

# The FLP Theorem

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# The Distributed Consensus Problem

Let's consider the problem of reaching agreement on a single value in a distributed, asynchronous system of processes.

For example, in a distributed database, all entities that have participated in the processing of a particular transaction need to agree whether to commit or rollback.

Whatever decision is made, all entities must make the same decision in order to preserve the consistency of the database.

# Faults

In this course we usually assumed that the participating processes and the network were completely reliable.

This is not the case with real systems, which are subject to a number of possible **faults**, such as process crashes, network partitioning, and lost, distorted, duplicated messages.

One can even consider **Byzantine** types of failures, in which faulty processes might go completely haywire, perhaps sending messages according to some malevolent plan [LSP82].

# The FLP Theorem

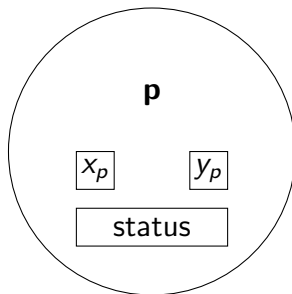
In 1985, Fischer, Lynch, and Paterson proved the surprising result that no completely asynchronous consensus protocol can tolerate even a single unannounced process death [FLP85].

Their proof did not consider Byzantine failures, and assumed that the message system is reliable — it delivers all messages correctly and exactly once.

Nevertheless, even with these assumptions, the stopping of a single process at an inopportune time can cause any distributed consensus protocol to fail to reach agreement.

# Consensus protocol

A **consensus protocol** is an asynchronous system of  $N$  processes ( $N \geq 2$ ). Each process  $p$  has a one-bit **input register**  $x_p$ , an **output register**  $y_p$  with values in  $\{\perp, 0, 1\}$  and an unbounded amount of internal storage.



# Decision states

**Initial states** prescribe fixed starting values for all but the input register; in particular, the output register starts with value  $\perp$ .

The states in which the output register has value 0 or 1 are distinguished as being **decision states**. The transition function cannot change the value of the output register once the process has reached a decision state; that is, the output register is “write-once”.

$p$  acts deterministically according to a **transition** function.

# Message system

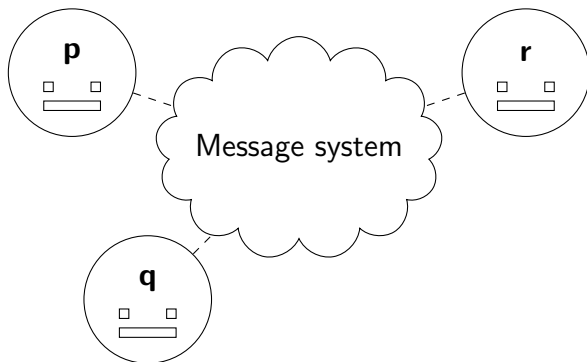
A **message** is a pair  $(p, m)$ , where  $p$  is the name of the destination process and  $m$  is a “message value” from a fixed universe  $M$ .

The **message system** maintains a multiset, called the **message buffer**, of messages that have been sent but not yet delivered. It supports two abstract operations:

- `send(p,m)`: Places  $(p, m)$  in the message buffer.
- `receive(p)`: Deletes some message  $(p, m)$  from the buffer and returns  $m$ , in which case we say  $(p, m)$  is **delivered**, or returns the special null marker  $\emptyset$  and leaves the buffer unchanged.

# Nondeterministic messaging

If  $\text{receive}(p)$  is performed infinitely many times, then every message  $(p, m)$  in the message buffer is eventually delivered.





# Configurations and steps

A **configuration**  $C$  of the system consists of the internal state of each process, together with the contents of the message buffer.

An **initial configuration** is one in which each process starts at an initial state and the message buffer is empty.

A **step** takes one configuration to another and consists of a primitive step by a single process  $p$ .

# Primitive steps and applicable events

A **primitive step** consists of two phases:

- 1 A `receive(p)` is performed on the message buffer in  $C$  to obtain a value  $m \in M \cup \{\emptyset\}$ .
- 2 Depending on  $p$ 's internal state in  $C$  and on  $m$ ,  $p$  enters a new internal state and sends a finite set of messages to other processes.

The step is completely determined by the pair  $e = (p, m)$ , which we call an **event**.

An event  $e$  that *could happen* at configuration  $C$  is called **applicable**, and the resulting configuration is denoted  $e(C)$ .

# Schedules and runs

A **schedule** from  $C$  is a finite or infinite sequence  $\sigma$  of applicable events, in turn, starting from  $C$ . The associated sequence of steps is called a **run**.

If  $\sigma$  is finite, we let  $\sigma(C)$  denote the resulting configuration, which is said to be **reachable** from  $C$ .

A configuration reachable from some initial configuration is said to be **accessible**.

# Partial correctness

A configuration  $C$  has **decision value**  $v$  if some process  $p$  is in a decision state with  $y_p = v$ .

## Definition (Partial correctness)

A consensus protocol is **partially correct** if:

- 1 No accessible configuration has more than one decision value.
- 2 For each  $v \in \{0, 1\}$ , some accessible configuration has decision value  $v$ .

# Total correctness in spite of one fault

A process  $p$  is **nonfaulty** in run if it takes infinitely many steps, otherwise it is **faulty**.

A run is **admissible** if at most one process is faulty and all messages sent to nonfaulty processes are eventually received.

A run is **deciding** if some process reaches a decision state.

## Definition (Total correctness in spite of one fault)

A consensus protocol  $P$  is **totally correct in spite of one fault** if it is partially correct and every admissible run is deciding.

# Main result

Theorem (Fischer, Lynch, Paterson 1985)

*No consensus protocol is totally correct in spite of one fault.*

A configuration is **bivalent** if the set of decision values of configurations reachable from it has 2 elements. It is instead **0-valent** or **1-valent** according to the corresponding value.

*Plan:* given an initial bivalent configuration, we construct an admissible run that at each stage results in another bivalent configuration.

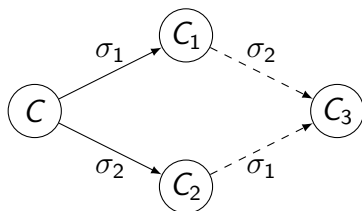
# Lemma 1

## Lemma

*Suppose that from some configuration  $C$ , the schedules  $\sigma_1, \sigma_2$  lead to configurations  $C_1, C_2$  respectively. If the sets of processes taking steps in  $\sigma_1$  and  $\sigma_2$ , respectively, are disjoint, then  $\sigma_2$  can be applied to  $C_1$  and  $\sigma_1$  can be applied to  $C_2$ , and both lead to the same configuration  $C_3$ .*

In other words: **schedules about disjoint processes commute with each other.**

# Proof of Lemma 1



Because the sets of processes are disjoint, an event in  $\sigma_1$  applicable to  $C$  is applicable to  $C_2$  as well.

Because of determinism, after all events are processed they must end up in the same state. □



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Let's call two initial configurations **adjacent** if they differ only in the initial value  $x_p$  of a single process  $p$ . Any two initial configurations are joined by a chain of initial configurations, each adjacent to the next.

# Proof of Lemma 2

Hence, there must exist a 0-valent initial configuration  $C_0$  adjacent to a 1-valent configuration  $C_1$ . Let  $p$  be the process in whose initial value they differ.

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Consider some admissible deciding run from  $C_0$  in which process  $p$  takes no steps, and let  $\sigma$  be the associated schedule. Then  $\sigma$  can be applied to  $C_1$  as well, and the corresponding configurations will be identical except for  $p$ 's internal state.

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Both runs reach the same decision value; if it is 1 then  $C_0$  was bivalent, if it is 0 then  $C_1$  was bivalent. □

# Lemma 3

## Lemma

*Let  $C$  be a bivalent configuration of  $P$ , and let  $e = (p, m)$  be an event that is applicable to  $C$ . Let  $\mathcal{C}$  be the set of configurations reachable from  $C$  without applying  $e$ , and let  $\mathcal{D} = e(\mathcal{C}) = \{e(E) \mid E \in \mathcal{C} \text{ and } e \text{ is applicable to } E\}$ . Then,  $\mathcal{D}$  contains a bivalent configuration.*

In other words: given a bivalent configuration and an event  $e$  applicable to it, **we construct another bivalent configuration having  $e$  as the last applied event.**

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Assume by contradiction that  $\mathcal{D}$  contains no bivalent configuration. We first show that  $\mathcal{D}$  contains both 0-valent and 1-valent configurations.



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Let  $E_i$  be an  $i$ -valent configuration reachable from  $C$  for  $i \in \{0, 1\}$ . If  $E_i \in \mathcal{C}$ , let  $F_i = e(E_i) \in \mathcal{D}$ . Otherwise  $e$  was applied in reaching  $E_i$ , and so there exists  $F_i \in \mathcal{D}$  from which  $E_i$  is reachable.

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In either case,  $F_i$  is  $i$ -valent since it is in  $\mathcal{D}$ , which contains no bivalent configuration by hypothesis.

# Proof of Lemma 3

We call two configurations **neighbors** if one results from the other in a single step. Then there exists neighbors  $C_0, C_1 \in \mathcal{C}$  such that  $D_i = e(C_i)$  is  $i$ -valent for  $i \in \{0, 1\}$ . We can assume that  $C_1 = e'(C_0)$  where  $e' = (p', m')$ .

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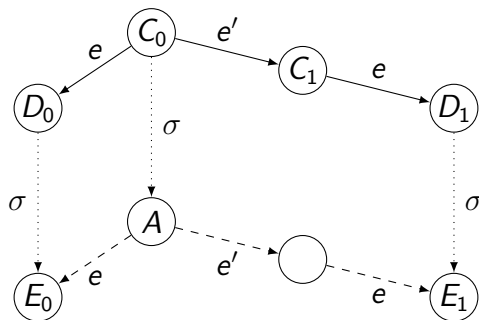
If  $p \neq p'$ , then  $D_1 = e'(D_0)$  by Lemma 1. This is impossible, since a successor of a 0-valent configuration is 0-valent.

If instead  $p = p'$ , then consider any finite deciding run from  $C_0$  in which  $p$  takes no steps. Let  $\sigma$  be the corresponding schedule, and let  $A = \sigma(C_0)$ .

# Proof of Lemma 3

By Lemma 1,  $\sigma$  is applicable to  $D_i$  and leads to an  $i$ -valent configuration  $E_i = \sigma(D_i)$  for  $i \in \{0, 1\}$ .

Also by Lemma 1,  $e(A) = E_0$  and  $e(e'(A)) = E_1$ . Hence,  $A$  is bivalent, a contradiction.



# Proof of main result

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Each stage consists of one or more process steps. A stage ends when the process that heads the queue takes a step in which, if its message queue was not empty at the start of the stage, its earliest message is received. This process is then moved to the back of the queue.



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In any infinite sequence of such stages every process takes infinitely many steps and receives every message sent to it. The run is therefore admissible.

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By Lemma 3, there is a bivalent configuration  $C'$  reachable from  $C$  by a schedule in which  $e$  is the last event applied.

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By Lemma 3, there is a bivalent configuration  $C'$  reachable from  $C$  by a schedule in which  $e$  is the last event applied.

Since each stage ends in a bivalent configuration, every stage in the construction of the infinite schedule succeeds.  $\square$



# Final remarks

We proved that no completely asynchronous consensus protocol can tolerate even a single *unannounced* process death.

This result doesn't apply to the case of *initially dead* processes: we can exhibit an algorithm that can tolerate any minority of initially dead processes.

Moreover, this result doesn't apply to the synchronous case, or the asynchronous case with failure detectors or timeouts.

# Bibliography

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-  Leslie Lamport, Robert Shostak, and Marshall Pease, *The byzantine generals problem*, ACM Transactions on Programming Languages and Systems (TOPLAS) **4** (1982), no. 3, 382–401.