A Distributed Strategy for Generalized Biconnectivity Maintenance in Open Multi-robot Systems

Master's Degree in Artificial Intelligence and Robotics

Analysis and Control of Multi-Robot Systems

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1 Introduction

Introduction and related work

- **Context**: robust communication in open multi-robot systems (OMRS) with a dynamic number of robots
- **Problem**: simple connectivity does not ensure robustness
- **Proposed solution**: distributed gradient-based control to enforce connectivity
- **Advantages**: support dynamic addition/removal of agents from a graph
- **Connectivity preservation by keeping all initial links** ([1]–[3]): ensures the original connections never break, but this limits flexibility and adaptability
- **Fault-tolerant and robust control methods** ([7]–[9]): focus on resilience of multi-agent systems in case of failures, but are not specifically designed for OMRS
- **Local connectivity maintenance for OMRS (only addition of agents)** ([10]): preserves connectivity under dynamic conditions but considers only joining agents, not removals
- **Biconnectivity maintenance algorithms** ([11]–[14]): ensure the graph stays connected even after removal of a node

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2 Preliminaries

Robot model

2 Preliminaries

- World frame $W = \{O_w, X_W, Y_W, Z_W\}$ with Z_W aligned with vertical (gravity) direction.
- Robot State
  - $x_i \in \mathbb{R}^d$ position w.r.t world frame with $d \in \{2, 3\}$.
  - $\psi_i \in S^1$ yaw angles around Z_W with S^1 representing unitary circumference.
  - $v_i \in \mathbb{R}^d$ body-frame linear velocity.
  - $w_i \in \mathbb{R}$ body-frame angular velocity / yaw rate.
- Kinematic model: $\dot{\eta}_i = J(\eta_i)v_i, \quad J(\eta_i) := \begin{bmatrix} R_i & 0 \\ 0 & 1 \end{bmatrix}$
- PHS model: $\begin{cases} \dot{p}_i = F_i^\lambda + F_i^e - B_i M_i^{-1} p_i \\ v_i = \frac{\partial \mathcal{K}_i}{\partial p_i} = M_i^{-1} p_i \end{cases} \qquad i = 1, \ldots, N(t)$

  $p_i = M_i v_i \in^{d+1}$ generalized momentum, $M_i \in^{(d+1)\times(d+1)}$ inertia matrix, $\mathcal{K}_i(p_i) := \frac{1}{2} p_i^T M_i^{-1} p_i$ kinetic energy, $B_i \in^{(d+1)\times(d+1)}$ damping term, $F_i^\lambda \in^{(d+1)}$ generalized biconnectivity force

Sensing and communication model

- **Sensors**: omnidirectional distance sensor to detect nearby robots and a camera-like sensor with limited FOV. If robot j is visible to robot i, the relative position is $x_{ij} = R_i^T(x_i - x_j)$, where R_i, x_i, and x_j are expressed in the world frame. These measurements enable robots to identify neighbors in the interaction graph.
- **Interaction graph**: time-varying graph $\mathcal{G}(\mathcal{V}(t), \mathcal{E}(t))$ has nodes $\mathcal{V}(t) = \{1, \ldots, N(t)\}$ and edges $\mathcal{E}(t) \subseteq \mathcal{V}(t)^2$ representing sensing/communication links. Set of neighbors of agent i is $\mathcal{N}_i(t) = \{j \in \mathcal{V}(t) : (i,j) \in \mathcal{E}(t)\}$, and the adjacency matrix A has $a_{ij} > 0$ if $j \in \mathcal{N}_i(t)$, otherwise $a_{ij} = 0$.
- **Laplacian matrix**: Laplacian $L = \text{diag}(A\mathbf{1}) - A$ is symmetric and positive semi-definite. Its second smallest eigenvalue λ_2, called algebraic connectivity, satisfies $\lambda_2 > 0$ if the graph is connected and $\lambda_2 = 0$ otherwise. The Laplacian and its eigenvalues thus capture the connectivity induced by the sensing/communication interactions among robots.

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3 Generalized connectivity

Generalized connectiviy

The adjacency matrix A encodes the presence of an interaction link between agents (i, j) and can also represent additional inter-agent behaviors and constraints.

$$a_{ij} = \gamma_{ij} f_{ij} \alpha_{ij} \beta_{ij}$$

- $\gamma_{ij} \geq 0$: maximum communication range / link quality
- $f_{ij} \in [0, 1]$: FOV constraint (0 if outside FOV)
- $\alpha_{ij} \geq 0$: hard requirements (e.g., collision avoidance)
- $\beta_{ij} \geq 0$: soft requirements (e.g., desired inter-agent distance)

By making a_{ij} smooth functions of agent states, the second smallest eigenvalue λ_2 becomes a smooth measure of graph connectivity.

Perturbed graph and perturbed algebraic connectivity

- Let $\mathcal{G}_{-i}$ be the graph remaining after the removal of node i, then $\mathcal{G}$ is said to be biconnected if for any $i \in \mathcal{V}$, the remaining graph $\mathcal{G}_{-i}$ is connected. if $\mathcal{G}_{-i}$ is disconnected, the node i is called articulation point.
- **Sufficient condition for biconnection**: If all nodes are locally biconnected then the entire graph is biconnected.
- Associate to each agent a parameter $\rho_i \in [0, 1-\delta]$,with $\delta > 0$ small constant, given by

$$\dot{\rho}_i = -k_1\rho_i + \frac{k_2}{2}(1 + sign(\sigma_\lambda - \lambda^l_{2,i}))$$

where k_1, k_2 are chosen such that $\frac{k_2}{k_1} = 1 - \delta$, σ_λ is a threshold parameters

- **Perturbed Adjiacency matrix** $\tilde{A}$ with $\tilde{a}_{ij} = \min\{\epsilon_i, \epsilon_j\}a_{ij}$ where $\epsilon_i = 1 - \rho_i$
- **Perturbed Laplacian matrix** $\tilde{L} = diag(\tilde{A}1) - \tilde{A}$ with $\hat{\lambda}_2 = \lim_{\delta\to 0^+} \frac{\tilde{\lambda}_2}{\delta} \to \hat{\lambda}_2 \approx \frac{\tilde{\lambda}_2}{\delta}$

Weights

Lets define the four sub-weights for $a_{ij} = \gamma_{ij} f_{ij} \alpha_{ij} \beta_{ij}$: considering $d_{ij} := \|x_{ij}\|$, we have

- $\gamma_{ij}(d_{ij}) = \begin{cases} 1 & 0 \leq d_{ij} \leq d_\gamma \\ \frac{1}{2} & d_\gamma < d_{ij} \leq D_\gamma \\ 0 & D_\gamma < d_{ij} \end{cases}$
- $B_{ij}(d_{ij}) = \exp(-\frac{(d_{ij} - d_\beta)^2}{\sigma}), \quad \sigma > 0$
- $\alpha_{ij}(d_{ij}) = (\prod_{k \in S_i} \alpha^*_{ij}(d_{ik}))(\prod_{k \in S_k/\{i\}} \alpha^*_{jk}(d_{jk}))$,

$$\alpha^*_{ij}(d_{ij}) = \begin{cases} 0 & 0 \leq d_{ij} \leq d_\alpha \\ \frac{1}{2}\alpha_k & d_\alpha < d_{ij} \leq D_\alpha \\ 1 & D_\alpha < d_{ij} \end{cases} , \quad \alpha_k / f_k = 1 - \cos(\frac{\pi(d/c_{ij} - d/c_\alpha)}{D_\alpha / c_M - d_\alpha / c_m})$$

- $f_{ij}(c_{ij}) = f^*_{ij}(c_{ij}) + f^*_{ji}(c_{ji}) - f^*_{ij}(c_{ij}) f^*_{ji}(c_{ji})$, $\quad f^*_{ij}(c_{ij}) = \begin{cases} 1 & 0 \leq c_{ij} \leq c_m \\ \frac{1}{2} f_k & c_m < c_{ij} \leq c_M \\ 0 & c_M < c_{ij}, \end{cases}$

Let's define

- $\mathcal{D} := \{\hat{\lambda}_2 \in R_{\geq 0} \mid \hat{\lambda}_2 > \bar{\lambda}\}$ feasible second smallest eigenvalues
- $V_\lambda : \mathcal{D} \to \mathcal{R}_{\geq 0}$ which is differentiable and had the property that $V_\lambda(\hat{\lambda}_2) \to \infty$ as $\hat{\lambda}_2 \to \partial\mathcal{D}$ so as it converge to the border of $\mathcal{D}$

Generalized biconnectivity force F_i^λ in the PHS is a simple integrator defined as

$$F_i^\lambda = -k\frac{\partial V_\lambda(\hat{\lambda}_2)}{\partial \eta_i} = -k\frac{\partial V_\lambda(\hat{\lambda}_2)}{\partial \hat{\lambda}_2}\begin{bmatrix}\frac{\partial \hat{\lambda}_2}{\partial \psi_i} \\ \frac{\partial \hat{\lambda}_2}{\partial \psi_i}\end{bmatrix}$$

Choosing $V_\gamma(\hat{\lambda}_2) = \coth\left(\hat{\lambda}_2 - \tilde{\lambda}^*\right) - 1 = \frac{e^{2(\hat{\lambda}_2 - \tilde{\lambda}^*)}+1}{e^{2(\hat{\lambda}_2 - \tilde{\lambda}^*)}-1} - 1 = \frac{2}{e^{2(\hat{\lambda}_2 - \tilde{\lambda}^*)}-1} > 0$ with $\lambda^* = \bar{\lambda}$ being the biconnectivity lower bound we have

$$\frac{\partial V_\lambda}{\partial \hat{\lambda}_2} = csch^2(\hat{\lambda}_2 - \lambda^*) = (\frac{2e^{(\lambda_2 - \lambda^*)}}{e^{2(\lambda_2 - \lambda^*)} - 1})^2 > 0$$

Closed-loop stability

3 Generalized connectivity

- **Control law:** $F_i^\lambda = -k\frac{\partial V_\lambda(\hat{\lambda}_2)}{\partial \eta_i}$
- **Key definitions:** $\mathcal{E}^* = \{e_1, \dots, e_{N(N-1)/2}\}$ (all possible edges), $E \in \mathbb{R}^{3N\times 3|\mathcal{E}^*|}$ (rotational incidence matrix), $\tilde{x}_k = x_i - x_j$, $\tilde{\psi}_k = \psi_i - \psi_j$, $\tilde{\eta}_k = [\tilde{x}_k^T\ \tilde{\psi}_k]^T$, $\tilde{v}_k = v_i - v_j$, $\frac{\partial \tilde{x}_k}{\partial x_i} = R_i^T$, $\tilde{\eta} = [\tilde{\eta}_1^T, \dots, \tilde{\eta}_{N(N-1)/2}^T]^T$, $\tilde{v} = [\tilde{v}_1^T, \dots, \tilde{v}_{N(N-1)/2}^T]^T$
- **Control in terms of relative poses:** $F_i^\lambda = -\sum_{k=1}^{3N(N-1)/2} \frac{\partial V^\lambda(\lambda_2)}{\partial \tilde{\eta}_k}$
- **System energy:** $H(p, \tilde{\eta}) = \sum_{i=1}^N (\mathcal{K}_i(p_i) + V^\lambda(\lambda_2(\tilde{\eta}))) \geq 0$
- **PH formulation:** $\begin{bmatrix} \dot{p} \\ \dot{\tilde{\eta}} \end{bmatrix} = \begin{bmatrix} B & -E \\ E^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \tilde{\eta}} \end{bmatrix} + \begin{bmatrix} F^e \\ 0 \end{bmatrix}, \quad \tilde{v} = \frac{\partial H}{\partial p}$
- **Energy derivative:** $\dot{H}(p, \tilde{\eta}) = -\frac{\partial H}{\partial p}^T B \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}^T F^e \leq \tilde{v}^T F^e$
- **Proposition:** If $\mathcal{G}(t_0)$ is biconnected, the system is passive w.r.t. $(F^e, \tilde{v})$ and the set $\mathcal{D}$ is forward invariant $\Rightarrow$ biconnectivity is maintained $\forall t \geq t_0$.

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4 Generalized biconnectivity for OMRS

- **Problem:** After agent join/leave, $\hat{\lambda}_2 \leq \bar{\lambda}$, V_λ and F_i^λ become undefined.
- **Solution: Time-varying lower bound** $\lambda^* : \mathbb{R}_{\geq 0} \to [0, \bar{\lambda} + \epsilon]$
  - $\lambda^* = 0$ when biconnectivity is lost, smoothly increases to guarantee reconnection.
  - Each agent keeps a local copy λ_i^* with dynamics:

$$\dot{\lambda}_i^* = f(\lambda_i^*) - k_c \sum_{j \in \mathcal{N}_i} (\lambda_i^* - \lambda_j^*) - k'_\lambda c_i(t) \lambda_i^*$$

  where $f(\lambda_i^*) = -k_\lambda(\lambda_i^* - (\bar{\lambda} + \epsilon))$ and $c_i(t) \in \{0, 1\}$ is a pinning variable.
- **Properties:**
  - Synchronized agents: $\lambda_i^* \to \bar{\lambda} + \epsilon$ if $c_i(t) = 0 \quad \forall i$
  - Pinned agents: $\lambda_i^* \to 0$
  - Passivity preserved:

$$V_\lambda(\hat{\lambda}_2) = \coth(\hat{\lambda}_2 - \tilde{\lambda}^*) - 1, \quad \dot{H} \leq \tilde{v}^T F^e - \frac{\partial V_\lambda}{\partial \hat{\lambda}_2} \dot{\tilde{\lambda}}^*$$

Biconnectivity for OMRS

- **Tank-Based Force Approximation:**
  - Tank state $x_{ti} \in \mathbb{R}$, energy $T_i = \frac{1}{2}x_{ti}^2 \geq 0$, dissipated energy $D_i = p_i^T M_i^{-1} B_i M_i^{-1} p_i$
  - Augmented dynamics: $\begin{cases} \dot{p}_i = F_i^e - w_i x_{ti} - B_i M_i^{-1} p_i \\ \dot{x}_{ti} = \frac{1}{x_{ti}} D_i + u_{ti} + w_i^T v_i, \quad v_i = M_i^{-1} p_i \end{cases}$
  - Set $w_i = -\frac{1}{x_{ti}} \hat{F}_i^{\lambda}$, $u_{ti} = \frac{c_i}{x_{ti}} \frac{\partial V_\lambda}{\partial \hat{\lambda}_2} \dot{\lambda}_i^*$
- **Total Energy:** $H(p, \tilde{\eta}, x_t) = \sum_i \mathcal{K}_i(p_i) + T_i(x_{ti}) + V^{\lambda}(\lambda_2(\tilde{\eta})) \geq 0$
- **PH Formulation:**

$$\begin{bmatrix} \dot{p} \\ \dot{\tilde{\eta}} \\ \dot{x}_t \end{bmatrix} = \left(\begin{bmatrix} 0 & E & \mathcal{W} \\ -E^T & 0 & 0 \\ -\mathcal{W}^T & 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathcal{B} & 0 & 0 \\ 0 & 0 & 0 \\ -\mathcal{P}\mathcal{B} & 0 & 0 \end{bmatrix} \right) \nabla H + G F^e + G_t u_t, \quad \tilde{v} = G^T \nabla H$$

- **Derivative:** $\dot{H} \leq \tilde{v}^T F^e + \epsilon_\lambda \Rightarrow$ passivity preserved
- **Key Point:** tanks compensate syncrhonization for errors due to the use of local λ_2^l
- Passivity ensured via PH, tank, and M^3D formulation, System remains passive at each switching instant

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System dynamics

- **M3D system dynamics:**
  - Nonlinear: $\dot{x}_{\sigma(t)}(t) = f_{\sigma(t)}(x_{\sigma(t)}(t))$
  - Linear: $\dot{x}_{\sigma(t)}(t) = (A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)})x_{\sigma(t)}(t)$
- $x_{\sigma(t)}(t) \in \mathbb{R}^{n_{\sigma(t)}}$, dimension varies with commutation signal $\sigma(t)$
- State may be discontinuous at switching instant t_k:

$$x_{\sigma(t_k^+)}(t_k^+) = \Xi_{\sigma(t_k^+),\sigma(t_k^-)}\, x_{\sigma(t_k^-)}(t_k^-) + \Phi_k$$

  - $\Xi_{\sigma(t_k^+),\sigma(t_k^-)} \in \{0,1\}^{n_{\sigma(t_k^+)} \times n_{\sigma(t_k^-)}}$: indicates dimension change (reduction/expansion)
  - $\Phi_k \in \mathbb{R}^{n_{\sigma(t_k^+)}}$: state jump
- Two types of impulse Φ_k:
  - **State-independent**: depends only on t_k, non-vanishing, prevents asymptotic stability, guarantees GUPS
  - **State-dependent**: depends on $x_{\sigma(t_k^-)}$, small state $\rightarrow$ small impulse

Impulse and stability

- **State-independent impulse:**
  - Nonzero at switching instant, independent of previous state
  - Prevents asymptotic convergence
  - Guarantees **GUPS (Global Uniform Practical Stability)**: state remains bounded around origin
- **State-dependent impulse:**
  - Scales with previous state magnitude
  - Allows asymptotic convergence
  - Guarantees **GUAS (Global Uniform Asymptotic Stability)**: state converges to 0
- **Summary:** type of impulse determines stability type in M3D systems:
  - State-independent $\rightarrow$ GUPS (bounded but not 0)
  - State-dependent $\rightarrow$ GUAS (converges to 0)

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Numerical example: paper

6 Numerical example

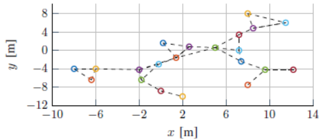


Fig. 1: Initial configuration and initial graph depicting also the field of view of the agents.

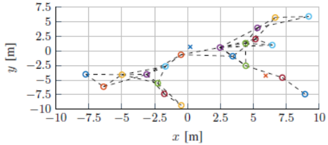


Fig. 2: Positions and graph at the switching time $t = 19.5$s. The agents that left are marked by the '$\times$'.

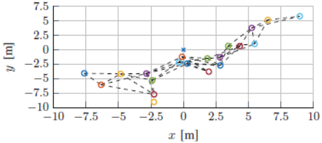


Fig. 3: Positions and graph at the switching time $t = 40$s. The agent that left is marked by the '$\times$'.

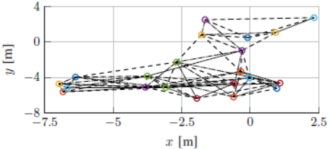


Fig. 4: Final positions and graph.

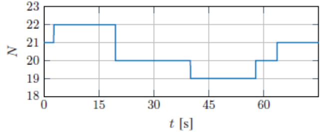


Fig. 5: Evolution of the number of agents in the network.

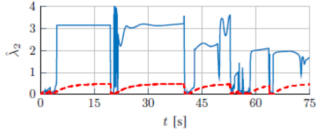
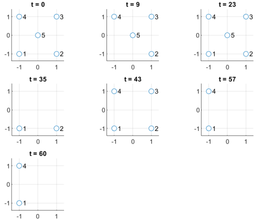
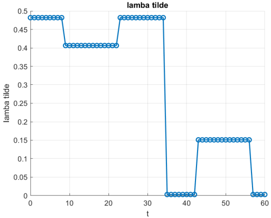


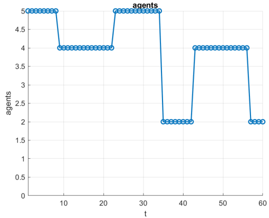
Fig. 6: Perturbed algebraic connectivity. The dashed lines represents the bounds $\lambda_i^*(t)$.

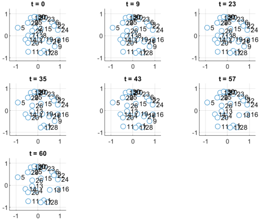
Numerical example: github

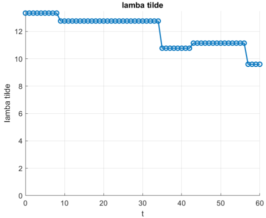
6 Numerical example

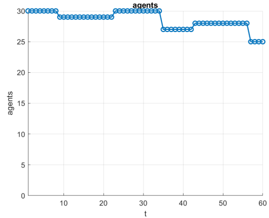












Simulation code and results in `https://github.com/jacopotdsc/cams`

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Conclusion and future works

- Distributed approach to acquire and preserve **generalized global biconnectivity** in OMRS
- Ensures the graph remains connected after agent addition/removal
- Interactions among robots are **sensor-based**
- Biconnectivity measure incorporates additional constraints:
  - Limited inter-robot communication range
  - Limited field-of-view (FOV)
  - Desired inter-agent distances
  - Collision avoidance
- Passivity established via **Port-Hamiltonian (PH) representation** w.r.t external inputs
- Focus on **persistent shared control** scenarios in OMRS

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- **Main papers**
  1 A Distributed Strategy for Generalized Biconnectivity Maintenance in Open Multi-robot Systems, 2024 IEEE 63rd Conference on Decision and Control (CDC), Dec. 16-19, 2024, MiCo, Milan, Italy. Authors: Esteban Restrepo and Paolo Robuffo Giordano
  2 Stability of Multi-Dimensional Switched Systems with an Application to Open Multi-Agent Systems, Mengqi Xue, Yang Tang, Wei Ren, Feng Qian. Affiliations: East China University of Science and Technology, Shanghai, China; University of California, Riverside, USA
- **References**

  1 M. Ji and M. Egerstedt, "Distributed coordination control of multiagent systems while preserving connectedness," IEEE Trans. Robot., vol. 23, no. 4, pp. 693–703, Aug. 2007.
  3 Y. Shang, "Resilient consensus in multi-agent systems with state constraints," Automatica, vol. 122, p. 109288, 2020.
  7 C. Ghedini, C. Ribeiro, and L. Sabattini, "Toward fault-tolerant multirobot networks," Networks, vol. 70, no. 4, pp. 388–400, 2017.
  9 J. Panerati, M. Minelli, C. Ghedini, L. Meyer, M. Kaufmann, L. Sabattini, and G. Beltrame, "Robust connectivity maintenance for fallible robots," Autonomous Robots, vol. 43, pp.

Thank you for your attention!