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A Rank-Based Reward between a Principal and a Field of Agents: Application to Energy Savings

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Section 1

Introduction

Context

Obligations imposed by governments:

- ◇ In France: electricity providers ("*Obligés*") have a target of Energy Saving Certificates[1] to hold at a predetermined horizon ($\simeq$ 3 years).
  If they fail, they face financial penalties.

Existing incentives "Provider $\rightarrow$ customers":

- ○ Comparison to similar customers
  - ◇ EDF, Total, Engie, . . .
- ○ Reward/Bonus when reduction compared to past consumption
  - ◇ "SimplyEnergy"[2], "Plüm énergie"[3], "OhmConnect"[4]

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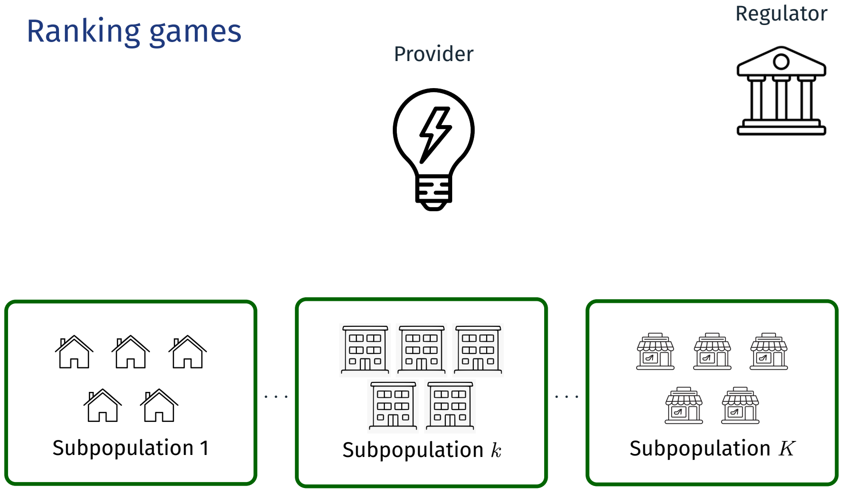
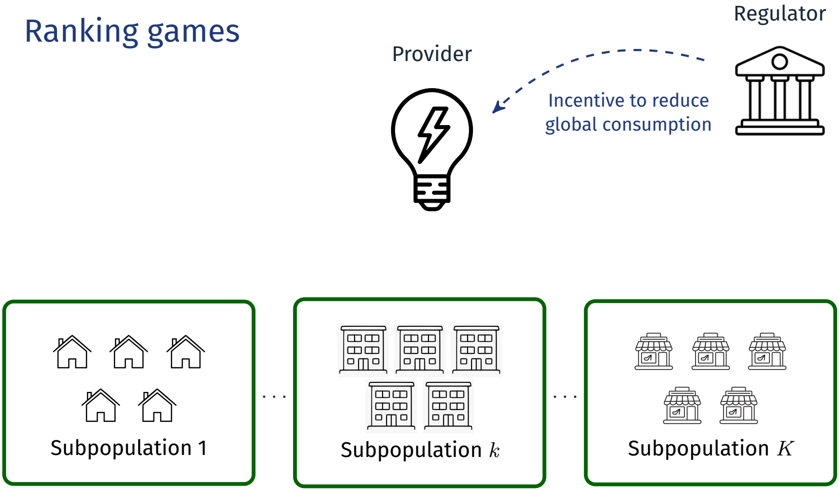
[1] www.powernext.com/french-energy-saving-certificates
[2] www.simplyenergy.com.au/residential/energy-efficiency/reduce-and-reward
[3] www.plum.fr/cagnotte/
[4] www.ohmconnect.com/

Ranking games



Mean-field assumption: Each subpopulation is composed of an *infinite* number of *indistinguishable* consumers

Ranking games



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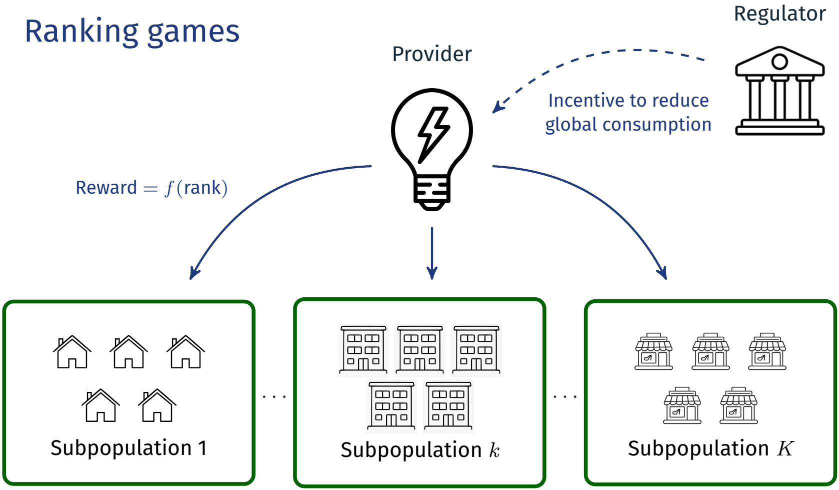

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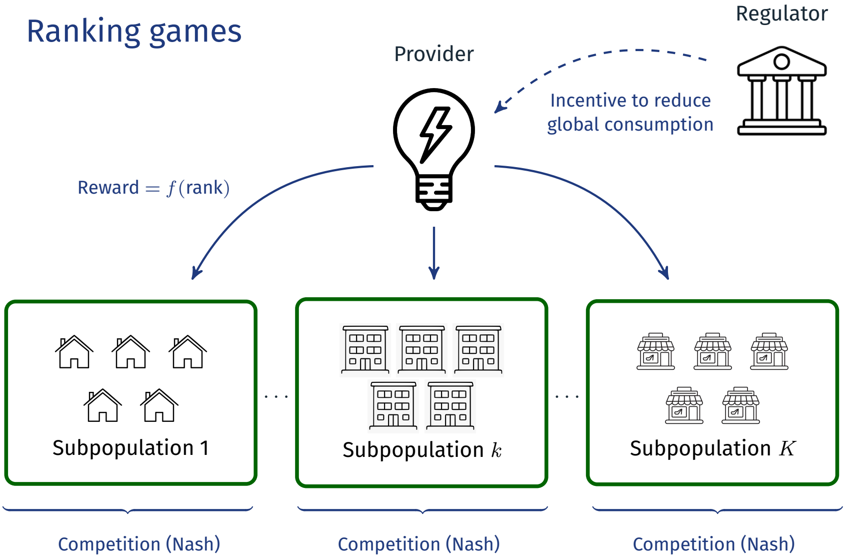

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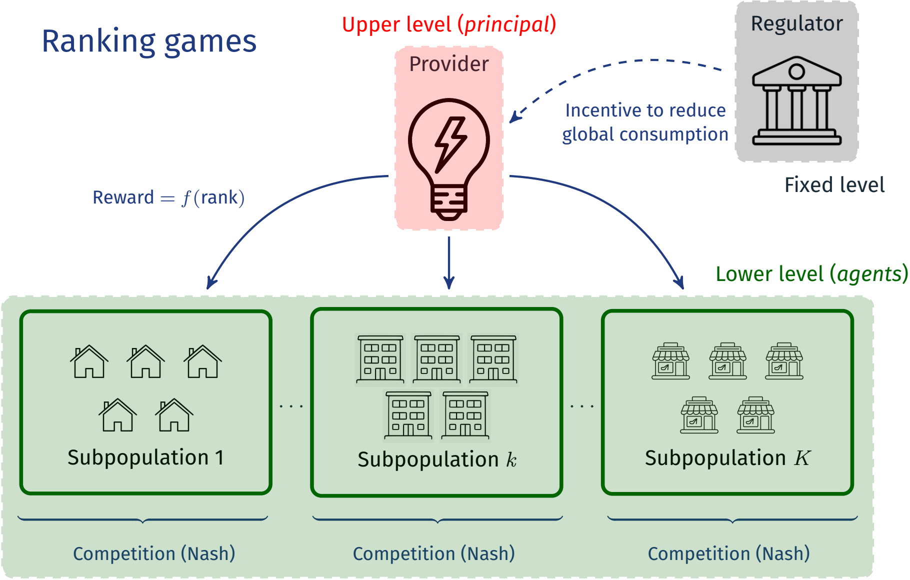

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Section 2

Agents' problem

A field of agents

- The population is divided into K clusters of *indistinguishable* consumers. Each cluster $k \in [K]$ represents a proportion ρ_k.
- $\{W_k\}_{1 \leq k \leq K}$ a family of K independent Brownian motions
- a_k a progressively measurable process satisfying $\mathbb{E} \int_0^T |a(s)| ds < \infty$
- $X_k^a(t)$ the *energy consumption* of a customer of k, forecasted at time t for consumption at $T > t$:

$$X_k^a(t) = X_k(0) + \int_0^t a_k(s) ds + \sigma_k \int_0^t dW_k(s), \quad X_k(0) = x_k^{\text{nom}}. \tag{1}$$

- $f_k^{\text{nom}}(\cdot)$ the p.d.f. of $X_k^a(T)$ under a zero effort ($a \equiv 0$):

$$f_k^{\text{nom}}(x) := \varphi\left(x;\, x_k^{\text{nom}}, \sigma_k \sqrt{T}\right) \quad .$$

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Interpretation:

- a_k is the consumer's effort to reduce his electricity consumption. Without effort, customers have a mean *nominal* consumption x_k^{nom}

Rank-based reward

Assumption: The reward R has the form

$$\mathbb{R} \times [0,1] \ni (x, r) \mapsto R(x, r) = B(r) - px \ , \tag{2}$$

- We call R the *total reward* and B the *additional reward*.
- When $R(x, r)$ is independent of x, the reward is *purely ranked-based*

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Interpretation:

- $-px$ represents the natural incentive to reduce the consumption, coming from the price p to consume one unit of energy
- $B(\cdot)$ is the additional financial reward based on their rank r.
- In the N-players game setting,
  - each cluster k contains N_k players
  - the ranking of a player i, consuming $X^i_k(T)$, is measured by

$$\frac{1}{N_k}\sum_{j=1}^{N_k} \mathbb{1}_{X^j_k(T) \leq X^i_k(T)}$$

$\Rightarrow$ the worst performer has the highest rank (highest consumption)

Mean-field game between consumers

Agents' problem:

$$V_k(R, \mu_k) := \sup_a \mathbb{E}\left[R_{\mu_k}(X_k^a(T)) - \underbrace{\int_0^T c_k a_k^2(t)\,dt}_{\text{cost of effort}} \right] , \qquad (P^{\text{cons}})$$

where $R_\mu(x) = R(x, F_\mu(x))$.

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Interpretation:

- The cost corresponds to the purchase of new equipment (new heating installation, isolation, ...).
- In exchange, the consumer receives $B(r)$, depending on his rank $r = F_{\mu_k}(x)$, where μ_k is the k-subpopulation's distribution.
- The quantity $V_k(R, \mu_k)$ is the *optimal utility* of an agent of k, *knowing* the provider's reward and the population distribution.

Nash Equilibrium

Theorem (Bayraktar and Zhang, 2021,Proposition 2.1)

Given $R \in \mathcal{R}$ and $\tilde{\mu}_k \in \mathcal{P}(\mathbb{R})$, let

$$\beta_k(\tilde{\mu}) = \int_{\mathbb{R}} f_k^{\text{nom}}(x) \exp\left(\frac{R_{\tilde{\mu}}(x)}{2c_k\sigma_k^2}\right) dx \quad (< \infty) \ . \tag{3}$$

Then, the optimal terminal distribution μ_k^* of the player of cluster k has p.d.f.

$$f_{\mu_k^*}(x) = \frac{1}{\beta(\tilde{\mu}_k)} f_k^{\text{nom}}(x) \exp\left(\frac{R_{\tilde{\mu}_k}(x)}{2c_k\sigma_k^2}\right) \quad , \tag{4}$$

and the optimal value is then $V_k(R, \tilde{\mu}_k) = 2c_k\sigma_k^2 \ln \beta_k(\tilde{\mu}_k)$.

Definition: $\mu_k \in \mathcal{P}(R)$ is an *equilibrium* if it is a fixed-point of

$$\Phi_k \ : \ \tilde{\mu}_k \mapsto \mu_k^* \ ,$$

with μ_k^* given by (4).

Nash Equilibrium

For purely ranked-based reward (Bayraktar and Zhang, 2021, Theorem 3.2)

The equilibrium ν_k is unique and the quantile is given by

$$q_{\nu_k}(r) = x_k^{\text{nom}} + \sigma_k \sqrt{T} N^{-1} \left(\frac{\int_0^r \exp\left(-\frac{B(z)}{2c_k\sigma_k^2}\right) dz}{\int_0^1 \exp\left(-\frac{B(z)}{2c_k\sigma_k^2}\right) dz} \right) . \tag{5}$$

Theorem

Let $R(x,r) = B(r) - px$. Then, the equilibrium μ_k is unique, and satisfies

$$q_{\mu_k}(r) = q_{\nu_k}(r) - \frac{pT}{2c_k} , \tag{6}$$

where ν_k is the (unique) equilibrium distribution for $p = 0$ (purely ranked-based reward), defined in (5).

$\Rightarrow$ add of a linear part in “x” acts as a shift on the probability density function.

Section 3

Principal's problem

Retailer's problem

For an equilibrium $(\mu_k)_{k\in[K]}$, the mean consumption is $m_{\mu_k} = \int_0^1 q_{\mu_k}(r)\,dr$, and the overall mean consumption is $m_\mu = \sum_{k\in[K]} \rho_k m_{\mu_k}$.

Principal's problem:

$$\max_{B\in\mathcal{R}_b^r} \left\{ s\left(m_\mu\right) + (p - c_r)m_\mu - \int_0^1 B(r)\,dr \;\middle|\; \begin{array}{l} \mu_k = \epsilon_k(B) \\ V_k(B) \geq V_k^{\text{pi}} \end{array} \right\} \qquad (P^{\text{ret}})$$

where

- $\mu_k = \epsilon_k(B)$ the *agents' equilibrium* with additional reward $B(\cdot)$,
- $s(\cdot)$ denotes the *valuation of the energy savings* (given by regulator),
- c_r denotes the *production cost* of energy,
- V^{pi} is the *reservation utility* (utility when $B \equiv 0$)

In the sequel, we denote by $g(\cdot)$ the function $g : m \mapsto s(m) - c_r m$.

Optimal reward – Homogeneous population ($K = 1$)

Assumption: The function $s : \mathbb{R} \to \mathbb{R}$ is supposed to be decreasing, concave and differentiable with $\|s'(m)\| \leq M_s$.

Using the characterization of the equilibrium,

$$B(r) = V^{\mathsf{pi}} + 2c\sigma^2 \ln\left(\zeta_\mu(q_\mu(r))\right) + pq_\mu(r) \ ,$$

with $\zeta_\mu := f_\mu / f^{\mathsf{nom}}$.

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Reformulation in the distribution space:

$$(P^{\text{ret}}) \begin{cases} \max\limits_{\mu} & g\left(\int_{-\infty}^{+\infty} y f_\mu(y)\,dy\right) - V^{\text{pi}} - 2c\sigma^2 \int_{-\infty}^{+\infty} \ln\left(\frac{f_\mu(y)}{f^{\text{nom}}(y)}\right) f_\mu(y)\,dy \\ \text{s.t.} & \int_{-\infty}^{+\infty} f_\mu(y)\,dy = 1 \\ & y \mapsto \ln\left(\frac{f_\mu(y)}{f^{\text{nom}}(y)}\right) + \frac{p}{2c\sigma^2} y \text{ bounded and decreasing} \end{cases}$$

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~~Reformulation~~ in the distribution space:

Relaxation

$$\begin{array}{l} \cancel{(P^{\mathsf{ret}})} \\ (\widetilde{P}^{\mathsf{ret}}) \end{array} \begin{cases} \max\limits_{\mu} \quad g\left(\int_{-\infty}^{+\infty} y f_\mu(y)\,dy\right) - V^{\mathsf{pi}} - 2c\sigma^2 \int_{-\infty}^{+\infty} \ln\left(\dfrac{f_\mu(y)}{f^{\mathsf{nom}}(y)}\right) f_\mu(y)\,dy \\ \text{s.t.} \quad \int_{-\infty}^{+\infty} f_\mu(y)\,dy = 1 \\ \qquad \cancel{y \mapsto \ln\left(\dfrac{f_\mu(y)}{f^{\mathsf{nom}}(y)}\right) + \dfrac{p}{2c\sigma^2}\, g \text{ bounded and decreasing}} \end{cases}$$

Optimal reward – Homogeneous population ($K = 1$)

Lemma

The optimal distribution μ^* for $(\widetilde{P}^{\text{ret}})$ satisfies the following equation:

$$f_\mu(y) \propto f^{\text{nom}}(y) \exp\left(y \frac{g'(m_\mu)}{2c\sigma^2} \right) \tag{7}$$

Sketch of proof: Use Karush-Kuhn-Tucker conditions, sufficient for $(\widetilde{P}^{\text{ret}})$

Theorem

Let $\delta(m) = p - c_r + s'(m)$. The distribution $\mu^* \hookrightarrow \mathcal{N}(m^*, \sigma\sqrt{T})$, where m^* satisfies

$$m - x^{\text{pi}} = \frac{T}{2c}\delta(m) \ , \tag{8}$$

is optimal for $(\widetilde{P}^{\text{ret}})$. Moreover, the associated reward B^* is

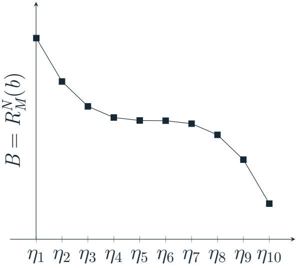
$$B^*(r) = \frac{c}{T}\left[(x^{\text{pi}})^2 - (m^*)^2 \right] + q_{\mu^*}(r)\delta(m^*) \ . \tag{9}$$

Remark: The function $\delta(\cdot)$ is viewed as the *reduction desire* of the provider.

Section 4

Numerical results

Algorithm



Restriction to piecewise linear reward:

- For $N \in \mathbb{N}$, $\Sigma_N := \{0 = \eta_1 < \eta_2 < \ldots < \eta_N = 1\}$.
- For $M \in \mathbb{R}_+$, we define the class of bounded piece-wise linear rewards adapted to Σ_N as

$$\widehat{\mathcal{R}}_M^N := \left\{ r \in [0,1] \mapsto \sum_{i=1}^{N-1} \mathbb{1}_{r \in [\eta_i, \eta_{i+1}[} \left[b_i + \frac{b_{i+1} - b_i}{\eta_{i+1} - \eta_i}(r - \eta_i) \right] \;\middle|\; \begin{array}{l} b \in [-M, M]^N \\ b_1 \geq \ldots \geq b_N \end{array} \right\} .$$

- $R_M^N(b)$ is the reward function obtained as a linear interpolation of b.

Optimization by a black-box solver:

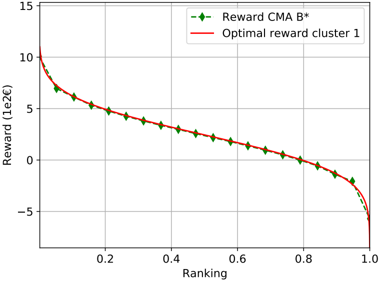
- We construct an oracle $b \in \mathbb{R}^N \mapsto \pi^{\text{ret}}(b)$, where $\pi^{\text{ret}}(b)$ is the retailer objective.
- We use a black-box solver, here CMA-ES (Hansen, 2006).

Instance

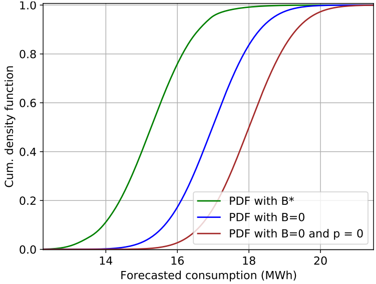
Parameter	Segment 1	Segment 2	Unit
T	3		years
p	0.17		€/kWh
c_r	0.15		€/kWh
$X(0)$	18	12	MWh
σ	0.6	0.3	MWh
c	2.5	5	€ [MWh]$^{-2}$ [years]^2
s	$m \mapsto 0.1m^2$		€
ρ	0.5	0.5	-

Table: Parameters of the instance

Results



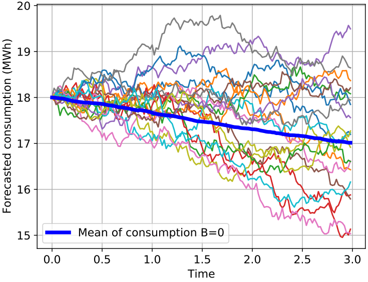
(a) Analytic optimal reward in red, compared to the reward function found by CMA



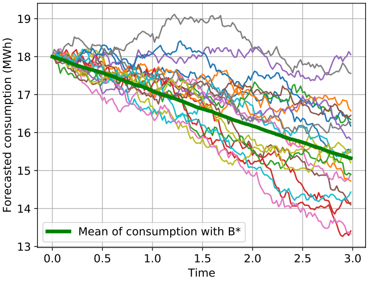
(b) Comparison of the three CDF: nominal, price incentive and with the optimal reward

Figure: Optimization in the homogeneous case

Results



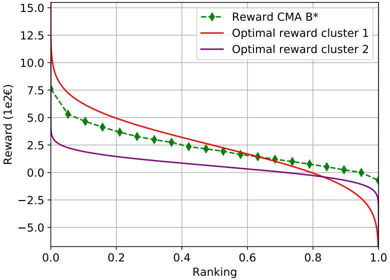
(a) Trajectories without additional reward



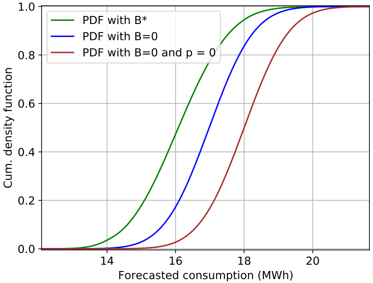
(b) Trajectories with optimal control from mean-field approximation

Figure: Trajectories for 20 consumers (homogeneous case)

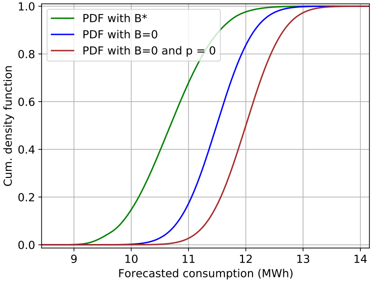
Results



(a) Red and purple rewards are the optimal reward in the homogeneous case. The reward function found by CMA is displayed in green.

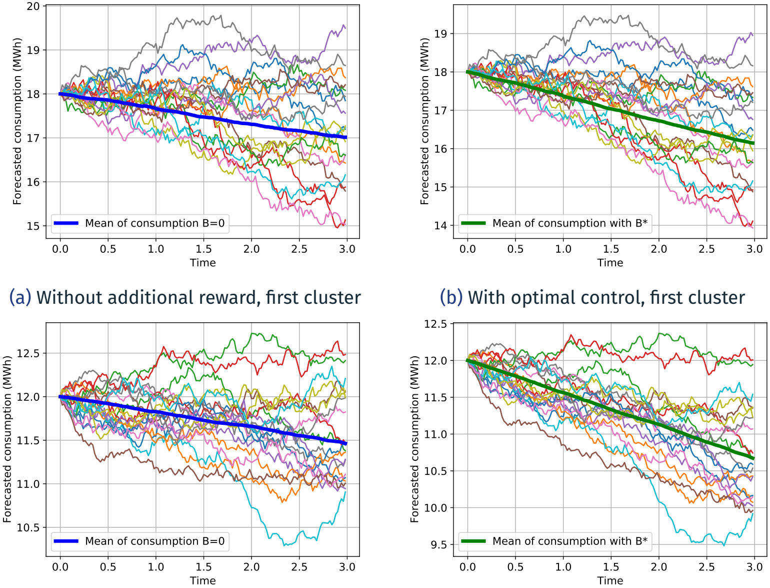


(b) Comparison of the three CDF (first cluster)



(c) Comparison of the three CDF (second cluster)

Figure: Optimization in the heterogeneous case



(a) Without additional reward, first cluster

(b) With optimal control, first cluster

(c) Without additional reward, second cluster

(d) With optimal control, second cluster

Figure: Trajectories for 20 consumers (heterogeneous case)

Section 5

Conclusion

Conclusion

Conclusion

- Characterization of mean-field equilibrium
- Closed-form formula of the optimal reward for homogeneous population
- Numerical computation of optimal reward for heterogeneous population
- Results on Energy Savings

Thank you for your attention !

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