Extensive form games

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Summary of the slides

1. Extensive form games
2. Perfect information
3. The tree of the game
4. Backward induction
5. Von Neumann theorem
6. Different (types of) solutions
7. Combinatorial games
8. The Nim game and Bouton theorem
9. Strategies

Extensive form

Three politicians must decide whether to raise their salaries or not. The vote is public and in sequence. They would prefer to have a salary increase, but they would also like to vote against it.

Main features

1. The moves are in sequence
2. Every possible situation is known to the players, at any time they know the whole past history, and the possible developments

This is called a game with perfect information: each player is informed of all the events that have previously occurred

How can we represent them?

How can we solve them?

The tree

The tree

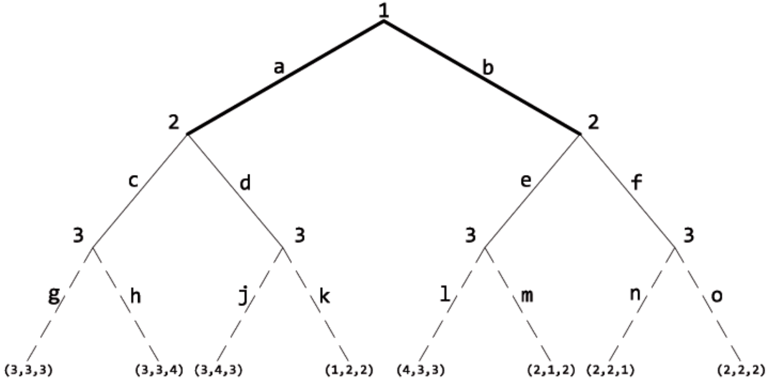
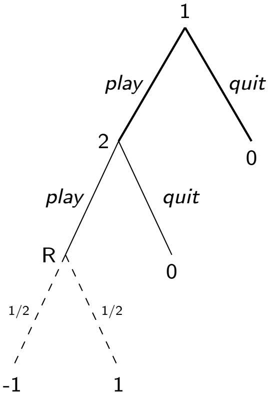


Figure: The game of the three politicians

A game with chance

The players must decide in sequence whether to play or not.
If both decide to play, a coin is tossed, and the first one wins with heads, the second one with tails.



Directed graphs

Definition

A finite directed graph is a pair (V, E) where

1. *V is a finite set, called the set of vertices*
2. *$E \subset V \times V$ is a set of ordered pairs of vertices called the set of the (directed) edges*

Definition

A path from a vertex v_1 to a vertex v_{k+1} is a finite sequence of vertices-edges $v_1, e_1, v_2, \dots, v_k, e_k, v_{k+1}$ such that $e_i \neq e_j$ if $i \neq j$ and $e_j = (v_j, v_{j+1})$. k is called the length of the path

Tree

Definition

An oriented graph is finite directed graph having no bidirected edges, that is for all j, k at most one between (v_j, v_k) and (v_k, v_j) may be arrows of the graph.

Definition

A tree is a triple (V, E, x_0) where (V, E) is an oriented graph and x_0 is a vertex in V such that there is a unique path from x_0 to x, where x is any vertex in V

Definition

A child of a vertex v is any vertex x such that $(v, x) \in E$. A vertex is called a leaf if it has no children. We say that the vertex x follows the vertex v if there is a path from v to x.

The tree of a game

In order to have a complete representation of a game with a tree, we need to add some requirements

Definition

An *Extensive form game with perfect information* *consists of*

1. *A finite set* $N = \{1, \ldots, n\}$[1] *of players*
2. *A game tree* (V, E, x_0)
3. *A partition* $\{P_1, P_2, \ldots, P_{n+1}\}$ *of the vertices which are not leaves*
4. *A probability distribution for each vertex in* P_{n+1}*, defined on the edges from the vertex to its children*
5. *An n-dimensional vector attached to each leaf*

[1] n will denote the cardinality of the set N.

Comments

1. The set P_i, for $i \leq n$, is the set of the nodes v where Player i must choose a child of v, representing a possible move form him at v
2. P_{n+1} is the set of the nodes where a chance move is present. P_{n+1} can be empty
3. When P_{n+1} is empty (i.e. no chance moves are present in the game) the players need to have only preferences on the leaves: a utility function is not required

Solving the game

We use the rationality axioms:

1. What one player does in positions leading to leaves can be determined by decision theory (rationality assumption 5)
2. This is known to all other players (r.a. 4) and this information can be used used by them
3. Thus players moving to vertices going to leaves can use decision theory (r.a. 5)
4. This is known to all other players (r.a. 4) and this information can be used used by them
5. ...
6. The player starting the game uses decision theory to make the first move

Backward induction

Definition

Define Length of the game as the length of the longest path in the game

Decision theory allows solving the games of length 1

Rationality assumption 4 allows solving a game of length $i+1$ if the games of length at most i are solved

Thus we can solve games of any finite length

This method is called backward induction

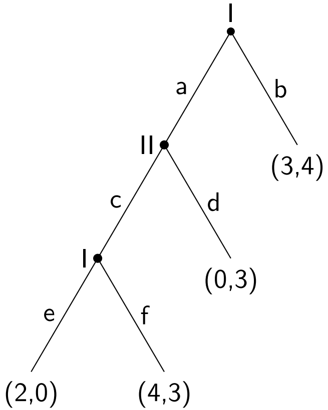
The first rationality theorem

Theorem

The rational outcomes of a finite, perfect information game are those given by the procedure of backward induction

Observe: this method can be applied since every vertex v of the game is the root of a new game, made by all followers of v in the initial game. This game is called a subgame of the original one.

Multiple solutions



The outcomes obtained by backward induction are: $(4,3)$ and $(3,4)$. Uniqueness is not guaranteed.

The chess theorem (von Neumann)

Theorem

In the game of chess one and only one of the following alternatives holds:

1. *The white has a way to win, no matter what the black does*
2. *The black has a way to win, no matter what the white does*
3. *The white has a way to force at least a draw, no matter what the black does, and the same holds for the black*

Main question: is the above a true theorem?

Why is it impossible to say more than that?

An interesting proof of the chess theorem

Here is a proof of the theorem:

Suppose the length of the game is $2K$ so each player has K choices to make. Call a_i the move of the White at her i-th stage and b_i that one of the Black.
The first alternative in the chess theorem can be expressed as

$$\exists a_1 : \forall b_1 \exists a_2 : \forall b_2 \ldots \exists a_K : \forall b_K \Rightarrow \text{white wins}$$

Now suppose this is not true. Then

$$\forall a_1 \exists b_1 : \forall a_2 : \exists b_2 : \ldots \forall a_K : \exists b_K :\Rightarrow \text{white does not win}$$

But this means exactly that Black has the possibility to get at least a draw.

Summarizing

If White does not have a strategy to win no matter what Black does, then Black has the possibility to get at least the draw.

Symmetrically, if Black does not have a strategy to win no matter what White does, then White has the possibility to get at least the draw

Thus if the first and the second alternatives in the chess theorem are not true, necessarily the third one is true!

Extending von Neumann theorem

The von Neumann theorem applies to every finite game of perfect information where the possible result is either the victory of one player or a tie. Thus the following corollary holds:

Corollary

Consider a finite perfect information game with two players, where the only possible outcomes are the victory of one or the other player. Then one and only one of the following alternative holds:

1. *The first player can win, no matter what the second one does*
2. *The second player can win, no matter what the first one does*

Different types of solutions

Very weak solution: The game has a rational outcome, but it is inaccessible, like in chess

Weak solution: The outcome of the game is known, but how to get it is not (in general)

Solution: It is possible to provide an algorithm to find a solution

Chomp

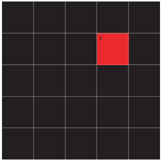


Figure: The first player removes the red square

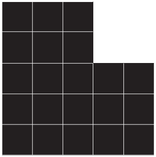
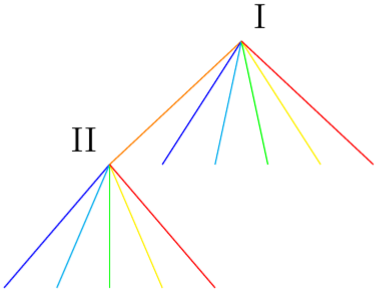


Figure: Now it is up to the second player to move

The player taking the most square at the bottom-left loses the game

Chomp



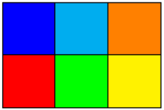


Figure: Edges are coloured according to the chosen square

Suppose the second wins. Then he has a winning action after the choice of the orange edge. Suppose the winning move for him is to choose the green edge. This means that the player starting at the node following the green edge will lose. But the tree starting at that node is exactly the same tree starting at the green edge relative to the first player. Thus the first player has a move available that guarantees the victorygame starting at the node following the green edge going out from the root of the tree (move of the first player) has the starting player (i.e. the second player) losing. Contradiction!

Impartial combinatorial games

Definition

An impartial combinatorial game is a game such that

1. *There are two players moving in alternate order*
2. *There is a finite number of positions in the game*
3. *The players follow the same rules*
4. *The game ends when no further moves are possible*
5. *The game does not involve chance*
6. *In the classical version the winner is the player leaving the other player with no available moves, in the misère version the opposite*

Examples of combinatorial games

1. k piles of cards. At her turn the player takes as many cards as she wants (at least one!) from one and only one pile
2. k piles of cards. At her turn the player takes as many cards as she wants (at least one!) from no more than $j < k$ piles
3. k cards in a row. At her turn the player takes either j_1 or ... or j_l cards

In the first two cases the positions are $(n_1, \dots, n_k)$ where n_i is a non negative integer for all i. In the last examples positions can be seen as all non negative integers smaller or equal to k.

How to solve these games

Partition the set of all possible positions into two sets:

1. P-positions
2. N-positions

Rules (for the classical version):

1. Terminal positions are P-positions (losing positions, the player does not have moves left)
2. From a P-position only N-positions are available
3. From an N-position it is possible to go to a P-position

The player playing from an N-position wins!

The Nim game

Nim game is defined as $(n_1, \dots, n_k)$ where for all i n_i is a positive integer. A player at her turn has to take one (and only one) n_i and substitute it with $\hat{n}_i < n_i$. The winner is the player arriving to the position $(0, \dots, 0)$.

Meaning: taking away cards form one pile. Goal: to clear the table.

A new operation on the non negative integers

Define an operation $\oplus$ on $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$ in the following way: for $n_1, n_2 \in \mathbb{N}$

1. Write n_1, n_2 in binary form, denoted by $[n_1]_2, [n_2]_2$
2. Write the sum $[n_1]_2 \oplus [n_2]_2$ in binary form where $\oplus$ is the (usual) sum, but without carry
3. What you get is the result in binary form

An example

Example

The $\oplus$ operation applied to 1,2,4, and 1.

$$
\begin{array}{lcccc}
[1]_2 & = & 0 & 0 & 1 \\
[2]_2 & = & 0 & 1 & 0 \\
[4]_2 & = & 1 & 0 & 0 \\
[1]_2 & = & 0 & 0 & 1 \\
\hline \\
[6]_2 & = & 1 & 1 & 0
\end{array}
$$

$$n_1 \oplus n_2 \oplus n_3 \oplus n_4 = 6$$

The group

Definition

A nonempty set A with an operation $\cdot$ on it is called a group provided:

1. *for $a, b \in A$ the element $a \cdot b \in A$*
2. *$\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$*
3. *there is an element (which will result unique, and called identity) e such that $a \cdot e = e \cdot a = a$ for all $a \in A$*
4. *for every $a \in A$ there is $b \in A$ such that $a \cdot b = b \cdot a = e$. Such an element is unique and called inverse of a*

If $a \cdot b = b \cdot a$ for all $a, b \in A$ the group is called abelian

Examples and properties

Example

Examples of groups are

1. *The integers* $\mathbb{Z}$ *with the usual sum*
2. *The real numbers without* 0, *with the usual product*
3. *The* $n \times n$ *matrices with non zero determinant*

Proposition

Let $(A, \cdot)$ *be a group. Then the cancelation law holds:*

$$a \cdot b = a \cdot c \Longrightarrow b = c$$

The Nim group

Proposition

The set of the natural numbers with $\oplus$ in an abelian group

Proof The identity element is 0, the inverse of n is n itself. Associativity and commutativity of $\oplus$ are easy. ∎

The cancelation law holds: $n_1 \oplus n_2 = n_1 \oplus n_3$ implies $n_2 = n_3$.

The Bouton theorem

Theorem (Bouton)

A $(n_1, n_2, \ldots, n_k)$ position in the Nim game is a P-position if and only if $n_1 \oplus n_2 \oplus \ldots \oplus n_k = 0$.

Proof

- Terminal states are P-positions This is obvious, the only terminal position is $(0, \ldots, 0)$
- Positions such that $n_1 \oplus n_2 \oplus \ldots \oplus n_N = 0$ go only to positions with Nim sum different from zero Suppose instead the new position is $(n_1', n_2, \ldots, n_N)$ and $n_1' \oplus n_2 \oplus \ldots \oplus n_N = 0 = n_1 \oplus n_2 \oplus \ldots \oplus n_N$; then by the cancelation law $n_1' = n_1$, which is impossible
- Positions with non zero nim sum can go to a position with Nim sum zero Let $z := n_1 \oplus n_2 \oplus \ldots \oplus n_N \neq 0$. Take a pile having 1 in the most left column where the expansion of z has 1, put there 0 and go right, leaving unchanged a digit corresponding to a 0 in the expansion of the sum, changing it otherwise. It is easy to check that the result is smaller than the original number

An example

Example

From

$$\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$

go to

$$\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$

Observe: there are three initial good moves.

Conclusions

Games with perfect information can be "solved" by using backward induction

However backward induction is a concrete solution method only for very simple games, because of limited rationality.

According to conclusions we can reach different level of solutions:

1. Very weak solutions: not even the outcome is predictable (chess. . .)
2. Weak solutions: a logical argument provides the outcome, but how to reach it is not known (chomp, in general)
3. Solutions: categories of games where it is possible to produce the way to get to the rational outcome

Strategies

In Backward induction a move must be specified at any node. P_i is the set of the nodes where player i is called to make a move

Definition

A pure strategy for player i is a function defined on the set P_i, associating to each node v in P_i a child w, or equivalently an edge (v,x)

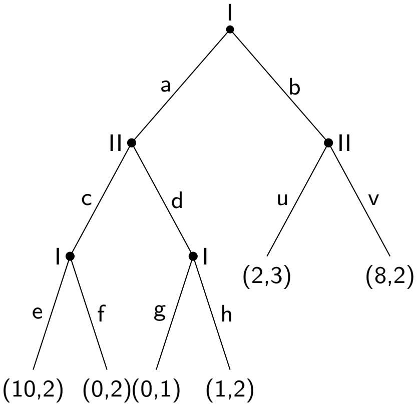
A mixed strategy is a probability distribution on the set of the pure strategies

When a player has n pure strategies, the set of her mixed strategies is

$$\Sigma_n := \{p = (p_1, \ldots, p_n) : p_i \geq 0 \sum p_i = 1\}$$

Σ_n is the fundamental simplex in n-dimensional space

Strategies in a tree



	cu	cv	du	dv
aeg	(10,2)	(10,2)	(0,1)	(0,1)
aeh	(10,2)	(10,2)	(1,2)	(1,2)
afg	(0,2)	(0,2)	(0,1)	(0,1)
afh	(0,2)	(0,2)	(1,2)	(1,2)
beg	(2,3)	(8,2)	(2,3)	(8,2)
beh	(2,3)	(8,2)	(2,3)	(8,2)
bfg	(2,3)	(8,2)	(2,3)	(8,2)
bfh	(2,3)	(8,2)	(2,3)	(8,2)

Observe: the table has pairs repeated several times: different strategies can lead to the same outcomes

Revisiting von Neumann

In terms of strategies here is von Neumann's theorem:

Theorem

In the chess game one of the following alternatives holds:

1. *the white has a winning strategy*
2. *the black has a winning strategy*
3. *both players have a strategy leading them at least to a tie*

Outcomes chess in strategic form 1

	1	2	3	...	K
a	...	...	...	...	...
b	...	...	...	...	...
c	W	W	W	...	W
...	...	...	...	...	...
K	...	...	...	...	...

The White has a winning strategy

Outcomes chess in strategic form 2

	1	2	3	...	K
a	...	...	B	...	...
b	...	...	B	...	...
c	...	...	B	...	...
...	...	...	...	...	...
K	...	...	B	...	...

The Black has a winning strategy

Outcomes chess in strategic form 3

	1	2	3	...	K
a	...	...	B	...	...
b	...	...	T	...	...
c	T	W	T	...	W
...	...	...	...	...	...
...	...	...	T	...	...

The outcome of the game is a Tie

This is excluded

T	B	W
W	T	B
B	W	T

This is the case excluded by von Neumann

Remark

If $P_i = \{v_1, \dots, v_k\}$ and v_j has n_j children then the number of strategies of Player i is $n_1 \cdot n_2 \cdot \dots \cdot n_k$

This shows that the number of strategies even in short games is usually very high. For instance, if Tic-Tac-Toe is stopped after three moves, the first player has (not exploiting symmetries) $9 \cdot 7^{72}$ strategies

Games with imperfect information

Sometimes players must make moves at the same time. This can be still represented with a tree.

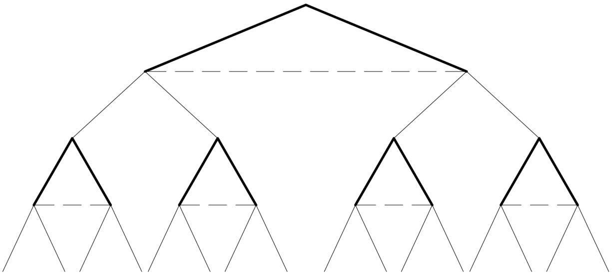


Figure: The prisoner dilemma repeated twice, after one shot the players see the result

Information set

Definition

An information set *for a player i is a pair $(U_i, A(U_i))$ with the following properties:*

1. $U_i \subset P_i$ *is a nonempty set of vertices* $v_1, \ldots, v_k$
2. *each* $v_j \in U_i$ *has the same number of children*
3. $A_i(U_i)$ *is a partition of the children of* $v_1 \cup \cdots \cup v_k$ *with the property that each element of the partition contains exactly one child of each vertex* v_j

Meaning: player i knows to be in U_i, but not in which vertex she is; the partition has the meaning that each set of the partition represents an available move for the player.

Formal definition

Definition

An Extensive form Game with imperfect information is constituted by

1) *A finite set* $N = \{1, \dots, n\}$ *of players*
2) *A game tree* (V, E, x_0)
3) *A partition made by sets* $P_1, P_2, \dots, P_{n+1}$ *of the vertices which are not leaves*
4) *A partition* $(U_i^j), j = 1, \dots, k_i$ *of the set* P_i*, for all* i*, with* (U_i^j, A_i^j) *information set for all* i *for all* j
5) *A probability distribution, for each vertex in* P_{n+1}*, defined on the edges going from the vertex to its children*
6) *An n-dimensional vector attached to each leaf*

Strategies for imperfect information games

Definition

A pure strategy for player i in an imperfect information game is a function defined on the collection $\mathcal{U}$ of his information sets and assigning to each U_i in $\mathcal{U}$ an element of the partition $A(U_i)$. A mixed strategy is a probability distribution over the pure strategies.

Remark

A game of perfect information is a particular game of imperfect information where all information sets of all players are singletons