Cooperative games (1)

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Summary of the slides

1. Cooperative game
2. TU game
3. Examples
4. Additive, superadditive and convex games
5. Imputation
6. Core
7. Nucleolus

Cooperative game

Definition

A *cooperative game* *is a pair* (N, V) *where*

$$V : \mathcal{P}(N) \to \mathbb{R}^n$$

and

$$V(A) \subseteq \mathbb{R}^A$$

- $\mathcal{P}(N)$ is the collection of all nonempty subsets of the finite set N, such that $|N| = n$, the set of the players
- $V(A)$, for a given $A \in \mathcal{P}(N)$ is the set of the aggregate utilities of the players in coalition A: $x = (x_i)_{i \in A} \in V(A)$ if the players in A can guarantee utility x_i to every $i \in A$
- Sometimes $V(A)$ represents costs rather utilities, in this case of course all inequalities must be reversed

TU game

Definition

A transferable utility game (TU game) is a function

$$v : 2^N \to \mathbb{R}$$

such that $v(\emptyset) = 0$

A TU game is (also) a cooperative game:

$$V(A) = \{x \in \mathbb{R}^A : \sum_{i \in A} x_i \leq v(A)\}$$

Seller and buyers

Example

There are one seller and two potential buyers for some item. Player one is the seller, and she evaluates the item a. Players two and three are the buyers, and they evaluate it b and c, respectively. Suppose that $a < b < c$

The TU game is

$$\begin{cases} v(\{1\}) = a, v(\{2\}) = v(\{3\}) = 0 \\ v(\{1,2\}) = b, v(\{1,3\}) = c, v(\{2,3\}) = 0, \\ v(N) = c \end{cases}$$

What can we expect it will happen?

Glove game

Example

N players are have a glove each, some of them a right glove, some other a left glove. The aim is to have pairs of gloves.

To formalize, we have a partition $\{L, R\}$ of N and

$$v(S) = \min\{|S \cap L|, |S \cap R|\}$$

How will the players make pairs of gloves?

Here is the case with players 1 and 2 with a right glove and player 3 with a left glove:

$$\begin{cases} v(\{1\}) = v(\{2\}) = v(\{3\}) = 0 \\ v(\{1,2\}) = 0, v(\{1,3\}) = v(\{2,3\}) = 1 \\ v(N) = 1 \end{cases}$$

Children game

Example

Three players vote one of their names. If one of them gets at least two votes, she gets 1000 *euros. They can make binding agreements about how to share the money. If no one gets at least two votes, the* 1000 *euros are lost.*

$$\begin{cases} v(A) = 1000 & \text{if } |A| \geq 2 \\ v(A) = 0 & \text{otherwise} \end{cases}$$

How the money could be divided among players?

Weighted majority game

Example

The game $[q; w_1, \dots, w_n]$ provides a model of the situation where n parties in a Parliament take a decision. Party i has w_i members; a proposal needs at least q votes to be approved

$$v(A) = \begin{cases} 1 & \sum_{I \in A} w_i \geq q \\ 0 & \mathit{otherwise} \end{cases}$$

In the UN council a decision needs the votes of the 5 permanent members plus at least 4 of the other 10 non permanent members.

$$v = [39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

Can we quantify the relative power of each party?

Bankruptcy game

Definition

A bankruptcy game is defined by the triple $B = (N, c, E)$, where $N = \{1, \dots, n\}$ is the set of creditors, $c = \{c_1, \dots, c_n\}$ where c_i represents the credit claimed by player i and E is the estate. The bankruptcy condition is then $E < \sum_{i \in N} c_i = C$

A pessimistic view of the game may be

$$v_P(S) = \max \left(0, E - \sum_{i \in N \setminus S} c_i \right) \quad S \subseteq N$$

An optimistic, but less realistic view may be

$$v_O(S) = \min \left(E, \sum_{i \in S} c_i \right) \quad S \subseteq N$$

What is the fair share of E among the creditors?

Airport game

Definition

A group of airlines who flies N airplanes needs a new runway close to some city. The set of airplanes is partitioned into groups of similar size $N_1, N_2, \ldots, N_k$, and to each N_j corresponds the cost c_j of the runway construction (the bigger the airplane, the longer and more expensive the runway).

$$v(S) = \max\{c_i : i \in S\}$$

How can we share the total cost c_k among the different groups?

Peer games

Let $N = \{1, \ldots, n\}$ be the set of players and $T = (N, A)$ a directed rooted tree. Each agent i has an individual potential v_i which represents the gain that player i can generate if all players at the higher levels of the hierarchy cooperate with him.

For every $i \in N$, we denote by $S(i)$ the set of all agents in the unique directed path connecting 1 to i, i.e. the set of superiors of i

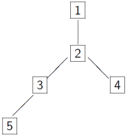
Definition

The peer game is the game v such that

$$v(S) = \sum_{i \in N : S(i) \subseteq S} a_i$$

How should we divide the value $v(N)$ among the players?

A peer game



$$v(A) = 0 \textit{ if } 1 \notin A, v(A) = v_1 \textit{ if } 2 \notin A$$

$$v(\{1,2\}) = v(\{1,2,5\}) = v_1 + v_2$$

$$v(\{1,2,4\}) = v(\{1,2,4,5\}) = v_1 + v_2 + v_4$$

$$v(\{1,2,3,4\}) = v_1 + v_2 + v_3 + v_4$$

$$v(\{1,2,3,5\}) = v_1 + v_2 + v_3 + v_5$$

$$v(N) = v_1 + v_2 + v_3 + v_4 + v_5$$

The set of the TU games

Let $\mathcal{G}(N)$ be the set of all cooperative games having N as set of players. Fix a list $S_1, \ldots, S_{2^n-1}$ of coalitions.
A vector $(v_1, \ldots, v_{2^n-1})$ represents a game, setting $v_i = v(S_i)$. Thus

Proposition

$\mathcal{G}(N)$ is isomorphic to $\mathbb{R}^{2^n-1}$

Proposition

the set $\{u_A : A \subseteq N\}$ of the unanimity games u_A

$$u_A(T) = \begin{cases} 1 & \text{if} \quad A \subseteq T \\ 0 & \textit{otherwise} \end{cases}$$

is a basis for the space $\mathcal{G}(N)$

Interesting subsets of $\mathcal{G}(N)$

Definition

A game is additive if

$$v(A \cup B) = v(A) + v(B)\,, \quad \textit{for all} \quad A \cap B = \emptyset$$

it is superadditive if

$$v(A \cup B) \geq v(A) + v(B)\,, \quad \textit{for all} \quad A \cap B = \emptyset$$

The set of the additive games is a vector space of dimension n. All games introduced as examples are superadditive. Superadditive games are games where the grand coalition forms.

Simple games

Definition

A game $v \in G$ is called simple *if*

- $v(S) \in \{0, 1\}$ *for every nonempty coalition* S
- $A \subseteq C$ *implies* $v(A) \leq v(C)$
- $v(N) = 1$

$v(A) = 1$ means that the coalition A wins
$v(A) = 0$ means that the coalition A loses

Weighted majority games and unanimity games are simple games.
Simple games are characterized by the list of all minimal winning coalitions

Definition

A coalition A in the simple game v is called minimal winning coalition *if*

- $v(A) = 1$
- $B \subsetneq A$ *implies* $v(B) = 0$

Solutions of cooperative games

Definition

A solution vector *for the game* $v \in \mathcal{G}(N)$ *is a vector* $(x_1, \ldots, x_n)$*. A* solution concept *(briefly,* solution*) for the game* $v \in \mathcal{G}(N)$ *is a multifunction*

$$S : \mathcal{G}(N) \to \mathbb{R}^n$$

The solution vector $x = (x_1, \ldots, x_n)$ assigns utility (or cost) x_i to player i

A solution assigns a set of solution vectors, possibly empty, to every game

Imputation

Definition

The solution $I : \mathcal{G}(N) \to \mathbb{R}^n$ such that $x \in I(v)$ if

1. $x_i \geq v(\{i\})$ *for all* i
2. $\sum_{i=1}^{n} x_i = v(N)$

is called imputation.

If a game satisfies $v(N) \geq \sum_i v(\{i\})$, then the imputation is nonempty.

If v is additive then $I(v) = \{(v(1), \dots, v(n))\}$

The structure of the imputation set

Proposition

The imputation set $I(v)$ is a polytope (i.e. the smallest closed convex set containing a finite number of points)

- Efficiency is a mandatory requirement: it makes a real difference with the non cooperative case
- The imputation set is nonempty if the game is superadditive (superadditivity of v is only only sufficient for nonemptyness of $I(v)$), it reduces to a singleton if the game is additive
- The imputation set lies in the hyperplane $H = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = v(N)\}$ and it is bounded since $x_i \geq v(\{i\})$ for all i. Since it is defined by linear inequalities, it is the intersection of half spaces.

The core

Definition

The core *is the solution $C : \mathcal{G}(N) \to \mathbb{R}^n$ such that*

$$C(v) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = v(N) \quad \wedge \quad \sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N \right\}$$

The core is a subset of the set of imputations. Imputations are efficient distributions of utilities accepted by all players individually, core vectors are efficient distributions of utilities accepted by all coalitions

The core of some games: seller-buyers

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0$$
$$v(\{1,2\}) = b, v(\{1,3\}) = c,\ v(N) = c$$

$$\begin{cases} x_1 \geq a, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq b, x_1 + x_3 \geq c, x_1 + x_3 \geq 0 \\ x_1 + x_2 + x_3 = c \end{cases}$$

$$C(v) = \{(x, 0, c - x) : \quad b \leq x \leq c\}$$

The glove game

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = 0, v(\{1,3\}) = v(\{2,3\}) = v(N) = 1$$

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 0, x_1 + x_3 \geq 1, x_2 + x_3 \geq 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$$C(v) = \{(0,0,1)\}$$

This can be extended to any glove game: if l people have left gloves and r people, $r > l$, have right gloves, then

$$C(v) = \{\underbrace{1,\dots,1}_{l \text{ times}}, \underbrace{0,\dots,0}_{r \text{ times}}\}$$

Children game

$v(A) = 1$ if $|A| \geq 2$, zero otherwise

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 1, x_1 + x_3 \geq 1, x_2 + x_3 \geq 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$C(v) = \emptyset$

The structure of the core

Proposition

The core $C(v)$ is a polytope (i.e. the smallest closed convex set containing a finite number of points)

Same proof as for the imputation set.

The core reduces to the singleton $(v(\{1\}), \dots, v(\{n\}))$ if v is additive.

Superadditive games can have an empty core.

The core in simple games

Definition

In a game v, a player i is a veto player if $v(A) = 0$ for all A such that $i \notin A$

Theorem

Let v be a simple game. Then $C(v) \neq \emptyset$ if and only if there is at least one veto player. When a veto player exists, the core is the closed convex polytope with extreme points the vectors $(0, \dots, 1, \dots, 0)$ where the 1 corresponds to a veto player

Proof If there is no veto player, then for every i $N \setminus \{i\}$ is a winning coalition. Suppose $(x_1, \dots, x_n) \in C(v)$

$$\sum_{j \neq i} x_j = 1; \quad i = 1, \dots, n$$

Summing up the above inequalities

$$(n-1) \sum_{j=1}^{n} x_j = n$$

a contradiction since $\sum_{j=1}^{n} x_j = 1$. Conversely, any imputation assigning zero to the non-veto players is in the core

Nonemptyness of the core: equivalent formulation

Given a game v, consider the following LP problem

$$\min \sum_{i=1}^{n} x_i \tag{1}$$

$$\sum_{i \in S} x_i \geq v(S) \quad \textit{for all } S \subseteq N \tag{2}$$

Theorem

The above LP problem has always a nonempty set of solutions C. The core $C(v)$ is nonempty and $C(v) = C$ if and only if the optimal value of the LP is $v(N)$

Remark

The value V of the LP problem is $V \geq v(N)$, due to the constraint $\sum_i x_i \geq v(N)$; thus for every x fulfilling (2) $\sum_{i=1}^{n} x_i \geq v(N)$

Dual formulation

Theorem

$C(v) \neq \emptyset$ if and only if every vector $(\lambda_S)_{S\subseteq N}$ such that

$$\lambda_S \geq 0 \; \forall S \subseteq N \quad \textit{and} \quad \sum_{S:i\in S\subseteq N} \lambda_S = 1 \quad \textit{for all } i = 1,\dots,n$$

satisfies also the following inequality

$$\sum_{S\subseteq N} \lambda_S v(S) \leq v(N)\,.$$

Proof

Proof The LP problem (1),(2) associated to the core problem has the following matrix form

$$\begin{cases} \min\langle c, x\rangle \\ Ax \geq b \end{cases}$$

where $c = 1_n$, $b = (v(\{1\}), v(\{2\}), \ldots, v(\{1,2\}), \ldots, v(N))$ and A is a $2^n - 1 \times n$ matrix. Each row of A represents a coalition, each column represents a player. $A_{ij} = 1$ if player j is in the coalition i, $A_{ij} = 0$ otherwise.
The dual of the problem takes the form

$$\begin{cases} \max \sum_{S\subseteq N} \lambda_S v(S) \\ \lambda_S \geq 0 \\ \sum_{S:i\in S\subseteq N} \lambda_S = 1 \qquad \textit{for all } i \end{cases}$$

Since the primal has (at least) a finite solution, the fundamental duality theorem states that the dual does too and there is no duality gap. Thus the core $C(v)$ is nonempty if and only if the value V of the dual problem is such that $V \leq v(N)$. ■

Example: three players

Primal:

$$\begin{cases} \min\ x_1 + x_2 + x_3 : \\ x_i \geq v(\{i\})\ i = 1, 2, 3 \\ x_1 + x_2 \geq v(\{1, 2\}) \\ x_1 + x_3 \geq v(\{1, 3\}) \\ x_2 + x_3 \geq v(\{2, 3\}) \\ x_1 + x_2 + x_3 \geq v(N) \end{cases}$$

Dual:

$$\begin{cases} \max[\lambda_{\{1\}} v(\{1\}) + \lambda_{\{2\}} v(\{2\}) + \lambda_{\{3\}} v(\{3\}) + \lambda_{\{1,2\}} v(\{1, 2\}) \\ +\lambda_{\{1,3\}} v(\{1, 3\}) + \lambda_{\{2,3\}} v(\{2, 3\}) + \lambda_N v(N)] \\ \lambda_S \geq 0\ \forall S \\ \lambda_{\{1\}} + \lambda_{\{1,2\}} + \lambda_{\{1,3\}} + \lambda_N = 1 \\ \lambda_{\{2\}} + \lambda_{\{1,2\}} + \lambda_{\{2,3\}} + \lambda_N = 1 \\ \lambda_{\{3\}} + \lambda_{\{1,3\}} + \lambda_{\{2,3\}} + \lambda_N = 1 \end{cases}$$

The three player case

The dual formulation of the problem implies that the condition

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

must be verified only on the extreme points of the polytope obtained by imposing the conditions:

$$\begin{cases} \lambda_S \geq 0 & \text{for all } S \subset N \\ \sum_{S: i \in S \subseteq N} \lambda_S = 1 & \text{for all } i \end{cases}$$

It can be shown that with three players and if the game is superadditive, only the following condition must be fulfilled

$$v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) \leq 2v(N)$$

Final remarks

- There are superadditive games v such that $C(v) = \emptyset$
- There are non superadditive games v such that $C(v) \neq \emptyset$

Excess of a coalition

Let v be some TU game.

Definition

The excess of a coalition A over the imputation x is

$$e(A, x) = v(A) - \sum_{i \in A} x_i$$

$e(A, x)$ is a measure of the dissatisfaction of the coalition A with respect to the assignment of the imputation x

Remark

An imputation x of the game v belongs to $C(v)$ if and only if $e(A, x) \leq 0$ for all A

Definition

The lexicographic vector attached to the imputation x is the $(2^n - 1)-th$ dimensional vector $\theta(x)$ such that

1. $\theta_i(x) = e(A, x)$, *for some* $A \subseteq N$
2. $\theta_1(x) \geq \theta_2(x) \geq \cdots \geq \theta_{2^n-1}(x)$

Definition

The nucleolus solution is the solution $\nu : \mathcal{G}(N) \to \mathbb{R}^n$ such that $\nu(v)$ is the set of the imputations x such that $\theta(x) \leq_L \theta(y)$, for all y imputations of the game v

Remark

$x \leq_L y$ *if* $x = y$ *or there exists* j *such that* $x_i = y_i$ *for all* $i < j$, *and* $x_j < y_j$.
$\leq_L$ *defines a total order in any Euclidean space*

An example

Example

Three players, $v(A) = 1$ *if* $|A| \geq 2$*, 0 otherwise.*
Suppose $x = (a, b, 1-a-b)$*, with* $a, b \geq 0$ *and* $a + b \leq 1$*. The coalitions* S *complaining (*$e(S, \emptyset) > 0$*) are those with two members.*

$$e(\{1,2\}) = 1 - (a+b), e(\{1,3\}) = b, e(\{2,3\}) = a$$

We must minimize

$$\max\{1-a-b, b, a\}$$

$\nu = (1/3, 1/3, 1/3)$

Remember $C(v) = \emptyset$

Nucleolus: one point solution

Theorem

For every TU game v with nonempty imputation set, the nucleolus $\nu(v)$ is a singleton

Thus the nucleolus is a one point solution

Nucleolus in the core

Proposition

Suppose v is such that $C(v) \neq \emptyset$. Then $\nu(v) \in C(v)$

Proof For all $x \in C(v)$, by definition of core $\theta_1(x) \leq 0$. Since the nucleolus minimizes the excesses, we have $\theta_1(\nu(v)) \leq 0$. Then $\nu(v)$ is in the core. ∎

Another example

$$v(\{1\}) = a, v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0\,,$$
$$v(\{1,2\}) = b, v(\{1,3\}) = c,\; v(N) = c$$

$$C(v) = \{(x, 0, c - x) :\; b \le x \le c\}$$

Must find x: $\nu(v) = (x, 0, c - x)$. The relevant excesses are

$$e(\{1,2\}) = b - x, \quad e(\{2,3\}) = x - c$$

Thus

$$\nu(v) = \{\frac{b+c}{2}, 0, \frac{c-b}{2}\}$$