1. Basics of Game Theory

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Decision theories

Variants with one decision maker:

1. scalar optimization
2. vector optimization
3. deterministic optimization
4. stochastic optimization
5. ...

Variants with many decision makers:

1. Game theory
2. Social choice
3. Mechanism design
4. Machine learning
5. ...

It is much easier to define what is the best choice when there is one decision maker only.

What is a game

Examples:

1. Chess, checkers,...
2. Two people bargaining how to divide a pie
3. A burglar and a guard
4. Parties in a Parliament
5. ...

Games are efficient models for an enormous amount of everyday life situations

A game is a process consisting in:

1. A set of players (at least two)
2. An initial situation
3. Rules that the players must follow
4. All possible final situations
5. The preferences of all players on the set of the final situations

Assumptions of the theory

Players are

1. Selfish
2. Rational

Selfish means that the players only care about their own preferences on the outcomes of the game

This is not an ethical issue, but a mathematical assumption. We need it to define what is the meaning of a rational choice.

Rationality is a much more involved issue.

Preferences

Definition

Let X be a set. A preference relation on X is a binary relation $\succeq$ such that for all $x, y, z \in X$:

1. $x \succeq x$ *(reflexive)*
2. $x \succeq y$ *or* $y \succeq x$ *or both (complete)*
3. *If* $x \succeq y$ *and* $y \succeq z$*, then* $x \succeq z$ *(transitive)*

The first rationality assumption is:

The players are able to provide a preference relation over the outcomes of the game.

Utility functions

Definition

Let $\succeq$ be a preference relation over X. A utility function representing $\succeq$ is a function $u : X \to \mathbb{R}$ such that

$$u(x) \geq u(y) \Longleftrightarrow x \succeq y.$$

1. A utility function may not exist in particular cases, however it exists in general setting, in particular if X is a finite set
2. When a utility function exists, then infinitely many utility functions exist, since any strictly increasing transformation of a utility function is a utility function.

The second rationality assumption reads:

The agents are able to provide a utility function representing their preferences relations, whenever necessary

Allais experiment 1

Alternative A

gain	probability
2500	33%
2400	66%
0	1%

Alternative B:

gain	probability
2500	0%
2400	100%
0	0%

In a sample of 72 people exposed to this experiment, 82% of them decided to play the Lottery B.

This is rational if $\frac{34}{100}u(2400) > \frac{33}{100}u(2500)$.

Allais experiment 2

Alternative C

gain	probability
2500	33%
0	67%

Alternative D:

gain	probability
2400	34%
0	66%

83% of the people interviewed selected lottery C.

This is rational if $\frac{34}{100}u(2400) < \frac{33}{100}u(2500)$.

Allais experiment shows that usually agents are not rational players!

Probability issues

The third rationality assumption reads:

The players use consistently the probability laws, in particular they are consistent with the computation of the expected utilities, they are able to update probabilities according to Bayes rule...

The beauty contest

Write an integer between 1 and 100.

The mean M is calculated.

Those writing the number at the minimum distance from qM win the game ($0 < q < 1$).

A rational player will answer 1, independently of q. And he will probably lose.

Deepness of the analysis

The fourth rationality assumption reads:

The players are able to understand the consequences of all their actions, the consequences of this information on any other player, the consequences of the consequences and so on.

Extending decision theory

Finally, the fifth rationality assumption reads:

The players are able to use decision theory, whenever it is possible

that is, given a set of alternatives X, and a utility function u on X, each player seeks an $\bar{x} \in X$ such that

$$u(\bar{x}) \geq u(x)\,, \qquad \forall x \in X.$$

Summary of the rationality assumptions

1. The players are able to rank the outcomes of the game
2. The players are able to provide a utility function for their ranking
3. The players use the expected value to build their utility function in presence of random events
4. The players are able to analyse all the consequences of their actions, and the consequences of the consequences and so on
5. The players use the tools of decision theory whenever possible

An immediate and important consequence of the axioms

A player does not take an action *a* it she has available an action *b* providing her a strictly better result, no matter what the other players do.

Principle of elimination of strictly dominated actions.

Example: Player one action set is $\{18, \ldots, 30\}$, player two action set is {accept, refuse}. If the preference of player two is passing the exam with any grade, rather than repeating it, the action *refuse* is strictly dominated. Observe that asking for a better grade is not an available action in this game

Bimatrices

Player 1 chooses a row, player 2 a column.
This results in a pair of numbers, respectively the utility of Player 1 and 2.

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}$$

Utilities of player 1:

$$\begin{pmatrix} 8 & 2 \\ 7 & 0 \end{pmatrix}$$

The second row is strictly dominated by the first, thus player 1 will select the first row

Even if this principle is usually not very informative, it has surprising consequences

Comparisons of games

Game 1:

$$\begin{pmatrix} (10,10) & (3,15) \\ (15,3) & (5,5) \end{pmatrix}$$

Game 2:

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}$$

Observe: in any outcome the players are better off in the first game rather than in the second:

However it is more convenient for them to play the second!

Less is better than more

The first game:

$$\begin{pmatrix} (10,10) & (3,5) \\ (5,3) & (1,1) \end{pmatrix}$$

The second game, containing all possible outcomes the first, and some further outcomes:

$$\begin{pmatrix} (1,1) & (11,0) & (4,0) \\ (0,11) & (10,10) & (3,5) \\ (0,4) & (5,3) & (1,1) \end{pmatrix}$$

Having less available actions can make the players better off!

Uniqueness issue

$$\begin{pmatrix} (0,0) & (1,1) \\ (1,1) & (0,0) \end{pmatrix}$$

Rational outcomes of this game?

We formally do not know but it is obvious that the rational outcomes will be $(1,1)$

(First row, second column) and (second row, first column) cannot be distinguished and this creates a coordination problem between the players

Elimination of dominated strategies

A vote. Three players, alternatives A, B, C. Players preferences:

$$A \quad \gneq_1 [1] \quad B \quad \gneq_1 \quad C$$

$$B \quad \gneq_2 \quad C \quad \gneq_2 \quad A$$

$$C \quad \gneq_3 \quad A \quad \gneq_3 \quad B$$

In case of three different votes, the alternative selected by player one is winning

What can we expect as rational outcome of the game?

Try with elimination of dominated actions...

[1] $A \gneq B$ means $A \succeq B$ and not $B \succeq A$

The voting game

- Alternative A is a weakly dominant strategy for Player 1
- Players 2 and 3 have as weakly dominated strategy to play their worst choice

Thus the game reduces to

A	A
C	A

The result is the worst one for the first player!