

EiR/CPR/CAMS 2024/2025

Analysis and Control of Multi-Robot Systems

Elements of Passivity Theory

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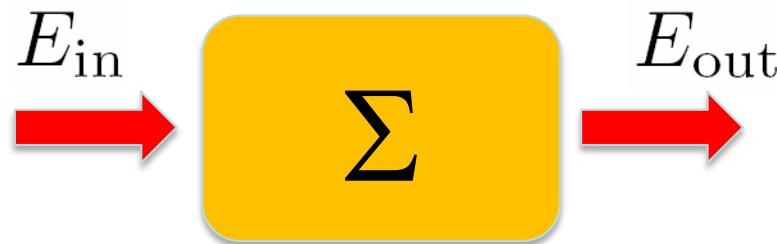
(Part of the slides by Paolo Robuffo Giordano)

DIPARTIMENTO DI INGEGNERIA INFORMATICA
AUTOMATICA E GESTIONALE ANTONIO RUBERTI

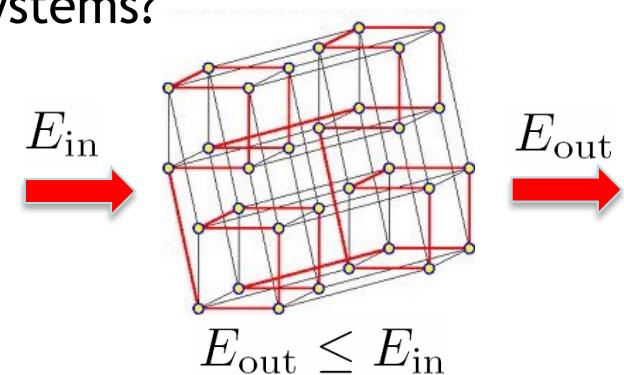


Introduction to Passivity

- What is **passivity**?
- Intuitively: something that **does not produce internal energy**



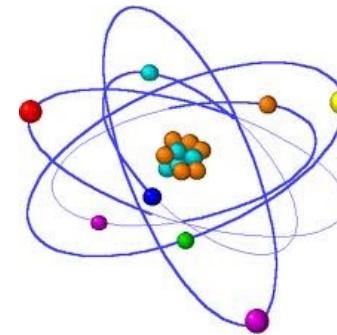
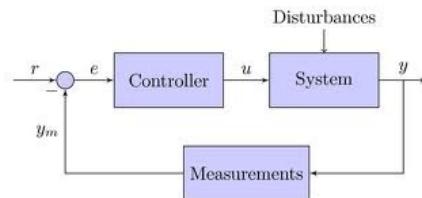
- Describes input/output (I/O) behaviors
- Seamlessly applies to linear and nonlinear systems
- Is this helpful for modeling and control of multi-robot systems?



Introduction to Passivity

- Passivity-like concepts are common to many scientific areas

- Mathematics
- Physics
- Electronics
- Control

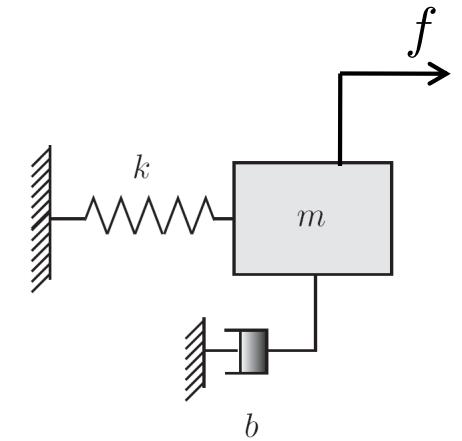


- Basic idea: most physical systems have common I/O characteristics dictated by
 - Energy conservation
 - Energy transportation
 - Energy dissipation
- Energy plays a fundamental role
 - Common unifying language across all physical domains

Introduction to Passivity

- Consider this **simple mechanical system** with dynamics

$$m\ddot{x} + b\dot{x} + kx = f$$



and **energy** $E(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \geq 0, \forall x, \dot{x}$

- How is the “**energy flowing**” within the system?

$$\frac{d}{dt}E(x, \dot{x}) = m\dot{x}\ddot{x} + kx\dot{x} = \boxed{f\dot{x}} - \boxed{b\dot{x}^2}$$

Input/Output Mechanical power Internal dissipated power

- Integrating back, we get

$$E(x(t), \dot{x}(t)) = \boxed{E(x(t_0), \dot{x}(t_0))} + \int_{t_0}^t f(\tau)\dot{x}(\tau)d\tau - \int_{t_0}^t b\dot{x}^2(\tau)d\tau$$

↑
Initial stored energy

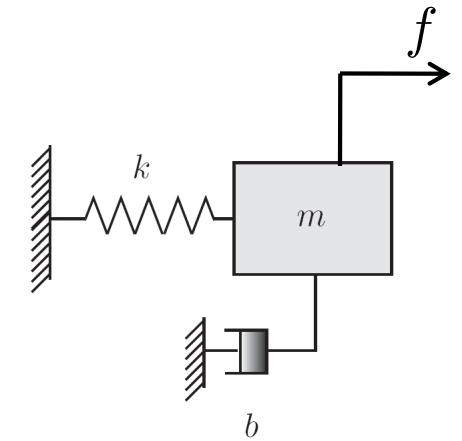
Introduction to Passivity

- if $f = 0, b = 0 \Rightarrow E = E(x(0), \dot{x}(0)) = \text{const}$

no I/O “energy flow”, but still an **internal dynamics**

$$m\ddot{x} + kx = 0$$

- if $b \geq 0, f \neq 0$



$$E(x(t), \dot{x}(t)) - E(x(t_0), \dot{x}(t_0)) = \int_{t_0}^t f(\tau) \dot{x}(\tau) d\tau - \int_{t_0}^t b \dot{x}^2(\tau) d\tau \leq \int_{t_0}^t f(\tau) \dot{x}(\tau) d\tau$$

and since $E(x(t), \dot{x}(t)) \geq 0$ we have

$$-E(x(t_0), \dot{x}(t_0)) \leq \int_{t_0}^t f(\tau) \dot{x}(\tau) d\tau$$

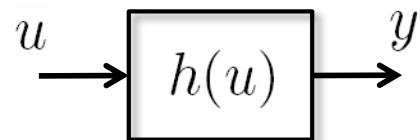
- The **total extractable energy** is limited by the **initial stored energy**

Introduction to Passivity

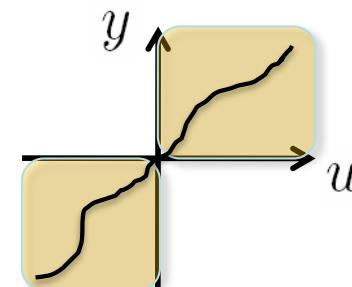
- Passivity: a property of a physical system (but also, more in general, of a linear/nonlinear dynamical system)
- Based on the concept of “**energy**”
- Describes the **energy flow (power)** through the system
- It is an **I/O** characterization
- Usually, passivity is a **robust property** (e.g., w.r.t. parametric variations)
- It is (of course) related to **classical Lyapunov stability** concepts
- **Proper compositions of passive systems are passive** -> very useful property (later)

Passivity: formal definitions

- A first definition of passivity can be given for **memoryless (static) functions** $y = h(u)$



- The function is said to be passive if $u^T y \geq 0, \forall u$
 - “Power” flowing into the system is never negative
 - The system does not produce energy (can only **absorb** and **dissipate**)
 - Example: the familiar **electrical resistance** $y = Ru, R > 0$, the power is $u^T y = Ru^2 \geq 0$
- For the scalar case, passivity imposes a constraint on the **graph** of $y = h(u)$
 - It must lie in the **first** and **third** quadrant
- But we are interested in **MIMO dynamical systems**



Passivity: formal definitions

- Consider a generic nonlinear system (affine in the input)

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

with state/input/output $x \in \mathbb{R}^n$, $u \in \boxed{\mathbb{R}^m}$, $y \in \boxed{\mathbb{R}^m}$

- The system is **dissipative** if there exists a continuous (differentiable) lower bounded function of the state (**storage function**)

$$V(x) \in \mathcal{C}^1 : \mathbb{R}^n \rightarrow \mathbb{R}^+$$

and a function of the input/output pair (**supply rate**) $w(u, y) : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$

such that (~equivalently)

$$\begin{cases} V(x(t)) - V(x(t_0)) \leq \int_{t_0}^t w(u(s), y(s)) ds \\ \dot{V}(x(t)) \leq w(u(t), y(t)) \end{cases}$$

Passivity: formal definitions

- When the supply rate is

$$w(u, y) = y^T u - \delta u^T u - \epsilon y^T y, \quad \delta, \epsilon \geq 0$$

the system is said **passive** (w.r.t. the **supply rate** w and with **storage function** V)

- In particular,

- **lossless** if $\delta = 0, \epsilon = 0$ and $\dot{V} = y^T u$
- **input strictly passive (ISP)** if $\delta > 0$
- **output strictly passive (OSP)** $\epsilon > 0$
- **very strictly passive (VSP)** $\delta > 0, \epsilon > 0$

- If there exists a **positive definite function** $S(x) : \mathbb{R}^n \rightarrow \mathbb{R}^+$ such that

$$\dot{V}(x) \leq y^T u - S(x)$$

then the system is said **strictly passive**, and $S(x)$ is called **dissipation rate**

Passivity: interpretation

- Some (physical) interpretation:
- The storage function $V(x)$ represents the **internal stored energy**
- The supply rate $y^T u$ is the **power (energy flow)** exchanged with the external world
- The basic passivity condition can be interpreted as

Current energy is at most equal to the initial energy + supplied energy from outside

$$V(x(t)) \leq V(x(t_0)) + \int_{t_0}^t y^T(s)u(s)ds$$

equivalent to “no internal generation of energy”

Passivity: interpretation

- Extractable energy is bounded from below

$$\int_{t_0}^t y^T(s)u(s)ds \geq V(x(t)) - V(x(t_0)) \geq -V(x(t_0)) \geq -c^2, \quad c \in \mathbb{R}$$

- One cannot extract an infinite amount of energy from a passive system
 - The maximum amount of extractable energy (net of the energy supplied from outside) is the initial stored energy (recall the example before)

- This yields an (additional) equivalent passivity condition: a system is passive if

$$\int_{t_0}^t y^T(\tau)u(\tau)d\tau \geq -c^2, \quad c \in \mathbb{R}, \quad \forall u, \forall t \geq t_0$$

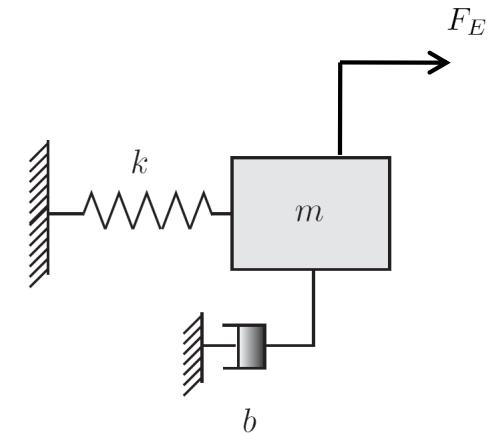
- This alternative definition is sometimes useful in proofs and general considerations on the system at hand
 - No formal need of a storage function

Passivity: review of the example

- Consider again the initial example: $m\ddot{x} + b\dot{x} + kx = f$

- Take $u = f$ as the **input** and $y = \dot{x}$ as the **output**

- Take the total energy $V = E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ as **storage function**



- Is the system **passive** w.r.t. the input/output pair (u, y) ?
- By differentiating V , we get $\dot{V} = f\dot{x} - b\dot{x}^2 = yu - by^2 \leq yu$
- Therefore, the system is **passive**, in particular **output strictly passive**

Passivity: another example

- The integrator

$$\begin{cases} \dot{x} = u \\ y = x \end{cases}$$

is **passive (lossless)** w.r.t. the storage function $V(x) = \frac{1}{2}x^2$ since

$$\dot{V} = xu = yu$$

- Similarly, the integrator with nonlinear output

$$\begin{cases} \dot{x} = u \\ y = h(x) \end{cases} \quad (\blacksquare)$$

with $\int_0^x h(\sigma)d\sigma \geq 0, \quad \forall x$

is **passive (lossless)** w.r.t. the storage function $V(x) = \int_0^x h(\sigma)d\sigma$

- This fact will be **heavily exploited** later on as (\blacksquare) will constitute the fundamental **energy storage element** with associated energy function $V(x)$

Passivity: what is it good for?

- Passivity, so far:
 - I/O characterization
 - Nice energetic interpretation
 - Used to describe how the “energy flows” within a system
 - Several equivalent definitions
- But what is it good for? How can we use it?
- Key features:
 - Strong link to Lyapunov stability
 - Proper (and useful) **interconnections** of passive systems are **passive (modularity)**
 - A system can **be made passive**
 - By a choice of the “right output”
 - By a feedback action
 - A passive system is “**easily stabilizable**” from the output
 - And... many **real-world systems are passive**

Passivity vs. Lyapunov

- Short summary about **Lyapunov stability**
- Given a system $\dot{x} = f(x) \quad f(0) = 0$ (■)

the equilibrium $x = 0$ is

- **Stable** if $\forall \epsilon > 0 \exists \delta(\epsilon) > 0 \mid \|x(t_0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon, \quad \forall t \geq t_0$
- **Unstable** if it is not stable
- **Asymptotically stable** if **stable** and $\|x(t_0)\| \leq \delta \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0$

- The Lyapunov Theorems allow to establish (asympt.) stability of (■) without explicitly computing the solution of (■)
- Pivotal is the concept of **Lyapunov function**, i.e., a positive definite function $V(x)$

$$V(x) \in \mathcal{C}^1 : D \rightarrow \mathbb{R}^+, \quad D \subset \mathbb{R}^n$$
$$V(0) = 0, \quad V(x) > 0 \text{ in } D - \{0\}$$

Passivity vs. Lyapunov

- If there exists a $V(x)$ such that
 - $\dot{V}(x) \leq 0$ in D then the system is **stable**
 - $\dot{V}(x) < 0$ in $D - \{0\}$ then the system is (locally) **asympt. stable (LAS)**
 - If $V(x)$ is **radially unbounded**, i.e., $D = \mathbb{R}^n$ and $\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$, and it still holds $\dot{V}(x) < 0$ in $D - \{0\}$ the system is **globally asympt. Stable (GAS)**
- Also in the case 1), let $S = \{x \in D \mid \dot{V}(x) = 0\}$
- **LaSalle Th.**: The system will converge towards M , the largest invariant set in S
- If $M = \{0\}$, i.e., only $x(t) \equiv 0$ can stay identically in S , then the system is **LAS (GAS)**

Passivity vs. Lyapunov

- Let us go back to the passivity conditions
- System dynamics $\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$ and there exists a **storage function** $V(x)$ such that $\dot{V}(x) \leq y^T u$
- Assume that $V(0) = 0$, then $V(x)$ is a Lyapunov candidate around 0 and
 - If $u \equiv 0$ then $\dot{V} \leq 0$, i.e., the system is **stable**
 - If $y \equiv 0$ then $\dot{V} \leq 0$, i.e., the **zero-dynamics** of the system is **stable**
- The system can be easily **stabilized by a static output feedback**

$$u = -\phi(y), \quad y^T \phi(y) > 0 \quad \forall y \neq 0$$

for instance $u = -ky, \quad k > 0$

Passivity: output feedback

- By setting $u = -\phi(y)$ we obtain
 - Non increasing storage function $\dot{V} \leq -y^T \phi(y) \leq 0 \rightarrow$ bounded state trajectories
 - Convergence to a manifold $\dot{V} \equiv 0 \Leftrightarrow y = h(x) \equiv 0$ (the set S of before)
- Remember LaSalle: if the system is zero-state observable
$$h(x(t)) \equiv 0 \Rightarrow x(t) \equiv 0$$
then $u = -\phi(y)$ provides asympt. stability (LAS)
- Global results (GAS) if the storage function $V(x)$ is radially unbounded

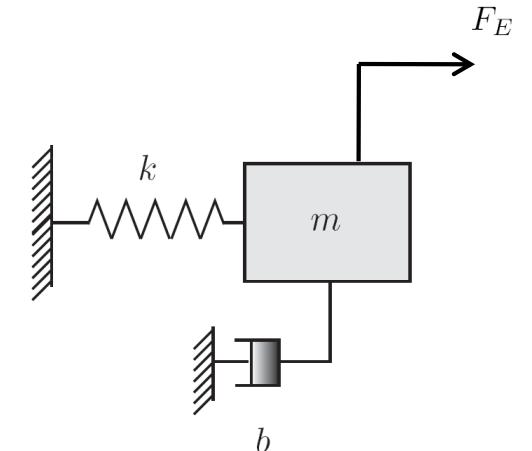
Passivity: output feedback

- Passivity of a system is w.r.t. an input/output pair
- One can also look for a **good output** w.r.t. which the system is passive
- Consider the state evolution $\dot{x} = f(x) + g(x)u$ and assume we can find a $V(x)$ such that $\frac{\partial V}{\partial x} f(x) \leq 0$, i.e., a **stable free evolution** ($u \equiv 0$)
- Then, the system is passive w.r.t. the output $y = \left[\frac{\partial V}{\partial x} g(x) \right]^T$
- The feedback $u = -ky = -k \left[\frac{\partial V}{\partial x} g(x) \right]^T$ makes the system LAS (GAS)

Passivity: output feedback

- Review of the example $m\ddot{x} + b\dot{x} + kx = f$

- Rewrite in canonical state-space form $(x_1, x_2) = (x, \boxed{m\dot{x}})$



$$\begin{cases} \dot{x}_1 = \frac{x_2}{m} \\ \dot{x}_2 = -\frac{b}{m}x_2 - kx_1 + u \\ y = \boxed{\frac{x_2}{m}} \end{cases}$$

with $u = f$

- Take the storage function (radially unbounded) $V = \frac{1}{2}kx_1^2 + \frac{1}{2}\frac{x_2^2}{m}$

- Passivity condition

$$\dot{V} = kx_1 \cancel{\frac{x_2}{m}} - b \frac{x_2^2}{m^2} - kx_1 \cancel{\frac{x_2}{m}} + \frac{x_2}{m}u = -b \frac{x_2^2}{m^2} + \frac{x_2}{m}u = -by^2 + yu$$

Passivity: output feedback

- by setting $u \equiv 0$ we obtain $\dot{V} = -by^2 \leq 0$, i.e.,
 - the state trajectories $(x_1(t), x_2(t))$ are bounded
 - the output (the velocity) will converge to 0: $y(t) \rightarrow 0$
- Let us check the **zero-state observability** (i.e., LaSalle)
- Zeroing the output means that $y(t) \equiv 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0$
- The dynamics restricted to this set become $0 = -kx_1 \Rightarrow x_1(t) \equiv 0$
- Therefore, the only possible solution is $(x_1(t), x_2(t)) \equiv (0, 0)$
- Or, in other words, **zeroing the output implies zeroing the complete state**
- One can still feed back the output $u = -ky$. This results in faster convergence

$$\dot{V} \leq -(b+k)y^2 < -by^2 \leq 0$$

Passivity: output feedback

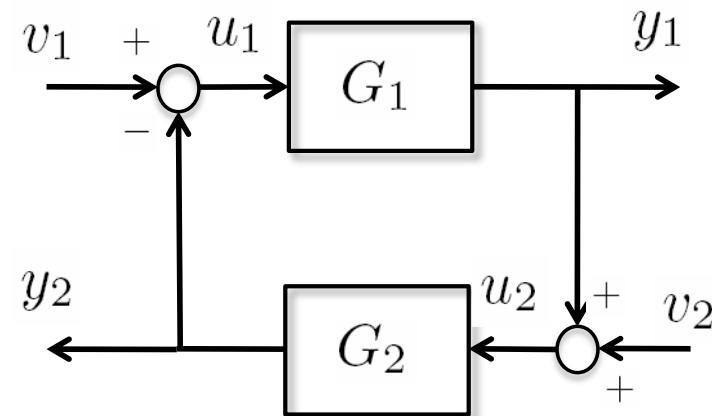
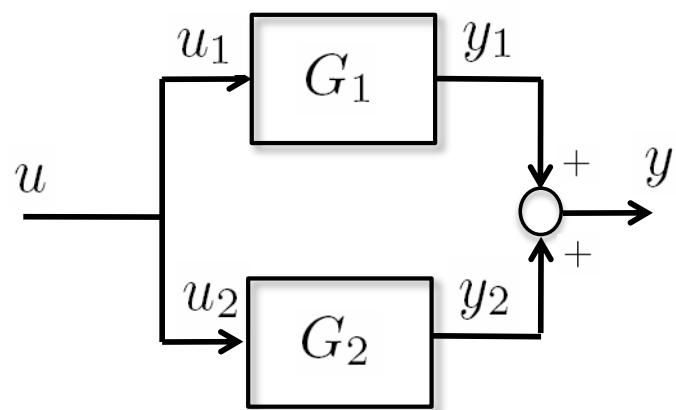
- Another example: consider the system without output

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 + u \end{cases}$$

- We want to look for an **output** $y = h(x)$ that makes the system **passive** w.r.t. the pair (u, y)
- Let us consider the (radially unbounded) Storage function $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$
- With $u \equiv 0$ it is $\dot{V} = 0$ (stable system). Remember slide 21...
- We can take $y = \frac{\partial V}{\partial x} g(x) = x_2$ and stabilize the system by $u = -kx_2$
- Is the system **zero-state observable?** (exercise)

Passivity: interconnection

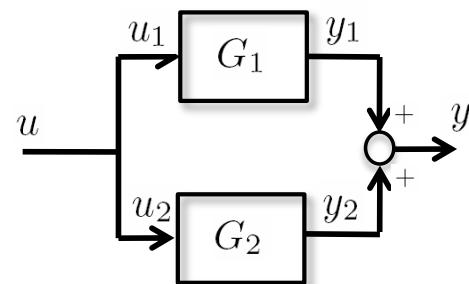
- Fundamental property: **proper interconnections of passive systems** are again **passive**
- This property opens the door to **modularity** (network modeling):
 - Identify subcomponents
 - Make them passive
 - Interconnect them in a “**proper way**”
 - The result will be a **passive system** (stable, etc.)
- We will address two possible interconnections: **parallel** and **feedback**



Passivity: interconnection

- Given two passive systems with **proper I/O dimensions** and storage functions $V_1(x_1)$ and $V_2(x_2)$

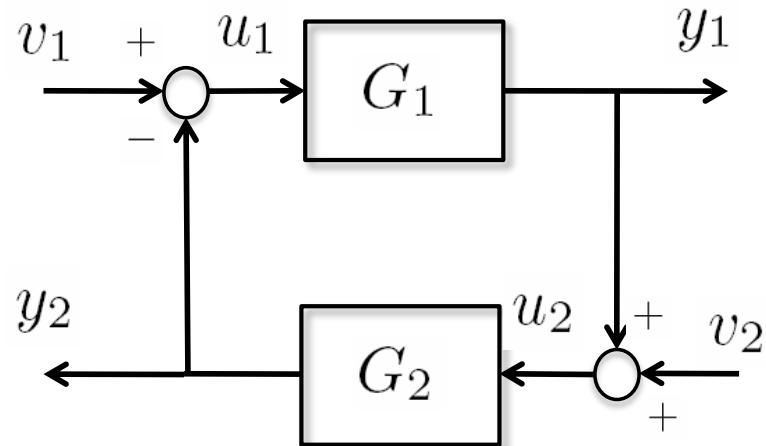
$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)u_1 \\ y_1 = h_1(x_1) \end{cases} \quad \begin{cases} \dot{x}_2 = f_2(x_2) + g_2(x_2)u_2 \\ y_2 = h_2(x_2) \end{cases}$$



- For the **parallel interconnection**, set $u_1 = u_2 = u$ and $y = y_1 + y_2$
- Let $x = (x_1, x_2)$ and $V(x) = V_1(x_1) + V_2(x_2)$ be the storage function
- Then $\dot{V} = \dot{V}_1 + \dot{V}_2 \leq y_1^T u_1 + y_2^T u_2 = (y_1 + y_2)^T u = y^T u$
- The new system is **passive** w.r.t. the pair $(y_1 + y_2, u) = (y, u)$

Passivity: interconnection

- Take $\begin{cases} u_1 = \pm y_2 + v_1 \\ u_2 = \mp y_1 + v_2 \end{cases}$ New (optional) inputs

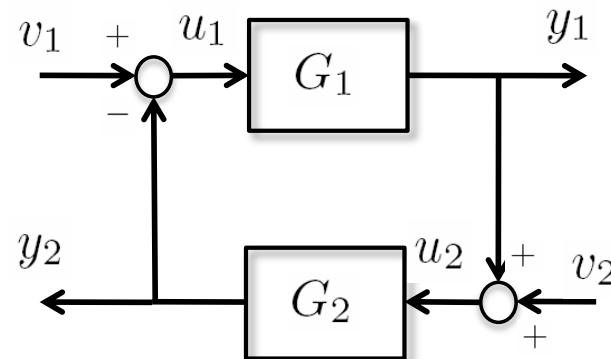


- Prove that the interconnected system is passive with storage function $V(x) = V_1(x_1) + V_2(x_2)$ w.r.t. the (composed) input/output pair

$$\left([y_1^T \ y_2^T]^T, [v_1^T \ v_2^T]^T \right)$$

Passivity: interconnection

- Note the particular structure of the **feedback interconnection**



$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

- The coupling matrix is **skew-symmetric**
- This is a fundamental property that allows to **retain passivity of the composed system**
- We will see later that this is an example of a **power-preserving interconnection**

Passivity: pre-post multiplication

- Assume G_1 is a **passive** system with **storage function** $V(x)$ w.r.t. the pair (u, y)
- Let $M(x)$ be a (possibly state-dependent) matrix, and let $u = M(x)\tilde{u}$ and $\tilde{y} = M^T(x)y$



- Passivity is preserved by a **pre-multiplication** of the input by $M(x)$ and a **post-multiplication** of the output by $M^T(x)$ (with the same storage function)

Summarizing

- **Passivity** is a I/O property of a dynamical system, intuitively equivalent to
 - No internal production of “energy”
 - Bounded extractable “energy”
- Seamlessly applies to **linear** and **nonlinear** systems
- Linked to **Lyapunov stability**
 - **Stability** of the origin in **free evolution** (asympt. with some observability properties)
 - Stable zero-dynamics
 - Easily stabilizable by **static output feedback**
- Can be enforced by
 - Finding the “**correct**” output
 - A proper **feedback (passifying) action**
- It is a **modular** property: proper interconnections of passive systems are passive

Review of consensus protocol

- Let us revisit the **consensus protocol** under the “**passivity light**”
- Take a **passive** (lossless) system: single integrators

$$\Sigma : \begin{cases} \dot{x} = u_1 \\ y_1 = x \end{cases}$$

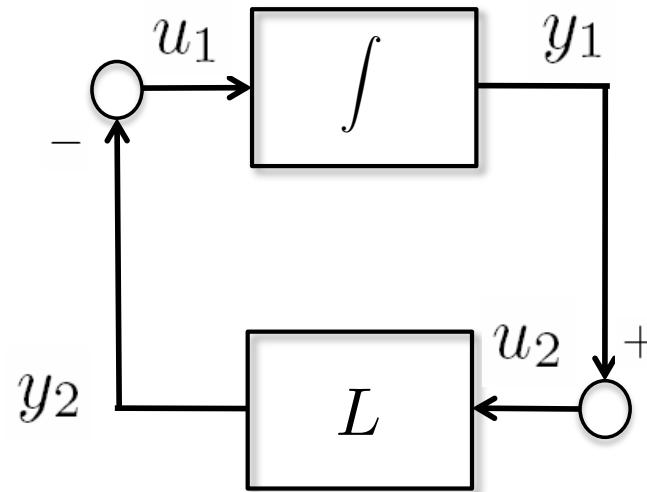
- Consider a **static function** $y_2(u_2) = Lu_2$
- This is a **passive static function** $u_2^T y_2 = u_2^T Lu_2 \geq 0$
- Interconnect these two passive systems by means of a “**feedback interconnection**”

$$\begin{cases} u_2 = y_1 \\ u_1 = -y_2 \end{cases}$$

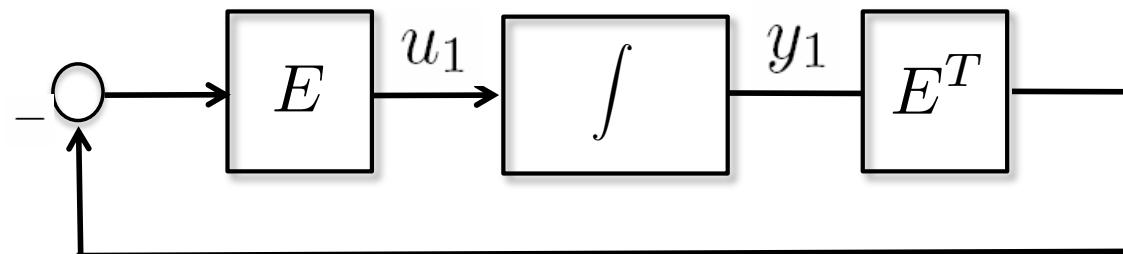
- The resulting system will necessarily be **passive**....
 - And is nothing but the consensus closed-loop dynamics $\dot{x} = -Lx$

Review of consensus protocol

- As a block scheme



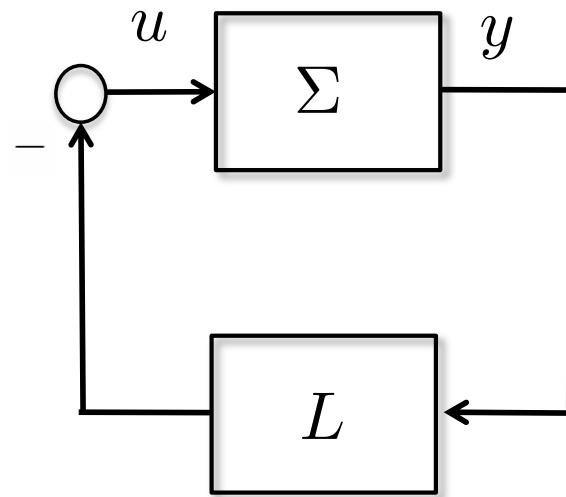
- Another point of view: recall that $L = EE^T$. Then, the consensus protocol is just



- Since the single integrator is passive and a pre-/post-multiplication preserves passivity, we are just closing the loop of a passive with a negative unitary output feedback

Review of consensus protocol

- This analysis also shows that a consensus-like approach extends to any **passive nonlinear dynamics**
- For example, if the **nonlinear system** $\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$ is **passive** w.r.t. (u, y) then



keeps being a (closed-loop) **passive system** (note that one could increase the “amount of passivity” by including an additional $u = -ky$)

- A possibility: $\begin{cases} \dot{x} = u \\ y = h(x) \end{cases}$ with $h(x) = \frac{\partial V(x)}{\partial x}$ and storage function $V(x)$

Kuramoto Model

- Consider the multi-agent system given by the Kuramoto model of n coupled oscillators interacting over a network \mathcal{G}

$$\dot{\theta}_i(t) = k \sum_{j \in N(i)} \sin(\theta_j(t) - \theta_i(t)), \quad i = 1, 2, \dots, n.$$

- $k=K/n$, where K represents the **coupling strength** between the oscillators

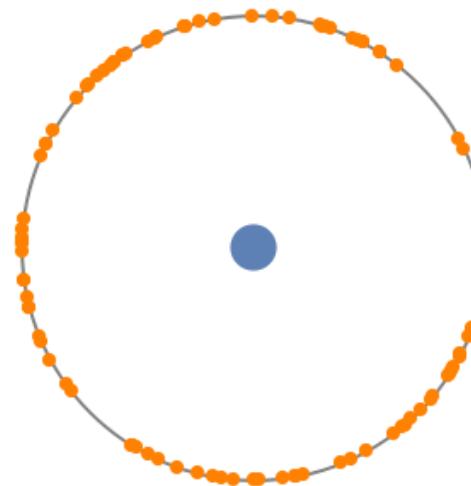
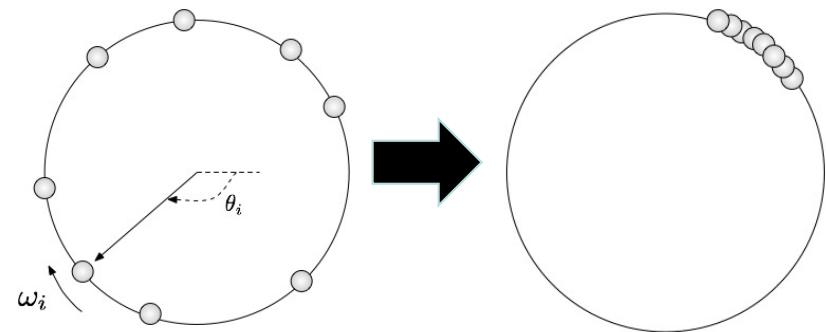


Image source: <https://mathematica.stackexchange.com/>

Passivity and Nonlinear Consensus

Kuramoto Model



$$\dot{\theta}(t) = -\frac{K}{n} E(\mathcal{G}) \sin(E(\mathcal{G})^T \theta(t))$$

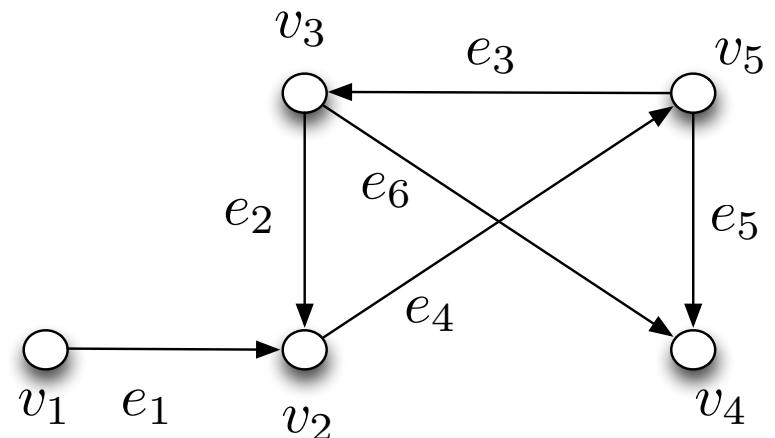
Theorem: *For an arbitrary connected graph \mathcal{G} , coupling strength $K > 0$, and almost all initial conditions in $(-\pi, \pi)^n$, the Kuramoto model will synchronize. Moreover, the rate of approach to synchronization is bounded by*

$$\frac{2K}{\pi n} \lambda_2(\mathcal{G}).$$

proof: passivity (Lyapunov) + LaSalle

Edge Laplacian

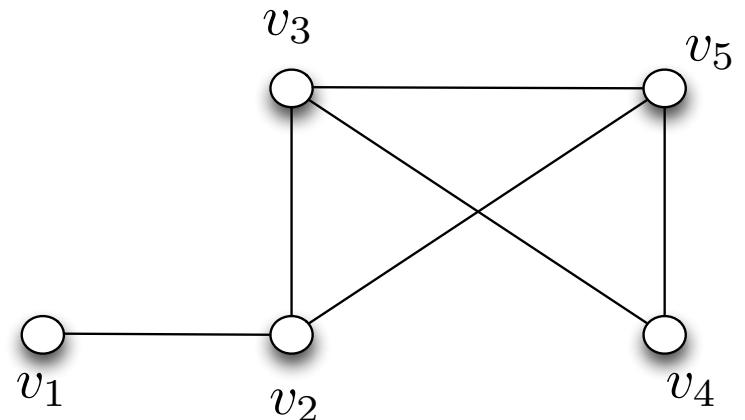
- Recall that the Laplacian is defined as $L = EE^T$, where E is the incidence matrix of the graph, for an arbitrary labeling of the edges



- We call **Edge Laplacian** the matrix $L_{\mathcal{E}} = E^T E$
- By construction, the nonzero eigenvalues of L and $L_{\mathcal{E}}$ are the same
- It can be interpreted as an edge adjacency matrix, where two edges are adjacent if they share a common vertex
- Every edge is considered to be adjacent to itself, hence the elements on the diagonal of $L_{\mathcal{E}}$ Laplacian are equal to 2

Edge Agreement

- Let us consider a graph, with **states defined over the edges** $x_{\mathcal{E}}(t)$ rather than over the vertexes (as in the standard Consensus setup)



- Assume that \mathcal{G} has n vertexes and m edges
- A state transformation can be used: $x_{\mathcal{E}}(t) = E^T x(t)$

- Differentiation yields:

$$\dot{x}_{\mathcal{E}}(t) = -L_{\mathcal{E}}x_{\mathcal{E}}(t)$$

Edge agreement protocol

Edge Agreement

- Agreement in the vertexes states translates to having $x_{\mathcal{E}}(t)=0$, when the graph is connected
- Suppose that the graph \mathcal{G} contains a spanning tree \mathcal{G}_τ . Then, up to a suitable permutation of edges labels, the incidence matrix can be decomposed as

$$E = [E_\tau \quad E_c]$$

where E_τ is the **incidence matrix of the given spanning tree**, and E_c accounts for the remaining edges (cycle edges).

- Accordingly, the Edge Laplacian can be decomposed as

$$L_{\mathcal{E}} = \begin{bmatrix} L_{\mathcal{E}_\tau} & E_\tau^T E_c \\ E_c^T E_\tau & L_{\mathcal{E}_c} \end{bmatrix}$$

Edge Agreement

- A nice feature: the Edge Laplacian of a spanning tree $L_{\mathcal{E}_\tau}$ is always **positive definite**

- A matrix R can be found such that

$$L_{\mathcal{E}} = R^T L_{\mathcal{E}_\tau} R$$

- In particular, the matrix can be obtained as $R = [I \ T]$, where

$$T = (E_\tau^T E_\tau)^{-1} E_\tau^T E_c$$

- Furthermore, the **edge agreement protocol can be reduced** by considering the restriction over the spanning tree

$$\dot{x}_\tau(t) = -L_{\mathcal{E}_\tau} R R^T x_\tau(t)$$

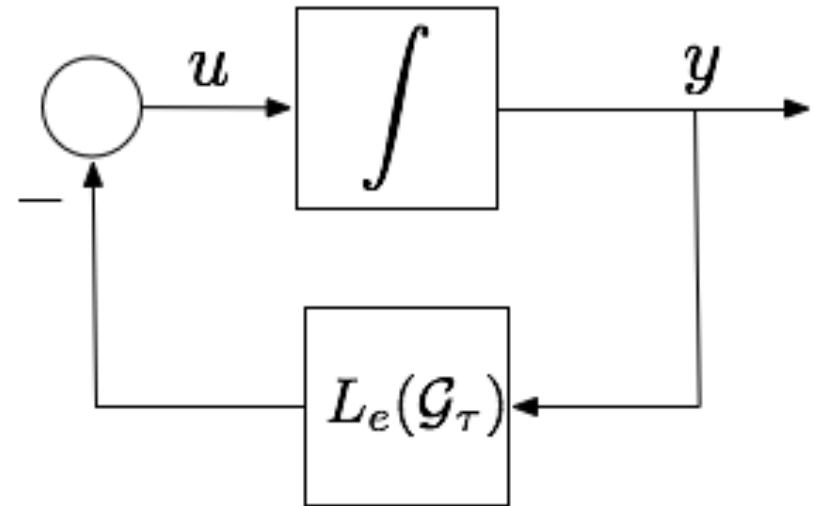
Passivity and the Edge Laplacian

Edge Agreement
over a Spanning Tree

Storage Function

$$V(x) = \frac{1}{2}x(t)^T x(t)$$

A strictly passive system!

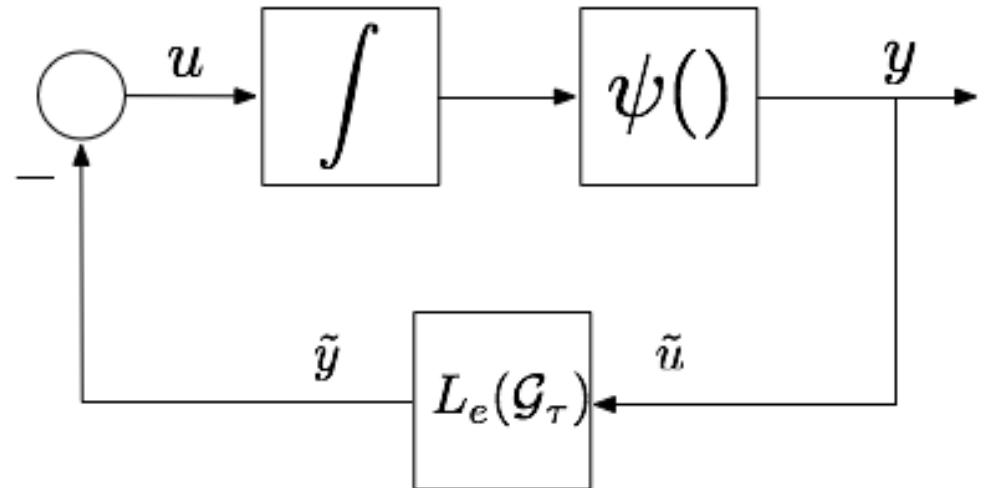


Positive-definite

$$u^T y = \dot{V} + y^T L_e(G_\tau) y$$

Passivity and the Edge Laplacian

Introduce non-linear element



Assume

$\exists V(x)$ positive definite, with $\nabla V(x) = \psi(x)$

→ “forward path” is passive w.r.t output

Let

$$\phi(v) = \lambda_2(\mathcal{G})v$$

$$\tilde{u}^T \tilde{y} \geq \tilde{u}^T \phi(\tilde{u}) = \lambda_2(\mathcal{G}) \tilde{u}^T \tilde{u} > 0$$



The closed loop map is asymptotically stable!

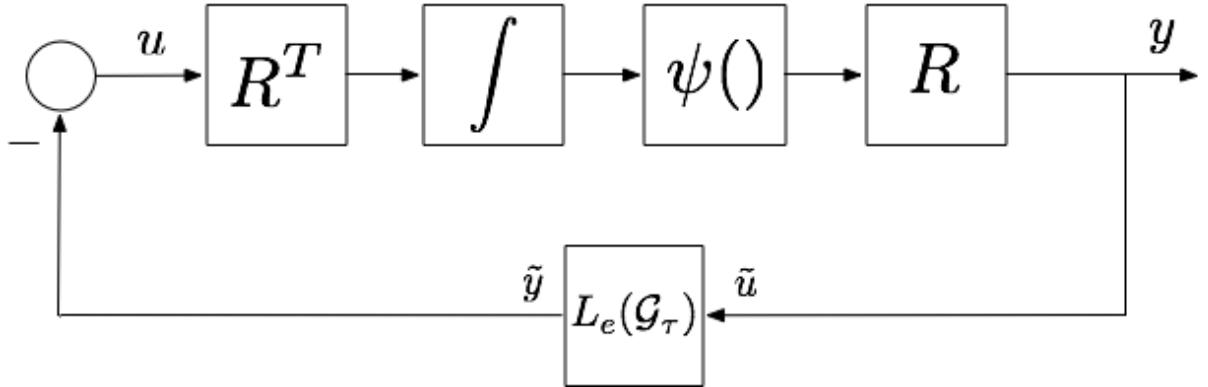
Passivity and the Edge Laplacian

more generally...

$$E(\mathcal{G}) = E(\mathcal{G}_\tau)R$$

$$L_e(\mathcal{G}) = E(\mathcal{G})^T E(\mathcal{G})$$

$$= R^T E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau)R$$



Pre/Post multiplication does not change
passivity properties of forward loop!



The closed loop map is
asymptotically stable!

Passivity and the Edge Laplacian

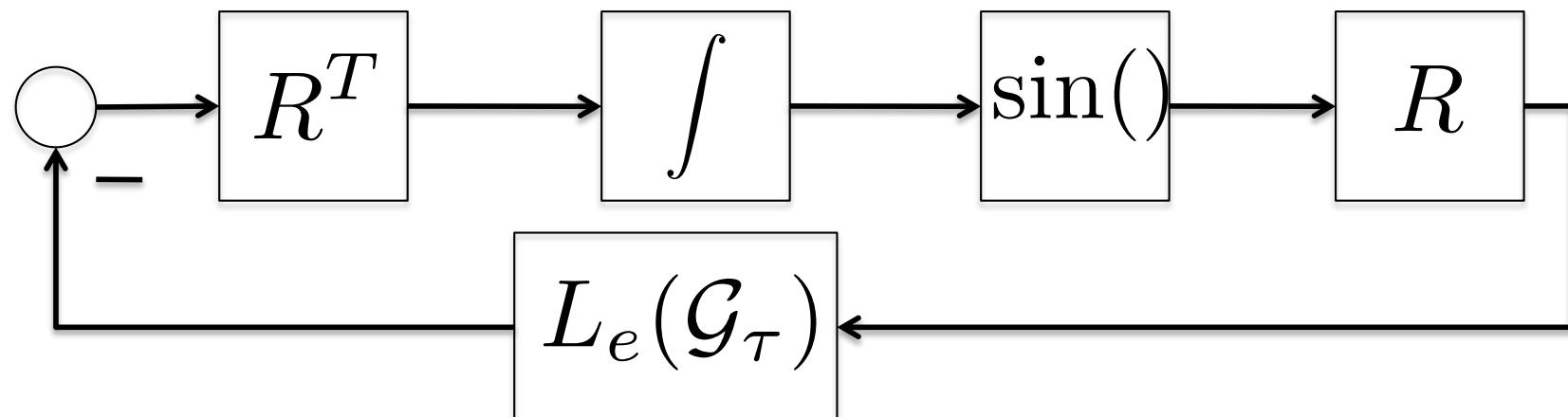
back to Kuramoto...

$$\dot{\theta}(t) = -k E(\mathcal{G}) \sin(E(\mathcal{G})^T \theta(t))$$

transform to edge states:

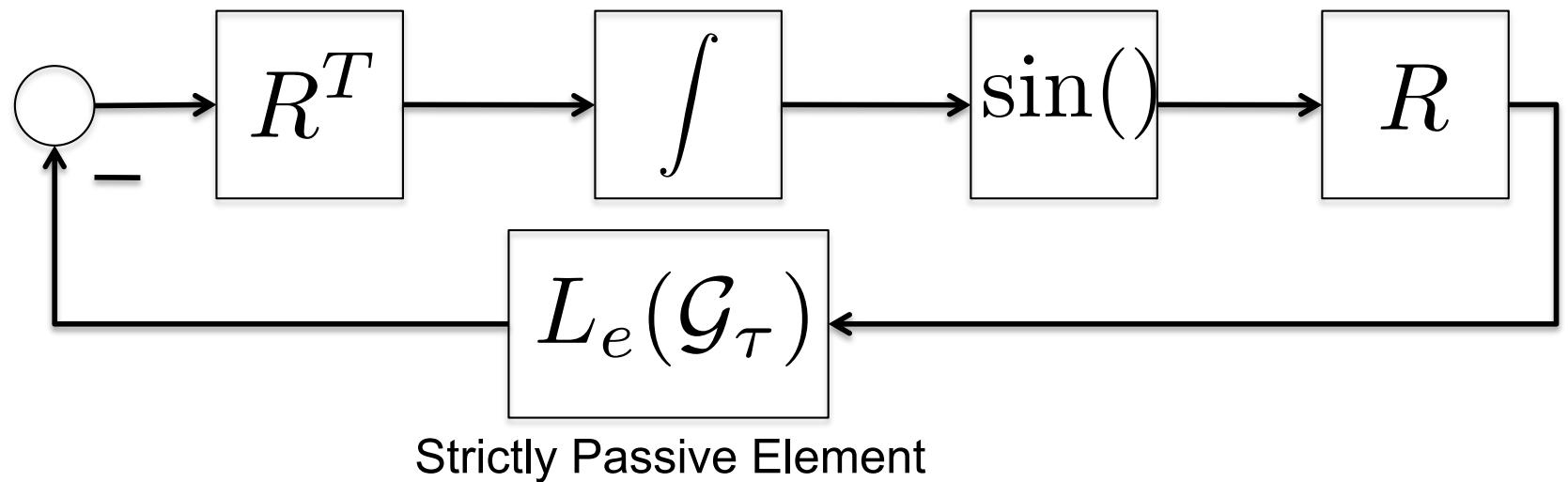
$$\theta_e(t) = E(\mathcal{G})^T \theta(t)$$

$$\dot{\theta}_e(t) = -k R^T L_e(\mathcal{G}_\tau) R \sin(\theta_e(t))$$



Passivity and the Edge Laplacian

$$\dot{\theta}_e(t) = -k R^T L_e(\mathcal{G}_\tau) R \sin(\theta_e(t))$$



$$V(\theta_e(t)) = \mathbf{1}^T (\mathbf{1} - \cos(\theta_e(t))) \quad \psi(\theta_e(t)) = \sin(\theta_e(t))$$

Using passivity machinery and Edge Laplacian, we conclude that synchronization state is asymptotically stable

