The Nash model

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Summary of the slides

Definition of non cooperative game

Definition

A two player noncooperative game in strategic form is $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$

X, Y are the strategy sets of the players, f, g their utility functions.

Equilibrium

A Nash equilibrium profile for the $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$ is a pair $(\bar{x}, \bar{y}) \in X \times Y$ such that:

- $f(\bar{x}, \bar{y}) \geq f(x, \bar{y})$ for all $x \in X$
- $g(\bar{x}, \bar{y}) \geq g(\bar{x}, y)$ for all $y \in Y$

A Nash equilibrium profile is a joint combination of strategies, stable w.r.t. unilateral deviations of a single player

The new rationality paradigm

Observe: new definition of rationality

Need to compare with former concepts

1) Suppose $\bar{x}$ is a (weakly) dominant strategy for Pl1:

$f(\bar{x}, y) \geq f(x, y)$ for all x, y.

Then, if $\bar{y}$ maximizes the function $y \rightarrowtail g(\bar{x}, y)$,

$(\bar{x}, \bar{y})$ is a NEp.

Nash equilibria in games with perfect information

2) Backward induction provides a Nash equilibrium profile for a game of perfect information. Is it possible that in games of perfect information there are more equilibria than that one(s) provided by backward induction?

Example

First, player 1 claims for himself $x \in [0,1]$*. Then, player 2 claims for herself* $y \in [0,1]$*. If* $x + y \leq 1$ *they get* (x, y)*, otherwise both players get 0.*

Backward induction provides strategies

- The first player proposes $x = 1$.
- The second player takes $y = 1 - x = 0$.

The outcome is $(1,0)$: the first player keeps all money.

On the contrary, any outcome $(x, 1-x)$ is the result of a NE profile.

Existence of Nash equilibria

Denote by BR the following multifunction:

$$BR_1 : Y \to X : \quad BR_1(y) = \mathrm{Arg\ Max}\,\{f(\cdot, y)\}$$

$$BR_2 : X \to Y : \quad BR_2(x) = \mathrm{Arg\ Max}\,\{g(x, \cdot)\}$$

and

$$BR : X \times Y \to X \times Y : \quad BR(x, y) = (BR_1(y), BR_2(x)).$$

$(\bar{x}, \bar{y})$ is a Nash equilibrium profile for the game if and only if

$$(\bar{x}, \bar{y}) \in BR(\bar{x}, \bar{y}).$$

Thus existence of a Nash equilibrium profile in a game is equivalent to existence of a fixed point for the Best Reply Multifunction.

The Nash theorem (2 players version)

Theorem

Given the game $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$, *suppose:*

1. X *and* Y *are compact convex subsets of some Euclidean space*
2. f, g *continuous*
3. $x \mapsto f(x, y)$ *is (quasi) concave for all* $y \in Y$
4. $y \mapsto g(x, y)$ *is (quasi) concave for all* $x \in X$

Then the game has at least one Nash equilibrium profile.

Quasi concavity for a real valued function h means that the sets

$$h_a = \{z : h(z) \geq a\}$$

are convex for all a (maybe empty for some a).

Finite games: notation

Suppose the sets of the strategies of the players are finite, $\{1, \dots, n\}$ for the first player, $\{1, \dots, m\}$ for the second player. Then the game can be represented by the bimatrix

$$\begin{pmatrix} (a_{11}, b_{11}) & \dots & (a_{1m}, b_{1m}) \\ \dots & \dots & \dots \\ (a_{n1}, b_{n1}) & \dots & (a_{nm}, b_{nm}) \end{pmatrix}$$

where a_{ij} (b_{ij}) is the utility of the row (column)player when row plays strategy i and column strategy j.

Denote by (A, B) such a game.

Finite games

Corollary

A finite game (A, B) admits always a Nash equilibrium profile in mixed strategies

In this case X and Y are simplexes, while $f(x, y) = x^t Ay, g(x, y) = x^t By$

and thus the assumption of the theorem are fulfilled.

Expliciting utilities:

$$f(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j a_{ij}, \quad g(x, y) = \sum_{i=1,\dots,n, j=1,\dots,m} x_i y_j b_{ij}$$

Remark

Once fixed the strategies of the other players, the utility function of one player is linear in its own variable.

Finding Nash equilibria

The game:

$$\begin{pmatrix} (1,0) & (0,3) \\ (0,2) & (1,0) \end{pmatrix}$$

Pl1 playing $(p, 1-p)$, PL2 playing $(q, 1-q)$:

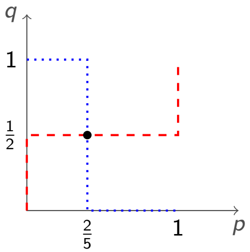
$$f(p,q) = pq + (1-p)(1-q) = p(2q-1) - q + 1$$

$$g(p,q) = 3p(1-q) + 2(1-p)q = q(2-5p) + 3p$$

The best reply multifunctions

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad 0 \leq q \leq \frac{1}{2} \\ p \in [0,1] & \text{if} \quad q = \frac{1}{2} \\ p = 1 & \text{if} \quad q > \frac{1}{2} \end{cases}$$

$$BR_2(p) = \begin{cases} q = 1 & \text{if} \quad 0 \leq p \leq \frac{2}{5} \\ q \in [0,1] & \text{if} \quad p = \frac{2}{5} \\ q = 0 & \text{if} \quad p > \frac{2}{5} \end{cases}$$



Indifference

Remark

Suppose $(\bar{x}, \bar{y})$ is a NE in mixed strategies. Suppose spt $\bar{x} = \{1, \dots, k\}$[1], spt $\bar{y} = \{1, \dots, l\}$, and $f(\bar{x}, \bar{y}) = v$. Then it holds:

$$\begin{cases} a_{11}\bar{y}_1 + a_{12}\bar{y}_2 + \dots + a_{1l}\bar{y}_l & = v \\ \dots & = v \\ a_{k1}\bar{y}_1 + a_{k2}\bar{y}_2 + \dots + a_{kl}\bar{y}_l & = v \\ a_{(k+1)1}\bar{y}_1 + a_{(k+1)2}\bar{y}_2 + \dots + a_{(k+1)l}\bar{y}_l & \leq v \\ \dots & \leq v \\ a_{n1}\bar{y}_1 + a_{n2}\bar{y}_2 + \dots + a_{nl}\bar{y}_l & \leq v \end{cases}$$

The above relations are due the the fact that rows used with positive probability must be all optimal (and thus they all give the same expected value), while the other ones are suboptimal

[1] spt $\bar{x} = \{i : \bar{x}_i > 0\}$

Cont'd

The above remark is useful to look for existence of fully mixed[2] Nash equilibria.

Suppose $(\bar{x}, \bar{y})$ is such a Nash equilibrium profile. Then it holds that

$$a_{i1}\bar{y}_1 + a_{i2}\bar{y}_2 + \cdots + a_{im}\bar{y}_m = a_{k1}\bar{y}_1 + a_{k2}\bar{y}_2 + \cdots + a_{km}\bar{y}_m$$

for all $i, k = 1, \dots n$, and similarly

$$b_{1r}\bar{x}_1 + b_{2r}\bar{x}_2 + \cdots + b_{nr}\bar{x}_n = b_{1s}\bar{x}_1 + b_{2s}\bar{x}_2 + \cdots + b_{ns}\bar{x}_n$$

for all $r, s = 1, \dots m$ with the further conditions

$$p_j, q_j \geq 0, \quad \sum_{i=1}^{n} p_i = 1, \quad \sum_{j=1}^{m} q_j = 1$$

[2] This means that all rows/columns are played with positive probabilities

Brute force algorithm

1. Guess the supports of the equilibria $spt(\bar{x})$ and $spt(\bar{y})$
2. Ignore the inequalities and find x, y, v, w by solving the linear system of $n + m + 2$ equations

$$\begin{cases} \sum_{i=1}^{n} x_i = 1 & \\ \sum_{j=1}^{m} a_{ij} y_j = v & \text{for all } i \in spt(\bar{x}) \\ x_i = 0 & \text{for all } i \notin spt(\bar{x}) \end{cases}$$

$$\begin{cases} \sum_{j=1}^{m} y_j = 1 & \\ \sum_{i=1}^{n} b_{ij} x_i = w & \text{for all } j \in spt(\bar{y}) \\ y_j = 0 & \text{for all } j \notin spt(\bar{y}) \end{cases}$$

3. Check whether the ignored inequalities are satisfied.
   If $x_i \geq 0, y_j \geq 0, \sum_{j=1}^{m} a_{ij} y_j \leq v$ and $\sum_{i=1}^{n} b_{ij} x_i \leq w$ then Stop: we have found a mixed equilibrium profile. Otherwise, go back to step 1 and try another guess of the supports.

Lemke-Howson Algorithm

Enumerating all the possible supports in the brute force algorithm quickly becomes computationally prohibitive: there are potentially $(2^n - 1)(2^m - 1)$ options!

For $n \times n$ games the number of combinations grow very quickly

n	# of potential supports
2	9
3	49
4	225
5	961
10	1.046.529
20	1.099.509.530.625

Lemke-Howson proposed a more efficient algorithm... though still with exponential running time in the worst case.

General strategic games

Consider an n-player game with strategy sets X_i and payoffs $f_i : X \to \mathbb{R}$ with $X = \prod_{j=1}^{n} X_j$.

Notation:
if $x = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ is a strategy profile, denote by x_{-i} the vector $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and write also $x = (x_i, x_{-i})$.

Then: $\bar{x} = (\bar{x}_i)_{i=1}^{n}$ is a NE p. if and only if for each player $i = 1, \ldots, n$ we have $\bar{x}_i \in BR_i(\bar{x}_{-i})$.

The Nash theorem

Theorem

Given a n-player game with strategy sets X_i and payoff functions $f_i : X \to \mathbb{R}$ where $X = \prod_{i=1}^n X_i$. Suppose:

- *each X_i is a closed bounded convex subset in a finite dimensional space $\mathbb{R}^{d_i}$*
- *each $f_i : X \to \mathbb{R}$ is continuous*
- *$x_i \mapsto f_i(x_i, x_{-i})$ is a (quasi) concave function for each fixed $x_{-i} \in X_{-i}$*

Then the game admits at least one Nash equilibrium profile.

Mixed equilibria for n-player finite games

Consider an n-person finite game with strategy sets A_i and payoffs $f_i(a_1, \ldots, a_n)$. In the *mixed extension* each player i choses a probability distribution $x^i \in \Sigma_{A_i}$, that is to say, $x^i_{a_i} \geq 0$ for all $a_i \in A_i$ and $\sum_{a_i \in A_i} x^i_{a_i} = 1$.

Denote $A = \prod_{i=1}^n A_i$ the set of pure strategy profiles. The probability of observing an outcome $(a_1, \ldots, a_n) \in A$ is the product $\prod_{i=1}^n x^i_{a_i}$ and the *expected* payoffs are:

$$\begin{aligned}
\bar{f}_i(x^1, \ldots, x^n) &= \sum_{(a_1,\ldots,a_n)\in A} f_i(a_1, \ldots, a_n) \prod_{j=1}^n x^j_{a_j} = \sum_{a_i \in A_i} x^i_{a_i} u_i(a_i, x^{-i}) \\
u_i(a_i, x^{-i}) &= \sum_{a_j \in A_j, j \neq i} f_i(a_1, \ldots, a_n) \prod_{j \neq i} x^j_{a_j}
\end{aligned}$$

Corollary

Every n-player finite game has at least one Nash equilibrium profile in mixed strategies.

Congestion games

A congestion game is defined as:

1. N players
2. A set R of resources
3. A collection of subsets of R (strategies)
4. for each r a function $d_r : \mathbb{N} \to \mathbb{R}$ (cost of using resource r)

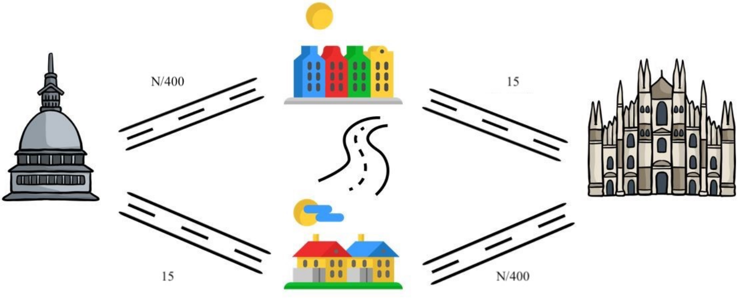
The game is built in this way: given $s = (s_1, \dots, s_n)$ strategy profile for the players, define $x(r, s)$ as the number of players such that $r \in s_i$. Cost for player i

$$c_i(s_1, \dots, s_n) = \sum_{r \in s_i} d_r(x(r, s))$$

A player decides which facilities to use. The cost to use the facility depends on the number of people using it.

Typical example, a road/web network.

The Braess paradox



4.000 people travel form Turin to Milan. Every driver wants to minimize time.
N is the number of people driving in the corresponding road

What are the Nash equilibria? What happens if the North-South street between the two small cities is made available to cars and time to travel on it is 2 minutes?

El Farol bar (3 people)

In Santa Fe there is nothing to do at night but look at the stars or go to the local bar, El Farol. Let us assume that the utility of looking at the stars as 0, the cost of walking over to the bar is 1, the utility from being at the bar is 2 if there are fewer than three people at the bar and $1/2$ if there are three or more people.

Suppose there are three people in Santa Fe. Then there are three pure-strategy Nash equilibria, in each of which two people go to the bar and the other one watches the stars. The average payoff per player in each of these equilibria is $2/3$. There is also a unique symmetric mixed-strategy Nash equilibrium (that is, each player uses the same mixed strategy) in which each resident goes to the bar with probability $p = \sqrt{2/3}$.

To see this, note that the probability that both the other two people go to the bar is p^2, so the expected payoff to going to the bar is $2(1-p^2) + \frac{p^2}{2} - 1$. Since this is a NEp, such payoff has to be the same as looking at the stars, therefore $p^2 = 2/3$.

El Farol bar (N people)

Suppose now that there are N people in Santa Fe, and the utility of visiting the bar is now $u(x) = x$ if $x \leq N/2$ and $u(x) = N/2 - x$ if $x > N/2$ (because the place is overcrowded). Let the utility of looking at the stars be 0.

We have now $\binom{N}{N/2}$ NEp in pure strategies.

Congestion games have always (pure) Nash equilibria, necessarily asymmetric!

A mixed symmetric Nash equilibrium profile is present in this case too.

It is much more difficult to compute!

Duopoly models

Two firms choose quantities of a good to produce.

- Firm 1 produces quantity q_1, firm 2 produces quantity q_2.
- The unitary cost of the good is $c > 0$ for both firms.
- A quantity $a > c$ of the good saturates the market.
- The price $p(q_1, q_2)$ is $p = \max\{a - (q_1 + q_2), 0\}$.

Payoffs:

$$u_1(q_1, q_2) = q_1 p(q_1, q_2) - cq_1 = q_1(a - (q_1 + q_2)) - cq_1,$$
$$u_2(q_1, q_2) = q_2 p(q_1, q_2) - cq_2 = q_2(a - (q_1 + q_2)) - cq_2.$$

The monopolist

Suppose $q_2 = 0$.

Firm 1 maximizes $u(q_1) = q_1(a - q_1) - cq_1$.

$$q_M = \frac{a-c}{2}, \quad p_M = \frac{a+c}{2} \quad u_M(q_M) = \frac{(a-c)^2}{4}$$

The duopoly

The utility functions are strictly concave and non positive at the endpoints of the domain, thus the first derivative must vanish:

$$a - 2q_1 - q_2 - c = 0, \qquad a - 2q_2 - q_1 - c = 0,$$

$$q_i = \frac{a-c}{3}, \quad p = \frac{a+2c}{3} \quad u_i(q_i) = \frac{(a-c)^2}{9}.$$

The case with a leader

One firm, the Leader, announces its strategy, and the other one, the Follower, acts taking for granted the announced strategy of the Leader.

$$\bar{q}_2(q_1) = \frac{a - q_1 - c}{2}.$$

The Leader maximizes

$$u_1(q_1, \frac{a - q_1 - c}{2})$$

$$\bar{q}_1 = \frac{a-c}{2},\ \bar{q}_2 = \frac{a-c}{4},\ u_1(\bar{q}_1, \bar{q}_2) = \frac{(a-c)^2}{8},\ u_2(\bar{q}_1, \bar{q}_2) = \frac{(a-c)^2}{16}$$

$$p = \frac{a+3c}{4}$$

Comparing the three cases

Monopoly

$$q_M = \frac{a-c}{2}, \quad p_M = \frac{a+c}{2} \quad u_M(q_M) = \frac{(a-c)^2}{4}$$

Duopoly

$$q_i = \frac{a-c}{3}, \quad p = \frac{a+2c}{3} \quad u_i(q_i) = \frac{(a-c)^2}{9}.$$

Leader

$$\bar{q}_1 = \frac{a-c}{2},\ \bar{q}_2 = \frac{a-c}{4},\ u_1(\bar{q}_1,\bar{q}_2) = \frac{(a-c)^2}{8},\ u_2(\bar{q}_1,\bar{q}_2) = \frac{(a-c)^2}{16}$$

$$p = \frac{a+3c}{4}$$

Comparing the three cases

Making a comparison with the case of a monopoly, we see that:

- the price is lower in the duopoly case;
- the total quantity of product in the market is superior in the duopoly case;
- the total payoff of the two firms is less than the payoff of the monopolist.

In particular, the two firm could consider the strategy of equally sharing the payoff of the monopolist, but this is not a NE profile! The result shows a very reasonable fact, the consumers are better off if there is no monopoly.