Repeated games-Correlated equilibria

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Summary of the slides

1. Repeated games: an example
2. Correlated equilibria: definition
3. Correlated equilibria: existence and geometry of the set
4. Correlated equilibria and strictly dominated strategies

The reference game

Example

$$\begin{pmatrix} (3,3) & (0,10) & (-2,-2) \\ (10,0) & (1,1) & (-1,-1) \\ (-2,-2) & (-1,-1) & (-2,-2) \end{pmatrix}$$

$(1,1)$ is the outcome at the equilibrium obtained with strictly dominant strategies

What happens if it played several times (days)?

Nash equilibria

Playing all days the dominant strategy is an obvious equilibrium

Are there other Nash equilibria? The more appealing outcome $(3, 3)$ is unavailable to the players?

We shall show that, for every $a > 0$, if the game is played a sufficiently large number of times, the players can get at least $3 - a$ each on average

The strategy profile

We say that the game is played once a day for N days. Consider the following strategy profile, with symmetric strategies:

Fix $k < N$. Each Player uses the first strategy (row/column) in the first $N - k$ days and the second strategy in the last k days, if the opponent plays the same, otherwise, if at one day the opponent deviates, from the subsequent stage the chooses the last strategy

Note that the strategy at every day $i > 1$ depends also from the choices of the players in the days $j < i$

It is a Nash equilibrium

What a player gets under the given strategy profile

$$\frac{(N-k)3+k1}{N}$$

What the player gets by deviating the last useful day (the best day for deviating)

$$\frac{(N-k-1)3+10+k(-1)}{N}$$

Thus the strategy profile is a NEp if and only if

$$\frac{(N-k)3+k1}{N} \geq \frac{(N-k-1)3+10+k(-1)}{N}$$

True provided $k > 3$

Payoffs

The payoffs at the NEp

$$\frac{(N-k)3+k1}{N}$$

For every k

$$\lim_{N\to\infty}\frac{(N-k)3+k1}{N}=3$$

On average the players can get at least $3-a$ each per day, if they play a sufficiently large number of days ■

Remarks

- Collaboration, even using strictly dominated strategies in the one shot game, can be based on rationality, provided the game is repeated
- In the example the NEp has a weakness: it is based on a mutual threat of the players, which could be considered not credible
- The number of the NEp in repeated games is very large, and thus hardly informative

Correlated equilibria: the battle of sexes

$$\begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix}$$

Three NEp. $[(1,0)(1,0)]$ $[(0,1)(0,1)]$ $[(\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3})]$

The first two NEp are in pure strategies and the outcomes are $(2,1)$ and $(1,2)$; the third one is fully mixed and the outcome is $(\frac{2}{3},\frac{2}{3})$

What if the players correlate their actions in the following way: they toss a fair coin, if the coin comes up heads, they play (F,F), and if it comes up tails, they play (C,C). The expected payoff is then $(\frac{3}{2},\frac{3}{2})$

It is easy to check that the correlated strategy is an equilibrium.

Correlated equilibria: another example

$$\begin{pmatrix} (6,6) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}$$

Three NEp. $[(1,0)(0,1)]$ $[(0,1)(1,0)]$ $[(\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3})]$

The first two NEp are in pure strategies and the outcomes are $(2,7)$ and $(7,2)$; third one fully mixed and outcome $(\frac{14}{3},\frac{14}{3})$

Can the players do any better?

Consider the following probability distribution over the outcomes

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$$

This provides better outcome ($\frac{15}{3}$) for both than the mixed NEp.

Can we convince the players to agree on this (i.e. is this an equilibrium)?

Partial information to the players

The solution is not as simple as in the battle of sexes (why?).

But suppose the player agree on the following mechanism. An external entity makes a random choice on the outcomes according to the probabilities on the outcomes given by the table, and tells them what to do, privately

With this partial information, the players do not have incentive to change strategy!

No incentive to change

- The random choice selects outcome $(7, 2)$. Player 1 is told to play second row, Player 2 first column. Player 1 now knows that Player 2 is told to play first column: he does not deviate since the outcome in NEp. Player 2 knows that the probability Player 1 is told to play first row is $\frac{1}{2}$. Thus his expected value following the suggestion is $\frac{1}{2}(6+2)$. If he deviates his expected value is $\frac{1}{2}(7+0)$: no interest to deviate for both
- The random choice selects outcome $(6, 6)$. Player 1 is told to play first row, Player 2 first column. Both players now now that the other player will play the two strategies with the same probability. Thus the expected value following the suggestion is $\frac{1}{2}(6+2)$. If the player deviates his expected value is $\frac{1}{2}(7+0)$: no interest to deviate for both
- The random choice selects outcome $(2, 7)$. Just as the first case (interchanging the role of the players): no interest to deviate for both

Toward the correlated equilibrium

In the above example the probability distribution over the outcomes is accepted by all players, since in any case they do not have incentive to deviate, given the information they have

Correlated equilibrium

Consider the game $(A, B) = (a_{ij}, b_{ij})$ $i = 1, \ldots, n$, $j = 1, \ldots, m$
Let $I = \{1, \ldots, n\}$, $J = \{1, \ldots, m\}$ and $X = I \times J$

Definition

A correlated equilibrium is a probability distribution $p = (p_{ij})$ on X such that, for all $i' \in I$

$$\sum_{j=1}^{m} p_{i'j} a_{i'j} \geq \sum_{j=1} p_{i'j} a_{ij} \qquad \forall i \in I$$

such that, for all $j' \in J$

$$\sum_{i=1}^{n} p_{ij'} b_{ij'} \geq \sum_{i=1}^{n} p_{ij'} b_{ij} \qquad \forall j \in J$$

The inequalities in the example

$$\begin{pmatrix} (6,6) & (2,7) \\ (7,2) & (0,0) \end{pmatrix} \quad \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

$$\begin{cases} 6x_1 + 2x_2 \geq 7x_1 \\ 7x_3 \geq 6x_3 + 2x_4 \\ 6x_1 + 2x_3 \geq 7x_1 \\ 7x_2 \geq 6x_2 + 2x_4 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ x_i \geq 0 \qquad\qquad i = 1, \dots, 4 \end{cases}$$

Existence

The following theorem guarantees that the set of the correlated equilibria of a finite game is nonempty

Theorem

A NEp profile generates a correlated equilibrium

Given the NEp $(\bar{x}, \bar{y})$ the probability distribution on the outcome matrix is $p = (p_{ij})$ with $p_{ij} = \bar{x}_i \bar{y}_j$

The proof

Proof

We have to prove that

$$\sum_{j=1}^{m} \bar{x}_{i'} \bar{y}_j a_{i'j} \geq \sum_{j=1}^{m} \bar{x}_{i'} \bar{y}_j a_{ij} \qquad \forall i \in I$$

Obvious if $\bar{x}_{i'} = 0$. If $\bar{x}_{i'} > 0$ we need to show that

$$\sum_{j=1}^{m} \bar{y}_j a_{i'j} \geq \sum_{j=1}^{m} \bar{y}_j a_{ij} \qquad \forall i \in I$$

The left (right) hand side is the expected utility of the first player if he plays row i'(row i) and the second plays his equilibrium strategy $\bar{y}$

The inequality holds since the pure strategy i' is played with positive probability so i' must be a best reaction to $\bar{y}$ ∎

The set of the correlated equilibria

Theorem

The set of the correlated equilibria of a finite game is a nonempty convex polytope

Proof Remember that a convex polytope is a closed bounded convex set which is the smallest convex set containing a finite number of points. The set of the correlated equilibria is the solution set of a system of $n^2 + m^2$ linear inequalities (n, m are the number of the pure strategies of the players), called incentive constraints, plus the conditions of being a probability distribution $(p_{ij} \geq 0, \sum p_{ij} = 1)$ ■

Dominated strategies

Proposition

If a row i' is strictly dominated, then $p_{i'j} = 0$ for every j

Proof

Suppose i' is strictly dominated by i. This implies $a_{i'j} - a_{ij} < 0$ for all j. Since $p_{i'j} \geq 0$ for every j and

$$\sum_{j=1}^{m} p_{i'j}(a_{i'j} - a_{ij}) \geq 0$$

it must be $p_{i'j} = 0$ for every j ■

Is the same true for a weakly dominated row?

Yet another example, with 3 players

$$
\begin{array}{ccc}
A & B & C \\
\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} (0,1,3) & (0,0,0) \\ (1,1,1) & (1,0,0) \end{pmatrix} \end{array}
&
\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} (2,2,2) & (0,0,0) \\ (2,2,0) & (2,2,2) \end{pmatrix} \end{array}
&
\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} (0,1,0) & (0,0,0) \\ (1,1,1) & (1,0,3) \end{pmatrix} \end{array}
\end{array}
$$

The strategies $(B, L, [\alpha A + (1-\alpha)C])$ are NEp, with payoff $(1,1,1)$.

Can the players get $(2,2,2)$ instead?