EiR/CPR/CAMS 2024/2025

Analysis and Control of Multi-Robot Systems

Elements of Port-Hamiltonian Modeling

Andrea Cristofaro

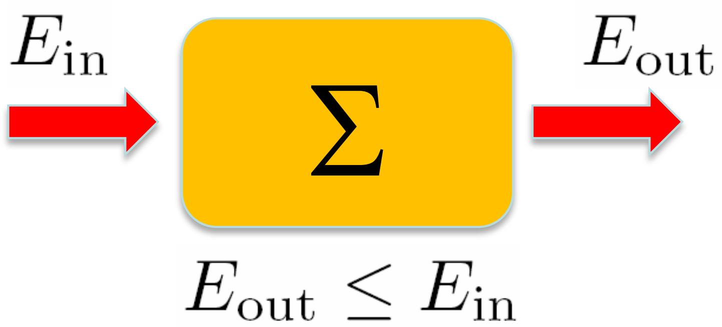
(Part of the slides by Paolo Robuffo Giordano)

Dipartimento di Ingegneria Informatica Automatica e Gestionale Antonio Ruberti

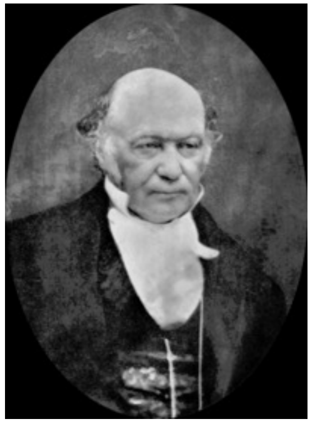
Sapienza Università di Roma

Introduction to Port-Hamiltonian Systems

- Port-Hamiltonian Systems (PHS): strong link with passivity



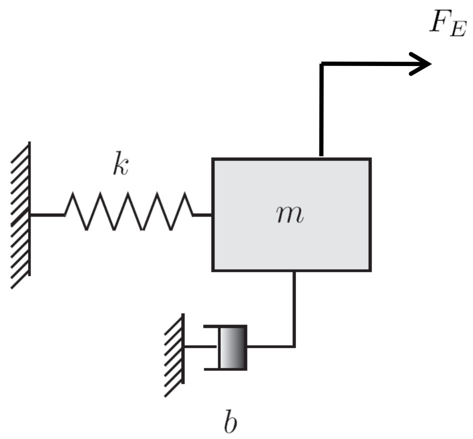
- Passivity:
  - I/O characterization
  - “Constraint” on the I/O energy flow
  - Many desirable properties
    - Stability of free-evolution
    - Stability of zero-dynamics
    - Easy stabilization with static output-feedback
    - Modularity: passivity is preserved under proper compositions



William Rowan Hamilton (1805–1865)

- However, no insights on the structure of a passive system
- PHS: focus on the structure behind passive systems

Mass-spring-damper vs. PHS



- Review of the mass-spring-damper example

$$\boxed{m\ddot{x} + b\dot{x} + kx = f}$$

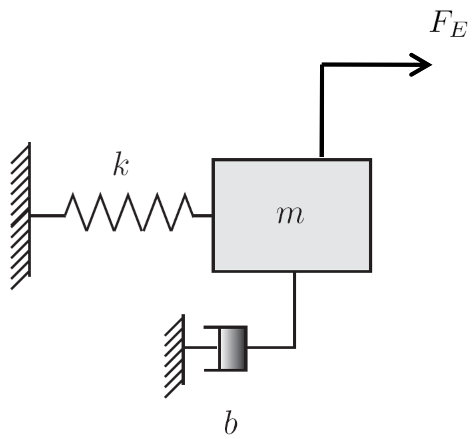
- This system was shown to be passive w.r.t. the pair $(u,\, y)$ with $u = f$, $y = \dot{x}$, and as storage function the total energy (kinetic + potential)

$$V = E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

- Indeed, it is $\boxed{\dot{V} = f\dot{x} - b\dot{x}^2 = yu - by^2 \leq yu}$

- But why is it passive? We must investigate its internal structure...

Mass-spring-damper vs. PHS



- The spring-mass system is made of 2 components (2 states)
  - Assume for now no damping $b = 0$
  - Mass = kinetic energy $K = \frac{1}{2}m\dot{x}^2 = \frac{p^2}{2m}$, $p = m\dot{x}$ → Linear momentum
  - Spring = elastic energy $V = \frac{1}{2}kx^2$

- Let us consider the 2 components separately

$$\mathcal{K} : \begin{cases} \dot{p} = f_p \\ v_p = \dfrac{\partial K}{\partial p} = \dfrac{p}{m} (= \dot{x}) \end{cases}$$

Kinetic energy storing

$$\mathcal{V} : \begin{cases} \dot{x} = v_x \\ f_x = \dfrac{\partial V}{\partial x} = kx \end{cases}$$

Potential energy storing

- Note that these (elementary) systems are "integrators with linear outputs"

- We know they are passive w.r.t. (v_p, f_p) and (v_x, f_x), respectively

Mass-spring-damper vs. PHS

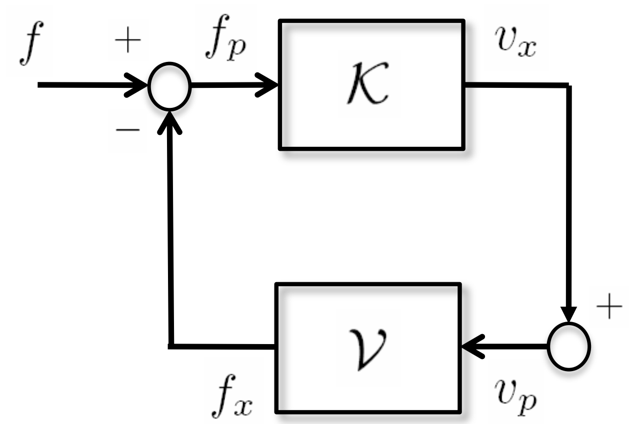
$$\mathcal{K}:\begin{cases}\dot{p} &= f_p\\ v_p &= \dfrac{\partial K}{\partial p} = \dfrac{p}{m}(=\dot{x})\end{cases}$$

Kinetic energy storing

$$\mathcal{V}:\begin{cases}\dot{x} &= v_x\\ f_x &= \dfrac{\partial V}{\partial x} = kx\end{cases}$$

Potential energy storing

- Let us interconnect them in "feedback" $v_x = v_p,\ f_p = -f_x + f$



- The resulting system can be written as

$$\begin{bmatrix}\dot{x}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\dfrac{\partial H}{\partial x}\\ \dfrac{\partial H}{\partial p}\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} f \qquad (\blacksquare)$$

where $H(x,p) = K(p) + V(x)$ is the total energy (Hamiltonian)

- **Prove that (■) is equivalent to** $m\ddot{x} + kx = f$

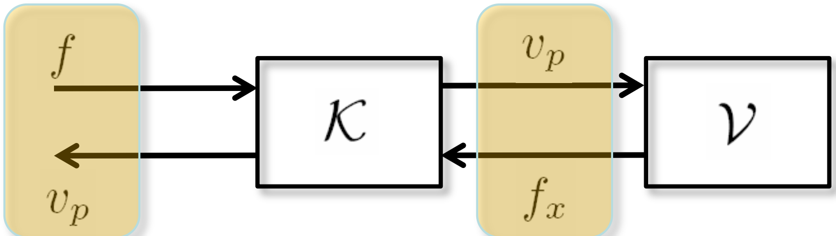
Mass-spring-damper vs. PHS

- How does the energy balance look like?

$$\dot{H} = \underbrace{\begin{bmatrix} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix}}_{\equiv 0} + \begin{bmatrix} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} f = \frac{\partial H^T}{\partial p} f = \boxed{f^T v_p}$$

Skew-symmetric

- We find again the passivity condition w.r.t. the pair $(f,\, v_p)$
- The subsystems $\mathcal{K}$ and $\mathcal{V}$ exchange energy in a power-preserving way – no energy is created/destroyed



- The subsystem $\mathcal{K}$ exchanges energy with the "external world" through the pair $(f,\, v_p)$
- Total energy H can vary only because of the power flowing through $(f,\, v_p)$

Mass-spring-damper vs. PHS

- What if a damping term $b > 0$ is present in the system?

- By interconnecting $\mathcal{K}$ and $\mathcal{V}$ as before (feedback interconnection), we get

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \left(\underbrace{\boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}}_{\text{Skew-symmetric}} - \underbrace{\boxed{\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}}}_{\text{Positive semi-def.}} \right) \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f \qquad (\blacksquare)$$

- **Prove that (■) is equivalent to** $\boxed{m\ddot{x} + b\dot{x} + kx = f}$

- The energy balance now reads

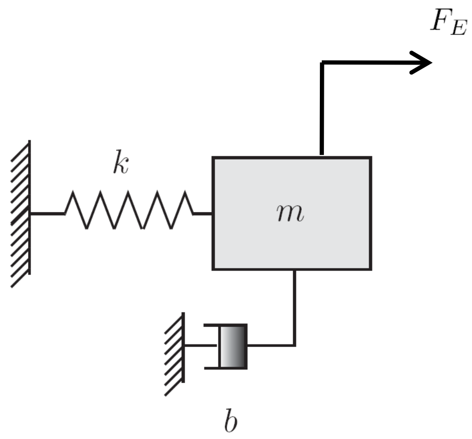
$$\dot{H} = \underbrace{-\begin{bmatrix} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix}}_{\leq 0} + \begin{bmatrix} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} f \boxed{\leq} \frac{\partial H^T}{\partial p} f = f^T v_p$$

Mass-spring-damper vs. PHS

$$\dot{H} = \underbrace{-\begin{bmatrix} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix}}_{\leq 0} + \begin{bmatrix} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} f \boxed{\leq} \frac{\partial H^T}{\partial p} f = f^T v_p$$

- Again the passivity condition w.r.t. the pair $(f,\, v_p)$
- Total energy H can now
  - vary only because of the power flowing through $(f,\, v_p)$
  - decrease because of internal dissipation
- But still, power-preserving exchange of energy between $\mathcal{K}$ and $\mathcal{V}$

Mass-spring-damper vs. PHS



- Summarizing, this particular passive system is made of:
  - Two atomic energy storing elements $\mathcal{K}$ and $\mathcal{V}$
  - A power-preserving interconnection among $\mathcal{K}$ and $\mathcal{V}$
  - An energy dissipation element b
  - A pair $(f,\ v_p)$ to exchange energy with the "external world"

- Why passivity of the complete system?

- $\mathcal{K}$ and $\mathcal{V}$ are passive (and "irreducible")
  - Their power-preserving interconnection is a feedback interconnection (thus, preserves passivity)
  - The element b dissipates energy
  - Therefore, any increase of the total energy H is due to the power flowing through $(f,\ v_p)$
  - For this reason, this pair is also called **power-port**

- How general are these results?

Introduction to Port-Hamiltonian Systems

- In the linear time-invariant case $\left\{ \begin{array}{lcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array} \right.$ (■)

  passivity implies existence of a storage function $H(x) = \frac{1}{2} x^T Q x,$ $\boxed{Q = Q^T \geq 0}$

  such that $A^T Q + QA \leq 0$ and $C = B^T Q$

- If $\ker Q \subset \ker A$ (always true if $\boxed{Q > 0}$)

  then (■) can be rewritten as

$$\left\{ \begin{array}{lcl} \dot{x} & = & \boxed{(J - R)} Qx + Bu, \quad \boxed{J = -J^T,} \quad \boxed{R = R^T \geq 0} \\ y & = & B^T Q x \end{array} \right.$$

  and energy balance $\boxed{\dot{H} = -x^T QRQx + x^T QBu \leq y^T u}$

- $H(x)$ is called the Hamiltonian function

Introduction to Port-Hamiltonian Systems

- Similarly, most nonlinear passive systems can be rewritten as

$$\begin{cases} \dot{x} &= [J(x) - R(x)]\dfrac{\partial H}{\partial x} + g(x)u, \quad J(x) = -J^T(x),\ R(x) \geq 0 \\ y &= g^T(x)\dfrac{\partial H}{\partial x} \end{cases}$$

with $H(x) \geq 0$ being the Hamiltonian function (storage function) and

$$\dot{H} = -\frac{\partial H^T}{\partial x}R(x)\frac{\partial H}{\partial x} + \frac{\partial H^T}{\partial x}g(x)u \leq y^T u$$

showing the passivity condition

- Roles:

$H(x)$ represents the energy stored by the system

$R(x)$ represents the internal dissipation in the system

$J(x)$ represents an internal power-preserving interconnection among different components

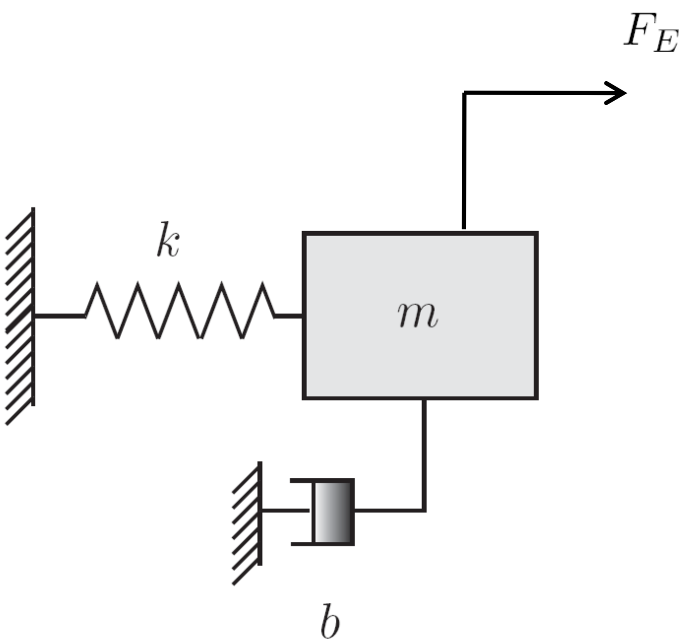
$(u,\ y)$ represents a "power-port", allowing energy exchange (in/out) with the external world

Introduction to Port-Hamiltonian Systems

- In the mass-spring-damper case, the generic Port-Hamiltonian formulation

$$\begin{cases} \dot{x} & = & [J(x) - R(x)]\dfrac{\partial H}{\partial x} + g(x)u, \quad J(x) = -J^T(x),\ R(x) \geq 0 \\ y & = & g^T(x)\dfrac{\partial H}{\partial x} \end{cases}$$

specializes into $J(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R(x) = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



Introduction to Port-Hamiltonian Systems

- In the (more abstract) example we have seen during the Passivity lectures, we showed that

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1^3 + u \end{cases}$$

is a passive system with passive output $y = x_2$ and Storage function

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \geq 0$$

- Can it be recast in PHS form with $H(x) = V(x)$ being the Hamiltonian?

- Yes:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dfrac{\partial H}{\partial x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y & = & \begin{bmatrix} 0 & 1 \end{bmatrix} \dfrac{\partial H}{\partial x} \end{cases}$$

General Mechanical System

- Any mechanical system (also constrained) described by the Euler-Lagrange equations can be recast in a Port-Hamiltonian form

- Start with a set of generalized coordinates $\boxed{q = [q_1^T \; \dots \; q_n^T]^T}$

- Define the Lagrangian $L = K(q,\, \dot{q}) - V(q)$ with $K(q,\, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$

being the kinetic energy, $V(q)$ the potential energy, and $M(q) > 0$ the positive definite Inertia matrix

- Apply a change of coordinates $\boxed{(q,\, \dot{q}) \longrightarrow (q,\, p)}$ where $p = M(q)\dot{q}$ are usually called "generalized momenta"

- The kinetic energy in the new coordinates is $K(q,\, p) = \frac{1}{2}p^T M^{-1}(q)p$

General Mechanical System

- Define the Hamiltonian (total energy) of the system as

$$\boxed{H(q,\, p) = K(q,\, p) + V(q)}$$

- The Euler-Lagrange equations for the system are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}(q,\, \dot{q})\right) - \frac{\partial L}{\partial q}(q,\, \dot{q}) = \tau \qquad (\blacksquare)$$

- Since $p = \dfrac{\partial L}{\partial \dot{q}} = \dfrac{\partial K}{\partial \dot{q}}$ we can rewrite (■) as

$$\begin{cases} \dot{q} &= \dfrac{\partial H}{\partial p} \\ \dot{p} &= -\dfrac{\partial H}{\partial q} + \tau \end{cases} \Rightarrow \left\{ \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tau \right.$$

General Mechanical System

- It follows that $\boxed{\dot{H} = \frac{\partial H^T}{\partial p}\tau = \dot{q}^T\tau}$

- If $H(q,\, p)$ (i.e., $V(q)$) is bounded from below, the system is passive w.r.t. the power port $(\dot{q},\, \tau)$

- Similarly, a mechanical system with collocated inputs and outputs (also underactuated) is generally described by

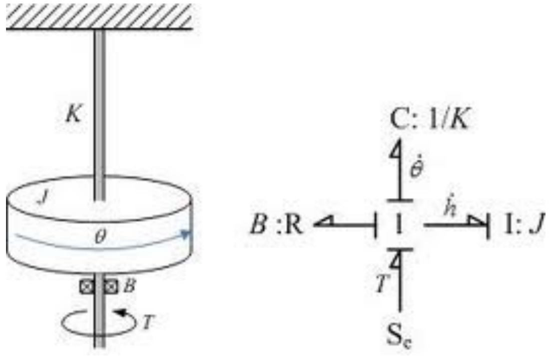
$$\begin{cases} \dot{q} &= \dfrac{\partial H}{\partial p} \\ \dot{p} &= -\dfrac{\partial H}{\partial q} + B(q)u \\ y &= B^T(q)\dfrac{\partial H}{\partial p} \qquad (= B^T(q)\dot{q}) \end{cases}$$

- Again, passivity w.r.t. $(y,\, u)$

Introduction to Port-Hamiltonian Systems

- What is then Port-Hamiltonian modeling?

- It is a cross-domain energy-based modeling philosophy, generalizing Bond Graphs
  - Historically, network modeling of lumped-parameter physical systems (e.g., circuit theory)



- Main insights: all the physical domains deal, in a way or another, with the concept of Energy storage and Energy flows
  - Electrical
  - Hydraulical
  - Mechanical
  - Thermodynamical

- Dynamical behavior comes from the exchange of energy
- The "energy paths" (power flows) define the internal model structure

Introduction to Port-Hamiltonian Systems

- Port-Hamiltonian modeling

- Most (**<u>passive</u>**) physical systems can be modeled as a set of simpler subsystems (modularity!) that either:

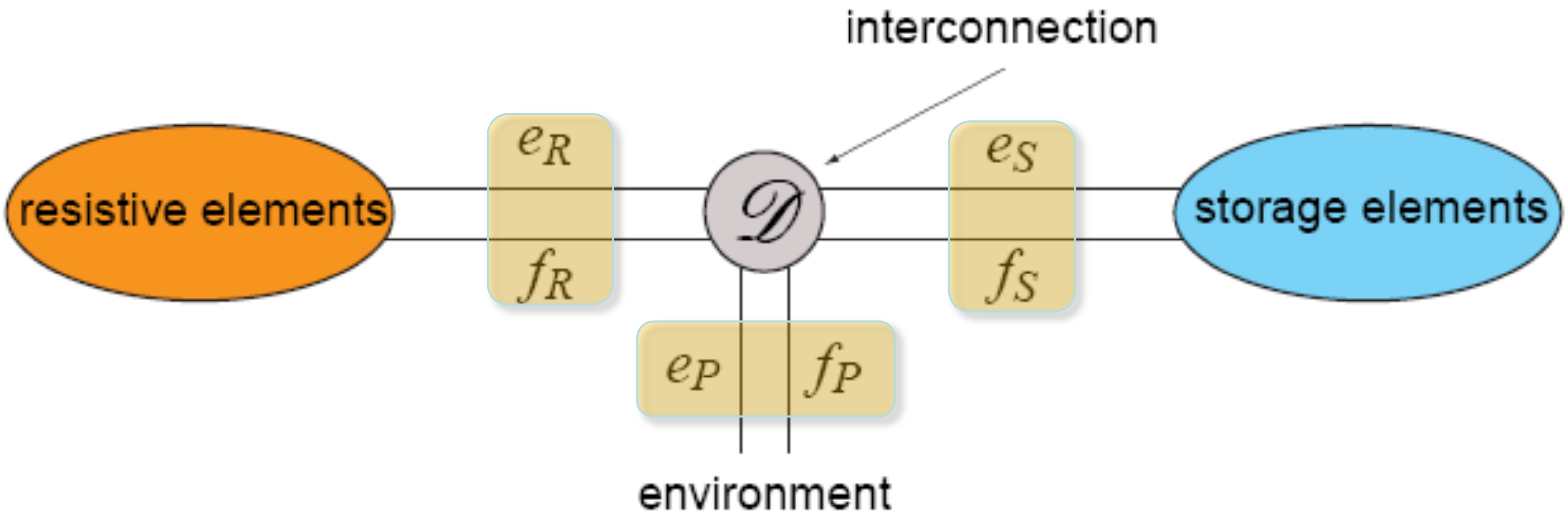
    - Store energy
    - Dissipate energy
    - Exchange energy (internally or with the external world) through power ports

- Role of energy and the interconnections between subsystems provide the basis for various control techniques

- Easily address complex nonlinear systems, especially when related to real "physical" ones

Introduction to Port-Hamiltonian Systems

- A generic port-Hamiltonian model is then
  - A set of energy storage elements (with their power ports $(e_S,\ f_S)$)
  - A set of resistive elements (with their power ports $(e_R,\ f_R)$)
  - A set of open power-ports (with their power ports $(e_P,\ f_P)$)
  - An internal <u>power-preserving interconnection</u> $\mathcal{D}$, called Dirac structure



- An explicit example of a "Dirac structure" is the power-preserving interconnection represented by the skew-symmetric matrix $J(x)$

Modularity

- As one can expect, the "proper" interconnection of a number of Port-Hamiltonian Systems

$$(\mathcal{M}_i,\, \mathcal{D}_i,\, H_i),\, i = 1 \dots k$$

  through a Dirac structure $\mathcal{D}_I$ is again a Port-Hamiltonian System $(\mathcal{M},\, \mathcal{D},\, H)$ with

- Hamiltonian $\boxed{H = H_1 + \dots H_k}$

- State manifold $\boxed{\mathcal{M} = \mathcal{M}_1 \times \dots \mathcal{M}_k}$

- Dirac structure $\boxed{\mathcal{D} = \mathcal{D}_1, \dots, \mathcal{D}_k,\, \mathcal{D}_I}$

- This allows for **modularity and scalability**

Modularity

- Example: given two Port-Hamiltonian System

$$\begin{cases} \dot{x}_1 & = & (J_1(x_1) - R_1(x_1))\dfrac{\partial H_1}{\partial x_1} + g_1(x_1)u_1 \\ y_1 & = & g_1^T(x_1)\dfrac{\partial H_1}{\partial x_1} \end{cases} \qquad \begin{cases} \dot{x}_2 & = & (J_2(x_2) - R_2(x_2))\dfrac{\partial H_2}{\partial x_2} + g_2(x_2)u_2 \\ y_2 & = & g_2^T(x_2)\dfrac{\partial H_2}{\partial x_2} \end{cases}$$

- Define an interconnecting Dirac structure $\mathcal{D}_I$ as (**<u>for example</u>**)

$$u_1 = y_2,\ u_2 = -y_1$$

- The composed system is again Port-Hamiltonian

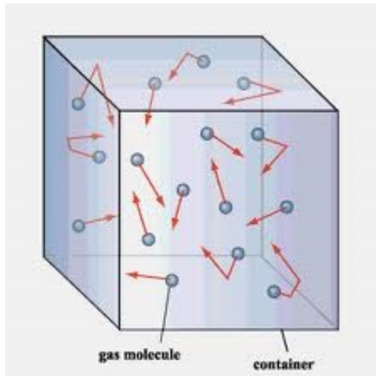
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left(\begin{bmatrix} J_1(x_1) & g_1(x_1)g_2^T(x_2) \\ -g_2(x_2)g_1^T(x_1) & J_2(x_2) \end{bmatrix} - \begin{bmatrix} R_1(x_1) & 0 \\ 0 & R_2(x_2) \end{bmatrix} \right) \begin{bmatrix} \dfrac{\partial H_1}{\partial x_1} \\ \dfrac{\partial H_2}{\partial x_2} \end{bmatrix}$$

with Hamiltonian function $H(x_1, x_2) = H_1(x_1) + H_2(x_2)$

Further generalizations

- **Much more** could be said on Port-Hamiltonian System….

- Can model distributed parameter physical systems (wherever energy plays a role)
  - Flexible beams
  - Wave equations
  - Gas/fluid dynamics



- Are modular (re-usability)
  - Network structure (…. -> multi-agent)

- Are flexible
  - State-dependent (time-varying) interconnection structure

$J(x)$

Summary

- PHS are a powerful way to model a very large class of physical systems
  - For instance, every physical system admitting an Energy concept (the whole physics?)

- In PHS, the emphasis in on the internal structure of a system. A PHS system is a network of
  - **Power ports**: medium to exchange energy
  - Elementary/irreducible **energy storing elements** endowed with their power ports
  - **Dissipating elements** endowed with their power ports
  - **"External world" power ports** for external interaction
  - **A power-preserving interconnection** structure (Dirac structure) among the internal power ports

- The total energy of a PHS is called Hamiltonian. If the Hamiltonian is bounded from below, a PHS is passive w.r.t. its external ports

- Proper compositions of PHS are PHS

Control of PHS

- How to control a Port-Hamiltonian System?

$$\begin{cases} \dot{x} &= [J(x) - R(x)]\dfrac{\partial H}{\partial x} + g(x)u \\ y &= g^T(x)\dfrac{\partial H}{\partial x} \end{cases}$$

- A PHS is still a dynamical system in the general form

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{cases}$$

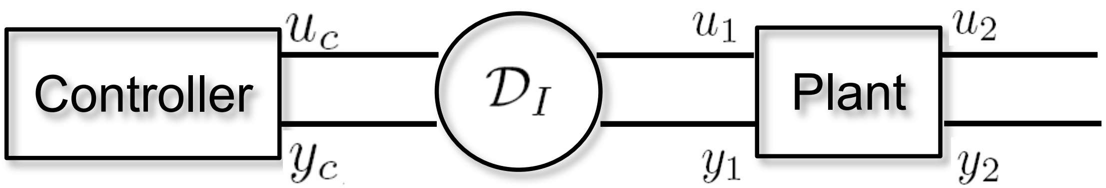
hence, one could use any of the available (nonlinear) control techniques

- However, in closed-loop, we want to retain and to exploit the PHS structure
  - PHS plant and controller
  - Power-preserving interconnection among them

Control of PHS

- The general idea is: assume a plant and controller in PHS form, and interconnected through a suitable $\mathcal{D}_I$

$$\begin{cases} \dot{x} &= [J(x) - R(x)]\dfrac{\partial H}{\partial x} + g(x)u \\ y &= g^T(x)\dfrac{\partial H}{\partial x} \end{cases} \qquad \begin{cases} \dot{x}_c &= (J_c(x_c) - R_c(x_c))\dfrac{\partial H_c}{\partial x_c} + g_c(x_c)u_c \\ y_c &= g_c^T(x_c)\dfrac{\partial H_c}{\partial x_c} \end{cases}$$



where we split the plant port $(u,\, y)$ into $(u_1,\, y_1)$ and $(u_2,\, y_2)$, and use $(u_1,\, y_1)$ for the interconnection with the controller port $(u_c,\, y_c)$

- In general, one can imagine two distinct control goals
  - Regulation to x^* or tracking of $x^*(t)$ for the plant state variables $x(t)$
  - Desired (closed-loop) behavior of the plant at the interaction port $(u_2,\, y_2)$
  - The latter is for instance the case of Impedance Control for robot manipulators

Control techniques for PHS

- Three important subclasses of the fundamental **Energy Shaping** problem
  - Energy Transfer Control
  - Energy Balancing
  - Interconnection and Damping assignment

- Note that, in general, a PHS system is passive w.r.t. the port $(u,\ y)$ if $H(x)$ is bounded from below (which we will assume from now on)

$$\begin{cases} \dot{x} &= [J(x) - R(x)]\dfrac{\partial H}{\partial x} + g(x)u \\ y &= g^T(x)\dfrac{\partial H}{\partial x} \end{cases}$$

- Thus, a simple termination of $(u,\ y)$ with a resistive element, e.g., $u = -ky,\ k > 0$ will yield a (asympt.) stable closed-loop system
  - Action called "damping injection"
  - See the slides on Passivity Theory
  - **<u>But we want much more!!</u>**

Energy Transfer Control

- Consider two PHS

$$\begin{cases} \dot{x}_1 & = & J_1(x_1)\dfrac{\partial H_1}{\partial x_1} + g_1(x_1)u_1 \\ y_1 & = & g_1^T(x_1)\dfrac{\partial H_1}{\partial x_1} \end{cases} \qquad \begin{cases} \dot{x}_2 & = & J_2(x_2)\dfrac{\partial H_2}{\partial x_2} + g_2(x_2)u_2 \\ y_2 & = & g_2^T(x_2)\dfrac{\partial H_2}{\partial x_2} \end{cases}$$

- And assume we want to transfer some amount of energy among them by keeping the total energy $H_1(x_1) + H_2(x_2)$constant

- This can be done by interconnecting the two PHS as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\alpha y_1(x_1)y_2^T(x_2) \\ \alpha y_2(x_2)y_1^T(x_1) & 0 \end{bmatrix}}_{\text{Skew-symmetric}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \alpha \in \mathbb{R}$$

- Note that this is an example of a state-modulated power preserving interconnection

$$J(x) = \begin{bmatrix} 0 & -\alpha y_1(x_1)y_2^T(x_2) \\ \alpha y_2(x_2)y_1^T(x_1) & 0 \end{bmatrix}$$

Energy Transfer Control

- Since the interconnection is power-preserving, it follows that the total Hamiltonian $H(x_1, x_2) = H(x_1) + H(x_2)$ stays constant, i.e.,

$$\boxed{\dot{H}(x_1, x_2) = 0}$$

- However, what happens to the individual energies?

- **Exercise: show that** $\boxed{\dot{H}_1(x_1) = -\alpha \|y_1\|^2 \|y_2\|^2 \quad \dot{H}_2(x_2) = \alpha \|y_1\|^2 \|y_2\|^2}$

- Thus, depending on the parameter α, energy is extracted/injected from system 1 to system 2 (no energy transfer with $\alpha = 0$)

- If $H_1(x_1)$ is lower-bounded, a finite amount of energy will be transferred to system 2. Indeed, at the minimum, $y_1 = 0 \implies \dot{H}_1 = 0$ and $\dot{H}_2 = 0$

- The same of course holds for $H_2(x_2)$

- We will use these ideas in some of the following developments

Energy Tanks

- Let us examine a concrete example of the Energy Transfer Control technique

- To this end, we introduce the **concept** of "**Energy Tank**"

- Assume the usual PHS

$$\begin{cases} \dot{x} &= [J(x) - R(x)] \dfrac{\partial H}{\partial x} + g(x)u \\ y &= g^T(x) \dfrac{\partial H}{\partial x} \end{cases}$$

- We know it is passive w.r.t. $(u,\, y)$ since

$$\dot{H} = -\frac{\partial H^T}{\partial x} R(x) \frac{\partial H}{\partial x} + \frac{\partial H^T}{\partial x} g(x) u \leq y^T u$$

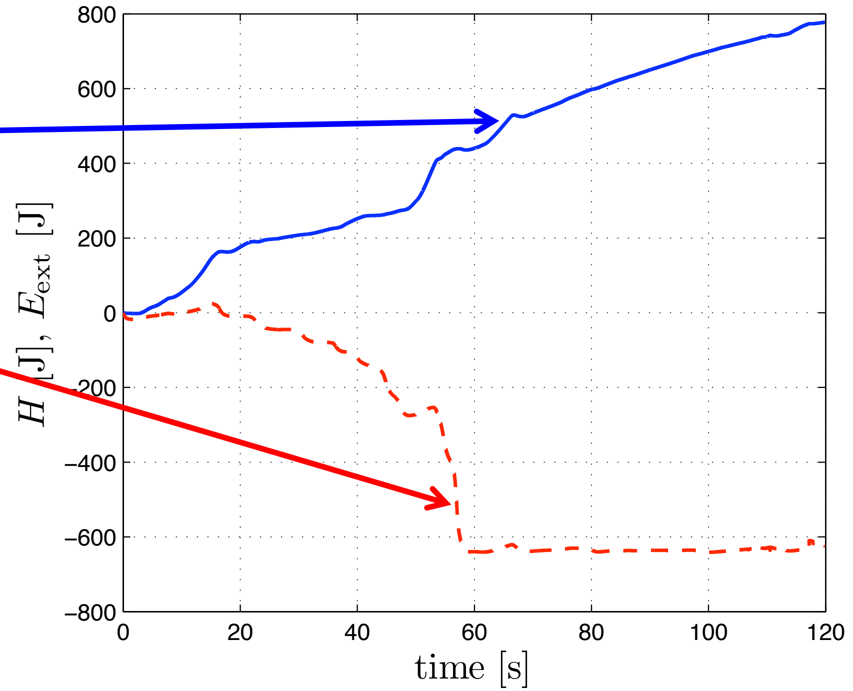
Energy Tanks

- In its integral form, the passivity condition reads

$$H(t) - H(t_0) = \int_{t_0}^{t} y^T u \, \mathrm{d}\tau \underbrace{- \int_{t_0}^{t} \frac{\partial H^T}{\partial x} R(x) \frac{\partial H}{\partial x} \mathrm{d}\tau}_{\leq 0}$$

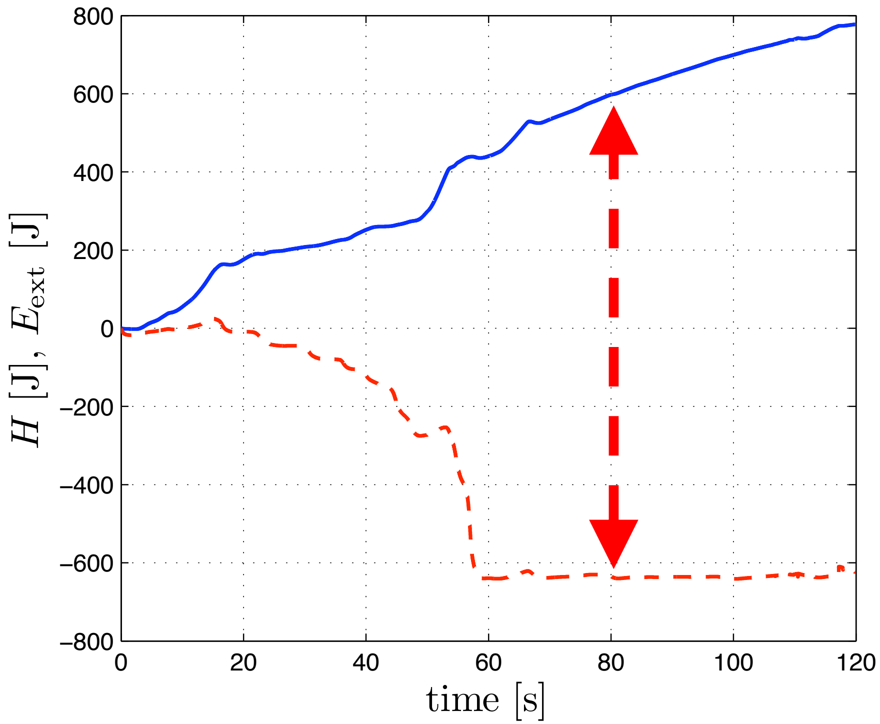
- Let $E_{\text{in}}(t) = H(t) - H(t_0)$ and $E_{\text{ext}}(t) = \int_{t_0}^{t} y^T u \, \mathrm{d}\tau$

- Over time, $E_{\text{in}}(t) \leq E_{\text{ext}}(t)$



Energy Tanks

- Why this gap over time between $E_{\text{ext}}(t)$ and $E_{\text{in}}(t)$?



- Because of the integral of the dissipation term

$$H(t) - H(t_0) = \int_{t_0}^{t} y^T u \, \mathrm{d}\tau \underbrace{- \int_{t_0}^{t} \frac{\partial H^T}{\partial x} R(x) \frac{\partial H}{\partial x} \mathrm{d}\tau}_{\leq 0}$$

- However, we would be happy (from the passivity point of view) by just ensuring a lossless energy balance

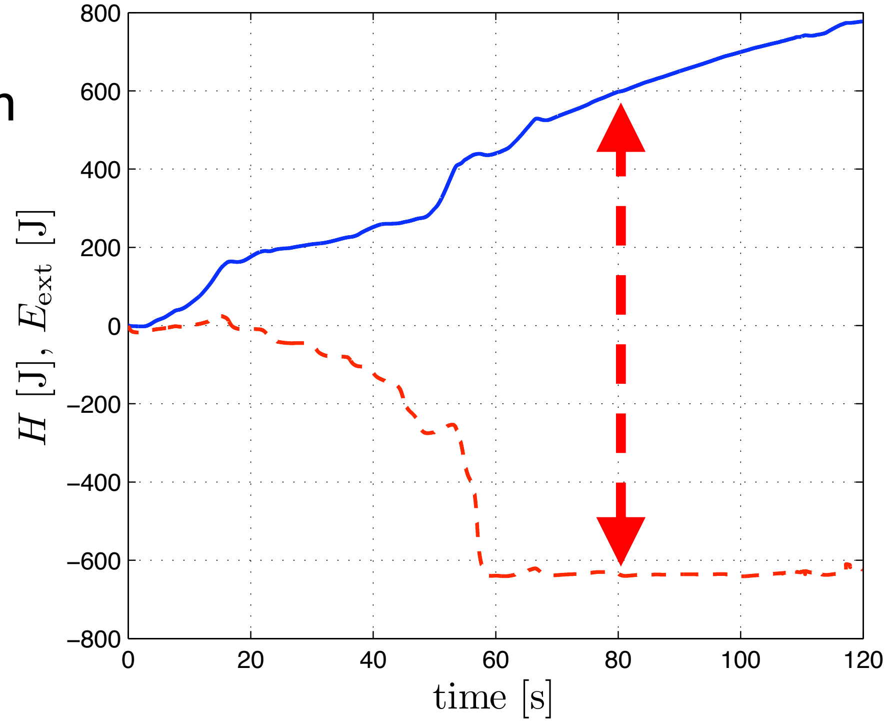
$$H(t) - H(t_0) = \int_{t_0}^{t} y^T u \, \mathrm{d}t \iff E_{\text{in}}(t) = E_{\text{ext}}(t)$$

Energy Tanks

- Dissipation term: passivity margin of the system

- Imagine we could recover this “passivity gap”

- This recovered energy can be freely used for whatever goal **without violating the passivity constraint**



- This idea is at the basis of the Energy Tank machinery

- Energy Tank: an atomic energy storing element with state $x_t \in \mathbb{R}$ and energy function $T(x_t) = \frac{1}{2}x_t^2 \geq 0$

$$\begin{cases} \dot{x}_t &= u_t \\ y_t &= \dfrac{\partial T}{\partial x_t} (= x_t) \end{cases}$$

Energy Tanks

- We want to exploit the tank for:
  - storing back the natural dissipation of a PHS
  - allowing to use the stored energy for implementing some action on the PHS
  - this tank-based action will necessarily meet the passivity constraint

- How to achieve these goals? Let us consider again the PHS and Tank Energy element

$$\begin{cases} \dot{x} = [J(x) - R(x)]\dfrac{\partial H}{\partial x} + g(x)u \\ y = g^T(x)\dfrac{\partial H}{\partial x} \end{cases} \qquad \begin{cases} \dot{x}_t = u_t \\ y_t = x_t \end{cases}$$

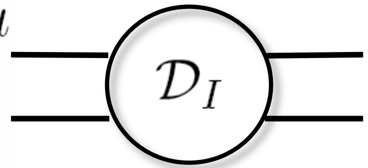
- Let $D(x) = \dfrac{\partial H^T}{\partial x} R(x) \dfrac{\partial H}{\partial x}$ represent the (scalar) dissipation rate of the PHS

- We start by choosing $u_t = \dfrac{1}{x_t} D(x) + \tilde{u}_t$ in the Tank dynamics

Energy Tanks

- The choice $u_t = \frac{1}{x_t} D(x) + \tilde{u}_t$ allows to store back the dissipated energy
- In fact, $\boxed{\dot{T}(x_t) = x_t \left(\frac{1}{x_t} D(x) + \tilde{u}_t \right) = D(x) + x_t \tilde{u}_t}$
- In order to exploit this stored energy to implement an action on the PHS system, we must design a suitable (power-preserving) interconnection among the PHS and Tank element

$$\begin{cases} \dot{x} &= [J(x) - R(x)] \dfrac{\partial H}{\partial x} + g(x)u \\ y &= g^T(x) \dfrac{\partial H}{\partial x} \end{cases}$$



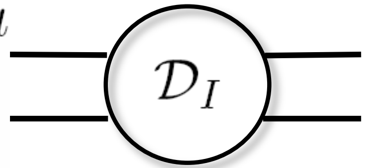
$$\begin{cases} \dot{x}_t &= \dfrac{1}{x_t} D(x) + \tilde{u}_t \\ y_t &= x_t \end{cases}$$

- We will make use of the ideas seen in the **Energy Transfer Control technique!**
- Implement the desired action as a "lossless energy transfer" between Tank and PHS
- **This action will always preserve passivity by construction**

Energy Tanks

- Assume we want to implement the action $w \in \mathbb{R}^m$ on the PHS ($m = \dim(u)$)

$$\begin{cases} \dot{x} &= [J(x) - R(x)] \dfrac{\partial H}{\partial x} + g(x)u \\ y &= g^T(x) \dfrac{\partial H}{\partial x} \end{cases}$$



$$\begin{cases} \dot{x}_t &= \dfrac{1}{x_t} D(x) + \tilde{u}_t \\ y_t &= x_t \end{cases}$$

- We then interconnect the PHS and the Tank element by means of this state-modulated power-preserving interconnection

$$\begin{bmatrix} u \\ \tilde{u}_t \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{w}{x_t} \\ -\dfrac{w^T}{x_t} & 0 \end{bmatrix} \begin{bmatrix} y \\ y_t \end{bmatrix}$$

- Since this coupling is skew-symmetric, no energy is created/lost during the transfer

Energy Tanks

- After this coupling the individual dynamics become

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x} + g(x)\left(\frac{w}{x_t}y_t\right) = [J(x) - R(x)]\frac{\partial H}{\partial x} + g(x)\boxed{w}$$

and

$$\dot{x}_t = \frac{1}{x_t}D(x) - \frac{w^T}{x_t}y = \frac{1}{x_t}D(x) - \frac{w^T}{x_t}g^T(x)\frac{\partial H}{\partial x}$$

- And altogether, a new Hamiltonian $\mathcal{H}(x,\, x_t) = H(x) + T(x_t)$

$$\begin{bmatrix}\dot{x}\\ \dot{x}_t\end{bmatrix} = \left(\underbrace{\begin{bmatrix} J(x) & \frac{w}{x_t} \\ -\frac{w^T}{x_t} & 0 \end{bmatrix}}_{\text{Skew-symmetric}} - \begin{bmatrix} R(x) & 0 \\ -\frac{1}{x_t}\frac{\partial \mathcal{H}^T}{\partial x}R(x) & 0 \end{bmatrix}\right)\begin{bmatrix}\frac{\partial \mathcal{H}}{\partial x}\\ \frac{\partial \mathcal{H}}{\partial x_t}\end{bmatrix}$$

Energy Tanks

- **Fact 1**: action w is correctly implemented on the original PHS

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x} + \boxed{g(x)w}$$

- **Fact 2**: the composite system is (altogether) a passive (lossless) system whatever the expression of w

- Proof: evaluating $\dot{\mathcal{H}}$ along the system trajectories, we obtain a lossless energy balance

$$\begin{bmatrix} \dot{x} \\ \dot{x}_t \end{bmatrix} = \left(\begin{bmatrix} J(x) & \frac{w}{x_t} \\ -\frac{w^T}{x_t} & 0 \end{bmatrix} - \begin{bmatrix} R(x) & 0 \\ -\frac{1}{x_t}\frac{\partial \mathcal{H}^T}{\partial x}R(x) & 0 \end{bmatrix} \right) \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial x} \\ \frac{\partial \mathcal{H}}{\partial x_t} \end{bmatrix}$$

$$\dot{\mathcal{H}} = -\frac{\partial \mathcal{H}^T}{\partial x}R(x)\frac{\partial \mathcal{H}}{\partial x} + \frac{\partial \mathcal{H}^T}{\partial x_t}\frac{1}{x_t}\frac{\partial \mathcal{H}}{\partial x}R(x)\frac{\partial \mathcal{H}}{\partial x} = 0$$

Energy Tanks

- **Fact 3**: the machinery proposed so far **becomes singular** when $x_t = 0$

- What does the condition $x_t = 0$ represent?

- From the definition of the Tank energy function $T(x_t) = \frac{1}{2}x_t^2 \geq 0$ we have that $x_t = 0 \iff$ the **Tank energy is depleted**

- Therefore, this singularity represents the impossibility of passively perform the desired action w

- One can always imagine some (safety) switching parameter $\alpha(t)$ such that

$$\begin{cases} \alpha = 1 & \text{if} \quad T(x_t) \geq \epsilon > 0 \\ \alpha = 0 & \text{if} \quad T(x_t) < \epsilon \end{cases}$$

and implement $\alpha(t)w$ instead of w (i.e., implement w only if you can in a "passive way"). If cannot implement w, **wait for better times (the Tank gets replenished)**

Energy Tanks

- Note that the Tank dynamics is made of two terms

$$\dot{x}_t = \boxed{\frac{1}{x_t} D(x)} - \boxed{\frac{w^T}{x_t} g^T(x) \frac{\partial H}{\partial x}}$$

- The first term is always non-negative, and represents the "refilling" action due to the dissipation present in the PHS plant

- The second term can have any sign, also **negative**. It is then possible for the action w to **actually refill the tank**!

- Finally, note that **no condition is present** on $x_t(t_0)$! This can be chosen as any $x_t(t_0) > 0$

- In other words, complete freedom in choosing the initial amount of energy in the tank $T(x_t(t_0))$

- In fact, passivity ultimately is: **bounded amount of extractable energy**, but for whatever **initial energy in the system** (only needs to be **finite**)

Energy shaping

$$\begin{cases} \dot{x} &= [J(x) - R(x)]\dfrac{\partial H}{\partial x} + g(x)u \\ y &= g^T(x)\dfrac{\partial H}{\partial x} \end{cases}$$

- Suppose $H(x)$ has a minimum at $x = 0$
- By applying the output feedback $u = -ky,\ k > 0$ we get

$$\dot{H} = -\frac{\partial H^T}{\partial x}R(x)\frac{\partial H}{\partial x} - k\|y\|^2 \leq 0$$

- Assuming $R(x) > 0$ and/or zero-state observability, $H(x)$ will asymptotically reach its minimum at $x = 0$

- However, what it we seek stabilization at a different $x = x^*$?

- Use a controller $u = \beta(x) + v$ to shape $H(x)$ into a new $H_d(x)$ with a minimum at $x = x^*$

- In closed-loop, this energy balance (w.r.t. new input v) must hold

$$H_d(x(t)) - H_d(x(t_0)) \leq \int_{t_0}^{t} v^T(\tau)y(\tau)\mathrm{d}\tau$$

Energy balancing

- A classical (state-feedback) formulation: **assume** one can find a function $\beta(x)$ such that, along the system trajectories,

$$\boxed{-\int_{t_0}^{t} \beta^T(x(\tau))y(\tau)\mathrm{d}\tau = H_c(x(t)) + k}$$

for some function of the state $H_c(x)$ and arbitrary constant k

- Then, the state-feedback $\boxed{u = \beta(x) + v}$ yields at closed-loop

$$H_d(x(t)) - H_d(x(t_0)) \leq \int_{t_0}^{t} v^T(\tau)y(\tau)\mathrm{d}\tau$$

that is, passivity of the pair $(v,\, y)$ w.r.t. the new (shaped) storage function

$$\boxed{H_d(x) = H(x) + H_c(x)}$$

if $H_d(x)$ is lower-bounded

Energy balancing

- If $H_d(x)$ has a minimum at $x = x^*$, then the system can be stabilized by the usual

$$\boxed{v = -ky,\ k > 0}$$

- "Energetic" interpretation (from which the name "Energy Balancing"):
  - the term $-\int_{t_0}^{t} \beta^T(x(\tau))y(\tau)\mathrm{d}\tau$ is the energy supplied by the controller to the plant, represented by the state function $H_c(x)$
  - this energy modifies the total energy (of the closed-loop) into $H_d(x) = H(x) + H_c(x)$
  - Furthermore, passivity of the closed-loop with $H_d(x)$ as storage function (if $H_d(x)$ is lower bounded)

- Let us consider again the mass-spring-damper system

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \right) \begin{bmatrix} \dfrac{\partial H}{\partial x} \\ \dfrac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial x} \\ \dfrac{\partial H}{\partial p} \end{bmatrix} \end{cases}$$

Energy balancing

- In this case we have $H(x, p) = H_1(x) + H_2(p)$ with $(x, p) = (0, 0)$ being the global minimum

- Assume we want stabilization to a different $(x, p) = (x^*, 0)$

- Take $\beta(x) = \dfrac{\partial H_1(x)}{\partial x} - \dfrac{\partial H_d(x)}{\partial x}$ with $H_d(x)$ having a minimum at $x = x^*$

- It follows that

$$-\int_{t_0}^{t} \beta^T(x(\tau))y(\tau)\mathrm{d}\tau = \int_{t_0}^{t} \left(-\frac{\partial H_1^T(x)}{\partial x} + \frac{\partial H_d^T(x)}{\partial x}\right) \frac{\partial H_2(p)}{\partial p}\mathrm{d}\tau = -H_1(x) + H_d(x) + c$$

as $\dfrac{\partial H_2(p)}{\partial p} = \dot{x}$. The closed-loop energy is $H_1(x) + \underbrace{(-H_1(x) + H_d(x) + c)}_{H_c(x)} = H_d(x) + c$

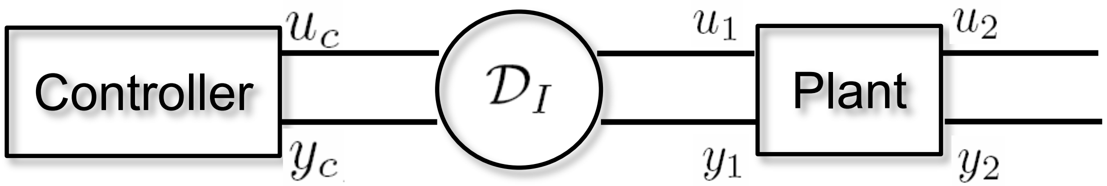
- Finally, choose (for instance) $H_d(x) = \dfrac{1}{2}\bar{k}(x - x^*)^2, \ \bar{k} > 0$

- The control action is nothing but $u = kx - \bar{k}(x - x^*) + v$ ....

Energy balancing

- How general are these results? Can this technique always be exploited?

- Can we reformulate (and generalize) this approach as an interconnection between a PHS plant and a PHS controller?



- Let us introduce the notion of Casimir functions for a PHS

$$C(x):\ \mathcal{M}\rightarrow\mathbb{R}$$

- Casimir functions are conserved quantities, i.e., $\dot{C}(x)=0$
  - along the PHS open-loop system trajectories, i.e., with $u\equiv 0$
  - independently of the Hamiltonian of the system $H(x)$

Intermezzo: Invariants of Motion

- Consider a PHS $\begin{cases} \dot{x} &= [J(x) - R(x)]\dfrac{\partial H}{\partial x} + g(x)u \\ y &= g^T(x)\dfrac{\partial H}{\partial x} \end{cases}$

- Assume it is evolving in free evolution $u(t) \equiv 0$

- Are there Invariants of Motion (Casimir functions), i.e., a function of the state $C(x)$ such that, along the system trajectories,

$$C(x(t)) \equiv const \Longleftrightarrow \dot{C}(x(t)) \equiv 0$$

- If such a function can be found, it then represents an "invariant of motion" of the system, i.e., a conserved quantity

- Since for PHS we speak about Energy storage and Energy flows, one can also look at "conserved quantities" like in classical physics

Intermezzo: Invariants of Motion

- The constraint $\dot{C}(x(t)) \equiv 0$ can be written $\dfrac{\partial C^T(x)}{\partial x}\dot{x} \equiv 0$ which, exploiting the PHS dynamics, becomes

$$\frac{\partial C^T(x)}{\partial x}[J(x) - R(x)]\frac{\partial H}{\partial x} \equiv 0$$

- For simplifying the analysis, we only consider the condition

$$\frac{\partial C^T(x)}{\partial x}[J(x) - R(x)] \equiv 0$$

- This is equivalent to ask: does matrix $J(x) - R(x)$ possess a non-void (left) nullspace?

- Equivalent: is the square matrix $J(x) - R(x)$ singular? (~$\det(J(x) - R(x)) = 0$)

- If yes, let $a(x)$ be a vector spanning its left nullspace, $a^T(x)[J(x) - R(x)] = 0$

Intermezzo: Invariants of Motion

- Having found $a(x)$, **one can hope** to solve the PDE $\frac{\partial C(x)}{\partial x} = a(x)$ and determine the Casimir function $C(x(t)) \equiv const$

- For the mass-spring-damper system, we have

$$[J(x) - R(x)] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix}$$

- Matrix $J(x) - R(x)$ is nonsingular => **no possible invariants of motion**

- What if no dissipation was present ($b = 0$)?

- Then $[J(x) - R(x)] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, but still a non-singular matrix

- So, **no invariants of motion** for the mass-spring damper

Intermezzo: Invariants of Motion

- Can we imagine a situation in which invariants of motion will **necessarily** be present for a PHS?

- **Yes:** when no dissipation is present $R(x) = 0$ and $\boxed{n = \dim(x) = 2k+1,\ k \geq 0}$

- In fact, in this case $\dot{x} = J(x)\dfrac{\partial H}{\partial x}$ and $\det(J(x)) = 0$ **for n odd**

- For $n = 1$, trivial case with $J(x) = 0$, $\dot{x} = 0$ and $C(x) = c$

- First interesting case for $n = 3$ since $J(x)$ will have a non-trivial left eigenvector spanning its null-space

- Example of a PHS for $n = 3$: **rigid body rotation dynamics**

Back to Energy balancing

- Formally, given a PHS $\begin{cases} \dot{x} & = & [J(x) - R(x)] \dfrac{\partial H}{\partial x} + g(x)u \\ y & = & g^T(x) \dfrac{\partial H}{\partial x} \end{cases}$

- Casimir functions must satisfy $\boxed{\dfrac{\partial C^T}{\partial x}(J(x) - R(x)) = 0}$

so that $\dot{C}(x) = \dfrac{\partial C^T}{\partial x}\dot{x} = 0$ whatever the Hamiltonian $H(x)$

- Therefore, Casimir functions are determined only by the internal geometry of the system
  - Interconnection structure $J(x)$
  - Dissipation structure $R(x)$

- And... definining $H_d(x) = H(x) + C(x)$ it follows $\dot{H}_d(x) = \dot{H}(x)$
  - **Can use $C(x)$ to shape $H(x)$ while retaining the same "convergence properties"!!**

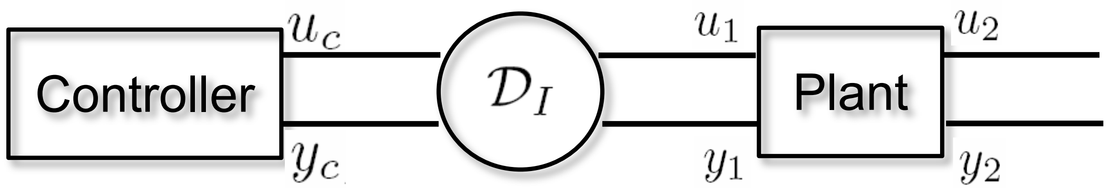
Energy balancing

- Consider the PHS plant (**given**) and controller (**to be determined**)

$$\begin{cases} \dot{x} = [J(x) - R(x)] \dfrac{\partial H}{\partial x} + g(x)u \\ y = g^T(x) \dfrac{\partial H}{\partial x} \end{cases} \qquad \begin{cases} \dot{x}_c = (J_c(x_c) - R_c(x_c)) \dfrac{\partial H_c}{\partial x_c} + g_c(x_c)u_c \\ y_c = g_c^T(x_c) \dfrac{\partial H_c}{\partial x_c} \end{cases}$$

and the interconnection structure $\mathcal{D}_I$ (**to be determined**) taken for simplicity as

$$u = -y_c + v_1, \; u_c = y + v_2$$



- The resulting PHS with Hamiltonian $H(x) + H_c(x_c)$ is

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \left(\begin{bmatrix} J(x) & -g(x)g_x^T(x_c) \\ g_c(x_c)g^T(x) & J_c(x_c) \end{bmatrix} - \begin{bmatrix} R(x) & 0 \\ 0 & R_c(x_c) \end{bmatrix} \right) \begin{bmatrix} \dfrac{\partial H}{\partial x} \\ \dfrac{\partial H_c}{\partial x_c} \end{bmatrix} + \begin{bmatrix} g(x) & 0 \\ 0 & g_c(x_c) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ \begin{bmatrix} y \\ y_c \end{bmatrix} = \begin{bmatrix} g^T(x) & 0 \\ 0 & g_c^T(x_c) \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial x} \\ \dfrac{\partial H_c}{\partial x_c} \end{bmatrix} \end{cases}$$

Energy balancing

- Now look for Casimir functions $C(x, x_c)$, that is functions satisfying

$$\begin{bmatrix} \frac{\partial C^T}{\partial x} & \frac{\partial C^T}{\partial x_c} \end{bmatrix} \left(\begin{bmatrix} J(x) & -g(x)g_x^T(x_c) \\ g_c(x_c)g^T(x) & J_c(x_c) \end{bmatrix} - \begin{bmatrix} R(x) & 0 \\ 0 & R_c(x_c) \end{bmatrix} \right) = 0$$

and yielding a Lyapunov function $\boxed{V(x, x_c) = H(x) + H_c(x_c) + C(x, x_c)}$
with a minimum at (x^*, x_c^*) where

- $x = x^*$ is the equilibrium we seek for the plant
- $x_c = x_c^*$ is a free parameter (the controller equilibrium)

- Optionally, can add extra damping to stabilize (damping injection), e.g., the usual output feedback

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -k \begin{bmatrix} y \\ y_c \end{bmatrix}, \quad k > 0$$

Energy balancing

- How to find useful Casimir functions? How to use them to shape $V(x,\, x_c)$?

- Hint again: since, along the system trajectories, $\dot{C}(x,\, x_c) = 0$, the system state is constrained to evolve on the invariant manifolds

$$\boxed{C(x,\, x_c) = c}$$

- This constraints the controller state to be a function of the plant state $x_c = \Gamma(x)$

- Thus, the total Hamiltonian takes the form $H_d(x) = H(x) + H_x(\Gamma(x)) + c$
  - A function of the plant state only

- We restrict the search within the family $C(x,\, x_c) = x_c - F(x)$ where $F(x)$ must be determined

Energy balancing

- The definition of Casimir functions $C(x,\, x_c) = x_c - F(x)$ yields

$$\frac{\partial F^T}{\partial x}[J(x) - R(x)] - g_c(x_c)g^T(x) = 0\,, \quad \frac{\partial F^T}{\partial x}g(x)g^T(x_c) + J_c(x_c) - R_c(x_c) = 0$$

which, after algebraic manipulations, result in

1) $$\boxed{\frac{\partial F^T}{\partial x}J(x)\frac{\partial F}{\partial x} = J_c(x_c)}$$

$$\boxed{R_c(x_c) = 0}\,, \quad \boxed{R(x)\frac{\partial F}{\partial x} = 0}\,, \quad \boxed{\frac{\partial F^T}{\partial x}J(x) = g_c(x_c)g^T(x)}$$

2) **3)** **4)**

- **1), 2), 3), 4)** are the conditions to be satisfied by functions $C(x,\, x_c) = x_c - F(x)$ and by the controller structure to obtain Casimir functions for the interconnected plant/controller PHS

Energy balancing

- If these conditions can be satisfied, then the controller state is completely determined by $\boxed{x_c = F(x) + c}$

- The (unforced) plant dynamics

$$\dot{x} = (J(x) - R(x))\frac{\partial H}{\partial x} - g(x)g_c^T(x_c)\frac{\partial H_c}{\partial x_c}$$

becomes $\boxed{\dot{x} = [J(x) - R(x)]\frac{\partial H_d}{\partial x}}$ where

$$\boxed{H_d(x) = H(x) + H_c(F(x) + c)}$$

- The interconnected plant/controller yields in closed-loop
  - The same internal structure of the plant, i.e., $J(x)$, $R(x)$
  - But a new "shaped" Hamiltonian $H_d(x)$
- However, note the constraints **2)** $R_c(x_c) = 0$ and **3)** $R(x)\frac{\partial F}{\partial x} = 0$ known as the **dissipation obstacle**

Energy balancing

- What is the dissipation obstacle?

- The controller cannot dissipate energy because of $\boxed{R_c(x_c) = 0}$

- The energy shaping cannot be performed on those coordinates affected by plant dissipation because of $\boxed{R(x)\frac{\partial F}{\partial x} = 0}$

- Physical reason: a passive controller can stabilize equilibria $(x^*,\ x_c^*)$ where
<u>no energy dissipation takes place</u>

- The controller cannot provide an infinite amount of energy at steady-state

- Not a problem when shaping the potential energy of mechanical systems!
  - Dissipation can be present only on the "velocity" variables defining the kinetic energy

Energy balancing

- How does the energy balance look like for the Casimir method?

- For the controller, since $R_c(x_c) = 0$, it is $\dot{H}_c = y_c^T u_c$
  - All the energy flowing through $(u_c,\ y_c)$ is stored (released) in $H_c(x_c)$

- The shaped (closed-loop) Hamiltonian evolves as $\dot{H}_d = \dot{H} + \dot{H}_c = \dot{H} - y^T u$ because of the interconnection $u = -y_c,\ u_c = y$

- Therefore, as expected $H_d(x(t)) = H(x(t)) - \int_{t_0}^{t} y^T(\tau)u(\tau)\mathrm{d}\tau + c$
  - That is: **Energy Balancing**

- The shaped energy is the difference between
  - the energy stored in the plant $H(x(t))$
  - The energy supplied by the controller $\int_{t_0}^{t} y^T(\tau)u(\tau)\mathrm{d}\tau = -H_c(x)$

Energy balancing

- Mass/spring/damper system again

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix} \end{cases}$$

- Take a controller $\begin{cases} \dot{x}_c &= u_c \\ y_c &= \dfrac{\partial H_c}{\partial x_c} \end{cases}$ where $H_c(x_c)$ must be determined

- We seek (like before) stabilization at some $(x,\, p) = (x^*,\, 0)$

- Look for Casimir functions $C = x_c - F(x,\, p)$

Energy balancing

- It can be shown that conditions 1), 2), 3), 4) yield $\boxed{C = x_c - x}$ as Casimir functions for the closed-loop system

- Therefore, along the system trajectories, it is $x_c - x = c$
- The controller state is constrained to be $x_c = x + c$

- Take as controller Hamiltonian $\boxed{H_c(x_c) = \frac{1}{2}kx_c^2 = \frac{1}{2}k(x+c)^2}$

- The closed-loop Hamiltonian becomes

$$H_d(x,\, p) = H(x,\, p) + H_c(x_c) = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kx^2 + \frac{1}{2}k(x+c)^2$$

- **Exercise: determine the value of** c **that makes** $(x^*,\, 0)$ **to be the minimum of** $H_d(x,\, p)$

- The control action is $u = -y_c = -\dfrac{\partial H_c}{\partial x_c} = -k(x+c)$