# Matching Problems

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# Background setting

Problems introduced in 1962 by Gale and Shapley for the study of two sided markets:

1) workers $\leftrightarrow$ employers;
2) interns $\leftrightarrow$ hospitals;
3) students $\leftrightarrow$ universities;
4) women $\leftrightarrow$ men;
5) $\cdots$

Basic question: what is the best definition of **stable placements**?

# First basic version

WOMEN & MEN

Two groups $\mathcal{A}$ and $\mathcal{B}$.

Each element of $\mathcal{A}$ ($\mathcal{B}$) has a ranking on the elements of $\mathcal{B}$ ($\mathcal{A}$).

Searching for pairs

One to one matching

Further initial assumptions

1) The two groups have the same number of elements;
2) All elements rank all elements of the other group.

# Basic definition

**Definition**

*Let $X$ be a set. A (strict) preference relation on $X$ is a binary relation $\succ$ fulfilling, for all $x, y, z \in X$:*

1) *if $x \neq y$ either $x \succ y$ or $y \succ x$ (completeness);*
2) *$x \not\succ x$ (irreflexivity);*
3) *$x \succ y \wedge y \succ z$ imply $x \succ z$ (transitivity).*

Meaning

1) Each pair is compared;
2) Preferences are strict;
3) Preferences are coherent.

# More definitions

### Definition

*A matching problem is given by:*

1) *a natural number $n$ (the common cardinality of two distinct sets $\mathcal{A}$ and $\mathcal{B}$, whose elements are called women and men);*
2) *a set of preferences such that each woman has a preference relation over the set of men and conversely.*

### Definition

*A matching is a bijection between the two sets.*

# An example

$$\mathcal{A} = \{\text{Anna}, \text{Giulia}, \text{Maria}\}$$

$$\mathcal{B} = \{\text{Bob}, \text{Frank}, \text{Emanuele}\}$$

A matching

$$[(\text{Anna}, \text{Emanuele}), (\text{Giulia}, \text{Bob}), (\text{Maria}, \text{Frank})] \, .$$

# Stable matching

**Definition**

*A pair man–woman $(m, W)$ objects to the matching $\Lambda$ if $m$ and $W$ both prefer each other to the person paired to them in the matching $\Lambda$.*

For instance:

$$\Lambda = \{(m, W), (b, Z), \dots\}$$

and

$$b \succ_W m \qquad \wedge \qquad W \succ_b Z.$$

**Definition**

*A matching $\Lambda$ is called stable provided there is no pair woman–man objecting to $\Lambda$.*

# Example

$$\mathcal{M} = \{a, b, c, d\}, \mathcal{W} = \{A, B, C, D\}.$$

Preferences men:

- $D \succ_a C \succ_a B \succ_a A$, $\quad B \succ_b A \succ_b C \succ_b D$
- $D \succ_c A \succ_c C \succ_c B$, $\quad A \succ_d D \succ_d C \succ_d B$.

Preferences women:

- $b \succ_A a \succ_A c \succ_A d$, $\quad b \succ_B d \succ_B a \succ_B c$
- $b \succ_C d \succ_C c \succ_C a$, $\quad b \succ_D c \succ_D d \succ_D a$.

Two matchings:

$\Lambda = \{(a, A), (b, C), (c, B), (d, D)\}$ $\Omega = \{(a, C), (b, B), (c, D), (d, A)\}$

$\Lambda$ unstable (a,B) $\Omega$ stable.

# Existence of stable matching

**Theorem**

*Every matching problem admits a stable matching*

# Proof

Constructive proof: algorithm night by night

1) Stage 1a) Every woman visits, the first night, her most preferred choice. Stage 1b) Every man chooses among the women he finds in front of his house (if any). If every woman is matched, the algorithm ends, otherwise go to stage 2
2) Stage 2a) Every woman dismissed at the previous stage visits her second choice. Stage 2b) Every man chooses among the women he finds in front of his house (if any), he dismisses the woman already in the house, if someone at the door is better for him. If every woman is matched, the algorithm ends, otherwise go to stage $k$
3) Stage $k$a) Every woman dismissed at the previous stage visits her choice after the man dismissing her. Stage $k$b) Every man chooses among the women he finds in front of his house (if any), he dismisses the woman already in the house, if someone at the door is better for him. If every woman is matched, the algorithm ends, otherwise go to stage $k$.

Claim. The algorithm ends and the resulting matching is stable

# Preliminary remarks

Easy to see

1) The women go down with their preferences along the algorithm;
2) The men go up with their preferences along the algorithm;
3) If a man is visited at stage $r$, then from stage $r+1$ on he will never be alone;
4) The algorithm generically provides two matchings.

# Proof: continued

Fact 1 The algorithm ends, and every man is matched to a woman.
Actually every woman can at most visit $n$ men, and this happens to every woman. Thus at most $n^2$ days are needed. Noticing that the first night all women are involved, so $n^2 - n + 1$ days are needed. Moreover, every man is visited at some stage: all women like better to be paired than to remain alone (Under the assumption of equality of the two groups and completeness of preferences). By second remark above once visited no man remains alone.

Fact 2 No woman can be part of an objecting pair for the resulting matching. Consider the woman $W$: she cannot be part of an objecting pair with a man she did not visit: she is married to a man preferred to all of them.
She cannot be part of an objecting pair with a man $m$ she already visited either: dismissed in favor of another woman, by transitivity she cannot be preferred to the woman matched to $m$. Hence, the matching resulting from the algorithm is stable. ∎

# Example revisited

Preferences men:

- $D \succ_a C \succ_a B \succ_a A$, $\quad B \succ_b A \succ_b C \succ_b D$
- $D \succ_c A \succ_c C \succ_c B$, $\quad A \succ_d D \succ_d C \succ_d B$.

Preferences women:

- $b \succ_A a \succ_A c \succ_A d$, $\quad b \succ_B d \succ_B a \succ_B c$
- $b \succ_C d \succ_C c \succ_C a$, $\quad b \succ_D c \succ_D d \succ_D a$.

Men visiting:

$$\{(a, C), (b, B), (c, D), (d, A)\}.$$

Women visiting

$$\{(a, A), (b, B), (c, D), (d, C)\}.$$

# One more example

Preferences men:

- $A \succ_a B \succ_a C, \quad C \succ_b A \succ_b B, \quad B \succ_c A \succ_c C.$

Preferences women:

- $c \succ_A a \succ_A b, \quad b \succ_B a \succ_B c, \quad a \succ_C b \succ_C c$

Men visiting:

$$\{(a, A), (b, C), (c, B)\}.$$

Women visiting:

$$\{(c, A), (b, B), (a, C)\}.$$

One more?

$$\{(a, B), (b, C), (c, A)\}.$$

# Some numbers

$|(\text{matching})| = n!$

$|(\text{stable matching})| = ?$

Usually not many!

However there is a method to build many stable matchings.

For instance:

$$n = 8 \qquad 269, \quad n = 16 \qquad 195472 \quad n = 32 \qquad 104310534400.$$

# Comparing matchings

Let us consider two matchings $\Delta$ and $\Theta$.

**Definition**

*Write $\Delta \succeq_m \Theta$ if every man is either associated to the same woman in the two matchings or associated to a preferred woman in $\Delta$. Write $\Delta \succeq_w \Theta$ if every woman is either associated to the same man in the two matchings or associated to a preferred man in $\Delta$.*

Remarks

$\Delta \succeq_m (\succeq_w)\Delta$ reflexivity;

$\Delta \succeq_m (\succeq_w)\Theta \quad \wedge \quad \Theta \succeq_m (\succeq_w)\Lambda$ imply $\Delta \succeq_m (\succeq_w)\Lambda$ transitivity;

This is not a complete ordering.

# Women versus men

**Theorem**

*Let $\Delta$ and $\Theta$ be stable matchings. Then $\Delta \succeq_m \Theta$ if and only if $\Theta \succeq_w \Delta$.*

**Proof** Suppose that $\Delta \succeq_m \Theta$. Let $(a, A) \in \Delta$ and $(b, A) \in \Theta$.
We have to prove that $b \succ_A a$.

Suppose that $(a, F) \in \Theta$. We have $A \succ_a F$ because $\Delta \succeq_m \Theta$.

So

$$\{(a, F), (b, A)\} \subset \Theta \,,$$

but $\Theta$ is stable matching,

therefore $b \succ_A a$, otherwise the pair $(a, A)$ would object.

# Ordering stable matching

**Theorem**

*Let $\Lambda_m(\Lambda_w)$ be the men (women) visiting matching and let $\Theta$ be another stable matching. Then*

$$\Lambda_m \succeq_m \Theta \succeq_m \Lambda_w \qquad \Lambda_w \succeq_w \Theta \succeq_w \Lambda_m.$$

Women visiting is the best algorithm for the women.
Men visiting is the best algorithm for the men.

# Proof

Proving that a woman cannot be rejected by a man available to her. Induction on the days of visit

First day. Suppose $A$ is rejected by $a$ in favor of $B$ and that there is a stable set $\Delta$ such that

$$\{(a, A), (b, B)\} \subset \Delta.$$

Then, since

$$B \succ_a A$$

then

$$b \succ_B a.$$

Impossible! By assumption $B$ is visiting $a$ the first day, thus $a$ is her preferred man!

# Proof continues

Suppose no woman was rejected by an available man the days $1, \dots, k-1$. See that no woman can be refused by an available man the day $k$.

By contradiction, suppose $A$ is rejected by $a$ in favor of $B$ in day $k$ and that there is a stable set $\Delta$ such that

$$\{(a, A), (b, B)\} \subset \Delta.$$

Then, since

$$B \succ_a A$$

implying

$$b \succ_B a.$$

Since $B$ is visiting $a$, but likes better $b$, then $B$ visited $b$ some day before and was rejected, against the inductive assumption.

# Proof ends

Hence, for every stable $\Theta$ we have

$$\Lambda_w \succeq_w \Theta\,.$$

To conclude is suffices to use the symmetry between men and women.

# Further result

### Definition

*Let $\Delta, \Theta$ be two matchings. Define a new "matching" $\Delta \vee_w \Theta$ by pairing each woman to the preferred man between those paired to her in $\Delta$ and $\Theta$.*

### Remark

*$\Delta \vee_w \Theta$ is not necessarily a matching, i.e. it may happen that two women are paired with the same man.*

### Theorem

*Let $\Delta, \Theta$ be two stable matching. Then $\Delta \vee_w \Theta$ is a stable matching.*

Then $\vee_w$ provides a lattice structure (i.e. a partial order such that every two elements have a unique maximum and a unique minimum) to the set of stable matchings

# An extension

One can consider the case when the number of men and women are different, for instance men are more.

**Definition**

*A matching is a function assigning to every man an element of the set* $\mathcal{W} \cup \{single\}$ *with every woman associated to a man.*

Assuming that for men remaining alone is worse than being paired to any woman, then $(A, a)$ objects to a matching if $a$ is either single or paired to a woman for him worse than $A$, and $A$ prefers $a$ to the man paired to her in the matching. A matching is stable if there are no objecting pairs.

**Theorem**

*The set of the men which are not married remains the same on all stable matchings.*

# A key issue

Is it possible that people lie on their true preferences?

It is intuitive, and true, that in the case of the women visiting, they do not have incentive to lie.

But what about men?

Examples show that in some cases for a man could be convenient to lie.

# Further extensions

1) Getting married, but not at any cost;
2) Polygamous matching;
3) Unisexual matching.

1) adaptation of the idea of stable matching, the algorithm works;
2) adaptation of the idea of stable matching, the algorithm works;
3) a stable matching need not to exist.