $C := (F^*B)^T \in \mathbb{C}^{n\times m}$;
2. $Y := 0$;
3. for $l = 1, \ldots, m$
   3.1. $D_l = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_k = \beta_{lk}\, y_l^*(\omega_k) C e_k$;
   3.2. $Y := Y + \operatorname{Re}(FD_l X_l^T)$.

The calculation of all the left $[y_l(\omega_k)]$ and right $[x_l(\omega_k)]$ eigenvectors and all the eigenvalues in (5) requires more arithmetic operations than solving the system (8) with the help of Algorithm 1. We need only $O(m^2 n \ln n)$ arithmetic operations for the entire algorithm if we use the fast Fourier transform twice at steps 1 and 3.2. Step 3.2 in Algorithm 1 requires the most arithmetical operations, because it consists of the fast Fourier transform

[$O(n \ln n)$ operations] for all m columns of the matrix $D_l X_l^T$ and is to be done m times. We need $O(mn)$ operations to compute D_l in Step 3.1 and $D_l X_l^T$ and a new Y in Step 3.2. The numerical solution of the eigenvalue problems (5) needs $O(m^3 n)$ arithmetical operations. The efficiency of the Algorithm 1 gives us the possibility to use CB matrices as preconditioning matrices for iterative solution of certain systems of linear equations with more general matrices; see Section 5.

We remark that it is not necessary to solve all n eigenvalue problems (5) numerically, because

$$A(\omega_{n-k+2}) = A(\overline{\omega_k}) = \overline{A(\omega_k)}, \qquad k = 2, \ldots, n/2 \tag{10}$$

and hence

$$\lambda_l(\omega_{n-k+2}) = \overline{\lambda_l(\omega_k)},$$

$$x_l(\omega_{n-k+2}) = \overline{x_l(\omega_k)},$$

$$y_l(\omega_{n-k+2}) = \overline{y_l(\omega_k)}, \qquad k = 2, \ldots, n/2.$$

So the number of eigenvalue problems is now reduced to $n/2 + 1$. To solve these problems, we transform the matrices $A(\omega_k)$ to Hessenberg form with the help of orthogonal Householder matrices and use the QR method (see [9, 11, 16]). The left eigenvectors can also be computed in this step. The efficiency of the algorithm increases if we have to solve more than one system (8) with the same matrix A for different right-hand sides (for example, if our CB matrix is the preconditioning matrix for some iterative method).

The numerical work needed to solve the eigenvalue problems (5) can be reduced if the CB matrix is symmetric (an SCB matrix):

$$\begin{aligned} A_{kk} &= A_{kk}^T, \qquad k = 1, \ldots, m, \\ A_{kl} &= A_{lk}^T, \qquad k \neq l. \end{aligned} \tag{11}$$

Each SCB matrix is defined by $nm^2/2 + m$ elements ($n/2 + 1$ elements per leading diagonal block, and n elements per block in the upper triangle of A), and satisfies the following important property.

LEMMA 5. *The matrices $A(\omega_k)$, $k = 1, \ldots, n$, are Hermitian.*

Proof. The eigenvalues $a_{ij}(\omega_k)$ and $a_{ji}(\omega_k)$ of the matrices A_{ij} and A_{ji} are complex conjugate:

$$a_{ij}(\omega_k) = \overline{a_{ji}(\omega_k)}, \tag{12}$$

because the matrix A_{ji} is the transpose of the matrix A_{ij} [see (11)]. ■

Therefore, it only remains to solve the following eigenvalue problems:

$$A(\omega_k)x_l(\omega_k) = \lambda_l(\omega_k)x_l(\omega_k), \qquad l = 1, \ldots, m, \quad k = 1, \ldots, n/2 + 1,$$

where the eigenvalues $\lambda_l(\omega_k)$ are real and the left eigenvectors are identical with the right eigenvectors. The algorithm for the numerical solution of the system of linear equations with a SCB matrix is very similar to that for an arbitrary matrix, but requires less arithmetic operations (β_{lk} are real) and less storage [$y_l(\omega_k) = x_l(\omega_k)$]. The matrices of the eigenvalue problems can be transformed into real symmetric tridiagonal form with the help of unitary reflections, after which the implicit QL algorithm is used.

Further decrease of numerical work is possible if the SCB matrix A has only symmetric blocks

$$A = A^T, \qquad A_{kl} = A_{kl}^T, \quad k, l = 1, \ldots, m$$

(is a SCBS matrix). It is easy to see that each SCBS matrix is defined by $(n + 1)m(m + 1)/4$ elements ($n/2 + 1$ elements per block).

LEMMA 6. *The matrices $A(\omega_k)$, $k = 1, \ldots, n$, are real and symmetric.*

Proof. The matrices $A(\omega_k)$, $k = 1, \ldots, n$, are at least Hermitian (Lemma 5). Furthermore, all $A(\omega_k)$ are real, because the elements $a_{ij}(\omega_k)$ are the eigenvalues of symmetric matrices A_{ij}. A real Hermitian matrix is symmetric. ■

Instead of (10) the matrices $A(\omega_k)$ now satisfy

$$A(\omega_k) = A(\omega_{n-k+2}), \qquad k = 2, \ldots, n/2.$$

Thus we only need to solve $n/2 + 1$ eigenvalue problems

$$A(\omega_k)x_l(\omega_k) = \lambda_l(\omega_k)x_l(\omega_k), \qquad l = 1, \ldots, m, \quad k = 1, \ldots, n/2 + 1,$$

where all the eigenvalues and eigenvectors are real. These eigenvalue problems can be solved by Housholder tridiagonalization and the implicit QL algorithm. Algorithm 1 can be used for the solution of a system of linear equations with a SCBS matrix if we denote $y_l^*(\omega_k) = x_l^T(\omega_k)$. As an example of a SCBS matrix we can consider the Galerkin approach for the numerical solution of certain boundary integral equations in a three-dimensional rotational domain [12].

4. NONDIAGONALIZABLE MATRICES

If the CB matrix of the system (8) is nondiagonalizable, we cannot obtain Equation (7) and hence Algorithm 1 cannot be used to solve (8). But if this matrix is real and positive definite,

$$(Ax, x) > 0 \qquad \forall x \in \mathbb{R}^N, \quad x \neq 0, \tag{13}$$

or is a complex symmetric matrix having positive definite real or imaginary part, (8) can be solved with the aid of the following observation.

Let A be a nonsymmetric real CB matrix satisfying the condition (13). We split this matrix as the sum of a symmetric matrix A^+ and an antisymmetric matrix A^-:

$$A = A^+ + A^-, \qquad A^+ = \tfrac{1}{2}(A + A^T), \quad A^- = \tfrac{1}{2}(A - A^T).$$

It is easy to see that A^+ and A^- are CB matrices. Further, the matrix A^+ is positive definite and therefore nonsingular:

$$(A^+z, z) = (Az, z) > 0 \qquad \forall z \in \mathbb{R}^N, \quad z \neq 0.$$

The inverse matrix A^{-1} can be now written in the form

$$A^{-1} = (A^+ + A^-)^{-1} = \left[I + (A^+)^{-1}A^-\right]^{-1}(A^+)^{-1}. \tag{14}$$

The generalized eigenvalue problem $\{A^-, A^+\}$ has (see Lemma 4) the following eigenvalues and eigenvectors:

$$A^-(J)x_l(\omega_k) \otimes f_k = \lambda_l(\omega_k)A^+(J)x_l(\omega_k) \otimes f_k, \qquad k = 1, \ldots, n,$$
$$l = 1, \ldots, m,$$

where $\lambda_l(\omega_k)$ are eigenvalues and $x_l(\omega_k)$ are eigenvectors of the generalized eigenvalue problems $\{A^-(\omega_k), A^+(\omega_k)\}$,

$$A^-(\omega_k)x_l(\omega_k) = \lambda_l(\omega_k)A^+(\omega_k)x_l(\omega_k),$$
$$l = 1,\ldots,m, \qquad k = 1,\ldots,n. \quad (15)$$

Existence of all the eigenvectors of the problems (15) is guaranteed, because the matrices $A^+(\omega_k)$ are Hermitian and positive definite and $A^-(\omega_k)$ are anti-Hermitian. The eigenvectors $x_l(\omega_k)$ are $A^+(\omega_k)$-orthogonal:

$$x_j(\omega_k)^*A^+(\omega_k)x_i(\omega_k) = \delta_{ij}, \qquad i,j = 1,\ldots,m, \quad k = 1,\ldots,n,$$

and the eigenvectors $x_l(\omega_k) \otimes f_k$ are $A^+(J)$-orthogonal:

$$[x_j(\omega_k) \otimes f_k]^*A^+(J)[x_i(\omega_l) \otimes f_l] = n\delta_{ij}\delta_{kl},$$
$$i,j = 1,\ldots,m, \qquad k,l = 1,\ldots,n.$$

Therefore $[A^+(J)]^{-1}A^-(J)$ can be given by

$$[A^+(J)]^{-1}A^-(J) = \frac{1}{n}\sum_{k,l}\lambda_l(\omega_k)[x_l(\omega_k) \otimes f_k][x_l(\omega_k) \otimes f_k]^*A^+(J), \quad (16)$$

and the solution of y of (8) can be computed with the help of (14) and (16) as

$$y = \frac{1}{n}\sum_{k,l}\frac{1}{1 + \lambda_l(\omega_k)}[x_l(\omega_k) \otimes f_k][x_l(\omega_k) \otimes f_k]^*b.$$

Thus Algorithm 1 can be used for the numerical solution of (8). Note that the values β_{lk} have the following form:

$$\beta_{lk} = \frac{1}{n[1 + \lambda_l(\omega_k)]}$$

and $y_l(\omega_k) = x_l(\omega_k)$.

This idea will also be useful for the solution of (8) with a diagonalizable positive definite matrix A, because we need less storage [the left eigenvectors $y_l(\omega_k)$ are dropped] and solving the eigenvalue problem (15) numerically is easier than (5). We can transform the generalized eigenvalue problems (15) to the usual eigenvalue problems with anti-Hermitian matrices:

$$\begin{aligned} A^+(\omega_k) &= L_k L_k^*, \\ L_k^{-1} A^-(\omega_k) L_k^{-*}\left[L_k^* x_l(\omega_k)\right] &= \lambda_l(\omega_k)\left[L_k^* x_l(\omega_k)\right], \end{aligned} \tag{17}$$

where L_k are lower triangular matrices. The Cholesky decomposition of $A^+(\omega_k)$ in (17) is guaranteed, because these matrices are Hermitian and positive definite.

Finally we consider the system of linear equations

$$Ay = b, \qquad A \in \mathbb{C}^{N\times N}, \quad y, b \in \mathbb{C}^N, \tag{18}$$

where A is a complex symmetric CB matrix for which

$$A = A_1 + iA_2, \qquad A_1, A_2 \in \mathbb{R}^{N\times N},$$

$$A_1 = A_1^T > 0, \qquad A_2 = A_2^T.$$

This last assumption guarantees the existence of all eigenvectors of the eigenvalue problems

$$A_2(\omega_k) x_l(\omega_k) = \lambda_l(\omega_k) A_1(\omega_k) x_l(\omega_k), \qquad k = 1, \ldots, n, \quad l = 1, \ldots, m. \tag{19}$$

The matrices $A_1(\omega_k)$ and $A_2(\omega_k)$ are Hermitian, and the $A_1(\omega_k)$ are positive definite. Furthermore, the eigenvectors $x_l(\omega_k)$ are $A_1(\omega_k)$-orthogonal, and the solution of the system (18) can be given by

$$u = \frac{1}{n} \sum_{k,l} \frac{1}{1 + i\lambda_l(\omega_k)} \left[x_l(\omega_k) \otimes f_k\right]\left[x_l(\omega_k) \otimes f_k\right]^* b,$$

where y is calculated via Algorithm 1 with $\mathrm{Re}(FD_l X_l^T)$ replaced by $FD_l X_l^T$ in Step 3.2, and β_{lk} replaced by $1/n[1 + i\lambda_l(\omega_k)]$ in Step 3.1.

The eigenvalue problems (19) can be transformed into eigenvalue problems with Hermitian matrices in the same way as in (17). We remark that a complex symmetric matrix A can be nondiagonalizable.

5. NUMERICAL EXPERIMENTS

In this section we consider as an example the numerical solution of the following integral equation of the first kind for an unknown function $v(x)$:

$$\frac{1}{4\pi}\int_{\Gamma}\frac{v(x)}{|x-y|}\,ds_x = f(y), \qquad x, y \in \Gamma \subset \mathbb{R}^3. \tag{20}$$

Here, Γ is a surface in $\mathbb{R}^3$ given by the parametric representation

$$\Gamma = \left\{ x \in \mathbb{R}^3,\ x = \frac{1}{2}\begin{pmatrix} R(z)\cos 2\pi t \\ R(z)\sin 2\pi t\,(2 - \frac{3}{2}\sin 2\pi t) \end{pmatrix},\right.$$

$$\left. 0 \leqslant z \leqslant 1,\ 0 \leqslant t < 1 \right\}, \tag{21}$$

with $R(z) = \sqrt{z(1-z)}$.

The solution of (20) will be approximated with the help of boundary elements using the collocation method [2] on the surface Γ_h. $\Gamma_h = \bigcup_{j=1}^{N} \Gamma_j$ is obtained by discretization of the domain $(z, t) \in [0, 1] \times [0, 1)$ into usual finite-element triangles using the nodes

$$(z_l, t_k) = (h_z(l-1), h_t(k-1)), \qquad k = 1, \ldots, n, \quad l = 1, \ldots, m,$$

where $h_t = 1/n$ and $h_z = 1/(m-1)$. The collocation method for (20) leads to the following problem: Find $(v_1, \ldots, v_N)^T \in \mathbb{R}^N$ such that the collocation equations

$$\frac{1}{4\pi}\sum_{j=1}^{N} v_j \int_{\Gamma_j}\frac{ds_x}{|x-y_i|} = f(y_i), \qquad i = 1, \ldots, N, \tag{22}$$

are satisfied.

If Γ is given by the rotation of the curve

$$\{R(z), 0 \leqslant z \leqslant 1\}$$

around the z-axis,

$$\Gamma = \left\{ x \in \mathbb{R}^2,\ x = \begin{pmatrix} R(z)\cos 2\pi t \\ R(z)\sin 2\pi t \\ z \end{pmatrix}, 0 \leqslant z \leqslant 1, 0 \leqslant t < 1 \right\}, \quad (23)$$

then the matrix A of the system (22) is a circulant-block matrix (see [15]). The elements of the matrix A,

$$a_{ij} = \int_{\Gamma_j} \frac{ds_x}{|x - y_i|},$$

can be computed analytically if we assume that Γ_j, $j = 1, \ldots, N$, are plane triangles. The circulant structure of the blocks is guaranteed if we enumerate the unknowns first in the t-direction for the fixed z and after that in the z-direction, using the 1-periodicity of the parametrization (23). The matrix A can be used for preconditioning the matrices arising from more general domains, e.g. (21).

Table 1 shows the number of iterations of the gradient method with and without preconditioning required for the accuracy of $\varepsilon = 10^{-8}$, and the corresponding computing times measured in CPU seconds on the INMOS T800 transputer. Also the computing time for Gaussian elimination is shown.

We remark that the Algorithm 1 was used in each iteration.

TABLE 1

	Without prec.		With prec.		Gaussian
N	It.	CPU	It.	CPU	CPU
32	50	0.58	14	1.03	0.11
64	69	2.02	19	3.45	0.75
128	75	11.43	19	12.98	5.50
256	140	81.76	31	54.67	42.13
512	167	382.72	32	159.73	329.55

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Received 10 March 1992; final manuscript accepted 25 January 1993